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VERONICA ANAYA, MOSTAFA BENDAHMANE,
DAVID MORA, MAURICIO SEPÚLVEDA

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A Virtual Element Method for a Nonlocal FitzHugh-Nagumo Model of Cardiac Electrophysiology

Verónica Anaya^a, Mostafa Bendahmane^b, David Mora^{a,c}, Mauricio Sepúlveda^{c,d}

^a*GIMNAP, Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile.*

^b*Institut de Mathématiques de Bordeaux, Université de Bordeaux, Talence, France.*

^c*Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, Concepción, Chile.*

^d*Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile.*

Abstract

We present a Virtual Element Method (VEM) for a nonlocal reaction-diffusion system of the cardiac electric field. To this system, we analyze an $H^1(\Omega)$ -conforming discretization by means of VEM which can make use of general polygonal meshes. Under standard assumptions on the computational domain, we establish the convergence of the discrete solution by considering a series of a priori estimates and by using a general L^p compactness criterion. Moreover, we obtain optimal order space-time error estimates in the L^2 norm. Finally, we report some numerical tests supporting the theoretical results.

Keywords: Virtual element method, FitzHugh–Nagumo equations, Convergence, Error estimates

AMS Subject Classification: 65M60, 65M15, 35Q92

1. Introduction

Reaction-diffusion systems appear in models of different areas such as medicine, engineering, biology, physics, etc. The study of this kind of models has attracted too much attention for many years, systems with different types of diffusion, for example: constant, nonlocal, cross. Mathematical models related with electrical activity in the heart (cardiac tissue) are becoming a powerful tools to study and understand many types of heart disease, as for example irregular heart rhythm.

The reaction-diffusion system of FitzHugh-Nagumo type [35, 45] is one of the most relevant and well-known generic model in physiology which describes complex wave phenomena in excitable or oscillatory media. This model is a reaction-diffusion system which is a simplification of the famous Hodgkin-Huxley model, which has been used to describe the propagation of the electrical potential in cardiac tissue [38, 46]. The FitzHugh-Nagumo reaction-diffusion system consists of one nonlinear parabolic partial differential equation (PDE) which describes the dynamic of the membrane potential, coupled with an ordinary differential equation which models the ionic currents associated with the reaction term. The main difficulties associated to solve this system are: the coupling of the equations, through a nonlinear term and the regularity of the solution of the system is low.

In this paper, we analyze a *Virtual Element Method* for a nonlinear parabolic problem arising in cardiac models (electrophysiology) with nonlocal diffusion (see system (2.1) below). In our study, the self-diffusion coefficient is assumed depending on the total of electrical potential in the heart. The Virtual Element

Email addresses: vanaya@ubiobio.cl (Verónica Anaya), mostafa.bendahmane@u-bordeaux.fr (Mostafa Bendahmane), dmora@ubiobio.cl (David Mora), mauricio@ing-mat.udec.cl (Mauricio Sepúlveda)

Method (VEM), recently introduced in [8, 10], is a generalization of the Finite Element Method which is characterized by the capability of dealing with very general polygonal/polyhedral meshes. In recent years, the interest in numerical methods that can make use of general polygonal/polyhedral meshes for the numerical solution of partial differential equations has undergone a significant growth; this because of the high flexibility that this kind of meshes allow in the treatment of complex geometries. Among the large number of papers on this subject, we cite as a minimal sample [12, 29, 34, 52, 53].

Although the VEM is very recent, it has been applied to a large number of problems; for instance, VEM for Stokes, Brinkman, Cahn-Hilliard, plates bending, advection-diffusion, Helmholtz, parabolic, and hyperbolic problems have been introduced in [4, 5, 15, 17, 24, 19, 21, 26, 27, 30, 51, 54, 55, 56], VEM for spectral problems in [18, 37, 42, 44], VEM for linear and non-linear elasticity in [6, 9, 13, 36, 57], whereas a posteriori error analysis have been developed in [16, 20, 28, 43].

Over the past years, some papers related to numerical tools for solving this model and its variations have appeared. For example, in [33] a continuous in space and discontinuous in time Galerkin method of arbitrary order has been developed, under minimal regularity assumptions, space-time error estimates are established in the natural norms. In [39] some estimates in the L^2 norm for semi-discrete Galerkin approximations for the FitzHugh-Nagumo model are derived. A finite difference method has been presented in [7], Chebyshev multidomain method has been presented in [49], fully space-time adaptive multiresolution methods based on the finite volume method and Barkleys method for simulating the complex dynamics of waves in excitable media in [25]. Finally, in [48] has been presented other methods related to the numerical analysis of general semilinear parabolic PDE.

Numerical methods to solve these kind of models have limitations in the range of applicable meshes. In particular, finite element methods rely on triangular (simplicial) or quadrilateral meshes. Moreover, the classical finite volume method has some restriction on the admissible meshes (for instance, orthogonality constraints). However, in complex simulations like fluid-structure interaction, phase change, medical applications, and many others, the geometrical complexity of the domain is a relevant issue when partial differential equations have to be solved on a good quality mesh; hence, it can be convenient to use more general polygonal/polyhedral meshes. Thus, in the present contribution, we are going to introduce and analyze a VEM which has the advantage of using general polygonal meshes to solve a nonlinear parabolic FitzHugh-Nagumo system, where the diffusion coefficient depends on a nonlocal quantity. The study of nonlocal diffusion problems has received considerable attention in recent years since they appear in important physical and biological applications [2, 3, 31, 32]. There are models of the FitzHugh-Nagumo type that also take into account the nonlocal diffusion phenomena, for example in [41] is considered a diffusive nonlocal term as fractional diffusion, in [50] is taken a nonlocal reactive term.

The aim of this paper is to introduce and analyze a conforming $H^1(\Omega)$ -VEM which applies to general polygonal meshes, for the two-dimensional nonlocal reaction-diffusion FitzHugh-Nagumo equations. We propose a space discretization by means of VEM, which is based on the discrete space introduced in [1] for the linear reaction-diffusion equation. We construct a proper L^2 -projection operator, that is used to approximate the bilinear form that appears for the time derivative discretization, which is obtained by a classical backward Euler method. We also use that projection to discretize the nonlocal term presented in the system. We prove that the fully discrete scheme is well posed and using standard space and time translates together with a priori estimates for the discrete solution, it is established convergence of the discrete scheme to the weak solution of the model. Under rather mild assumptions on the polygonal meshes, we establish optimal order space-time error estimates in the L^2 norm.

The structure of the paper is organized as follows: in Section 2, we give some preliminaries and assumptions on the data. Moreover, we introduce the concept of weak solution. In Section 3, we propose the semi-discrete and fully-discrete virtual element method. In Section 4, we prove the existence and convergence of the discrete solution. In Section 5, we give error estimates, and finally, in Section 6, some numerical results.

2. Model problem and weak solution

Fix a final time $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary Σ and outer unit normal vector \mathbf{n} . For all $(x, t) \in \Omega_T := \Omega \times (0, T)$, $v = v(x, t)$ and $w = w(x, t)$ stand for the transmembrane potential and the gating variable, respectively. The governing equations of the nonlocal reaction-diffusion FitzHugh-Nagumo system are:

$$\begin{cases} \partial_t v - D \left(\int_{\Omega} v(x, t) dx \right) \Delta v + I_{\text{ion}}(v, w) = I_{\text{app}}(x, t) & (x, t) \in \Omega_T, \\ \partial_t w - H(v, w) = 0 & (x, t) \in \Omega_T, \\ D \left(\int_{\Omega} v(x, t) dx \right) \nabla v \cdot \mathbf{n} = 0 & (x, t) \in \Sigma_T := \Sigma \times (0, T), \\ v(x, 0) = v_0(x) & x \in \Omega, \\ w(x, 0) = w_0(x) & x \in \Omega. \end{cases} \quad (2.1)$$

Herein, I_{app} is the stimulus. In this work, the diffusion rate $D > 0$ is supposed to depend on the whole of the transmembrane potential in the domain rather than on the local diffusion, i.e. the diffusion of the transmembrane potential is guided by the global state of the potential in the medium. We assume that $D : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the following: there exist constants $d_1, d_2 > 0$ such that

$$d_1 \leq D \quad \text{and} \quad |D(I_1) - D(I_2)| \leq d_2 |I_1 - I_2| \quad \text{for all } I_1, I_2 \in \mathbb{R}. \quad (2.2)$$

Now, we make some assumptions on the data of the nonlocal FitzHugh-Nagumo model. For the ionic current $I_{\text{ion}}(v, w)$, we assume that it can be decomposed into $I_{1,\text{ion}}(v)$ and $I_{2,\text{ion}}(w)$, where $I_{\text{ion}}(v, w) = I_{1,\text{ion}}(v) + I_{2,\text{ion}}(w)$. We assume that $I_{1,\text{ion}}, I_{2,\text{ion}} : \mathbb{R} \rightarrow \mathbb{R}$ and $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions, and that there exist constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ such that

$$\begin{aligned} a) \quad & \frac{1}{\alpha_1} |v|^4 \leq |I_{1,\text{ion}}(v)v| \leq \alpha_2 (|v|^4 + 1), \\ b) \quad & |I_{2,\text{ion}}(w)| \leq \alpha_3 (|w| + 1), \\ c) \quad & \forall z, s \in \mathbb{R} \quad (I_{1,\text{ion}}(z) - I_{1,\text{ion}}(s))(z - s) \geq -C_h |z - s|^2, \\ d) \quad & |H(v, w)| \leq \alpha_4 (|v| + |w| + 1). \end{aligned} \quad (2.3)$$

It is well known that the above assumptions are fulfilled if the functions are specified as follows:

$$H(v, w) = av - bw, \quad (2.4)$$

and

$$I_{\text{ion}}(v, w) = -\lambda(w - v(1 - v)(v - \theta)), \quad (2.5)$$

where a, b, λ, θ are given parameters.

Next, we will use the following spaces: by $H^m(\Omega)$, we denote the usual Sobolev space of order m . Given $T > 0$ and $1 \leq p \leq \infty$, $L^p(0, T; \mathbb{R})$ denotes the space of L^p integrable functions from the interval $[0, T]$ into \mathbb{R} . The weak solution to the model (2.1) is defined as follows.

Definition 2.1 (Weak solution). A weak solution to the system (2.1) is a double function (v, w) such that $v \in L^2(0, T; H^1(\Omega)) \cap L^4(\Omega_T)$, $\partial_t v \in L^2(0, T; (H^1(\Omega))') + L^{\frac{4}{3}}(\Omega_T)$, $w \in C([0, T]; L^2(\Omega))$, and satisfying the following weak formulation

$$\begin{aligned} \iint_{\Omega_T} \partial_t v \varphi + \int_0^T D \left(\int_{\Omega} v(x, t) dx \right) \int_{\Omega} \nabla v \cdot \nabla \varphi + \iint_{\Omega_T} I_{\text{ion}}(v, w) \varphi &= \iint_{\Omega_T} I_{\text{app}}(x, t) \varphi, \\ \iint_{\Omega_T} \partial_t w \phi - \iint_{\Omega_T} H(v, w) \phi &= 0, \end{aligned} \quad (2.6)$$

for all $\varphi \in L^2(0, T; H^1(\Omega)) \cap L^4(\Omega_T)$ and $\phi \in C([0, T]; L^2(\Omega))$.

Remark 2.1. Note that, in view of the conditions stated in (2.3), we can easily check that Definition 2.1 makes sense. Furthermore, observe that Definition 2.1 implies $v \in C([0, T]; L^2(\Omega))$ (see [47]).

3. Virtual element scheme and main result

In this section, first we recall the mesh construction and the assumptions consider to introduce the discrete virtual element space. Then, we present the virtual element approximation of the FitzHugh-Nagumo model. In the sequel, the existence and uniqueness is proved.

3.1. The VEM semi-discrete problem

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K . Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K \in \mathcal{T}_h} h_K$. In what follows, we denote by N_K the number of vertices of K , by e a generic edge of \mathcal{T}_h and for all $e \in \partial K$, we define a unit normal vector \mathbf{n}_K^e that points outside of K .

For the analysis, we will make the following assumptions as in [8, 18]: there exists a positive real number $C_{\mathcal{T}}$ such that, for every h and every $K \in \mathcal{T}_h$,

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;

A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}}h_K$.

For any subset $S \subseteq \mathbb{R}^2$ and nonnegative integer k , we indicate by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on S .

Now, we consider a simple polygon K (meaning open simply connected set whose boundary is a non-intersecting line made of a finite number of straight line segments), and we start by introducing a preliminary virtual element space. For all $K \in \mathcal{T}_h$, the local space $V_{k|K}$ is defined by

$$V_{k|K} := \{\varphi \in H^1(K) \cap C^0(K) : \varphi|_e \in \mathbb{P}_k(e) \forall e \in \partial K, \Delta \varphi \in \mathbb{P}_k(K)\}.$$

Now, we introduce the following set of linear operators from $V_{k|K}$ into \mathbb{R} . For all $\varphi \in V_{k|K}$:

- D_1 : The values of φ at the vertices of K ;
- D_2 : Values of φ at $k - 1$ distinct points in e , for all $e \in \partial K$;
- D_3 : All moments $\int_K \varphi p \, dx$, for all $p \in \mathbb{P}_{k-2}(K)$.

