UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

A priori and a posteriori error analysis of an augmented mixed-FEM for the Navier-Stokes-Brinkman problem

Luis F. Gatica, Ricardo Oyarzúa, Nestor Sánchez

PREPRINT 2016-27

SERIE DE PRE-PUBLICACIONES

A priori and a posteriori error analysis of an augmented mixed-FEM for the Navier-Stokes-Brinkman problem *

Luis F. Gatica[†], Ricardo Oyarzúa[‡] and Nestor Sánchez[§]

Abstract

We introduce and analyze an augmented mixed finite element method for the Navier-Stokes-Brinkman problem. We employ a technique previously applied to the stationary Navier-Stokes equation, which consists of the introduction of a modified pseudostress tensor relating the gradient of the velocity and the pressure with the convective term, and propose a pseudostress-velocity formulation for the model problem. Since the convective term forces the velocity to live in a smaller space than usual, we augment the variational formulation with suitable Galerkin type terms. The resulting augmented scheme is then written equivalently as a fixed point equation, so that the well-known Banach fixed point theorem, combined with the Lax-Milgram lemma, are applied to prove the unique solvability of the continuous and discrete systems. We point out that no discrete inf-sup conditions are required for the solvability analysis, and hence, in particular for the Galerkin scheme, arbitrary finite element subspaces of the respective continuous spaces can be utilized. For instance, given an integer $k \geq 0$, the Raviart-Thomas spaces of order k and continuous piecewise polynomials of degree $\leq k+1$ constitute feasible choices of discrete spaces for the pseudostress and the velocity, respectively, yielding optimal convergence. In addition, we derive a reliable and efficient residual-based a posteriori error estimator for the augmented mixed method. The proof of reliability makes use of a global inf-sup condition, a Helmholtz decomposition, and local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, approximation properties of the L^2-orthogonal projector, and known results from previous works, are the main tools for proving the efficiency of the estimator. Finally, some numerical results illustrating the performance of the augmented mixed method, confirming the theoretical rate of convergence and properties of the estimator, and showing the behaviour of the associated adaptive algorithms, are reported.

Key words: Navier-Stokes-Brinkman, mixed finite element method, augmented formulation, Raviart-Thomas elements

Mathematics Subject Classifications (1991): 65N15, 65N30, 76D05, 76M10

*This research was partially supported by CONICYT-Chile through projects Fondecyt 11121347 and 1161325, project Anillo ACT1118 (ANANUM); and by Dirección de Investigación of the Universidad Católica de la Santísima Concepción, through project DIN 14/2016.

[†]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: lgatica@ucsc.cl

[‡]Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl

[§]Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, email: nsanchez@ubiobio.cl

1 Introduction

The devising of appropriate numerical methods to simulate fluid flows in porous media has gained considerably attention during the last decades due to its diverse applications in different applied sciences, such as petroleum, agricultural, and biomedical engineering, to name a few. This area of research is also of great importance in the designing and manufacturing of several industrial pieces involving filtration (see e.g.[13, 37]). Depending on the phenomenon, different mathematical models may be used to obtain accurate and suitable results. For instance, one of the most studied approaches is the Stokes-Darcy model (or Navier-Stokes-Darcy model), which consists of a coupled system of equations where the Stokes (or Navier-Stokes) and Darcy equations, both set in disjoint domains, are coupled through a common interface by a mass conservation condition, balance of normal forces, and the so called Beaver–Joseph–Safmann law (see e.g. [16, 31]). The other widely accepted model in the community is the Brinkman equation ([3, 4]). This model, which is more suitable for mixture of materials, consists roughly speaking, of a combination of the Stokes and Darcy equations in a common domain. In the literature can be also found several generalizations of the aforementioned models, including nonlinear problems such as the Forchheimer and the Navier-Stokes-Brinkman (NSB) models. Precisely, in this paper we are interested in introducing a new pseudostress-based finite element method for the latter.

In the context of fluid flow problems, the idea of introducing a pseudostress tensor as a further unknown allows, on the one hand, besides the original unknowns, to obtain direct approximations of several variables of interest, and on the other hand, to unify the analysis for Newtonian and non-Newtonian flows (see for instance [7, 5, 18, 26, 30, 36]). In particular, the method proposed in [7], which has been already extended to the Navier-Stokes problem with variable viscosity [6], to the Boussinesq problem [12], and to the Navier-Stokes/Darcy problem with variable viscosity [11], is the first Raviart-Thomas-based mixed method for the Navier-Stokes problem providing optimal convergence for all the unknowns (see e.g. [5, 18, 36] for previous results). This optimal convergence is attained thanks to an augmentation procedure consisting in the introduction of residual terms arising from the constitutive and equilibrium equations (see, e.g., [20, 19, 21, 23] for more details on this procedure of augmentation). Now, specifically for the Brinkman problem, the first contribution in proposing and analyzing a pseudostress-based method is [26]. This work, which was later extended to the Brinkman problem with variable viscosity in [27], provides an optimal convergent Raviart-Thomas-based method for the model problem, where the pseudostress is the only unknown of the resulting formulation. The velocity, along with the pressure, can be easily recovered through a simple post processing procedure.

Concerning our problem of interest, in [38] has been recently studied a primal finite element method for a NSB model with nonselenoidal velocity and inhomogeneous Dirichlet boundary condition. This non-Darcian model (see also [34]), which describes the behaviour of a fluid flow with high velocity through complex geometries with relative large pores, has diverse applications in the design of complex engineering systems, such as wind farms with closely placed turbines and porous breakwaters, like rubble-mound breakwaters in ports and coastal areas. Considering discontinuous parameters, due to the complexity of the phenomena's geometry, in [38] it is proved well-posedness of the continuous and discrete formulations, as well as the corresponding optimal convergence, by means of a suitable smallness assumption on the data.

In this paper we attempt to contribute to the development of new numerical methods for porous media flows problems, and propose and analyze an augmented mixed method for the NSB model studied in [38] considering, for the sake of simplicity, constants parameters. More precisely, we adopt the methodology developed in [7] and, by introducing an auxiliary unknown relating the gradient of the velocity and the pressure with the convective term (pseudostress tensor), and then eliminating the pressure from the system, we propose and analyze a conforming augmented pseudostress-velocity mixed method for the NSB model, providing optimal convergence when considering Raviart-Thomas elements of order $k \geq 0$ for the pseudostress and continuous piecewise polynomial elements of degree $k+1$ for the velocity.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of boundary layers and singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities Θ that are expressed in terms of local indicators Θ_T defined on each element T of a given triangulation $\mathcal{T}_h$. The estimator Θ is said to be efficient (resp. reliable) if there exists $C_{\text{eff}} > 0$ (resp. $C_{\text{rel}} > 0$), independent of the meshsizes, such that

$$C_{\text{eff}}\Theta + \text{h.o.t.} \leq \|\text{error}\| \leq C_{\text{rel}}\Theta + \text{h.o.t.},$$

where h.o.t. is a generic expression denoting one or several terms of higher order. According to the above, and with the purpose of complementing our contribution, we apply the techniques developed in [8, 9, 25, 32, 33] and derive a reliable and efficient residual-based a posteriori error estimator for our augmented mixed method.

The rest of this work is organized as follows. In Section 2 we introduce the model problem and derive the augmented mixed variational formulation. In Section 3 we analyze the well-posedness of the continuous problem by means of a fixed-point argument. Next, in Section 4 we define the Galerkin scheme considering generic finite dimensional subspaces and provide its unique solvability together with the corresponding Cea's estimate. A specific choice of finite element subspaces is introduced in Section 4.3. In Section 5 we employ a global continuous inf-sup condition, a Helmholtz decomposition, the local approximation properties of the Clément and Raviart-Thomas operators and derive a reliable and efficient residual-based a posteriori error estimator. Finally, several numerical results, illustrating the performance of the proposed mixed finite element method, confirming the reliability and efficiency of the a posteriori estimators, and showing the good behaviour of the associated adaptive algorithms, are provided in Section 6.

We end this section by introducing some definitions and fixing some notations. Given the vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, with $n \in \{2, 3\}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div}\mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

In addition, for any tensor field $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} := (\zeta_{ij})_{i,j=1,n}$, we let

$$\mathbf{div}\,\boldsymbol{\tau} = (\operatorname{div}(\tau_{i1}, \cdots, \tau_{in}))_{1\leq i\leq n},$$

and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\text{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\text{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathrm{I},$$

where I is the identity matrix in $\mathbb{R}^{n\times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Moreover, given an integer $k\geq 0$ and a set $S\subseteq\mathbb{R}^n$, $P_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, we set $\mathbf{P}_k(S) := [P_k(S)]^n$ and $\mathbb{P}_k(S) := [P_k(S)]^{n\times n}$.

