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A mixed virtual element method for the Navier-Stokes equations*

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Abstract

A mixed virtual element method (mixed-VEM) for a pseudostress-velocity formulation of the two-dimensional Navier-Stokes equations with Dirichlet boundary conditions is proposed and analyzed in this work. More precisely, we employ a dual-mixed approach based on the introduction of a non-linear pseudostress linking the usual linear one for the Stokes equations and the convective term. In this way, the aforementioned new tensor together with the velocity constitute the only unknowns of the problem, whereas the pressure is computed via a postprocessing formula. In addition, the resulting continuous scheme is augmented with Galerkin type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, all them multiplied by suitable stabilization parameters, so that the Banach fixed-point and Lax-Milgram theorems are applied to conclude the well-posedness of the continuous and discrete formulations. Next, we describe the main VEM ingredients that are required for our discrete analysis, which, besides projectors commonly utilized for related models, include, as the main novelty, the simultaneous use of virtual element subspaces for $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$ in order to approximate the velocity and the pseudostress, respectively. Then, the discrete bilinear and trilinear forms involved, their main properties and the associated mixed virtual scheme are defined, and the corresponding solvability analysis is performed using again appropriate fixed-point arguments. Moreover, Strang-type estimates are applied to derive the *a priori* error estimates for the two components of the virtual element solution as well as for the fully computable projections of them and the postprocessed pressure. As a consequence, the corresponding rates of convergence are also established. Finally, we follow the same approach employed in previous works by some of the authors and introduce an element-by-element post-processing formula for the fully computable pseudostress, thus yielding an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm.

Key words: Navier-Stokes problem, pseudostress-based formulation, augmented formulation, mixed virtual element method, high-order approximations

1 Introduction

The utilization of virtual element methods (VEM) in fluid mechanics has become a very active research subject in recent years. Indeed, regarding the Stokes equations, we begin by referring to [3], [8], and [19], where stream function-based, divergence free, and non-conforming virtual element methods, respectively, have been developed for the classical velocity-pressure formulation of this problem. In

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particular, a new family of virtual elements for the Stokes problem on polygonal meshes, in which the discrete velocity is pointwise divergence-free, is provided in [8]. Moreover, the associated virtual scheme is shown to be equivalent to a problem with less degrees of freedom, thus yielding a more efficient method. In turn, the virtual element method proposed in [19] approximates the pressure using discontinuous piecewise polynomials, whereas the components of the velocity are approximated using a globally nonconforming virtual element space. In fact, the virtual element functions are locally defined as the solution of Poisson problems with polynomial Neumann boundary conditions. More recently, a family of virtual element methods for the two-dimensional Navier-Stokes equations is introduced and analyzed in [9], which constitutes, up to our knowledge, the first paper applying the VEM technique to that nonlinear model. As in [8], pointwise divergence-free discrete velocities are also obtained in [9], and hence the virtual element scheme suggested there can be seen as a natural extension of the approach provided in [8].

Furthermore, other contributions in the aforementioned direction have concentrated on the combined use of pseudostress-based dual-mixed variational formulations and virtual element methods, thus yielding the first mixed-VEM schemes known so far for the Stokes and related models in fluid mechanics (see [13], [14], and [15]). Before describing the main aspects of these works in what follows, we notice that the name *dual-mixed* refers here to those formulations in which the main unknown of the resulting saddle point problem lives in either a vectorial $\mathbf{H}(\mathrm{div})$ or a tensorial $\mathbb{H}(\mathbf{div})$ space, which is precisely the case when the stress or the pseudostress is employed. Having said the above, we now recall that a mixed-VEM for the pseudostress-velocity formulation of the Stokes problem, in which the pressure is computed via a postprocessing formula, was introduced and analized in [13]. In particular, in order to derive the explicitly computable discrete bilinear form, a new local projector onto a suitable polynomial space, which takes into account the main features of the continuous solution and allows the explicit integration of terms involving deviatoric tensors, is proposed there. Then, the analysis in [13] is extended in [14] to derive two mixed virtual element methods for the two-dimensional Brinkman problem. Proceeding as in [27], the equilibrium equation and the incompressibility condition are utilized in [14] to eliminate the velocity and the pressure, respectively, whence the pseudostress becomes the only unknown of the resulting dual-mixed formulation. In this way, the aforementioned two schemes arise from the use of one from two different projectors, the ad-hoc one introduced in [13] and the L^2-orthogonal one analyzed in [7] (see also [5]). Another virtual element method for the Brinkman equations, not employing the aforementioned dual-mixed approach, is proposed in [33]. We end this paragraph by remarking that the analysis and results from [13] and [14] were extended in [15] to the case of quasi-Newtonian Stokes flows, for which the nonlinear model studied in [28] (see also [29]) was considered.

In addition to the above, it is important to highlight that the incorporation of the pseudostress as one of the main unknowns of a dual-mixed variational formulation in continuum mechanics, is mainly motivated by the need of finding new ways of circumventing the symmetry requirement of the usual stress-based methods. In particular, and since in this paper we are interested in developing a VEM scheme for a dual-mixed formulation of the Navier-Stokes equations, we begin the corresponding bibliographic discussion with [30], where a new mixed finite element method for that model was introduced and analyzed. More precisely, the main unknowns of the approach in [30] are given by the velocity, its gradient, and a modified nonlinear pseudostress tensor linking the usual stress and the convective term. A fixed-point argument and the Babuška-Brezzi theory are applied there to derive the well-posedness of the resulting continuous formulation. Then, the procedure from [30] is modified in [18] through the introduction of a new nonlinear tensor linking now the pseudostress (instead of the stress) and the convective term, which, together with the velocity, constitute the only unknowns. Suitable Galerkin type terms arising from the constitutive and equilibrium equations, and the boundary condition, are then incorporated into the formulation of [18], so that the Lax-Milgram and Banach fixed-point theorems are employed to prove the well-posedness of both the continuous and discrete schemes. In turn, the approach from [18] has been further extended to other boundary value problems, including the development of new dual-mixed formulations for the stationary Boussinesq problem (see [21], [22], [23], [24]), and for the Navier-Stokes equations with constant density and variable viscosity (see [16], [17]). Besides the methods and tools utilized in [18], the foregoing extensions also make use of the Brouwer fixed-point theorem and the Babuška-Brezzi theory.

According to the above discussion, and in order to additionally contribute in the direction drawn by [15] and [9], we now aim to continue extending the applicability of the VEM technique to nonlinear problems in fluid mechanics. More precisely, we consider the same variational formulation from [18] (see also [16], [17]), and develop, up to our knowledge, the first dual-mixed virtual element method for the Navier Stokes equations. The rest of this work is organized as follows. At the end of the present section we provide some useful notations. In Section 2 we describe our nonlinear model, recall from [18] the derivation of the augmented pseudostress-velocity formulation to be employed, and state the corresponding well-posedness result. Then, in Section 3 we introduce the virtual element subspaces approximating the velocity and the pseudostress in $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$, respectively, state their approximation properties, and define the projectors and remaining ingredients that are needed for the discrete analysis. In turn, computable discrete versions of the bilinear and trilinear forms involved, and of the corresponding functional on the right-hand side of the formulation, are locally and then globally defined in Section 4. In addition, the main mapping properties connecting them with their continuous versions are also proved in this section. Next, in Section 5 we define the associated mixed virtual element scheme, perform the solvability analysis by using suitable fixed-point arguments (as done in [18] and its further extensions), and apply Strang-type estimates to derive the *a priori* error estimates for both the virtual element solution and the fully computable projections of its components. The corresponding rates of convergence are then readily established by using the approximation properties given in Section 3. Finally, following previous works by some of the authors, an element-by-element postprocessing formula for the fully computable pseudostress is suggested at the last part of Section 5, which leads to an optimally convergent approximation of this unknown with respect to the broken $\mathbb{H}(\mathbf{div})$-norm.

Notations

We end the present section by providing some notations to be used along the paper, including those already employed above. Firstly, for any vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$ we set the gradient, divergence and tensor product operators as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \qquad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,2}\,,$$

respectively. In addition, denoting by $\mathbb{I}$ the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathbf{t}} := (\tau_{ji})\,, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathbf{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}\,, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij},$$

which corresponds, respectively, to the transpose, the trace, and the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. Next, given a bounded domain $\mathcal{O} \subseteq \mathrm{R}^2$, with boundary $\partial\mathcal{O}$, we let $\mathbf{n}$ be the outward unit normal vector on $\partial\mathcal{O}$. Also, given $r \geq 0$ and $p > 1$, we let $\mathrm{W}^{r,p}(\mathcal{O})$ be the standard Sobolev space with norm $\|\cdot\|_{r,p,\mathcal{O}}$ and seminorm $|\cdot|_{r,p,\mathcal{O}}$. In particular, for $r = 0$ we let $\mathrm{L}^p(\mathcal{O}) := \mathrm{W}^{0,p}(\mathcal{O})$ be the usual Lebesgue space, and for $p = 2$ we let $\mathrm{H}^s(\mathcal{O}) := \mathrm{W}^{r,2}(\mathcal{O})$ be the classical Hilbertian Sobolev space with norm $\|\cdot\|_{s,\mathcal{O}}$ and seminorm $|\cdot|_{s,\mathcal{O}}$. Furthermore, given a generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts, respectively, whose norms and seminorms are denoted exactly as those of M. On the other hand, letting $\mathbf{div}$ (resp. $\mathbf{rot}$) be the usual divergence operator div (resp. rotational operator rot) acting along the rows of a given tensor, we recall that the spaces

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \Big\{\tau \in \mathbf{L}^2(\mathcal{O}): \quad \mathrm{div}(\tau) \in \mathrm{L}^2(\mathcal{O})\Big\}\,,$$

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})\Big\}\,,$$

$$\mathbf{H}(\mathrm{rot};\mathcal{O}) := \Big\{\tau \in \mathbf{L}^2(\mathcal{O}): \quad \mathrm{rot}(\tau) \in \mathrm{L}^2(\mathcal{O})\Big\}\,,$$

and

$$\mathbb{H}(\mathbf{rot};\mathcal{O}) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}): \quad \mathbf{rot}(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})\Big\}\,,$$

equipped with the usual norms

$$\begin{aligned}
\|\tau\|^2_{\mathrm{div};\mathcal{O}} &:= \|\tau\|^2_{0,\mathcal{O}} + \|\mathrm{div}(\tau)\|^2_{0,\mathcal{O}} \qquad \forall\,\tau \in \mathbf{H}(\mathrm{div};\mathcal{O})\,,\\
\|\boldsymbol{\tau}\|^2_{\mathbf{div};\mathcal{O}} &:= \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\mathcal{O}} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})\,,\\
\|\tau\|^2_{\mathrm{rot};\mathcal{O}} &:= \|\tau\|^2_{0,\mathcal{O}} + \|\mathrm{rot}(\tau)\|^2_{0,\mathcal{O}} \qquad \forall\,\tau \in \mathbf{H}(\mathrm{rot};\mathcal{O})\,,
\end{aligned}$$

and

$$\|\boldsymbol{\tau}\|^2_{\mathbf{rot};\mathcal{O}} := \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{rot}(\boldsymbol{\tau})\|^2_{0,\mathcal{O}} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{rot};\mathcal{O})\,,$$

are Hilbert spaces. Finally, in what follows we employ $\mathbf{0}$ to denote a generic null vector, null tensor or null operator, and use C to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The Navier-Stokes equations

2.1 The model problem

Let Ω be a bounded polygonal domain in R^2 with boundary Γ. In what follows we consider the stationary Navier-Stokes equations with constant viscosity $\mu > 0$. In other words, given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and a Dirichlet datum $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek the velocity $\mathbf{u}$ and the pressure p of a fluid occupying the region Ω, such that

$$\begin{gathered}
-\mu\Delta\mathbf{u} + (\nabla\mathbf{u})\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega\,, \qquad \mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \int_\Omega p = 0\,,
\end{gathered} \tag{2.1}$$

where the last equation in (2.1) is imposed to guarantee the uniqueness of the pressure solution. Notice here that, due to the incompressibility condition given by the second equation of (2.1), $\mathbf{g}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0\,. \tag{2.2}$$

Then, following [18] (see also [17, 22]), we introduce a pseudostress tensor defined by

$$\boldsymbol{\sigma} := \mu\nabla\mathbf{u} - \mathbf{u}\otimes\mathbf{u} - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{2.3}$$

which establishes that the first equation in (2.1) can be written as the equilibrium equation

$$-\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,.$$

Next, it is not difficult to see that (2.3) and the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω, are equivalent to the pair of equations given by

$$\boldsymbol{\sigma}^{\mathbf{d}} \;=\; \mu\nabla\mathbf{u} \,-\, (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} \quad \text{in} \quad \Omega \quad \text{and} \quad p \;=\; -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}+\mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega\,, \tag{2.4}$$

whence (2.1) can be rewritten as: Find the pseudostress $\boldsymbol{\sigma}$ and the velocity $\mathbf{u}$ such that

$$\begin{gathered} \boldsymbol{\sigma}^{\mathbf{d}} \;=\; \mu\nabla\mathbf{u} \,-\, (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} \quad \text{in} \quad \Omega\,, \qquad -\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,,\\ \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \text{and} \quad \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}+\mathbf{u}\otimes\mathbf{u}) \;=\; 0\,. \end{gathered} \tag{2.5}$$

We stress here that we have eliminated the pressure from the original model (2.1). However, using the second equation in (2.4) we can recover p by a postprocessing formula in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$.

2.2 The augmented mixed formulation

In what follows we derive a weak formulation of (2.5). To this end, and proceeding as in [18, 17], we multiply the first equation in (2.5) by $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$, integrate by parts in Ω, and use the Dirichlet boundary condition to deduce that

$$\int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} \,+\, \mu\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} : \boldsymbol{\tau} \;=\; \mu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{g}\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)\,.$$

In turn, the equilibrium equation, which is given by the second equation of (2.5), is rewritten as

$$\mu\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \;=\; -\mu\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{L}^2(\Omega)\,.$$

In this way, we arrive at first instance at the following weak formulation of (2.5): Find $\boldsymbol{\sigma}\in\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{u}$ in a suitable space, such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} \,+\, \mu\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} : \boldsymbol{\tau} \;&=\; \mu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{g}\rangle_\Gamma && \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)\,,\\ \mu\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \;&=\; -\mu\int_\Omega \mathbf{f}\cdot\mathbf{v} && \forall\,\mathbf{v}\in\mathbf{L}^2(\Omega)\,,\\ \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;&=\; -\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,. \end{aligned} \tag{2.6}$$

We now define

$$\mathbb{H}_0(\mathbf{div};\Omega) \;:=\; \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega) \;:\; \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\}\,,$$

and recall (see [12, 26]) that there holds the decomposition

$$\mathbb{H}(\mathbf{div};\Omega) \;=\; \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,. \tag{2.7}$$

More precisely, for each $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$ there exist unique $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $c := \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau})\in\mathrm{R}$, where $|\Omega|$ denotes the measure of Ω, such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathbb{I}$. In particular, the third equation of (2.6) yields $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}$, with $\boldsymbol{\sigma}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and the constant c given explicitly in terms of $\mathbf{u}$ by

$$c \;=\; -\frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) \;=\; -\frac{1}{2|\Omega|}\,\|\mathbf{u}\|_{0,\Omega}^2\,. \tag{2.8}$$

In this way, replacing $\boldsymbol{\sigma}$ by the expression $\boldsymbol{\sigma}_0 + c\mathbb{I}$ in (2.6), using that $\boldsymbol{\sigma}^{\mathbf{d}} = \boldsymbol{\sigma}_0^{\mathbf{d}}$ and $\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{div}(\boldsymbol{\sigma}_0)$, taking into account the condition (2.2), and denoting from now on the remaining unknown $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$ simply by $\boldsymbol{\sigma}$, we deduce that the weak formulation of (2.5) can be written as: Find $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$ and $\mathbf{u}$ in a suitable space, such that

$$
\begin{aligned}
\int_\Omega \boldsymbol{\sigma}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} + \mu \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}} : \boldsymbol{\tau} &= \mu\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{g}\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,, \\
\mu \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) &= -\mu \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega)\,.
\end{aligned}
$$

On the other hand, we notice that the third term in the first equation of the foregoing system requires $\mathbf{u}$ to lie in a smaller space than $\mathbf{L}^2(\Omega)$. In fact, applying the Cauchy-Schwarz inequality, and employing the compact (and hence continuous) injection

$$
\mathbf{i}_c \,:\, \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega) \tag{2.9}
$$

(cf. the Rellich-Kondrachov theorem in [1, Theorem 6.3] or [31, Theorem 1.3.5]), we arrive at

$$
\left|\int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} : \boldsymbol{\zeta}\right| \;\le\; \|\mathbf{w}\|_{0,4,\Omega}\|\mathbf{z}\|_{0,4,\Omega}\|\boldsymbol{\zeta}\|_{0,\Omega} \;\le\; \|\mathbf{i}_c\|^2\,\|\mathbf{w}\|_{1,\Omega}\|\mathbf{z}\|_{1,\Omega}\|\boldsymbol{\zeta}\|_{0,\Omega}\,, \tag{2.10}
$$

for all $\mathbf{w},\mathbf{z} \in \mathbf{H}^1(\Omega)$, and $\boldsymbol{\zeta} \in \mathbb{L}^2(\Omega)$, which suggests to look for the unknown $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$ and to restrict the set of corresponding test functions $\mathbf{v}$ to the same space. Consequently, and in order to be able to analyze the present variational formulation of (2.5), we follow [18] and incorporate the following redundant Galerkin terms:

$$
\begin{aligned}
\kappa_1 \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) &= -\kappa_1 \int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,, \\
\kappa_2 \int_\Omega \Big\{\mu\nabla\mathbf{u} - \boldsymbol{\sigma}^{\mathbf{d}} - (\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}}\Big\} : \nabla\mathbf{v} &= 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,, \\
\kappa_3 \int_\Gamma \mathbf{u}\cdot\mathbf{v} &= \kappa_3 \int_\Gamma \mathbf{g}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,,
\end{aligned}
$$

where κ_1, κ_2 and κ_3 are positive parameters to be specified later. In this way, we obtain the following augmented mixed formulation: Find $\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{H} := \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ such that

$$
\mathbf{A}(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) \;+\; \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}) \;=\; \mathbf{F}(\vec{\boldsymbol{\tau}}) \qquad \forall\, \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H}\,, \tag{2.11}
$$

where $\mathbf{A} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ is the bilinear form

$$
\begin{aligned}
\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;:=\; & \int_\Omega \boldsymbol{\zeta}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}} + \kappa_1 \int_\Omega \mathbf{div}(\boldsymbol{\zeta})\cdot\mathbf{div}(\boldsymbol{\tau}) + \kappa_2\mu \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{v} + \kappa_3 \int_\Gamma \mathbf{w}\cdot\mathbf{v} \\
& - \mu \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta}) + \mu \int_\Omega \mathbf{w}\cdot\mathbf{div}(\boldsymbol{\tau}) - \kappa_2 \int_\Omega \boldsymbol{\zeta}^{\mathbf{d}} : \nabla\mathbf{v}
\end{aligned} \tag{2.12}
$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta},\mathbf{w})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H}$, $\mathbf{F} : \mathbf{H} \to \mathrm{R}$ is the linear functional

$$
\mathbf{F}(\vec{\boldsymbol{\tau}}) \;:=\; \mu\langle \boldsymbol{\tau}\mathbf{n},\mathbf{g}\rangle_\Gamma - \kappa_1 \int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) + \mu \int_\Omega \mathbf{f}\cdot\mathbf{v} + \kappa_3 \int_\Gamma \mathbf{g}\cdot\mathbf{v} \tag{2.13}
$$

for all $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H}$, and given $\mathbf{z} \in \mathbf{H}^1(\Omega)$, $\mathbf{B}(\mathbf{z};\cdot,\cdot) : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ is the bilinear form

$$
\mathbf{B}(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;:=\; \int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} : \big\{\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\big\} \tag{2.14}
$$

for all $\vec{\zeta} := (\zeta, \mathbf{w})$, $\vec{\tau} := (\tau, \mathbf{v}) \in \mathbf{H}$. We notice here, according to (2.10) and (2.14), that there holds

$$|\,\mathbf{B}(\mathbf{z};\vec{\zeta},\vec{\tau})\,| \;\le\; \|\mathbf{i}_c\|^2\,(1+\kappa_2^2)^{1/2}\,\|\mathbf{z}\|_{1,\Omega}\,\|\vec{\zeta}\|_{\mathbf{H}}\,\|\vec{\tau}\|_{\mathbf{H}} \qquad \forall\,\mathbf{z}\in\mathbf{H}^1(\Omega),\ \forall\,\vec{\zeta},\ \vec{\tau}\in\mathbf{H}\,. \tag{2.15}$$