Now, we split the bilinear form $a(\cdot, \cdot) := (\nabla \cdot, \nabla \cdot)_{0,\Omega}$,

$$a(v, \varphi) := \sum_{K \in \mathcal{T}_h} a^K(v, \varphi), \quad \forall v, \varphi \in H^1(\Omega),$$

where

$$a^K(v, \varphi) := \int_K \nabla v \cdot \nabla \varphi, \quad \forall v, \varphi \in H^1(\Omega).$$

For the analysis we will introduce the following broken seminorm:

$$|\varphi|_{1,h} := \left(\sum_{K \in \mathcal{T}_h} |\varphi|_{1,K}^2 \right)^{1/2}.$$

Let $\Pi_{K,k} : V_{k|K} \rightarrow \mathbb{P}_k(K)$ be the projection operator defined by

$$\begin{cases} a^K(\Pi_{K,k}v, q) = a^K(v, q) & \forall q \in \mathbb{P}_k(K), \\ P_0(\Pi_{K,k}v) = P_0v, \end{cases}$$

where P_0 can be taken as

$$\begin{cases} P_0v := \frac{1}{N_K} \sum_{i=1}^{N_K} v(V_i) & k = 1, \\ P_0v := \frac{1}{|K|} \int_K v dx & k > 1, \end{cases}$$

with V_i the vertices of K , $1 \leq i \leq N_K$ where N_K is the number of vertices in K .

Using an integration by parts, it is easy to check that, for any $\varphi \in V_{k|K}$, the values of the linear operators D_1, D_2 and D_3 given before are sufficient in order to compute $\Pi_{K,k}$. As a consequence, the projection operator $\Pi_{K,k}$ depends only on the values of the operators D_1, D_2 and D_3 .

Now, we introduce our virtual local space

$$W_{k|K} := \left\{ \varphi \in V_{k|K} : \int_K (\Pi_{K,k}\varphi)q dx = \int_K \varphi q dx \quad \forall q \in \mathbb{P}_k/\mathbb{P}_{k-2}(K) \right\},$$

where the symbol $\mathbb{P}_k/\mathbb{P}_{k-2}(K)$ denotes the polynomials of degree k living on K that are L^2 -orthogonal to all polynomials of degree $k-2$ on K . We observe that, since $W_{k|K} \subset V_{k|K}$, the operator $\Pi_{K,k}$ is well defined on $W_{k|K}$ and computable only on the basis of the values of the operators D_1, D_2 and D_3 .

The global discrete space will be

$$W_h := \{\varphi \in H^1(\Omega) : \varphi|_K \in W_{k|K}, \quad \forall K \in \mathcal{T}_h\}.$$

In agreement with the local choice of the degrees of freedom, in W_h we choose the following degrees of freedom:

- DG_1 : The values of φ at the vertices of \mathcal{T}_h ;
- DG_2 : Values of φ at $k-1$ distinct points in e , for all $e \in \mathcal{T}_h$;
- DG_3 : All moments $\int_K \varphi p dx$, for all $p \in \mathbb{P}_{k-2}(K)$ on each element $K \in \mathcal{T}_h$.

On the other hand, let $S^K(\cdot, \cdot)$ and $S_0^K(\cdot, \cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy

$$c_0 a^K(\varphi_h, \varphi_h) \leq S^K(\varphi_h, \varphi_h) \leq c_1 a^K(\varphi_h, \varphi_h) \quad \forall \varphi_h \in V_{k|K} \quad \text{with} \quad \Pi_{K,k}\varphi_h = 0, \quad (3.1)$$

$$\tilde{c}_0(\varphi_h, \varphi_h)_{0,K} \leq S_0^K(\varphi_h, \varphi_h) \leq \tilde{c}_1(\varphi_h, \varphi_h)_{0,K} \quad \forall \varphi_h \in V_{k|K}, \quad (3.2)$$

for some positive constants c_0, c_1, \tilde{c}_0 and \tilde{c}_1 independent of K .

We define the local discrete bilinear and trilinear forms:

$$\begin{aligned} a_h^K(\cdot, \cdot) : W_h \times W_h &\rightarrow \mathbb{R}, & m_h^K(\cdot, \cdot) : W_h \times W_h &\rightarrow \mathbb{R}, \\ b_h^K(\cdot, \cdot, \cdot) : W_h \times W_h \times W_h &\rightarrow \mathbb{R}, & c_h^K(\cdot, \cdot, \cdot) : W_h \times W_h \times W_h &\rightarrow \mathbb{R}, \end{aligned}$$

as follow, for all $v_h, w_h, \varphi_h \in W_{k|K}$:

$$a_h^K(v_h, \varphi_h) := a^K(\Pi_{K,k}v_h, \Pi_{K,k}\varphi_h) + S^K(v_h - \Pi_{K,k}v_h, \varphi_h - \Pi_{K,k}\varphi_h),$$

$$m_h^K(v_h, \varphi_h) := (\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 \varphi_h) + S_0^K(v_h - \Pi_{K,k}^0 v_h, \varphi_h - \Pi_{K,k}^0 \varphi_h),$$

$$b_h^K(v_h, w_h, \varphi_h) := \int_K I_{ion}(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h) \Pi_{K,k}^0 \varphi_h,$$

$$c_h^K(v_h, w_h, \varphi_h) := \int_K H(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h) \Pi_{K,k}^0 \varphi_h,$$

where $\Pi_{K,k}^0 : W_{k|K} \rightarrow \mathbb{P}_k(K)$ is the standard L^2 -projection operator which is computable on the basis of the degrees of freedom (see [1, 56]).

We observe that for all $K \in \mathcal{T}_h$ it holds:

- k -consistency: for all $p \in \mathbb{P}_k(K)$ and for all $\varphi_h \in W_{k|K}$

$$\begin{aligned} a_h^K(p, \varphi_h) &= a^K(p, \varphi_h), \\ m_h^K(p, \varphi_h) &= (p, \varphi_h)_{0,K}. \end{aligned} \quad (3.3)$$

- stability: there exist four positive constants $\alpha', \alpha'', \beta', \beta''$, independent of h , such that for all $\varphi_h \in W_{k|K}$

$$\begin{aligned} \alpha' a^K(\varphi_h, \varphi_h) &\leq a_h^K(\varphi_h, \varphi_h) \leq \alpha'' a^K(\varphi_h, \varphi_h), \\ \beta' (\varphi_h, \varphi_h)_{0,K} &\leq m_h^K(\varphi_h, \varphi_h) \leq \beta'' (\varphi_h, \varphi_h)_{0,K}. \end{aligned} \quad (3.4)$$

Then, we set for all $v_h, w_h, \varphi_h \in W_h$,

$$a_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} a_h^K(v_h, \varphi_h), \quad m_h(v_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} m_h^K(v_h, \varphi_h),$$

$$b_h(v_h, w_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} b_h^K(v_h, w_h, \varphi_h), \quad c_h(v_h, w_h, \varphi_h) := \sum_{K \in \mathcal{T}_h} c_h^K(v_h, w_h, \varphi_h).$$

We discretize the nonlocal diffusion term using the L^2 -projection as follows:

$$J(v_h) := \int_{\Omega} v_h = \sum_{K \in \mathcal{T}_h} \int_K \Pi_{K,k}^0 v_h, \quad v_h \in W_h. \quad (3.5)$$

For the right-hand side, we assume $I_{app}(x, t) \in L^2(\Omega_T)$ and we set

$$I_{app,h}(t) = \Pi_k^0 I_{app}(\cdot, t) \quad \text{for a.e. } t \in (0, T),$$

where we have introduced Π_k^0 as the following operator which is defined in L^2 by

$$(\Pi_k^0 g)|_K := \Pi_{K,k}^0 g \quad \text{for all } K \in \mathcal{T}_h \quad (3.6)$$

with $\Pi_{K,k}^0$ the $L^2(K)$ -projection.

Now, we note that from the symmetry of $a_h(\cdot, \cdot)$ and $m_h(\cdot, \cdot)$ and the stability conditions stated before imply the continuity of a_h and m_h . In fact, for all $v_h, \varphi_h \in W_h$:

$$\begin{aligned} |a_h(v_h, \varphi_h)| &\leq C \|v_h\|_{H^1(\Omega)} \|\varphi_h\|_{H^1(\Omega)}, \\ |m_h(v_h, \varphi_h)| &\leq C \|v_h\|_{L^2(\Omega)} \|\varphi_h\|_{L^2(\Omega)}. \end{aligned} \quad (3.7)$$

The semidiscrete VEM formulation reads as follows. For all $t > 0$, find $v_h, w_h \in L^2(0, T; W_h)$ with $\partial_t v_h, \partial_t w_h \in L^2(0, T; W_h)$ such that

$$\begin{cases} m_h(\partial_t v_h(t), \varphi_h) + D(J(v_h(t))) a_h(v_h(t), \varphi_h) + b_h(v_h(t), w_h(t), \varphi_h) = (I_{app,h}(t), \varphi_h)_{0,\Omega} \\ m_h(\partial_t w_h(t), \phi_h) - c_h(v_h(t), w_h(t), \phi_h) = 0, \end{cases} \quad (3.8)$$

for all $\varphi_h, \phi_h \in W_h$. Additionally, we set $v_h(0) = v_h^0$ and $w_h(0) = w_h^0$. A classical backward Euler integration method is employed for the time discretization of (3.8) with time step $\Delta t = T/N$. This results in the following fully discrete method: find $v_h^n, w_h^n \in W_h$ such that

$$\begin{cases} m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + D(J(v_h^n)) a_h(v_h^n, \varphi_h) + b_h(v_h^n, w_h^n, \varphi_h) = (I_{app,h}^n, \varphi_h)_{0,\Omega} \\ m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h(v_h^n, w_h^n, \phi_h) = 0, \end{cases} \quad (3.9)$$

for all $\varphi_h, \phi_h \in W_h$, for all $n \in \{1, \dots, N\}$; the initial condition takes the form v_h^0, w_h^0 and $I_{app,h}^n := I_{app,h}(t_n)$ with $t_n := n\Delta t$, for $n = 0, \dots, N$.

Our main result is the following theorem:

Theorem 3.1. *Assume that (2.2) and (2.3) hold. If $v_0(x) \in L^2(\Omega)$, $w_0(x) \in L^2(\Omega)$, and $I_{app}(x, t) \in L^2(\Omega_T)$, then the virtual element solution $\mathbf{u}_h^n = (v_h^n, w_h^n)$, generated by (3.9), converges along a subsequence to $\mathbf{u} = (v, w)$ as $h \rightarrow 0$, where \mathbf{u} is a weak solution of (2.1). Moreover, the weak solution is unique.*

In the next section, we prove Theorem 3.1 by establishing the convergence of the virtual element solution (v_h^n, w_h^n) , based on a priori estimates and the compactness method. Moreover, we provide error estimates in Section 5.

4. Existence of solution for the virtual element scheme

The existence result for the virtual element scheme is given in the following proposition.

Proposition 4.1. *Assume that (2.2) and (2.3) hold. Then, the problem (3.9) admits a discrete solution $\mathbf{u}_h^n = (v_h^n, w_h^n)$.*

Proof. The existence of \mathbf{u}_h^n is shown by induction on $n = 0, \dots, N$. For $n = 0$, solution is given by $\mathbf{u}_h^0 = (v_h(0), w_h(0)) = (v_h^0, w_h^0)$. Assume that \mathbf{u}_h^{n-1} exists. Choose $[\![\cdot, \cdot]\!]$ as the scalar product on $H^1(\Omega) \times L^2(\Omega)$. We are looking for a solution \mathbf{u}_h^n to $[\![L(\mathbf{u}_h^n), \Phi_h]\!] = 0$, where the operator $L : W_h \times W_h \rightarrow W_h \times W_h$ is given by

$$\begin{aligned} [\![L(\mathbf{u}_h^n), \Phi_h]\!] &= m_h\left(\frac{v_h^n - v_h^{n-1}}{\Delta t}, \varphi_h\right) + D(J(v_h^n)) a_h(v_h^n, \varphi_h) + b_h(v_h^n, w_h^n, \varphi_h) - (I_{app,h}(t_n), \varphi_h)_{0,\Omega} \\ &\quad + m_h\left(\frac{w_h^n - w_h^{n-1}}{\Delta t}, \phi_h\right) - c_h(v_h^n, w_h^n, \phi_h), \end{aligned}$$

for all $\Phi_h := (\varphi_h, \phi_h) \in W_h \times W_h$. Note that the continuity of the operator L is a consequence of the continuity of m_h, a_h, b_h and c_h . Moreover, the following bound holds from the discrete Hölder and Sobolev inequalities (recall that $H^1(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq 6$):

$$[\![L(\mathbf{u}_h^n), \Phi_h]\!] \leq C(\|v_h^n\|_{H^1(\Omega)} + \|w_h^n\|_{L^2(\Omega)} + 1)(\|\varphi_h\|_{H^1(\Omega)} + \|\phi_h\|_{L^2(\Omega)}),$$

for all \mathbf{u}_h and Φ_h in $W_h \times W_h$. Moreover, from (2.3) and Young inequality, we get

$$\left[L(\mathbf{u}_h^n), \mathbf{u}_h^n \right] \geq C(\|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2) + C'$$

for some constant $C, C' > 0$. Finally, we conclude that $\left[L(\mathbf{u}_h^n), \mathbf{u}_h^n \right] \geq 0$ for $\|\mathbf{u}_h^n\|^2 := \|v_h^n\|_{H^1(\Omega)}^2 + \|w_h^n\|_{L^2(\Omega)}^2$ sufficiently large. The existence of \mathbf{u}_h^n follows by the standard Brouwer fixed point argument (see [40, Lemma 4.3]). \blacksquare

4.1. A priori estimates

In this section, we establish several a priori (discrete energy) estimates for the virtual element scheme, which eventually will imply the desired convergence results.