In the sequel, we will utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, Γ is an open or closed Lipschitz curve (respectively surface in $\mathbb{R}^3$), and $s\in\mathbb{R}$, we define

$$\mathbf{H}^s(\mathcal{O}) := [H^s(\mathcal{O})]^n, \quad \mathbb{H}^s(\mathcal{O}) := [H^s(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^s(\Gamma) := [H^s(\Gamma)]^n.$$

However, when $s=0$ we usually write $\mathbf{L}^2(\mathcal{O}), \mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O}), \mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$ for $H^s(\mathcal{O})$, $\mathbf{H}^s(\mathcal{O})$ and $\mathbb{H}^s(\mathcal{O})$, and $\|\cdot\|_{s,\Gamma}$ for $H^s(\Gamma)$ and $\mathbf{H}^s(\Gamma)$. For $s\geq 0$, we write $|\cdot|_{s,\mathcal{O}}$ for the $\mathbf{H}^s$-seminorm. In addition, we recall that

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{\mathbf{w}\in\mathbf{L}^2(\mathcal{O}): \quad \mathrm{div}\,\mathbf{w}\in L^2(\mathcal{O})\right\},$$

is a standard Hilbert space in the realm of mixed problems (see, e.g. [2, 22]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$. Note that if $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\mathcal{O})$, then $\mathbf{div}\,\boldsymbol{\tau}\in\mathbf{L}^2(\mathcal{O})$ and $\boldsymbol{\tau}\mathbf{n}\in\mathbf{H}^{-1/2}(\partial\mathcal{O})$, where $\mathbf{n}$ denotes the outward unit vector normal to the boundary $\partial\mathcal{O}$. Note also that $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ can be characterized as the space of matrix valued functions $\boldsymbol{\tau}$ such that $\mathbf{c}^t\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ for any constant column vector $\mathbf{c}$. The norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div}\,;\mathcal{O}}$, respectively. In addition, we have the following decomposition:

$$\mathbb{H}(\mathbf{div}\,;\mathcal{O}) \,=\, \mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) \,\oplus\, P_0(\mathcal{O})\,\mathrm{I}\,, \tag{1.1}$$

where

$$\mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) \,:=\, \left\{\boldsymbol{\tau}\,\in\,\mathbb{H}(\mathbf{div}\,;\mathcal{O}): \quad \int_{\mathcal{O}}\mathrm{tr}\,\boldsymbol{\tau} = 0\right\}. \tag{1.2}$$

More precisely, each $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\mathcal{O})$ can be decomposed uniquely as:

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 \,+\, c\,\mathrm{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0\,\in\,\mathbb{H}_0(\mathbf{div}\,;\mathcal{O}) \quad \text{and} \quad c \,:=\, \frac{1}{n\,|\mathcal{O}|}\int_{\mathcal{O}}\mathrm{tr}\,\boldsymbol{\tau}\,\in\,\mathbb{R}\,. \tag{1.3}$$

For the sake of simplicity, in what follows we will use the notation

$$(u,v)_\Omega := \int_\Omega uv, \quad (\mathbf{u},\mathbf{v})_\Omega := \int_\Omega \mathbf{u}\cdot\mathbf{v}, \quad (\mathbf{u},\mathbf{v})_\Gamma := \int_\Gamma \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_\Omega := \int_\Omega \boldsymbol{\sigma}:\boldsymbol{\tau}.$$

Finally, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 Continuous problem

In this section we introduce the model problem, rewrite it as a first order set of equations, and derive and analyze the corresponding weak formulation.

2.1 The model problem

Let Ω be a bounded and simply connected open polyhedral domain in $\mathbb{R}^n$, $n \in \{2,3\}$ with boundary $\Gamma := \partial\Omega$ and denote by $\mathbf{n}$ the unit outward normal on Γ. In this work we are interested in deriving and analyzing an augmented mixed formulation for the Navier-Stokes-Brinkman problem consisting in finding a velocity vector field $\mathbf{u}$, and a pressure scalar field p, satisfying the set of partial differential equations:

$$\begin{aligned} \alpha\,\mathbf{u} - \nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= g && \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on} \quad \Gamma, \\ \int_\Omega p &= 0, \end{aligned} \tag{2.1}$$

where $\nu > 0$ is the dynamic fluid viscosity, $\alpha > 0$ is the fluid viscosity divided by the permeability, and $\mathbf{f}$ and g are given data representing external body forces and sources and/or sinks in Ω, respectively. Notice that, according to the nonselenoidal condition $\operatorname{div}\mathbf{u} = g$, the data $\mathbf{u}_D$ and g must satisfy the compatibility condition

$$\langle \mathbf{n}, \mathbf{u}_D\rangle_\Gamma - (g,1)_\Omega = 0, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ with respect to the $\mathbf{L}^2(\Gamma)$-inner product.

In order to derive our weak formulation, we first rewrite (2.1) as an equivalent first-order set of equations. To that end we introduce the pseudostress tensor

$$\boldsymbol{\sigma} := \nu\,\nabla\mathbf{u} \,-\, p\,\mathrm{I} \,-\, \mathbf{u}\otimes\mathbf{u} \quad \text{in} \quad \Omega, \tag{2.3}$$

and observe that the nonselenoidal condition $\operatorname{div}\mathbf{u} = \operatorname{tr}(\nabla\mathbf{u}) = g$ in Ω, implies

$$\mathbf{div}\,(\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\cdot\nabla)\,\mathbf{u} + g\,\mathbf{u} \qquad \text{in} \quad \Omega \quad \text{and} \quad \operatorname{tr}(\boldsymbol{\sigma}) = \nu g \,-\, np - \operatorname{tr}(\mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega. \tag{2.4}$$

In particular, from the second equation in (2.4) we observe that the pressure p can be written in terms of the velocity $\mathbf{u}$ and the tensor $\boldsymbol{\sigma}$, as

$$p := -\frac{1}{n}\left(\operatorname{tr}(\boldsymbol{\sigma}) + \operatorname{tr}(\mathbf{u}\otimes\mathbf{u}) - \nu\,g\right) \quad \text{in} \quad \Omega. \tag{2.5}$$

Then, eliminating the pressure p from the system, and proceeding similarly as in [7], we observe that (2.1) can be rewritten equivalently as the following first-order set of equations

$$\begin{aligned} \nu\,\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} - \frac{\nu}{n}\,g\,\mathrm{I} &= \boldsymbol{\sigma}^{\mathrm{d}} && \text{in} \quad \Omega, \\ \alpha\mathbf{u} - g\,\mathbf{u} - \mathbf{div}\,\boldsymbol{\sigma} &= \mathbf{f} && \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on} \quad \Gamma, \\ (\operatorname{tr}(\boldsymbol{\sigma}) + \operatorname{tr}(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega &= \nu\,(g,1)_\Omega. \end{aligned} \tag{2.6}$$

Notice that the pressure p can be easily computed as a postprocess of the solution by using (2.5). Notice also that further variables of interest, such as such the shear stress tensor $\tilde{\boldsymbol{\sigma}} =$

$\nu\,(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm t})-p\,\mathrm{I}$, the velocity gradient $\nabla\mathbf{u}$ and the vorticity $\boldsymbol{\omega}:=\frac{1}{2}\,(\nabla\mathbf{u}-(\nabla\mathbf{u})^{\mathrm t})$, can be easily computed, respectively, as follows

$$\begin{aligned}\tilde{\boldsymbol{\sigma}} &= \boldsymbol{\sigma}^{\mathrm d}\,+\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm d}+\boldsymbol{\sigma}^{\mathrm t}\,+\,\mathbf{u}\otimes\mathbf{u}\,+\,\frac{\nu}{n}\,g\,\mathrm{I},\\ \nabla\mathbf{u} &= \frac{1}{\nu}\Big(\boldsymbol{\sigma}^{\mathrm d}\,+\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm d}+\frac{\nu}{n}\,g\,\mathrm{I}\Big),\\ \boldsymbol{\omega} &= \frac{1}{2\,\nu}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathrm t}).\end{aligned} \tag{2.7}$$

2.2 The augmented mixed variational problem

In this section we proceed analogously to [7] and derive our weak formulation for (2.1). To do that we first test the first equation of (2.1) with arbitrary $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\Omega)$ integrate by parts, utilize the Dirichlet boundary condition $\mathbf{u}=\mathbf{u}_D$ on Γ and the identity $\boldsymbol{\sigma}^{\mathrm d}:\boldsymbol{\tau}=\boldsymbol{\sigma}^{\mathrm d}:\boldsymbol{\tau}^{\mathrm d}$, to obtain

$$(\boldsymbol{\sigma}^{\mathrm d},\boldsymbol{\tau}^{\mathrm d})_\Omega+\nu\,(\mathbf{u},\mathbf{div}\;\boldsymbol{\tau})_\Omega+(\mathbf{u}\otimes\mathbf{u},\boldsymbol{\tau}^{\mathrm d})_\Omega=-\frac{\nu}{n}\,(g,\mathrm{tr}\,(\boldsymbol{\tau}))_\Omega+\nu\,\langle\boldsymbol{\tau}\,\mathbf{n},\mathbf{u}_D\rangle_\Gamma\,. \tag{2.8}$$

In addition, differently from [7], with the purpose of avoiding the incorporation of new terms in the resulting variational equation, instead of incorporating the equilibrium equation weakly, we proceed similarly as in [26] and replace the identity

$$\mathbf{u}=\frac{1}{\alpha}\,(g\mathbf{u}+\mathbf{f}+\mathbf{div}\,\boldsymbol{\sigma}) \tag{2.9}$$

in the second term of (2.8), to obtain, consequently, the variational problem: Find $\boldsymbol{\sigma}\in\mathbb{H}(\mathbf{div}\,;\Omega)$ and $\mathbf{u}$ in a suitable space (to be specified below), such that $(\mathrm{tr}\,(\boldsymbol{\sigma})+\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1)_\Omega\;=\;\nu\,(g,1)_\Omega$, and

$$\begin{aligned}(\boldsymbol{\sigma}^{\mathrm d},\boldsymbol{\tau}^{\mathrm d})_\Omega\;+\;\frac{\nu}{\alpha}(\mathbf{div}\,\boldsymbol{\sigma},\mathbf{div}\,\boldsymbol{\tau})_\Omega\;+\;\frac{\nu}{\alpha}(g\mathbf{u},\mathbf{div}\;\boldsymbol{\tau})_\Omega\;+\;(\mathbf{u}\otimes\mathbf{u},\boldsymbol{\tau}^{\mathrm d})_\Omega\\ =\;-\frac{\nu}{\alpha}(\mathbf{f},\mathbf{div}\,\boldsymbol{\tau})_\Omega\;-\;\frac{\nu}{n}(g,\mathrm{tr}\,(\boldsymbol{\tau}))_\Omega\;+\;\nu\,\langle\boldsymbol{\tau}\,\mathbf{n},\mathbf{u}_D\rangle_\Gamma\,,\end{aligned} \tag{2.10}$$

for all $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}\,;\Omega)$.

