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ELECTROMAGNETIC STEKLOFF EIGENVALUES IN INVERSE SCATTERING

JESSIKA CAMAÑO*, CHRISTOPHER LACKNER†, AND PETER MONK‡

Abstract. In [5] it was suggested to use Stekloff eigenvalues for the Helmholtz equation to detect changes in a scatterer using remote measurements of the scattered wave. This paper investigates the use of Stekloff eigenvalues for Maxwell's equations for the same purpose. Because the Stekloff eigenvalue problem for Maxwell's equations is not a standard eigenvalue problem for a compact operator, we propose a modified Stekloff problem that restores compactness. In order to measure the modified Stekloff eigenvalues of a domain from far field measurements we perturb the usual far field equation of the Linear Sampling Method by using the far field pattern of an auxiliary impedance problem related to the modified Stekloff problem. We are then able to show 1) existence of modified Stekloff eigenvalues, 2) well-posedness of the corresponding auxiliary exterior impedance problem and 3) provide theorems that support our claim to be able to detect modified Stekloff eigenvalues from far field measurements. Preliminary numerical results show that for some simple domains it is possible to measure a few modified Stekloff eigenvalues (as for the Helmholtz equation, not all eigenvalues can be measured). In addition the modified Stekloff eigenvalues are changed by perturbations of the scatterer. An open problem is to obtain a proof of the existence of modified Stekloff eigenvalues for absorbing media.

Key words: Stekloff eigenvalues, inverse problem, non-destructive testing, Herglotz wave function.

Mathematics subject classifications: 35R30, 35J25, 35P25, 35P05

1. Introduction. In a recent paper [5] it was suggested to use Stekloff eigenvalues for the Helmholtz equation as a novel "target signature" for non-destructive testing via inverse scattering. In particular it was shown that it is possible to measure Stekloff eigenvalues for a bounded inhomogeneous scatterer by solving a sequence of modified far field equations (for a general discussion of the far field operator and associated equation, see for example [9]). By numerical examples it was shown that even in the presence of noise on the far field data it is possible to identify a few Stekloff eigenvalues. The number of eigenvalues that can be identified depends on the noise level, the shape of the scatterer and the wavenumber of the incident field. It was argued that shifts in these eigenvalues can be used to monitor changes in a medium (or the shape of the scatterer). As an alternative, as mentioned above, the eigenvalues could be compared to a dictionary of possible values to determine which of the possible targets is present. In this paper we shall continue this research program by considering the determination of Stekloff eigenvalues from far field data for Maxwell's equations.

As is well known, the Stekloff eigenvalue problem for the Helmholtz equation (or more commonly Laplace equation) for a bounded domain is equivalent to the

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determination of eigenvalues of the corresponding Neumann-to-Dirichlet map. For the Helmholtz equation the compactness of this map allowed the authors of [5] to verify the existence of Stekloff eigenvalues for the Helmholtz equation even in the case of an absorbing medium where the problem is not self adjoint. In our case, for Maxwell's equations, compactness of the corresponding electric current to magnetic current map (still referred to as the Neumann-to-Dirichlet or NtD map here) is unlikely as we shall show. However there is no need for us to use the "natural" NtD map because it is only used as an auxiliary problem for modifying the far field operator. So we propose a new modified Stekloff eigenvalue problem that does give rise to a compact and self adjoint eigenvalue problem in a dielectric medium.

Of course the standard NtD map (or its inverse the Dirichlet-to-Neumann or DtN map) is widely used for the analysis of inverse problems in electrical impedance tomography [1] and inverse scattering [3]. For Maxwell's equations this work goes back at least to the work of Sun and Uhlmann [18] who used the DtN map to prove a local uniqueness result for the inverse problem of determining coefficients in Maxwell's equations from boundary measurements. In a similar vein the determination of almost constant electromagnetic properties is considered in [17]. The uniqueness problem for coefficient identification for the scattering problem was then improved in [15]. More recently Joshi and Lionheart considered general DtN type maps for the Hodge Laplacian [11]. In this setting they verified that the operator is pseudo-differential of order -1.

For Laplace's equation, Stekloff eigenvalues have a long history and many interesting properties [12]. For Maxwell's equations we are not aware of any work on Stekloff eigenvalues.

The contributions of this paper are threefold. First we pose the electromagnetic Stekloff eigenvalue and a new modified Stekloff eigenvalue problems. We point out that the first problem cannot, in general, be reduced to analyzing a symmetric compact operator, while the second problem can be analyzed in this way. Second, we prove, using similar arguments to those in [5] but suitably modified for Maxwell's equations, results that suggest that Stekloff eigenvalues (if they exist) and modified Stekloff eigenvalues can be determined from far field data using a modified electromagnetic far field operator. Thirdly we show some limited numerical results to illustrate and support the theory. In the future, modifying the far field operator by using the far field pattern of other scattering problems is also possible, and these will lead to other target signatures perhaps better suited to particular scatterers.

We now describe the standard forward scattering problem that is the basis of our study. Suppose D is a bounded domain containing the origin such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and such that the boundary of D denoted ∂D is smooth. The forward electromagnetic scattering problem is to find the electric field $\boldsymbol{E} \in \boldsymbol{H}_{\mathtt{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{D})$ and the magnetic field $\boldsymbol{H} \in \boldsymbol{H}_{\mathtt{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{D})$ such that

$$\mathbf{curl}\,\boldsymbol{E} - i\kappa\boldsymbol{H} = \mathbf{0}\,, \qquad \mathbf{curl}\,\boldsymbol{H} + i\kappa\epsilon_r\boldsymbol{E} = \mathbf{0} \tag{1.1}$$

in $\mathbb{R}^3$ where the wavenumber κ is real and positive. In contrast to the case when transmission eigenvalues are used as target signatures [4], in this paper $\kappa > 0$ is fixed so that the method is applicable to data at a single frequency.

The relative permittivity ϵ_r is assumed to be piecewise smooth. If the medium is conducting, ϵ_r is complex valued whereas for a dielectric ϵ_r is real valued. In this paper we assume ϵ_r is real valued in order to allow a simple proof of existence of appropriate eigenvalues. We also assume that scatterer is bounded so that $\epsilon_r(\boldsymbol{x}) = 1$

for $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D}$. To assert certain regularity results later in the paper, we assume that D may be decomposed into M subdomains denoted D_m, $m = 1, ..., M$, such that $\overline{D} = \bigcup_{m=1}^{M} \overline{D}_m$; $D_m \cap D_p = \emptyset$, if $m \neq p$; each subdomain D_m, $m = 1, ..., p$, is connected and has a Lipschitz boundary; there is a constant $\alpha > 0$ such that for each m, $m = 1, ..., M$, $\mathrm{Re}(\epsilon_r) \geq \alpha$ on D_m and $\mathrm{Im}(\epsilon_r) \geq 0$. The total fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are given by

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s\,,$$
$$\boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s\,,$$

where $(\boldsymbol{E}^s, \boldsymbol{H}^s)$ is the scattered field satisfying the Silver–Müller radiation condition

$$\lim_{r\to\infty} (\boldsymbol{H}^s \times \boldsymbol{x} - r\boldsymbol{E}^s) = \boldsymbol{0} \tag{1.2}$$

uniformly in $\widehat{\boldsymbol{x}} := \boldsymbol{x}/|\boldsymbol{x}|$ where $r = |\boldsymbol{x}|$. The incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is assumed to be a plane wave given by

$$\begin{aligned} \boldsymbol{E}^i(\boldsymbol{x}) &= \frac{i}{\kappa}\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p}e^{-i\kappa\boldsymbol{x}\cdot\boldsymbol{d}} \;=\; i\kappa(\boldsymbol{d}\times\boldsymbol{p})\times\boldsymbol{d}e^{-i\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\,,\\ \boldsymbol{H}^i(\boldsymbol{x}) &= \mathbf{curl}\;\boldsymbol{p}e^{-i\kappa\boldsymbol{x}\cdot\boldsymbol{d}} \;=\; i\kappa\boldsymbol{d}\times\boldsymbol{p}e^{-i\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\,. \end{aligned} \tag{1.3}$$

Here $\boldsymbol{d}$ is a unit vector giving the direction of propagation of the plane wave and $\boldsymbol{p}$ is the polarization vector assumed real and non zero. The scattered field has the following asymptotic expansion in r [9]:

$$\boldsymbol{E}^s(\boldsymbol{x}) \;=\; \frac{e^{i\kappa r}}{r}\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}; \boldsymbol{p}) \;+\; \mathcal{O}\left(\frac{1}{r^2}\right)\,, \quad r\to\infty\,,$$

$$\boldsymbol{H}^s(\boldsymbol{x}) \;=\; \frac{e^{i\kappa r}}{r}\boldsymbol{H}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}; \boldsymbol{p}) \;+\; \mathcal{O}\left(\frac{1}{r^2}\right)\,, \quad r\to\infty\,,$$

where $\boldsymbol{E}_\infty$ (respectively $\boldsymbol{H}_\infty$) is called the far field pattern of the scattered wave $\boldsymbol{E}^s$ (respectively $\boldsymbol{H}^s$) depending, as indicated, on the measurement direction $\hat{\boldsymbol{x}}$, the incident direction $\boldsymbol{d}$, and the polarization $\boldsymbol{p}$. Note that both far field patterns are linear functions of $\boldsymbol{p}$. It is well known that under the restrictions on ϵ_r mentioned previously, and with the given incident field, there is a unique solution to equations (1.1)-(1.2) [9].

Let $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \mid |\boldsymbol{x}| = 1\}$. Now we can state the inverse problem we wish to solve: given the far field pattern for all $\hat{\boldsymbol{x}} \in \mathbb{S}^2$, $\boldsymbol{d} \in \mathbb{S}^2$ and $\boldsymbol{p} \in \mathbb{R}^3$, $\boldsymbol{p} \neq \boldsymbol{0}$, we wish to compute approximations of modified Stekloff eigenvalues (that we shall define shortly) for a domain B such that either $B = D$ (in non-destructive testing we may know the shape of the object and wish to monitor its interior for changes in ϵ_r) or B is a ball containing D in its interior (for example if the shape of D is not a priori known).

An outline of the paper is as follows. We end this section by defining some notation. Then in Section 2 we consider the Stekloff eigenvalue problem corresponding to the standard NtD map for Maxwell's equations. We show by example that this does not arise from an eigenvalue problem for a symmetric compact operator. In Section 3 we define a new modified Stekloff eigenvalue problem that we can show is equivalent to an eigenvalue problem for a compact self adjoint operator when ϵ_r is real, thus proving existence and discreteness of modified Stekloff eigenvalues. Next in Section 4 we prove theorems analogous to those in [5] that suggest that Stekloff eigenvalues or modified Stekloff eigenvalues can be determined by solving a suitable modified far field equation. The results in this section can be proved for either Stekloff eigenvalue problem assuming such eigenvalues exist (even if ϵ_r is complex) or modified Stekloff eigenvalues. Next in Section 5 we provide preliminary numerical results concerning the determination of modified Stekloff eigenvalues from far field data. Finally we draw some conclusions in Section 6.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded open simply connected domain with Lipschitz continuous boundary $\partial\mathcal{O}$ and $\boldsymbol{\nu}$ the unit outward normal to $\mathcal{O}$. We consider the space $L^2(\mathcal{O})$ with its corresponding norm $\|\cdot\|_{0,\mathcal{O}}$. For convenience, we denote $\|\cdot\|_{0,\mathcal{O}}$ the norm of $L^2(\mathcal{O})^3$, too.

Let us introduce the following spaces:

$$\boldsymbol{H}(\mathbf{curl},\mathcal{O}) := \{\boldsymbol{u} \in L^2(\mathcal{O})^3 \,:\, \mathbf{curl}\,\boldsymbol{u} \in L^2(\mathcal{O})^3\}\,,$$

$$\boldsymbol{L}_t^2(\partial\mathcal{O}) := \{\boldsymbol{u} \in L^2(\partial\mathcal{O})^3 \,:\, \boldsymbol{\nu}\cdot\boldsymbol{u} = 0 \text{ a.e. on } \partial\mathcal{O}\}\,,$$

$$\boldsymbol{\mathcal{X}}(\mathcal{O}) := \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\mathcal{O}) \,:\, \boldsymbol{u}_T \in \boldsymbol{L}_t^2(\partial\mathcal{O})\}\,,$$

where $\boldsymbol{u}_T := (\boldsymbol{\nu}\times\boldsymbol{u})\times\boldsymbol{\nu}$ corresponds to the tangential component of $\boldsymbol{u}$. The spaces $\boldsymbol{H}(\mathbf{curl},\mathcal{O})$ and $\boldsymbol{\mathcal{X}}(\mathcal{O})$ are respectively endowed with the norms defined by

$$\|\boldsymbol{u}\|^2_{\mathbf{curl},\mathcal{O}} := \|\boldsymbol{u}\|^2_{0,\mathcal{O}} + \|\,\mathbf{curl}\,\boldsymbol{u}\|^2_{0,\mathcal{O}} \quad \text{and} \quad \|\boldsymbol{u}\|^2_{\boldsymbol{\mathcal{X}}(\mathcal{O})} := \|\boldsymbol{u}\|^2_{\mathbf{curl},\mathcal{O}} + \|\boldsymbol{u}_T\|^2_{0,\partial\mathcal{O}}\,.$$

For the exterior domain $\mathbb{R}^3\setminus\overline{\mathcal{O}}$ we define the above spaces in the same way for every $(\mathbb{R}^3\setminus\overline{\mathcal{O}})\cap B_R$, with B_R a ball containing $\mathcal{O}$ in its interior, and denote the spaces by $\boldsymbol{H}_{\mathtt{loc}}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\mathcal{O}})$ and $\boldsymbol{\mathcal{X}}_{\mathtt{loc}}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\mathcal{O}})$, respectively.