Up to minor changes caused by the present non-homogeneous Dirichlet boundary condition for $\mathbf{u}$, the unique solvability of (2.11) was basically derived in [18]. In particular, it was proved there (cf. [18, Lemma 3.1]) that for κ_1, $\kappa_3 > 0$ and $0 < \kappa_2 < 2\mu$, there exists $\alpha_{\mathbf{A}} > 0$, depending on κ_1, κ_2, κ_3, μ, and the constants $c_1(\Omega)$ and $c_2(\Omega)$ (cf. Lemma 4.10 below), such that

$$\mathbf{A}(\vec{\tau},\vec{\tau}) \;\ge\; \alpha_{\mathbf{A}}\,\|\vec{\tau}\|_{\mathbf{H}}^2 \qquad \forall\,\vec{\tau}\in\mathbf{H}\,, \tag{2.16}$$

which, together with (2.15), yielded the $\mathbf{H}$-ellipticity of the bilinear form $\mathbf{A}+\mathbf{B}(\mathbf{z};\cdot,\cdot)$ for sufficiently small $\mathbf{z}$. More precisely, for each $\mathbf{z}\in\mathbf{H}^1(\Omega)$ such that $\|\mathbf{z}\|_{1,\Omega} \le \frac{\alpha_{\mathbf{A}}}{2\,\|\mathbf{i}_c\|^2\,(1+\kappa_2^2)^{1/2}}$, there holds (cf. [18, eq. (3.16)])

$$\mathbf{A}(\vec{\tau},\vec{\tau}) \,+\, \mathbf{B}(\mathbf{z};\vec{\tau},\vec{\tau}) \;\ge\; \frac{\alpha_{\mathbf{A}}}{2}\,\|\vec{\tau}\|_{\mathbf{H}}^2 \qquad \forall\,\vec{\tau}\in\mathbf{H}\,. \tag{2.17}$$

In addition, letting $\gamma_0 : \mathbf{H}^1(\Omega)\to\mathbf{L}^2(\Omega)$ be the usual trace operator, it was shown (cf. [18, eqs. (3.6) and (3.9)]) that there exist $C_{\mathbf{A}}$, $M_{\mathbf{F}} > 0$, depending on κ_1, κ_2, κ_3, μ, and $\|\gamma_0\|$, such that

$$|\,\mathbf{A}(\vec{\zeta},\vec{\tau})\,| \;\le\; C_{\mathbf{A}}\,\|\vec{\zeta}\|_{\mathbf{H}}\,\|\vec{\tau}\|_{\mathbf{H}} \qquad \forall\,\vec{\zeta},\ \vec{\tau}\in\mathbf{H}\,, \tag{2.18}$$

and

$$|\mathbf{F}(\vec{\tau})| \;\le\; M_{\mathbf{F}}\Big\{\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\Big\}\,\|\vec{\tau}\|_{\mathbf{H}} \qquad \forall\,\vec{\tau}\in\mathbf{H}\,. \tag{2.19}$$

In this way, reformulating (2.11) as a fixed-point operator equation, and assuming that $\mathbf{f}$ and $\mathbf{g}$ are suitably bounded, the well-posedness of (2.11) was established thanks to the Lax-Milgram Lemma and the Banach fixed-point Theorem. The corresponding result is stated as follows.

Theorem 2.1. *Let κ_1, $\kappa_3 > 0$, $0 < \kappa_2 < 2\mu$, and given $\rho \in \Big(0, \frac{\alpha_{\mathbf{A}}}{2\,\|\mathbf{i}_c\|^2\,(1+\kappa_2^2)^{1/2}}\Big)$ (cf. (2.9) and (2.16)), set $W_\rho := \big\{\mathbf{z}\in\mathbf{H}^1(\Omega) :\ \|\mathbf{z}\|_{1,\Omega} \le \rho\big\}$. In addition, assume that the data $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma)$ satisfy (cf. (2.19))*

$$M_{\mathbf{F}}\Big\{\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\Big\} \;\le\; \frac{\alpha_{\mathbf{A}}}{2}\,\rho\,.$$

Then, there exists a unique $\vec{\sigma} := (\sigma, \mathbf{u}) \in \mathbf{H}$ solution of (2.11), with $\mathbf{u}\in W_\rho$, and there holds

$$\|\vec{\sigma}\|_{\mathbf{H}} \;\le\; C_{\mathbf{T}}\Big\{\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\Big\}$$

with the constant $C_{\mathbf{T}} := \frac{2\,M_{\mathbf{F}}}{\alpha_{\mathbf{A}}}$.

Proof. We omit details and refer to [18, Theorem 3.4] (see also [17, Theorem 3.9] for a similar proof). □

3 The virtual element subspaces

3.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements. For each $K\in\mathcal{T}_h$ we denote its barycenter, diameter, and number of edges by $\mathbf{x}_K$, h_K, and d_K, respectively, and define, as usual, $h := \max\{h_K : K\in\mathcal{T}_h\}$. Furthermore, in what follows we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K\in\mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$, that is , for each $x_0 \in B$, all the line segments joining x_0 with any $x \in K$ are contained in K, or equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in K.

As consequence of the above hypotheses, one can show that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that $d_K \le N_{\mathcal{T}} \quad \forall\, K \in \mathcal{T}_h$.

Now, given an integer $\ell \ge 0$ and $\mathcal{O} \subseteq \mathrm{R}^2$, we let $\mathrm{P}_\ell(\mathcal{O})$ be the space of polynomials on $\mathcal{O}$ of degree up to ℓ, and according to the notations introduced at the end of Section 1, we set $\mathbf{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^2$ and $\mathbb{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^{2\times 2}$. Also, in what follows we use the multi-index notation, that is, given $\mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \mathrm{R}^2$ and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)^{\mathbf{t}}$, with non-negative integers α_1, α_2, we let $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2$. Furthermore, given $K \in \mathcal{T}_h$ and an edge $e \in \partial K$ with barycentric x_e and diameter h_e, we introduce the following sets of $(\ell+1)$ normalized monomials on e

$$\mathcal{B}_\ell(e) := \left\{ \left(\frac{x - x_e}{h_e} \right)^j \right\}_{0 \le j \le \ell},$$

and $\frac{1}{2}(\ell+1)(\ell+2)$ normalized monomials on K

$$\mathcal{B}_\ell(K) := \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^{\boldsymbol{\alpha}} \right\}_{0 \le |\boldsymbol{\alpha}| \le \ell},$$

which constitute basis of $\mathrm{P}_\ell(e)$ and $\mathrm{P}_\ell(K)$, respectively. In addition, denoting $\widetilde{\mathcal{B}}_1(K) := \mathcal{B}_1(K)$, we define for each integer $\ell \ge 2$,

$$\widetilde{\mathcal{B}}_\ell(K) := \mathcal{B}_\ell(K) \setminus \mathcal{B}_{\ell-2}(K),$$

which is a basis of the subspace of polynomials on K of degree exactly $\ell - 1$ or ℓ. In turn, the corresponding vector and tensor versions of the foregoing sets of monomials are given by

$$\begin{aligned}
\boldsymbol{\mathcal{B}}_\ell(e) &:= \Big\{ (q,0)^{\mathbf{t}} : \quad q \in \mathcal{B}_\ell(e) \Big\} \cup \Big\{ (0,q)^{\mathbf{t}} : \quad q \in \mathcal{B}_\ell(e) \Big\}, \\
\boldsymbol{\mathcal{B}}_\ell(K) &:= \Big\{ (\mathbf{q},0)^{\mathbf{t}} : \quad \mathbf{q} \in \mathcal{B}_\ell(K) \Big\} \cup \Big\{ (0,\mathbf{q})^{\mathbf{t}} : \quad \mathbf{q} \in \mathcal{B}_\ell(K) \Big\},
\end{aligned}$$

and

$$\widetilde{\boldsymbol{\mathcal{B}}}_\ell(K) := \Big\{ (\mathbf{q},0)^{\mathbf{t}} : \quad \mathbf{q} \in \widetilde{\mathcal{B}}_\ell(K) \Big\} \cup \Big\{ (0,\mathbf{q})^{\mathbf{t}} : \quad \mathbf{q} \in \widetilde{\mathcal{B}}_\ell(K) \Big\}.$$

On the other hand, for each integer $\ell \ge 0$, we let $\mathcal{G}_\ell(K)$ be a basis of $\big(\nabla \mathrm{P}_{\ell+1}(K)\big)^\perp \cap \mathbf{P}_\ell(K)$, which is the $\mathbf{L}^2(K)$-orthogonal of $\nabla \mathbf{P}_{\ell+1}(K)$ in $\mathbf{P}_\ell(K)$, and denote its vectorial counterparts as follow:

$$\boldsymbol{\mathcal{G}}_\ell(K) := \left\{ \begin{pmatrix} \mathbf{q} \\ \mathbf{0} \end{pmatrix} : \mathbf{q} \in \mathcal{G}_\ell(K) \right\} \cup \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{q} \end{pmatrix} : \mathbf{q} \in \mathcal{G}_\ell(K) \right\}.$$

Finally, we let

$$\mathbf{H}^1(\mathcal{T}_h) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}|_K \in \mathbf{H}^1(K) \quad \forall\; K \in \mathcal{T}_h \Big\},$$

and consider the $\mathbf{H}^1$–broken seminorm

$$|\mathbf{v}|_{1,h} := \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v}\|_{0,K}^2 \right\}^{1/2} \qquad \forall\; \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h).$$

3.2 The virtual element subspace of $\mathbf{H}^1(\Omega)$

In this section we present a suitable choice for the virtual element subspace of $\mathbf{H}^1(\Omega)$. To this end, given $K \in \mathcal{T}_h$ and an integer $k \geq 1$, we first let $\mathcal{R}_k^K : \mathbf{H}^1(K) \to \mathbf{P}_k(K)$ be the projection operator defined for each $\mathbf{v} \in \mathbf{H}^1(K)$ as the unique polynomial $\mathcal{R}_k^K(\mathbf{v}) \in \mathbf{P}_k(K)$ satisfying (cf. [6])

$$\begin{aligned} \int_K \nabla \mathcal{R}_k^K(\mathbf{v}) : \nabla \mathbf{q} \;&=\; \int_K \nabla \mathbf{v} : \nabla \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_k(K)\,, \\ \int_{\partial K} \mathcal{R}_k^K(\mathbf{v}) \;&=\; \int_{\partial K} \mathbf{v}\,. \end{aligned} \tag{3.1}$$

Notice, however, that a modified version of $\mathcal{R}_k^K$ can be found in [2]. Also, it is readily seen from the first equation of (3.1) that

$$|\mathcal{R}_k^K(\mathbf{v})|_{1,K} \;\leq\; |\mathbf{v}|_{1,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(K)\,. \tag{3.2}$$

In addition, we recall from [6, Lemma 5.1] that for integers $s \in [1, k+1]$ and $m \in [1, s]$, there holds

$$\|\mathbf{v} - \mathcal{R}_k^K(\mathbf{v})\|_{m,K} \;\leq\; C\, h_K^{s-m}\, |\mathbf{v}|_{s,K} \qquad \forall \mathbf{v} \in \mathbf{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{3.3}$$

Furthermore, we now consider the finite-dimensional subspace of $\mathbf{C}(\partial K)$ given by

$$\mathbb{B}_k(\partial K) \;:=\; \Big\{ \mathbf{v} \in \mathbf{C}(\partial K) : \quad \mathbf{v}|_e \in \mathbf{P}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K \Big\}\,,$$

define the following local virtual element space of order k (see, e.g. [2])

$$\begin{aligned} V_k^K \;:=\; \Big\{ \mathbf{v} \in \mathbf{H}^1(K) : \quad & \mathbf{v}|_{\partial K} \in \mathbb{B}_k(\partial K)\,, \quad \Delta \mathbf{v}|_K \in \mathbf{P}_k(K)\,, \\ \text{and} \quad & \int_K \big\{ \mathcal{R}_k^K(\mathbf{v}) - \mathbf{v} \big\} \cdot \mathbf{p} \;=\; 0 \qquad \forall\, \mathbf{p} \in \widetilde{\boldsymbol{\mathcal{B}}}_k(K) \Big\}\,, \end{aligned} \tag{3.4}$$

and recall from [2] the following degrees of freedom for a given $\mathbf{v} \in V_k^K$

$$\begin{aligned} m_{i,v}^V(\mathbf{v}) \;&:=\; \text{value of } \mathbf{v} \text{ at the } i\text{th vertex of } K\,, \quad \forall\, i \text{ vertex of } K \\ m_e^V(\mathbf{v}) \;&:=\; \text{values of } \mathbf{v} \text{ at } k-1 \text{ uniformly spaced points on } e\,,\ \forall\, e \in \partial K\,,\ \text{for } k \geq 2\,, \\ m_{\mathbf{q},K}^V(\mathbf{v}) \;&:=\; \text{value of } \int_K \mathbf{v} \cdot \mathbf{q}\,,\ \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_{k-2}(K)\,,\ \text{for } k \geq 2\,. \end{aligned} \tag{3.5}$$

Then, the following result summarizes the unisolvency of (3.5) with respect to V_k^K.

Lemma 3.1. *Let $k \geq 1$ be an integer. Then the amount of degrees of freedom defined by (3.5) is given by $n_{k,K}^V \;:=\; \dim V_k^K \;=\; 2k\, d_K + k(k-1)$. In addition, they are unisolvent with respect to V_k^K.*

Proof. We refer to [2, Propositions 1 and 2] for details. $\square$

In what follows we show that for each $\mathbf{v} \in V_k^K$ its projection $\mathcal{R}_k^K(\mathbf{v})$ can be computed explicitly by using the degrees of freedom defined in (3.5). In fact, we begin by noticing that, given $\mathbf{v} \in V_k^K$ and $\mathbf{q} \in \mathbf{P}_k(K)$, the right-hand side of the first equation of (3.1) can be integrated by parts to yield

$$\int_K \nabla \mathbf{v} : \nabla \mathbf{q} \;=\; -\int_K \mathbf{v} \cdot \Delta \mathbf{q} \;+\; \int_{\partial K} (\nabla \mathbf{q})\mathbf{n} \cdot \mathbf{v}\,.$$

Since $\Delta\mathbf{q} \in \mathbf{P}_{k-2}(K)$, the first term on the right-hand side of the foregoing equation can be computed by using the moments $m^V_{\mathbf{q},K}(\mathbf{v})$, whereas for the second one the degrees of freedom given by $m^V_{i,v}(\mathbf{v})$ and $m^V_e(\mathbf{v})$ are employed. In turn, it is straightforward to see that the right-hand side of the second equation of (3.1) can be calculated using also $m^V_{i,v}(\mathbf{v})$ and $m^V_e(\mathbf{v})$.

Furthermore, we now denote by $\{m^V_{j,K}(\mathbf{v})\}_{j=1}^{n^V_{k,K}}$ the degrees of freedom defined by (3.5), and let $\Pi^K_k : \mathbf{H}^1(K) \to V^K_k$ be the associated local interpolation operator, that is, given $\mathbf{v} \in \mathbf{H}^1(K)$, $\Pi^K_k(\mathbf{v})$ is the unique element in V^K_k such that

$$m^V_{j,K}(\mathbf{v} - \Pi^K_k(\mathbf{v})) \;=\; 0 \qquad \forall\, j \in \big\{1, 2, \ldots, n^V_{k,K}\big\}\,.$$

The following lemma establishes the approximation properties of Π^K_k.

Lemma 3.2. *Let k, m and s be integers such $0 \le m \le 1$ and $2 \le s \le k+1$. Then, there exists a constant $C > 0$, independent of K, such that for each $K \in \mathcal{T}_h$, there holds*

$$|\mathbf{v} - \Pi^K_k(\mathbf{v})|_{m,K} \;\le\; C\, h_K^{s-m}\, |\mathbf{v}|_{s,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^s(K)\,.$$

Proof. See [2, Proposition 4]. $\square$

We end this section by establishing the virtual element space on the whole Ω. Indeed, for every polygonal decomposition $\mathcal{T}_h$ of Ω, and for every integer $k \ge 1$, we consider the following virtual element subspace of $\mathbf{H}^1(\Omega)$

$$V^h_k \;:=\; \Big\{\mathbf{v} \in \mathbf{H}^1(\Omega) : \quad \mathbf{v}|_K \in V^K_k \quad \forall\, K \in \mathcal{T}_h\Big\}\,. \tag{3.6}$$

In addition, we remark here that, for a given $\mathbf{v} \in V^h_k$, the local degrees of freedom defined by $m^V_{i,v}(\mathbf{v})$ and $m^V_e(\mathbf{v})$ in (3.5), together with the fact that $\mathbf{v}|_e \in \mathbf{P}_k(e) \quad \forall\, e \in \mathcal{T}_h$ (cf. (3.4)), guarantee the continuity of the trace of $\mathbf{v}$ across the edges e of $\mathcal{T}_h$. It follows that $\mathbf{v} \in \mathbf{H}^1(\Omega)$, which confirms that V^h_k is in fact a $\mathbf{H}^1(\Omega)$-conforming subspace. According to this discussion and Lemma 3.2, the approximation property of V^h_k is given by:

$(\mathbf{AP}^{\mathbf{u}}_h)$: there exists $C > 0$, independent of h, such that for each integer $s \in [2, k+1]$ there holds

$$\mathrm{dist}(\mathbf{v}, V^h_k) \;\le\; C h^{s-1} \Big\{ \sum_{K \in \mathcal{T}_h} |\mathbf{v}|^2_{s,K} \Big\}^{1/2} \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \text{ such that } \mathbf{v}|_K \in \mathbf{H}^s(K) \quad \forall\, K \in \mathcal{T}_h\,.$$

3.3 The virtual element subspace of $\mathbb{H}_0(\mathbf{div};\Omega)$

Throughout this section we consider an integer $k \ge 1$. Then, given $K \in \mathcal{T}_h$, we introduce the local virtual element space H^K_k of order k as follows (see, e.g. [5, 6, 14])

$$\begin{aligned} H^K_k \;:=\; \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; K) \cap \mathbb{H}(\mathbf{rot}; K) : \quad & \boldsymbol{\tau}\mathbf{n}|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \in \partial K\,, \\ & \mathbf{div}(\boldsymbol{\tau})|_K \in \mathbf{P}_{k-1}(K)\,, \quad \text{and} \quad \mathbf{rot}(\boldsymbol{\tau})|_K \in \mathbf{P}_{k-1}(K)\Big\}\,, \end{aligned} \tag{3.7}$$

whose local degrees of freedom are given by (see [5])

$$\begin{aligned} m^H_{\mathbf{q},\mathbf{n}}(\boldsymbol{\tau}) \;&:=\; \int_e \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_k(e)\,, \quad \forall \text{ edge } e \in \partial K\,, \\ m^H_{\mathbf{q},\mathbf{div}}(\boldsymbol{\tau}) \;&:=\; \int_K \boldsymbol{\tau} : \nabla\mathbf{q} \qquad \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_{k-1}(K) \setminus \{(1,0)^{\mathbf{t}}, (0,1)^{\mathbf{t}}\}\,, \\ m^H_{\boldsymbol{\rho},\mathbf{rot}}(\boldsymbol{\tau}) \;&:=\; \int_K \boldsymbol{\tau} : \boldsymbol{\rho} \qquad \forall\, \boldsymbol{\rho} \in \boldsymbol{\mathcal{G}}_k(K)\,. \end{aligned} \tag{3.8}$$

The unisolvency of (3.8) in H_k^K is summarized as follows.