Notice that the term $(\mathbf{u}\otimes\mathbf{u},\boldsymbol{\tau}^{\mathrm d})_\Omega$ in (2.10) requires the velocity $\mathbf{u}$ to live in a smaller space than $\mathbf{L}^2(\Omega)$. In fact, by applying the Cauchy–Schwarz and Hölder inequalities and then the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we find that

$$|(\mathbf{u}\otimes\mathbf{w},\mathbf{r}^{\mathrm d})_\Omega|\le\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^4(\Omega)}\|\mathbf{r}\|_{0,\Omega}\le\|\mathbf{i}_c\|^2\|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}\|\mathbf{r}\|_{0,\Omega}$$

for all $\mathbf{u},\mathbf{w}\in\mathbf{H}^1(\Omega)$ and $\mathbf{r}\in\mathbf{L}^2(\Omega)$. According to this, we propose to look for the unknown $\mathbf{u}\in\mathbf{H}^1(\Omega)$. Notice also from (2.10) that the lack of a test function in the space where $\mathbf{u}$ lives (now in $\mathbf{H}^1(\Omega)$), makes the well-posedness analysis of (2.10) non-viable. Then, in order to be able to carry out the existence and uniqueness analysis of (2.10) we propose to enrich our formulation with the following residual terms arising from the constitutive equation (first equation of (2.6)) and the Dirichlet boundary condition:

$$\kappa_1\,\Big(\nu\,\nabla\mathbf{u}\,-\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm d}\,-\,\boldsymbol{\sigma}^{\mathrm d},\nabla\mathbf{v}\Big)_\Omega=\kappa_1\,\frac{\nu}{n}\,(g,\mathrm{div}\,\mathbf{v})_\Omega, \tag{2.11}$$

$$\kappa_2\,\left(\mathbf{u},\mathbf{v}\right)_\Gamma=\kappa_2\,\left(\mathbf{u}_D,\mathbf{v}\right)_\Gamma, \tag{2.12}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, where κ_1, κ_2 are positive parameters to be specified later. Therefore, from (2.10), (2.11) and (2.12) we arrive at the variational problem: Find $\underline{\boldsymbol{\phi}} := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X} := \mathbb{H}(\mathbf{div}\,;\Omega) \times \mathbf{H}^1(\Omega)$, such that $(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega \;=\; \nu\,(g,1)_\Omega$, and

$$\mathbf{A}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}}) \;+\; \mathbf{C}_{\mathbf{u}}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}}) \;=\; \mathbf{F}(\underline{\boldsymbol{\psi}}) \qquad \forall\; \underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}, \tag{2.13}$$

where, the product space $\mathbf{X}$ is endowed with the norm

$$\|\underline{\boldsymbol{\psi}}\|^2_{\mathbf{X}} := \|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega} \;+\; \|\mathbf{v}\|^2_{1,\Omega} \quad \forall\; \underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X},$$

and given $\mathbf{z} \in \mathbf{H}^1(\Omega)$, the forms $\mathbf{A} : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$, $\mathbf{C}_{\mathbf{z}} : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$, and the functional $\mathbf{F} : \mathbf{X} \to \mathbb{R}$ are defined as

$$\begin{aligned}
\mathbf{A}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}}) :=\;\; & (\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega + \frac{\nu}{\alpha}\,(g\mathbf{u}, \mathbf{div}\,\boldsymbol{\tau})_\Omega + \frac{\nu}{\alpha}\,(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{div}\,\boldsymbol{\tau})_\Omega \\
& + \nu\,\kappa_1\,(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega \;-\; \kappa_1\,(\boldsymbol{\sigma}^{\mathrm{d}}, \nabla\mathbf{v})_\Omega + \kappa_2\,(\mathbf{u},\mathbf{v})_\Gamma \\
\mathbf{C}_{\mathbf{z}}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}}) \;\; & := (\mathbf{u}\otimes\mathbf{z}, \boldsymbol{\tau}^{\mathrm{d}})_\Omega - \kappa_1\,((\mathbf{u}\otimes\mathbf{z})^{\mathrm{d}}, \nabla\mathbf{v})_\Omega,
\end{aligned} \tag{2.14}$$

for all $\boldsymbol{\phi} = (\boldsymbol{\sigma}, \mathbf{u})$, $\underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v})$ in $\mathbf{X}$, and

$$\begin{aligned}
\mathbf{F}(\underline{\boldsymbol{\psi}}) \;\;=\;\; & -\frac{\nu}{\alpha}\,(\mathbf{f}, \mathbf{div}\,\boldsymbol{\tau})_\Omega - \frac{\nu}{n}\,(g, \mathrm{tr}\,(\boldsymbol{\tau}))_\Omega + \nu\,\langle\boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \\
& + \frac{\kappa_1\,\nu}{n}(g, \mathrm{div}\,\mathbf{v})_\Omega + \kappa_2\,(\mathbf{u}_D, \mathbf{v})_\Gamma, \quad \forall\,\underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \;\in\; \mathbf{X}.
\end{aligned} \tag{2.15}$$

3 Analysis of the continuous problem

Before beginning with the well-posedness analysis of (2.13), let us assume for a moment that (2.13) posses a solution $\underline{\boldsymbol{\phi}} := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}$. Then defining the tensor

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \frac{1}{n\,|\Omega|}\,((\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega - \nu\,(g,1)_\Omega)\,\mathrm{I}, \tag{3.1}$$

we observe that $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$, if and only if,

$$(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega = \nu\,(g,1)_\Omega. \tag{3.2}$$

Then, similarly to [7], after simple computations it can be proved that if (3.2) holds, then $\underline{\boldsymbol{\phi}}_0 := (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0 := \mathbb{H}_0(\mathbf{div}\,;\Omega) \times \mathbf{H}^1(\Omega)$ is a solution to the problem: Find $\underline{\boldsymbol{\phi}}_0 := (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$, such that

$$\mathbf{A}(\underline{\boldsymbol{\phi}}_0, \underline{\boldsymbol{\psi}}) \;+\; \mathbf{C}_{\mathbf{u}}(\underline{\boldsymbol{\phi}}_0, \underline{\boldsymbol{\psi}}) \;=\; \mathbf{F}(\underline{\boldsymbol{\psi}}) \qquad \forall\; \underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}_0. \tag{3.3}$$

Even more, problems (2.13) and (3.3) are equivalent. This result, which is similar to [7, Lemma 2.2], is established now.

Lemma 3.1 *Problems* (2.13) *and* (3.3) *are equivalent in the following sense:*
(i) If $\underline{\boldsymbol{\phi}} = (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}$ *is a solution of* (2.13) *satisfying* (3.2)*, then* $\underline{\boldsymbol{\phi}}_0 = (\boldsymbol{\sigma}_0, \mathbf{u})$*, with* $\boldsymbol{\sigma}_0$ *defined by* (3.1)*, is a solution of* (3.3)*.*
(ii) If $\underline{\boldsymbol{\phi}}_0 = (\boldsymbol{\sigma}_0, \mathbf{u})$ *is a solution of* (3.3)*, and* $\boldsymbol{\sigma}$ *is defined by the relation* (3.1)*, then* $\underline{\boldsymbol{\phi}} = (\boldsymbol{\sigma}, \mathbf{u})$*, is a solution of* (2.13)*.*

Proof. The proof follows straightforwardly by applying the same arguments utilized in the proof of [7, Lemma 2.2]. We omit further details. □

As a consequence of the previous lemma, in what follows we focus on analyzing problem (3.3). To that end we first review the stability properties of the forms and functional involved.

3.1 Stability

We start the analysis by establishing the continuity of the forms $\mathbf{A}$, $\mathbf{C}_{\mathbf{z}}$, and the functional $\mathbf{F}$. First, for the continuity of $\mathbf{A}$ we recall the following estimate, which is a direct consequence of the well known generalized Poincaré inequality (see e.g. [14] or Chapter I in [22]),

$$C_p\,\|v\|_{1,\Omega}^2 \le |v|_{1,\Omega}^2 + \|v\|_{0,\Gamma}^2 \le \tilde{C}_p\,\|v\|_{1,\Omega}^2 \quad \forall\, v \in \mathrm{H}^1(\Omega), \tag{3.4}$$

with C_p and $\tilde{C}_p$, only depending on Ω. Then, from (3.4), the Hölder's inequality, the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, and the fact that $\|\boldsymbol{\sigma}_0^{\mathrm{d}}\|_{0,\Omega} \le \|\boldsymbol{\sigma}_0\|_{0,\Omega}$, it is not difficult to see that

$$|\mathbf{A}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}})| \le C_{\mathbf{A}} \|\underline{\boldsymbol{\phi}}\|_{\mathbf{X}} \|\underline{\boldsymbol{\psi}}\|_{\mathbf{X}}, \tag{3.5}$$

for all $\underline{\boldsymbol{\phi}}$, $\underline{\boldsymbol{\psi}} \in \mathbf{X}$ with $C_{\mathbf{A}} = 2\max\left\{\left(1+\frac{\nu}{\alpha}\right), \|\mathbf{i}_c\|\,\frac{\nu}{\alpha}\,\|g\|_{L^4(\Omega)}, \kappa_1, \tilde{C}_p\,\max\{\nu\kappa_1, \kappa_2\}\right\} > 0$.