Proposition 4.2. *Let $\mathbf{u}_h^n = (v_h^n, w_h^n)$ be a solution of the virtual element scheme (3.9). Then, there exist constants $C > 0$, depending on Ω, T, v_h^0, w_h^0 and I_{app} such that*

$$\begin{aligned} \|v_h\|_{L^\infty(0,T;L^2(\Omega))} + \|w_h\|_{L^\infty(0,T;L^2(\Omega))} &\leq C, \\ \|\nabla v_h\|_{L^2(\Omega_T)} &\leq C, \\ \|\Pi_k^0 v_h\|_{L^4(\Omega_T)} &\leq C, \end{aligned}$$

where Π_k^0 has been introduced in (3.6).

Proof. We use (3.9) with $\varphi_h = v_h^n, \phi_h = w_h^n$, and we sum over $n = 1, \dots, \kappa$ for all $1 < \kappa \leq N$.

$$\begin{aligned} \sum_{n=1}^{\kappa} m_h(v_h^n - v_h^{n-1}, v_h^n) + \sum_{n=1}^{\kappa} m_h(w_h^n - w_h^{n-1}, w_h^n) + \int_0^{\kappa\Delta t} D(J(v_h^n)) a_h(v_h^n, v_h^n) \\ + \int_0^{\kappa\Delta t} b_h(v_h^n, w_h^n, v_h^n) = \int_0^{\kappa\Delta t} c_h(v_h^n, w_h^n, w_h^n) + \int_0^{\kappa\Delta t} (I_{\text{app},h}, v_h^n)_{0,\Omega}. \end{aligned}$$

Observe that an application of Hölder and Young inequalities, we get

$$\begin{aligned} \sum_{n=1}^{\kappa} m_h(v_h^n - v_h^{n-1}, v_h^n) &= \sum_{n=1}^{\kappa} m_h(v_h^n, v_h^n) - \sum_{n=1}^{\kappa} m_h(v_h^{n-1}, v_h^n) \\ &\geq \sum_{n=1}^{\kappa} m_h(v_h^n, v_h^n) - \sum_{n=1}^{\kappa} \left(m_h(v_h^n, v_h^n) \right)^{1/2} \left(m_h(v_h^{n-1}, v_h^{n-1}) \right)^{1/2} \\ &\geq \sum_{n=1}^{\kappa} m_h(v_h^n, v_h^n) - \frac{1}{2} \sum_{n=1}^{\kappa} m_h(v_h^n, v_h^n) - \sum_{n=1}^{\kappa} \frac{1}{2} m_h(v_h^{n-1}, v_h^{n-1}) \\ &= \sum_{n=1}^{\kappa} \left(\frac{1}{2} m_h(v_h^n, v_h^n) - \frac{1}{2} m_h(v_h^{n-1}, v_h^{n-1}) \right) \\ &= \frac{1}{2} m_h(v_h^\kappa, v_h^\kappa) - \frac{1}{2} m_h(v_h^0, v_h^0). \end{aligned}$$

Using the last inequality, the definition of the forms b_h, c_h , the assumption (2.2) and (3.4) we get

$$\begin{aligned} \frac{1}{2} \beta'(v_h^\kappa, v_h^\kappa)_{0,\Omega} + \frac{1}{2} \beta'(w_h^\kappa, w_h^\kappa)_{0,\Omega} + d_1 \alpha' \int_0^{\kappa\Delta t} a(v_h^n, v_h^n) + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K I_{1,\text{ion}}(\Pi_{K,k}^0 v_h^n) \Pi_{K,k}^0 v_h^n \right) \\ \leq \frac{1}{2} \beta''(v_h^0, v_h^0)_{0,\Omega} + \frac{1}{2} \beta''(w_h^0, w_h^0)_{0,\Omega} + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K H(\Pi_{K,k}^0 v_h^n, \Pi_{K,k}^0 w_h^n) \Pi_{K,k}^0 w_h^n \right) \\ - \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K I_{2,\text{ion}}(\Pi_{K,k}^0 w_h^n) \Pi_{K,k}^0 v_h^n \right) + \int_0^{\kappa\Delta t} (I_{\text{app},h}, v_h^n)_{0,\Omega}. \end{aligned}$$

Now, using the definition of bilinear form $a(\cdot, \cdot)$, and (2.3)(a) on the left hand side. Moreover, we use (2.3)(b), (2.3)(c) and Cauchy-Schwarz inequality and the fact that $I_{\text{app}}(x, t) \in L^2(\Omega_T)$, on the right hand side, we obtain

$$\begin{aligned} & \frac{1}{2}\beta' \|v_h^\kappa\|_{0,\Omega}^2 + \frac{1}{2}\beta' \|w_h^\kappa\|_{0,\Omega}^2 + d_1\alpha' \int_0^{\kappa\Delta t} |v_h^n|_{1,\Omega}^2 + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K \frac{1}{\alpha_1} |\Pi_{K,k}^0 v_h^n|^4 \right) \\ & \leq \frac{1}{2}\beta'' \|v_h^0\|_{0,\Omega}^2 + \frac{1}{2}\beta'' \|w_h^0\|_{0,\Omega}^2 + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K |\Pi_{K,k}^0 v_h^n| |\Pi_{K,k}^0 w_h^n| + |\Pi_{K,k}^0 w_h^n|^2 \right) \\ & \quad + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K |\Pi_{K,k}^0 w_h^n| |\Pi_{K,k}^0 v_h^n| \right) + \int_0^{\kappa\Delta t} \|v_h^n\|_{0,\Omega}^2 + C. \end{aligned}$$

An application of the Cauchy-Schwarz and Young inequalities, the continuity of $\Pi_{K,k}^0$ with respect to $\|\cdot\|_{0,K}$, yields

$$\begin{aligned} & \frac{1}{2}\beta' \|v_h^\kappa\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta' \|w_h^\kappa\|_{L^2(\Omega)}^2 + d_1\alpha' \int_0^{\kappa\Delta t} |v_h^n|_{H^1(\Omega)}^2 + \int_0^{\kappa\Delta t} \left(\sum_{K \in \mathcal{T}_h} \int_K \frac{1}{\alpha_1} |\Pi_{K,k}^0 v_h^n|^4 \right) \\ & \leq \frac{1}{2}\beta'' \|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta'' \|w_h^0\|_{L^2(\Omega)}^2 + \int_0^{\kappa\Delta t} \|v_h^n\|_{L^2(\Omega)}^2 + \int_0^{\kappa\Delta t} \|w_h^n\|_{L^2(\Omega)}^2 + C \\ & \leq \frac{1}{2}\beta'' \|v_h^0\|_{L^2(\Omega)}^2 + \frac{1}{2}\beta'' \|w_h^0\|_{L^2(\Omega)}^2 + \|v_h^n\|_{L^2(\Omega_T)}^2 + \|w_h^n\|_{L^2(\Omega_T)}^2 + C, \end{aligned} \quad (4.1)$$

thus, for some constants $C_1, C_2, C_3 > 0$. This implies

$$\frac{1}{2}\beta' \|v_h^\kappa\|_{L^2(\Omega)} + \frac{1}{2}\beta' \|w_h^\kappa\|_{L^2(\Omega)} \leq C_1 \|v_h\|_{L^2(\Omega_T)}^2 + C_2 \|w_h\|_{L^2(\Omega_T)}^2 + C_3. \quad (4.2)$$

Therefore, by the discrete Gronwall inequality, yields from (4.2)

$$\|v_h\|_{L^\infty(0,T;L^2(\Omega))} + \|w_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C_4, \quad (4.3)$$

for some constant $C_4 > 0$. Finally, using (4.3) in (4.1) and (2.3), we get

$$\|\Pi_k^0 v_h\|_{L^4(\Omega_T)} + \|\nabla v_h\|_{L^2(\Omega_T)} \leq C_5, \quad (4.4)$$

for some constant $C_5 > 0$. This concludes the proof of Lemma 4.2. ■

4.2. Compactness argument and convergence

In this section, we will use time continuous approximation of our discrete solution to obtain compactness in $L^2(\Omega_T)$. For this, we introduce \bar{v}_h and \bar{w}_h the piecewise affine in t functions in $W^{1,\infty}([0, T]; W_h)$ interpolating the states $(v_h^n)_{n=0,\dots,N} \subset W_h$ and $(w_h^n)_{n=0,\dots,N} \subset W_h$ at the points $(n\Delta t)_{n=0,\dots,N}$. Then, we have

$$\begin{cases} m_h(\partial_t \bar{v}_h(t), \varphi_h) + D(J(v_h(t))) a_h(v_h(t), \varphi_h) + b_h(v_h(t), w_h(t), \varphi_h) = (I_{\text{app},h}(t), \varphi_h)_{0,\Omega}, \\ m_h(\partial_t \bar{w}_h(t), \phi_h) = c_h(v_h(t), w_h(t), \phi_h), \end{cases} \quad (4.5)$$

for all φ_h and $\phi_h \in W_h$.

Lemma 4.1. *There exists a positive constant $C > 0$ depending on Ω, T, v_0 and I_{app} such that*

$$\iint_{\Omega_r \times (0,T)} m_h(v_h(x + \mathbf{r}, t) - v_h(x, t), v_h(x + \mathbf{r}, t) - v_h(x, t)) \leq C |\mathbf{r}|^2, \quad (4.6)$$

for all $\mathbf{r} \in \mathbb{R}^2$ with $\Omega_{\mathbf{r}} := \{x \in \Omega \mid x + \mathbf{r} \in \Omega\}$, and

$$\iint_{\Omega \times (0, T-\tau)} m_h(v_h(x, t+\tau) - v_h(x, t), v_h(x, t+\tau) - v_h(x, t)) dx dt \leq C(\tau + \Delta t), \quad (4.7)$$

for all $\tau \in (0, T)$.

Proof. In the first step, we provide the *proof of estimate (4.6)*. In this regard, we start with the uniform estimate of space translate of v_h from the uniform $L^2(\Omega_T)$ estimate of ∇v_h . Observe that from $L^2(0, T; H^1(\Omega))$ estimate of v_h , we get easily the bound:

$$m_h^{\mathbf{r}}(v_h(x + \mathbf{r}, t) - v_h(x, t), v_h(x + \mathbf{r}, t) - v_h(x, t)) \leq C \int_0^T \int_{\Omega_{\mathbf{r}}} |v_h(x + \mathbf{r}, \cdot) - v_h(x, \cdot)|^2 \leq C|\mathbf{r}|^2, \quad (4.8)$$

for some constant $C > 0$, where $m_h^{\mathbf{r}}(\cdot, \cdot)$ is the restriction of the bilinear form $m_h(\cdot, \cdot)$ on $\Omega_{\mathbf{r}}$. It is clear that the right-hand side in (4.8) vanishes as $|\mathbf{r}| \rightarrow 0$, uniformly in h .

Now, we furnish the proof of estimate (4.7). Observe that for all $t \in [0, T-\tau]$, the function φ_h^v such that $\varphi_h^v(x, t) = v_h(x, t+\tau) - v_h(x, t)$ takes value in W_h for $(x, t) \in \Omega_T$. Therefore, we can use this function as a test function in the weak formulations (3.9). Moreover, we previously proved uniform in h bounds on v_h and ∇v_h in $L^2(\Omega_T)$ and on $\Pi_k^0 v_h$ in $L^4(\Omega_T)$. This implies the analogous bounds for the translates φ_h^v and $\nabla \varphi_h^v$ in $L^2(\Omega \times (0, T-\tau))$ and $\Pi_k^0 \varphi_h^v$ in $L^4(\Omega \times (0, T-\tau))$.

