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Convergence of a lowest-order finite element method for the transmission eigenvalue problem

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Abstract The transmission eigenvalue problem arises in scattering theory. The main difficulty in its analysis is the fact that, depending on the chosen formulation, it leads either to a quadratic eigenvalue problem or to a non-classical mixed problem. In this paper we prove the convergence of a mixed finite element approximation. This approach, which is close to the Ciarlet-Raviart discretization of biharmonic problems, is based on Lagrange finite elements and is one of the less expensive methods in terms of the amount of degrees of freedom. The convergence analysis is based on classical abstract spectral approximation result and the theory of mixed finite element methods for solving the stream function-vorticity formulation of the Stokes problem. Numerical experiments are reported in order to assess the efficiency of the method.

Keywords Transmission eigenvalues · Mixed formulation · Lagrange finite element approximation

Mathematics Subject Classification (2000) 65N25 · 65N30

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1 Introduction

We consider the interior transmission eigenvalue problem corresponding to the scattering problem for an isotropic inhomogeneous medium for the Helmholtz equation. Let us assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex Lipschitz domain. The transmission eigenvalue problem is to find $k^2 \in \mathbb{C}$ and non vanishing $w, z \in \mathrm{L}^2(\Omega)$ such that $w - z \in \mathrm{H}^2(\Omega)$ and

$$\Delta w + k^2 n w = 0 \qquad \text{in } \Omega, \tag{1a}$$

$$\Delta z + k^2 z = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$w - z = 0 \qquad \text{on } \partial\Omega, \tag{1c}$$

$$\frac{\partial}{\partial \nu}(w - z) = 0 \qquad \text{on } \partial\Omega, \tag{1d}$$

where ν is the unit outward normal to the boundary $\partial\Omega$. Here, n is the index of refraction, which is a positive coefficient that may vary along Ω but is always either greater or less than the index of refraction of the background medium. More precisely, we assume that there exists two positive numbers n_* and n^* such that n has to satisfy either one of the following assumptions for all $x \in \Omega$:

$$1 < n_* \leq n(x) \leq n^* < \infty, \tag{2a}$$

$$0 < n_* \leq n(x) \leq n^* < 1. \tag{2b}$$

Values of $k > 0$ such that there exists a nontrivial solution to problem (1) are called transmission eigenvalues. This problem has important applications in inverse scattering. It can be used to obtain estimates for the material properties of the scattering object and have a theoretical importance for the analysis of reconstruction in inverse scattering theory. For this reason, this problem has attracted the attention of many researchers.

From the mathematical point of view, transmission eigenvalue problems are nonstandard and difficult to treat. This is mainly because their different formulations lead either to quadratic or to non standard mixed eigenvalue problems and, thus, classical theory can not be directly applied. Despite this, the following theorem about the existence of infinitely many transmission eigenvalues has been proved in [2, Theorem 2.5]:

Theorem 1 *Let* $n \in \mathrm{L}^\infty(\Omega)$ *satisfy either* (2a) *or* (2b). *Then, there exists an infinite set of transmission eigenvalues with* $+\infty$ *as the only accumulation point.*

The numerical computation of transmission eigenvalues has been studied for many researchers. Several numerical schemes based on finite elements have been proposed during the last years. Very likely, the first reference in this respect is [5], where the authors introduced three different formulations of the problem and approximations of each of them based on Argyris H^2-conforming elements, Raviart-Thomas elements and Lagrange finite elements, respectively.

Computational results were reported in this reference but not a theoretical approximation analysis. Rigorous numerical analyses were done in [3,9] for variants of the method based on Argyris elements, which are significantly expensive because of the high degree of these elements (see also [14,7]). More recently, H^2-nonconforming methods applied to similar formulations have been also theoretically studied in [10,16]. On the other hand, according to our numerical experiments, the method based on Raviart-Thomas elements allows approximating the transmission eigenvalues, but also introduces spurious modes.

The aim of this paper is to analyze theoretically the method based on Lagrange elements introduced in [5] and to prove its spectral convergence. Let us remark that this is not the only method based on Lagrange elements proposed for the computation of transmission eigenvalues. In fact, an alternative approach based on a formulation in terms of three scalar fields that can be approximated also by Lagrange elements has been studied in [15]. In this case, convergence has been proved, but only for elements of degree $l \geq 2$.

In this paper, we resort to the formulation proposed in [8] discretized by piecewise linear continuous elements and provide a theoretical analysis. When $k \neq 0$, this method is equivalent to the one proposed in [5, Section 4.2]. However, one advantage of this approach as compared with the Lagrange formulation introduced in [5], is that the zero transmission eigenvalue, which has an infinitely dimensional space, is eliminated (see [8]). The analysis presented here follows the theory of spectral approximation of compact operators (see, for instance, [1, Chapter 2]) and the error analysis of a mixed finite element method for solving the Stokes problem in stream function-vorticity formulation studied, for instance, in [6, Section 3].

The article is organized as follows. In Section 2, we recall the mixed formulation proposed in [8], introduce the solution operator and characterize its spectrum. Next, in Section 3, we introduce the numerical approach based on piecewise linear elements and prove its convergence. Finally, in Section 4, we report some numerical experiments.