Next, for $\mathbf{C}_{\mathbf{z}}$ we apply the Cauchy–Schwarz and Hölder inequalities, and the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, to easily obtain

$$\begin{aligned} |\mathbf{C}_{\mathbf{z}}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}})| \quad &\le (1+\kappa_1^2)^{1/2}\,\|\mathbf{z}\|_{\mathbf{L}^4(\Omega)}\,\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\,\|\underline{\boldsymbol{\psi}}\|_{\mathbf{X}} \\ &\le (1+\kappa_1^2)^{1/2}\,\|\mathbf{i}_c\|^2\,\|\mathbf{z}\|_{1,\Omega}\,\|\mathbf{u}\|_{1,\Omega}\,\|\underline{\boldsymbol{\psi}}\|_{\mathbf{X}} \\ &\le C_{\mathbf{C}}\,\|\mathbf{z}\|_{1,\Omega}\,\|\underline{\boldsymbol{\phi}}\|_{\mathbf{X}}\,\|\underline{\boldsymbol{\psi}}\|_{\mathbf{X}}, \end{aligned} \tag{3.6}$$

for all $\mathbf{z} \in \mathbf{H}^1(\Omega)$, and for all $\underline{\boldsymbol{\phi}} = (\boldsymbol{\sigma}, \mathbf{u})$, $\underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$, where $C_{\mathbf{C}} := (1+\kappa_1^2)^{1/2}\,\|\mathbf{i}_c\|^2$.

Finally, for $\mathbf{F}$ we apply again Hölder's inequality, the trace inequality $\|\mathbf{v}\|_{0,\Gamma} \le C_\Gamma \|\mathbf{v}\|_{1,\Omega}$, the continuity of the normal trace $|\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma| \le \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\|\mathbf{u}_D\|_{1/2,\Gamma}$, and the fact that $\|\mathrm{div}\,\mathbf{v}\|_{0,\Omega}^2 \le n\|\mathbf{v}\|_{1,\Omega}^2$ and $\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}^2 \le n\|\boldsymbol{\tau}\|_{0,\Omega}^2$, to obtain

$$\begin{aligned} |\mathbf{F}(\underline{\boldsymbol{\psi}})| \le \frac{\nu}{\alpha}\,\|\mathbf{f}\|_{0,\Omega}\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} + \frac{\nu}{n}\,\|g\|_{0,\Omega}\,\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega} + \nu\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,\|\mathbf{u}_D\|_{1/2,\Gamma} \\ + \frac{\kappa_1\,\nu}{n}\,\|g\|_{0,\Omega}\,\|\mathrm{div}\,\mathbf{v}\|_{0,\Omega} + \kappa_2 C_\Gamma\,\|\mathbf{u}_D\|_{0,\Gamma}\|\mathbf{v}\|_{1,\Omega}, \\ \le C_{\mathbf{F}}(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{u}_D\|_{0,\Gamma})\|\underline{\boldsymbol{\psi}}\|_{\mathbf{X}} \end{aligned} \tag{3.7}$$

for all $\underline{\boldsymbol{\psi}} = (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$, with

$$C_{\mathbf{F}} = \sqrt{2}\max\left\{\frac{\nu}{\alpha}, \frac{\nu}{\sqrt{n}}(1+\kappa_1^2), \nu, C_\Gamma\kappa_2\right\}.$$

Let us now recall the following well known inequality. For its proof we refer to Lemma 3.1 in [1] or chapter IV in [2].

$$C_d\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \le \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega), \tag{3.8}$$

with C_d only depending on Ω. Owing to the inequality above, and proceeding analogously to [7, Lemma 3.1], it can be proved the ellipticity of $\mathbf{A}$.

Lemma 3.2 *Assume that* $0 < \kappa_1 < 2\nu,\ \kappa_2 > 0$. *Assume further that* $g \in L^4(\Omega)$ *and satisfies*

$$\|g\|^2_{L^4(\Omega)} \leq \frac{C_p \alpha}{\nu \|\mathbf{i}_c\|^2} \min\left\{\frac{\kappa_1}{2}(2\nu - \kappa_1), \kappa_2\right\}. \tag{3.9}$$

Then there exists $\alpha_{\mathbf{A}} > 0$, *such that*

$$\mathbf{A}(\underline{\psi}, \underline{\psi}) \geq \alpha_{\mathbf{A}} \|\underline{\psi}\|^2_{\mathbf{X}} \qquad \forall\ \underline{\psi}\ \in\ \mathbf{X}_0. \tag{3.10}$$

Proof. Given $\underline{\psi} := (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$, we utilize the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, and the Hölder's inequality, to obtain

$$\begin{aligned}
\mathbf{A}(\underline{\psi}, \underline{\psi}) &= \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \frac{\nu}{\alpha}(g\,\mathbf{v}, \mathbf{div}\ \boldsymbol{\tau})_\Omega + \frac{\nu}{\alpha}\|\mathbf{div}\ \boldsymbol{\tau}\|^2_{0,\Omega} + \nu\,\kappa_1\,|\mathbf{v}|^2_{1,\Omega} - \kappa_1\,(\boldsymbol{\tau}^{\mathrm{d}}, \nabla\mathbf{v})_\Omega + \kappa_2\,\|\mathbf{v}\|^2_{0,\Gamma} \\
&\geq \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} - \left(\sqrt{\nu\alpha^{-1}}\,\|\mathbf{i}_c\|\,\|g\|_{L^4(\Omega)}\,\|\mathbf{v}\|_{1,\Omega}\right)\left(\sqrt{\nu\alpha^{-1}}\,\|\mathbf{div}\ \boldsymbol{\tau}\|_{0,\Omega}\right) + \frac{\nu}{\alpha}\,\|\mathbf{div}\ \boldsymbol{\tau}\|^2_{0,\Omega} \\
&\quad + \nu\,\kappa_1\,\|\nabla\mathbf{v}\|^2_{0,\Omega} - \left(\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\right)\left(\kappa_1\,\|\nabla\mathbf{v}\|_{0,\Omega}\right) + \kappa_2\,\|\mathbf{v}\|^2_{0,\Gamma} \\
&\geq \frac{1}{2}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \frac{\nu}{2\,\alpha}\,\|\mathbf{div}\ \boldsymbol{\tau}\|^2_{0,\Omega} + \frac{\kappa_1}{2}\,(2\,\nu - \kappa_1)\,\|\nabla\mathbf{v}\|^2_{0,\Omega} + \kappa_2\,\|\mathbf{v}\|^2_{0,\Gamma} \\
&\quad - \frac{\nu}{2\,\alpha}\,\|\mathbf{i}_c\|^2\,\|g\|^2_{L^4(\Omega)}\,\|\mathbf{v}\|^2_{1,\Omega}\,,
\end{aligned}$$

which together to the estimates (3.4), (3.8) and assumption (3.9), implies (3.10) with

$$\alpha_{\mathbf{A}} = \frac{1}{2}\min\left\{C_d \min\left\{1, \frac{\nu}{2\,\alpha}\right\}, \frac{\nu}{2\alpha}, C_p \min\left\{\frac{\kappa_1}{2}(2\nu - \kappa_1), \kappa_2\right\}\right\}.$$

□

3.2 Existence and uniqueness of solution

In this section we prove the well-posedness of problem (3.3) by means of a fixed-point strategy. More precisely, in what follows we rewrite problem (3.3) as an equivalent fixed-point problem and apply the Banach's fixed point theorem to conclude the desired unique solvability.

We begin by introducing the bounded set

$$\mathbf{K} := \left\{\mathbf{z} \in \mathbf{H}^1(\Omega)\ :\ \|\mathbf{z}\|_{1,\Omega}\ \leq\ \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right)\right\}, \tag{3.11}$$

and the mapping

$$\mathcal{J} : \mathbf{K} \to \mathbf{K}, \quad \mathbf{z} \to \mathcal{J}(\mathbf{z}) = \mathbf{u}, \tag{3.12}$$

where $\mathbf{u} \in \mathbf{H}^1(\Omega)$ is the second component of $\underline{\phi}_0 = (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$, satisfying

$$\mathbf{A}(\underline{\phi}_0, \underline{\psi})\ +\ \mathbf{C}_{\mathbf{z}}(\underline{\phi}_0, \underline{\psi}) = \mathbf{F}(\underline{\psi}) \quad \forall\ \underline{\psi} \in \mathbf{X}_0. \tag{3.13}$$

It is clear that $\underline{\phi}_0 \in \mathbf{X}_0$ is a solution to (3.3), if and only if, $\mathcal{J}(\mathbf{u}) = \mathbf{u}$. Accordingly, to prove the well-posedness of problem (2.13), it suffices prove that $\mathcal{J}$ has a unique fixed point in $\mathbf{K}$.

Before continuing, we first need to verify that $\mathcal{J}$ is a well defined operator. This result is established now under a sufficiently small data assumption.

Lemma 3.3 *Assume that the assumptions of Lemma 3.2 hold. Assume further that*

$$\frac{4\,C_{\mathbf{C}}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2}\,\Big(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big) \leq 1 \tag{3.14}$$

with $\mathbf{C}_{\mathbf{C}}$ *and* $\mathbf{C}_{\mathbf{F}}$ *being the continuity constants of* $\mathbf{C}$ *and* $\mathbf{F}$*, respectively (cf.* (3.6) *and* (3.7)*). Then, given* $\mathbf{z} \in \mathbf{K}$*, there exists a unique* $\mathbf{u} \in \mathbf{K}$ *such that* $\mathcal{J}(\mathbf{z}) = \mathbf{u}$*.*

Proof. Let $\mathbf{z} \in \mathbf{K}$. We first notice that, owing to assumption (3.14), there holds

$$\|\mathbf{z}\|_{1,\Omega} \leq \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\,\big(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \leq \frac{\alpha_{\mathbf{A}}}{2\,C_{\mathbf{C}}}. \tag{3.15}$$

In turn, from inequalities (3.6) and (3.10), we easily obtain

$$\begin{aligned} \mathbf{A}(\underline{\psi},\underline{\psi}) + \mathbf{C}_{\mathbf{z}}(\underline{\psi},\underline{\psi}) \geq &\quad \mathbf{A}(\underline{\psi},\underline{\psi}) \,-\, |\mathbf{C}_{\mathbf{z}}(\underline{\psi},\underline{\psi})| \qquad \forall\, \underline{\psi} \in \mathbf{X}_0,\\ \geq &\quad \big(\alpha_{\mathbf{A}} \,-\, \mathbf{C}_{\mathbf{C}}\,\|\mathbf{z}\|_{1,\Omega}\big)\,\|\underline{\psi}\|_{\mathbf{X}}^2 \qquad \forall\, \underline{\psi} \in \mathbf{X}_0. \end{aligned} \tag{3.16}$$