In addition, for the case in which the boundary $\partial\mathcal{O}$ is smooth, we introduce the following spaces:

$$\boldsymbol{H}_t^s(\partial\mathcal{O}) := \{\boldsymbol{u} \in H^s(\partial\mathcal{O})^3 \,:\, \boldsymbol{\nu}\cdot\boldsymbol{u} = 0 \text{ a.e. on } \partial\mathcal{O}\}\,,$$

$$\boldsymbol{H}^s(\operatorname{div}_{\partial\mathcal{O}},\partial\mathcal{O}) := \{\boldsymbol{\mu} \in \boldsymbol{H}_t^s(\partial\mathcal{O}) \,:\, \operatorname{div}_{\partial B}\boldsymbol{\mu} \in H^s(\partial\mathcal{O})\}\,,$$

$$\boldsymbol{H}^{-1/2}(\operatorname{curl}_{\partial\mathcal{O}},\partial\mathcal{O}) := \{\boldsymbol{\mu} \in \boldsymbol{H}_t^{-1/2}(\partial\mathcal{O}) \,:\, \operatorname{curl}_{\partial B}\boldsymbol{\mu} \in H^{-1/2}(\partial\mathcal{O})\}\,,$$

where $\operatorname{curl}_{\partial\mathcal{O}}$ and $\operatorname{div}_{\partial\mathcal{O}}$ are the surface scalar curl and divergence operator, respectively, and $s\in\mathbb{R}$. In addition we will denote by $\mathbf{curl}_{\partial\mathcal{O}}$ the surface vectorial curl. We rename the spaces $\boldsymbol{H}_t^0(\partial\mathcal{O})$ and $\boldsymbol{H}^0(\operatorname{div}_{\partial\mathcal{O}},\partial\mathcal{O})$ by $\boldsymbol{L}_t^2(\partial\mathcal{O})$ and $\boldsymbol{H}(\operatorname{div}_{\partial\mathcal{O}},\partial\mathcal{O})$, respectively. The space $\boldsymbol{H}_t^s(\partial\mathcal{O})$ is equipped with the standard norm (see, for instance, [13]). In addition, the spaces $\boldsymbol{H}^s(\operatorname{div}_{\partial\mathcal{O}},\partial\mathcal{O})$ and $\boldsymbol{H}^{-1/2}(\operatorname{curl}_{\partial\mathcal{O}},\partial\mathcal{O})$ are endowed with their respective natural norms

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\operatorname{div}_{\partial\mathcal{O}},\partial\mathcal{O})} := \|\boldsymbol{u}\|^2_{s,\partial\mathcal{O}} + \|\operatorname{div}_{\partial\mathcal{O}}\boldsymbol{u}\|^2_{s,\partial\mathcal{O}}$$

and

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{-1/2}(\operatorname{curl}_{\partial\mathcal{O}},\partial\mathcal{O})} := \|\boldsymbol{u}\|^2_{-1/2,\partial\mathcal{O}} + \|\operatorname{curl}_{\partial\mathcal{O}}\boldsymbol{u}\|^2_{-1/2,\partial\mathcal{O}}\,.$$

For more details about the norms and properties of this operators, see, for instance [13].

To study the far field operator we shall also need the following space of tangential vector fields

$$\boldsymbol{L}_t^2(\mathbb{S}^2) := \{\boldsymbol{u} : \mathbb{S}^2 \to \mathbb{R}^3 : \boldsymbol{u} \in L^2(\mathbb{S}^2)^3, \ \boldsymbol{u}(\boldsymbol{d}) \cdot \boldsymbol{d} = 0, \ \boldsymbol{d} \in \mathbb{S}^2\}$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin.

2. The Standard Stekloff Boundary Condition. Following the ideas in [5], we are interested in studying the inverse problem by considering a modified far field operator whose kernel is the difference between the measured far field pattern due to the scattering object and the far field pattern of an auxiliary scattering problem involving a Stekloff (also called the Steklov or impedance) boundary condition on the boundary of a domain B containing D (possibly $B = D$). In particular, we introduce the following auxiliary scattering problem: Let $B \subset \mathbb{R}^3$ be a ball centered at the origin containing D in its interior and let $\boldsymbol{E}_0 \in \boldsymbol{\mathcal{X}}_{\mathbf{loc}}(\mathbb{R}^3 \setminus \overline{B})$ and $\boldsymbol{H}_0 = (1/i\kappa)\, \mathbf{curl}\, \boldsymbol{E}_0$ be such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_0 - \kappa^2 \boldsymbol{E}_0 &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}, \\ \boldsymbol{E}_0 &= \boldsymbol{E}^i + \boldsymbol{E}_0^s \quad \text{in} \quad \mathbb{R}^3 \setminus B, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{E}_0 - \lambda\, \boldsymbol{E}_{0,T} &= \mathbf{0} \quad \text{on} \quad \partial B, \\ \lim_{r\to\infty} (\mathbf{curl}\ \boldsymbol{E}_0^s \times \boldsymbol{x} - ikr\boldsymbol{E}_0^s) &= \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\nu}$ is the unit outward normal to B, λ is a real constant, $\lambda \neq 0$, and the incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is given by (1.3). We denote by $(\boldsymbol{E}_{0,\infty}, \boldsymbol{H}_{0,\infty})$ the far field patterns of $(\boldsymbol{E}_0^s, \boldsymbol{H}_0^s)$. Uniqueness of the solution of this problem will be proved later on. Following the ideas in [6] it is possible to prove existence of solution for this problem when $\lambda < 0$.

The standard electric far field operator $\mathcal{F} : \boldsymbol{L}_t^2(\mathbb{S}^2) \to \boldsymbol{L}_t^2(\mathbb{S}^2)$ is defined by

$$(\mathcal{F}\boldsymbol{g})(\hat{\boldsymbol{x}}) := \int_{\mathbb{S}^2} \boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))\, ds_{\boldsymbol{d}}$$

for all $\boldsymbol{g} \in \boldsymbol{L}_t^2(\mathbb{S}^2)$. Then the modified far field operator using far field pattern $\boldsymbol{E}_{0,\infty}$ of the standard exterior Stekloff problem (2.1) is given by

$$(\mathcal{F}\boldsymbol{g})(\hat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} [\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d})) - \boldsymbol{E}_{0,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))]\, ds_{\boldsymbol{d}}$$

In order to understand the connection of this modified far field operator with the interior Stekloff problem we now study the injectivity of $\mathcal{F}$. Thus we consider the modified homogeneous integral equation $\mathcal{F}\boldsymbol{g} = 0$ and suppose $\boldsymbol{g} \in \boldsymbol{L}_t^2(\mathbb{S}^2)$ is a non-trivial solution so that

$$\int_{\mathbb{S}^2} [\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d})) - \boldsymbol{E}_{0,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))]\, ds_{\boldsymbol{d}} = 0\,. \tag{2.2}$$

Defining the Herglotz wave function $\boldsymbol{v}_{\boldsymbol{g}}$ by

$$\boldsymbol{v}_{\boldsymbol{g}}(\boldsymbol{x}) := -i\kappa \int_{\mathbb{S}^2} \boldsymbol{g}(\boldsymbol{d})\, e^{-i\kappa \boldsymbol{x}\cdot\boldsymbol{d}}\, ds_{\boldsymbol{d}} \tag{2.3}$$

we obtain that

$$\boldsymbol{w}_\infty(\widehat{\boldsymbol{x}}) := \int_{\mathbb{S}^2} \boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))\, ds_{\boldsymbol{d}}$$

is the far field pattern (see [9, Lemma 6.35]) for the solution $\boldsymbol{w}$ of

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - \kappa^2\epsilon_r\boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3\,, \\ \boldsymbol{w} &= \boldsymbol{v}_{\boldsymbol{g}} + \boldsymbol{w}^s\,, \\ \lim_{r\to\infty}(\mathbf{curl}\,\boldsymbol{w}^s \times \boldsymbol{x} - ikr\boldsymbol{w}^s) &= \mathbf{0}. \end{aligned} \tag{2.4}$$

In addition,

$$\boldsymbol{w}_{0,\infty}(\widehat{\boldsymbol{x}}) := \int_{\mathbb{S}^2} \boldsymbol{E}_{0,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))\, ds_{\boldsymbol{d}}$$

is the far field pattern (see [9, Lemma 6.35]) for the solution $\boldsymbol{w}_0$ of

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}_0 - \kappa^2\boldsymbol{w}_0 &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3\setminus\overline{B}\,, \\ \boldsymbol{w}_0 &= \boldsymbol{v}_{\boldsymbol{g}} + \boldsymbol{w}_0^s \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w}_0 - \lambda\,\boldsymbol{w}_{0,T} &= \mathbf{0} \quad \text{on} \quad \partial B\,, \\ \lim_{r\to\infty}(\mathbf{curl}\,\boldsymbol{w}_0^s \times \boldsymbol{x} - r\boldsymbol{w}_0^s) &= \mathbf{0}\,. \end{aligned} \tag{2.5}$$

Following the argument in [5] for the Helmholtz equation, we obtain from (2.2) that $\boldsymbol{w}_\infty(\widehat{\boldsymbol{x}}) = \boldsymbol{w}_{0,\infty}(\widehat{\boldsymbol{x}})$. Then by Rellich's Lemma $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}_0(\boldsymbol{x})$ in $\mathbb{R}^3\setminus D$ where we have used the fact that if B is a ball the solution (2.1) can be extended as a solution of $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_0 - \kappa^2\boldsymbol{E}_0 = \mathbf{0}$ in $\mathbb{R}^3\setminus\{\mathbf{0}\}$. This can be shown by writing the solution as a series involving spherical Hankel functions as in the Helmholtz case [9].

From the above argument we obtain that $\boldsymbol{w}$ is a non-trivial solution of the homogeneous interior problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - \kappa^2\epsilon_r\boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w} - \lambda\,\boldsymbol{w}_T &= \mathbf{0} \quad \text{on} \quad \partial B\,. \end{aligned} \tag{2.6}$$

Before continuing with the analysis, we introduce the following definition.

Definition 2.1. *For fixed k, $\lambda := \lambda(k) \in \mathbb{C}$ is called a Stekloff eigenvalue if the homogeneous problem* (2.6) *has a nontrivial solution $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}(B)$.* Following the above argument, we see that the far field operator is injective if λ is not a Stekloff eigenvalue. If λ is a Stekloff eigenvalue, then $\mathcal{F}$ is still injective unless $\boldsymbol{w} = \boldsymbol{v}_{\boldsymbol{g}} + \boldsymbol{w}^s$ for some Herglotz wave function $\boldsymbol{v}_{\boldsymbol{g}}$ and radiating field $\boldsymbol{w}^s$. Since this is an analogue of the result for the Helmholtz equation in [5], this suggests that it may be possible to identify Stekloff eigenvalues using the far field equation and we shall shortly prove that this is the case. Furthermore since (2.6) involves ϵ_r it may be possible to infer properties of ϵ_r from the Stekloff eigenvalues. Numerical results suggest this is correct.

With this in mind, we are interested in studying the existence of Stekloff eigenvalues. In order to simplify the problem, we start by analyzing the case in which ϵ_r is constant in which case we may choose $\epsilon_r(\boldsymbol{x}) = 1$. Multiplying by suitable test functions (2.6) and integrating by parts, we obtain the following variational formulation for the Stekloff eigenvalue problem: Find λ and $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}(B)$, $\boldsymbol{w} \neq \boldsymbol{0}$, such that

$$\int_B \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \overline{\boldsymbol{z}}\, d\boldsymbol{x} \,-\, \kappa^2 \int_B \boldsymbol{w} \cdot \overline{\boldsymbol{z}}\, d\boldsymbol{x} \;=\; -\lambda \int_{\partial B} \boldsymbol{w}_T \cdot \overline{\boldsymbol{z}_T}\, ds_{\boldsymbol{x}}\,, \tag{2.7}$$

for all $\boldsymbol{z} \in \boldsymbol{\mathcal{X}}(B)$. Notice that it is very easy to prove that problems (2.6) and (2.7) are equivalent.

Provided κ^2 is not an eigenvalue of the problem:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} \,-\, \kappa^2 \boldsymbol{u} \;&=\; \boldsymbol{0} \qquad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{u} \;&=\; \boldsymbol{0} \qquad \text{on} \quad \partial B\,, \end{aligned}$$

for the analysis of problem (2.7), we can consider the Neumann-to-Dirichlet map:

$$\begin{aligned} \boldsymbol{T} \,:\, \boldsymbol{L}^2_t(\partial B) \quad &\longrightarrow \boldsymbol{L}^2_t(\partial B)\,, \\ \boldsymbol{f} \quad &\longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{u}_T\,, \end{aligned}$$

with $\boldsymbol{u} \in \boldsymbol{\mathcal{X}}(B)$ such that

$$\int_B \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{z}}\, d\boldsymbol{x} \,-\, \kappa^2 \int_B \boldsymbol{u} \cdot \overline{\boldsymbol{z}}\, d\boldsymbol{x} \;=\; \int_{\partial B} \boldsymbol{f} \cdot \overline{\boldsymbol{z}_T}\, ds_{\boldsymbol{x}}\,, \tag{2.8}$$

for all $\boldsymbol{z} \in \boldsymbol{\mathcal{X}}(B)$. Assuming $\boldsymbol{T}$ is well defined the Stekloff eigenvalue problem can be rewritten as the problem of finding $\mu \in \mathbb{C}$ and non-trivial $\boldsymbol{f} \in \boldsymbol{L}^2_t(\partial B)$ such that

$$\mu \boldsymbol{f} = \boldsymbol{T}\boldsymbol{f}$$

Here $\mu = -1/\lambda$. In order to study the eigenvalue problem (2.7), we should start by analyzing the well-posedness of (2.8) and hope that the Neumann-to-Dirichlet operator is self-adjoint and compact. We will start by analyzing the last condition.

Let $\boldsymbol{u} := (1/i\kappa)\, \mathbf{curl}\, \boldsymbol{w}$ in B. Then, from problem (2.6), we obtain that $\boldsymbol{w} = (-1/i\kappa)\, \mathbf{curl}\, \boldsymbol{u}$ in B and, in particular, we can rewritte the problem (2.6) in terms of $\boldsymbol{u}$ as follows:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} \,-\, \kappa^2 \boldsymbol{u} \quad &=\; \boldsymbol{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{u} \,-\, \frac{\kappa^2}{\lambda}\, \boldsymbol{u}_T \quad &=\; \boldsymbol{0} \quad \text{on} \quad \partial B\,. \end{aligned}$$

Thus, if $\lambda \neq 0$ is a Stekloff eigenvalue of problem (2.6), then so is $-\kappa^2/\lambda$. So if there are countably many eigenvalues $\lambda_n \to \infty$, then there are also countably many eigenvalues $\mu_n = -\kappa^2/\lambda_n \to 0$.

Thus either

- The Neumann-to-Dirichlet map is not compact

  or

- There are finitely many eigenvalues.