Lemma 3.3. *The amount of local degrees of freedom defined in* (3.8) *is given by*

$$n_{k,K}^H \;:=\; \dim H_k^K \;=\; 2\left\{(k+1)(d_K+k)-1\right\},$$

where d_K is the number of edges of $K \in \mathcal{T}_h$. In addition, the local degrees of freedom (3.8) *are unisolvent in H_k^K.*

Proof. See [5, Theorem 1] for details. □

We now gather all the degrees of freedom (3.8) in the set $\{m_{j,K}^H(\boldsymbol{\tau})\}_{j=1}^{n_{k,K}^H}$, and then, we introduce the interpolation operator $\boldsymbol{\Pi}_k^K : \mathbb{H}^1(K) \to H_k^K$, which is defined for each $\boldsymbol{\tau} \in \mathbb{H}^1(K)$ as the unique $\boldsymbol{\Pi}_k^K(\boldsymbol{\tau})$ in H_k^h such that

$$m_{j,K}^H(\boldsymbol{\tau} - \boldsymbol{\Pi}_k^K(\boldsymbol{\tau})) \;=\; 0 \qquad \forall\, j \in \{1,2,\dots,n_{k,K}^H\}.$$

Concerning the approximation properties of $\boldsymbol{\Pi}_k^K$, we first recall from [7, eq. (3.19)] that for each integer $s \in [1,k+1]$ there exists $C>0$, independent of K, such that

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k^K(\boldsymbol{\tau})\|_{0,K} \;\le\; C\,h_K^s\,|\boldsymbol{\tau}|_{s,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K). \tag{3.9}$$

In addition, similarly to [14, eq. (3.14)], it is easy to check that

$$\mathbf{div}(\boldsymbol{\Pi}_k^K(\boldsymbol{\tau})) \;=\; \mathcal{P}_{k-1}^K(\mathbf{div}(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^1(K),$$

where $\mathcal{P}_{k-1}^K : \mathbf{L}^2(K) \to \mathbf{P}_{k-1}(K)$ is the orthogonal projector (see Section 3.4 below). In this way, applying (3.14) we deduce that for each integer $s \in [0,k]$ there exists $C>0$, independent of K, such that

$$\|\mathbf{div}(\boldsymbol{\tau}) - \mathbf{div}(\boldsymbol{\Pi}_k^K(\boldsymbol{\tau}))\|_{0,K} \;\le\; C\,h_K^s\,|\mathbf{div}(\boldsymbol{\tau})|_{s,K} \qquad \forall \boldsymbol{\tau} \in \mathbb{H}^1(K) \text{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^s(K). \tag{3.10}$$

The foregoing estimate together with (3.9) yields the following result.

Lemma 3.4. *For each integer $s \in [1,k]$ there exists $C>0$, independent of K, such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k^K(\boldsymbol{\tau})\|_{\mathbf{div};K} \;\le\; C\,h_K^s\left\{|\boldsymbol{\tau}|_{s,K} \,+\, |\mathbf{div}(\boldsymbol{\tau})|_{s,K}\right\} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K) \textit{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^s(K).$$

Proof. It follows straightforwardly from (3.9) and (3.10). □

Finally, for every integer $k \ge 1$ we define the global virtual element subspaces of $\mathbb{H}_0(\mathbf{div};\Omega)$ as

$$H_k^h \;:=\; \left\{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}|_K \in H_k^K \quad \forall\, K \in \mathcal{T}_h\right\}. \tag{3.11}$$

Note here that given $\boldsymbol{\tau} \in H_k^h$, the local degrees of freedom defined by $m_{\mathbf{q},\mathbf{n}}^H(\boldsymbol{\tau})$ in (3.8), along with the fact that $\boldsymbol{\tau}\mathbf{n}|_e \in \mathbf{P}_k(e)\ \ \forall$ edge $e \in \mathcal{T}_h$ (cf. (3.7)), guarantee the continuity of the normal components of $\boldsymbol{\tau}$ across the edges e of $\mathcal{T}_h$. It follows that $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, which confirms that H_k^h is in fact contained in $\mathbb{H}(\mathbf{div};\Omega)$. Also, using this and Lemma 3.4, it is easy to obtain the following approximation property:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ For each integer $s \in [1,k]$ there exists $C>0$, independent of h, such that

$$\operatorname{dist}(\boldsymbol{\tau}, H_k^h) \;\le\; C\,h^s\left\{\sum_{K\in\mathcal{T}_h}\Big(|\boldsymbol{\tau}|_{s,K}^2 \,+\, |\mathbf{div}(\boldsymbol{\tau})|_{s,K}^2\Big)\right\}^{1/2}$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)$ such that $\boldsymbol{\tau}|_K \in \mathbb{H}^s(K)$ and $\mathbf{div}(\boldsymbol{\tau})|_K \in \mathbf{H}^s(K)$, for all $K \in \mathcal{T}_h$.

3.4 $\mathbf{L}^2$-orthogonal projections

We now let $\mathcal{P}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$ and $\boldsymbol{\mathcal{P}}_k^K : \mathbb{L}^2(K) \to \mathbb{P}_k(K)$ be the vectorial and tensorial versions of the $\mathrm{L}^2(K)$-orthogonal projector, respectively, which, given $\mathbf{v} \in \mathbf{L}^2(K)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(K)$, are characterized by

$$\mathcal{P}_k^K(\mathbf{v}) \in \mathbf{P}_k(K) \quad \text{and} \quad \int_K \mathcal{P}_k^K(\mathbf{v}) \cdot \mathbf{q} \,=\, \int_K \mathbf{v} \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_k(K) \tag{3.12}$$

and

$$\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) \in \mathbb{P}_k(K) \quad \text{and} \quad \int_K \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) : \mathbf{p} \,=\, \int_K \boldsymbol{\tau} : \mathbf{p} \qquad \forall\, \mathbf{p} \in \mathbb{P}_k(K)\,, \tag{3.13}$$

respectively. In addition, it is well-known (see, e.g. [13, Lemma 3.4]) that, given integers k, s, and m such that $k \geq 0$, $s \in [1, k+1]$, and $m \in [0, s]$, there hold the following approximation properties

$$\|\mathbf{v} - \mathcal{P}_k^K(\mathbf{v})\|_{m,K} \;\leq\; C\, h_K^{s-m}\, |\mathbf{v}|_{s,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,, \tag{3.14}$$

and

$$\|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{m,K} \;\leq\; C\, h_K^{s-m}\, |\boldsymbol{\tau}|_{s,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K)\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{3.15}$$

In addition, we remark here that the degrees of freedom given by (3.5) do allow the explicit calculation of the right-hand side of (3.12) (and hence of $\mathcal{P}_k^K(\mathbf{v})$) for each $\mathbf{v} \in V_k^K$. Indeed, it is easy to see first that the degrees of freedom given by $m_{\mathbf{q},K}^V(\mathbf{v})$ (cf. (3.5)) yields the computation of $\int_K \mathbf{v} \cdot \mathbf{q}$ when $\mathbf{q} \in \boldsymbol{\mathcal{B}}_{k-2}(K)$, whereas for $\mathbf{q} \in \widetilde{\boldsymbol{\mathcal{B}}}_k(K)$ we recall from (3.4) that

$$\int_K \mathbf{v} \cdot \mathbf{q} \;=\; \int_K \mathcal{R}_k^K(\mathbf{v}) \cdot \mathbf{q}\,,$$

and then use that $\mathcal{R}_k^K(\mathbf{v})$ is explicitly computable for each $\mathbf{v} \in V_k^K$.

On the other hand, we now aim to derive additional approximation properties for the projection $\mathcal{P}_k^K$. The goal is to extend the estimate (3.14) to the case of general Sobolev spaces. To this end, we need to recall from [13, Section 3.3] some preliminary notations and technical results. Indeed, for each element $K \in \mathcal{T}_h$ we first define $\widetilde{K} := T_K(K)$, where $T_K : \mathrm{R}^2 \to \mathrm{R}^2$ is the bijective affine mapping defined by

$$T_K(\mathbf{x}) \;:=\; \frac{\mathbf{x} - \mathbf{x}_B}{h_K} \quad \forall\, \mathbf{x} \in \mathrm{R}^2\,.$$

Then, as it was remarked in [13, Section 3.3], it is easy to see that the diameter $h_{\widetilde{K}}$ of $\widetilde{K}$ is 1, the shortest edge of $\widetilde{K}$ is bigger than $C_{\mathcal{T}}$ (which follows from assumptions a) and b) in Section 3.1), and $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. Then, by connecting each vertex of $\widetilde{K}$ to the center of $\widetilde{B}$, that is to the origin, we generate a partition of $\widetilde{K}$ into $d_{\widetilde{K}}$ triangles $\widetilde{\Delta}_i$, $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $d_{\widetilde{K}} \leq N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\widetilde{h}_i(\widetilde{\rho}_i)^{-1} \leq c_{\mathcal{T}} \quad \forall\, i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $\widetilde{h}_i$ is the diameter of $\widetilde{\Delta}_i$ and $\widetilde{\rho}_i$ is the diameter of the largest ball contained in $\widetilde{\Delta}_i$. We also let $\widehat{\Delta}$ be the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$, and for each $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$ we let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i \mathbf{x} \quad \forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \widetilde{\Delta}_i$. We remark that the fact that the origin is a vertex of each triangle $\widetilde{\Delta}_i$ allows to choose F_i as indicated.

In what follows, given $K \in \mathcal{T}_h$ and $\mathbf{v} \in \mathbf{L}^2(K)$, we let $\widetilde{\mathbf{v}} := \mathbf{v} \circ T_K^{-1} \in \mathbf{L}^2(\widetilde{K})$. Also, we recall from the Introduction that given $r \geq 0$, $p > 1$, $p \neq 2$, and an arbitrary domain $\mathcal{O} \subseteq \mathrm{R}^2$, $\|\cdot\|_{r,p,\mathcal{O}}$ and $|\cdot|_{r,p,\mathcal{O}}$ stand for the norm and seminorm, respectively, of the Sobolev space $\mathrm{W}^{r,p}(\mathcal{O})$ and its vectorial and tensorial versions. Then, we have the following result.

Lemma 3.5. *Given an integer $\ell \geq 0$, there holds $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v} \in \mathbf{L}^2(K)$. In addition, for integers r, $s \geq 0$ and for $p \geq 2$, there holds $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(\mathbf{W}^{r,p}(\widetilde{K}), \mathbf{W}^{s,p}(\widetilde{K}))$, with $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{W}^{r,p}(\widetilde{K}),\mathbf{W}^{s,p}(\widetilde{K}))}$ independent of $\widetilde{K}$.*

Proof. We begin by recalling from [13, Lemma 3.2] that $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v} \in \mathbf{L}^2(K)$. Next, denoting $N_\ell := (\ell+1)(\ell+2)$, we let $\{\widetilde{\boldsymbol{\varphi}}_1, \widetilde{\boldsymbol{\varphi}}_2, \ldots, \widetilde{\boldsymbol{\varphi}}_{N_\ell}\}$ be a $\mathbf{L}^2(\widetilde{K})$-orthonormal basis of $\mathbf{P}_\ell(\widetilde{K})$, which yields

$$\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}}) \;=\; \sum_{j=1}^{N_\ell} \langle \widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\varphi}}_j \rangle_{0,\widetilde{K}}\, \widetilde{\boldsymbol{\varphi}}_j \qquad \forall\, \widetilde{\mathbf{v}} \in \mathbf{L}^2(\widetilde{K})\,.$$

Now, given $p \geq 2$, we let q be the conjugate of p, that is $q := \frac{p}{p-1}$. Thus, using the triangle and Hölder inequalities, we find that for any pair of non-negative integers r and s there holds

$$\begin{aligned}\|\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})\|_{s,p,\widetilde{K}} \;&\leq\; \left\{ \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,q,\widetilde{K}} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,p,\widetilde{K}} \right\} \|\,\widetilde{\mathbf{v}}\|_{0,p,\widetilde{K}} \\ &\leq\; \left\{ \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,q,\widetilde{K}} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,p,\widetilde{K}} \right\} \|\widetilde{\mathbf{v}}\|_{r,p,\widetilde{K}} \qquad \forall\, \widetilde{\mathbf{v}} \in \mathbf{W}^{r,p}(\widetilde{K})\,,\end{aligned}$$

which proves that $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(\mathbf{W}^{r,p}(\widetilde{K}), \mathbf{W}^{s,p}(\widetilde{K}))$ with

$$\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{W}^{r,p}(\widetilde{K}),\mathbf{W}^{s,p}(\widetilde{K}))} \;\leq\; \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,q,\widetilde{K}} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,p,\widetilde{K}}\,.$$

In this way, applying the same arguments from the last part of the proof of [13, Lemma 3.2], we deduce that $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{W}^{r,p}(\widetilde{K}),\mathbf{W}^{s,p}(\widetilde{K}))}$ is bounded independently of $\widetilde{K}$. In fact, using the afore described decomposition of $\widetilde{K}$, we can write

$$\|\widetilde{\boldsymbol{\varphi}}_j\|_{s,p,\widetilde{K}} \;=\; \left\{ \sum_{t=0}^{s} |\widetilde{\boldsymbol{\varphi}}_j|_{t,p,\widetilde{K}}^p \right\}^{1/p} \;=\; \left\{ \sum_{t=0}^{s} \sum_{i=1}^{d_{\widetilde{K}}} |\widetilde{\boldsymbol{\varphi}}_j|_{t,p,\widetilde{\Delta}_i}^p \right\}^{1/p}.$$

In turn, applying the usual scaling properties, we deduce the existence of a constant $\widetilde{C}_t > 0$, depending only on t, such that

$$|\widetilde{\boldsymbol{\varphi}}_j|_{t,p,\widetilde{\Delta}_i} \;\leq\; \widetilde{C}_t\, \widetilde{h}_i^{-t+2/p} |\widehat{\boldsymbol{\varphi}}_{j,i}|_{t,p,\widehat{\Delta}}\,,$$

where $\widehat{\boldsymbol{\varphi}}_{j,i} := \widetilde{\boldsymbol{\varphi}}_j|_{\widetilde{\Delta}_i} \circ F_i \in \mathbf{P}_l(\widehat{\Delta})$. Then, according to the equivalence of norms in $\mathbf{P}_l(\widehat{\Delta})$, we find that

$$|\widetilde{\boldsymbol{\varphi}}_j|_{t,p,\widetilde{\Delta}_i} \;\leq\; \widetilde{C}_t\, \widetilde{h}_i^{-t+2/p} \|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{t,p,\widehat{\Delta}} \;\leq\; \widetilde{C}_t\, \widetilde{h}_i^{-t+2/p}\, \widehat{C}\, \|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{0,\widehat{\Delta}}\,,$$

with a constant $\widehat{C} > 0$ depending on t, p, ℓ, and $\widehat{\Delta}$. Moreover, applying again the scaling properties, we have that

$$\|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{0,\widehat{\Delta}} \;\leq\; C_0\, \widetilde{h}_i^{-1}\, \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{\Delta}_i} \;\leq\; C_0\, \widetilde{h}_i^{-1}\, \|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{K}} \;=\; C_0\, \widetilde{h}_i^{-1}\,,$$

with a constant $C_0 > 0$ depending only on $\widehat{\Delta}$, and using that $1 \;\leq\; \widetilde{h}_i^{-1} \;\leq\; C_{\mathcal{T}}^{-1}$, we get

$$|\widetilde{\boldsymbol{\varphi}}_j|_{t,p,\widetilde{\Delta}_i} \;\leq\; \widetilde{C}_t\, \widehat{C}\, C_0\, \widetilde{h}_i^{-t+2/p-1} \;\leq\; \widetilde{C}_t\, \widehat{C}\, C_0\, \widehat{C}_{\mathcal{T}}\,,$$

where

$$\widehat{C}_{\mathcal{T}} := \begin{cases} 1 & \text{if} \quad -t + 2/p - 1 \geq 0 \\ C_{\mathcal{T}}^{t+1-2/p} & \text{if} \quad -t + 2/p - 1 < 0 \end{cases}.$$

Finally, since $d_{\widetilde{K}} \leq N_{\mathcal{T}}$, we conclude that $\|\widetilde{\boldsymbol{\varphi}}_j\|_{s,p,\widetilde{K}}$ is bounded by a constant depending only on s, p, $N_{\mathcal{T}}$, $C_{\mathcal{T}}$, $\widehat{\Delta}$, and ℓ. The estimate for $\|\widetilde{\boldsymbol{\varphi}}_j\|_{0,q,\widetilde{K}}$ proceeds similarly, and hence further details are omitted, thus concluding the proof. $\square$

The next result is taken from [10, Lemma 4.3.8].

Lemma 3.6. *Let $\mathcal{O}$ be star-shaped with respect to a ball B with radius $\rho > \frac{1}{2}\rho_{\max}$, where $\rho_{\max} := \max\{\rho : \mathcal{O}$ is star-shaped with respect to a ball of radius $\rho\}$. In addition, given an integer $s \geq 0$, $p \geq 1$, and $\mathbf{v} \in \mathbf{W}^{s,p}(\mathcal{O})$, we let $\mathbf{Q}^s(\mathbf{v})$ be the Taylor polynomial of degree s of $\mathbf{v}$ averaged over B. Then, there holds*

$$|\mathbf{v} - \mathbf{Q}^s(\mathbf{v})|_{m,p,\mathcal{O}} \leq C\, d^{s-m}\, |\mathbf{v}|_{s,p,\mathcal{O}} \qquad \forall\, m \in \{0, 1, \ldots, s\},$$

where $d = \operatorname{diam}(\mathcal{O})$ and $C > 0$ depends on s and the chunkiness parameter $d/\rho_{\max}$.