Then, combining (3.15) and (3.16), it readily follows that

$$\mathbf{A}(\underline{\psi},\underline{\psi}) + \mathbf{C}_{\mathbf{z}}(\underline{\psi},\underline{\psi}) \geq \frac{\alpha_{\mathbf{A}}}{2}\,\|\underline{\psi}\|_{\mathbf{X}}^2 \quad \underline{\psi} \in \mathbf{X}_0, \tag{3.17}$$

which implies that the bilinear form $\mathbf{A}(\cdot,\cdot) + \mathbf{C}_{\mathbf{z}}(\cdot,\cdot)$ is elliptic on $\mathbf{X}_0$. Therefore, applying Lax-Milgram lemma we obtain that there exists a unique $\underline{\phi}_0 = (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$ satisfying (3.13), or equivalently, there exists a unique $\mathbf{u} \in \mathbf{H}^1(\Omega)$, such that $\mathcal{J}(\mathbf{z}) = \mathbf{u}$. In addition, from (3.7), (3.13) and (3.17), it is easy to see that

$$\begin{aligned} \frac{\alpha_{\mathbf{A}}}{2}\,\|\underline{\phi}_0\|_{\mathbf{X}}^2 \leq &\quad \mathbf{A}(\underline{\phi}_0,\underline{\phi}_0) + \mathbf{C}_{\mathbf{z}}(\underline{\phi}_0,\underline{\phi}_0) = \mathbf{F}(\underline{\phi}_0)\\ \leq &\quad C_{\mathbf{F}}\,\big(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big)\,\|\underline{\phi}_0\|_{\mathbf{X}}, \end{aligned}$$

from which

$$\|\mathbf{u}\|_{1,\Omega} \leq \|\underline{\phi}_0\|_{\mathbf{X}} \leq \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\,\big(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big)\,,$$

which implies that $\mathbf{u}$ belongs to $\mathbf{K}$ and concludes the proof. □

We are now in position of establishing the main result of this section, namely, the well-posedness of problem (3.3).

Theorem 3.4 *Let* $\mathbf{f} \in \mathbf{L}^2(\Omega)$*,* $g \in L^4(\Omega)$ *and* $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$*. Assume that hypotheses of Lemma 3.2 hold. Assume further that*

$$\frac{4\,C_{\mathbf{C}}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2}\,\Big(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big) \leq \frac{1}{2}. \tag{3.18}$$

Then, there exists a unique $\underline{\phi}_0 \in \mathbf{X}_0$ solution to (3.3)*. In addition, the solution $\underline{\phi}_0$ satisfies the a priori estimate*

$$\|\underline{\phi}_0\|_{\mathbf{X}} \,\le\, \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right). \tag{3.19}$$

Proof. First, let us observe that assumption (3.18) clearly implies (3.14). Consequently, from Lemma 3.3 it follows that operator $\mathcal{J}$ is well defined.

Now, since the solvability of problem (3.3) is equivalent to the existence and uniqueness of a fixed point to $\mathcal{J}$, to conclude the required solvability it suffices to prove that $\mathcal{J}$ is a contraction mapping and apply the classical Banach's fixed point theorem.

Let $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{u}_1$, $\mathbf{u}_2 \,\in\, \mathbf{K}$, such that $\mathbf{u}_1 = \mathcal{J}(\mathbf{z}_1)$ and $\mathbf{u}_2 = \mathcal{J}(\mathbf{z}_2)$, and let $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$, such that $\underline{\phi}_1 = (\boldsymbol{\sigma}_1, \mathbf{u}_1) \in \mathbf{X}_0$ and $\underline{\phi}_2 = (\boldsymbol{\sigma}_2, \mathbf{u}_2) \in \mathbf{X}_0$ satisfy

$$\mathbf{A}(\underline{\phi}_i, \underline{\psi}) \,+\, \mathbf{C}_{\mathbf{z}_i}(\underline{\phi}_i, \underline{\psi}) = \mathbf{F}(\underline{\psi}) \quad \forall\, \underline{\psi} \in \mathbf{X}_0, \quad i = 1, 2. \tag{3.20}$$

Notice that for $i = 1, 2$, the bilinear forms $\mathbf{A}(\cdot,\cdot) + \mathbf{C}_{\mathbf{z}_i}(\cdot,\cdot)$ are elliptic on $\mathbf{X}_0$ and the pairs $\underline{\phi}_i = (\boldsymbol{\sigma}_i, \mathbf{u}_i)$ satisfy the estimates

$$\|\underline{\phi}_i\|_{\mathbf{X}} \le \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right). \tag{3.21}$$

In turn, from (3.20) we readily obtain

$$\mathbf{A}(\underline{\phi}_1 - \underline{\phi}_2, \underline{\psi}) \,+\, \mathbf{C}_{\mathbf{z}_1}(\underline{\phi}_1, \underline{\psi}) - \mathbf{C}_{\mathbf{z}_2}(\underline{\phi}_2, \underline{\psi}) \,=\, 0 \qquad \forall\, \underline{\psi} \in \mathbf{X}_0,$$

from which, after adding and subtracting suitable terms, and since $\underline{\phi}_1 - \underline{\phi}_2 \in \mathbf{X}_0$, we arrive at

$$\mathbf{A}(\underline{\phi}_1 - \underline{\phi}_2, \underline{\phi}_1 - \underline{\phi}_2) + \mathbf{C}_{\mathbf{z}_2}(\underline{\phi}_1 - \underline{\phi}_2, \underline{\phi}_1 - \underline{\phi}_2) \,=\, -\mathbf{C}_{\mathbf{z}_1 - \mathbf{z}_2}(\underline{\phi}_1, \underline{\phi}_1 - \underline{\phi}_2). \tag{3.22}$$

Then, utilizing the ellipticity of $\mathbf{A}(\cdot,\cdot) + \mathbf{C}_{\mathbf{z}_2}(\cdot,\cdot)$ (cf. (3.17)), the continuity of $\mathbf{C}$ (cf. (3.6)) and the estimate (3.21), from (3.22) we get

$$\begin{aligned}\|\underline{\phi}_1 - \underline{\phi}_2\|_{\mathbf{X}} &\le \frac{2\,C_{\mathbf{C}}}{\alpha_{\mathbf{A}}}\, \|\underline{\phi}_1\|_{\mathbf{X}}\, \|\underline{\zeta}_1 - \underline{\zeta}_2\|_{\mathbf{X}} \\ &\le \frac{4\,C_{\mathbf{C}}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2}\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right)\|\underline{\zeta}_1 - \underline{\zeta}_2\|_{\mathbf{X}},\end{aligned} \tag{3.23}$$

which together to assumption (3.18) implies that $\mathcal{J}$ is a contraction mapping. In this way, applying the Banach's fixed point theorem we straightforwardly obtain the existence unique solvability of problem (2.13).

Finally, since $\mathbf{u}$ belongs to $\mathbf{K}$, by applying the same steps at the end of the proof of Lemma 3.3 we can easily obtain that $\underline{\phi}_0$ satisfies (3.19), which concludes the proof. □

We conclude this section by establishing the well-posedness of problem (2.13).

Corollary 3.5 *Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^4(\Omega)$ and $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$. Assume that hypotheses of Theorem 3.4 hold. Then, there exists a unique $\underline{\phi} \in \mathbf{X}$ solution to* (2.13)*. In addition, the solution $\underline{\phi}$ satisfies the a priori estimate*

$$\|\underline{\phi}\|_{\mathbf{X}} \,\le\, \frac{2^{3/2}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right) + \sqrt{2}\left(\frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}} + \nu\sqrt{n}|\Omega|\right)\|g\|_{0,\Omega}. \tag{3.24}$$

Proof. It is clear that the existence and uniqueness of solution of problem (2.13) follows straightforwardly from Lemma 3.1 and Theorem 3.4. Now for the estimate (3.24), let $c = \frac{1}{n|\Omega|}\left((\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1)_\Omega - \nu\,(g,1)_\Omega\right)$, and let $\underline{\boldsymbol{\phi}} = (\boldsymbol{\sigma}_0 - c\mathrm{I}, \mathbf{u}) \in \mathbf{X}$ be the solution of (2.13), with $\underline{\boldsymbol{\phi}}_0 = (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$ being the solution of (3.3). After simple computations it is easy to see that

$$|c| \le \frac{1}{n|\Omega|^{1/2}}\|\mathbf{u}\|_{0,\Omega} + \nu|\Omega|^{1/2}\|g\|_{0,\Omega} \quad \text{and} \quad \|\boldsymbol{\sigma}_0 + c\mathrm{I}\|^2_{\mathbf{div}\,;\Omega} = \|\boldsymbol{\sigma}_0\|^2_{\mathbf{div}\,;\Omega} + n|\Omega|c^2,$$

and both combined imply

$$\|\boldsymbol{\sigma}_0 + c\mathrm{I}\|^2_{\mathbf{div}\,;\Omega} \le \|\underline{\boldsymbol{\phi}}_0\|^2_{\mathbf{X}} + 2n\nu^2|\Omega|^2\|g\|^2_{0,\Omega}.$$

The latter inequality and estimate (3.19) imply (3.24). □

4 The Galerkin scheme

In this section we introduce and analyze the Galerkin scheme associated to problem (3.3). We notice in advance that most of the details are omitted since the analysis of the corresponding discrete problem follows straightforwardly by adapting the fixed-point strategy introduced and analyzed in Section 3.