2 A mixed variational formulation

For the sake of definiteness, we restrict our attention to the case in which the refractive index n satisfies (2a). The other case, (2b), can be treated in a similar manner. We write $n = 1 + q$, so that from (2a) we have that $0 < n_* - 1 \leq q(x) \leq n^* - 1 < \infty$ for all $x \in \Omega$.

We first rewrite the transmission eigenvalue problem (1) as in [8,15,3]. We consider the change of variables $u := w - z \in \mathrm{H}_0^2(\Omega)$, $v := -k^2 z \in \mathrm{L}^2(\Omega)$ and obtain the following equations:

$$-\Delta u + qv = k^2 (1 + q) u \qquad \text{in } \Omega, \tag{3a}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } \Omega. \tag{3b}$$

It is easy to check that, for $k \neq 0$, problems (1) and (3) are actually equivalent.

If we further assume that the index of refraction n is smooth, then it can be proved that the solutions of problem (3) have some additional regularity. From now on, we assume that

$$n = 1 + q \in \mathrm{W}^{2,\infty}(\Omega).$$

Hence, dividing by q and taking Laplacian in (3a), by using (3b) we obtain that u satisfies

$$-\Delta^2 u = q\left[k^2 v + \Delta\left(\frac{1+q}{q}\,k^2 u\right) + \Delta\left(\frac{1}{q}\right)\Delta u + 2\nabla\left(\frac{1}{q}\right)\cdot\nabla\Delta u\right] \in \mathrm{H}^{-1}(\Omega), \tag{4a}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{4b}$$

Thus, $u \in \mathrm{H}^3(\Omega) \cap \mathrm{H}^2_0(\Omega)$ (see, for instance, [6, Theorem I.1.12]), which together with (3a) imply that $v \in \mathrm{H}^1(\Omega)$. Then, multiplying equations (3) by suitable test functions and integrating by parts, we obtain the following weak formulation: Find $k^2 \in \mathbb{C}$ and non vanishing $(u, v) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$(\nabla u, \nabla\varphi) + (qv, \varphi) = k^2\left((1+q)\,u, \varphi\right) \qquad \forall\varphi \in \mathrm{H}^1(\Omega), \tag{5a}$$

$$(\nabla v, \nabla\psi) = k^2\,(v, \psi) \qquad \forall\psi \in \mathrm{H}^1_0(\Omega). \tag{5b}$$

This problem is actually equivalent to problem (1), as is shown in what follows.

Lemma 1 *If (k^2, w, z) is a solution to problem (1) with $k \neq 0$, then $(k^2, w - z, -k^2 z)$ is a solution to problem (5). Conversely, if (k^2, u, v) is a solution to problem (5), then $k \neq 0$ and $(k^2, u - v/k^2, -v/k^2)$ is a solution to problem (1).*

Proof. We have just proved the first assertion. For the converse, let (k^2, u, v) be a solution to problem (5). Then, $k \neq 0$ (in fact, otherwise, taking $\varphi = v$ and $\psi = u$ would lead to $v = 0$ and, consequently, $u = 0$). On the other hand, integrating by parts the equations of problem (5), we obtain that (k^2, u, v) satisfies (3) and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Hence, since Ω is convex, $u \in \mathrm{H}^2_0(\Omega)$. Thus, (k^2, u, v) is a solution to problem (3), which, as was stated above, is equivalent to problem (1). $\square$

The following solution operator will be used to study the numerical approximation of problem (5):

$$\begin{aligned} T:\ \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega) &\longrightarrow \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega), \\ (f, g) &\longmapsto T(f, g) := (u, v), \end{aligned}$$

with $(u, v) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega)$ satisfying

$$(\nabla u, \nabla\varphi) + (qv, \varphi) = \left((1+q)\,f, \varphi\right) \qquad \forall\varphi \in \mathrm{H}^1(\Omega), \tag{6a}$$

$$(\nabla v, \nabla\psi) = (g, \psi) \qquad \forall\psi \in \mathrm{H}^1_0(\Omega). \tag{6b}$$

Next, we prove that this solution operator is well-posed and regularizing.

Lemma 2 *For each* $(f,g)\in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega)$*, problem* (6) *has a unique solution* $(u,v)\in \mathrm{H}_0^1(\Omega)\times\mathrm{H}^1(\Omega)$*. Moreover,* $u\in\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^2(\Omega)$ *and there exists* $C>0$*, independent of* f *and* g*, such that*

$$\|u\|_{3,\Omega}+\|v\|_{1,\Omega}\le C\left(\|f\|_{1,\Omega}+\|g\|_{0,\Omega}\right). \tag{7}$$

Proof. We start by proving the existence of solution. To this end, for $(f,g)\in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega)$, let $u\in\mathrm{H}_0^2(\Omega)$ be such that

$$\int_\Omega \frac{1}{q}\Delta u\,\Delta\varphi=-\int_\Omega g\varphi+\int_\Omega\nabla\left(\frac{1+q}{q}f\right)\cdot\nabla\varphi\qquad\forall\varphi\in\mathrm{H}_0^2(\Omega).$$