The following lemma establishes the approximation properties of the projector $\mathcal{P}_k^K : \mathbf{L}^2(K) \to \mathbf{P}_k(K)$ with respect to more general Sobolev norms.

Lemma 3.7. *Let $K \in \mathcal{T}_h$ and k, s, m, and p be integers such that $k \geq 0$, $0 \leq m \leq s \leq k+1$, and $p \geq 2$. Then, there exists a constant $C > 0$, independent of K, such that*

$$|\mathbf{v} - \mathcal{P}_k^K(\mathbf{v})|_{m,p,K} \leq C\, h_K^{s-m}\, |\mathbf{v}|_{s,p,K} \qquad \forall\, \mathbf{v} \in \mathbf{W}^{s,p}(K). \tag{3.16}$$

Proof. Given $K \in \mathcal{T}_h$ and $\mathbf{v} \in \mathbf{W}^{s,p}(K)$, we first observe, thanks to the scaling properties, that there hold

$$|\widetilde{\mathbf{v}}|_{m,p,\widetilde{K}} \leq C_m\, h_K^{m-2/p}\, |\mathbf{v}|_{m,p,K} \quad \text{and} \quad |\mathbf{v}|_{m,p,K} \leq \widetilde{C}_m\, h_K^{-m+2/p}\, |\widetilde{\mathbf{v}}|_{m,p,\widetilde{K}}, \tag{3.17}$$

where C_m and $\widetilde{C}_m$ are positive constants depending only on m. In turn, letting $\widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}})$ be the Taylor polynomial of order s of $\widetilde{\mathbf{v}}$ averaged over a ball of radius $> \frac{1}{2}\widetilde{\rho}_{\max}$, we have that $\widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}}) \in \mathbf{P}_{s-1}(\widetilde{K}) \subseteq \mathbf{P}_k(\widetilde{K})$, which certainly yields

$$\mathcal{P}_k^{\widetilde{K}}(\widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}})) = \widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}}). \tag{3.18}$$

Recall here that $h_{\widetilde{K}} := \operatorname{diam}(\widetilde{K}) = 1$ and that $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. It follows, using (3.17), that

$$|\mathbf{v} - \mathcal{P}_k^K(\mathbf{v})|_{m,p,K} \leq \widetilde{C}_m\, h_K^{-m+2/p}\, |\widetilde{\mathbf{v}} - \widetilde{\mathcal{P}_k^K(\mathbf{v})}|_{m,p,\widetilde{K}} = \widetilde{C}_m\, h_K^{-m+2/p}\, |\widetilde{\mathbf{v}} - \mathcal{P}_k^{\widetilde{K}}(\widetilde{\mathbf{v}})|_{m,p,\widetilde{K}},$$

and along with (3.18), we obtain from Lemmas 3.5 (with $r = s = m$) and 3.6 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
|\mathbf{v} - \mathcal{P}_k^K(\mathbf{v})|_{m,p,K} &\leq \widetilde{C}_m\, h_K^{-m+2/p}\, |(I - \mathcal{P}_k^{\widetilde{K}})(\widetilde{\mathbf{v}} - \widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}}))|_{m,p,\widetilde{K}} \\
&\leq \widetilde{C}_m\, \|I - \mathcal{P}_k^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{W}^{m,p}(\widetilde{K}),\mathbf{W}^{m,p}(\widetilde{K}))} h_K^{-m+2/p} \|\widetilde{\mathbf{v}} - \widetilde{\mathbf{Q}}^s(\widetilde{\mathbf{v}})\|_{m,p,\widetilde{K}} \\
&\leq C\, h_K^{-m+2/p}\, h_{\widetilde{K}}^{s-m}\, |\widetilde{\mathbf{v}}|_{s,p,\widetilde{K}} = C\, h_K^{-m+2/p}\, |\widetilde{\mathbf{v}}|_{s,p,\widetilde{K}} \\
&\leq C\, h_K^{-m+2/p}\, h_K^{s-2/p}\, |\mathbf{v}|_{s,p,K} = C\, h_K^{s-m}\, |\mathbf{v}|_{s,p,K},
\end{aligned}$$

which completes the proof of the lemma. $\square$

As a consequence of the previous lemma, we have the following result.

Lemma 3.8. *Let $K \in \mathcal{T}_h$ and k, s, and p be integers such that $k \geq 0$, $0 \leq s \leq k+1$, and $p \geq 2$. Then, there exists a constant $M_k \geq 1$, independent of K, such that*

$$|\mathcal{P}_k^K(\mathbf{v})|_{s,p,K} \ \leq \ M_k \, |\mathbf{v}|_{s,p,K} \qquad \forall \ \mathbf{v} \in \mathbf{W}^{s,p}(K)\,.$$

Proof. It follows by adding and subtracting $\mathbf{v}$ and then employing the triangle inequality and Lemma 3.7 with $m = s$. □

We end this section by remarking that for the case $s = 0$ and $p = 4$, Lemma 3.8 yields

$$\|\mathcal{P}_k^K(\mathbf{v})\|_{0,4,K} \ \leq \ M_k \, \|\mathbf{v}\|_{0,4,K} \qquad \forall \ \mathbf{v} \in \mathbf{L}^4(K)\,, \quad \forall \ K \in \mathcal{T}_h\,. \tag{3.19}$$

4 The discrete forms

We begin by introducing the global virtual element subspaces of $\mathbf{H} := \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$. More precisely, given $k \geq 1$, we set

$$\mathbf{H}_k^h \ := \ H_k^h \times V_k^h\,,$$

where H_k^h and V_k^h have been defined in (3.11) and (3.6), respectively. Hence, the main goal of this section is to propose computable discrete versions $\mathbf{A}_h : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$ and $\mathbf{B}_h(\mathbf{z};\,\cdot\,,\,\cdot\,) : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$, for each $\mathbf{z} \in V_k^h$, of the bilinear forms $\mathbf{A}$ (cf. (2.12)) and $\mathbf{B}(\mathbf{z};\cdot,\cdot)$ (cf. (2.14)), respectively. Additionally, a computable discrete version of the functional $\mathbf{F}$ (cf. (2.13)) is also presented here.

4.1 The discrete bilinear form $\mathbf{A}_h$

According to the definition of the spaces V_k^h (cf. (3.6)) and H_k^h (cf. (3.11)), we observe from (2.12) that, given $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$, $\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})$ is not explicitly computable because of the following three terms:

$$A^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \ := \ \int_K \boldsymbol{\zeta}^{\mathbf{d}} : \boldsymbol{\tau}^{\mathbf{d}}\,, \qquad A^{K,\nabla}(\mathbf{w},\mathbf{v}) \ := \ \kappa_2\mu \int_K \nabla\mathbf{w} : \nabla\mathbf{v}\,, \quad \text{and} \quad \kappa_2 \int_K \boldsymbol{\zeta}^{\mathbf{d}} : \nabla\mathbf{v}\,, \tag{4.1}$$

in which the tensors $\boldsymbol{\zeta}^{\mathbf{d}}$, $\boldsymbol{\tau}^{\mathbf{d}}$, $\nabla\mathbf{w}$ and $\nabla\mathbf{v}$ are not known on each $K \in \mathcal{T}_h$. This is the reason why in what follows we define discrete computable versions of the forms in (4.1), all them in terms of some suitable projection operators.

We begin by defining the discrete local bilinear form $A_h^{K,\mathbf{d}} : H_k^K \times H_k^K \to \mathrm{R}$ as follows

$$A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \ := \ A^{K,\mathbf{d}}(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})) \ + \ \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})) \qquad \forall \ \boldsymbol{\zeta},\boldsymbol{\tau} \in H_k^K\,, \tag{4.2}$$

where $\mathcal{S}^{K,\mathbf{d}} : H_k^K \times H_k^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_{k,K}^H \times n_{k,K}^H}$ with respect to a basis of H_k^K. Equivalently, we set

$$\mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \ := \ \sum_{i=1}^{n_{k,K}^H} m_{i,K}^H(\boldsymbol{\zeta}) m_{i,K}^H(\boldsymbol{\tau}) \qquad \forall \ \boldsymbol{\zeta},\boldsymbol{\tau} \in H_k^K\,,$$

where $m_{i,K}^H$, $i \in \{1,\ldots,n_{k,K}^H\}$, are the degrees of freedom defined in (3.8).

The following estimates concerning $\mathcal{S}^{K,\mathbf{d}}$ will be very useful in what follows.

Lemma 4.1. *There exist constants $\widehat{c}_0, \widehat{c}_1 > 0$, depending only on $C_{\mathcal{T}}$, such that*

$$\widehat{c}_0 \, \|\boldsymbol{\zeta}\|_{0,K}^2 \;\le\; \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \;\le\; \widehat{c}_1 \, \|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\zeta} \in H_k^K \,, \quad \forall\, K \in \mathcal{T}_h \,. \tag{4.3}$$

Proof. See [7, eqs. (3.36) and (6.2)] (see also [11, eq. (5.8)] and [13, Lemma 4.5]). $\square$

As a consequence of Lemma 4.1 and the properties of $\boldsymbol{\mathcal{P}}_k^K$, we have the following result.

Lemma 4.2. *For each $K \in \mathcal{T}_h$, there holds*

$$A_h^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau}) \;=\; A^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau}) \qquad \forall\, \mathbf{p} \in \mathbb{P}_k(K)\,, \quad \forall\, \boldsymbol{\tau} \in H_k^K \,. \tag{4.4}$$

In addition, there exist constants $\alpha_1,\ \alpha_2 > 0$, independent of h and K, such that

$$|A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \;\le\; \alpha_2 \, \|\boldsymbol{\zeta}\|_{0,K} \|\boldsymbol{\tau}\|_{0,K} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_k^K \,, \tag{4.5}$$

and

$$\alpha_1 \, \|\boldsymbol{\zeta}^{\mathbf{d}}\|_{0,K}^2 \;\le\; A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\zeta}) \;\le\; \alpha_2 \, \|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\zeta} \in H_k^K \,. \tag{4.6}$$

Proof. Let $\mathbf{p} \in \mathbb{P}_k(K)$ and $\boldsymbol{\tau} \in H_k^K$. Then, bearing in mind the definitions of $A_h^{K,\mathbf{d}}$ (cf. (4.2)), $A^{K,\mathbf{d}}$ (cf. (4.1)), and $\boldsymbol{\mathcal{P}}_k^K$ (cf. (3.13)), and using that certainly $\boldsymbol{\mathcal{P}}_k^K(\mathbf{p}) = \mathbf{p}$ and $\mathbf{p}^{\mathbf{d}} \in \mathbb{P}_k(K)$, we deduce that

$$A_h^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau}) \;=\; \int_K \mathbf{p}^{\mathbf{d}} : \big(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big)^{\mathbf{d}} \;=\; \int_K \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) : \mathbf{p}^{\mathbf{d}} \;=\; \int_K \boldsymbol{\tau} : \mathbf{p}^{\mathbf{d}} \;=\; A^{K,\mathbf{d}}(\mathbf{p},\boldsymbol{\tau})\,,$$

which establishes (4.4). In turn, applying the Cauchy-Schwarz inequality and the upper bound in (4.3), we find that for each $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in H_k^K$ there holds

$$\begin{aligned}
|A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \;\le\; & \|\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K} \|\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K} \\
+ \quad & \Big\{\mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))\Big\}^{1/2} \Big\{\mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}), \boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}))\Big\}^{1/2} \\
\le \quad & \|\boldsymbol{\zeta}\|_{0,K} \|\boldsymbol{\tau}\|_{0,K} \,+\, \widehat{c}_1 \, \|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K} \|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K} \\
\le \quad & (1 + \widehat{c}_1) \, \|\boldsymbol{\zeta}\|_{0,K} \|\boldsymbol{\tau}\|_{0,K} \,,
\end{aligned}$$

which is (4.5) with $\alpha_2 := 1 + \widehat{c}_1$. Next, employing triangle inequality, the definition of $A^{K,\mathbf{d}}$ (cf. (4.1)), and the lower bound in (4.3), we deduce that

$$\begin{aligned}
\|\boldsymbol{\zeta}^{\mathbf{d}}\|_{0,K}^2 \;\le\; & 2 \, \|(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}}\|_{0,K}^2 \,+\, 2 \, \|(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}}\|_{0,K}^2 \\
\le \quad & 2 \, \|(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}}\|_{0,K}^2 \,+\, \frac{2}{\widehat{c}_0} \Big\{\widehat{c}_0 \, \|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}^2\Big\} \\
\le \quad & 2 \, A^{K,\mathbf{d}}(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})) \,+\, \frac{2}{\widehat{c}_0} \, \mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}), \boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))\,,
\end{aligned}$$

which yields the lower bound in (4.6) with $\alpha_1 := \Big(2 \max\big\{1, (\widehat{c}_0)^{-1}\big\}\Big)^{-1}$. Finally, we remark that the upper bound in (4.6) follows straightforwardly from (4.5). $\square$

On the other hand, we define the discrete local bilinear form $A_h^{K,\nabla} : V_k^K \times V_k^K \to \mathrm{R}$ as

$$A_h^{K,\nabla}(\mathbf{w},\mathbf{v}) \;:=\; A^{K,\nabla}(\mathcal{R}_k^K(\mathbf{w}), \mathcal{R}_k^K(\mathbf{v})) \,+\, \mathcal{S}^{K,\nabla}(\mathbf{w} - \mathcal{R}_k^K(\mathbf{w}), \mathbf{v} - \mathcal{R}_k^K(\mathbf{v})) \quad \forall\, \mathbf{w},\mathbf{v} \in V_k^K \,, \tag{4.7}$$

where $\mathcal{R}_k^K$ is defined in (3.1), and $\mathcal{S}^{K,\nabla} : V_k^K \times V_k^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_{k,K}^V \times n_{k,K}^V}$ with respect to a basis of V_k^K, that is,

$$\mathcal{S}^{K,\nabla}(\mathbf{w},\mathbf{v}) \;:=\; \sum_{i=1}^{n_{k,K}^V} m_{i,K}^V(\mathbf{w})\, m_{i,K}^V(\mathbf{v}) \qquad \forall\, \mathbf{w},\mathbf{v} \in V_k^K\,,$$

where, as indicated in Section 3.2, $m_{i,K}^V$, $i \in \{1,\dots,n_{k,K}^V\}$, are the degrees of freedom defined by (3.5).

Similarly to Lemma 4.1, the following result establishes the estimates for $\mathcal{S}^{K,\nabla}$.

Lemma 4.3. *There exist constants $\widetilde{c}_0, \widetilde{c}_1 > 0$, depending only on $C_{\mathcal{T}}$, such that*

$$\widetilde{c}_0\, |\mathbf{w}|_{1,K}^2 \;\le\; \mathcal{S}^{K,\nabla}(\mathbf{w},\mathbf{w}) \;\le\; \widetilde{c}_1\, |\mathbf{w}|_{1,K}^2 \qquad \forall\, \mathbf{w} \in V_k^K\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{4.8}$$

Proof. It follows from [4, eq. (4.20) and Section 4.6]. □

Now, as a consequence of the previous lemma and the properties of the projector $\mathcal{R}_k^K$ (cf. (3.1)), we deduce the following result.

Lemma 4.4. *For each $K \in \mathcal{T}_h$ there holds*

$$A_h^{K,\nabla}(\mathbf{q},\mathbf{v}) \;=\; A^{K,\nabla}(\mathbf{q},\mathbf{v}) \quad \forall\, \mathbf{q} \in \mathbf{P}_k(K)\,, \quad \forall\, \mathbf{v} \in V_k^K\,, \tag{4.9}$$

and there exist positive constants β_1, β_2, independent of h and K, such that

$$|A_h^{K,\nabla}(\mathbf{w},\mathbf{v})| \;\le\; \beta_2\, |\mathbf{w}|_{1,K} |\mathbf{v}|_{1,K} \tag{4.10}$$

and

$$\beta_1\, |\mathbf{w}|_{1,K}^2 \;\le\; A_h^{K,\nabla}(\mathbf{w},\mathbf{w}) \;\le\; \beta_2\, |\mathbf{w}|_{1,K}^2 \tag{4.11}$$

for all $\mathbf{w},\mathbf{v} \in V_k^K$.

Proof. Given $\mathbf{q} \in \mathbf{P}_k(K)$, it is clear from (3.1) that $\mathcal{R}_k^K(\mathbf{q}) = \mathbf{q}$. In addition, given $\mathbf{v} \in V_k^K$, it follows from the definitions of $A_h^{K,\nabla}$ (cf. (4.7)), $A^{K,\nabla}$ (cf. (4.1)), and $\mathcal{R}_k^K$ (cf. (3.1)), that

$$A_h^{K,\nabla}(\mathbf{q},\mathbf{v}) \;=\; \kappa_2\mu \int_K \nabla\mathbf{q} : \nabla\mathcal{R}_k^K(\mathbf{v}) \;=\; \kappa_2\mu \int_K \nabla\mathcal{R}_k^K(\mathbf{v}) : \nabla\mathbf{q} \;=\; \kappa_2\mu \int_K \nabla\mathbf{v} : \nabla\mathbf{q} = A^{K,\nabla}(\mathbf{q},\mathbf{v})\,,$$

which proves (4.9). Now, for the estimate (4.10) we apply the Cauchy-Schwarz inequality, the upper bound in (4.8), and (3.2), to establish that for each $\mathbf{w},\mathbf{v} \in V_k^K$ there holds

$$\begin{aligned}
|A_h^{K,\nabla}(\mathbf{w},\mathbf{v})| \;&\le\; \kappa_2\mu\, |\mathcal{R}_k^K(\mathbf{w})|_{1,K} |\mathcal{R}_k^K(\mathbf{v})|_{1,K} \\
&\quad +\; \big\{\mathcal{S}^{K,\nabla}(\mathbf{w}-\mathcal{R}_k^K(\mathbf{w}),\mathbf{w}-\mathcal{R}_k^K(\mathbf{w}))\big\}^{1/2} \big\{\mathcal{S}^{K,\nabla}(\mathbf{v}-\mathcal{R}_k^K(\mathbf{v}),\mathbf{v}-\mathcal{R}_k^K(\mathbf{v}))\big\}^{1/2} \\
&\le\; \kappa_2\mu\, |\mathbf{w}|_{1,K} |\mathbf{v}|_{1,K} \;+\; \widetilde{c}_1\, |\mathbf{w}-\mathcal{R}_k^K(\mathbf{w})|_{1,K} |\mathbf{v}-\mathcal{R}_k^K(\mathbf{v})|_{1,K} \\
&\le\; \beta_2\, |\mathbf{w}|_{1,K} |\mathbf{v}|_{1,K}\,,
\end{aligned}$$

with $\beta_2 := \kappa_2\mu + 4\,\widetilde{c}_1$. Next, using the lower bound in (4.8), we easily obtain

$$\begin{aligned}
|\mathbf{w}|_{1,K}^2 \;&\le\; 2\,|\mathcal{R}_k^K(\mathbf{w})|_{1,K}^2 \;+\; 2\,|\mathbf{w}-\mathcal{R}_k^K(\mathbf{w})|_{1,K}^2 \\
&\le\; 2\,|\mathcal{R}_k^K(\mathbf{w})|_{1,K}^2 \;+\; \frac{2}{\widetilde{c}_0}\Big\{\widetilde{c}_0\, |\mathbf{w}-\mathcal{R}_k^K(\mathbf{w})|_{1,K}^2\Big\} \\
&\le\; 2\, A^{K,\nabla}(\mathcal{R}_k^K(\mathbf{w}),\mathcal{R}_k^K(\mathbf{w})) \;+\; \frac{2}{\widetilde{c}_0}\, \mathcal{S}^{K,\nabla}(\mathbf{w}-\mathcal{R}_k^K(\mathbf{w}),\mathbf{w}-\mathcal{R}_k^K(\mathbf{w}))\,,
\end{aligned}$$

which establishes (4.11) with $\beta_1 := \big(2\max\{1,(\widetilde{c}_0)^{-1}\}\big)^{-1}$. Finally, the upper bound of (4.11) follows straightforwardly from (4.10). $\square$

The following two lemmas compare $A^{K,\mathbf{d}}$ and $A^{K,\nabla}$ with their computable versions $A_h^{K,\mathbf{d}}$ and $A_h^{K,\nabla}$, respectively.