4.1 Discrete problem

We start by considering the generic finite dimensional subspaces

$$\mathbf{H}_h(\mathrm{div}\,;\Omega) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega), \qquad \mathrm{H}^1_h(\Omega) \subseteq \mathrm{H}^1(\Omega), \tag{4.1}$$

the discrete spaces

$$\begin{aligned} \mathbb{H}_h :=& \;\; \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}\,;\Omega) \,:\, \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h \in \mathbf{H}_h(\mathrm{div}\,;\Omega) \quad \forall\, \mathbf{c} \in \mathbb{R}^n\right\}, \\ \mathbb{H}_{h,0} :=& \;\; \mathbb{H}_h \cap \mathbb{H}_0(\mathbf{div}\,;\Omega), \\ \mathbf{H}^1_h :=& \;\; [\mathrm{H}^1_h(\Omega)]^n, \end{aligned} \tag{4.2}$$

and the global discrete spaces

$$\mathbf{X}_h := \mathbb{H}_h \times \mathbf{H}^1_h \quad \text{and} \quad \mathbf{X}_{h,0} := \mathbb{H}_{h,0} \times \mathbf{H}^1_h.$$

Hereafter, h stands for the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra (when $n = 3$) of diameter h_K, defined as $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, in order to have meaningful subspace $\mathbb{H}_{h,0}$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h$. This request is certainly satisfied if we assume that:

$$[P_0(\Omega)]^{n\times n} \subseteq \mathbb{H}_h. \tag{4.3}$$

In this way, the Galerkin scheme of (3.3) reads: Find $\underline{\boldsymbol{\phi}}_{h,0} = (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_h$, such that

$$\mathbf{A}(\underline{\boldsymbol{\phi}}_{h,0}, \underline{\boldsymbol{\psi}}_h) \;+\; \mathbf{C}_{\mathbf{u}_h}(\underline{\boldsymbol{\phi}}_{h,0}, \underline{\boldsymbol{\psi}}_h) \;=\; \mathbf{F}(\underline{\boldsymbol{\psi}}_h) \qquad \forall\, \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_{h,0}. \tag{4.4}$$

As for the continuous case, it is not difficult to see that problem (4.4) is equivalent to the discrete version of (2.13): Find $\underline{\phi}_h = (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_h$, such that $(\mathrm{tr}\,(\boldsymbol{\sigma}_h) + \mathrm{tr}\,(\mathbf{u}_h \otimes \mathbf{u}_h), 1)_\Omega = \nu\,(g, 1)_\Omega$, and

$$\mathbf{A}(\underline{\phi}_h, \underline{\psi}_h) \;+\; \mathbf{C}_{\mathbf{u}_h}(\underline{\phi}_h, \underline{\psi}_h) \;=\; \mathbf{F}(\underline{\psi}_h) \qquad \forall\, \underline{\psi}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_h, \tag{4.5}$$

through the relation:

$$\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_{h,0} - \frac{1}{n\,|\Omega|}(\mathrm{tr}\,(\mathbf{u}_h \otimes \mathbf{u}_h) - \nu\,(g, 1)_\Omega, 1)_\Omega \mathrm{I}. \tag{4.6}$$

This result, which is analogous to Lemma 3.1, is established now.

Lemma 4.1 *If $\underline{\phi}_h = (\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is a solution of (4.5), then $\underline{\phi}_{h,0} = (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h)$, with $\boldsymbol{\sigma}_{h,0}$ defined through (4.6) is a solution of (4.4). Conversely, if $\underline{\phi}_{h,0} = (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h)$ is a solution of (4.4), then $\underline{\phi}_h = (\boldsymbol{\sigma}_h, \mathbf{u}_h)$, with $\boldsymbol{\sigma}_h$ given by the relation (4.6) is a solution of (4.5).*

According to the latter result, after computing the solution of problem (4.4) one can easily recover the solution of problem by using the formula (4.6).

4.2 Well-posedness of the discrete problem

In this section we address the solvability analysis of problem (4.4) by adapting the fixed-point strategy introduced and analyzed in Section 3.2. We start by observing that the boundedness of the forms $\mathbf{A}$, $\mathbf{C}_{\mathbf{z}}$, and the functional $\mathbf{F}$, namely (3.5), (3.6) and (3.7), are clearly inherited from the continuous case. In turn, the ellipticity of $\mathbf{A}$ on $\mathbf{X}_{h,0}$, whose proof is almost verbatim to that of the corresponding continuous estimate provided by Lemma 3.2, holds with the same constant $\alpha_{\mathbf{A}} > 0$.

Lemma 4.2 *Assume that the assumptions of Lemma 3.2 hold. Then there exists $\alpha_{\mathbf{A}} > 0$, such that*

$$\mathbf{A}(\underline{\psi}_h, \underline{\psi}_h) \geq \alpha_{\mathbf{A}} \|\underline{\psi}_h\|^2_{\mathbf{X}} \qquad \forall\, \underline{\psi}_h \in \mathbf{X}_{h,0}. \tag{4.7}$$

Now, similarly to the continuous case, let us define the bounded finite dimensional set

$$\mathbf{K}_h := \left\{ \mathbf{z}_h \in \mathbf{H}^1_h \,:\, \|\mathbf{z}_h\|_{1,\Omega} \,\leq\, \frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}} \left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right) \right\}, \tag{4.8}$$

and the mapping

$$\mathcal{J}_h : \mathbf{K}_h \to \mathbf{K}_h, \quad \mathbf{z}_h \to \mathcal{J}_h(\mathbf{z}_h) = \mathbf{u}_h, \tag{4.9}$$

where $\mathbf{u}_h \in \mathbf{H}^1_h$ is the second component of $\underline{\phi}_{h,0} = (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_{h,0}$, satisfying

$$\mathbf{A}(\underline{\phi}_{h,0}, \underline{\psi}_h) \,+\, \mathbf{C}_{\mathbf{z}_h}(\underline{\phi}_{h,0}, \underline{\psi}_h) = \mathbf{F}(\underline{\psi}_h) \quad \forall\, \underline{\psi} \in \mathbf{X}_{h,0}. \tag{4.10}$$

As for the continuous case, it can be proved that $\mathcal{J}_h$ is well defined. The proof of this result is analogous to that of its continuous counterpart given by Lemma 3.3.

Lemma 4.3 *Assume that the assumptions of Lemma 3.2 hold. Assume further that*

$$\frac{4\,C_{\mathbf{C}}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2}\,\Big(\|\mathbf{f}\|_{0,\Omega}+\|g\|_{0,\Omega}+\|\mathbf{u}_D\|_{0,\Gamma}+\|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\le 1 \tag{4.11}$$

with $C_{\mathbf{C}}$ *and* $C_{\mathbf{F}}$ *being the continuity constants of* $\mathbf{C}$ *and* $\mathbf{F}$*, respectively (cf.* (3.6) *and* (3.7)*). Then, given* $\mathbf{z}_h\in\mathbf{K}_h$*, there exists a unique* $\mathbf{u}_h\in\mathbf{K}_h$ *such that* $\mathcal{J}_h(\mathbf{z}_h)=\mathbf{u}_h$*.*

Proof. Given $\mathbf{z}_h\in\mathbf{K}_h$, it suffices to see that the following estimate holds

$$\mathbf{A}(\underline{\psi}_h,\underline{\psi}_h)\;+\;\mathbf{C}_{\mathbf{z}_h}(\underline{\psi}_h,\underline{\psi}_h)\ge\frac{\alpha_{\mathbf{A}}}{2}\|\underline{\psi}_h\|_{\mathbf{X}}^2,\qquad\forall\,\underline{\psi}_h\in\mathbf{X}_{h,0}, \tag{4.12}$$

which implies that $\mathbf{A}(\cdot,\cdot)\;+\;\mathbf{C}_{\mathbf{z}_h}(\cdot,\cdot)$ is elliptic on $\mathbf{X}_{h,0}$, and repeat the steps of the proof of Lemma 3.3. □

Let us now establish the well-posedness of problem (4.5).

Theorem 4.4 *Let* $\mathbf{f}\in\mathbf{L}^2(\Omega)$*,* $g\in L^4(\Omega)$ *and* $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$*. Assume that hypotheses of Lemma 3.2 hold. Assume further that*

$$\frac{4\,C_{\mathbf{C}}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2}\,\Big(\|\mathbf{f}\|_{0,\Omega}+\|g\|_{0,\Omega}+\|\mathbf{u}_D\|_{0,\Gamma}+\|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\le \frac{1}{2}. \tag{4.13}$$

Then, there exists a unique $\underline{\phi}_{h,0}\in\mathbf{X}_{h,0}$ *solution to* (3.3)*. In addition, the solution* $\underline{\phi}_{h,0}$ *satisfies the a priori estimate*

$$\|\underline{\phi}_{h,0}\|_{\mathbf{X}}\;\le\;\frac{2\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega}+\|g\|_{0,\Omega}+\|\mathbf{u}_D\|_{0,\Gamma}+\|\mathbf{u}_D\|_{1/2,\Gamma}\right). \tag{4.14}$$

Proof. Analogously to the continuous case, it readily follows that $\underline{\phi}_{h,0}=(\boldsymbol{\sigma}_{h,0},\mathbf{u}_h)\in\mathbf{X}_{h,0}$ is a solution to (4.4), if and only if, $\mathcal{J}_h(\mathbf{u}_h)=\mathbf{u}_h$. As a result, to prove the well-posedness of problem (2.13), it suffices to repeat the arguments utilized in the proof of Theorem 3.4 and prove that $\mathcal{J}_h$ has a unique fixed point in $\mathbf{K}_h$ by means of the Banach fixed point theorem. In addition, analogously to the continuous case, to derive the estimate (4.14) we simply notice that $\mathbf{u}_h\in\mathbf{K}_h$ and apply the same arguments employed at the end of the proof of Lemma 3.3. □

Finally, we provide the well-posedness of problem (4.5). Its proof is analogous to the proof of Corollary 3.5.