However for the sphere we know that there are infinitely many Stekloff eigenvalues (as we shall show shortly) and so in that case the Neumann-to-Dirichlet map is not compact. In order to prove the existence of countably many eigenvalues and understand this phenomenon, we will study the case of the sphere in detail. To that end, we need to recall some results. First, we know from [9, Theorem 6.26] that the pair

$$M_n(\boldsymbol{x}) = \mathbf{curl}\left(\boldsymbol{x} j_n(\kappa|\boldsymbol{x}|) Y_n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right)\,, \quad \frac{1}{i\kappa}\, \mathbf{curl}\, M_n(\boldsymbol{x}) \tag{2.9}$$

is an entire solution to Maxwell's equations, where Y_n is a spherical harmonic of order $n \geq 1$ and j_n is the spherical Bessel function of order n (see, for instance, [9]). Notice that M_n can be written equivalently as follows:

$$M_n(\boldsymbol{x}) = j_n(\kappa|\boldsymbol{x}|)\nabla_S Y_n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,, \tag{2.10}$$

where ∇_S corresponds to the surface gradient on a sphere. From the interior analogue of [9, Theorem 6.27], we know that we may write the expansion of the electric field $\boldsymbol{E}$ in the form

$$\boldsymbol{E}(\boldsymbol{x}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_n^m M_n(\boldsymbol{x}) + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_n^m \,\mathbf{curl}\, M_n(\boldsymbol{x})\,, \tag{2.11}$$

where a_n^m and b_n^m are Fourier coefficients.

From (2.9), (2.10) and (2.11) and after some computations, it is possible to obtain that

$$\begin{aligned}
&\widehat{\boldsymbol{x}} \times \mathbf{curl}\,\boldsymbol{E}(\boldsymbol{x}) \,-\, \lambda\,(\widehat{\boldsymbol{x}} \times \boldsymbol{E}(\boldsymbol{x})) \times \widehat{\boldsymbol{x}} = \\
&\sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_n^m\, \widehat{\boldsymbol{x}} \times \nabla_S Y_n^m(\widehat{\boldsymbol{x}}) \left[\frac{1}{|\boldsymbol{x}|}\left\{j_n(\kappa|\boldsymbol{x}|) + \kappa|\boldsymbol{x}|j_n'(\kappa|\boldsymbol{x}|)\right\} + \lambda j_n(\kappa|\boldsymbol{x}|) + \right] \\
&\sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_n^m\, \widehat{\boldsymbol{x}} \times (\nabla_S Y_n^m(\widehat{\boldsymbol{x}}) \times \widehat{\boldsymbol{x}}) \left[\kappa^2 j_n(\kappa|\boldsymbol{x}|) - \frac{\lambda}{|\boldsymbol{x}|}\left\{j_n(\kappa|\boldsymbol{x}|) + \kappa|\boldsymbol{x}|j_n'(\kappa|\boldsymbol{x}|)\right\}\right]\,,
\end{aligned}$$

where $\widehat{\boldsymbol{x}} := \boldsymbol{x}/|\boldsymbol{x}|$. Using the impedance boundary condition, we see that for a sphere of radius one, the previous expression could be zero if one of these two situations happens, for each $n \geq 1$:

(a) $\lambda_n = -\dfrac{j_n(\kappa) + \kappa j_n'(\kappa)}{j_n(\kappa)}$ and the rest of the terms are zero.

(b) $\lambda_n = \dfrac{\kappa^2 j_n(\kappa)}{j_n(\kappa) + \kappa j_n'(\kappa)}$ and the rest of the terms are zero.

In particular there are countably many eigenvalues with accumulation points at $-\infty$ and 0 (see also Fig. 2.1). So the Neumann-to-Dirichet map cannot be compact which defeats an easy proof of the existence of Stekloff eigenvalues in the this case (and presumably in general).

In addition, notice that it can be proved that the surface scalar curl of the term

$$\widehat{\boldsymbol{x}} \times (\nabla_S Y_n^m(\widehat{\boldsymbol{x}}) \times \widehat{\boldsymbol{x}}) \left[\kappa^2 j_n(\kappa|\boldsymbol{x}|) - \lambda\frac{1}{|\boldsymbol{x}|}\left\{j_n(\kappa|\boldsymbol{x}|) + \kappa|\boldsymbol{x}|j_n'(\kappa|\boldsymbol{x}|)\right\}\right]$$

is zero and the surface divergence of the term

$$\widehat{\boldsymbol{x}} \times \nabla_S Y_n^m(\widehat{\boldsymbol{x}}) \left[\frac{1}{|\boldsymbol{x}|}\left\{j_n(\kappa|\boldsymbol{x}|) + \kappa|\boldsymbol{x}|j_n'(\kappa|\boldsymbol{x}|)\right\} + \lambda j_n(\kappa|\boldsymbol{x}|)\right]$$

also is zero. After this analysis arises the idea of considering just a part of the sum and to try to obtain an eigenvalue problem with an associated compact operator. Recall that we have arrived to the problem (2.6) from the auxiliary problem defined in (2.5) which was introduced with the aim of studying the inverse electromagnetic scattering problem defined at the beginning of this section. We are thus free to choose a different auxiliary problem. In the following section, we will prove that at least when we consider the case in which the terms have vanishing surface divergence, we are able to define a compact operator.

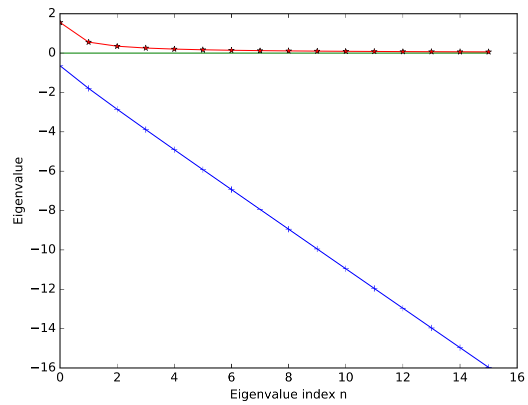

FIG. 2.1. *The first 16 eigenvalues of type a) and b) (see text). One set of eigenvalues tends to* $-\infty$ *and the other to* 0.

3. Modified Stekloff Boundary Condition. In this section we introduce a linear operator which will allow us to define a convenient auxiliary problem in order to analyze the corresponding modified Stekloff eigenvalue problem. We start by defining the linear operator $\boldsymbol{\mathcal{S}}$ as follows:

$$\begin{aligned} \boldsymbol{\mathcal{S}} \,:\, \boldsymbol{H}^{-1/2}(\mathrm{curl}_{\partial B}, \partial B) \quad &\longrightarrow \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B) \\ \boldsymbol{\mu} \quad &\longmapsto \boldsymbol{\mathcal{S}}\boldsymbol{\mu} := \mathbf{curl}_{\partial B}\, q\,, \end{aligned} \tag{3.1}$$

where $q \in H^1(\partial B)/\mathbb{C}$ is the solution of the problem $\Delta_{\partial B}\, q = \mathrm{curl}_{\partial B}\, \mathbf{curl}_{\partial B}\, q = \mathrm{curl}_{\partial B}\, \boldsymbol{\mu}$. By using an eigensystem expansion (e.g. [14]) we see that $\mathbf{curl}_{\partial B}\, q \in \boldsymbol{H}^{1/2}_t(\partial B)$. Thus, $\boldsymbol{\mathcal{S}}\boldsymbol{\mu} \in \boldsymbol{H}^{1/2}_t(\partial B)$, $\mathrm{div}_{\partial B}\, \boldsymbol{\mu} = 0$ and

$$\|\boldsymbol{\mathcal{S}}\boldsymbol{\mu}\|_{\boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B)} = \|\boldsymbol{\mathcal{S}}\boldsymbol{\mu}\|_{1/2,\partial B} = \|\,\mathrm{curl}_{\partial \mathrm{B}}\, q\|_{1/2,\partial B} \le C_{\boldsymbol{\mathcal{S}}} \|\mathrm{curl}_{\partial B}\, \boldsymbol{\mu}\|_{-1/2,\partial B}.$$

In addition, since $\mathrm{curl}_{\partial B}\,(\mathbf{curl}_{\partial B}\; q - \boldsymbol{\mu}) = 0$, then we can find $v \in H^{1/2}(\partial B)$ such that $\mathbf{curl}_{\partial B}\; q - \boldsymbol{\mu} = \nabla_{\partial \mathrm{B}}\, v$. Thus, for all $\boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(\mathrm{curl}_{\partial B}, \partial B)$, there exist q and v such that $\boldsymbol{\mu} = \mathbf{curl}_{\partial B}\; q - \nabla_{\partial \mathrm{B}}\, v$, or, equivalently,

$$\boldsymbol{\mathcal{S}}\boldsymbol{\mu} \;=\; \boldsymbol{\mu} \,+\, \nabla_{\partial \mathrm{B}}\, v.$$

For simplicity of notation, we represent duality pairs as integrals on the boundary and state the following result that shows that $\boldsymbol{\mathcal{S}}$ is self-adjoint.

LEMMA 3.1. *The operator* $\boldsymbol{\mathcal{S}}$ *satisfies:*

$$\int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T \cdot \overline{\boldsymbol{z}_T}\, ds \;=\; \int_{\partial B} \boldsymbol{u}_T \cdot \overline{\boldsymbol{\mathcal{S}}\boldsymbol{z}_T}\, ds \;=\; \int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T \cdot \overline{\boldsymbol{\mathcal{S}}\boldsymbol{z}_T}\, ds\,, \tag{3.2}$$

for all $\boldsymbol{u}$, $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$.

Proof. Let $\boldsymbol{u}$, $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$. Then, there exist $v, w \in H^{1/2}(\partial B)$ such that $\boldsymbol{\mathcal{S}}\boldsymbol{u}_T = \nabla_{\partial \mathrm{B}}\, v + \boldsymbol{u}_T$ and $\boldsymbol{\mathcal{S}}\boldsymbol{z}_T = \nabla_{\partial \mathrm{B}}\, w + \boldsymbol{z}_T$. Hence and thanks to the fact that $\boldsymbol{\mathcal{S}}\left(\boldsymbol{H}^{-1/2}(\mathrm{curl}_{\partial B}, \partial B)\right) \subset \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B)$, we obtain that

$$\int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T \cdot \overline{\boldsymbol{z}_T}\, ds = \int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T \cdot (\overline{\boldsymbol{\mathcal{S}}\boldsymbol{z}_T} - \nabla_{\partial \mathrm{B}}\, \overline{w})\, ds = \int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T \cdot \overline{\boldsymbol{\mathcal{S}}\boldsymbol{z}_T}\, ds.$$

Analogously,

$$\int_{\partial B} \boldsymbol{u}_T \cdot \overline{\boldsymbol{\mathcal{S}} \boldsymbol{z}_T}\, ds = \int_{\partial B} (\boldsymbol{\mathcal{S}} \boldsymbol{u}_T - \nabla_{\partial \mathrm{B}}\, v) \cdot \overline{\boldsymbol{\mathcal{S}} \boldsymbol{z}_T}\, ds = \int_{\partial B} \boldsymbol{\mathcal{S}} \boldsymbol{u}_T \cdot \overline{\boldsymbol{\mathcal{S}} \boldsymbol{z}_T}\, ds,$$

which completes the proof. ◻

We now assume that the ball B contains D in its interior or $B = D$. Next we define the exterior problem for the modified Stekloff boundary condition. Let $\boldsymbol{E}_S \in \boldsymbol{H}_{\mathtt{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{B})$ satisfy

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_S \,-\, \kappa^2 \boldsymbol{E}_S \,=\, \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}\,, \\ \boldsymbol{E}_S \,=\, \boldsymbol{E}^i + \boldsymbol{E}_S^s \quad \text{in} \quad \mathbb{R}^3 \setminus B, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{E}_S \,-\, \lambda \boldsymbol{\mathcal{S}} \boldsymbol{E}_{S,T} \,=\, \mathbf{0} \quad \text{on} \quad \partial B\,, \\ \lim_{r\to\infty} (\mathbf{curl}\,\boldsymbol{E}_S^s \times \boldsymbol{x} - ikr\boldsymbol{E}_S^s) = \mathbf{0}\,, \end{aligned} \tag{3.3}$$

where λ is a real constant. Note that the corresponding magnetic field is given by $\boldsymbol{H}_S = (1/i\kappa)\ \mathbf{curl}\,\boldsymbol{E}_S$. We denote by $(\boldsymbol{E}_{S,\infty}, \boldsymbol{H}_{S,\infty})$ the far field pattern of $(\boldsymbol{E}_S^s, \boldsymbol{H}_S^s)$.

THEOREM 3.2. *Suppose* $\lambda \in \mathbb{R}$. *If* $\boldsymbol{\mathcal{S}} = I$ *or* $\boldsymbol{\mathcal{S}}$ *is as given in* (3.1)*, the exterior problem* (3.3) *has at most one solution.*

Proof. Let $\boldsymbol{E}_1$, $\boldsymbol{H}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{H}_2$ denote two solution pairs to problem (3.3). Then $\boldsymbol{E}_0 = \boldsymbol{E}_1 - \boldsymbol{E}_2$, $\boldsymbol{H}_0 = \boldsymbol{H}_1 - \boldsymbol{H}_2$ satisfies (3.3) with $\boldsymbol{E}^i = \boldsymbol{H}^i = \mathbf{0}$, and in particular the radiation condition. To prove uniqueness of this exterior problem we consider the domain $B_R \setminus \overline{B}$, where $B_R = \{\boldsymbol{x} \in \mathbb{R}^3 \,:\, |\boldsymbol{x}| < R\}$ is a ball containing to B in its interior. Integrating over $B_R \setminus \overline{B}$ and then using integration by parts, we obtain

$$\int_{B_R\setminus\overline{B}} (|\,\mathbf{curl}\,\boldsymbol{E}_0|^2 - \kappa^2 |\boldsymbol{E}_0|^2)\, d\boldsymbol{x} - i\kappa \int_{\partial B_R} \boldsymbol{\nu} \times \overline{\boldsymbol{E}_0} \cdot \boldsymbol{H}_0\, ds - \lambda \int_{\partial B} |\boldsymbol{\mathcal{S}} \boldsymbol{E}_{0,T}|^2\, ds = 0.$$

Hence,

$$\begin{aligned} &\mathrm{Re} \int_{\partial B_R} \boldsymbol{\nu} \times \overline{\boldsymbol{E}_0} \cdot \boldsymbol{H}_0\, ds \\ &= \mathrm{Re} \left(\frac{-1}{i\kappa} \left\{ \int_{B_R\setminus\overline{B}} (|\,\mathbf{curl}\,\boldsymbol{E}_0|^2 - \kappa^2 |\boldsymbol{E}_0|^2)\, d\boldsymbol{x} - \lambda \int_{\partial B} |\boldsymbol{\mathcal{S}} \boldsymbol{E}_{0,T}|^2\, ds \right\} \right) = 0. \end{aligned}$$

and therefore, uniqueness follows from [9, Theorem 6.11] and the unique continuation principle. ◻

To prove existence of a solution by variational means, we need to truncate the exterior domain. Thus we consider the domain $B_R \setminus \overline{B}$ where B_R is such that $\overline{B}$ is contained in the interior of B_R. One way to impose the boundary condition on the artificial surface ∂B_R is by incorporating the capacity operator (see, for instance, [14]). Here we only present the result.