As a consequence of Lax-Milgram lemma, there exists a unique u that satisfies the equation above and $\|u\|_{2,\Omega}\le C(\|f\|_{1,\Omega}+\|g\|_{0,\Omega})$. In addition, u satisfies

$$-\Delta\left(\frac{1}{q}\Delta u\right)=g+\Delta\left(\frac{1+q}{q}f\right)\in\mathrm{H}^{-1}(\Omega), \tag{8}$$

Then, proceeding as to derive (4), we obtain

$$\begin{aligned}&-\Delta^2u=q\left[g+\Delta\left(\frac{1+q}{q}f\right)+\Delta\left(\frac{1}{q}\right)\Delta u+2\nabla\left(\frac{1}{q}\right)\cdot\nabla\Delta u\right]\in\mathrm{H}^{-1}(\Omega),\\&u=\frac{\partial u}{\partial\nu}=0\quad\text{on }\partial\Omega.\end{aligned}$$

Thus, $u\in\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^2(\Omega)$ (cf. [6, Theorem I.1.12], again) and $\|u\|_{3,\Omega}\le C(\|f\|_{1,\Omega}+\|g\|_{0,\Omega})$, too.

Next, we define

$$v:=\frac{1+q}{q}f+\frac{1}{q}\Delta u.$$

Since $u\in\mathrm{H}^3(\Omega)$, we have that $v\in\mathrm{H}^1(\Omega)$ and $\|v\|_{1,\Omega}\le C(\|f\|_{1,\Omega}+\|g\|_{0,\Omega})$, as well. On the other hand, from the definition of v, we conclude that (u,v) satisfies (6a) and, from (8),

$$\Delta v=\Delta\left(\frac{1+q}{q}f+\frac{1}{q}\Delta u\right)=-g,$$

which leads to (6b). Thus, (u,v) as defined above is a solution of problem (6).

To prove that this solution is unique, let us consider $(f,g)=(0,0)$. In such a case, by taking $\varphi=v$ in (6a) and $\psi=u$ in (6b), it follows that $v=0$. Then, by taking $\varphi=u$ in (6a), we conclude that $u=0$. Hence, problem (6) has a unique solution and we conclude the proof of the lemma. □

From the previous lemma, T is a bounded linear operator and, in addition, compact. The latter is a consequence of the fact that $T(\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega))\subset\left[\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^2(\Omega)\right]\times\mathrm{H}^1(\Omega)\hookrightarrow\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega)$, the second inclusion being compact. Thus, the spectrum of T, $\sigma(T)$, has 0 as the only possible accumulation point and any nonzero $\mu\in\sigma(T)$ is an eigenvalue of finite multiplicity. Moreover, it is easy to check that $\mu=0$ is not an eigenvalue of T.

The spectrum of the solution operator T is related with the solutions of problem (5) and, *a fortiori*, with those of problem (1) (cf. Lemma 1). In fact, the following result is easy to check.

Lemma 3 *$T(u,v)=\mu(u,v)$ with $\mu\neq 0$ if and only if (k^2,u,v) is a solution of problem* (5) *with $k^2=1/\mu$.*

As a consequence of the previous lemma, Lemma 1 and Theorem 1, we know that $\sigma(T)$ contains a sequence of nonzero finite-multiplicity real eigenvalues converging to zero.

3 Finite element approximation

In this section we consider a Galerkin approximation of problem (5) and prove convergence for the computed eigenvalues and eigenfunctions. With this end, we proceed as in [8] and follow the approach from [4], which is based on discretizing the problem with standard Lagrange finite elements.

We consider a regular family of partitions of Ω in triangles $\{\mathcal{T}_h\}_{h>0}$. The corresponding discrete spaces to approximate $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^1_0(\Omega)$ are

$$\mathcal{L}_h := \{\varphi_h \in \mathcal{C}(\Omega):\ \varphi_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h\} \quad \text{and} \quad \mathcal{L}_h^0 := \mathcal{L}_h \cap \mathrm{H}^1_0(\Omega),$$

respectively. The Galerkin approximation of problem (5) reads as follows: Find $k_h^2 \in \mathbb{C}$ and non vanishing $(u_h, v_h) \in \mathcal{L}_h^0 \times \mathcal{L}_h$ such that

$$(\nabla u_h, \nabla \varphi_h) + (q v_h, \varphi_h) = k_h^2\left((1+q)\,u_h, \varphi_h\right) \qquad \forall \varphi_h \in \mathcal{L}_h, \tag{9a}$$

$$(\nabla v_h, \nabla \psi_h) = k_h^2\,(v_h, \psi_h) \qquad \forall \psi_h \in \mathcal{L}_h^0. \tag{9b}$$

Let us remark that problem (5) does not fit into the classical theoretical framework for mixed eigenvalue problems analyzed in [12]. Indeed, according to the terminology used in this reference, this problem is neither of type (Q1) nor of type (Q2).