Lemma 4.5. *Let α_2 be the constant from* (4.5) *(cf. proof of Lemma* 4.2*). Then, for each $K \in \mathcal{T}_h$ there holds*

$$|A^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) - A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \;\le\; \alpha_2\,\|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\|\boldsymbol{\tau}\|_{0,K} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \in H_k^K\,. \tag{4.12}$$

Proof. Let $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in H_k^K$. Then, from the definitions of $A^{K,\mathbf{d}}$ (cf. (4.1)), $A_h^{K,\mathbf{d}}$ (cf. (4.2)), and $\boldsymbol{\mathcal{P}}_k^K$ (cf. (3.13)), along with Lemma 4.1, it follows that

$$\begin{aligned}
|A^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) - A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \;\le\; & \left|\int_K \boldsymbol{\zeta}^{\mathbf{d}} : \boldsymbol{\tau} - \int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\right| \\
+ \;& \Big\{\mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}),\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))\Big\}^{1/2}\,\Big\{\mathcal{S}^{K,\mathbf{d}}(\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}),\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}))\Big\}^{1/2} \\
\le \;& \left|\int_K \big(\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathbf{d}} : \boldsymbol{\tau}\right| + \widehat{c}_1\|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K} \\
\le \;& \|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\|\boldsymbol{\tau}\|_{0,K} \,+\, \widehat{c}_1\|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} \\
= \;& \alpha_2\,\|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K}\,,
\end{aligned}$$

which yields (4.12), thus completing the proof. $\square$

Lemma 4.6. *Let $\beta_2 > 0$ be the constant from* (4.10) *(cf. proof of Lemma* 4.4*). Then, for each $K \in \mathcal{T}_h$ there holds*

$$|A^{K,\nabla}(\mathbf{w},\mathbf{v}) - A_h^{K,\nabla}(\mathbf{w},\mathbf{v})| \;\le\; \beta_2\,|\mathbf{w} - \mathcal{R}_k^h(\mathbf{w})|_{1,K}|\mathbf{v}|_{1,K} \qquad \forall\,\mathbf{w},\mathbf{v} \in V_k^K\,. \tag{4.13}$$

Proof. Let $\mathbf{v},\mathbf{w} \in V_k^h$. Then, according to the definitions of $A^{K,\nabla}$ (cf. (4.1)), $A_h^{K,\nabla}$ (cf. (4.7)), and $\mathcal{R}_k^K$ (cf. (3.1)), and using (4.8), we find that

$$\begin{aligned}
|A^{K,\nabla}(\mathbf{w},\mathbf{v}) - A_h^{K,\nabla}(\mathbf{w},\mathbf{v})| \;\le\; & \kappa_2\mu\left|\int_K \nabla\mathbf{w} : \nabla\mathbf{v} - \int_K \nabla\mathcal{R}_k^K(\mathbf{w}) : \nabla\mathcal{R}_k^K(\mathbf{v})\right| \\
+ & \Big\{\mathcal{S}^{K,\nabla}(\mathbf{w} - \mathcal{R}_k^K(\mathbf{w}),\mathbf{w} - \mathcal{R}_k^K(\mathbf{w}))\Big\}^{1/2}\Big\{\mathcal{S}^{K,\nabla}(\mathbf{v} - \mathcal{R}_k^K(\mathbf{v}),\mathbf{v} - \mathcal{R}_k^K(\mathbf{v}))\Big\}^{1/2} \\
\le \;& \kappa_2\mu\left|\int_K \nabla\big(\mathbf{w} - \mathcal{R}_k^K(\mathbf{w})\big) : \nabla\mathbf{v}\right| + \widetilde{c}_1\,|\mathbf{w} - \mathcal{R}_k^K(\mathbf{w})|_{1,K}|\mathbf{v} - \mathcal{R}_k^K(\mathbf{v})|_{1,K} \\
\le \;& \kappa_2\mu\,|\mathbf{w} - \mathcal{R}_k^K(\mathbf{w})|_{1,K}|\mathbf{v}|_{1,K} \,+\, 2\,\widetilde{c}_1\,|\mathbf{w} - \mathcal{R}_k^K(\mathbf{w})|_{1,K}|\mathbf{v}|_{1,K} \\
\le \;& \beta_2\,|\mathbf{w} - \mathcal{R}_k^K(\mathbf{w})|_{1,K}|\mathbf{v}|_{1,K}\,,
\end{aligned}$$

which shows (4.13), thus finishing the proof. $\square$

Having provided the above analysis, we now introduce the complete local discrete bilinear form $\mathbf{A}_h^K : \mathbf{H}_k^K \times \mathbf{H}_k^K \to \mathrm{R}$ in terms of $A_h^{K,\mathbf{d}}$, $A_h^{K,\nabla}$, and $\boldsymbol{\mathcal{P}}_k^K$, as

$$\begin{aligned}
\mathbf{A}_h^K(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;:=\; & A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,+\, \kappa_1\int_K \mathbf{div}(\boldsymbol{\zeta})\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, A_h^{K,\nabla}(\mathbf{w},\mathbf{v}) \,+\, \kappa_3\int_{\partial K\cap\Gamma} \mathbf{w}\cdot\mathbf{v} \\
& -\,\mu\int_K \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta}) \,+\, \mu\int_K \mathbf{w}\cdot\mathbf{div}(\boldsymbol{\tau}) \,-\, \kappa_2\int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v})
\end{aligned} \tag{4.14}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{v})$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{w}) \in \mathbf{H}_k^K := H_k^K \times V_k^K$. It is important to remark here that the integrals $\displaystyle\int_K \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\zeta})$ and $\displaystyle\int_K \mathbf{w} \cdot \mathbf{div}(\boldsymbol{\tau})$ are both computable. Indeed, using the fact that $\mathbf{div}(\boldsymbol{\zeta})$, $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(K)$, it readily follows that

$$\int_K \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\zeta}) \;=\; \int_K \mathcal{P}_{k-1}^K(\mathbf{v}) \cdot \mathbf{div}(\boldsymbol{\zeta})$$

and

$$\int_K \mathbf{w} \cdot \mathbf{div}(\boldsymbol{\tau}) \;=\; \int_K \mathcal{P}_{k-1}^K(\mathbf{w}) \cdot \mathbf{div}(\boldsymbol{\tau})\,.$$

Similarly, integrating by parts we observe that

$$\begin{aligned}\int_K \nabla\mathbf{v} : \mathbf{p} \;&=\; -\int_K \mathbf{v} \cdot \mathbf{div}(\mathbf{p}) \,+\, \int_{\partial K} \mathbf{p}\mathbf{n} \cdot \mathbf{v} \\ &=\; -\int_K \mathcal{P}_{k-1}^K(\mathbf{v}) \cdot \mathbf{div}(\mathbf{p}) \,+\, \int_{\partial K} \mathbf{p}\mathbf{n} \cdot \mathbf{v} \qquad \forall\, \mathbf{p} \in \mathbb{P}_k(K)\,,\end{aligned}$$

which yields the explicit computation of $\boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) \quad \forall\, \mathbf{v} \in V_k^K$.

Hence, we define the global discrete bilinear form $\mathbf{A}_h : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$ as

$$\mathbf{A}_h(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) \;:=\; \sum_{K\in\mathcal{T}_h} \mathbf{A}_h^K(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) \qquad \forall\, \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h\,. \tag{4.15}$$

In turn, in what follows we denote by $\mathcal{R}_k^h$, $\mathcal{P}_k^h$, and $\boldsymbol{\mathcal{P}}_k^h$, the global counterparts of the projections $\mathcal{R}_k^K$ (cf. (3.1)), $\mathcal{P}_k^K$ (cf. (3.12)), and $\boldsymbol{\mathcal{P}}_k^K$ (cf. (3.13)), respectively. In other words, for each $K \in \mathcal{T}_h$ we let

$$\mathcal{R}_k^h(\mathbf{z})|_K \;:=\; \mathcal{R}_k^K(\mathbf{z}|_K)\,, \quad \mathcal{P}_k^h(\mathbf{v})|_K \;:=\; \mathcal{P}_k^K(\mathbf{v}|_K)\,, \quad \text{and} \quad \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau})|_K \;:=\; \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}|_K)\,,$$

for all $\mathbf{z} \in \mathbf{H}^1(\Omega)$, $\mathbf{v} \in \mathbf{L}^2(\Omega)$, and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$.

The following two lemmas, which refer to the relationship between $\mathbf{A}_h$ and $\mathbf{A}$, can be seen as the global analogues of (4.4) - (4.9) (cf. Lemmas 4.2 and 4.4) and Lemmas 4.5 - 4.6.

Lemma 4.7. *There holds*

$$\mathbf{A}_h(\vec{\mathbf{p}}, \vec{\boldsymbol{\tau}}) \;=\; \mathbf{A}(\vec{\mathbf{p}}, \vec{\boldsymbol{\tau}})$$

for all $\vec{\mathbf{p}} \in \mathbb{P}_k(\Omega) \times \mathbf{P}_k(\Omega)$ *and* $\vec{\boldsymbol{\tau}} \in \mathbf{H}_k^K$.

Proof. Let $\vec{\mathbf{p}} := (\mathbf{p}, \mathbf{q}) \in \mathbb{P}_k(\Omega) \times \mathbf{P}_k(\Omega)$ and $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^K$. Then, noting that $\boldsymbol{\mathcal{P}}_k^K(\mathbf{p}) = \mathbf{p}$ and $\mathbf{p}^{\mathbf{d}} \in \mathbb{P}_k(\Omega)$, and employing the characterization of $\boldsymbol{\mathcal{P}}_k^K$ (cf. (3.13)), we observe that

$$\int_K (\boldsymbol{\mathcal{P}}_k^K(\mathbf{p}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) \;=\; \int_K \mathbf{p}^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) \;=\; \int_K \mathbf{p}^{\mathbf{d}} : \nabla\mathbf{v}\,.$$

which, together with (4.4), (4.9), and the definitions of $\mathbf{A}_h$ (cf. (4.15)) and $\mathbf{A}_h^K$ (cf. (4.14)), yield the required identity. □

Lemma 4.8. *There exists a constant $L_{\mathbf{A}} > 0$, depending only on α_2, β_2 and κ_2, such that*

$$|\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) - \mathbf{A}_h(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})| \;\le\; L_{\mathbf{A}}\Big\{\|\boldsymbol{\zeta} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\zeta})\|_{0,\Omega} \,+\, |\mathbf{w} - \mathcal{R}_k^h(\mathbf{w})|_{1,h}\Big\}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \tag{4.16}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^K$.

Proof. It follows from the definitions of $\mathbf{A}$ (cf. (2.12)) and $\mathbf{A}_h$ (cf. (4.15)) that

$$\begin{aligned}\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})-\mathbf{A}_h(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;=\; \sum_{K\in\mathcal{T}_h}\Big\{ & A^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) - A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,+\, A^{K,\nabla}(\mathbf{w},\mathbf{v}) \,-\, A_h^{K,\nabla}(\mathbf{w},\mathbf{v}) \\ & -\,\kappa_2\int_K \boldsymbol{\zeta}^{\mathbf{d}} : \nabla\mathbf{v} \,+\, \kappa_2\int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v})\Big\}.\end{aligned}$$

Next, thanks to the fact that $(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} \in \mathbb{P}_k(K)$, along with (3.13), we have

$$\int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^K(\nabla\mathbf{v}) \;=\; \int_K (\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}))^{\mathbf{d}} : \nabla\mathbf{v}\,,$$

which implies that

$$\begin{aligned}|\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})-\mathbf{A}_h(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\; \sum_{K\in\mathcal{T}_h}\Big\{ & |A^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau}) - A_h^{K,\mathbf{d}}(\boldsymbol{\zeta},\boldsymbol{\tau})| \\ + \Big|A^{K,\nabla}(\mathbf{w},\mathbf{v}) \,-\, A_h^{K,\nabla}(\mathbf{w},\mathbf{v})\Big| & \,+\, \kappa_2\left|\int_K \big(\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathbf{d}} : \nabla\mathbf{v}\right|\Big\}.\end{aligned}$$

Now, employing the Cauchy-Schwarz inequality, (4.12), and (4.13), we find that

$$\begin{aligned}|\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})-\mathbf{A}_h(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\; & \sum_{K\in\mathcal{T}_h}\Big\{\alpha_2\,\|\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\|\boldsymbol{\tau}\|_{0,K} \\ & + \beta_2\,|\mathbf{w}-\mathcal{R}_k^K(\mathbf{w})|_{1,K}|\mathbf{v}|_{1,K} \,+\, \kappa_2\,\|\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}|\mathbf{v}|_{1,K}\Big\} \\ \le\; & \max\{\alpha_2+\kappa_2,\beta_2\}\left(\sum_{K\in\mathcal{T}_h}\Big\{\|\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}^2 \,+\, |\mathbf{w}-\mathcal{R}_k^K(\mathbf{w})|_{1,K}^2\Big\}\right)^{1/2}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}\,,\end{aligned}$$

which is (4.16) with $L_{\mathbf{A}} := \max\{\alpha_2+\kappa_2,\beta_2\}$. □

Next, we prove the boundedness and ellipticity properties of $\mathbf{A}_h$. We begin with the first of them.

Lemma 4.9. *There exists a constant $\widetilde{C}_{\mathbf{A}} > 0$, independent of h, such that*

$$|\mathbf{A}_h(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\; \widetilde{C}_{\mathbf{A}}\,\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \qquad \forall\,\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}\in\mathbf{H}_k^h\,.$$

Proof. The result follows straightforwardly from the definition of $\mathbf{A}_h$ (cf. (4.15)), the estimates (4.5) and (4.10) (cf. Lemmas 4.2 and 4.4), the Cauchy-Schwarz and trace inequalities, and the boundedness of $\boldsymbol{\mathcal{P}}_k^h$. We omit further details and just mention that $\widetilde{C}_{\mathbf{A}}$ depends on μ, α_2, β_2, κ_1, κ_3, and $\|\boldsymbol{\gamma}_0\|$. □

In turn, for the second property we require the estimates provided by the following lemma.

Lemma 4.10. *There exist constants $c_1(\Omega)$, $c_2(\Omega) > 0$, independent of h, such that*

$$c_1(\Omega)\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \;\le\; \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega)$$

and

$$c_2(\Omega)\,\|\mathbf{v}\|_{1,\Omega}^2 \;\le\; |\mathbf{v}|_{1,\Omega}^2 \,+\, \|\mathbf{v}\|_{0,\Gamma}^2 \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega)\,.$$

Proof. See [12, Proposition 3.1, Chapter IV] and [25, Lemma 3.3], respectively. □

In this way, the $\mathbf{H}_k^h$-ellipticity of $\mathbf{A}_h$ is proved as follows.

Lemma 4.11. *Assume that $\kappa_1, \kappa_3 > 0$ and $0 < \kappa_2 < 2\min\{\alpha_1, \beta_1\}$, where α_1 and β_1 are the positive constants from (4.6) and (4.11), respectively. Then, there exists a constant $\alpha(\Omega) > 0$, independent of h, such that*

$$\mathbf{A}_h(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \;\geq\; \alpha(\Omega)\, \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}^2 \qquad \forall\, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h\,. \tag{4.17}$$

Proof. Let $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$. Then, bearing in mind the definition of the bilinear form $\mathbf{A}_h$ (cf. (4.14) and (4.15)), and employing the estimates (4.6) and (4.11), the Cauchy-Schwarz inequality, the fact that $(\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}))^{\mathbf{d}} = \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}^{\mathbf{d}}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, and the boundedness of $\boldsymbol{\mathcal{P}}_k^h$, we find that

$$\begin{aligned}
\mathbf{A}_h(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \;&\geq\; \alpha_1 \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 + \kappa_1\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; \beta_1\, |\mathbf{v}|_{1,\Omega}^2 + \kappa_3\, \|\mathbf{v}\|_{0,\Gamma}^2 - \kappa_2 \|(\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}))^{\mathbf{d}}\|_{0,\Omega}\, \|\boldsymbol{\mathcal{P}}_k^h(\nabla \mathbf{v})\|_{0,\Omega} \\
&=\; \alpha_1 \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \;+\; \kappa_1\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; \beta_1\, |\mathbf{v}|_{1,\Omega}^2 \;+\; \kappa_3\, \|\mathbf{v}\|_{0,\Gamma}^2 \;-\; \kappa_2 \|\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}^{\mathbf{d}})\|_{0,\Omega} \|\boldsymbol{\mathcal{P}}_k^h(\nabla \mathbf{v})\|_{0,\Omega} \\
&\geq\; \alpha_1 \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \;+\; \kappa_1\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; \beta_1\, |\mathbf{v}|_{1,\Omega}^2 \;+\; \kappa_3\, \|\mathbf{v}\|_{0,\Gamma}^2 \;-\; \kappa_2 \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega} |\mathbf{v}|_{1,\Omega}\,,
\end{aligned}$$

which, using the Young inequality, yields

$$\mathbf{A}_h(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \;\geq\; \Big(\alpha_1 - \frac{\kappa_2}{2}\Big)\, \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \;+\; \kappa_1\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; \Big(\beta_1 - \frac{\kappa_2}{2}\Big)\, |\mathbf{v}|_{1,\Omega}^2 \;+\; \kappa_3\, \|\mathbf{v}\|_{0,\Gamma}^2\,.$$

Then, assuming the stipulated hypotheses on κ_1, κ_2 and κ_3, applying the estimates provided by Lemma 4.10, and defining the constant $\widetilde{\alpha}(\Omega) := \min\big\{\alpha_1 - \frac{\kappa_2}{2}, \frac{\kappa_1}{2}, \beta_1 - \frac{\kappa_2}{2}, \kappa_3\big\}$, it follows that

$$\begin{aligned}
\mathbf{A}_h(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \;&\geq\; \widetilde{\alpha}(\Omega) \Big\{ \|\boldsymbol{\tau}^{\mathbf{d}}\|_{0,\Omega}^2 \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; |\mathbf{v}|_{1,\Omega}^2 \;+\; \|\mathbf{v}\|_{0,\Gamma}^2 \Big\} \\
&\geq\; \widetilde{\alpha}(\Omega) \Big\{ c_1(\Omega)\, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;+\; c_2(\Omega) \|\mathbf{v}\|_{1,\Omega}^2 \Big\}
\end{aligned}$$

which yields (4.17) with $\alpha(\Omega) := \widetilde{\alpha}(\Omega)\, \min\{1, c_1(\Omega), c_2(\Omega)\}$, thus completing the proof. $\square$

4.2 The discrete trilinear form $\mathbf{B}_h$

Similarly as in the previous section, we now introduce a computable discrete version of the form $\mathbf{B}$ defined in (2.14). More precisely, for each $\mathbf{z} \in V_k^h$ we let $\mathbf{B}_h(\mathbf{z}; \cdot\,, \cdot\,) : \mathbf{H}_k^h \times \mathbf{H}_k^h \to \mathrm{R}$ be the bilinear form defined by

$$\mathbf{B}_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) \;:=\; \int_\Omega \big(\mathcal{P}_k^h(\mathbf{w}) \otimes \mathcal{P}_k^h(\mathbf{z})\big)^{\mathbf{d}} : \big\{\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}) - \kappa_2 \boldsymbol{\mathcal{P}}_k^h(\nabla \mathbf{v})\big\} \tag{4.18}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$.