THEOREM 3.3. *Suppose* $\lambda \in \mathbb{R}$*,* $\lambda < 0$ *and* $\boldsymbol{\mathcal{S}} = I$ *or* $\lambda \in \mathbb{R}$ *and* $\boldsymbol{\mathcal{S}}$ *as given in* (3.1)*. Then the exterior problem* (3.3) *has a unique solution that satisfies*

$$\|\boldsymbol{E}_S^s\|_{\boldsymbol{H}(B_R\setminus\overline{B})} \leq C \|\boldsymbol{f}\|_{\boldsymbol{H}^{1/2}(\mathrm{div}_{\,\partial B}, \partial B)}$$

for some positive constant C *depending on* R *where* $\boldsymbol{f} = -(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{E}^i \,-\, \lambda \boldsymbol{\mathcal{S}} \boldsymbol{E}_T^i)$.

From now on we use the auxiliary problem (3.3), instead of problem (2.1) $\boldsymbol{\mathcal{S}} \neq I$). The new modified far field operator $\mathcal{F}_S : \boldsymbol{L}_t^2(\mathbb{S}^2) \to \boldsymbol{L}_t^2(\mathbb{S}^2)$ uses the far field pattern $\boldsymbol{E}_{S,\infty}$ of the modified exterior Stekloff problem (3.3) and is given by

$$(\mathcal{F}_S \boldsymbol{g})(\hat{\boldsymbol{x}}) = \int_{\mathbb{S}^2} [\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d})) - \boldsymbol{E}_{S,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))] \, ds_{\boldsymbol{d}} \,. \tag{3.4}$$

We can now repeat the argument given in the last section concerning injectivity of the far field operator, using the new auxiliary function. Suppose that $\mathcal{F}_S \boldsymbol{g} = 0$ has a non-trivial solution so that $\boldsymbol{g} \in \boldsymbol{L}_t^2(\mathbb{S}^2)$ is a non-trivial solution of

$$\int_{\mathbb{S}^2} [\boldsymbol{E}_\infty(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d})) - \boldsymbol{E}_{S,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d}))] \, ds_{\boldsymbol{d}} = 0$$

and from the definition of the Herglotz wave function $\boldsymbol{v}_{\boldsymbol{g}}$ in (2.3) we now obtain that

$$\boldsymbol{w}_{S,\infty}(\widehat{\boldsymbol{x}}) := \int_{\mathbb{S}^2} \boldsymbol{E}_{S,\infty}(\widehat{\boldsymbol{x}}, \boldsymbol{d}, \boldsymbol{g}(\boldsymbol{d})) \, ds_{\boldsymbol{d}}$$

is the far field pattern (see [9, Lemma 6.35]) for

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{w}_S - \kappa^2 \boldsymbol{w}_S &= \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}\,, \\ \boldsymbol{w}_S &= \boldsymbol{v}_{\boldsymbol{g}} + \boldsymbol{w}_S^s\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{w}_S - \lambda \boldsymbol{\mathcal{S}} \boldsymbol{w}_{S,T} &= \mathbf{0} \quad \text{on} \quad \partial B\,, \\ \lim_{r\to\infty} (\mathbf{curl}\, \boldsymbol{w}_S^s \times \boldsymbol{x} - ikr\boldsymbol{w}_S^s) &= \mathbf{0}\,. \end{aligned} \tag{3.5}$$

Proceeding as in Section 3, we obtain now that the solution $\boldsymbol{w}$ of (2.4) (using the new $\boldsymbol{v}_{\boldsymbol{g}}$) satisfies the boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{w} - \kappa^2 \epsilon_r \boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{w} - \lambda \boldsymbol{\mathcal{S}} \boldsymbol{w}_T &= \mathbf{0} \quad \text{on} \quad \partial B\,. \end{aligned} \tag{3.6}$$

Thus the modified Stekloff eigenvalue problem serves to study the new modified far field operator $\mathcal{F}_S$.

Before proving the existence of modified Stekloff eigenvalues we first show eigenvalues for the unit sphere as a function of the wave number k when $\epsilon_r = 1$ in Fig. 3.1. As expected from the two dimensional case [5] there are vertical asymptotes (at interior Maxwell eigenvalues). For low wave number the eigenvalues are negative asymptote to $-\infty$.

We now wish to prove that modified Stekloff eigenvalues exist. So far we are only able to do this for real $\epsilon_r(\boldsymbol{x})$. We start by deriving a weak form of the eigenvalue problem. Multiplying (3.6) by suitable test functions and integrating by parts, we obtain the following variational problem: Find λ and non-trivial $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}, B)$ such that

$$\int_B \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \overline{\boldsymbol{z}} \, d\boldsymbol{x} - \kappa^2 \int_B \epsilon_r(\boldsymbol{x}) \boldsymbol{w} \cdot \overline{\boldsymbol{z}} \, d\boldsymbol{x} = -\lambda \int_{\partial B} \boldsymbol{\mathcal{S}} \boldsymbol{w}_T \cdot \overline{\boldsymbol{z}_T} \, ds_{\boldsymbol{x}}\,, \tag{3.7}$$

for all $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$. This eigenvalue problem can be rewritten in the following convenient way: Provided κ^2 is not an eigenvalue of the problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \kappa^2 \epsilon_r(\boldsymbol{x}) \boldsymbol{u} &= \mathbf{0} \qquad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{u} &= \mathbf{0} \qquad \text{on} \quad \partial B\,, \end{aligned}$$

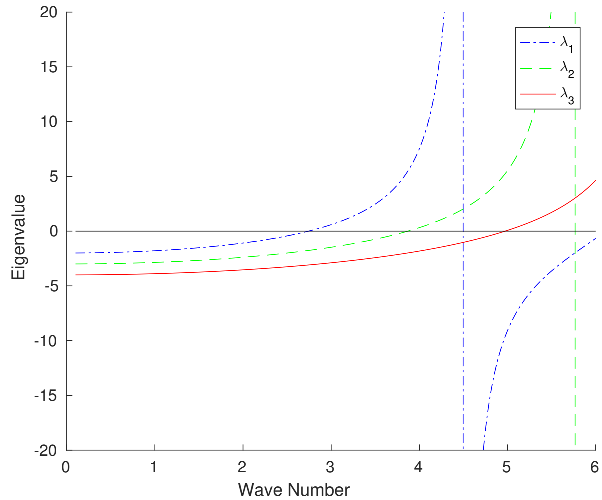

FIG. 3.1. *The first three modified Stekloff eigenvalues for the unit sphere as a function of wave number κ. The eigenvalues are numbered to the index of the Bessel function representing the corresponding eigenfunction.*

we can define the map

$$\begin{aligned} \boldsymbol{T}\, : \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B) \quad &\longrightarrow \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B)\,, \\ \boldsymbol{f} \quad &\longmapsto \boldsymbol{T}\boldsymbol{f} := \boldsymbol{\mathcal{S}}\boldsymbol{u}_T\,, \end{aligned}$$

with $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, B)$ such that

$$\int_B \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{z}}\,d\boldsymbol{x} \;-\; \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\boldsymbol{u} \cdot \overline{\boldsymbol{z}}\,d\boldsymbol{x} \;=\; \int_{\partial B} \boldsymbol{f} \cdot \overline{\boldsymbol{z}_T}\,ds_{\boldsymbol{x}}\,, \tag{3.8}$$

for all $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$. Thus if $(\lambda, \boldsymbol{w}) \in \mathbb{R} \times \boldsymbol{H}(\mathbf{curl}, B)$ solves (3.7) with $\lambda \neq 0$ then by design $\boldsymbol{T}\boldsymbol{\mathcal{S}}\boldsymbol{w}_T = \mu\,\boldsymbol{\mathcal{S}}\boldsymbol{w}_T$ with $\mu = -1/\lambda \neq 0$. On the other hand if $(\mu, \xi) \in \mathbb{R} \times \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B)$, $\xi \neq 0$, satisfies $\boldsymbol{T}\xi = \mu\,\xi$ then choosing $\lambda = -1/\mu$ we can compute $\tilde{\boldsymbol{w}} \in \boldsymbol{H}(\mathbf{curl}, B)$ such that

$$\int_B \mathbf{curl}\,\tilde{\boldsymbol{w}} \cdot \mathbf{curl}\,\overline{\boldsymbol{z}}\,d\boldsymbol{x} \;-\; \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\tilde{\boldsymbol{w}} \cdot \overline{\boldsymbol{z}}\,d\boldsymbol{x} \;=\; \int_{\partial B} \xi \cdot \overline{\boldsymbol{z}_T}\,ds_{\boldsymbol{x}}.$$

Then by definition $\mu\xi = \boldsymbol{T}\xi = \boldsymbol{\mathcal{S}}\tilde{\boldsymbol{w}}_T$. Substituting for ξ above

$$\int_B \mathbf{curl}\,\tilde{\boldsymbol{w}} \cdot \mathbf{curl}\,\overline{\boldsymbol{z}}\,d\boldsymbol{x} \;-\; \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\tilde{\boldsymbol{w}} \cdot \overline{\boldsymbol{z}}\,d\boldsymbol{x} \;=\; \frac{1}{\mu}\int_{\partial B} \boldsymbol{\mathcal{S}}\tilde{\boldsymbol{w}}_T \cdot \overline{\boldsymbol{z}_T}\,ds_{\boldsymbol{x}}\,,$$

Setting $\lambda = -1/\mu$ we see that $(\lambda, \boldsymbol{\mathcal{S}}\tilde{\boldsymbol{w}}_T)$ is an eigenpair for (3.7). We have thus established that finding eigenpairs for (3.7) is equivalent to finding eigenpairs for the modified Neumann-to-Dirichlet map $\boldsymbol{T}$. It remains to analyze $\boldsymbol{T}$. The definition of $\boldsymbol{T}$ was designed to prove the next result:

LEMMA 3.4. *The operator $\boldsymbol{T}\colon \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B) \to \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B)$ is compact.*

Proof. Let $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\operatorname{div}^0_{\partial B}, \partial B)$. Notice that $\boldsymbol{u}$ is solution of the problem (3.8) if and only if $\boldsymbol{u}$ satisfies:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} \,-\, \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{u} \;&=\; \mathbf{0} \qquad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{u} \,+\, \boldsymbol{f} \;&=\; \mathbf{0} \qquad \text{on} \quad \partial B\,. \end{aligned}$$

From the first equation, we obtain that $\mathbf{curl}\,\boldsymbol{u} \in L^2(B)^3$ and, therefore, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, B) \cap H(\mathrm{div}^0, B)$ where $H(\mathrm{div}^0, B) := \{\boldsymbol{z} \in H(\mathrm{div}, B) : \mathrm{div}(\boldsymbol{z}) = 0 \text{ in } B\}$. In addition, from the second equality, we have that $\boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}_t^{1/2}(\partial B)$. These two conditions together with the last remark in [10] imply that $\boldsymbol{\nu}\cdot\mathbf{curl}\,\boldsymbol{u} \in L^2(\partial B)$. Hence, we obtain that $\mathrm{curl}_{\partial B}\,\boldsymbol{u}_T = \boldsymbol{\nu}\cdot\mathbf{curl}\,\boldsymbol{u} \in L^2(\partial B)$. On the other hand, by construction, we know that there exists q such that $\boldsymbol{\mathcal{S}}\boldsymbol{u}_T = \mathbf{curl}_{\partial B}\, q \in \boldsymbol{H}^{1/2}(\mathrm{div}^0_{\partial B}, \partial B)$. Since $\mathrm{curl}_{\partial B}\,\mathbf{curl}_{\partial B}\, q = \mathrm{curl}_{\partial B}\,\boldsymbol{\mathcal{S}}\boldsymbol{u}_T = \mathrm{curl}_{\partial B}\,\boldsymbol{u}_T \in L^2(\partial B)$, then we obtain that $\mathbf{curl}_{\partial B}\, q \in \boldsymbol{H}^1_t(\partial B)$ and, therefore, $\boldsymbol{\mathcal{S}}\boldsymbol{u}_T \in \boldsymbol{H}^1(\mathrm{div}_{\partial B}{}^0, \partial B)$. Hence and the fact that $\boldsymbol{H}^1(\mathrm{div}_{\partial B}{}^0, \partial B)$ is compactly included in $\boldsymbol{H}^{1/2}(\mathrm{div}^0_{\partial B}, \partial B)$, we can conclude the proof. □

LEMMA 3.5. *If ϵ_r is real, the operator $\boldsymbol{T} : \boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B) \longrightarrow \boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B)$ is self-adjoint.*

Proof. Given $\boldsymbol{f}, \boldsymbol{h} \in \boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B)$, let $\boldsymbol{\mathcal{S}}\boldsymbol{u}_T := \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{\mathcal{S}}\boldsymbol{v}_T := \boldsymbol{T}\boldsymbol{h}$. Since $\mathrm{div}_{\partial B}\,\boldsymbol{f} = \mathrm{div}_{\partial B}\,\boldsymbol{h} = 0$, then we only have to prove that

$$\int_{\partial B} \boldsymbol{T}\boldsymbol{f}\cdot\overline{\boldsymbol{h}}\, ds = \int_{\partial B} \boldsymbol{f}\cdot\boldsymbol{T}\overline{\boldsymbol{h}}\, ds\,.$$

In fact,

$$\begin{aligned}
\int_{\partial B} \boldsymbol{T}\boldsymbol{f}\cdot\overline{\boldsymbol{h}}\, ds &= \int_{\partial B} \boldsymbol{\mathcal{S}}\boldsymbol{u}_T\cdot\overline{\boldsymbol{h}}\, ds = \int_{\partial B} \boldsymbol{u}_T\cdot\overline{\boldsymbol{h}}\, ds = \overline{\int_{\partial B} \boldsymbol{h}\cdot\overline{\boldsymbol{u}_T}\, ds} \\
&= \overline{\int_B \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\, d\boldsymbol{x} - \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\boldsymbol{v}\cdot\overline{\boldsymbol{u}}\, d\boldsymbol{x}} \\
&= \int_B \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\, d\boldsymbol{x} - \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\, d\boldsymbol{x} \\
&= \int_{\partial B} \boldsymbol{f}\cdot\overline{\boldsymbol{v}_T}\, ds = \int_{\partial B} \boldsymbol{f}\cdot\boldsymbol{\mathcal{S}}\overline{\boldsymbol{v}_T}\, ds \\
&= \int_{\partial B} \boldsymbol{f}\cdot\overline{\boldsymbol{T}\boldsymbol{h}}\, ds.
\end{aligned}$$

□

Notice that it is easy to prove that $\boldsymbol{T}(\boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B)) \subset \boldsymbol{H}^{1/2}(\mathrm{div}^0_{\partial B}, \partial B)$ following the ideas in the proof of Lemma 3.4. Hence, the spectrum of T as a map from $\boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B) \longrightarrow \boldsymbol{H}(\mathrm{div}_{\partial B}{}^0, \partial B)$ is the same that the spectrum corresponding to $\boldsymbol{T} : \boldsymbol{H}^{1/2}(\mathrm{div}^0_{\partial B}, \partial B) \longrightarrow \boldsymbol{H}^{1/2}(\mathrm{div}^0_{\partial B}, \partial B)$. Then, by Lemmas 3.4 and 3.5 we can conclude the following theorem.