On the other hand, the discrete problem (9) can also be seen as a finite element approximation of the following variationally formulated eigenvalue problem:

$$a((u,v),(\psi,\varphi)) = k^2 b((u,v),(\psi,\varphi)) \qquad \forall (\psi,\varphi) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1(\Omega),$$

where $a((u,v),(\psi,\varphi)) := (\nabla u, \nabla\varphi) + (qv,\varphi) + (\nabla v, \nabla\psi)$ and $b((u,v),(\psi,\varphi)) := ((1+q)\,u,\varphi) + (v,\psi)$. Therefore, at first glance, one could try to study the numerical approximation of this problem by using the approximation theory for variationally posed eigenvalue problems analyzed in [11] (see also [1, Section II.8]). However, to apply the results from these references, it would be necessary to prove that there exists a constant β (which in principle could depend on h) such that

$$\sup_{(u_h,v_h)\in\mathcal{L}_h^0\times\mathcal{L}_h} \frac{a((u_h,v_h),(\psi_h,\varphi_h))}{\|u_h\|_{1,\Omega} + \|v_h\|_{1,\Omega}} \geq \beta\left(\|\psi_h\|_{1,\Omega} + \|\varphi_h\|_{1,\Omega}\right) > 0$$

and

$$\lim_{h\to 0}\left[\frac{1}{\beta}\inf_{(\psi_h,\varphi_h)\in\mathcal{L}_h^0\times\mathcal{L}_h}\Big(\|u-\psi_h\|_{1,\Omega}+\|v-\varphi_h\|_{1,\Omega}\Big)\right]=0 \qquad \forall (u,v)\in \mathrm{H}_0^1(\Omega)\times\mathrm{H}^1(\Omega),$$

which, to the best of the authors' knowledge, is not known to hold true.

Instead, our approach is to define the corresponding discrete solution operator, to prove that it converges in norm to T and to resort to the abstract spectral approximation theory for compact operators (cf. [1, Chapter 2]). With this aim, consider the discrete solution operator, which is defined by

$$\begin{aligned} T_h:\ \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega) &\longrightarrow \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega),\\ (f,g)&\longmapsto T_h(f,g):=(u_h,v_h),\end{aligned}$$

with $(u_h,v_h)\in\mathcal{L}_h^0\times\mathcal{L}_h$ satisfying

$$(\nabla u_h,\nabla\varphi_h)+(qv_h,\varphi_h)=((1+q)\,f,\varphi_h) \qquad \forall\varphi_h\in\mathcal{L}_h, \tag{10a}$$

$$(\nabla v_h,\nabla\psi_h)=(g,\psi_h) \qquad \forall\psi_h\in\mathcal{L}_h^0. \tag{10b}$$

Uniqueness of solution follows as in the continuous problem. Then, T_h is a well-defined linear operator. Moreover, as in the continuous case, its spectrum is related with the solutions of the eigenvalue problem (9). In fact, the following discrete version of Lemma 3 is easy to check.

Lemma 4 *$T_h(u_h,v_h)=\mu_h(u_h,v_h)$ with $\mu_h\neq 0$ if and only if (k_h^2,u_h,v_h) is a solution of problem* (9) *with $k_h^2=1/\mu_h$.*

Our next step is to analyze the convergence of T_h to the continuous operator T. With this aim, we study the convergence of the proposed Galerkin scheme (10). First, we notice that problem (6) and its discrete version, problem (10), are similar to two problems introduced for the analysis of the stream function-vorticity-pressure formulation of the Stokes problem in [6, Section 3] (see (2.20) and (2.29) from this reference, respectively). However, the right-hand side of the problems differ and this prevents us to use directly the error estimates from [6] to prove the convergence of T_h to T.

The following analysis is based on Theorem III.2.6 and Lemma III.3.1 from [6]. With this in mind, let us first introduce the elliptic projector $P_h:\mathrm{H}^1(\Omega)\to\mathcal{L}_h$ defined by

$$\begin{aligned}&(\nabla P_h v-\nabla v,\nabla\theta_h)=0 \qquad \forall\theta_h\in\mathcal{L}_h,\\ &(P_h v-v,1)=0.\end{aligned}$$

We also introduce the following sets:

$$\begin{aligned}V(f):=\big\{&(\psi,\varphi)\in\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega):\\ &(\nabla\psi,\nabla\omega)+(q\varphi,\omega)=((1+q)\,f,\omega)\ \ \forall\omega\in\mathrm{H}^1(\Omega)\big\},\end{aligned}$$

$$V_h(f) := \left\{(\psi_h, \varphi_h) \in \mathcal{L}_h^0 \times \mathcal{L}_h : \right.$$
$$\left. (\nabla\psi_h, \nabla\omega_h) + (q\varphi_h, \omega_h) = ((1+q)\, f, \omega_h) \;\; \forall \omega_h \in \mathcal{L}_h \right\}.$$

Our first step is to prove the following auxiliary error estimate.

Lemma 5 *Given $(f,g) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$, let $(u,v) := T(f,g)$ and $(u_h, v_h) := T_h(f,g)$. Then, there exists $C > 0$, independent of h, f and g, such that*

$$\|u - u_h\|_{1,\Omega} + \|v - v_h\|_{0,\Omega}$$
$$\leq C \inf_{(\psi_h,\varphi_h)\in V_h(f)} \Big(\|u - \psi_h\|_{1,\Omega} + \|v - \varphi_h\|_{0,\Omega} \Big) + C \, \|P_h v - v\|_{0,\Omega}. \quad (11)$$

Proof. Let $(\psi_h, \varphi_h) \in V_h(f)$. The sketch of the proof is as follows: we estimate $\|u - u_h\|_{1,\Omega}$ and $\|v - v_h\|_{0,\Omega}$ by adding and subtracting ψ_h and φ_h, respectively, bounding $\|u_h - \psi_h\|_{1,\Omega}$ and $\|v_h - \varphi_h\|_{0,\Omega}$ in terms of $\|v - \varphi_h\|_{0,\Omega}$ and $\|P_h v - v\|_{0,\Omega}$, and using triangle inequality.