Corollary 4.5 *Let* $\mathbf{f}\in\mathbf{L}^2(\Omega)$*,* $g\in L^4(\Omega)$ *and* $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$*. Assume that hypotheses of Theorem 3.4 hold. Then, there exists a unique* $\underline{\phi}_h\in\mathbf{X}_h$ *solution to* (4.5)*. In addition, the solution* $\underline{\phi}$ *satisfies the a priori estimate*

$$\|\underline{\phi}_h\|_{\mathbf{X}}\;\le\;\frac{2^{3/2}\,C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega}+\|\mathbf{u}_D\|_{0,\Gamma}+\|\mathbf{u}_D\|_{1/2,\Gamma}\right)+\sqrt{2}\left(\frac{2C_{\mathbf{F}}}{\alpha_{\mathbf{A}}}+\nu\sqrt{n}|\Omega|\right)\|g\|_{0,\Omega}. \tag{4.15}$$

4.3 A particular choice of discrete spaces

For each integer $k \geq 0$ and for each $K \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order k (see, for instance [2]):

$$\mathbf{RT}_k(K) := [P_k(K)]^n \,+\, P_k(K)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_n)^{\mathrm{t}}$ is a generic vector of $\mathbb{R}^n$. Then, we specify the discrete spaces in (4.1) by:

$$\begin{aligned} \mathbf{H}_h(\mathrm{div}\,;\Omega) :=& \quad \{\tau \in \mathbf{H}(\mathrm{div}\,;\Omega) : \quad \tau|_K \in \mathbf{RT}_k(K), \quad \forall\, K \in \mathcal{T}_h\}\,, \\ \mathrm{H}^1_h(\Omega) :=& \quad \left\{v \in C(\bar{\Omega}) : \quad v|_K \in P_{k+1}(K), \quad \forall\, K \in \mathcal{T}_h\right\}. \end{aligned} \tag{4.16}$$

It is well known that these subspaces satisfy the following approximation properties (see, e.g. [2], [14], [24]):

For each $s > 0$ and for each $\tau \in \mathbf{H}^s(\Omega)$, with $\mathrm{div}\,\tau \in \mathrm{H}^s(\Omega)$, there exists $\tau_h \in \mathbf{H}_h$, such that

$$\|\tau - \tau_h\|_{\mathrm{div}\,,\Omega} \leq C\, h^{\min\{s,k+1\}}\, \{\|\tau\|_{s,\Omega} + \|\mathrm{div}\,\tau\|_{s,\Omega}\}\,. \tag{4.17}$$

For each $s > 0$ and for each $v \in \mathrm{H}^{s+1}(\Omega)$ there exists $v_h \in \mathrm{H}^1_h(\Omega)$ such that

$$\|v - v_h\|_{1,\Omega} \leq C\, h^{\min\{s,k+1\}}\, \|v\|_{s+1,\Omega}. \tag{4.18}$$

4.4 A priori error analysis

In this section, we carry out the error analysis for our Galerkin scheme (4.4). We first deduce the corresponding Céa estimate by considering the generic finite dimensional subspaces (4.2), and then we apply it to derive the theoretical rates of convergence when using the specific discrete spaces provided in Section 4.3. We begin with the Céa estimate. For its proof we proceed similarly to the proof of [7, Theorem 4.6].

Theorem 4.6 *Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^4(\Omega)$ and $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ and assume that hypotheses of Theorem 3.4 hold. Let $\underline{\boldsymbol{\phi}}_0 := (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}$ and $\underline{\boldsymbol{\phi}}_{h,0} := (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_h$ be the unique solutions of the continuous and discrete problems* (3.3) *and* (4.4)*. Then, there exists $C_{cea} > 0$, independent of h, such that*

$$\|\underline{\boldsymbol{\phi}}_0 - \underline{\boldsymbol{\phi}}_{h,0}\|_{\mathbf{X}} \,\leq\, C_{cea} \inf_{\underline{\boldsymbol{\psi}}_h \in \mathbf{X}_{h,0}} \|\underline{\boldsymbol{\phi}}_0 - \underline{\boldsymbol{\psi}}_h\|_{\mathbf{X}}. \tag{4.19}$$

Proof. For the sake of simplicity, in what follows we drop the subscript 0 from the global solutions $\underline{\boldsymbol{\phi}}_0$ and $\underline{\boldsymbol{\phi}}_{h,0}$.

Let $\underline{\boldsymbol{\phi}} = (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$ and $\underline{\boldsymbol{\phi}}_h = (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_{h,0}$ be the unique solutions of problems (3.3) and (4.4), respectively, and let $\mathbf{e}_{\underline{\boldsymbol{\phi}}} = \underline{\boldsymbol{\phi}} - \underline{\boldsymbol{\phi}}_h = (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h)$. Given $\hat{\underline{\boldsymbol{\psi}}}_h := (\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h) \in \mathbf{X}_{h,0}$, let us decompose this error into

$$\mathbf{e}_{\underline{\boldsymbol{\phi}}} = \boldsymbol{\xi}_{\underline{\boldsymbol{\phi}}} + \boldsymbol{\chi}_{\underline{\boldsymbol{\phi}}} = (\underline{\boldsymbol{\phi}} - \hat{\underline{\boldsymbol{\psi}}}_h) + (\hat{\underline{\boldsymbol{\psi}}}_h - \underline{\boldsymbol{\phi}}_h) = (\boldsymbol{\sigma}_0 - \hat{\boldsymbol{\tau}}_h, \mathbf{u} - \hat{\mathbf{v}}_h) + (\hat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_{h,0}, \hat{\mathbf{v}}_h - \mathbf{u}_h). \tag{4.20}$$

Then, from (2.13) and (4.5), it is not difficult to see that

$$\mathbf{A}(\mathbf{e}_{\underline{\boldsymbol{\phi}}}, \underline{\boldsymbol{\psi}}_h) \,+\, [\mathbf{C}_{\mathbf{u}}(\underline{\boldsymbol{\phi}}, \underline{\boldsymbol{\psi}}_h) - \mathbf{C}_{\mathbf{u}_h}(\underline{\boldsymbol{\phi}}_h, \underline{\boldsymbol{\psi}}_h)] \,=\, 0 \qquad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{X}_{h,0}, \tag{4.21}$$

which, after a simple computation, implies

$$\mathbf{A}(\mathbf{e}_{\underline{\phi}},\underline{\psi}_h) + [\mathbf{C}_{\mathbf{u}_h}(\mathbf{e}_{\underline{\phi}},\underline{\psi}_h) + \mathbf{C}_{\mathbf{u}-\mathbf{u}_h}(\underline{\phi},\underline{\psi}_h)] = 0 \qquad \forall\, \underline{\psi}_h \in \mathbf{X}_{h,0},$$

or equivalently

$$\begin{aligned}\mathbf{A}(\boldsymbol{\chi}_{\underline{\phi}},\underline{\psi}_h) + \mathbf{C}_{\mathbf{u}_h}(\boldsymbol{\chi}_{\underline{\phi}},\underline{\psi}_h) \quad &= -\mathbf{A}(\boldsymbol{\xi}_{\underline{\phi}},\underline{\psi}_h) - \mathbf{C}_{\mathbf{u}-\hat{\mathbf{v}}_h}(\underline{\phi},\underline{\psi}_h) - \mathbf{C}_{\hat{\mathbf{v}}_h-\mathbf{u}_h}(\underline{\phi},\underline{\psi}_h)\\ &\quad - \mathbf{C}_{\mathbf{u}_h}(\boldsymbol{\xi}_{\underline{\phi}},\underline{\psi}_h),\end{aligned}$$

for all $\underline{\psi}_h \in \mathbf{X}_{h,0}$. In particular, for $\underline{\psi}_h = \boldsymbol{\chi}_{\underline{\phi}}$, from (4.12), and the continuity of $\mathbf{A}$ and $\mathbf{C}$, we obtain

$$\begin{aligned}\frac{\alpha_{\mathbf{A}}}{2}\|\boldsymbol{\chi}_{\underline{\phi}}\|_{\mathbf{X}} \leq \quad & C_{\mathbf{A}}\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}} + C_{\mathbf{C}}\|\underline{\phi}\|_{\mathbf{X}}\|\mathbf{u}-\hat{\mathbf{v}}_h\|_{1,\Omega} + C_{\mathbf{C}}\,\|\mathbf{u}_h\|_{1,\Omega}\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}}\\ & + C_{\mathbf{C}}\,\|\hat{\mathbf{v}}_h-\mathbf{u}_h\|_{1,\Omega}\,\|\underline{\phi}\|_{\mathbf{X}}\\ \leq \quad & (C_{\mathbf{A}} + C_{\mathbf{C}}\|\underline{\phi}\|_{\mathbf{X}} + C_{\mathbf{C}}\|\mathbf{u}_h\|_{1,\Omega})\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}} + C_{\mathbf{C}}\|\boldsymbol{\chi}_{\underline{\phi}}\|_{\mathbf{X}}\|\underline{\phi}\|_{\mathbf{X}}.\end{aligned}$$

Then, since $\underline{\phi}$ satisfies (3.19) and $\mathbf{u}_h$ belongs to $\mathbf{K}_h$, from the inequality above, we readily see that

$$\begin{aligned}&\frac{\alpha_{\mathbf{A}}}{2}\left(1 - \frac{4C_{\mathbf{F}}C_{\mathbf{C}}}{\alpha_{\mathbf{A}}^2}\left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right)\right)\|\boldsymbol{\chi}_{\underline{\phi}}\|_{\mathbf{X}}\\ &\leq \left(C_{\mathbf{A}} + \frac{4C_{\mathbf{F}}C_{\mathbf{C}}}{\alpha_{\mathbf{A}}}\left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right)\right)\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}},\end{aligned}$$

which together to (3.18), implies

$$\frac{\alpha_{\mathbf{A}}}{4}\|\boldsymbol{\chi}_{\underline{\phi}}\|_{\mathbf{X}} \leq \left(C_{\mathbf{A}} + \frac{1}{2}\right)\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}}. \tag{4.22}$$

Therefore, from (4.20), (4.22) and the triangle inequality we obtain that there exists $C_{cea} > 0$, independent of h, such that

$$\|\underline{\phi} - \underline{\phi}_h\|_{\mathbf{X}} = \|\mathbf{e}_{\underline{\phi}}\|_{\mathbf{X}} \leq \|\boldsymbol{\chi}_{\underline{\phi}}\|_{\mathbf{X}} + \|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}} \leq C_{cea}\,\|\boldsymbol{\xi}_{\underline{\phi}}\|_{\mathbf{X}} = C_{cea}\,\|\underline{\phi} - \underline{\hat{\psi}}_h\|_{\mathbf{X}},$$

for all $\underline{\hat{\psi}}_h \in \mathbf{X}_{h,0}$, which concludes the proof. □

Now, we provide the rate of convergence of our Galerkin scheme with the specific finite element subspaces introduced in Section 4.3.