THEOREM 3.6. *When ϵ_r is real, a countable discrete set of modified Stekloff eigenvalues (eigenvalues of* (3.7)*) exist. They are real and accumulate at ∞.*

REMARK 3.7. *When ϵ_r is complex, modified Stekloff eigenvalues, if they exist, would all be complex. It remains to prove existence of modified Stekloff eigenvalues in this case.*

4. Determination of eigenvalues from far field data. We now want to show that modified Stekloff eigenvalues can be determined from far field data. Note that these results are independent the choice of ϵ_r (i.e. it could be complex valued) or the type of Stekloff eigenvalues used, provided they exist. For simplicity, we shall assume ϵ_r is real and that modified Stekloff eigenvalues are sought. With this aim in mind, we start analyzing the following auxiliary result. Let $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\mathrm{div}_{\partial B}, \partial B)$ and consider the problem of finding $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}, B)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - \kappa^2\epsilon_r\boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w} - \lambda\boldsymbol{\mathcal{S}}\boldsymbol{w}_T &= \boldsymbol{f} \quad \text{on} \quad \partial B\,. \end{aligned} \tag{4.1}$$

LEMMA 4.1. *Assume that $\lambda \neq 0$ is not a modified Stekloff eigenvalue. Then (4.1) has a unique solution $\boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl}, B)$ that satisfies*

$$\|\boldsymbol{w}\|_{\mathbf{curl},B} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}^{1/2}(\mathrm{div}_{\,\partial B},\partial B)}$$

for some positive constant C. In addition, this solution can be decomposed as $\boldsymbol{w} = \boldsymbol{w}^i + \boldsymbol{w}^s$ where $\boldsymbol{w}^i \in \boldsymbol{H}(\mathbf{curl}, B)$ solves the Maxwell's equations in B and $\boldsymbol{w}^s \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$ is a radiating field (i.e. satisfies Maxwell's equations with $\epsilon_r = 1$ outside B together with the Silver–Müller radiation condition).

Proof. Thanks to the previous section, we know that $\boldsymbol{T} : \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B) \longrightarrow \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B)$ is a compact operator. We assume that $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B)$. Then, if we set $\alpha = \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{u}$ on ∂B, and using the definition of $\boldsymbol{T}$ we have

$$\alpha - \lambda\boldsymbol{T}\alpha = \boldsymbol{f}.$$

We have uniqueness provided λ is not an eigenvalue, and so by the Fredholm alternative we have existence of α. Then we can extend inside the domain by solving the problem:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w} &= \alpha \quad \text{on} \quad \partial B\,, \end{aligned}$$

since k is not an eigenvalue.

Now, let $\boldsymbol{f} \in \boldsymbol{H}^{1/2}(\mathrm{div}_{\,\partial B}, \partial B)$ the source term of the problem (4.1) and $\boldsymbol{w}_0 \in \boldsymbol{H}(\mathbf{curl}, B)$ solution of the problem:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}_0 - \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{w}_0 &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w}_0 &= \boldsymbol{f} \quad \text{on} \quad \partial B\,. \end{aligned}$$

Decomposing $\boldsymbol{w} = \boldsymbol{w}_0 + \boldsymbol{z}$, we can rewrite the problem (4.1) in terms of the new unknown $\boldsymbol{z}$: Find $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{z} - \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{z} &= \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{z} - \lambda\boldsymbol{\mathcal{S}}\boldsymbol{z}_T &= F \quad \text{on} \quad \partial B\,. \end{aligned} \tag{4.2}$$

where $F := \lambda\boldsymbol{\mathcal{S}}\boldsymbol{w}_{0,T} \in \boldsymbol{H}^{1/2}(\mathrm{div}^{\,0}_{\partial B}, \partial B)$. Then, thanks to the previous analysis and the fact that $\lambda \neq 0$ is not a Stekloff eigenvalue, there exists a unique solution $\boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, B)$ of the problem (4.2).

On the other hand, thanks to the Stratton–Chu formula (see, for instance, [2, Theorem 3.2]) we see that $\boldsymbol{w}$ can be decomposed as $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}^i(\boldsymbol{x}) + \boldsymbol{w}^s(\boldsymbol{x})$ where

$$\begin{aligned} \boldsymbol{w}^i(\boldsymbol{x}) \quad := \quad & -\,\mathbf{curl}_{\boldsymbol{x}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{w}(\boldsymbol{y})\,\Phi(\boldsymbol{x},\boldsymbol{y})\,ds_{\boldsymbol{y}} \\ & +\nabla_{\boldsymbol{x}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{y})\cdot\boldsymbol{w}(\boldsymbol{y})\,\Phi(\boldsymbol{x},\boldsymbol{y})\,ds_{\boldsymbol{y}} \\ & - \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{y})\times\mathbf{curl}\,\boldsymbol{w}(\boldsymbol{y})\,\Phi(\boldsymbol{x},\boldsymbol{y})\,ds_{\boldsymbol{y}} \qquad \boldsymbol{x}\in B\,, \end{aligned}$$

where

$$\Phi(\boldsymbol{x},\boldsymbol{z}) := \frac{e^{i\kappa|\boldsymbol{x}-\boldsymbol{z}|}}{4\pi|\boldsymbol{x}-\boldsymbol{z}|}$$

is the fundamental solution to the Helmholtz equation. Then we can rewrite $\boldsymbol{w}^i$ as

$$\begin{aligned}\boldsymbol{w}^i(\boldsymbol{x}) \quad := \quad & -\,\mathbf{curl}_{\boldsymbol{x}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{y}) \times \boldsymbol{w}(\boldsymbol{y})\,\Phi(\boldsymbol{x},\boldsymbol{y})\,ds_{\boldsymbol{y}} \\ & -\frac{1}{\kappa^2}\,\mathbf{curl}_{\boldsymbol{x}}\;\mathbf{curl}_{\boldsymbol{x}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{y}) \times \mathbf{curl}\,\boldsymbol{w}(\boldsymbol{y})\,\Phi(\boldsymbol{x},\boldsymbol{y})\,ds_{\boldsymbol{y}} \qquad \boldsymbol{x}\in B\,.\end{aligned}$$

It is then easy to prove that $\boldsymbol{w}^i$ satisfies the Maxwell's equations with $\epsilon_r(\boldsymbol{x}) = 1$in B.

In addition,

$$\begin{aligned}\boldsymbol{w}^s(\boldsymbol{x}) \quad := \quad & \kappa^2 \int_B (\epsilon_r(\boldsymbol{y})-1)\boldsymbol{w}(\boldsymbol{y})\Phi(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} \\ & -\nabla_{\boldsymbol{x}} \int_B \operatorname{div}_{\boldsymbol{y}}(\boldsymbol{w}(\boldsymbol{y}))\,\Phi(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y} \qquad \boldsymbol{x}\in\mathbb{R}^3\end{aligned}$$

satisfies the Silver–Müller radiation condition. Furthermore since $\epsilon_r(\boldsymbol{x}) = 1$ outside D, $\boldsymbol{w}$ satisfies $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - \kappa^2\boldsymbol{w} = \mathbf{0}$ in there.□

Now, we can continue with our analysis of the new modified far field operator $\mathcal{F}_S$ defined in (3.4). We know that $\mathcal{F}_S$ is injective unless λ is a modified Stekloff eigenvalue with eigenfunction of the form $\boldsymbol{w} = \boldsymbol{v}_{\boldsymbol{g}} + \boldsymbol{w}^s$ (see our previous argument on injectivity). This can be proved similarly to the proof of [9, Theorem 10.42]). Since $\boldsymbol{E}_\infty$ and $\boldsymbol{E}_{S,\infty}$ satisfy the reciprocity relation (see [9, Theorem 6.30]) $\mathcal{F}_S$ is injective with dense range if and only if there does not exist a Stekloff eigenfunction for B which has the above decomposition (see [9, Theorems 6.36, 6.37]).

Let us recall that an electric dipole with polarization $\boldsymbol{q}$ is defined by

$$\begin{aligned}\boldsymbol{E}_e(\boldsymbol{x},\boldsymbol{z},\boldsymbol{q}) \quad &:= \quad \frac{i}{\kappa}\,\mathbf{curl}_{\boldsymbol{x}}\,\mathbf{curl}_{\boldsymbol{x}}\;\boldsymbol{q}\Phi_\kappa(\boldsymbol{x},\boldsymbol{z})\,, \\ \boldsymbol{H}_e(\boldsymbol{x},\boldsymbol{z},\boldsymbol{q}) \quad &:= \quad \mathbf{curl}_{\boldsymbol{x}}\;\boldsymbol{q}\Phi_\kappa(\boldsymbol{x},\boldsymbol{z})\,.\end{aligned} \tag{4.3}$$

In particular, $\boldsymbol{E}_e(\cdot,\boldsymbol{z},\boldsymbol{q})$ is a radiating solution to Maxwell's equations outside a neighborhood of $\boldsymbol{z}$ and the corresponding far field pattern is given by

$$\boldsymbol{E}_{e,\infty}(\widehat{\boldsymbol{x}},\boldsymbol{z},\boldsymbol{q}) := \frac{i\kappa}{4\pi}(\widehat{\boldsymbol{x}}\times\boldsymbol{q})\times\widehat{\boldsymbol{x}}\,e^{-i\kappa\,\widehat{\boldsymbol{x}}\cdot\boldsymbol{z}}\,.$$

Now, we will show that Stekloff eigenvalues can now we obtained by solving the following modified far field equation: Find $\boldsymbol{g}\in\boldsymbol{L}^2_t(\mathbb{S}^2)$ such that

$$(\mathcal{F}_S\boldsymbol{g})(\widehat{\boldsymbol{x}}) = \boldsymbol{E}_{e,\infty}(\widehat{\boldsymbol{x}},\boldsymbol{z},\boldsymbol{q})\,. \tag{4.4}$$

In general this problem cannot be solved for any $\boldsymbol{z}\in B$. Indeed, if $\boldsymbol{g}_{\boldsymbol{z}}\in\boldsymbol{L}^2_t(\mathbb{S}^2)$ satisfies (4.4), then by Rellich's Lemma

$$\boldsymbol{w}_{\boldsymbol{z}}(\boldsymbol{x}) - \boldsymbol{w}_{0_{\boldsymbol{z}}}(\boldsymbol{x}) = \boldsymbol{w}^s_{\boldsymbol{z}}(\boldsymbol{x}) - \boldsymbol{w}^s_{0_{\boldsymbol{z}}}(\boldsymbol{x}) = \boldsymbol{E}_e(\boldsymbol{x},\boldsymbol{z},\boldsymbol{q}) \quad \text{ for } \boldsymbol{x}\in\mathbb{R}^3\setminus\overline{B}$$

where $\boldsymbol{w}_{\boldsymbol{z}}$ and $\boldsymbol{w}_{0_{\boldsymbol{z}}}$ are defined by (2.4) and (3.5), respectively, with $\boldsymbol{v}_{\boldsymbol{g}} = \boldsymbol{v}_{\boldsymbol{g}_{\boldsymbol{z}}}$. Therefore, $\boldsymbol{w}_{\boldsymbol{z}}$ must solve the problem

$$\begin{aligned}\mathbf{curl}\,\mathbf{curl}\;\boldsymbol{w}_{\boldsymbol{z}} - \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{w}_{\boldsymbol{z}} \quad &= \quad \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{w}_{\boldsymbol{z}} - \lambda\boldsymbol{\mathcal{S}}\boldsymbol{w}_{\boldsymbol{z},T} \quad &= \quad \boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{E}_e(\cdot,\boldsymbol{z},\boldsymbol{q}) \\ &\qquad -\lambda\boldsymbol{\mathcal{S}}\boldsymbol{E}_{e,T}(\cdot,\boldsymbol{z},\boldsymbol{q}) \quad \text{on} \quad \partial B\,,\end{aligned} \tag{4.5}$$

where the last equation comes from the fact that $\boldsymbol{w}_{0_z}$ satisfies the third equation of (3.5). In addition,

$$\boldsymbol{w}_{\boldsymbol{z}} = \boldsymbol{v}_{\boldsymbol{g}_z} + \boldsymbol{w}^s_{\boldsymbol{z}} \quad \text{in } \mathbb{R}^3\,. \tag{4.6}$$