The following result establishes the comparison between $\mathbf{B}$ and $\mathbf{B}_h$.

Lemma 4.12. *Let $\mathbf{i}_c$ and $M_k \geq 1$ be specified in (2.9) and (3.19), respectively. Then, there holds*

$$\begin{aligned}
|\mathbf{B}(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}}) - \mathbf{B}_h(\mathbf{z}; \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}})| \;&\leq\; (1 + \kappa_2^2)^{1/2} \Big\{ \|\mathbf{w} \otimes \mathbf{z} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w} \otimes \mathbf{z})\|_{0,\Omega} \\
&+ \|\mathbf{i}_c\|\, M_k \Big(\|\mathbf{w}\|_{1,\Omega} \|\mathbf{z} - \mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega} \;+\; \|\mathbf{z}\|_{1,\Omega}\, \|\mathbf{w} - \mathcal{P}_k^h(\mathbf{w})\|_{0,4,\Omega} \Big) \Big\} \|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}
\end{aligned} \tag{4.19}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$ and $\mathbf{z} \in V_k^h$.

Proof. Let $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$ and $\mathbf{z} \in V_k^h$. Then, from the definitions of $\mathbf{B}$ (cf. (2.14)) and $\mathbf{B}_h$ (cf. (4.18)), and after adding and subtracting suitable terms, we obtain

$$
\begin{aligned}
\mathbf{B}(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) - \mathbf{B}_h(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;=\;& \int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} : \{\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\} \;-\; \int_\Omega \big(\mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\big)^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}) \\
=\;& \int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} : \{\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\} \;-\; \int_\Omega (\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}) \\
&+ \int_\Omega \big(\mathbf{w}\otimes\mathbf{z} - \mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\big)^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}) \\
=\;& \int_\Omega \Big\{(\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}} - \boldsymbol{\mathcal{P}}_k^h\big((\mathbf{w}\otimes\mathbf{z})^{\mathbf{d}}\big)\Big\} : \{\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\} \\
&+ \int_\Omega \big(\mathbf{w}\otimes\mathbf{z} - \mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\big)^{\mathbf{d}} : \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v})\,,
\end{aligned}
$$

which, along with the Cauchy-Schwarz inequality and the fact that $\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau}^{\mathbf{d}}) = \big(\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau})\big)^{\mathbf{d}} \;\; \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, leads to

$$
\begin{aligned}
|\mathbf{B}(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) - \mathbf{B}_h(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\;& \|\mathbf{w}\otimes\mathbf{z} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w}\otimes\mathbf{z})\|_{0,\Omega}\,\|\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v}\|_{0,\Omega} \\
&+ \|\mathbf{w}\otimes\mathbf{z} - \mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega}\,\|\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v})\|_{0,\Omega} \\
\le\;& (1+\kappa_2^2)^{1/2}\Big\{\|\mathbf{w}\otimes\mathbf{z} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w}\otimes\mathbf{z})\|_{0,\Omega} \,+\, \|\mathbf{w}\otimes\mathbf{z} - \mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega}\Big\}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}\,.
\end{aligned}
\tag{4.20}
$$

In turn, adding and subtracting $\mathcal{P}_k^h(\mathbf{z})$, employing the Cauchy-Schwarz inequality, and applying (2.9) and the estimate (3.19), we find that

$$
\begin{aligned}
\|\mathbf{w}\otimes\mathbf{z} - \mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega} \;=\;& \|\mathbf{w}\otimes(\mathbf{z} - \mathcal{P}_k^h(\mathbf{z})) \,+\, (\mathbf{w} - \mathcal{P}_k^h(\mathbf{w}))\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega} \\
\le\;& \|\mathbf{w}\|_{0,4,\Omega}\,\|\mathbf{z} - \mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega} \,+\, \|\mathbf{w} - \mathcal{P}_k^h(\mathbf{w})\|_{0,4,\Omega}\,\|\mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega} \\
\le\;& \|\mathbf{w}\|_{0,4,\Omega}\,\|\mathbf{z} - \mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega} \,+\, M_k\,\|\mathbf{z}\|_{0,4,\Omega}\,\|\mathbf{w} - \mathcal{P}_k^h(\mathbf{w})\|_{0,4,\Omega} \\
\le\;& \|\mathbf{i}_c\|\,M_k\Big\{\|\mathbf{w}\|_{1,\Omega}\,\|\mathbf{z} - \mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega} \,+\, \|\mathbf{z}\|_{1,\Omega}\,\|\mathbf{w} - \mathcal{P}_k^h(\mathbf{w})\|_{0,4,\Omega}\Big\}\,.
\end{aligned}
$$

Finally, replacing the foregoing estimate into (4.20) we arrive to (4.19) and complete the proof. $\square$

On the other hand, the boundedness of the bilinear form $\mathbf{B}_h$ is established in the following result.

Lemma 4.13. *There holds*

$$
|\mathbf{B}_h(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\; \|\mathbf{i}_c\|^2\,M_k^2\,(1+\kappa_2^2)^{1/2}\,\|\mathbf{z}\|_{1,\Omega}\,\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \tag{4.21}
$$

for all $\mathbf{z} \in V_k^h$ *and* $\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}} \in \mathbf{H}_k^h$.

Proof. Let $\mathbf{z} \in V_k^h$, $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}, \mathbf{w}), \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_k^h$. Then, applying the Cauchy-Schwarz inequality in (4.18), and then employing the boundedness of $\boldsymbol{\mathcal{P}}_k^h$, the Cauchy-Schwarz inequality, the boundedness of $\mathbf{i}_c$ (cf. (2.9)), and the estimate (3.19), we readily obtain

$$
\begin{aligned}
|\mathbf{B}_h(\mathbf{z};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})| \;\le\;& \|\mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega}\,\|\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\tau} - \kappa_2\nabla\mathbf{v})\|_{0,\Omega} \\
\le\;& (1+\kappa_2^2)^{1/2}\,\|\mathcal{P}_k^h(\mathbf{w})\otimes\mathcal{P}_k^h(\mathbf{z})\|_{0,\Omega}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \\
\le\;& (1+\kappa_2^2)^{1/2}\,\|\mathcal{P}_k^h(\mathbf{w})\|_{0,4,\Omega}\,\|\mathcal{P}_k^h(\mathbf{z})\|_{0,4,\Omega}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \\
\le\;& M_k^2\,(1+\kappa_2^2)^{1/2}\,\|\mathbf{z}\|_{0,4,\Omega}\,\|\mathbf{w}\|_{0,4,\Omega}\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}} \\
\le\;& \|\mathbf{i}_c\|^2\,M_k^2\,(1+\kappa_2^2)^{1/2}\,\|\mathbf{z}\|_{1,\Omega}\,\|\mathbf{w}\|_{1,\Omega}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{H}}\,,
\end{aligned}
\tag{4.22}
$$

which, noticing that $\|\mathbf{w}\|_{1,\Omega} \leq \|\vec{\boldsymbol{\zeta}}\|_{\mathbf{H}}$, yields (4.21) and completes the proof. $\square$

4.3 The discrete linear form $\mathbf{F}_h$

In this section we introduce a computable discrete version $\mathbf{F}_h : \mathbf{H}_k^h \to \mathrm{R}$ of the functional $\mathbf{F}$ (cf. (2.13)). More precisely, we define

$$\mathbf{F}_h(\vec{\boldsymbol{\tau}}) \;:=\; \mu\langle\boldsymbol{\tau}\mathbf{n},\mathbf{g}\rangle_\Gamma \,-\, \kappa_1\int_\Omega \mathbf{f}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \mu\int_\Omega \mathcal{P}_{k-1}^h(\mathbf{f})\cdot\mathbf{v} \,+\, \kappa_3\int_\Gamma \mathbf{g}\cdot\mathbf{v}\,, \tag{4.23}$$

which can be calculated using the fact that $\mathbf{div}(\boldsymbol{\tau})|_K \in \mathbf{P}_{k-1}(K)$ and $\boldsymbol{\tau}\mathbf{n}|_e \in \mathbf{P}_k(e)$, for each $e \in \partial K$ and for all $K \in \mathcal{T}_h$. Similarly, the degrees of freedom of $\mathbf{v}$ allow us to compute the boundary integrals $\int_\Gamma \mathbf{g}\cdot\mathbf{v}$ and the term

$$\int_\Omega \mathcal{P}_{k-1}^h(\mathbf{f})\cdot\mathbf{v} \;=\; \int_\Omega \mathcal{P}_{k-1}^h(\mathbf{f})\cdot\mathcal{P}_{k-1}^h(\mathbf{v})\,.$$

In addition, we have the following lemma comparing $\mathbf{F}$ and $\mathbf{F}_h$.

Lemma 4.14. *There exists a constant $C_{\mathrm{F}} > 0$, independent of h, such that*

$$|\mathbf{F}(\vec{\boldsymbol{\tau}}) - \mathbf{F}_h(\vec{\boldsymbol{\tau}})| \;\leq\; C_{\mathrm{F}}\, h\, \|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\, |\mathbf{v}|_{1,\Omega} \tag{4.24}$$

for all $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H}_k^h$.

Proof. It suffices to observe from the definitions of $\mathbf{F}$ (cf. (2.13)), $\mathbf{F}_h$ (cf. (4.23)), and $\mathcal{P}_{k-1}^h$ (cf. (3.12)), that

$$\begin{aligned}|\mathbf{F}(\vec{\boldsymbol{\tau}}) - \mathbf{F}_h(\vec{\boldsymbol{\tau}})| \;&=\; \mu\left|\int_\Omega \big\{\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\big\}\cdot\mathbf{v}\right| \;=\; \mu\left|\int_\Omega \big\{\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\big\}\cdot\big\{\mathbf{v} - \mathcal{P}_{k-1}^h(\mathbf{v})\big\}\right| \\ &\leq\; \mu\,\|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\,\|\mathbf{v} - \mathcal{P}_{k-1}^h(\mathbf{v})\|_{0,\Omega}\,,\end{aligned}$$

from which we arrive to (4.24) after applying (3.14) with $m = 0$ and $s = 1$. $\square$

5 The virtual element scheme

We now use the discrete forms analyzed in the previous section to introduce our mixed virtual element scheme associated with (2.11), which reads: Find $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{H}_k^h$ such that

$$\mathbf{A}_h(\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) \,+\, \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\sigma}}_h,\vec{\boldsymbol{\tau}}_h) \;=\; \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h) \qquad \forall\, \vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h\,, \tag{5.1}$$

where $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{F}_h$ are the forms defined by (4.15), (4.18), and (4.23), respectively.

5.1 The solvability analysis

In this section we follow the approach from [17, Section 3.2] and employ a fixed-point strategy to analyze the solvability and stability of the Galerkin scheme (5.1). To this end, we first define the discrete operator $\mathbf{T}_h : V_k^h \to V_k^h$ as

$$\mathbf{T}_h(\mathbf{z}_h) \;:=\; \mathbf{w}_h \qquad \forall\, \mathbf{z}_h \in V_k^h\,,$$

where $\mathbf{w}_h$ is the second component of the unique solution (to be confirmed below) of the discrete problem: Find $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{H}_k^h$ such that

$$\mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{B}_h(\mathbf{z}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) = \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h) \qquad \forall\, \vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h . \tag{5.2}$$

In this way, we realize that the augmented mixed-VEM formulation (5.1) can be rewritten as the fixed-point problem: Find $\mathbf{u}_h \in V_k^h$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h . \tag{5.3}$$

Now, before studying the solvability of (5.3), we need to prove that $\mathbf{T}_h$ is well-defined, which is equivalent to the well-posedness of (5.2). Indeed, the following lemma shows that $\mathbf{T}_h$ makes sense only in a closed ball of V_k^h.

Lemma 5.1. *Suppose that the parameters κ_1, κ_2 and κ_3, satisfy the conditions required by Lemma 4.11. Then, there exists $\rho_0 > 0$, independent of h, such that for each $\rho \in (0, \rho_0)$, problem (5.2) has a unique solution $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{H}_k^h$ for each $\mathbf{z}_h \in V_k^h$ such that $\|\mathbf{z}_h\|_{1,\Omega} \le \rho$. In addition, there exists a constant $c_{\mathbf{T}} > 0$, independent of $\mathbf{z}_h$, $\mathbf{f}$, $\mathbf{g}$, and h, such that*

$$\|\mathbf{T}_h(\mathbf{z}_h)\|_{1,\Omega} = \|\mathbf{w}_h\|_{1,\Omega} \le \|\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \le c_{\mathbf{T}} \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\} . \tag{5.4}$$

Proof. Let $\mathbf{z}_h \in V_k^h$. Then, thanks to the boundedness properties of $\mathbf{A}_h$ and $\mathbf{B}_h$ (cf. Lemmas 4.9 and 4.13), and defining $C_{\mathbf{AB}}(\mathbf{z}_h) := \widetilde{C}_{\mathbf{A}} + \|\mathbf{i}_c\|^2 M_k^2 (1+\kappa_2^2)^{1/2} \|\mathbf{z}_h\|_{1,\Omega}$, we find that

$$|\mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{B}_h(\mathbf{z}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)| \le C_{\mathbf{AB}}(\mathbf{z}_h) \, \|\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \, \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h ,$$

which shows that $\mathbf{A}_h + \mathbf{B}_h(\mathbf{z}_h; \cdot, \cdot)$ is bounded. Next, according to the hypotheses on κ_1, κ_2, and κ_3, we know from Lemma 4.11 that $\mathbf{A}_h$ becomes elliptic with constant $\alpha(\Omega)$, and hence, employing (4.21), we deduce that

$$\mathbf{A}_h(\vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{B}_h(\mathbf{z}_h; \vec{\boldsymbol{\tau}}_h, \vec{\boldsymbol{\tau}}_h) \ge \Big\{ \alpha(\Omega) - \|\mathbf{i}_c\|^2 M_k^2 (1+\kappa_2^2)^{1/2} \|\mathbf{z}_h\|_{1,\Omega} \Big\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}^2 \ge \frac{\alpha(\Omega)}{2} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}^2 \tag{5.5}$$

for all $\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h$, provided

$$\|\mathbf{i}_c\|^2 M_k^2 (1+\kappa_2^2)^{1/2} \|\mathbf{z}_h\|_{1,\Omega} \le \frac{\alpha(\Omega)}{2} .$$

Therefore, given $\mathbf{z}_h \in V_k^h$, the ellipticity of the bilinear form $\mathbf{A}_h + \mathbf{B}_h(\mathbf{z}_h; \cdot, \cdot)$ is ensured with the constant $\frac{\alpha(\Omega)}{2}$, independent of $\mathbf{z}_h$, by requiring

$$\|\mathbf{z}_h\|_{1,\Omega} \le \rho_0 := \frac{\alpha(\Omega)}{2\, \|\mathbf{i}_c\|^2 M_k^2 (1+\kappa_2^2)^{1/2}} . \tag{5.6}$$

In turn, it is easy to see that, with the same constant $M_{\mathbf{F}} > 0$ from (2.19), which is independent of $\mathbf{z}_h$, h, and the data $\mathbf{f}$ and $\mathbf{g}$, there holds

$$|\mathbf{F}_h(\vec{\boldsymbol{\tau}}_h)| \le M_{\mathbf{F}} \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h ,$$

which shows that $\mathbf{F}_h$ is bounded with

$$\|\mathbf{F}_h\| \le M_{\mathbf{F}} \Big\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Gamma} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\} . \tag{5.7}$$

Hence, a straightforward application of the Lax-Milgram lemma implies the existence of a unique solution $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{H}_k^h$ of (5.2). Moreover, the corresponding continuous dependence result establishes that

$$\|\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \le \frac{2}{\alpha(\Omega)} \|\mathbf{F}_h\| ,$$

from which, utilizing (5.7), we conclude (5.4) with $c_{\mathbf{T}} := \frac{2 M_{\mathbf{F}}}{\alpha(\Omega)}$, which is clearly independent of $\mathbf{z}_h$. $\square$

Having proved that $\mathbf{T}_h : \mathbf{H}_k^h \to \mathbf{H}_k^h$ is well defined, we now employ the Banach theorem to establish the existence of a unique fixed-point of this operator. We begin with the following result, from which a Lipschitz-continuity property of $\mathbf{T}_h$ will be derived later on (see the proof of Theorem 5.1 below).

Lemma 5.2. *Given $\rho \in (0, \rho_0)$, with ρ_0 defined by (5.6), we let*

$$W_\rho^h \; := \; \Big\{ \mathbf{z}_h \in V_k^h : \quad \|\mathbf{z}_h\|_{1,\Omega} \, \le \, \rho \Big\} . \tag{5.8}$$

Then, there holds

$$\|\mathbf{T}_h(\mathbf{z}_{1,h}) - \mathbf{T}_h(\mathbf{z}_{2,h})\|_{1,\Omega} \; \le \; \frac{1}{\rho_0} \, \|\mathbf{T}_h(\mathbf{z}_{1,h})\|_{1,\Omega} \, \|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{1,\Omega} \qquad \forall \, \mathbf{z}_{1,h}, \mathbf{z}_{2,h} \in W_\rho^h . \tag{5.9}$$

Proof. Given $\rho \in (0, \rho_0)$ and $\mathbf{z}_{1,h}$, $\mathbf{z}_{2,h} \in W_\rho^h$, we let

$$\mathbf{u}_{1,h} \; = \; \mathbf{T}_h(\mathbf{z}_{1,h}) \qquad \text{and} \qquad \mathbf{u}_{2,h} \; = \; \mathbf{T}_h(\mathbf{z}_{2,h})$$

be the second components of the corresponding solutions $\vec{\boldsymbol{\sigma}}_{1,h}$ and $\vec{\boldsymbol{\sigma}}_{2,h}$ of the problems

$$\mathbf{A}_h(\vec{\boldsymbol{\sigma}}_{1,h}, \vec{\boldsymbol{\tau}}_h) \, + \, \mathbf{B}_h(\mathbf{z}_{1,h}; \vec{\boldsymbol{\sigma}}_{1,h}, \vec{\boldsymbol{\tau}}_h) \; = \; \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h) \tag{5.10}$$

and

$$\mathbf{A}_h(\vec{\boldsymbol{\sigma}}_{2,h}, \vec{\boldsymbol{\tau}}_h) \, + \, \mathbf{B}_h(\mathbf{z}_{2,h}; \vec{\boldsymbol{\sigma}}_{2,h}, \vec{\boldsymbol{\tau}}_h) \; = \; \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h) \, , \tag{5.11}$$

for all $\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h$, respectively. Then, applying the ellipticity of $\mathbf{A}_h + \mathbf{B}_h(\mathbf{z}_{2,h}, \cdot, \cdot)$ (cf. (5.5)) with $\vec{\boldsymbol{\tau}}_h := \vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}$, and employing (5.10) and (5.11), we find that

$$\begin{aligned} \frac{\alpha(\Omega)}{2} \|\vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}\|_{\mathbf{H}}^2 \, &\le \, \mathbf{A}_h(\vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}, \vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}) \, + \, \mathbf{B}_h(\mathbf{z}_{2,h}; \vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}, \vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}) \\ &= - \quad \mathbf{B}_h(\mathbf{z}_{1,h} - \mathbf{z}_{2,h}; \, \vec{\boldsymbol{\sigma}}_{1,h}, \, \vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}) \, , \end{aligned}$$

which, together with the estimate obtained at the end of (4.22), yield

$$\frac{\alpha(\Omega)}{2} \|\vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}\|_{\mathbf{H}}^2 \; \le \; \|\mathbf{i}_c\|^2 \, M_k^2 \, (1 + \kappa_2^2)^{1/2} \, \|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{1,\Omega} \, \|\mathbf{u}_{1,h}\|_{1,\Omega} \, \|\vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}\|_{\mathbf{H}} \, . \tag{5.12}$$

Finally, recalling the definition of ρ_0 (cf. (5.6)), we see that (5.12) can be rewritten as

$$\|\vec{\boldsymbol{\sigma}}_{1,h} - \vec{\boldsymbol{\sigma}}_{2,h}\|_{\mathbf{H}} \; \le \; \frac{1}{\rho_0} \, \|\mathbf{u}_{1,h}\|_{1,\Omega} \, \|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{1,\Omega} \, ,$$

which gives (5.9) and finishes the proof. □

The main result of this section is stated as follows.