We integrate the first approximation equation of (4.5) with respect to the time parameter $s \in [t, t+\tau]$ (with $0 < \tau < T$). In the resulting equations, we take the test function as the corresponding translate φ_h^v . The result is

$$\begin{aligned} & \int_0^{T-\tau} \int_{\Omega} m_h(v_h(x, t+\tau) - v_h(x, t), v_h(x, t+\tau) - v_h(x, t)) dx dt \\ &= \int_0^{T-\tau} \int_{\Omega} \int_t^{t+\tau} m_h(\partial_s \bar{v}_h(x, s), v_h(x, t+\tau) - v_h(x, t)) ds dx dt \\ &= - \int_0^{T-\tau} \int_{\Omega} \int_t^{t+\tau} D(J(v_h(x, s))) a_h(v_h(x, s), v_h(x, t+\tau) - v_h(x, t)) ds dx dt \\ &\quad - \int_0^{T-\tau} \int_{\Omega} \int_t^{t+\tau} b_h(v_h(x, s), w_h(x, s), v_h(x, t+\tau) - v_h(x, t)) ds dx dt \\ &\quad + \int_0^{T-\tau} \int_{\Omega} \int_t^{t+\tau} (I_{app, h}, v_h(x, t+\tau) - v_h(x, t)) ds dx dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now, we bound these integrals separately. For the term I_1 , we have

$$\begin{aligned} |I_1| &\leq C \left[\int_0^{T-\tau} \int_{\Omega} \left(\int_t^{t+\tau} |\nabla v_h(x, s)|^2 ds \right)^2 dx dt \right]^{\frac{1}{2}} \times \left[\int_0^{T-\tau} \int_{\Omega} |\nabla(v_h(x, t+\tau) - v_h(x, t))|^2 dx dt \right]^{\frac{1}{2}} \\ &\leq C \tau. \end{aligned}$$

for some constant $C > 0$. Herein, we used the Fubini theorem (recall that $\int_t^{t+\tau} ds = \tau = \int_{s-\tau}^s dt$), the Hölder inequality and the bounds in L^2 of ∇v_h . Keeping in mind the growth bound of the nonlinearity I_{ion} , we apply the Hölder inequality (with $p = 4$, $p' = 4/3$ in the ionic current term and with $p = p' = 2$ in the

other ones) to deduce

$$\begin{aligned}
|I_2| &\leq C \left(\left[\int_0^{T-\tau} \int_{\Omega} \left(\int_t^{t+\tau} |\Pi_k^0 v_h(x, s)|^4 ds \right)^2 dx dt \right]^{\frac{3}{4}} \times \left[\int_0^{T-\tau} \int_{\Omega} |\Pi_k^0 \varphi_h^v(x, t)|^4 dx dt \right]^{\frac{1}{4}} \right. \\
&\quad \left. + \left[\int_0^{T-\tau} \int_{\Omega} \left(\int_t^{t+\tau} |w_h(x, s)|^2 ds \right)^2 dx dt \right]^{\frac{1}{2}} \times \left[\int_0^{T-\tau} \int_{\Omega} |\varphi_h^v(x, t)|^2 dx dt \right]^{\frac{1}{2}} \right) \\
&\leq C \tau,
\end{aligned}$$

for some constant $C > 0$, where we have used that v_h, φ_h^v and w_h are uniformly bounded in L^2 , and $\Pi_k^0 v_h, \Pi_k^0 \varphi_h^v$ are bounded in L^4 , and the continuity of $\Pi_{K,k}^0$ with respect to $\|\cdot\|_{0,K}$.

Analogously we obtain

$$|I_3| \leq C \tau,$$

for some constant $C > 0$. Collecting the previous inequalities, we readily deduce

$$\int_0^{T-\tau} \int_{\Omega} m_h(v_h(x, t+\tau) - v_h(x, t), v_h(x, t+\tau) - v_h(x, t)) \leq C \tau.$$

Note that, it is easily seen from the definition of (\bar{v}_h, \bar{w}_h) and from the discrete weak formulation (3.9) and estimates in Proposition 4.2 that

$$\|\bar{v}_h - v_h\|_{L^2(\Omega_T)}^2 \leq \sum_{n=1}^N \Delta t \|v_h^n - v_h^{n-1}\|_{L^2(\Omega)}^2 \leq C(\Delta t) \rightarrow 0 \text{ as } \Delta t \rightarrow 0.$$

This concludes the proof of Lemma 4.1. ■

4.3. Convergence of the virtual element scheme

For convergence of our numerical scheme we need the following estimate

$$\|\Pi_k^0 u - u\|_{L^2(\Omega)} \leq Ch^{k+1} \|u\|_{H^{k+1}(\Omega)} \text{ for all } u \in H^{k+1}(\Omega), \quad (4.9)$$

for some constant $C > 0$. This result follows from standard approximation results (see [22]).

Note that from Lemma 4.1 and the stability condition (3.4), we get

$$\iint_{\Omega_r \times (0, T)} |v_h(x + \mathbf{r}, t) - v_h(x, t)|^2 dx dt \leq \frac{C}{\beta'} |\mathbf{r}|^2,$$

and

$$\iint_{\Omega \times (0, T-\tau)} |v_h(x, t+\tau) - v_h(x, t)|^2 dx dt \leq \frac{C}{\beta'} (\tau + \Delta t).$$

Therefore, the next lemma is a consequence of (4.9), Lemma 4.1 and Kolmogorov's compactness criterion (see, e.g., [23], Theorem IV.25).

Lemma 4.2. *There exists a subsequence of $\mathbf{u}_h = (v_h, w_h)$, not relabeled, such that, as $h \rightarrow 0$,*

$$\begin{aligned}
v_h, \Pi_k^0 v_h &\rightarrow v \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T, \\
w_h, \Pi_k^0 w_h &\rightarrow w \text{ weakly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T, \\
v_h &\rightharpoonup v \text{ weakly in } L^2(0, T; H^1(\Omega)), \\
\Pi_k^0 v_h &\rightharpoonup v \text{ weakly in } L^4(\Omega_T).
\end{aligned} \quad (4.10)$$

Now, we are going to show that the limit functions $\mathbf{u} := (v, w)$ constructed in Lemma 4.2 constitute a weak solution of the nonlocal system defined in (2.6).

For that we let $\varphi \in \mathcal{D}(\Omega \times [0, T])$. We approximate φ by $\varphi_h \in C[0, T; L^2(\Omega)]$ such that $\varphi_h|_{(t^{n-1}, t^n)} \in \mathcal{P}_k[t^{n-1}, t^n; W_h]$ and $\varphi_h(T) = 0$, where $\mathcal{P}_k[t^{n-1}, t^n; W_h]$ denotes the space of polynomials of degree k or less having values in W_h .

Let $\mathbf{u}_h := (v_h, w_h)$ be the unique solution of the fully discrete method (3.9). The proof is based on the convergence to zero as h goes to zero of each term of the problems.

We start with the convergence of the nonlocal diffusion term. Observe that

$$\begin{aligned} |D(J(v_h))a_h(v_h, \varphi_h) - D(J(v))a(v, \varphi)| &\leq |D(J(v))[a_h(v_h, \varphi_h) - a(v, \varphi)]| \\ &\quad + |D(J(v_h)) - D(J(v))||a_h(v_h, \varphi_h)| \\ &:= A_1 + A_2. \end{aligned} \quad (4.11)$$

For A_2 , we have

$$\begin{aligned} A_2 &= |D(J(v_h)) - D(J(v))||a_h(v_h, \varphi_h)| \leq C|J(v_h) - J(v)||a_h(v_h, \varphi_h)| \\ &\leq C(\|v_h - v\|_{L^2(\Omega)} + \|v - \Pi_k^0 v\|_{L^2(\Omega)})|v_h|_{H^1(\Omega)}|\varphi_h|_{H^1(\Omega)} \\ &\leq C(\|v_h - v\|_{L^2(\Omega)} + h|v|_{H^1(\Omega)})|v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)} \end{aligned}$$

where we have used the assumption (2.2), the definition of $J(v_h)$ in (3.5), then we add and subtract an appropriate polynomial function and finally the continuity of bilinear form $a_h(\cdot, \cdot)$ in (3.7). Thus, using (4.10), we have that (recall that $\varphi \in \mathcal{D}(\Omega \times [0, T])$)

$$\lim_{h \rightarrow 0} \int_0^T A_2 dt = 0.$$

Now, we bound the term A_1 in (4.11). Using the definition of bilinear form $a_h(\cdot, \cdot)$, the assumption (2.2), we have

$$\begin{aligned} A_1 &= |D(J(v))[a_h(v_h, \varphi_h) - a(v, \varphi)]| \leq |J(v)| \sum_{K \in \mathcal{T}_h} |a_h^K(v_h, \varphi_h) - a^K(v, \varphi)| \\ &\leq |J(v)| \left[\sum_{K \in \mathcal{T}_h} |a^K(\Pi_{K,k} v_h, \Pi_{K,k} \varphi_h) - a^K(v, \varphi)| + \sum_{K \in \mathcal{T}_h} |S^K(v_h - \Pi_{K,k} v_h, \varphi_h - \Pi_{K,k} \varphi_h)| \right] \\ &\leq C\|v\|_{L^2(\Omega)} \left[\sum_{K \in \mathcal{T}_h} |a^K(\Pi_{K,k} v_h - v, \Pi_{K,k} \varphi_h)| + \sum_{K \in \mathcal{T}_h} |a^K(v, \Pi_{K,k} \varphi_h - \varphi)| \right. \\ &\quad \left. + \sum_{K \in \mathcal{T}_h} |a^K(v_h - \Pi_{K,k} v_h, \varphi_h - \Pi_{K,k} \varphi_h)| \right], \end{aligned}$$

where we have added and subtracted $a^K(v, \Pi_{K,k} \varphi_h)$ and used (3.1). Defining

$$\Theta(h) = \sum_{K \in \mathcal{T}_h} |a^K(\Pi_{K,k} v_h - v, \Pi_{K,k} \varphi_h)|$$

Now, using this, and the Cauchy-Schwarz inequality, we obtain,

$$A_1 \leq C\|v\|_{L^2(\Omega)} \left[\Theta(h) + \sum_{K \in \mathcal{T}_h} |v|_{H^1(K)} |\Pi_{K,k} \varphi_h - \varphi|_{H^1(K)} + \sum_{K \in \mathcal{T}_h} |v_h - \Pi_{K,k} v_h|_{H^1(K)} |\varphi_h - \Pi_{K,k} \varphi_h|_{H^1(K)} \right],$$

Next, we add and substract an appropriate polynomial φ_Π in the second term, and we add and substract φ in the last term. Thus, we have

$$\begin{aligned} A_1 &\leq C\|v\|_{L^2(\Omega)} \left[\Theta(h) + \sum_{K \in \mathcal{T}_h} |v|_{H^1(K)} (|\Pi_{K,k}(\varphi_h - \varphi_\Pi)|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)}) \right. \\ &\quad \left. + \sum_{K \in \mathcal{T}_h} |v_h - \Pi_{K,k}v_h|_{H^1(K)} (|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \Pi_{K,k}\varphi_h|_{H^1(K)}) \right] \\ &\leq C\|v\|_{L^2(\Omega)} \left[\Theta(h) + \sum_{K \in \mathcal{T}_h} |v|_{H^1(K)} (|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)}) \right. \\ &\quad \left. + \sum_{K \in \mathcal{T}_h} |v_h|_{H^1(K)} (|\varphi_h - \varphi|_{H^1(K)} + |\varphi - \varphi_\Pi|_{H^1(K)}) \right]. \end{aligned}$$

Now, using (4.10), standard approximation results for polynomials, and the regularity of φ , we obtain

$$\lim_{h \rightarrow 0} \int_0^T A_1 dt = 0.$$

Finally, we get

$$\int_0^T |D(J(v_h))a_h(v_h, \varphi_h) - D(J(v))a(v, \varphi)| dt \rightarrow 0 \text{ as } h \rightarrow 0.$$

Now, we prove

$$\left| \int_0^T m_h(v_h, \partial_t \varphi_h) - (v, \partial_t \varphi)_{0,\Omega} \right| \rightarrow 0 \text{ as } h \rightarrow 0. \quad (4.12)$$

In fact, using the definition of the bilinear form $m_h(\cdot, \cdot)$, we obtain

$$\begin{aligned} \left| \int_0^T m_h(v_h, \partial_t \varphi_h) - (v, \partial_t \varphi)_{0,\Omega} \right| &\leq \left| \sum_{K \in \mathcal{T}_h} (\Pi_K^0 v_h, \Pi_K^0 \partial_t \varphi_h)_{0,K} - (v, \partial_t \varphi)_{0,K} \right| \\ &\quad + |S_0^K(v_h - \Pi_K^0 v_h, \partial_t \varphi_h - \Pi_K^0 \partial_t \varphi_h)| \\ &\leq \left| \sum_{K \in \mathcal{T}_h} (\Pi_K^0 v_h - v, \Pi_K^0 \partial_t \varphi_h)_{0,K} \right| + |(v, \Pi_K^0 \partial_t \varphi_h - \partial_t \varphi)_{0,K}| \\ &\quad + |S_0^K(v_h - \Pi_K^0 v_h, \partial_t \varphi_h - \Pi_K^0 \partial_t \varphi_h)| \\ &\leq \|v_h - v\|_{L^2(\Omega)} \|\partial_t \varphi\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \|\partial_t \varphi_h - \partial_t \varphi\|_{L^2(\Omega)} \\ &\quad + \|v_h\|_{L^2(\Omega)} (\|\partial_t \varphi_h - \partial_t \varphi_\Pi\|_{L^2(\Omega)} + \|\partial_t \varphi - \partial_t \varphi_\Pi\|_{L^2(\Omega)}). \end{aligned}$$

Using this, (4.10), standard approximation results for polynomials and the regularity of φ , we arrive to (4.12). Now, we prove

$$\int_0^T |b_h(v_h, w_h, \varphi_h) - (I_{\text{ion}}(v, w), \varphi)_{0,\Omega}| dt \rightarrow 0 \text{ as } h \rightarrow 0.$$