As for the Helmholtz case (c.f. [5]), the unique solution of (4.5) (provided that λ is not a Stekloff eigenvalue for fixed κ^2), in general cannot be decomposed of the form (4.6) with $\boldsymbol{v}_{\boldsymbol{g}_z}$ being a Herglotz function. However according to Lemma 4.1 $\boldsymbol{w}_{\boldsymbol{z}}$ can be decomposed as $\boldsymbol{w}_{\boldsymbol{z}}(\boldsymbol{x}) = \boldsymbol{w}^i_{\boldsymbol{z}}(\boldsymbol{x}) + \boldsymbol{w}^s_{\boldsymbol{z}}(\boldsymbol{x})$ for $\boldsymbol{x} \in B$ where $\boldsymbol{w}^i_{\boldsymbol{z}} \in \boldsymbol{H}(\mathbf{curl}, B)$ satisfies Maxwell's equations in B, and $\boldsymbol{w}^s_{\boldsymbol{z}} \in \boldsymbol{H}_{\mathtt{loc}}(\mathbf{curl}, \mathbb{R}^3)$ is a radiating solution to

$$\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{w}^s_{\boldsymbol{z}} \,-\, \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{w}^s_{\boldsymbol{z}} \,=\, \kappa^2(\epsilon_r(\boldsymbol{x})-1)\boldsymbol{w}^i_{\boldsymbol{z}} \quad \text{in} \quad \mathbb{R}^3\,,$$
$$\lim_{r\to\infty}(\boldsymbol{u}^s_{\boldsymbol{z}} \times \boldsymbol{x} - r\boldsymbol{w}^s_{\boldsymbol{z}}) = \mathbf{0}\,,$$

where $\boldsymbol{u}^s_{\boldsymbol{z}} := (1/i\kappa)\,\mathbf{curl}\,\boldsymbol{w}^s_{\boldsymbol{z}}$. Note that $1-\epsilon_r$ is supported inside B. Hence the kernel $\boldsymbol{g}$ of the Herglotz function $\boldsymbol{v}_{\boldsymbol{g}}$ such that $\boldsymbol{v}_{\boldsymbol{g}}$ approximates the above $\boldsymbol{w}^i_{\boldsymbol{z}}$ in the $\boldsymbol{H}(\mathbf{curl}, B)$ norm (c.f. [8]) satisfies

$$\|\mathcal{F}_S\boldsymbol{g} - \boldsymbol{E}_{e,\infty}(\cdot, \boldsymbol{z}, \boldsymbol{q})\|_{\boldsymbol{L}^2_t(\mathbb{S}^2)} < \epsilon.$$

Let us define the following space

$$\boldsymbol{H}_{\mathtt{inc}}(\mathbf{curl}, B) := \{\boldsymbol{u}^i \in \boldsymbol{H}(\mathbf{curl}, B)\,:\, \mathbf{curl}\ \mathbf{curl}\,\boldsymbol{u}^i - \kappa^2\boldsymbol{u}^i = \mathbf{0}\}\,.$$

Notice that the operator $\mathcal{F}_S\,:\,\boldsymbol{L}^2_t(\mathbb{S}^2) \to \boldsymbol{L}^2_t(\mathbb{S}^2)$ can be written as $\mathcal{F}_S\boldsymbol{g} = \mathcal{B}\boldsymbol{v}_{\boldsymbol{g}}$ where $\mathcal{B}\,:\,\boldsymbol{H}_{\mathtt{inc}}(\mathbf{curl}, B) \to \boldsymbol{L}^2_t(\mathbb{S}^2)$ is defined as

$$\mathcal{B} = \mathcal{C}_{\boldsymbol{w}}\mathcal{D} - \mathcal{C}_{\boldsymbol{w}_0}$$

with $\mathcal{C}_{\boldsymbol{w}_0}\,:\,\boldsymbol{H}_{\mathtt{inc}}(\mathbf{curl}, B) \to \boldsymbol{L}^2_t(\mathbb{S}^2)$, $\boldsymbol{v}^i \mapsto \mathcal{C}_{\boldsymbol{w}_0}(\boldsymbol{v}^i) := \boldsymbol{w}_{0,\infty}$ with $\boldsymbol{w}_{0,\infty}$ being the far field patters of $\boldsymbol{w}^s_0$ that solves

$$\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{w}^s_0 \,-\, \kappa^2\boldsymbol{w}^s_0 \,=\, \mathbf{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}\,,$$
$$\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{w}^s_0 \,-\, \lambda\boldsymbol{\mathcal{S}}\,\boldsymbol{w}^s_{0,T} \,=\, -\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{v}^i \,+\, \lambda\boldsymbol{\mathcal{S}}\,\boldsymbol{v}^i_T \quad \text{on} \quad \partial B\,,$$
$$\lim_{r\to\infty}((1/i\kappa)\ \mathbf{curl}\ \boldsymbol{w}^s_0 \times \boldsymbol{x} - r\boldsymbol{w}^s_0) = \mathbf{0}\,,$$

$\mathcal{D}\,:\,\boldsymbol{H}_{\mathtt{inc}}(\mathbf{curl}, B) \to \boldsymbol{L}^2(D)$, $\boldsymbol{v}^i \mapsto \mathcal{D}(\boldsymbol{v}^i) := \kappa^2(1-\epsilon_r)\boldsymbol{v}^i$, and $\mathcal{C}_{\boldsymbol{w}}\,:\,\boldsymbol{L}^2(D) \to \boldsymbol{L}^2_t(\mathbb{S}^2)$, $\boldsymbol{f} \mapsto \mathcal{C}_{\boldsymbol{w}}(\boldsymbol{f}) := \boldsymbol{w}_\infty$ where $\boldsymbol{w}_\infty$ is the far field patters of $\boldsymbol{w}^s$ that solves

$$\begin{aligned}\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{w}^s_{\boldsymbol{z}} \,-\, \kappa^2\epsilon_r(\boldsymbol{x})\boldsymbol{w}^s_{\boldsymbol{z}} \,=\, \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^3\,,\\ \lim_{r\to\infty}((1/i\kappa)\ \mathbf{curl}\ \boldsymbol{w}^s_{\boldsymbol{z}} \times \boldsymbol{x} - r\boldsymbol{w}^s_{\boldsymbol{z}}) = \mathbf{0}\,.\end{aligned} \tag{4.7}$$

By construction we have that

$$\mathcal{B}\boldsymbol{v}^i = \boldsymbol{E}_{e,\infty}(\cdot, \boldsymbol{z}, \boldsymbol{q})\,, \qquad \boldsymbol{z} \in B$$

has a solution which is $\boldsymbol{v}^i := \boldsymbol{w}^i_{\boldsymbol{z}}$.

Notice that the operator $\mathcal{B}$ clearly is a compact operator since the solution operators $\mathcal{C}_{\boldsymbol{w}_0}$ and $\mathcal{C}_{\boldsymbol{w}}$ are bounded and compact and $\mathcal{D}$ is bounded.

As a consequence of the previous discussion, we conclude the following theorem.

THEOREM 4.2. *Assume that λ is a not Stekloff eigenvalue and let $\boldsymbol{z} \in D$. Then, for every $\epsilon > 0$ there exists a $\boldsymbol{g}_\epsilon^{\boldsymbol{z}} \in \boldsymbol{L}^2_t(\mathbb{S}^2)$ that satisfies*

$$\lim_{\epsilon \to 0} \|\mathcal{F}_S \boldsymbol{g}_\epsilon^{\boldsymbol{z}} - \boldsymbol{E}_{e,\infty}(\cdot, \boldsymbol{z}, \boldsymbol{q})\|_{\boldsymbol{L}^2_t(\mathbb{S}^2)} = 0 \tag{4.8}$$

such that $\lim_{\epsilon\to 0} \|\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}} - \boldsymbol{v}^i\|_{\mathbf{curl},B} = 0$ and hence $\|\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}}\|_{\mathbf{curl},B}$ is bounded as $\epsilon \to 0$, where $\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}}$ is the Herglotz wave function with kernel $\boldsymbol{g}_\epsilon^{\boldsymbol{z}}$.

Now we focus on providing a method to compute Stekloff eigenvalues from far field measured scattering data at a given fixed frequency κ. To this end, first we need to introduce the following technical result.

LEMMA 4.3. *For all $\boldsymbol{z} \in B$, $\boldsymbol{q} \in \mathbb{R}^3$ and for all regular function $\boldsymbol{u}$, we have*

$$\begin{aligned} & \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{E}_e(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{q})\, ds_{\boldsymbol{x}} \\ = \;& i\kappa \boldsymbol{q} \cdot \nabla_{\boldsymbol{z}} \mathrm{div}\,_{\boldsymbol{z}} \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \times \boldsymbol{\nu}(\boldsymbol{x})\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\, ds_{\boldsymbol{x}} \\ & + i\kappa \boldsymbol{q} \cdot \int_{\partial B} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\; \mathbf{curl}\, \boldsymbol{u}(x) \times \boldsymbol{\nu}(\boldsymbol{x})\, ds_{\boldsymbol{x}} \end{aligned}$$

and

$$\int_{\partial B} \boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x}) \cdot \mathbf{curl}_{\boldsymbol{x}}\, \boldsymbol{E}_e(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{q})\, ds_{\boldsymbol{x}} = i\kappa \boldsymbol{q} \cdot \mathbf{curl}_{\boldsymbol{z}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x})\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\, ds_{\boldsymbol{x}}.$$

Proof.

Let $\boldsymbol{z} \in B$ and $\boldsymbol{q} \in \mathbb{R}^3$. First, from the equality $\mathbf{curl}\ \mathbf{curl} = -\Delta + \nabla \mathrm{div}$, we remark that

$$\begin{aligned} \mathbf{curl}_x\ \mathbf{curl}_x(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) &= -\Delta_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) + \nabla_{\boldsymbol{x}} \mathrm{div}\,_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) \\ &= \kappa^2\, \boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z}) + \nabla_{\boldsymbol{x}} \mathrm{div}\,_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) \end{aligned} \tag{4.9}$$

and

$$\mathbf{curl}_{\boldsymbol{x}}\ \mathbf{curl}_{\boldsymbol{x}}\ \mathbf{curl}_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) = \kappa^2\ \mathbf{curl}_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z}))\,. \tag{4.10}$$

In addition, it is easy to prove that

$$\nabla_{\boldsymbol{x}} \mathrm{div}\,_{\boldsymbol{x}}(\boldsymbol{q} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) = \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\, \boldsymbol{q} \quad \text{and} \quad \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})^{\mathrm{t}} = \nabla_{\boldsymbol{z}} \nabla_{\boldsymbol{z}} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z}). \tag{4.11}$$

From (4.3), (4.9) and (4.11) we have

$$\begin{aligned} & \int_{\partial B} \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{\nu} \times \boldsymbol{E}_e\, ds_{\boldsymbol{x}} \\ = \;& \frac{i}{\kappa} \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \times \mathbf{curl}_{\boldsymbol{x}}\ \mathbf{curl}_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})) ds_{\boldsymbol{x}} \\ = \;& i\kappa \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\, ds_{\boldsymbol{x}} \\ & + \frac{i}{\kappa} \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \times \nabla_{\boldsymbol{x}} \mathrm{div}\,_{\boldsymbol{x}}(\boldsymbol{q}\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z}))\, ds_{\boldsymbol{x}} \\ = \;& i\kappa\, \boldsymbol{q} \cdot \Bigg(\int_{\partial B} \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\ \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \times \boldsymbol{\nu}(\boldsymbol{x})\, ds_{\boldsymbol{x}} \\ & + \frac{1}{\kappa^2} \nabla_{\boldsymbol{z}} \mathrm{div}\,_{\boldsymbol{z}} \int_{\partial B} \mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}) \times \boldsymbol{\nu}(\boldsymbol{x})\, \Phi_\kappa(\boldsymbol{x}, \boldsymbol{z})\, ds_{\boldsymbol{x}} \Bigg) \end{aligned}$$

which yields the first equality.

On the other hand, from the definition of $\boldsymbol{E}_e$ (cf. (4.3)), (4.10) and

$$\boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x}) \cdot \mathbf{curl}_{\boldsymbol{x}}(\boldsymbol{q}\,\Phi_\kappa(\boldsymbol{x},\boldsymbol{z})) \;=\; \boldsymbol{q} \cdot \mathbf{curl}_{\boldsymbol{z}}(\boldsymbol{\nu}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x})\Phi_\kappa(\boldsymbol{x},\boldsymbol{z}))$$

we obtain the second equality. □

Now we show that the term $\|\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}}\|_{\mathbf{curl},B}$ explodes when λ is a Stekloff eigenvalue.

Theorem 4.4. *Assume that λ is a Stekloff eigenvalue and $\boldsymbol{g}_\epsilon^{\boldsymbol{z}} \in \boldsymbol{L}_t^2(\mathbb{S}^2)$ satisfies (4.8). Then $\|\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}}\|_{\mathbf{curl},B}$ can not be bounded as $\epsilon \to 0$ for almost every $\boldsymbol{z} \in B_\rho$ where $B_\rho \subset D$ is an arbitrary ball of radius ρ.*

Proof. Assume to the contrary that for some $B_\rho \subset D$ and all $\boldsymbol{z} \in B_\rho$, $\|\boldsymbol{v}^{\boldsymbol{z}}_{\boldsymbol{g}_\epsilon}\|_{\mathbf{curl},B}$ is bounded as $\epsilon \to 0$, i.e. up to a subsequence $\boldsymbol{v}^{\boldsymbol{z}}_{g_\epsilon}$ converge weakly to a $\boldsymbol{v}^i \in \boldsymbol{H}_{\texttt{inc}}(\mathbf{curl},B)$. By compactness of $\mathcal{B}$ we conclude that

$$\lim_{\epsilon\to 0} \|\mathcal{B}\boldsymbol{v}_{\boldsymbol{g}_\epsilon^{\boldsymbol{z}}} - \mathcal{B}\boldsymbol{v}^i\|_{\boldsymbol{L}_t^2(\mathbb{S}^2)} \;=\; \lim_{\epsilon\to 0} \|\mathcal{F}_S\boldsymbol{g}_\epsilon^{\boldsymbol{z}} - \mathcal{B}\boldsymbol{v}^i\|_{\boldsymbol{L}_t^2(\mathbb{S}^2)} \;=\; 0\,.$$

From here $\mathcal{B}\boldsymbol{v}^i = \boldsymbol{E}_{e,\infty}(\cdot,\boldsymbol{z},\boldsymbol{q})$ and from the Rellich's Lemma and the definition of $\mathcal{B}$ we can conclude that $\boldsymbol{w} := \boldsymbol{w}^s + \boldsymbol{v}^i$, where $\boldsymbol{w}^s$ satisfies (4.7) with $\boldsymbol{w}^i_{\boldsymbol{z}}$ replaced by $\boldsymbol{v}^i$ satisfies (4.5). But from Lemma 4.1 and the Fredholm alternative, (4.5) is solvable if and only if

$$\int_{\partial B} (\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{E}_e(\cdot,\boldsymbol{z},\boldsymbol{q}) - \lambda\boldsymbol{\mathcal{S}}\boldsymbol{E}_{e,T}(\cdot,\boldsymbol{z},\boldsymbol{q}))\,\overline{\boldsymbol{w}_{\lambda,T}}\,ds \;=\; 0 \tag{4.12}$$

with $\boldsymbol{w}_\lambda \in \boldsymbol{H}(\mathbf{curl},B)$ satisfying

$$\begin{aligned} \mathbf{curl}\;\mathbf{curl}\;\boldsymbol{w}_\lambda \;-\; \kappa^2\overline{\epsilon_r}\boldsymbol{w}_\lambda \;&=\; \mathbf{0} \quad \text{in} \quad B\,, \\ \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{w}_\lambda \;-\; \overline{\lambda}\boldsymbol{\mathcal{S}}\boldsymbol{w}_{\lambda,T} \;&=\; \mathbf{0} \quad \text{on} \quad \partial B\,. \end{aligned} \tag{4.13}$$

From the boundary condition satisfied by $\boldsymbol{\mathcal{S}}\overline{\boldsymbol{w}_\lambda}$ on ∂B and (3.2) we obtain that (4.12) can be rewritten as follows:

$$\int_{\partial B} (\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{E}_e(\cdot,\boldsymbol{z},\boldsymbol{q}) \cdot \overline{\boldsymbol{w}_{\lambda,T}} - \boldsymbol{E}_{e,T}(\cdot,\boldsymbol{z},\boldsymbol{q}) \cdot \boldsymbol{\nu} \times \mathbf{curl}\,\overline{\boldsymbol{w}_\lambda})\;ds \;=\; 0\,, \tag{4.14}$$

for all $\boldsymbol{z} \in B_\rho$ and $\boldsymbol{q} \in \mathbb{R}^3$.