Our first step is to bound $\|v_h - \varphi_h\|_{0,\Omega}$. With this aim, we recall that $q(x) \geq n_* - 1 > 0$ in Ω and write

$$(n_* - 1) \, \|v_h - \varphi_h\|_{0,\Omega}^2 \leq (q v_h - q\varphi_h, v_h - \varphi_h)$$
$$= (q v_h - q\varphi_h, v_h - P_h v) + (q v_h - q\varphi_h, P_h v - \varphi_h). \quad (12)$$

We will show that the first term on the right-hand side above vanishes. In fact, since (ψ_h, φ_h) and (u_h, v_h) belong to $V_h(f)$ (cf. (10a) for the latter), it is easy to check that

$$(\nabla u_h - \nabla\psi_h, \nabla\omega_h) + (q v_h - q\varphi_h, \omega_h) = 0 \qquad \forall \omega_h \in \mathcal{L}_h. \quad (13)$$

In particular, for $\omega_h := v_h - P_h v \in \mathcal{L}_h$, we have

$$(q v_h - q\varphi_h, v_h - P_h v) = -\left(\nabla u_h - \nabla\psi_h, \nabla v_h - \nabla P_h v\right). \quad (14)$$

On the other hand, since from (6b) and (10b) $(\nabla v, \nabla\theta_h) = (\nabla v_h, \nabla\theta_h)$ for all $\theta_h \in \mathcal{L}_h^0$, we have that

$$(\nabla v_h - \nabla P_h v, \nabla\theta_h) = (\nabla v - \nabla P_h v, \nabla\theta_h) = 0 \qquad \forall \theta_h \in \mathcal{L}_h^0,$$

the last equality because of the definition of the projector P_h. In particular, taking $\theta_h := u_h - \psi_h \in \mathcal{L}_h^0$,

$$(\nabla v_h - \nabla P_h v, \nabla u_h - \nabla\psi_h) = 0$$

and, substituting this in (14),

$$(q v_h - q\varphi_h, v_h - P_h v) = 0.$$

Now, using the equation above in (12) and Cauchy–Schwarz inequality, we write

$$
\begin{aligned}
(n_* - 1) \left\| v_h - \varphi_h \right\|_{0,\Omega}^2 &\leq (q v_h - q\varphi_h, P_h v - \varphi_h) \\
&\leq (n^* - 1) \left\| v_h - \varphi_h \right\|_{0,\Omega} \left\| P_h v - \varphi_h \right\|_{0,\Omega} .
\end{aligned}
$$

Therefore,

$$
\left\| v_h - \varphi_h \right\|_{0,\Omega} \leq C \left\| P_h v - \varphi_h \right\|_{0,\Omega} \leq C \left(\left\| v - \varphi_h \right\|_{0,\Omega} + \left\| P_h v - v \right\|_{0,\Omega} \right) \tag{15}
$$

with the positive constant C only depending on the upper and lower bounds of q.

Next step is to bound $\left\| u_h - \psi_h \right\|_{1,\Omega}$. Taking now $\omega_h := u_h - \psi_h$ in (13), Poincaré's inequality yields

$$
\begin{aligned}
\left\| \nabla u_h - \nabla \psi_h \right\|_{0,\Omega}^2 &\leq \left\| q v_h - q\varphi_h \right\|_{0,\Omega} \left\| u_h - \psi_h \right\|_{0,\Omega} \\
&\leq (n^* - 1) \left\| v_h - \varphi_h \right\|_{0,\Omega} C_P \left\| \nabla u_h - \nabla \psi_h \right\|_{1,\Omega} ,
\end{aligned}
$$

where C_P is the constant in Poincaré's inequality. Therefore, using again Poincaré's inequality,

$$
\left\| u_h - \psi_h \right\|_{1,\Omega} \leq C \left\| v_h - \varphi_h \right\|_{0,\Omega} \tag{16}
$$

with C only depending on the upper and lower bounds of q and the constant C_P.

To end the proof, we use triangle inequality, (15) and (16) to write

$$
\begin{aligned}
&\left\| u - u_h \right\|_{1,\Omega} + \left\| v - v_h \right\|_{0,\Omega} \\
&\qquad \leq \left\| u - \psi_h \right\|_{1,\Omega} + \left\| u_h - \psi_h \right\|_{1,\Omega} + \left\| v - \varphi_h \right\|_{0,\Omega} + \left\| v_h - \varphi_h \right\|_{0,\Omega} \\
&\qquad \leq C \left(\left\| u - \psi_h \right\|_{1,\Omega} + \left\| v - \varphi_h \right\|_{0,\Omega} \right) + C \left\| P_h v - v \right\|_{0,\Omega} .
\end{aligned}
$$

Since this inequality holds for all $(\psi_h, \varphi_h) \in V_h(f)$ with the same constant C, we conclude the proof. □

Next, we estimate each of the two terms on the right hand side of (11). We begin with the first one.