Theorem 5.1. *Suppose that the parameters κ_1, κ_2 and κ_3, satisfy the conditions required by Lemma 4.11. In addition, given $\rho \in (0, \rho_0)$, with ρ_0 defined by (5.6), we let W_ρ^h as in (5.8), and assume that the data satisfy*

$$c_{\mathbf{T}} \Big\{ \|\mathbf{f}\|_{0,\Omega} \, + \, \|\mathbf{g}\|_{0,\Gamma} \, + \, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \; \le \; \rho \, , \tag{5.13}$$

with $c_{\mathbf{T}}$ given by Lemma 5.1. Then, the mixed virtual element scheme (5.1) has a unique solution $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_k^h$ with $\mathbf{u}_h \in W_\rho^h$, and there holds

$$\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \; \le \; c_{\mathbf{T}} \Big\{ \|\mathbf{f}\|_{0,\Omega} \, + \, \|\mathbf{g}\|_{0,\Gamma} \, + \, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} . \tag{5.14}$$

Proof. We first notice, thanks to (5.4), that the assumption (5.13) guarantees that $\mathbf{T}_h(W^h_\rho) \subseteq W^h_\rho$. Next, using (5.9) along with (5.4) and (5.13), we obtain

$$\|\mathbf{T}_h(\mathbf{z}_{1,h}) - \mathbf{T}_h(\mathbf{z}_{2,h})\|_{1,\Omega} \;\le\; \frac{\rho}{\rho_0}\,\|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{1,\Omega} \qquad \forall\, \mathbf{z}_{1,h}, \mathbf{z}_{2,h} \in W^h_\rho\,,$$

which proves that $\mathbf{T}_h : W^h_\rho \to W^h_\rho$ is a contraction, that is a Lipschitz-continuous mapping with corresponding constant in $(0,1)$. Hence, a simple application of the Banach theorem implies the existence of a unique fixed-point $\mathbf{u}_h \in W^h_\rho$ of (5.3). In this way, the equivalence between (5.3) and the Galerkin scheme (5.1) shows that (5.1) has a unique solution $\vec{\boldsymbol{\sigma}}_h \in \mathbf{H}^h_k$, whose stability (5.14) follows directly from (5.4). $\square$

5.2 The a priori error analysis

We now aim to derive the *a priori* estimates for the error $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$, where $\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H} := \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ and $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}^h_k := H^h_k \times V^h_k$ are the unique solutions of the continuous and discrete schemes (2.11) and (5.1), respectively. In this regard, and as suggested by Theorems 2.1 and 5.1, we first define

$$\widetilde{\rho}_0 \;:=\; \min\left\{ \frac{\alpha_{\mathbf{A}}}{2\,\|\mathbf{i}_c\|^2\,(1+\kappa_2^2)^{1/2}},\; \rho_0 \right\},$$

with $\mathbf{i}_c$, $\alpha_{\mathbf{A}}$, and ρ_0, given by (2.9), (2.16), and (5.6), respectively, and observe that the existence of $\vec{\boldsymbol{\sigma}}$ and $\vec{\boldsymbol{\sigma}}_h$ is guaranteed within the respective balls centered at the origin and with radius $\rho \in (0, \widetilde{\rho}_0)$, and under the assumptions that $\kappa_1,\ \kappa_3 > 0$, and $0 < \kappa_2 < 2\min\{\mu, \alpha_1, \beta_1\}$. In particular, we know from Theorem 2.1 that there holds

$$\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} \;\le\; C_{\mathbf{T}}\left\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \right\} \;\le\; \rho\,. \tag{5.15}$$

Furthermore, we now recall from [20, Theorem 4.1.1] (see also [32, Theorem 11.1]) the first Strang lemma for linear problems, which will be utilized to obtain the main result of this section.

Lemma 5.3. *Let H be a Hilbert space, $F \in H'$, and $A : H \times H \to \mathrm{R}$ a bounded and H-elliptic bilinear form. In addition, let $\{H_h\}_{h>0}$ be a sequence of finite-dimensional subspaces of H, and for each $h > 0$ consider a functional $F_h \in H_h'$ and a bounded bilinear form $A_h : H_h \times H_h \to \mathrm{R}$. Assume that the family $\{A\} \cup \{A_h\}_{h>0}$ is uniformly bounded and uniformly elliptic with constants L_{B} and L_{E}, respectively. In turn, let $u \in H$ and $u_h \in H_h$ such that*

$$A(u,v) \;=\; F(v) \quad \forall\, v \in H$$

and

$$A_h(u_h,v_h) \;=\; F_h(v_h) \quad \forall\, v_h \in H_h\,.$$

Then, for all $h > 0$ there holds

$$\begin{aligned}\|u - u_h\|_H \;\le\; C_{\mathrm{ST}}\Bigg\{ &\sup_{\substack{w_h\in H_h\\ w_h\neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_H} \\ &+\; \inf_{v_h\in H_h}\Bigg(\|u - v_h\|_H \,+\, \sup_{\substack{w_h\in H_h\\ w_h\neq 0}} \frac{|A(v_h,w_h) - A_h(v_h,w_h)|}{\|w_h\|_H} \Bigg)\Bigg\},\end{aligned}$$

with $C_{\mathrm{ST}} := L_{\mathrm{E}}^{-1}\max\{1, L_{\mathrm{E}} + L_{\mathrm{B}}\}$.

We begin the analysis with a preliminary estimate for $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$.

Lemma 5.4. *There exists a positive constant C_{p}, independent of h, such that*

$$
\begin{aligned}
\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \;\le\; C_{\mathrm{p}} \Bigg\{ & \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, |\mathbf{u} - \mathcal{R}_k^h(\mathbf{u})|_{1,h} \,+\, h\,\|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \\
& + \inf_{\vec{\boldsymbol{\zeta}}_h \in \mathbf{H}_k^h} \left(\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \right) \Bigg\}
\end{aligned}
\tag{5.16}
$$

Proof. It reduces to apply Lemma 5.3 to the context given by (2.11) and (5.1). In fact, we first set

$$
\begin{aligned}
A(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \;&:=\; \mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}}) \,+\, \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}})\,, \qquad F(\vec{\boldsymbol{\tau}}) \;:=\; \mathbf{F}(\vec{\boldsymbol{\tau}})\,, \\
A_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) \;&:=\; \mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) \,+\, \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)\,, \qquad \text{and} \qquad F_h(\vec{\boldsymbol{\tau}}_h) \;:=\; \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h)\,,
\end{aligned}
$$

for all $\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\tau}} \in H := \mathbf{H}$ and $\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h \in H_h := \mathbf{H}_k^h$. Then, employing the bounds provided by (2.15), (2.18), Lemmas 4.9 and 4.13, and (5.15), and recalling that $M_k \ge 1$ (cf. (3.19)), we deduce that the family $\{A\} \cup \{A_h\}_{h>0}$ is uniformly bounded with a constant, independent of h, given by

$$
L_{\mathrm{B}} \;:=\; \max\left\{C_{\mathbf{A}}, \widetilde{C}_{\mathbf{A}}\right\} \,+\, \|\mathbf{i}_c\|^2\, M_k^2\, (1+\kappa_2^2)^{1/2}\, \widetilde{\rho}_0\,.
$$

In turn, using now (2.17) and (5.5), we obtain that $\{A\} \cup \{A_h\}_{h>0}$ is uniformly elliptic with the constant

$$
L_{\mathrm{E}} \;:=\; \frac{1}{2}\min\{\alpha_{\mathbf{A}}, \alpha(\Omega)\}\,.
$$

Hence, a straightforward application of Lemma 5.3 yields

$$
\begin{aligned}
\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \;\le\; C_{\mathrm{ST}} \Bigg\{ & \sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{F}(\vec{\boldsymbol{\tau}}_h) - \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \,+\, \inf_{\vec{\boldsymbol{\zeta}}_h \in \mathbf{H}_k^h} \Bigg(\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \\
& + \sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{A}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) + \mathbf{B}(\mathbf{u};\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}_h;\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \Bigg) \Bigg\}\,,
\end{aligned}
\tag{5.17}
$$

where $C_{\mathrm{ST}} \;:=\; L_{\mathrm{E}}^{-1}\, \max\left\{1, L_{\mathrm{E}} \,+\, L_{\mathrm{B}}\right\}$. Next, thanks to (4.24) (cf. Lemma 4.14), we find that

$$
\sup_{\substack{\vec{\boldsymbol{\tau}}_h \in \mathbf{H}_k^h \\ \vec{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{|\mathbf{F}(\vec{\boldsymbol{\tau}}_h) - \mathbf{F}_h(\vec{\boldsymbol{\tau}}_h)|}{\|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}} \;\le\; C_{\mathrm{F}}\, h \|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\,,
\tag{5.18}
$$

whereas, setting $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h)$, (4.16) (cf. Lemma 4.8) gives

$$
|\mathbf{A}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)| \;\le\; L_{\mathbf{A}} \left\{ \|\boldsymbol{\zeta}_h - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\zeta}_h)\|_{0,\Omega} \,+\, |\mathbf{w}_h - \mathcal{R}_k^h(\mathbf{w}_h)|_{1,h} \right\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}\,.
$$

Thus, adding and subtracting $\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})$ and $\mathbf{u} - \mathcal{R}_k^h(\mathbf{u})$, respectively, in the first and second expressions on the right-hand side of the foregoing equation, and using the boundedness of $\boldsymbol{\mathcal{P}}_k^h$ and $\mathcal{R}_k^h$ (cf. (3.2)), we deduce that

$$
\begin{aligned}
& |\mathbf{A}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h) - \mathbf{A}_h(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)| \\
& \le\; L_{\mathbf{A}} \left\{ 2\|\boldsymbol{\sigma} - \boldsymbol{\zeta}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, 2|\mathbf{u} - \mathbf{w}_h|_{1,\Omega} \,+\, |\mathbf{u} - \mathcal{R}_k^h(\mathbf{u})|_{1,h} \right\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \\
& \le\; L_{\mathbf{A}} \left\{ 3\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, |\mathbf{u} - \mathcal{R}_k^h(\mathbf{u})|_{1,h} \right\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}}\,.
\end{aligned}
\tag{5.19}
$$

Finally, replacing (5.18) and (5.19) back into (5.17), we arrive at (5.16) with C_{p} depending on C_{ST}, C_{F}, and $L_{\mathbf{A}}$, thus completing the proof. □

We now aim to bound the supremum in (5.16). For this purpose, we first observe that, while the estimates (4.19) (cf. Lemma 4.12) and (4.21) (cf. Lemma 4.13) were proved for $\mathbf{z} \in V_k^h$, it is easily seen that they are also valid for $\mathbf{z} \in \mathbf{H}^1(\Omega)$. Then, we have the following result.

Lemma 5.5. *There exists $\widetilde{C}_{\mathrm{p}} > 0$, independent of h, but depending on κ_2, $\|\mathbf{i}_c\|$, and M_k, such that*

$$
\begin{aligned}
|\mathbf{B}(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)| \,\le\, \widetilde{C}_{\mathrm{p}} \bigg\{ & \Big(\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} + \|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \Big) \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \\
+ \quad & \|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} \Big(\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} + \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \Big) \\
+ \quad & \|\mathbf{u} \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{u} \otimes \mathbf{u})\|_{0,\Omega} \bigg\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \,,
\end{aligned}
\tag{5.20}
$$

for all $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{H}_k^h$.

Proof. Let $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{H}_k^h$. Then, adding and subtracting $\mathbf{B}_h(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)$, we find that

$$
|\mathbf{B}(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)| \,\le\, \Big|\mathbf{B}(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)\Big| + \Big|\mathbf{B}_h(\mathbf{u} - \mathbf{u}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)\Big| \,. \tag{5.21}
$$

For the second term on the right-hand side of (5.21) we apply (4.21) (cf. Lemma 4.13) and obtain

$$
\Big|\mathbf{B}_h(\mathbf{u} - \mathbf{u}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)\Big| \,\le\, \|\mathbf{i}_c\|^2 M_k^2 (1 + \kappa_2^2)^{1/2} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \,,
$$

which, adding and subtracting $\vec{\boldsymbol{\sigma}}$, yield

$$
\begin{aligned}
|\mathbf{B}_h(\mathbf{u} - \mathbf{u}_h; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)| \,&\le\, \|\mathbf{i}_c\|^2 M_k^2 (1 + \kappa_2^2)^{1/2} \Big\{ \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} \Big\} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \\
&\le\, \|\mathbf{i}_c\|^2 M_k^2 (1 + \kappa_2^2)^{1/2} \Big\{ \Big(\|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u}_h\|_{1,\Omega} \Big) \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}} + \|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \Big\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \,.
\end{aligned}
\tag{5.22}
$$

In addition, thanks to (4.19) (cf. Lemma 4.12), the corresponding first term is bounded as follows

$$
\begin{aligned}
\Big|\mathbf{B}(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) - \mathbf{B}_h(\mathbf{u}; \vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)\Big| \le\, & (1 + \kappa_2^2)^{1/2} \bigg\{ \|\mathbf{w}_h \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w}_h \otimes \mathbf{u})\|_{0,\Omega} \\
+ \quad & \|\mathbf{i}_c\| M_k \Big(\|\mathbf{w}_h\|_{1,\Omega} \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} + \|\mathbf{u}\|_{1,\Omega} \|\mathbf{w}_h - \mathcal{P}_k^h(\mathbf{w}_h)\|_{0,4,\Omega} \Big) \bigg\} \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{H}} \,.
\end{aligned}
\tag{5.23}
$$

Now, adding and subtracting $\mathbf{u}$, it follows that

$$
\mathbf{w}_h \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w}_h \otimes \mathbf{u}) \,=\, (\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h\big((\mathbf{w}_h - \mathbf{u}) \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{u}\big) \,,
$$

from which, using the $\mathbf{L}^2(\Omega)$-boundedness of $\boldsymbol{\mathcal{P}}_k^h$, the Cauchy-Schwarz inequality, and (2.9), we deduce that

$$
\begin{aligned}
\|\mathbf{w}_h \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{w}_h \otimes \mathbf{u})\|_{0,\Omega} \,&\le\, 2 \|\mathbf{u} - \mathbf{w}_h\|_{0,4,\Omega} \|\mathbf{u}\|_{0,4,\Omega} + \|\mathbf{u} \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{u} \otimes \mathbf{u})\|_{0,\Omega} \\
&\le\, 2 \|\mathbf{i}_c\|^2 \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u} \otimes \mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{u} \otimes \mathbf{u})\|_{0,\Omega} \,.
\end{aligned}
\tag{5.24}
$$

In turn, similar reasonings, but employing now the $\mathbf{L}^4(\Omega)$-boundedness of $\boldsymbol{\mathcal{P}}_k^h$ (cf. (3.19)), yield

$$
\|\mathbf{w}_h\|_{1,\Omega} \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \,\le\, \|\mathbf{i}_c\| (1 + M_k) \|\mathbf{u}\|_{1,\Omega} \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u}\|_{1,\Omega} \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \tag{5.25}
$$

and

$$
\|\mathbf{w}_h - \mathcal{P}_k^h(\mathbf{w}_h)\|_{0,4,\Omega} \,\le\, \|\mathbf{i}_c\| (1 + M_k) \|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega} + \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \,. \tag{5.26}
$$

In this way, replacing (5.24), (5.25), and (5.26) back into (5.23), and then using the resulting estimate together with (5.22) in (5.21), we are lead to (5.20) after bounding $\|\mathbf{u} - \mathbf{w}_h\|_{1,\Omega}$, $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$, $\|\mathbf{u}\|_{1,\Omega}$, and $\|\mathbf{u}_h\|_{1,\Omega}$ by $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}$, $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$, $\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}$, and $\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$, respectively. □

As a consequence of Lemmas 5.4 and 5.5, we are able to establish the following definite *a priori* estimate for $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$.

Theorem 5.2. *Let $C_{\mathbf{T}}$, C_{p}, and $\widetilde{C}_{\mathrm{p}}$ be the constants from Theorem 2.1, Lemma 5.4, and Lemma 5.5, respectively, and assume that the data $\mathbf{f}$ and $\mathbf{g}$ satisfy*

$$C_{\mathbf{T}}\, C_{\mathrm{p}}\, \widetilde{C}_{\mathrm{p}} \Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \;\le\; \frac{1}{2}\,. \tag{5.27}$$

Then there exists a positive constant $\widehat{C}_{\mathrm{p}}$, independent of h, such that

$$\begin{aligned} \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \;\le\; \widehat{C}_{\mathrm{p}} \Bigg\{ & \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, |\mathbf{u} - \mathcal{R}_k^h(\mathbf{u})|_{1,h} \,+\, h\, \|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \\ + \quad & \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \,+\, \|\mathbf{u}\otimes\mathbf{u} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} \,+\, \mathrm{dist}(\vec{\boldsymbol{\sigma}}, \mathbf{H}_k^h) \Bigg\}\,. \end{aligned} \tag{5.28}$$

Proof. It suffices to replace (5.20) back into (5.16), and then proceed to estimate the resulting terms in a suitable manner. In particular, the expressions $\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}$ and $\|\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$ multiplying $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\|_{\mathbf{H}}$ or $\|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega}$ are bounded by $\widetilde{\rho}_0$, whereas (5.15) is used to bound $\|\vec{\boldsymbol{\sigma}}\|_{\mathbf{H}}$ in terms of the data when it multiplies the exact error $\|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}$. In this way, and after taking the infimum on $\vec{\boldsymbol{\zeta}}_h \in \mathbf{H}_k^h$, which yields $\mathrm{dist}(\vec{\boldsymbol{\sigma}}, \mathbf{H}_k^h)$, we are lead on the right hand side to the remaining expression

$$C_{\mathbf{T}}\, C_{\mathrm{p}}\, \widetilde{C}_{\mathrm{p}} \Big\{ \|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{0,\Gamma} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\}\, \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}\,,$$

which is handled according to the assumption (5.27). Other details are omitted. □

Having established Theorem 5.2, we now provide the corresponding rates of convergence.