Let $\mathcal{G}$ a function defined in $\mathbb{R}^3 \setminus B$ by

$$\begin{aligned} \mathcal{G}(\boldsymbol{z}) \;\;=\;\; & \int_{\partial B} \Phi_\kappa(\boldsymbol{x},\boldsymbol{z})\;\mathbf{curl}\;\overline{\boldsymbol{w}_\lambda(\boldsymbol{x})} \times \boldsymbol{\nu}(\boldsymbol{x})\,ds_{\boldsymbol{x}} \\ & + \frac{1}{\kappa^2}\nabla_{\boldsymbol{z}}\mathrm{div}_{\,\boldsymbol{z}} \int_{\partial B} \mathbf{curl}\,\overline{\boldsymbol{w}_\lambda(\boldsymbol{x})} \times \boldsymbol{\nu}(\boldsymbol{x})\,\Phi_\kappa(\boldsymbol{x},\boldsymbol{z})\,ds_{\boldsymbol{x}} \\ & - \mathbf{curl}_{\boldsymbol{z}} \int_{\partial B} \boldsymbol{\nu}(\boldsymbol{x}) \times \overline{\boldsymbol{w}_\lambda(\boldsymbol{x})}\,\Phi_\kappa(\boldsymbol{x},\boldsymbol{z})\,ds_{\boldsymbol{x}}\,. \end{aligned}$$

Note that $\mathcal{G}$ is a radiating solution to Maxwell's equations. From Lemma 4.3, (4.14) is equivalent to $i\kappa\boldsymbol{q}\cdot\mathcal{G}(\boldsymbol{z}) = 0$, for all $\boldsymbol{z} \in B_\rho$ and all $\boldsymbol{q} \in \mathbb{R}^3$. As a consequence, using the unique continuation principle, $\mathcal{G} = \mathbf{0}$ in B. Hence, we obtain that $\boldsymbol{\nu} \times \mathcal{G}^- = \mathbf{0}$ and $\boldsymbol{\nu} \times \mathbf{curl}\;\mathcal{G}^- = \mathbf{0}$ on ∂B, where, by the superscript + and −, we distinguish the limit obtained by approaching the boundary ∂B from $\mathbb{R}^3 \setminus \overline{B}$ and B, respectively. Therefore if $\boldsymbol{z} \to \partial B$, the following jump relations (cf. [9, Theorem 6.12]) hold,

$$\boldsymbol{\nu} \times \mathcal{G}^+ = -\boldsymbol{\nu} \times \overline{\boldsymbol{w}_\lambda} \quad \text{and} \quad \boldsymbol{\nu} \times \mathbf{curl}\, \mathcal{G}^+ = -\boldsymbol{\nu} \times \mathbf{curl}\, \overline{\boldsymbol{w}_\lambda} \quad \text{on } \partial B\,. \tag{4.15}$$

Combining (4.15) and the second equation of (4.13), we obtain

$$\boldsymbol{\nu} \times \mathbf{curl}\, \mathcal{G}^+ - \lambda \boldsymbol{\mathcal{S}} \mathcal{G}_T^+ = -(\boldsymbol{\nu} \times \mathbf{curl}\, \overline{\boldsymbol{w}_\lambda} - \lambda \boldsymbol{\mathcal{S}} \overline{\boldsymbol{w}_{\lambda,T}}) = \mathbf{0} \quad \text{on } \partial B\,. \tag{4.16}$$

Therefore, $\mathcal{G}$ is a radiating solution to Maxwell's equations satisfying (4.16). Defining $\mathcal{H} := (1/i\kappa)\, \mathcal{G}$ and following the ideas in the proof of Theorem 3.2, we obtain that

$$\mathrm{Re}\left(\int_{\partial B} \boldsymbol{\nu} \times \mathcal{G} \cdot \overline{\mathcal{H}}\, ds \right) = 0\,,$$

and from here and [9, Theorem 6.11], we conclude that $\mathcal{G} = \mathcal{H} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{B}$. Then $\boldsymbol{\nu} \times \mathcal{G}^+ = \mathbf{0}$ and $\boldsymbol{\nu} \times \mathbf{curl}\, \mathcal{G}^+ = \mathbf{0}$ on ∂B and therefore, $\boldsymbol{\nu} \times \boldsymbol{w}_\lambda = \mathbf{0}$ and $\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{w}_\lambda = \mathbf{0}$ on ∂B. Finally, the unique continuation principle let us conclude that $\boldsymbol{w}_\lambda = \mathbf{0}$ in B which contradicts the fact that $\boldsymbol{w}_\lambda$ is a eigenfunction. □

5. Numerical experiments.

5.1. Detection of modified Stekloff eigenvalues. We start by showing that for two simple scatterers it is possible to detect a few modified Stekloff eigenvalues from the far field pattern even at low frequency. In particular all results are computed using $\kappa = 1$ so that the wavelength of light in the exterior is 2π.

For the two domains (a unit sphere and a unit cube) we computed multistatic scattering data for several incoming waves using quadratic conforming edge finite elements and the Perfectly Matched Layer (see for example [13]) provided by the Netgen/NGSolve package [16]. For both the penetrable scattering problem (1.1)-(1.2) and the modified Stekloff problem (3.3) we used quadratic edge elements and a spherical PML, and approximate curved boundaries or interfaces by fifth order polynomials. For the modified Stekloff problem we used cubic H^1 conforming elements on the surface of the unit sphere ∂B_R to implement $\mathcal{S}$ (see the upcoming discussion of the modified Stekloff eigenvalue problem).

In both cases we used a requested mesh size of $2\pi/(12k\sqrt{\epsilon_r})$ from Netgen in each subdomain of the problem (we take ϵ_r piecewise constant). The far field pattern is computed using the procedure from [13] implemented in Python.

The incoming wave directions and measurement directions are taken to be the vertices of a finite element mesh on $\mathbb{S}^2$ computed by Netgen using requested mesh size $h = 0.4$ (89 incoming waves) or $h = 0.3$ (159 incoming waves). The far field equation (4.4) is then approximated by quadrature using the nodes of the surface grid on $\mathbb{S}^2$ as quadrature points. Two independent polarizations are chosen for each incident direction resulting in a far field matrix $\mathbb{F}_S$ approximating $\mathcal{F}_S$ defined in (3.4) of size 178×178 or 318×318.

Then the discrete far field equation

$$\mathbb{F}_S \vec{g} = \vec{b}$$

approximating (4.4) is solved for each λ by Tikhonov regularization using

$$\vec{g} = (\mathbb{F}^H \mathbb{F} + \alpha I)^{-1} (\mathbb{F}^H \vec{b})$$

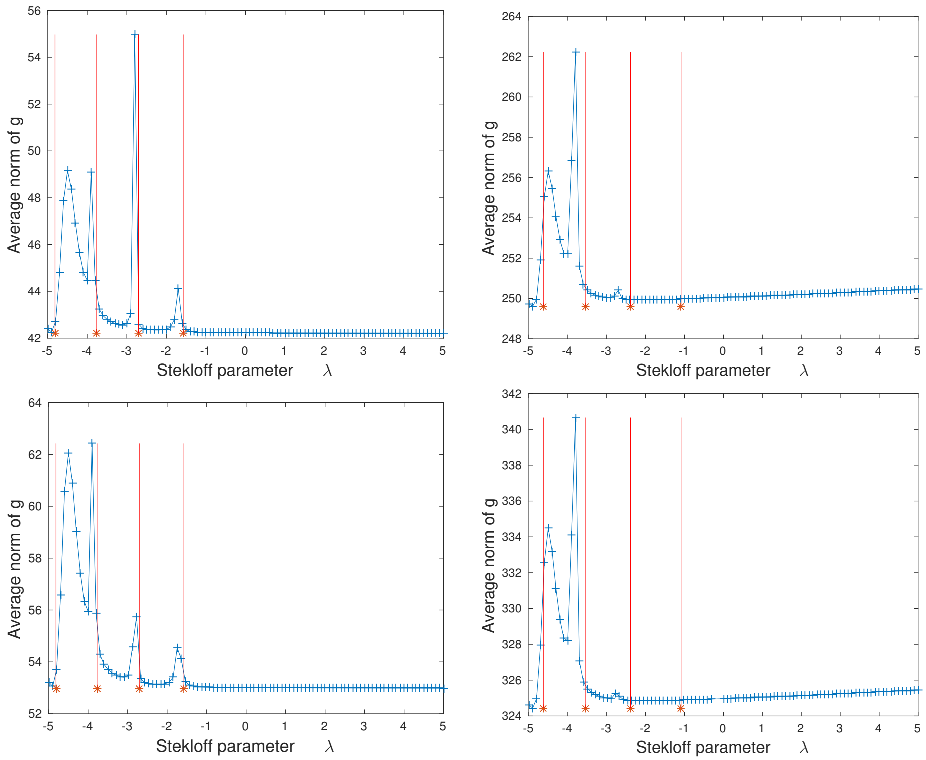

FIG. 5.1. *Average norm of the discrete approximation to $\boldsymbol{g}_{\boldsymbol{z}}$ against Stekloff parameter λ. Top row: 89 incoming waves. Bottom row: 159 incident waves. Left column: $\epsilon_r|_D = 2$. Right column: $\epsilon_r|D = 4$. Stars mark exact values of the Stekloff eigenvalues.*

where $\mathbb{F}^H$ is the conjugate transpose of $\mathbb{F}$, $\alpha = 10^{-8}$ and $\vec{b}$ depends on $\boldsymbol{z}$ and the artificial polarization $\boldsymbol{q}$ (see [5, 7] for more details on the discretization and solution of the discrete problem). This part of the study was performed in Matlab.

We use 10 randomly chosen points $\boldsymbol{z}$ in a cube of side length $1/5$ centered at the origin and all three unit vectors for polarization. This results in 30 discrete approximations to $\vec{g}$ and we average these to produce the approximate results shown in Fig. 5.1 and Fig, 5.2.

The results in Fig. 5.1 show the averaged norm of $\vec{g}$ against λ for the unit sphere in which $\epsilon_r = 2$ or $\epsilon_r = 4$ (the second case is a 3D analogue of results in [5]). We also tested two surface triangulations on $\mathbb{S}^2$. When $\epsilon_r = 2$ we can detect three eigenvalues and a fourth less accurately. For $\epsilon_r = 4$ only two eigenvalues in the range being analyzed could be detected. There was little improvement moving to the finer surface mesh, and so we did not use that mesh again.

In Fig. 5.2 we show corresponding results for the unit cube as scatterer, but using the unit sphere for the modified Stekloff problem (i.e. ∂B is the surface of the unit sphere) so that the outer boundary is smooth. Results are broadly similar to the case of the sphere. In both cases four modified Stekloff eigenvalues can be approximated.

5.2. Modified Stekloff eigenvalues. Next we show that modified Stekloff eigenvalues are effected both by bulk changes in the permittivity of the scatterer as well as by more localized changes. We do this by computing modified Stekloff

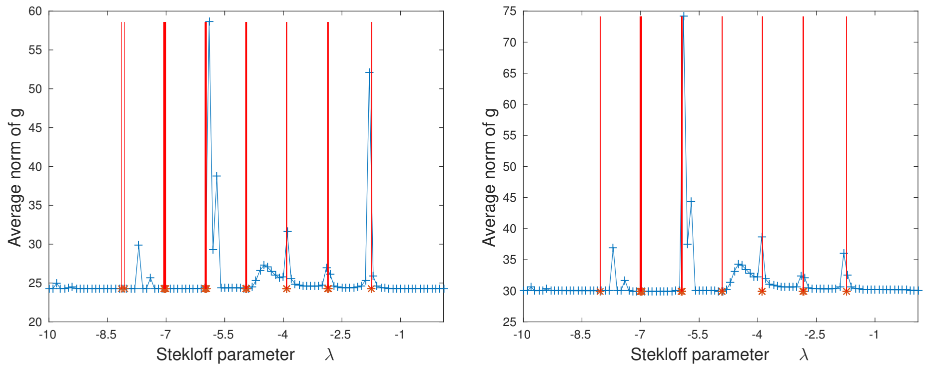

FIG. 5.2. *Average norm of the discrete approximation to g_z against Stekloff parameter λ for scattering by a cube. Left: $\epsilon_r|_D = 2$. Right: $\epsilon_r|D = 4$. Stars mark exact values of the Stekloff eigenvalues.*

eigenvalues directly in the following way.