Lemma 6 *There exists a constant $C > 0$ such that, for all $(\psi, \varphi) \in V(f)$,*

$$
\begin{aligned}
&\inf_{(\psi_h,\varphi_h)\in V_h(f)} \left(\left\| \psi - \psi_h \right\|_{1,\Omega} + \left\| \varphi - \varphi_h \right\|_{0,\Omega} \right) \\
&\leq C \inf_{(\theta_h,\omega_h)\in \mathcal{L}_h^0 \times \mathcal{L}_h} \left[\left\| \psi - \theta_h \right\|_{1,\Omega} + \left\| \varphi - \omega_h \right\|_{0,\Omega} + \sup_{\xi_h \in \mathcal{L}_h} \frac{(\nabla\psi - \nabla\theta_h, \nabla\xi_h)}{\left\| \xi_h \right\|_{0,\Omega}} \right] .
\end{aligned}
\tag{17}
$$

Proof. Let $(\psi,\varphi)\in V(f)$. For each $(\theta_h,\omega_h)\in\mathcal{L}_h^0\times\mathcal{L}_h$, let $\tau_h\in\mathcal{L}_h$ be defined by

$$(q\tau_h,\xi_h)=(q\varphi-q\omega_h,\xi_h)+(\nabla\psi-\nabla\theta_h,\nabla\xi_h)\qquad\forall\xi_h\in\mathcal{L}_h.$$

Then,

$$\|\tau_h\|_{0,\Omega}\le C\,\|\varphi-\omega_h\|_{0,\Omega}+C\sup_{\xi_h\in\mathcal{L}_h}\frac{(\nabla\psi-\nabla\theta_h,\nabla\xi_h)}{\|\xi_h\|_{0,\Omega}},$$

with C only depending on the upper and lower bounds of q. Now, the definition of τ_h and the fact that $(\psi,\varphi)\in V(f)$ yield

$$(\nabla\theta_h,\nabla\xi_h)+(q\omega_h+q\tau_h,\xi_h)=(\nabla\psi,\nabla\xi_h)+(q\varphi,\xi_h)=((1+q)\,f,\xi_h)$$

for all $\xi_h\in\mathcal{L}_h$. Hence, if we define $(\psi_h,\varphi_h):=(\theta_h,\omega_h+\tau_h)$, then $(\psi_h,\varphi_h)\in V_h(f)$. Thus, the result is a consequence of the following triangular inequality,

$$\|\psi-\psi_h\|_{1,\Omega}+\|\varphi-\varphi_h\|_{0,\Omega}\le\|\psi-\theta_h\|_{1,\Omega}+\|\varphi-\omega_h\|_{0,\Omega}+\|\tau_h\|_{0,\Omega}\,,$$

and the estimate of $\|\tau_h\|_{0,\Omega}$. □

In order to estimate the last term on the right hand side of (17), we introduce the following elliptic projector: $P_{0,h}:\mathrm{H}_0^1(\Omega)\to\mathcal{L}_h^0$, defined by

$$(\nabla P_{0,h}v-\nabla v,\nabla\theta_h)=0\qquad\forall\theta_h\in\mathcal{L}_h^0,$$

We have the following estimate.

Lemma 7 *For each $\varepsilon\in\left(0,\frac{1}{2}\right)$, there exists a constant $C(\varepsilon)>0$ such that, for all $\psi\in\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^1(\Omega)$,*

$$\sup_{\omega_h\in\mathcal{L}_h}\frac{(\nabla\psi-\nabla P_{0,h}\psi,\nabla\xi_h)}{\|\xi_h\|_{0,\Omega}}\le C(\varepsilon)h^{1/2-\varepsilon}\,\|\psi\|_{3,\Omega}$$

Proof. Let $\psi\in\mathrm{H}^3(\Omega)\cap\mathrm{H}_0^1(\Omega)$. According to Lemma III.3.2 from [6], for each $p>2$, there exists $C>0$ such that

$$\sup_{\omega_h\in\mathcal{L}_h}\frac{(\nabla\psi-\nabla P_{0,h}\psi,\nabla\omega_h)}{\|\omega_h\|_{0,\Omega}}\le Ch^{1/2-1/p}\,\|\psi\|_{2,p,\Omega}\,.$$

Since from the Sobolev's embedding theorem, $\|\psi\|_{2,p,\Omega}\le C\,\|\psi\|_{3,\Omega}$ with a positive constant C depending on p (cf. [6, Theorem I.1.3]), the result follows from these estimates by taking $p:=1/\varepsilon$. □

Putting together the last three lemmas, we derive the following estimate.

Lemma 8 *For each $\varepsilon\in\left(0,\frac{1}{2}\right)$, there exists a constant $C(\varepsilon)>0$ such that, for all $(f,g)\in\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega)$, if $(u,v)=T(f,g)$ and $(u_h,v_h)=T_h(f,g)$, then*

$$\|u-u_h\|_{1,\Omega}+\|v-v_h\|_{0,\Omega}\le C(\varepsilon)h^{1/2-\varepsilon}\,\|u\|_{3,\Omega}+\|P_{0,h}u-u\|_{1,\Omega}+\|P_hv-v\|_{0,\Omega}\,.$$

Proof. The result follows by applying Lemma 6 to the first term on the right hand side of (11), choosing $\omega_h=P_hv$ and $\theta_h=P_{0,h}u$ and using Lemma 7. □

The following result is an immediate consequence of the previous lemma, estimate (7) and standard error estimates for P_h and $P_{0,h}$ (see, for instance, [6, Theorem A.2]).