Using the definition of the form $b_h(\cdot, \cdot, \cdot)$ and the decomposition of the ionic current $I_{\text{ion}}(v, w)$ we have

$$\begin{aligned} |b_h(v_h, w_h, \varphi_h) - (I_{\text{ion}}(v, w), \varphi)_{0,\Omega}| &= \left| \sum_{K \in \mathcal{T}_h} (I_{\text{ion}}(\Pi_K^0 v_h, \Pi_K^0 w_h), \Pi_K^0 \varphi_h)_{0,K} - (I_{\text{ion}}(v, w), \varphi)_{0,K} \right| \\ &= \left| \sum_{K \in \mathcal{T}_h} (I_{1,\text{ion}}(\Pi_K^0 v_h), \Pi_K^0 \varphi_h)_{0,K} + (I_{2,\text{ion}}(\Pi_K^0 w_h), \Pi_K^0 \varphi_h)_{0,K} - (I_{1,\text{ion}}(v), \varphi)_{0,K} - (I_{2,\text{ion}}(w), \varphi)_{0,K} \right| \\ &\leq \sum_{K \in \mathcal{T}_h} |(I_{1,\text{ion}}(\Pi_K^0 v_h), \Pi_K^0 \varphi_h)_{0,K} - (I_{1,\text{ion}}(v), \varphi)_{0,K}| + |(I_{2,\text{ion}}(\Pi_K^0 w_h), \Pi_K^0 \varphi_h)_{0,K} - (I_{2,\text{ion}}(w), \varphi)_{0,K}| \\ &=: B_1 + B_2. \end{aligned}$$

Note that since the function $I_{2,\text{ion}}$ is a linear function, we get easily

$$\int_0^T B_2 dt \rightarrow 0 \text{ as } h \text{ goes to } 0.$$

Now, we turn to the term B_1 , we have the following estimation

$$\begin{aligned} B_1 &\leq \sum_{K \in \mathcal{T}_h} |(I_{1,\text{ion}}(\Pi_K^0 v_h), \Pi_K^0 \varphi_h)_{0,K} - (I_{1,\text{ion}}(\Pi_K^0 v_h), \varphi)_{0,K}| \\ &\quad + \sum_{K \in \mathcal{T}_h} |(I_{1,\text{ion}}(\Pi_K^0 v_h), \varphi)_{0,K} - (I_{1,\text{ion}}(v), \varphi)_{0,K}| \\ &\leq \|\varphi_h - \varphi\|_{L^\infty(\Omega)} \|I_{1,\text{ion}}(\Pi_K^0 v_h)\|_{L^1(\Omega)} + \text{Const}(v, \Pi_K^0 v_h, v_h) \|\Pi_K^0 v_h - v\|_{L^2(\Omega)}, \end{aligned}$$

where $\text{Const}(v, \Pi_K^0 v_h, v_h) > 0$ is a constant. This implies that

$$\int_0^T B_1 dt \rightarrow 0 \text{ as } h \text{ goes to } 0.$$

Similarly, we get

$$\int_0^T |(I_{\text{app},h}, \varphi_h)_{0,\Omega} - (I_{\text{app}}(x, t), \varphi)_{0,\Omega}| dt \rightarrow 0 \text{ as } h \rightarrow 0.$$

With the above convergences and by density, we are ready to identify the limit $\mathbf{u} = (v, w)$ as a (weak) solution of the system (2.1). Finally, let $\varphi \in L^2(0, T; H^1(\Omega)) \cap L^4(\Omega_T)$ and $\phi \in C([0, T]; L^2(\Omega))$, then by passing to the limit $h \rightarrow 0$ in the following weak formulation (with the help of Lemma 4.2)

$$\begin{aligned} - \int_0^T m_h(v_h(t), \partial_t \varphi_h) + \int_0^T D(J(v_h(t))) a_h(v_h(t), \varphi_h) + \int_0^T b_h(v_h(t), w_h(t), \varphi_h) &= \int_0^T (I_{\text{app},h}(t), \varphi_h)_{0,\Omega} \\ \int_0^T m_h(\partial_t w_h(t), \phi_h) &= \int_0^T c_h(v_h(t), w_h(t), \phi_h), \end{aligned}$$

we obtain the limit $\mathbf{u} = (v, w)$ which is a solution of system (2.1) in the sense of Definition 2.1.

5. Error estimates analysis

In this section, an error estimates will be developed to our model (2.1). For technical reason (because of the nonlinearity of I_{ion}), we need to relax the assumptions (2.3). For the error estimates analysis, we will use the following assumption on I_{ion} : we assume that I_{ion} is a linear function on v and w satisfying

$$\forall s_1, s_2, z_1, z_2 \in \mathbb{R} \quad |I_{\text{ion}}(s_1, z_1) - I_{\text{ion}}(s_2, z_2)| \leq \alpha_7(|s_1 - s_2| + |z_1 - z_2|), \quad (5.1)$$

for some constant $\alpha_7 > 0$.

First, we introduce the projection $\mathcal{P}^h : H^1(\Omega) \rightarrow W_h$ as the solution of the following well-posed problem:

$$\begin{cases} \mathcal{P}^h u \in W_h, \\ a_h(\mathcal{P}^h u, \varphi_h) = a(u, \varphi_h) \text{ for all } \varphi_h \in W_h. \end{cases}$$

We have the following lemma, the proof can be found in [11, Lemma 3.1].

Lemma 5.1. *Let $u \in H^1(\Omega)$. Then, there exist $C, \tilde{C} > 0$, independent of h , such that*

$$|\mathcal{P}^h u - u|_{H^1(\Omega)} \leq Ch^k |u|_{H^{k+1}(\Omega)},$$

Moreover, if the domain is convex, then

$$\|\mathcal{P}^h u - u\|_{L^2(\Omega)} \leq \tilde{C}h^{k+1} |u|_{H^{k+1}(\Omega)}.$$

Our main result in this section is the following theorem.

Theorem 5.2. *Let (v, w) be the solution of system (2.1) and let $(v_h(t), w_h(t))$ be the solution of the problem (3.8). Then, for all $t \in (0, T)$, we have*

$$\begin{aligned} & \|v_h(\cdot, t) - v(\cdot, t)\|_{L^2(\Omega)} + \|w_h(\cdot, t) - w(\cdot, t)\|_{L^2(\Omega)} \\ & \leq C \left[\|v^0 - v_h^0\|_{L^2(\Omega)} + \|w^0 - w_h^0\|_{L^2(\Omega)} + h^{k+1} \left(|v^0|_{H^{k+1}(\Omega)} + |w^0|_{H^{k+1}(\Omega)} \right. \right. \\ & \quad \left. \left. + \int_0^t \left(|I_{\text{app}}|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t v|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)} \right) dt \right) \right] \\ & \quad \times \exp \left(\int_0^t (1 + |v|_2) dt \right), \end{aligned} \quad (5.2)$$

for some constant $C > 0$. Moreover, let $\mathbf{u}_h^n = (v_h^n, w_h^n)$ be the virtual element solution generated by (3.9). Then, for $n = 1, \dots, N$

$$\begin{aligned} & \|v_h^n - v(\cdot, t_n)\|_{L^2(\Omega)} + \|w_h^n - w(\cdot, t_n)\|_{L^2(\Omega)} \\ & \leq C \left[\|v^0 - v_h^0\|_{L^2(\Omega)} + \|w^0 - w_h^0\|_{L^2(\Omega)} + \Delta t \int_0^{t_n} (|\partial_{tt}^2 v| + |\partial_{tt}^2 w|) dt \right. \\ & \quad \left. + h^{k+1} \left(|v^0|_{H^{k+1}(\Omega)} + |w^0|_{H^{k+1}(\Omega)} \right. \right. \\ & \quad \left. \left. + \int_0^{t_n} \left(|I_{\text{app}}|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t v|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)} \right) dt \right) \right] \\ & \quad \times \exp \left(\int_0^{t_n} (1 + |v|_2) dt \right), \end{aligned} \quad (5.3)$$

Proof. We start with the proof of bound (5.2). First note that

$$U_h(\cdot, t) - U(\cdot, t) = (U_h(\cdot, t) - \mathcal{P}^h U(\cdot, t)) + (\mathcal{P}^h U(\cdot, t) - U(\cdot, t)) \quad \text{for } U = v, w.$$

Observe that from Lemma 5.1, we get easily for $U = v, w$

$$\begin{aligned} \|\mathcal{P}^h U(\cdot, t) - U(\cdot, t)\|_{L^2(\Omega)} & \leq Ch^{k+1} |U|_{H^{k+1}(\Omega)} \\ & \leq Ch^{k+1} \left(|U_0|_{H^{k+1}(\Omega)} + \int_0^t |\partial_t U(\cdot, s)|_{H^{k+1}(\Omega)} ds \right) \\ & = Ch^{k+1} (|U_0|_{H^{k+1}(\Omega)} + \|\partial_t U\|_{L^1(0, t; H^{k+1}(\Omega))}), \end{aligned} \quad (5.4)$$

for all $t \in (0, T)$.

Observe that, using the continuous and semidiscrete problems (cf. (2.1) and (3.8)), the definition of the projector \mathcal{P}^h and the fact that the derivative with respect to time commutes with this projector, we obtain

$$\begin{aligned} & m_h(\partial_t(v_h - \mathcal{P}^h v), \varphi_h^v) + D(J(v_h)) a_h((v_h - \mathcal{P}^h v), \varphi_h^v) \\ & = (I_{\text{app}h}, \varphi_h^v)_{0, \Omega} - b_h(v_h, w_h, \varphi_h^v) - m_h(\partial_t \mathcal{P}^h v, \varphi_h^v) - D(J(v_h)) a_h(\mathcal{P}^h v, \varphi_h^v) \\ & = (I_{\text{app}h}, \varphi_h^v)_{0, \Omega} - b_h(v_h, w_h, \varphi_h^v) - m_h(\mathcal{P}^h \partial_t v, \varphi_h^v) - D(J(v_h)) a(v, \varphi_h^v) \\ & = \left[(I_{\text{app}h}, \varphi_h^v)_{0, \Omega} - (I_{\text{app}}, \varphi_h^v)_{0, \Omega} \right] - \left[b_h(v_h, w_h, \varphi_h^v) - (I_{\text{ion}}(v, w), \varphi_h^v)_{0, \Omega} \right] \\ & \quad + \left[(\partial_t v, \varphi_h^v)_{0, \Omega} - m_h(\mathcal{P}^h \partial_t v, \varphi_h^v) \right] + \left[(D(J(v)) - D(J(v_h))) a(v, \varphi_h^v) \right] \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (5.5)$$

Now, we are going to bound each term I_1, \dots, I_4 . Regarding the first term I_1 , we have

$$I_1 = (\Pi_k^0 I_{\text{app}} - I_{\text{app}}, \varphi_h^v)_{0,\Omega} \leq Ch^{k+1} |I_{\text{app}}|_{H^{k+1}(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)}, \quad (5.6)$$

for some constant $C > 0$, where we have used the definition of I_{app_h} . Next, for I_2 , using the definition of the form $b_h(\cdot, \cdot, \cdot)$ and adding and subtracting adequate terms, we have

$$\begin{aligned} I_2 &= - \left[b_h(v_h, w_h, \varphi_h^v) - (\mathcal{P}^h I_{\text{ion}}(v, w), \varphi_h^v)_{0,\Omega} \right] - \left[(\mathcal{P}^h I_{\text{ion}}(v, w), \varphi_h^v)_{0,\Omega} - (I_{\text{ion}}(v, w), \varphi_h^v)_{0,\Omega} \right] \\ &= - \left[\sum_{K \in \mathcal{T}_h} (I_{\text{ion}}(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h), \Pi_{K,k}^0 \varphi_h^v)_{0,K} - (I_{\text{ion}}(\mathcal{P}^h v, \mathcal{P}^h w), \varphi_h^v)_{0,K} \right] \\ &\quad - \left[(\mathcal{P}^h I_{\text{ion}}(v, w) - I_{\text{ion}}(v, w), \varphi_h^v)_{0,\Omega} \right] \\ &= - \left[\sum_{K \in \mathcal{T}_h} (I_{\text{ion}}(\Pi_{K,k}^0 v_h, \Pi_{K,k}^0 w_h) - I_{\text{ion}}(\mathcal{P}^h v, \mathcal{P}^h w), \varphi_h^v)_{0,K} \right] - \left[(\mathcal{P}^h I_{\text{ion}}(v, w) - I_{\text{ion}}(v, w), \varphi_h^v)_{0,\Omega} \right] \\ &\leq C \left[\sum_{K \in \mathcal{T}_h} (\|\Pi_{K,k}^0 v_h - \mathcal{P}^h v\|_{L^2(K)} + \|\Pi_{K,k}^0 w_h - \mathcal{P}^h w\|_{L^2(K)}) \|\varphi_h^v\|_{L^2(K)} \right] \\ &\quad + Ch^{k+1} (|v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)}) \|\varphi_h^v\|_{L^2(\Omega)} \\ &\leq C \left(\|\Pi_k^0 v_h - \mathcal{P}^h v\|_{L^2(\Omega)} + \|\Pi_k^0 w_h - \mathcal{P}^h w\|_{L^2(\Omega)} \right) \|\varphi_h^v\|_{L^2(\Omega)} \\ &\quad + Ch^{k+1} (|v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)}) \|\varphi_h^v\|_{L^2(\Omega)}, \end{aligned}$$

for some constant $C > 0$, where we have used that I_{ion} is a linear function together with (5.1), the properties of projectors Π_k^0 and \mathcal{P}^h and finally Lemma 5.1.