Theorem 5.3. *Let $\vec{\boldsymbol{\sigma}} \in \mathbf{H}$ and $\vec{\boldsymbol{\sigma}}_h \in \mathbf{H}$ be the unique solutions of the continuous and discrete schemes (2.11) and (5.1), respectively. Assume that for integers $r \in [1,k]$, $s \in [2,k+1]$, and $\ell \in [1,k+1]$, there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$, $\mathbf{u}|_K \in \mathbf{H}^s(K)$, and $(\mathbf{u}\otimes\mathbf{u})|_K \in \mathbb{H}^\ell(K)$, for each $K \in \mathcal{T}_h$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned} \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} &:= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \\ &\le\; C\, h^{\min\{r,s-1,\ell\}} \Bigg\{ \sum_{K\in\mathcal{T}_h} \Big(|\boldsymbol{\sigma}|_{r,K}^2 \,+\, |\mathbf{div}(\boldsymbol{\sigma})|_{r,K}^2 \,+\, |\mathbf{u}|_{s,K}^2 \,+\, |\mathbf{u}\otimes\mathbf{u}|_{\ell,K}^2 \Big) \Bigg\}^{1/2} \\ &+ \quad C\, h^{s-1} \Bigg\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u}|_{s-1,4,K}^4 \Bigg\}^{1/4}\,. \end{aligned} \tag{5.29}$$

Proof. It follows from (5.28) and the approximation properties provided along the paper. In fact, employing ($\mathbf{AP}_h^{\mathbf{u}}$) (cf. Section 3.2) and ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) (cf. Section 3.3), we obtain

$$\mathrm{dist}(\mathbf{u}, V_k^h) \;\le\; C\, h^{s-1} \Bigg\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u}|_{s,K}^2 \Bigg\}^{1/2}$$

and

$$\mathrm{dist}(\boldsymbol{\sigma}, H_k^h) \;\le\; C\, h^r \Bigg\{ \sum_{K\in\mathcal{T}_h} \Big(|\boldsymbol{\sigma}|_{r,K}^2 \,+\, |\mathbf{div}(\boldsymbol{\sigma})|_{r,K}^2 \Big) \Bigg\}^{1/2},$$

respectively, whereas straighforward applications of (3.14) and (3.15) imply

$$h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\,\le\,C\,h^{r+1}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{f}|_{r,K}^2\right\}^{1/2}\,=\,C\,h^{r+1}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{div}(\boldsymbol{\sigma})|_{r,K}^2\right\}^{1/2},$$

$$\|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,\le\,C\,h^{r}\left\{\sum_{K\in\mathcal{T}_h}|\boldsymbol{\sigma}|_{r,K}^2\right\}^{1/2},$$

and

$$\|\mathbf{u}\otimes\mathbf{u}-\boldsymbol{\mathcal{P}}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega}\,\le\,C\,h^{\ell}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}\otimes\mathbf{u}|_{\ell,K}^2\right\}^{1/2}.$$

In turn, (3.3) gives

$$|\mathbf{u}-\mathcal{R}_k^h(\mathbf{u})|_{1,h}\,\le\,C\,h^{s-1}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{s,K}^2\right\}^{1/2},$$

and the fact that $\mathbf{H}^s(K)\subseteq\mathbf{W}^{s-1,4}(K)$ together with (3.16) (cf. Lemma 3.7) yield

$$\|\mathbf{u}-\mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega}\,\le\,C\,h^{s-1}\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{s-1,4,K}^4\right\}^{1/4}.$$

The foregoing estimates and a simple algebraic inequality lead to (5.29), thus concluding the proof. $\square$

5.3 Computable approximations of $\boldsymbol{\sigma}$, $\mathbf{u}$, and p

We first introduce the fully computable approximations of $\boldsymbol{\sigma}_h$ and $\mathbf{u}_h$ given by

$$\widehat{\boldsymbol{\sigma}}_h\;:=\;\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma}_h)\qquad\text{and}\qquad\widehat{\mathbf{u}}_h\;:=\;\mathcal{P}_k^h(\mathbf{u}_h)\,,\tag{5.30}$$

and establish the corresponding *a priori* error estimates in the $\mathbb{L}^2(\Omega)$-norm for $\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h$, and in the $\mathbf{L}^2(\Omega)$-norm and broken $\mathbf{H}^1$-seminorm for $\mathbf{u}-\widehat{\mathbf{u}}_h$. As shown below in Theorem 5.6, they yield exactly the same rate of convergence given by Theorem 5.3.

We begin the analysis with the following result.

Theorem 5.4. *There exists a positive constant $C>0$, independent of h, such that*

$$\begin{aligned}\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,+\,\|\mathbf{u}-\widehat{\mathbf{u}}_h\|_{0,\Omega}\,+\,\left\{\sum_{K\in\mathcal{T}_h}|\mathbf{u}-\widehat{\mathbf{u}}_h|_{1,K}^2\right\}^{1/2}&\\ \le\;C\left\{\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}\,+\,\|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,+\,\left\{\sum_{K\in\mathcal{T}_h}\|\mathbf{u}-\mathcal{P}_k^K(\mathbf{u})\|_{1,K}^2\right\}^{1/2}\right\}.&\end{aligned}\tag{5.31}$$

Proof. In order to bound $\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}$, we add and subtract $\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})$, and then employ the boundedness of $\boldsymbol{\mathcal{P}}_k^h$, which gives

$$\begin{aligned}\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,&\le\,\|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,+\,\|\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega}\\ &\le\;\;\|\boldsymbol{\sigma}-\boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,+\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}\,.\end{aligned}\tag{5.32}$$

Similarly, adding and subtracting $\mathcal{P}_k^K(\mathbf{u})$, and using now the boundedness of $\mathcal{P}_k^K$ (cf. Lemma 3.8), we are lead to

$$\|\mathbf{u}-\widehat{\mathbf{u}}_h\|_{1,K}\,\le\,\|\mathbf{u}-\mathcal{P}_k^K(\mathbf{u})\|_{1,K}\,+\,M_k\,\|\mathbf{u}-\mathbf{u}_h\|_{1,K}\qquad\forall\,K\in\mathcal{T}_h\,,$$

from which, taking square and summing over $K \in \mathcal{T}_h$, it follows that

$$\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega} + \left\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u} - \widehat{\mathbf{u}}_h|_{1,K}^2 \right\}^{1/2} \le C \left\{ \left\{ \sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \mathcal{P}_k^K(\mathbf{u})\|_{1,K}^2 \right\}^{1/2} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right\}. \quad (5.33)$$

In this way, (5.32) and (5.33) yield (5.31), which ends the proof. □

Next, according to the second equation in (2.4) and the decomposition of $\boldsymbol{\sigma}$ provided by (2.7) and (2.8), we suggest the following computable approximation of the pressure:

$$\widehat{p}_h := -\frac{1}{2}\,\mathrm{tr}\Big(\widehat{\boldsymbol{\sigma}}_h + \widehat{c}_h \mathbb{I} + \widehat{\mathbf{u}}_h \otimes \widehat{\mathbf{u}}_h\Big) \quad \text{in} \quad \Omega, \qquad \text{with} \qquad \widehat{c}_h := -\frac{1}{2|\Omega|}\,\|\widehat{\mathbf{u}}_h\|_{0,\Omega}^2 . \quad (5.34)$$

The following lemma establishes the corresponding *a priori* error estimate.

Theorem 5.5. *There exists a positive constant $C > 0$, independent of h, such that*

$$\|p - \widehat{p}_h\|_{0,\Omega} \le C \Big\{ \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} + \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,\Omega} + \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} \Big\}. \quad (5.35)$$

Proof. According to (2.4), (2.7), (2.8), and (5.34), we have that

$$p - \widehat{p}_h = -\frac{1}{2}\mathrm{tr}\Big((\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h) + (\mathbf{u}\otimes\mathbf{u} - \widehat{\mathbf{u}}_h \otimes \widehat{\mathbf{u}}_h)\Big) + \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u} - \widehat{\mathbf{u}}_h \otimes \widehat{\mathbf{u}}_h),$$

which, applying the Cauchy-Schwarz inequality, yields

$$\|p - \widehat{p}_h\|_{0,\Omega} \le C \Big\{ \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{u}\otimes\mathbf{u} - \widehat{\mathbf{u}}_h \otimes \widehat{\mathbf{u}}_h\|_{0,\Omega} \Big\}. \quad (5.36)$$

Then, adding and subtracting $\widehat{\mathbf{u}}_h$, and using the triangle and Cauchy-Schwarz inequalities, the boundedness of $\mathcal{P}_k^h$ (cf. (3.19)), (2.9), and the fact that $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{u}_h\|_{1,\Omega}$ are bounded by $\widetilde{\rho}_0$, we find that

$$\begin{aligned}
\|\mathbf{u}\otimes\mathbf{u} - \widehat{\mathbf{u}}_h \otimes \widehat{\mathbf{u}}_h\|_{0,\Omega} &= \|\mathbf{u}\otimes(\mathbf{u} - \widehat{\mathbf{u}}_h)\|_{0,\Omega} + \|(\mathbf{u} - \widehat{\mathbf{u}}_h)\otimes\widehat{\mathbf{u}}_h\|_{0,\Omega} \\
&\le \Big(\|\mathbf{u}\|_{0,4,\Omega} + \|\widehat{\mathbf{u}}_h\|_{0,4,\Omega}\Big)\,\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,4,\Omega} = \Big(\|\mathbf{u}\|_{0,4,\Omega} + \|\mathcal{P}_k^h(\mathbf{u}_h)\|_{0,4,\Omega}\Big)\,\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,4,\Omega} \\
&\le \Big(\|\mathbf{u}\|_{0,4,\Omega} + M_k\,\|\mathbf{u}_h\|_{0,4,\Omega}\Big)\,\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,4,\Omega} \le (1 + M_k)\,\|\mathbf{i}_c\|\,\widetilde{\rho}_0\,\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,4,\Omega} .
\end{aligned} \quad (5.37)$$

In turn, adding and subtracting $\mathcal{P}_k^h(\mathbf{u})$, and employing again the boundedness of $\mathcal{P}_k^h$ and (2.9), we readily obtain

$$\begin{aligned}
\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,4,\Omega} &\le \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} + \|\mathcal{P}_k^h(\mathbf{u} - \mathbf{u}_h)\|_{0,4,\Omega} \\
&\le \|\mathbf{u} - \mathcal{P}_k^h(\mathbf{u})\|_{0,4,\Omega} + \|\mathbf{i}_c\|\,M_k\,\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} .
\end{aligned} \quad (5.38)$$

Finally, (5.36), (5.37), and (5.38), together with (5.31) imply (5.35) and finish the proof. □

We end this section by providing the theoretical rates of convergence for $\widehat{\boldsymbol{\sigma}}_h$, $\widehat{\mathbf{u}}_h$, and $\widehat{p}_h$.

Theorem 5.6. *Let $\vec{\boldsymbol{\sigma}} \in \mathbf{H}$ and $\vec{\boldsymbol{\sigma}}_h \in \mathbf{H}$ be the unique solutions of the continuous and discrete schemes* (2.11) *and* (5.1)*, respectively. In addition, let $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h)$ and $\widehat{p}_h$ be the discrete approximations introduced in* (5.30) *and* (5.34)*, respectively. Assume that for integers $r \in [1,k]$, $s \in [2,k+1]$, and $\ell \in [1,k+1]$,* *there hold* $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$, $\mathbf{u}|_K \in \mathbf{H}^s(K)$, *and* $(\mathbf{u}\otimes\mathbf{u})|_K \in \mathbb{H}^\ell(K)$, *for each* $K \in \mathcal{T}_h$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega} \,+\, \left\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u} - \widehat{\mathbf{u}}_h|_{1,K}^2 \right\}^{1/2} \,+\, \|p - \widehat{p}_h\|_{0,\Omega} \\
&\quad\leq\; C\, h^{\min\{r,s-1,\ell\}} \left\{ \sum_{K\in\mathcal{T}_h} \Big(|\boldsymbol{\sigma}|_{r,K}^2 \,+\, |\mathbf{div}(\boldsymbol{\sigma})|_{r,K}^2 \,+\, |\mathbf{u}|_{s,K}^2 \,+\, |\mathbf{u}\otimes\mathbf{u}|_{\ell,K}^2 \Big) \right\}^{1/2} \\
&\quad+\;\; C\, h^{s-1} \left\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u}|_{s-1,4,K}^4 \right\}^{1/4}.
\end{aligned} \tag{5.39}$$

Proof. It follows from (5.31), (5.35), Theorem 5.3, and the approximation properties provided along the paper. In particular, applying (3.16) (cf. Lemma 3.7), we readily find that

$$\left\{ \sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \mathcal{P}_k^K(\mathbf{u})\|_{1,K}^2 \right\}^{1/2} \;\leq\; C\, h^{s-1} \left\{ \sum_{K\in\mathcal{T}_h} |\mathbf{u}|_{s,K}^2 \right\}^{1/2}.$$

Further details, being similar to those shown in the proof of Theorem 5.3, are omitted. $\square$

5.4 A convergent approximation of $\boldsymbol{\sigma}$ in the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm

In what follows we proceed as in [14, Section 5.3] and propose a second approximation $\widetilde{\boldsymbol{\sigma}}_h$ of the pseudostress $\boldsymbol{\sigma}$, which yields the same rate of convergence from Theorems 5.3 and 5.6 in the broken $\mathbb{H}(\mathbf{div};\Omega)$-norm. For this purpose, we now consider for each $K \in \mathcal{T}_h$ an arbitrary but explicitly known finite dimensional subspace $\mathbb{U}(K)$ of $\mathbb{H}(\mathbf{div};K)$, to be specified later on, and let $(\cdot,\cdot)_{\mathbf{div};K}$ be the usual $\mathbb{H}(\mathbf{div};K)$-inner product with induced norm $\|\cdot\|_{\mathbf{div};K}$. Then, we let $\widetilde{\boldsymbol{\sigma}}_h \in \mathbb{L}^2(\Omega)$ be the tensor defined locally as $\widetilde{\boldsymbol{\sigma}}_h\,|_K := \widetilde{\boldsymbol{\sigma}}_{h,K}$, where $\widetilde{\boldsymbol{\sigma}}_{h,K} \in \mathbb{U}(K)$ is the unique solution of the problem

$$(\widetilde{\boldsymbol{\sigma}}_{h,K}, \boldsymbol{\tau}_h)_{\mathbf{div};K} \;=\; \int_K \widehat{\boldsymbol{\sigma}}_h : \boldsymbol{\tau}_h \;+\; \int_K \mathbf{div}(\boldsymbol{\sigma}_h)\cdot\mathbf{div}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{U}(K)\,. \tag{5.40}$$

Note here that the right-hand side of (5.40), and hence $\widetilde{\boldsymbol{\sigma}}_{h,K}$, is fully computable since both $\widehat{\boldsymbol{\sigma}}_h$ and $\mathbf{div}(\boldsymbol{\tau}_h)$ are. In addition, it is important to remark that $\widetilde{\boldsymbol{\sigma}}_{h,K}$ can be calculated for each $K \in \mathcal{T}_h$, independently, which certainly suggests a parallel implementation of these computations. Next, denoting by $\Pi_U^K : \mathbb{H}(\mathbf{div};K) \to \mathbb{U}(K)$ the orthogonal projector with respect to $(\cdot,\cdot)_{\mathbf{div};K}$, we have the following result establishing the *a priori* estimate for the local error $\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_{h,K}\|_{\mathbf{div};K}$.

Lemma 5.6. *For each* $K \in \mathcal{T}_h$ *there holds*

$$\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_{h,K}\|_{\mathbf{div};K} \;\leq\; \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,K} \,+\, \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,K} \,+\, \|\boldsymbol{\sigma} - \Pi_U^K(\boldsymbol{\sigma})\|_{\mathbf{div};K}\,. \tag{5.41}$$

Proof. It proceeds exactly as the proof of [14, Lemma 5.3], and hence we refer to that work and omit details here. $\square$

In this way, since we know from (5.29) (cf. Theorem 5.3) and (5.39) (cf. Theorem 5.6) that the errors $\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ and $\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}$ converge at most with order $O(h^k)$, which holds when $r = k$, $s = k+1$, and $\ell = k$, it follows from (5.41) that we need to guarantee at least the same rate for $\left\{ \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\sigma} - \Pi_U^K(\boldsymbol{\sigma})\|_{\mathbf{div};K}^2 \right\}^{1/2}$. Thus, in order to achieve this goal, we take for simplicity $\mathbb{U}(K) := \mathbb{P}_k(K)$, which means that Π_U^K becomes $\boldsymbol{\mathcal{P}}_k^K$, whence (3.15) yields

$$\|\boldsymbol{\tau} - \Pi_U^K(\boldsymbol{\tau})\|_{\mathbf{div};K} \;\leq\; \|\boldsymbol{\tau} - \Pi_U^K(\boldsymbol{\tau})\|_{1,K} \;\leq\; C\, h_K^r\, |\boldsymbol{\tau}|_{r+1,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^{r+1}(K)\,, \quad \forall\, K \in \mathcal{T}_h\,.$$

Therefore, according to the foregoing analysis, we are able to state the following theorem.

Theorem 5.7. *Let* $\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}$ *and* $\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}$ *be the unique solutions of the continuous and discrete schemes* (2.11) *and* (5.1)*, respectively. In addition, let* $\widehat{\boldsymbol{\sigma}}_h$ *and* $\widetilde{\boldsymbol{\sigma}}_h$ *be the discrete approximations of* $\boldsymbol{\sigma}$ *introduced in* (5.30) *and* (5.40)*, respectively. Assume that for integers* $r \in [1, k]$*,* $s \in [2, k+1]$*, and* $\ell \in [1, k+1]$*, there hold* $\boldsymbol{\sigma}|_K \in \mathbb{H}^{r+1}(K)$*,* $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{H}^r(K)$*,* $\mathbf{u}|_K \in \mathbf{H}^s(K)$*, and* $(\mathbf{u} \otimes \mathbf{u})|_K \in \mathbb{H}^\ell(K)$*, for each* $K \in \mathcal{T}_h$*. Then, there holds*

$$\left\{ \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_{h,K}\|^2_{\mathbf{div};K} \right\}^{1/2} = O(h^{\min\{r, s-1, \ell\}}) \,.$$

We end this paper by remarking that full details on the computational implementation of (5.1), and several numerical results confirming the theoretical rates of convergence, will be provided in a forthcoming work.

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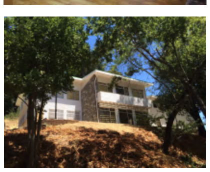

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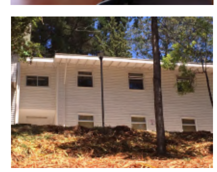

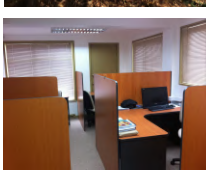

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