Lemma 9 *The family of operators* $T_h : \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ *converge in norm to the operator* $T : \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega) \longrightarrow \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$, *namely*

$$\|T - T_h\|_{\mathcal{L}(\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega))} \longrightarrow 0 \qquad \text{as } h \to 0.$$

Now, we are in a position to write the main result of this paper, which yields analogous convergence properties of the proposed scheme, as a consequence of Lemmas 3 and 4.

Theorem 2 *Let* $\mu \in \sigma(T)$ *be an eigenvalue of multiplicity* m*. Then, there exists* $h_0 > 0$ *such that, for all* $h < h_0$, $\sigma(T_h)$ *contains exactly* m *eigenvalues* $\mu_h^{(1)}, \dots, \mu_h^{(m)}$ *(repeated according to their respective multiplicity) that converges to* μ *as* $h \to 0$.

Let $\mathcal{E}$ *be the invariant subspace of* T *associated to* μ*. Let* $\mathcal{E}_h$ *be the direct sum of the invariant subspaces of* T_h *associated to* $\mu_h^{(1)}, \dots, \mu_h^{(m)}$*. Let* $\widehat{\delta}(\mathcal{E}, \mathcal{E}_h)$ *be the gap between these subspaces, defined by* $\widehat{\delta}(\mathcal{E}, \mathcal{E}_h) := \max\{\delta(\mathcal{E}, \mathcal{E}_h), \delta(\mathcal{E}_h, \mathcal{E})\}$ *with*

$$\delta(\mathcal{E}, \mathcal{E}_h) := \sup_{\substack{(\psi,\varphi)\in\mathcal{E} \\ \|\psi\|_{1,\Omega}^2 + \|\varphi\|_{0,\Omega}^2 = 1}} \left[\inf_{(\psi_h,\varphi_h)\in\mathcal{E}_h} \left(\|\psi - \psi_h\|_{1,\Omega}^2 + \|\varphi - \varphi_h\|_{0,\Omega}^2 \right)^{1/2} \right]$$

and analogously for $\delta(\mathcal{E}_h, \mathcal{E})$*. Then,*

$$\widehat{\delta}(\mathcal{E}, \mathcal{E}_h) \longrightarrow 0 \qquad \text{as } h \to 0.$$

Proof. It is a direct consequence of Lemma 9 and Theorems 7.1 and 7.2 from [1]. □

4 Numerical tests

We report in this section the results of a couple of numerical tests obtained with the method analyzed above. With this purpose, we have implemented the numerical solution of problem (9) in a MATLAB code.

For the first test, we have considered the transmission eigenvalue problem in a disk. In such a geometry, when the index of refraction n is constant, the eigenvalues can be semi-analytically computed by numerically solving an algebraic equation involving Bessel functions (see [5]).

We have used a disk of radius 1/2 and a refractive index $n = 16$ as in [5]. We have solved the problem with our code on several meshes $\mathcal{T}_h$ with different levels of refinement; each mesh is identified by its respective number of triangles N_h. We have computed the five smallest real positive eigenvalues: $k_{h,1}, \dots, k_{h,5}$. We report in Table 1 the computed transmission eigenvalues as well as those determined in a semi-analytical manner, $k_{\mathrm{ex},j}$ (see [5]). The table also includes the estimated rates of convergence computed for each eigenvalue by means of a least-squares fitting.

Table 1 Computed and exact eigenvalues, $k_{h,j}$ and $k_{\mathrm{ex},j}$, respectively, and estimated rates of convergence.

N_h	256	$1,024$	$4,096$	$16,384$	$36,864$	$k_{\mathrm{ex},j}$	Rates
$k_{h,1}$	2.0269	1.9978	1.9905	1.9886	1.9883	1.9880	1.992
$k_{h,2}$	2.6834	2.6306	2.6174	2.6141	2.6134	2.6129	1.992
$k_{h,3}$	2.6837	2.6308	2.6174	2.6141	2.6134	2.6129	1.988
$k_{h,4}$	3.3586	3.2599	3.2351	3.2288	3.2276	3.2267	1.988
$k_{h,5}$	3.3791	3.2644	3.2360	3.2290	3.2277	3.2267	2.016

Problem (1) may have also complex eigenvalues. In fact, in the present setting, it is shown in [5] that they can be computed by finding complex zeros of the same algebraic equation as above. In particular, it is reported in [5] that $k_{\mathrm{ex}} = 4.901 \pm 0.578\mathrm{i}$ are complex eigenvalues of the same problem (radius $1/2$ and $n = 16$). According to our theoretical results (cf. Theorem 2), our code should also provide approximations of them. We report in Table 2 the same results as above for these complex eigenvalues.

Table 2 Disk of radius $1/2$. Computed and exact complex eigenvalues and estimated rates of convergence.