For I_3 , we use the consistency and stability properties of the bilinear for $m_h(\cdot, \cdot)$ to get

$$\begin{aligned} I_3 &= \sum_{K \in \mathcal{T}_h} \left[(\partial_t v - \Pi_{K,k}^0 \partial_t v, \varphi_h^v)_{0,K} + m_h^K(\Pi_{K,k}^0 \partial_t v - \mathcal{P}^h \partial_t v, \varphi_h^v) \right] \\ &\leq C \sum_{K \in \mathcal{T}_h} \left[\|\partial_t v - \Pi_{K,k}^0 \partial_t v\|_{L^2(K)} + \|\Pi_{K,k}^0 \partial_t v - \mathcal{P}^h \partial_t v\|_{L^2(K)} \right] \|\varphi_h^v\|_{L^2(K)} \\ &\leq Ch^{k+1} |\partial_t v|_{H^{k+1}(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)}, \end{aligned}$$

for some constant $C > 0$. Moreover, by using the assumption on D , an integration by parts, the Cauchy-Schwarz inequality and the continuity of projector Π_k^0 , we get

$$I_4 \leq C (\|v - v_h\|_{L^2(\Omega)} + \|v - \Pi_k^0 v\|_{L^2(\Omega)}) \|\Delta v\|_{L^2(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)},$$

for some constant $C > 0$.

On the other hand, similarly for w_h , we obtain

$$\begin{aligned} m_h(\partial_t(w_h - \mathcal{P}^h w), \varphi_h^w) &= (c_h(v_h, w_h, \varphi_h^w) - m_h(\partial_t \mathcal{P}^h w, \varphi_h^w)) \\ &= c_h(v_h, w_h, \varphi_h^w) - m_h(\mathcal{P}^h \partial_t w, \varphi_h^w) - (H(v, w), \varphi_h^w)_{0,\Omega} + (\partial_t w, \varphi_h^w)_{0,\Omega} \\ &\leq \left[c_h(v_h, w_h, \varphi_h^w) - (\mathcal{P}^h H(v, w), \varphi_h^w)_{0,\Omega} \right] \\ &\quad + \left[(\mathcal{P}^h H(v, w), \varphi_h^w)_{0,\Omega} - (H(v, w), \varphi_h^w)_{0,\Omega} \right] + \left[(\partial_t w, \varphi_h^w) - m_h(\partial_t \mathcal{P}^h w, \varphi_h^w) \right]. \end{aligned}$$

Now, assuming that H is a linear function satisfying (5.1), repeating the arguments used to bound I_2 and I_3 , and using once again the properties of projectors Π_k^0 and \mathcal{P}^h and finally Lemma 5.1, we readily obtain,

$$\begin{aligned} m_h(\partial_t(w_h - \mathcal{P}^h w), \varphi_h^w) \leq & C \left(\|\Pi_k^0 v_h - \mathcal{P}^h v\|_{L^2(\Omega)} + \|\Pi_k^0 w_h - \mathcal{P}^h w\|_{L^2(\Omega)} \right) \|\varphi_h^w\|_{L^2(\Omega)} \\ & + Ch^{k+1} (|v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)}) \|\varphi_h^w\|_{L^2(\Omega)}, \end{aligned} \quad (5.7)$$

for some constant $C > 0$.

Collecting the previous results (5.5)-(5.7), and using the approximation properties of projectors Π_k^0 and \mathcal{P}^h , we get

$$\begin{aligned} & m_h(\partial_t(v_h - \mathcal{P}^h v), \varphi_h^v) + m_h(\partial_t(w_h - \mathcal{P}^h w), \varphi_h^w) \\ & \leq Ch^{k+1} \left[|I_{\text{app}}|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t v|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)} \right. \\ & \quad \left. + C(1 + \|\Delta v\|_{L^2(\Omega)}) \left(\|v - v_h\|_{L^2(\Omega)} + \|w - w_h\|_{L^2(\Omega)} \right) \right] (\|\varphi_h^v\|_{L^2(\Omega)} + \|\varphi_h^w\|_{L^2(\Omega)}) \end{aligned} \quad (5.8)$$

Now, we set $\varphi_h^v := (v_h - \mathcal{P}^h v) \in W_h$ and $\varphi_h^w := (w_h - \mathcal{P}^h w) \in W_h$ in (5.8), we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(m_h(v_h - \mathcal{P}^h v, v_h - \mathcal{P}^h v) + m_h(w_h - \mathcal{P}^h w, w_h - \mathcal{P}^h w) \right) \\ & \leq Ch^{k+1} \left[|I_{\text{app}}|_{H^{k+1}(\Omega)} + |v|_{H^{k+1}(\Omega)} + |w|_{H^{k+1}(\Omega)} + |\partial_t v|_{H^{k+1}(\Omega)} + |\partial_t w|_{H^{k+1}(\Omega)} \right. \\ & \quad \left. + C(1 + \|\Delta v\|_{L^2(\Omega)}) \left(\|v - v_h\|_{L^2(\Omega)} + \|w - w_h\|_{L^2(\Omega)} \right) \right] \\ & \quad \times (\|(v_h - \mathcal{P}^h v)\|_{L^2(\Omega)} + \|(w_h - \mathcal{P}^h w)\|_{L^2(\Omega)}). \end{aligned}$$

Herein, we used the equivalence of the norm $\|\cdot\|_h := m_h(\cdot, \cdot)$ with the L^2 norm, integrating the previous bound on $(0, t)$ and an application of Gronwall inequality, we get

$$\begin{aligned} & (\|(v_h - \mathcal{P}^h v)\|_{L^2(\Omega)} + \|(w_h - \mathcal{P}^h w)\|_{L^2(\Omega)}) \\ & \leq C \left[\|v_0 - v_{0,h}\|_0 + \|w_0 - w_{0,h}\|_0 + h^{k+1} \left(|v_0|_{k+1} + |w_0|_{k+1} \right. \right. \\ & \quad \left. \left. + \int_0^t \left(|I_{\text{app}}|_{k+1} + |v|_{k+1} + |w|_{k+1} + |\partial_t v|_{k+1} + |\partial_t w|_{k+1} \right) dt \right) \int_0^t (1 + |v|_2) dt \right]. \end{aligned}$$

Using this and (5.4), we get (5.2).

Proof of (5.3): Similarly to (5.2), observe that for $n = 1, \dots, N$

$$U_h^n - U(\cdot, t_n) = (U_h^n - \mathcal{P}^h U(\cdot, t_n)) + (\mathcal{P}^h U(\cdot, t_n) - U(\cdot, t_n)) \quad \text{for } U = v, w.$$

and from Lemma 5.1, we get easily for $U = v, w$ and for all $t \in (0, T)$

$$\|\mathcal{P}^h U(\cdot, t_n) - U(\cdot, t_n)\|_{L^2(\Omega)} \leq Ch^{k+1} (|U_0|_{H^{k+1}(\Omega)} + \|\partial_t U\|_{L^1(0,t;H^{k+1}(\Omega))}),$$

for some constant $C > 0$. Next, we bound the term $(U_h^n - \mathcal{P}^h U(\cdot, t_n))$ for $U = v, w$. Note that using the

continuous and fully discrete problems (cf. (2.1) and (3.9)), the definition of the projector \mathcal{P}^h , we obtain

$$\begin{aligned}
& m_h \left(\frac{(v_h^n - \mathcal{P}^h v(\cdot, t_n)) - (v_h^{n-1} - \mathcal{P}^h v(\cdot, t_{n-1}))}{\Delta t}, \varphi_h^v \right) + D(J(v_h^n)) a_h((v_h^n - \mathcal{P}^h v(\cdot, t_n)), \varphi_h^v) \\
&= (I_{app,h}^n, \varphi_h^v)_{0,\Omega} - b_h(v_h^n, w_h^n, \varphi_h^v) - m_h \left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v \right) - D(J(v_h^n)) a_h(\mathcal{P}^h v(\cdot, t_n), \varphi_h^v) \\
&= (I_{app,h}^n, \varphi_h^v)_{0,\Omega} - b_h(v_h^n, w_h^n, \varphi_h^v) - (I_{app}(\cdot, t_n), \varphi_h^v)_{0,\Omega} - (I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} + (\partial_t v(\cdot, t_n), \varphi_h^v) \\
&\quad - m_h \left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v \right) + (D(J(v(\cdot, t_n))) - D(J(v_h^n))) a(v(\cdot, t_n), \varphi_h^v) \\
&= \left[(I_{app,h}^n, \varphi_h^v)_{0,\Omega} - (I_{app}(\cdot, t_n), \varphi_h^v)_{0,\Omega} \right] - \left[b_h(v_h^n, w_h^n, \varphi_h^v) - (I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} \right] \\
&\quad + \left[(\partial_t v(\cdot, t_n), \varphi_h^v) - m_h \left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v \right) \right] \\
&\quad + \left[(D(J(v(\cdot, t_n))) - D(J(v_h^n))) a(v(\cdot, t_n), \varphi_h^v) \right] \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned} \tag{5.9}$$

Now, we will bound the terms $\mathcal{I}_1, \dots, \mathcal{I}_4$. Note that the first term \mathcal{I}_1 can be estimated like (5.6)

$$\mathcal{I}_1 \leq Ch^{k+1} |I_{app}(\cdot, t_n)|_{H^{k+1}(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)},$$

for some constant $C > 0$. Next, for \mathcal{I}_2 , using the definition of the form $b_h(\cdot, \cdot, \cdot)$, adding and subtracting adequate terms, we have

$$\begin{aligned}
\mathcal{I}_2 &= - \left[b_h(v_h^n, w_h^n, \varphi_h^v) - (\mathcal{P}^h I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} \right] \\
&\quad - \left[(\mathcal{P}^h I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} - (I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} \right] \\
&= - \left[\sum_{K \in \mathcal{T}_h} (I_{ion}(\Pi_{K,k}^0 v_h^n, \Pi_{K,k}^0 w_h^n, \Pi_{K,k}^0 \varphi_h^v)_{0,K} - (I_{ion}(\mathcal{P}^h v(\cdot, t_n), \mathcal{P}^h w(\cdot, t_n)), \varphi_h^v)_{0,K}) \right] \\
&\quad - \left[(\mathcal{P}^h I_{ion}(v(\cdot, t_n), w(\cdot, t_n)) - I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} \right] \\
&= - \left[\sum_{K \in \mathcal{T}_h} (I_{ion}(\Pi_{K,k}^0 v_h^n, \Pi_{K,k}^0 w_h^n) - I_{ion}(\mathcal{P}^h v(\cdot, t_n), \mathcal{P}^h w(\cdot, t_n)), \varphi_h^v)_{0,K} \right] \\
&\quad - \left[(\mathcal{P}^h I_{ion}(v(\cdot, t_n), w(\cdot, t_n)) - I_{ion}(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^v)_{0,\Omega} \right] \\
&\leq C \left[\sum_{K \in \mathcal{T}_h} (\|\Pi_{K,k}^0 v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_{L^2(K)} + \|\Pi_{K,k}^0 w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_{L^2(K)}) \|\varphi_h^v\|_{L^2(K)} \right] \\
&\quad + Ch^{k+1} (|v(\cdot, t_n)|_{H^{k+1}(\Omega)} + |w(\cdot, t_n)|_{H^{k+1}(\Omega)}) \|\varphi_h^v\|_{L^2(\Omega)} \\
&\leq C \left(\|\Pi_k^0 v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_{L^2(\Omega)} + \|\Pi_k^0 w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_{L^2(\Omega)} \right) \|\varphi_h^v\|_{L^2(\Omega)} \\
&\quad + Ch^{k+1} (|v(\cdot, t_n)|_{H^{k+1}(\Omega)} + |w(\cdot, t_n)|_{H^{k+1}(\Omega)}) \|\varphi_h^v\|_{L^2(\Omega)},
\end{aligned}$$

for some constant $C > 0$, where we have used that I_{ion} is a linear function together with (5.1), the properties of projectors Π_k^0 and \mathcal{P}^h and finally Lemma 5.1.