First we describe in detail our procedure for computing modified Stekloff eigenvalues. We note that for the modified Stekloff eigenvalue problem (3.7), because $\boldsymbol{\mathcal{S}}\boldsymbol{w}_T \in H^{1/2}(\partial B)$, Costabel's regularity result shows that $\boldsymbol{w} \in \boldsymbol{H}^{1/2+s}(\partial B)$ for some $s > 0$ and so $\boldsymbol{w}_T \in \boldsymbol{L}^2_t(\partial B)$. It thus suffices to seek solutions $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}(B)$. In this case, for $\boldsymbol{w}_T \in \boldsymbol{L}^2_t(\partial B)$ the operator $\boldsymbol{\mathcal{S}}$ has a variational definition. In particular

$$\boldsymbol{\mathcal{S}}\boldsymbol{w}_T = \boldsymbol{w}_T + \nabla_{\partial B}\phi$$

where $\phi \in H^1(\partial B)$ satisfies

$$\int_{\partial B} (\boldsymbol{w}_T + \nabla_{\partial B}\phi) \cdot \nabla_{\partial B}\overline{\psi}\, ds_{\boldsymbol{x}} = 0, \qquad \int_{\partial B} \phi\, ds_{\boldsymbol{x}} = 0,$$

for all $\psi \in H^1(\partial B)$. Thus

$$\lambda \int_{\partial B} (\boldsymbol{w}_T + \nabla_{\partial B}\phi) \cdot \nabla_{\partial B}\overline{\psi}\, ds_{\boldsymbol{x}} = 0, \qquad \int_{\partial B} \phi\, ds_{\boldsymbol{x}} = 0.$$

Subtracting this equation from (3.7) and expanding $\boldsymbol{\mathcal{S}}$ we obtain the problem of finding $(\boldsymbol{w}, \phi) \in \boldsymbol{\mathcal{X}}(B) \times H^1(\partial B)/\mathbb{C}$, $\boldsymbol{w} \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$\int_B \mathbf{curl}\, \boldsymbol{w}\cdot\mathbf{curl}\, \overline{\boldsymbol{z}}\, d\boldsymbol{x} - \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\boldsymbol{w}\cdot\overline{\boldsymbol{z}}\, d\boldsymbol{x} \;=\; -\lambda \int_{\partial B} (\boldsymbol{w}_T + \nabla_{\partial B}\phi)\cdot(\overline{\boldsymbol{z}_T} + \nabla_{\partial B}\overline{\psi})\, ds_{\boldsymbol{x}}\,,$$

for all $(\boldsymbol{z}, \psi) \in \boldsymbol{\mathcal{X}}(B) \times H^1(\partial B)/\mathbb{C}$. This formulation would degenerate badly if $\lambda = 0$ is a generalized Stekloff eigenvalue, or equivalently if κ^2 is an interior Neumann eigenvalue for Maxwell's equations. We shall assume that such values of κ are excluded from our study.

In practice we add a small regularizing term in place of the constraint on the average value to guarantee that ϕ is well defined and solve the problem of finding $(\boldsymbol{w}, \phi) \in \boldsymbol{\mathcal{X}}(B) \times H^1(\partial B)$, $\boldsymbol{w} \neq 0$, and $\lambda \in \mathbb{R}$ such that

$$\int_B \mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \overline{\boldsymbol{z}}\, d\boldsymbol{x} \;-\; \kappa^2 \int_B \epsilon_r(\boldsymbol{x})\boldsymbol{w} \cdot \overline{\boldsymbol{z}}\, d\boldsymbol{x}$$
$$= -\lambda \int_{\partial B} \left[(\boldsymbol{w}_T + \nabla_{\partial B}\phi) \cdot (\overline{\boldsymbol{z}_T} + \nabla_{\partial B}\overline{\psi}) + \gamma\phi\,\overline{\psi})\right]\, ds_{\boldsymbol{x}}\,,$$

for all $(\boldsymbol{x}, \psi) \in \boldsymbol{\mathcal{X}}(B) \times H^1(\partial B)$ where $\gamma > 0$ is chosen, in our calculations, to be 10^{-8}.

After discretization by curl conforming finite elements for $\boldsymbol{z}$ and H^1 conforming elements for ϕ we have the block matrix problem

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c} \vec{z} \\ \vec{\phi} \end{array}\right) = -\lambda\left(\begin{array}{c|c} M_{\text{curl}} & B \\ \hline B^* & M_{H^1} \end{array}\right)\left(\begin{array}{c} \vec{z} \\ \vec{\phi} \end{array}\right)$$

where A is the usual Maxwell matrix with natural boundary conditions, and $\vec{z}$ and $\vec{\phi}$ are vectors of degrees of freedom. Then since $\lambda \neq 0$, as long as we are not at an interior Maxwell eigenvalue and the mesh is fine enough,

$$\vec{\phi} = -M_{H^1}^{-1} B^* \vec{z}$$

and we arrive at a more standard Stekloff type eigenvalue problem

$$A\vec{z} = -\lambda(M_{\text{curl}} - BM_{H^1}^{-1}B^*)\vec{z}$$

This still involves interior unknowns but then we can further decompose into interior and boundary degrees of freedom $\vec{z} = [\vec{z}_{\text{bnd}}, \vec{z}_{\text{int}}]$ then

$$\vec{z}_{\text{bnd}} = -\lambda[A^{-1}(M_{\text{curl}} - BM_{H^1}^{-1}B^*)\vec{z}_{\text{bnd}}]_{\text{bnd}}$$

This reduces to a large dense eigenvalue problem involving only boundary degrees of freedom. Unfortunately this algorithm doesn't scale well. For the upcoming example of a sphere when $h = 0.25$ the dense matrix $A^{-1}(M_{\text{curl}} - BM_{H^1}^{-1}B^*)$ (using quadratic edge elements) is 900×900 while when $h = .125$ it is $11,700 \times 11,700$.

We solve the eigenvalue problem using quadratic full degree edge elements to approximate $\boldsymbol{\mathcal{X}}(B)$ and cubic Lagrange elements to approximate $H^1(\partial B)$ (tests with quadratic Lagrange elements showed a failure to converge while quartic Lagrange elements showed little improvement on the computed eigenvalues as is to be expected from consideration of the discrete de Rham diagram [13]). In addition we use curved elements of degree 5 to approximate the boundary.

To check convergence of our method we first computed the modified Stekloff eigenvalues for the unit sphere with $\kappa = 2$ and $\epsilon_r = 1$ as shown Table 5.1. The reported mesh size h is the maximum element size requested from Netgen. The actual mesh size may differ from this. Considering the 4th and 5th eigenvalue we see that the order of convergence is $O(h)$ and $O(h^{5/2})$ respectively and this is lower than we might expect for a quadratic edge element approximation to a self adjoint eigenvalue problem. However this problem involves boundary and interior terms and so the actual optimal rate is unknown. This troubling issue needs to be investigated further.

Now we can investigate changes in eigenvalues for the unit cube. First we consider how changes to the bulk permittivity ϵ_r in D change the Stekloff eigenvalues. We simply compute the modified Stekloff eigenvalues for the cube inside the unit sphere, and for the cube alone, as ϵ_r is varied. In these cases their are obvious multiple eigenvalues and since the shape of the scatterer is unchanged this multiplicity persists for each ϵ_r. So we show the average of the eigenvalues in each cluster. Results are shown in Fig. 5.3. Two things are evident: first, over this range of parameters, the change appears linear. We have no explanation for this! The second is that the eigenvalues for the cube alone are much more sensitive to changes in ϵ_r than for the cube in the sphere.

Exact Eigenvalue	Computed Eigenvalue			Multiplicity
	$h = 0.5$	$h = 0.25$	$h = 0.125$	
-1.0884	-1.0920	-1.0886	-1.0884	3
-2.3880	-2.3993	-2.3884	-2.3880	5
-3.5363	-3.5714	-3.5392	-3.5364	7
-4.6257	-4.8882	-4.6416	-4.6264	9
-5.6857	-6.5180	-5.7370	-5.6885	11
-6.7290	-8.5890	-6.8530	-6.7367	13

TABLE 5.1

Modified Stekloff eigenvalues for the unit sphere, $k = 2$ and $\epsilon_r|_D = 1$.

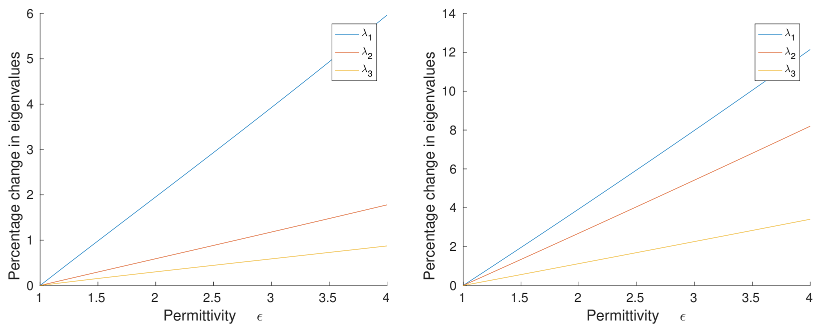

FIG. 5.3. *Relative change in the Modified Stekloff eigenvalues as ϵ_r changes. Left: Unit cube inside the unit sphere. Right: Unit cube alone*

Our second example in Fig. 5.4 shows changes in the first five eigenvalues (with no averaging) as a function of the radius of a spherical inclusion positioned at $(x, y, z) = (0.25, 0, 0)$. The inclusion is assumed to have $\epsilon_r = 1$ inside. Again the sensitivity of the eigenvalues to the presence of this inclusion is better for eigenvalues of the unit cube alone, rather than the cube inside the unit sphere.

6. Conclusion. While much more numerical testing on realistic geometries is need, the results here suggest that modified Stekloff eigenvalues can be detected from far field data. In addition these eigenvalues change as the internal permittivity of the scatter changes either due to bulk changes of more localized changes. However the sensitivity of the eigenvalues to changes in the scatterer is decreased when the surface $\partial B \neq \partial D$.

The major open problem suggested by this study is whether standard Stekloff eigenvalues exist. If they do exist it should be possible to detect them from scattering data. In either case (Stekloff or modified Stekloff) improved understanding of the convergence of the discrete eigenvalue problem and better methods for computing the eigenvalues would allow a more complete study.

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REFERENCES

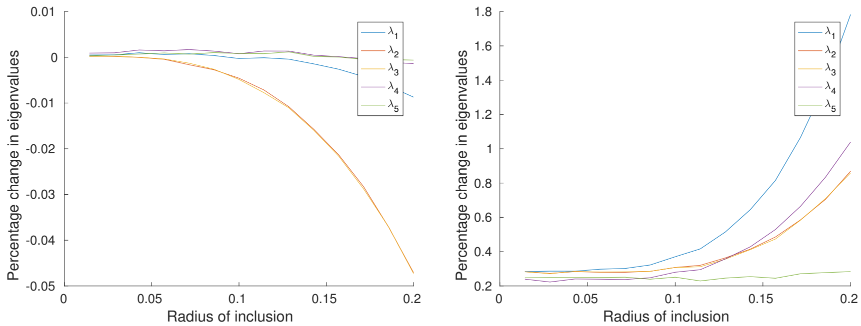

FIG. 5.4. *Relative change in the Modified Stekloff eigenvalues as the radius of the spherical inclusion changes. Left: Unit cube inside the unit sphere. Right: Unit cube alone*

[1] L. Borcea, *Electrical impedance tomography*, Inverse Problems, 18 (2002), pp. R99–R136.

[2] A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Math., 92 (2002), pp. 679–710.

[3] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory: An Introduction*, Springer-Verlag, New York, 2006.

[4] F. Cakoni, D. Colton, and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, 88, SIAM Publications, 2016.

[5] F. Cakoni, D. Colton, S. Meng, and P. Monk, *Stekloff eigenvalues in inverse scattering.* to appear in *SIAM J. Appl. Math.*, 2016.

[6] F. Cakoni, D. Colton, and P. Monk, *The electromagnetic inverse scattering problem for partly coated Lipschitz domains*, Proc. Royal Soc. Edinburgh, 134A (2004), pp. 661–682.

[7] ———, *The Linear Sampling Method in Inverse Electromagnetic Scattering*, vol. 80 of CBMS, SIAM, Philadelphia, 2011.

[8] D. Colton and R. Kress, *On the denseness of Herglotz wave functions and electromagnetic Herglotz pairs in Sobolev spaces*, Math. Meth. Appl. Sci., 24 (2001), pp. 1289–303.

[9] ———, *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93 of Applied Mathematical Sciences, Springer, New York, third ed., 2013.

[10] M. Costabel, *A remark on the regularity of solutions of Maxwells equations on Lipschitz domains*, Math. Methods Appl. Sci., 12 (1990), pp. 365–268.

[11] M.S. Joshi and W.R.B. Lionheart, *An inverse boundary value problem for harmonic differential forms*, Asymptotic Analysis, 41 (2005), pp. 93–106.

[12] N. Kuznetsov, T. Kulczycki, M. Kwaśnicki, A. Nazarov, S. Poborchi, I. Polterovich, and B. Siudeja, *The legacy of Vladimir Andreevich Steklov*, Notices of the AMS, 61 (2014), pp. 9–22.

[13] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford, 2003.

[14] J.-C. Nédélec, *Acoustic and Electromagnetic Equations*, no. 144 in Applied Mathematical Sciences, Springer, New York, 2001.

[15] O. Ola, L. Päivärinta, and E. Sommersalo, *An inverse boundary value problem in electrodynamics*, Duke Math. Journal, 70 (1993), pp. 617–653.

[16] J. Schoeberl, *Ngsolve.* from https://gitlab.asc.tuwien.ac.at/jschoeberl/ngsolve-docu/wikis/home. Accessed: 25 October 2016.

[17] E. Somersalo, D. Issacson, and M. Cheney, *A linearized inverse boundary value problem for Maxwell's equations*, J. Comp. Appl. Math., 42 (1992), pp. 123–136.

[18] Z. Sun and G. Uhlmann, *An inverse boundary value problem for Maxwell's equations*, Arch. Rational Mech. Anal., 119 (1992), pp. 71–93.

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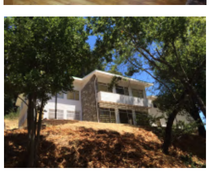

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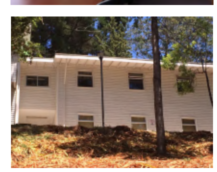
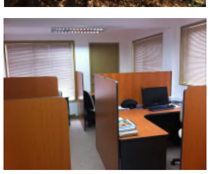

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