$N_h = 256$	$N_h = 1,024$	$N_h = 4,096$	$N_h = 16,384$	k_{ex}	Rate
$5.025 \pm 0.541\mathrm{i}$	$4.936 \pm 0.578\mathrm{i}$	$4.910 \pm 0.579\mathrm{i}$	$4.903 \pm 0.578\mathrm{i}$	$4.901 \pm 0.578\mathrm{i}$	1.970

For the second test, we have considered a different domain, a unit square, and the same refractive index, $n = 16$. In this case, in absence of an analytical solution, we have estimated for each eigenvalue the rate of convergence t and a more accurate approximation of the exact eigenvalue $\hat{k}_{\mathrm{ex},j}$, by means of a least-squares fitting of the model: $k_{h,j} \approx \hat{k}_{\mathrm{ex},j} + Ch^t$.

We report in Table 3 the five smallest eigenvalues computed on several meshes. As in the previous test, N_h denotes the corresponding number of triangles for each mesh. We also report the more accurate approximation $\hat{k}_{\mathrm{ex},j}$ and the estimated rates of convergence.

Table 3 Unit square. Computed eigenvalues $\hat{k}_{h,j}$, more accurate approximations $k_{\mathrm{ex},j}$ and estimated rates of convergence.

N_h	512	$2,048$	$8,192$	$32,768$	$131,072$	$\hat{k}_{\mathrm{ex},j}$	Rates
$k_{h,1}$	1.9073	1.8865	1.8813	1.8800	1.8797	1.8796	2.004
$k_{h,2}$	2.4857	2.4546	2.4468	2.4449	2.4444	2.4442	1.996
$k_{h,3}$	2.5089	2.4603	2.4483	2.4452	2.4445	2.4442	2.006
$k_{h,4}$	2.9635	2.8909	2.8726	2.8680	2.8668	2.8664	1.988
$k_{h,5}$	3.2392	3.1650	3.1463	3.1417	3.1405	3.1401	1.994

In all the tables, a quadratic rate of convergence can be clearly observed, which is consistent with the numerical results reported in [5] and [13].

Finally, Figures 1 and 2 show the eigenfunctions w (left) and z (right) from problem (1) corresponding to the two lowest transmission eigenvalues $k_{h,1}$ and $k_{h,2}$, respectively.

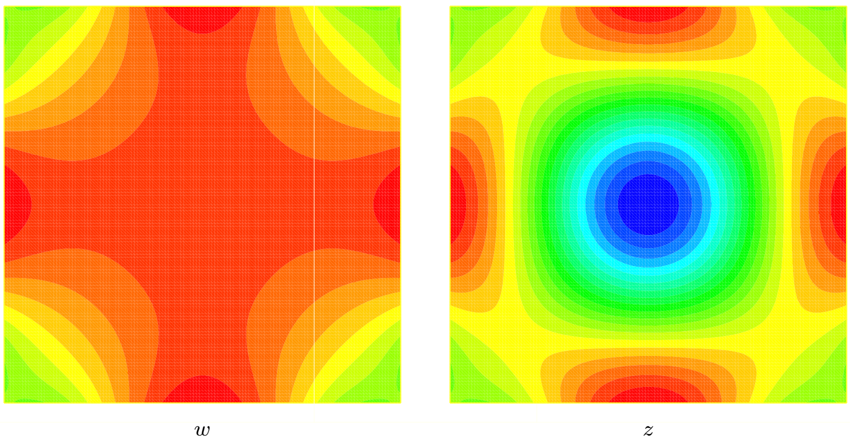

Fig. 1 Unit square. Eigenfunctions corresponding to $k_{h,1}$.

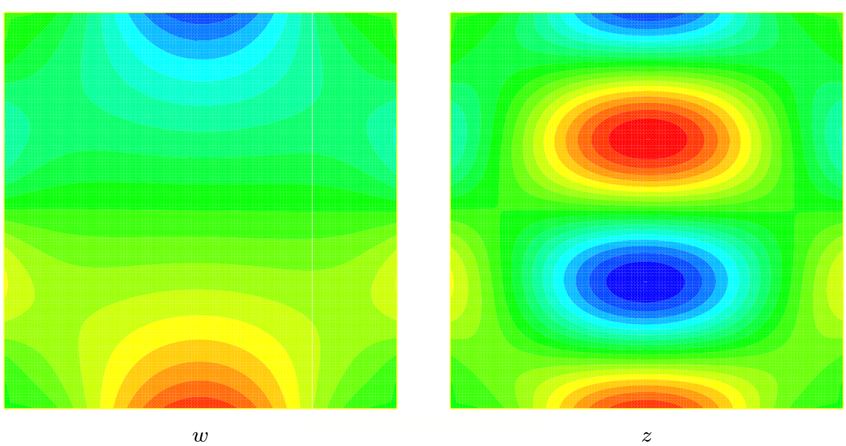

Fig. 2 Unit square. Eigenfunctions corresponding to $k_{h,2}$.

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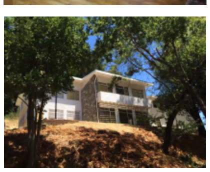

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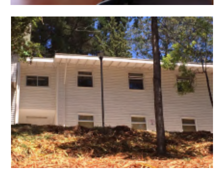

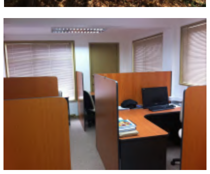

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