Regarding \mathcal{I}_3 , we use the consistency and stability properties of the bilinear form m_h to get

$$\begin{aligned}
\mathcal{I}_3 &= \sum_{K \in \mathcal{T}_h} \left[(\partial_t v(\cdot, t_n), \varphi_h^v)_{0,K} - m_h^K \left(\frac{\mathcal{P}^h v(\cdot, t_n) - \mathcal{P}^h v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v \right) \right] \\
&= \sum_{K \in \mathcal{T}_h} \left[\left(\partial_t v(\cdot, t_n) - \frac{v(\cdot, t_n) - v(\cdot, t_{n-1})}{\Delta t}, \varphi_h^v \right)_{0,K} \right. \\
&\quad + \left(\frac{v(\cdot, t_n) - v(\cdot, t_{n-1})}{\Delta t} - \frac{\Pi_{K,k}^0(v(\cdot, t_n) - v(\cdot, t_{n-1}))}{\Delta t}, \varphi_h^v \right)_{0,K} \\
&\quad \left. + m_h^K \left(\frac{\Pi_{K,k}^0(v(\cdot, t_n) - v(\cdot, t_{n-1}))}{\Delta t} - \frac{\mathcal{P}^h(v(\cdot, t_n) - v(\cdot, t_{n-1}))}{\Delta t}, \varphi_h^v \right) \right] \\
&\leq \frac{C}{\Delta t} \sum_{K \in \mathcal{T}_h} \left[\|\Delta t \partial_t v(\cdot, t_n) - (v(\cdot, t_n) - v(\cdot, t_{n-1}))\|_{L^2(K)} \right. \\
&\quad + \|(v(\cdot, t_n) - v(\cdot, t_{n-1})) - \Pi_{K,k}^0(v(\cdot, t_n) - v(\cdot, t_{n-1}))\|_{L^2(K)} \\
&\quad \left. + \|\Pi_{K,k}^0(v(\cdot, t_n) - v(\cdot, t_{n-1})) - \mathcal{P}^h(v(\cdot, t_n) - v(\cdot, t_{n-1}))\|_{L^2(K)} \right] \|\varphi_h^v\|_{L^2(K)} \\
&\leq \frac{C}{\Delta t} \left[\|\Delta t \partial_t v(\cdot, t_n) - (v(\cdot, t_n) - v(\cdot, t_{n-1}))\|_{L^2(\Omega)} + h^{k+1} \|v(\cdot, t_n) - v(\cdot, t_{n-1})\|_{H^{k+1}(\Omega)} \right] \|\varphi_h^v\|_{L^2(\Omega)} \\
&\leq \frac{C}{\Delta t} \left[\Delta t \int_{t_{n-1}}^{t_n} \|\partial_{tt}^2 v(\cdot, s)\|_{L^2(\Omega)} ds + h^{k+1} \int_{t_{n-1}}^{t_n} \|v_t(\cdot, s)\|_{H^{k+1}(\Omega)} ds \right] \|\varphi_h^v\|_{L^2(\Omega)}
\end{aligned}$$

for some constant $C > 0$, where we have used Cauchy-Schwarz inequality, and the approximation properties of Π_k^0 and \mathcal{P}^h , finally Lemma 5.1. Moreover, for \mathcal{I}_4 by using an integration by parts, the assumption on D , Cauchy-Schwarz inequality, and the continuity of projector Π_k^0 , we get

$$\mathcal{I}_4 \leq C \left(\|v(\cdot, t_n) - v_h^n\|_{L^2(\Omega)} + \|v(\cdot, t_n) - \Pi_k^0 v(\cdot, t_n)\|_{L^2(\Omega)} \right) \|\Delta v(\cdot, t_n)\|_{L^2(\Omega)} \|\varphi_h^v\|_{L^2(\Omega)},$$

for some constant $C > 0$. On the other hand, similarly for w_h , we obtain

$$\begin{aligned}
m_h \left(\frac{(w_h^n(\cdot) - \mathcal{P}^h w(\cdot, t_n)) - (w_h^{n-1}(\cdot) - \mathcal{P}^h w(\cdot, t_{n-1}))}{\Delta t}, \varphi_h^w \right) &= c_h(v_h^n, w_h^n, \varphi_h^w) \\
&\quad - m_h \left(\frac{\mathcal{P}^h w(\cdot, t_n) - \mathcal{P}^h w(\cdot, t_{n-1})}{\Delta t}, \varphi_h^w \right) \\
&= c_h(v_h^n, w_h^n, \varphi_h^w) - m_h \left(\frac{\mathcal{P}^h w(\cdot, t_n) - \mathcal{P}^h w(\cdot, t_{n-1})}{\Delta t}, \varphi_h^w \right) \\
&\quad - (H(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^w)_{0,\Omega} + (\partial_t w(\cdot, t_n), \varphi_h^w)_{0,\Omega} \\
&\leq \left[c_h(v_h^n, w_h^n, \varphi_h^w) - (\mathcal{P}^h H(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^w)_{0,\Omega} \right] \\
&\quad + \left[(\mathcal{P}^h H(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^w)_{0,\Omega} - (H(v(\cdot, t_n), w(\cdot, t_n)), \varphi_h^w)_{0,\Omega} \right] \\
&\quad + \left[(\partial_t w(\cdot, t_n), \varphi_h^w)_{0,\Omega} - m_h \left(\frac{\mathcal{P}^h w(\cdot, t_n) - \mathcal{P}^h w(\cdot, t_{n-1})}{\Delta t}, \varphi_h^w \right) \right].
\end{aligned}$$

Now, assuming that H is a linear function satisfying (5.1), repeating the arguments used to bound \mathcal{I}_2 and

\mathcal{I}_3 , and using once again the properties of projectors Π_k^0 and \mathcal{P}^h and finally Lemma 5.1, we readily obtain,

$$\begin{aligned}
m_h \left(\frac{(w_h^n(\cdot) - \mathcal{P}^h w(\cdot, t_n)) - (w_h^{n-1}(\cdot) - \mathcal{P}^h w(\cdot, t_{n-1}))}{\Delta t}, \varphi_h^w \right) &\leq C \left(\|\Pi_k^0 v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_{L^2(\Omega)} \right. \\
&+ \|\Pi_k^0 w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_{L^2(\Omega)} \\
&+ Ch^{k+1} (|v(\cdot, t_n)|_{H^{k+1}(\Omega)} + |w(\cdot, t_n)|_{H^{k+1}(\Omega)}) \Big) \|\varphi_h^w\|_{L^2(\Omega)} \\
&+ \frac{C}{\Delta t} \left[\Delta t \int_{t_{n-1}}^{t_n} \|\partial_{tt}^2 w(\cdot, s)\|_{L^2(\Omega)} ds + h^{k+1} \int_{t_{n-1}}^{t_n} |w_t(\cdot, s)|_{H^{k+1}(\Omega)} ds \right] \|\varphi_h^v\|_{L^2(\Omega)},
\end{aligned} \tag{5.10}$$

for some constant $C > 0$.

Collecting the previous results (5.9) - (5.10), and using the approximation properties of projectors Π_k^0 and \mathcal{P}^h , after substituting $\varphi_h^v = v_h - \mathcal{P}^h v$ and $\varphi_h^w = w_h - \mathcal{P}^h w$ in (5.9) and (5.10), respectively, we deduce

$$\begin{aligned}
&\left(m_h(v_h^n - \mathcal{P}^h v(\cdot, t_n), v_h^n - \mathcal{P}^h v(\cdot, t_n)) + m_h(w_h^n - \mathcal{P}^h w(\cdot, t_n), w_h^n - \mathcal{P}^h w(\cdot, t_n)) \right) \\
&\leq \left(m_h(v_h^{n-1} - \mathcal{P}^h v(\cdot, t_{n-1}), v_h^n - \mathcal{P}^h v(\cdot, t_n)) + m_h(w_h^{n-1} - \mathcal{P}^h w(\cdot, t_{n-1}), w_h^n - \mathcal{P}^h w(\cdot, t_n)) \right) \\
&+ C\Delta t \left[(1 + |v(\cdot, t_n)|_2) \|\Pi_k^0 v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_{L^2(\Omega)} + \|\Pi_k^0 w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_{L^2(\Omega)} \right. \\
&+ h^{k+1} (|v(\cdot, t_n)|_{H^{k+1}(\Omega)} + |w(\cdot, t_n)|_{H^{k+1}(\Omega)}) + \int_{t_{n-1}}^{t_n} (\|\partial_{tt}^2 v(\cdot, s)\|_{L^2(\Omega)} + \|\partial_{tt}^2 w(\cdot, s)\|_{L^2(\Omega)}) ds \\
&+ h^{k+1} \int_{t_{n-1}}^{t_n} (|v_t(\cdot, s)|_{H^{k+1}(\Omega)} + |w_t(\cdot, s)|_{H^{k+1}(\Omega)}) ds + h^{k+1} |I_{\text{app}}(\cdot, t_n)|_{H^{k+1}(\Omega)} \Big] \times \\
&\left(\|v_h^n - \mathcal{P}^h v(\cdot, t_n)\| + \|w_h^n - \mathcal{P}^h w(\cdot, t_n)\| \right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\left(\|v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_h + \|w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_h \right) \\
&\leq \left(\|v_h^{n-1} - \mathcal{P}^h v(\cdot, t_{n-1})\|_h + \|w_h^{n-1} - \mathcal{P}^h w(\cdot, t_{n-1})\|_h \right) \\
&+ C\Delta t \left[(1 + |v(\cdot, t_n)|_2) \|\Pi_k^0 v_h^n - \mathcal{P}^h v(\cdot, t_n)\|_{L^2(\Omega)} + \|\Pi_k^0 w_h^n - \mathcal{P}^h w(\cdot, t_n)\|_{L^2(\Omega)} \right. \\
&+ h^{k+1} (|v(\cdot, t_n)|_{H^{k+1}(\Omega)} + |w(\cdot, t_n)|_{H^{k+1}(\Omega)}) + \int_{t_{n-1}}^{t_n} (\|\partial_{tt}^2 v(\cdot, s)\|_{L^2(\Omega)} + \|\partial_{tt}^2 w(\cdot, s)\|_{L^2(\Omega)}) ds \\
&+ h^{k+1} \int_{t_{n-1}}^{t_n} (|v_t(\cdot, s)|_{H^{k+1}(\Omega)} + |w_t(\cdot, s)|_{H^{k+1}(\Omega)}) ds + h^{k+1} |I_{\text{app}}(\cdot, t_n)|_{H^{k+1}(\Omega)} \Big]
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&\leq \left(\|v_h^0 - \mathcal{P}^h v(\cdot, 0)\|_h + \|w_h^0 - \mathcal{P}^h w(\cdot, 0)\|_h \right) \\
&\quad + C\Delta t \sum_{l=1}^n \left[(1 + |v(\cdot, t_l)|_2) (\|v_h^l - \mathcal{P}^h v(\cdot, t_l)\|_h + \|w_h^l - \mathcal{P}^h w(\cdot, t_l)\|_h) \right. \\
&\quad + \int_{t_{l-1}}^{t_l} (\|\partial_{tt}^2 v(\cdot, s)\|_{L^2(\Omega)} + \|\partial_{tt}^2 w(\cdot, s)\|_{L^2(\Omega)}) ds + \|v_0 - v_{0,h}\|_{L^2(\Omega)} + \|w_0 - w_{0,h}\|_{L^2(\Omega)} \\
&\quad + h^{k+1} \left(|v_0|_{H^{k+1}(\Omega)} + |w_0|_{H^{k+1}(\Omega)} + |v(\cdot, t_l)|_{H^{k+1}(\Omega)} + |w(\cdot, t_l)|_{H^{k+1}(\Omega)} \right. \\
&\quad \left. \left. + \int_{t_{l-1}}^{t_l} (|v(\cdot, s)|_{H^{k+1}(\Omega)} + |w(\cdot, s)|_{H^{k+1}(\Omega)}) ds + |I_{\text{app}}(\cdot, t_l)|_{H^{k+1}(\Omega)} \right) \right]
\end{aligned}$$

Finally, we use the equivalence of the norm $\|\cdot\|_h := m_h(\cdot, \cdot)$ with the L^2 norm and an application of discrete Gronwall inequality to (5.11) to get (5.3). This concludes the proof of Theorem 5.2. \blacksquare

6. Numerical results

In the present section, we report some numerical examples of the proposed virtual element method. With this aim, we have implemented in a MATLAB code the lowest-order VEM ($k = 1$) on arbitrary polygonal meshes following the ideas proposed in [10]. Moreover, we solve the nonlinear problem derived from (3.9) by a classical Picard-type iteration.

To complete the choice of the VEM, we have to choose the bilinear forms $S^K(\cdot, \cdot)$ and $S_0^K(\cdot, \cdot)$ satisfying (3.1) and (3.2), respectively. In this respect, we have proceeded as in [8, Section 4.6]: for each polygon K with vertices P_1, \dots, P_{N_K} , we have used

$$\begin{aligned}
S^K(u, v) &:= \sum_{r=1}^{N_K} u(P_r) v(P_r), \quad u, v \in W_{1|K}, \\
S_0^K(u, v) &:= h_K^2 \sum_{r=1}^{N_K} u(P_r) v(P_r), \quad u, v \in W_{1|K}.
\end{aligned}$$

A proof of (3.1)-(3.2) for the above (standard) choices could be derived following the arguments in [1, 8, 14]. The choices above are standard in the Virtual Element Literature and correspond to a scaled identity matrix in the space of the degrees of freedom values.

In all the numerical examples we have considered $H(v, w)$ and $I_{\text{ion}}(v, w)$ as in (2.4) and (2.5), respectively. Moreover, we have tested the method by using different families of meshes (see Figure 1).

6.1. Example 1

The aim of this numerical example is to test the convergence properties of the proposed VEM. With this objective, we introduce the following discrete relative L^2 norm of the difference between the interpolant w_I of a reference solution obtain on an extremely fine mesh and the numerical solution w_h at the final time T , that is,

$$E_{h,\Delta t}^2 := \frac{m_h(w_I(\cdot, T) - w_h(\cdot, T), w_I(\cdot, T) - w_h(\cdot, T))}{m_h(w_I(\cdot, T), w_I(\cdot, T))}.$$

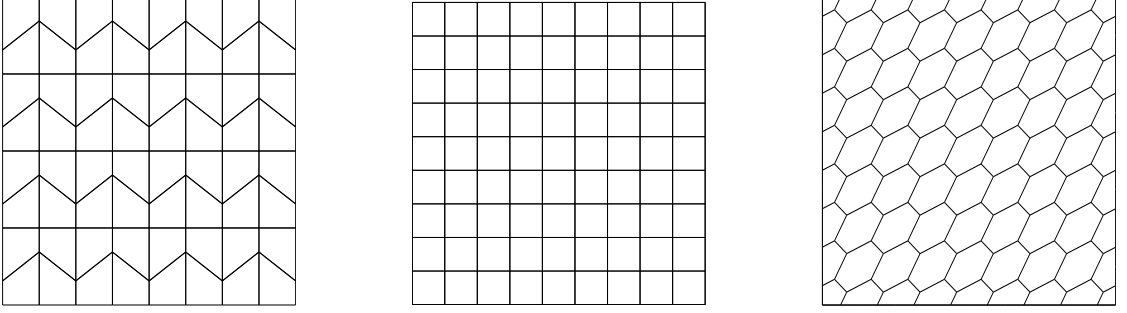


Figure 1: Sample meshes: \mathcal{T}_h^1 (left), \mathcal{T}_h^2 (center), \mathcal{T}_h^3 (right).

For this example, the domain will be $\Omega = (0, 1)^2$ and the time interval $[0, 1]$, we will take the model constants as follow $a = 0.2232, b = 0.9, \lambda = -1, \theta = 0.004$. We also take $I_{app} = 0$ and $D(x) = 0.01x$. Moreover, we consider the following initial data:

$$v^0(x, y) = \left(1 + 0.5 \cos(4\pi x) \cos(4\pi y)\right), \quad w^0(x, y) = \left(1 + 0.5 \cos(8\pi x) \cos(8\pi y)\right).$$

Due to the lack of exact solution for this example, we compute errors using a numerical solution on an extremely fine mesh ($h = 1/512$) and time step ($\Delta t = 1/200$) as reference v_{ref}, w_{ref} .

We report in Table 1 the relative errors in the discrete L^2 -norm of the variable v , for the family of meshes \mathcal{T}_h^2 and different refinement levels and time steps.

$h \setminus \Delta t$	$\Delta t = 1/10$	$\Delta t = 1/20$	$\Delta t = 1/40$	$\Delta t = 1/80$
1/8	0.039090164250364	0.022444307847497	0.017875789409797	0.016570898457504
1/16	0.032397693528292	0.013142430630831	0.007095750525537	0.005512035102741
1/32	0.031718009281784	0.011763920419019	0.004646804404667	0.002076778899273
1/64	0.031626299496412	0.011596060410183	0.004354704245183	0.001528339183782

Table 1: Test 1: Computed error in the discrete L^2 norm for v .

It can be seen from Table 1 that the error in the discrete L^2 norm reduced with a quadratic order with respect to h , which is the expected order of convergence for $k = 1$.

We show in Figure 2 the profiles of the computed quantities.

6.2. Example 2

In this test, we consider a benchmark example. We solve the FitzHugh-Nagumo equation using meshes \mathcal{T}_h^1 (with $h = 1/128$) on the unit square and time interval $[0, 5]$ (with $\Delta t = 1/100$) and with the following model constants: $a = 0.16875, b = 1, \lambda = -100, \theta = 0.25$. Moreover, we consider the following initial data:

$$v^0(x, y) = \left(1 - \frac{1}{1 + e^{-50(x^2+y^2)^{1/2}-0.1}}\right), \quad w^0(x, y) = 0.$$

After $4ms$, an instantaneous stimulus is applied in $(x_0, y_0) = (0.5, 0.5)$ to the transmembrane potential v ,

$$I_{app} = \begin{cases} 1 & mV & \text{if } (x - x_0)^2 + (y - y_0)^2 < 0.04 \\ 0 & mV & \text{otherwise.} \end{cases} \quad cm^2,$$

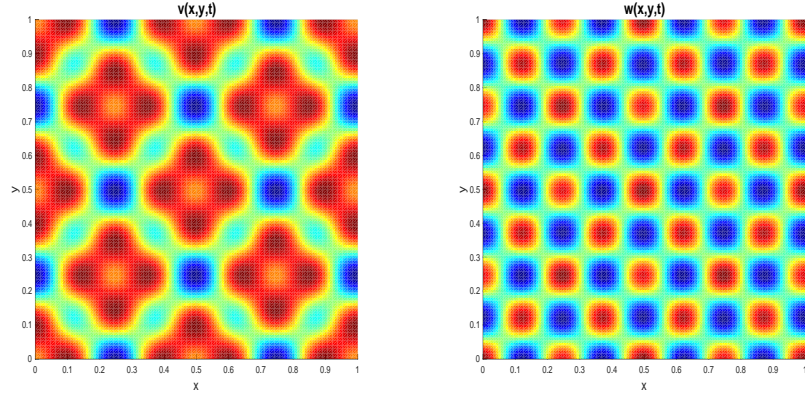


Figure 2: Variables v and w for $h = 1/64$ and $\Delta t = 1/80$.

We show in Figure 3 the evolution of the transmembrane potential v for different times.

6.3. Example 3

The aim of this test is to obtain the well-known periodic spiral wave (see Figure 4). For this example, we use meshes \mathcal{T}_h^3 (with $h = 1/128$) on the domain $\Omega := (0, 1)^2$, and time interval $[0, 15]$ (with $\Delta t = 1/200$). We will take the model constants as follow $a = 0.16875, b = 1, \lambda = -100, \theta = 0.25$. Moreover, we consider the following initial data:

$$v^0(x, y) = \begin{cases} 1.4 & \text{if } x < 0.5 \text{ and } y < 0.5 \\ 0 & \text{otherwise,} \end{cases}$$

$$w^0(x, y) = \begin{cases} 0.15 & \text{if } x > 0.5 \text{ and } y < 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

As expected the initial data evolves to a spiral wave; see Figure 4.

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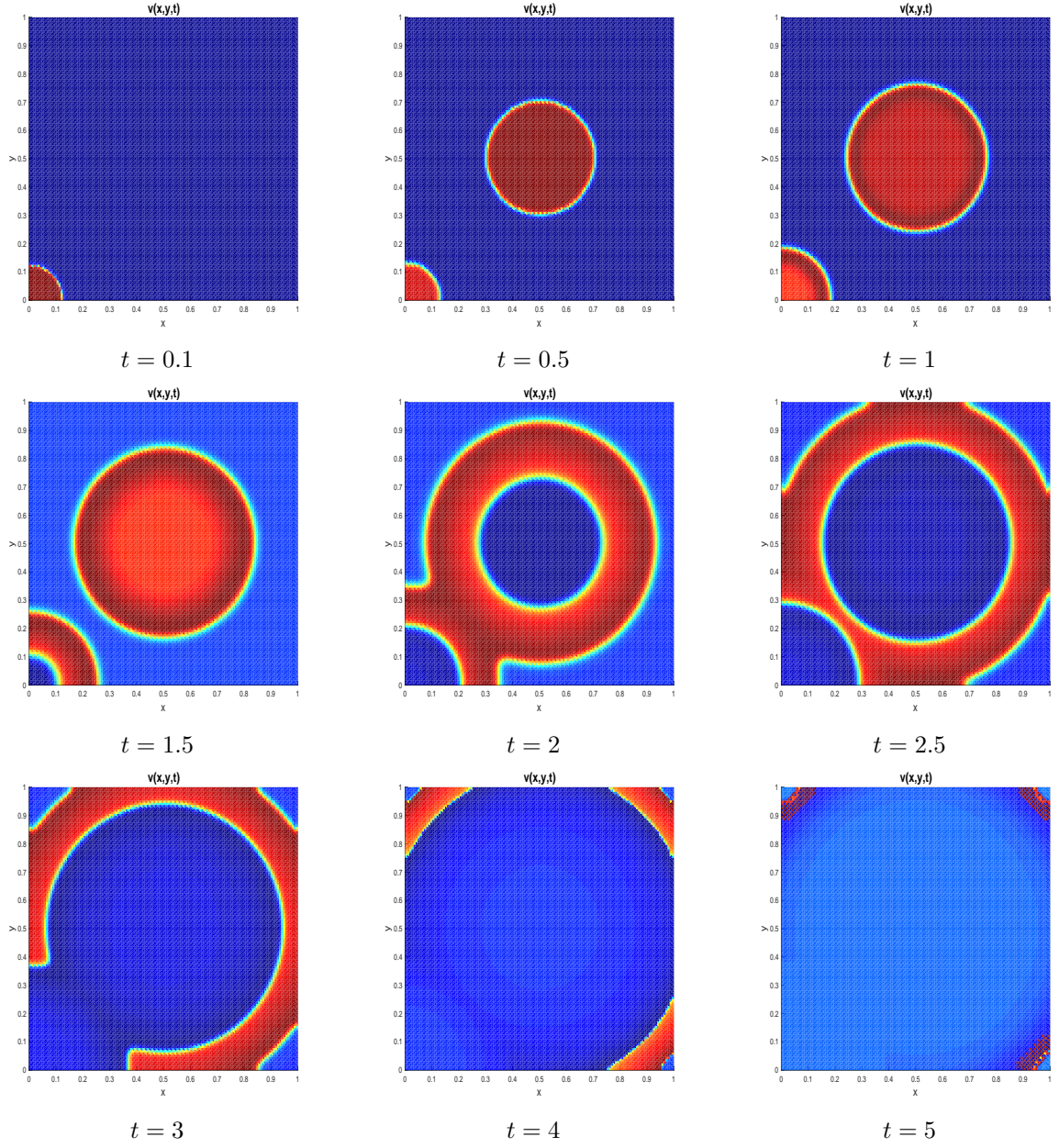


Figure 3: Numerical solution of the transmembrane potential v for different times.

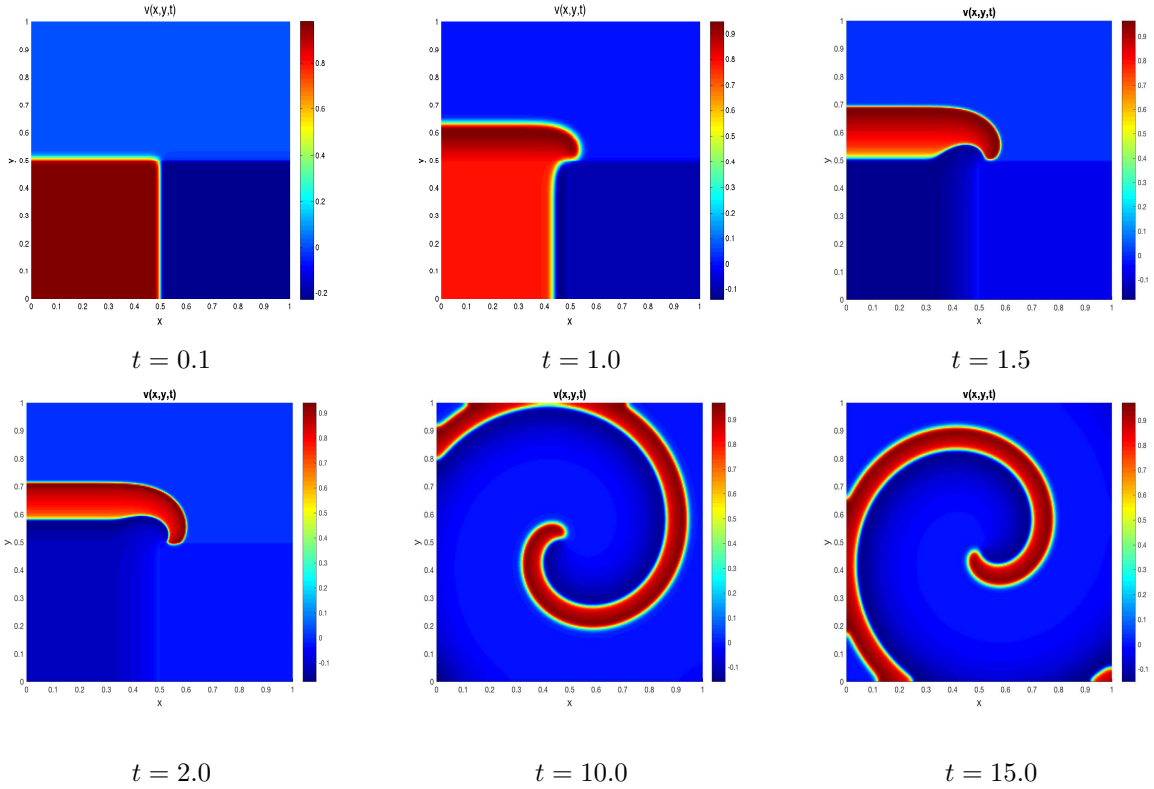


Figure 4: Numerical solution of the transmembrane potential v for different times.

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