Theorem 4.7 *In addition to the hypotheses of Theorem 4.6, assume that $\boldsymbol{\sigma}_0 \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma}_0 \in \mathbf{H}^s(\Omega)$, and $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, for some $s > 0$ and that the finite element subspaces $\mathbb{H}_{h,0}$ and $\mathbf{H}_h^1$ are defined using (4.16). Then there exists $C_{rate} > 0$, independent of h, such that*

$$\|\underline{\phi}_0 - \underline{\phi}_{h,0}\|_{\mathbf{X}} \leq C_{rate}h^{\min\{s,k+1\}}\left\{\|\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{u}\|_{s+1,\Omega}\right\} \tag{4.23}$$

Proof. The result is a straightforward application of Theorem 4.6, and properties (4.17) and (4.18). □

We complete our a priori error analysis with the following result providing the rate of convergence for the error $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}$, with $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$ and $\boldsymbol{\sigma}_h \in \mathbb{H}_h$ defined by (3.1) and (4.6), respectively.

Corollary 4.8 *Let $\underline{\phi}_0 := (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$ and $\underline{\phi}_{h,0} := (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_{h,0}$ be the unique solutions of the continuous and discrete problems* (3.3) *and* (4.4)*, respectively. Assume that $\boldsymbol{\sigma}_0 \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma}_0 \in \mathbf{H}^s(\Omega)$, and $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, for some $s > 0$ and that the finite element subspaces $\mathbb{H}_h$ and $\mathbf{H}^1_h$ are defined using* (4.16)*. Let $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}\,;\Omega)$ and $\boldsymbol{\sigma}_h \in \mathbb{H}_h$ defined by* (3.1) *and* (4.6)*, respectively. Then there exists $\tilde{C} > 0$, independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}\,;\Omega} \,\leq\, \tilde{C} h^{\min\{s,k+1\}} \left\{\|\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{u}\|_{s+1,\Omega}\right\} \tag{4.24}$$

Proof. Using the definition of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ given by (3.1) and (4.6), respectively, the result is a straightforward application of Theorem 4.7. □

Remark 4.9 *Provided the solution $(\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_{h,0}$ of problem* (4.4)*, and after simple computations, it is not difficult to see that the pressure can be approximated through the following formula*

$$p_h := -\frac{1}{n}\left(\mathrm{tr}\,(\boldsymbol{\sigma}_{h,0}) + \mathrm{tr}\,(\mathbf{u}_h \otimes \mathbf{u}_h) - \nu\, g\right) + \frac{1}{n|\Omega|}\left(\mathrm{tr}\,(\mathbf{u}_h \otimes \mathbf{u}_h) - \nu g, 1\right)_\Omega \quad in \quad \Omega. \tag{4.25}$$

Moreover, using the identities (2.5) *and* (3.1)*, and the estimate* (4.23) *it is easy to prove that the following estimate holds*

$$\|p - p_h\|_{0,\Omega} \,\leq\, \tilde{C} h^{\min\{s,k+1\}} \left\{\|\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}_0\|_{s,\Omega} + \|\mathbf{u}\|_{s+1,\Omega}\right\}, \tag{4.26}$$

with $C > 0$ independent of h.

5 A residual-based a posteriori error estimate

In this section, we derive and analyze a reliable and efficient residual-based a posteriori error estimate for our discrete problem (4.4), with the discrete spaces introduced in Section 4. We restrict our analysis to the two-dimensional case since the extension to three dimensions should be quite straightforward.

We begin by introducing some standard notations. For each $K \in \mathcal{T}_h$, we let $\mathcal{E}(K)$ be the set of edges of K, and we denoted by $\mathcal{E}_h$ the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $\mathcal{E}_h := \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the diameter of a given edge e. Also, for each edge $e \in \mathcal{E}_h$ we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathrm{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\mathrm{t}}$ be the corresponding fixed and tangential vector along e. Then, given $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, such that $\boldsymbol{\tau}\big|_K \in [C(K)]^{2\times 2}$, for each $K \in \mathcal{T}_h$, we let $[\boldsymbol{\tau}\,\mathbf{t}_e]$ be the tangential jump across e of $\boldsymbol{\tau}$, that is, $[\boldsymbol{\tau}\,\mathbf{t}_e] := \left\{(\boldsymbol{\tau}\big|_{K'})\Big|_e - (\boldsymbol{\tau}\big|_{K''})\Big|_e\right\}\mathbf{t}_e$, where K' and K'' are the triangles of $\mathcal{T}_h$ having e as an edge. From now on, when no confusion arises, we will simply write $\mathbf{n}$ and $\mathbf{t}$ instead of $\mathbf{n}_e$ and $\mathbf{t}_e$, respectively. Finally, for sufficiently smooth vector and tensor fields $\mathbf{v} := (v_1, v_2)^t$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let:

$$\mathbf{curl}\,\mathbf{v} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\[2ex] \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \mathbf{rot}\;\boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathrm{t}}.$$

Now, let $\underline{\phi} := (\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}$ and $\underline{\phi}_h := (\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_h$ be the unique solutions of the continuous and discrete problems (2.13) and (4.5), respectively. Then, we introduce the global a posteriori error estimator

$$\Theta := \left\{ \sum_{K\in\mathcal{T}_h} \Theta_K^2 \right\}^{1/2} \tag{5.1}$$

where for each $K \in \mathcal{T}_h$:

$$\begin{aligned}
\Theta_K^2 \quad &= (1 + h_K^2) \left\| \frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\, \nabla \mathbf{u}_h \right\|_{0,K}^2 \\
&+ \|\mathbf{f} + g\, \mathbf{u}_h + \mathbf{div}\, \boldsymbol{\sigma}_{h,0} - \alpha\, \mathbf{u}_h\|_{0,K}^2 + h_K^2 \left\| \mathbf{rot}\, \left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \right\|_{0,K}^2 \\
&+ \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Omega)} h_e \left\| \left[\!\left[\left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} \right]\!\right] \right\|_{0,e}^2 \\
&+ \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Gamma)} \left(h_e \left\| \left(\frac{\nu}{2} g \mathrm{I} + \boldsymbol{\sigma}_h^{\mathrm{d}} + (\mathbf{u}_{h,0} \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} - \nu\, \mathbf{u}_D' \right\|_{0,e}^2 + (1 + h_e)\, \|\mathbf{u}_D - \mathbf{u}_h\|_{0,e}^2 \right)
\end{aligned}$$

with $(\cdot)'$ denoting the tangential derivative along Γ. Note that the term $\mathbf{u}_D'$ requires that $\mathbf{u}_D \in \mathbf{H}^1(\Gamma)$, while the terms

$$\left\| \mathbf{rot}\, \left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \right\|_{0,K} \quad \text{and} \quad \left\| \left[\!\left[\left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \cdot \mathbf{t} \right]\!\right] \right\|_{0,e}$$

are well defined if $g\big|_K \in H^1(K), \forall\, K \in \mathcal{T}_h$.

5.1 Reliability of the a posteriori error estimator

In this section we prove the reliability of the a posteriori error estimator given by (5.1). This result is established now whose proof will be addressed in several steps.

Theorem 5.1 *Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^4(\Omega)$ and $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$. Assume that hypotheses of Lemma 3.2 hold. Assume further that*

$$\frac{4\, C_{\mathbf{C}}\, C_{\mathbf{F}}}{\alpha_{\mathbf{A}}^2} \left(\|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma} \right) \leq \frac{1}{2}. \tag{5.2}$$

Then, there exists $C_{rel} > 0$, independent of h, such that

$$\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div}\,;\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \leq C_{rel}\, \Theta \tag{5.3}$$

We begin by recalling from the proof of Lemma 3.3 that if $\mathbf{z} \in \mathbf{K}$, then the bilinear form $\mathbf{A}(\cdot,\cdot) + \mathbf{C}_{\mathbf{z}}(\cdot,\cdot)$ is elliptic on $\mathbf{X}_0$ with ellipticity constant $\frac{\alpha_{\mathbf{A}}}{2}$, which certainly implies that

$$\frac{\alpha_{\mathbf{A}}}{2} \|\underline{\psi}\|_{\mathbf{X}}^2 \leq \sup_{\underline{\psi}\in\mathbf{X}_0\setminus\{\mathbf{0}\}} \frac{\mathbf{A}(\underline{\zeta}, \underline{\psi}) + \mathbf{C}_{\mathbf{z}}(\underline{\zeta}, \underline{\psi})}{\|\underline{\psi}\|_{\mathbf{X}}}, \quad \forall\, \underline{\zeta} \in \mathbf{X}_0.$$

In particular, taking $\mathbf{z} = \mathbf{u} \in \mathbf{K}$ and $\underline{\zeta} = \underline{\phi}_0 - \underline{\phi}_{h,0} = (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h) \in \mathbf{X}_0$ in the estimate above, we obtain

$$\|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{X}} \leq \frac{2}{\alpha_{\mathbf{A}}} \sup_{(\boldsymbol{\tau},\mathbf{v})\,\in\,\mathbf{X}_0\setminus\{\mathbf{0}\}} \left\{ \frac{\mathcal{R}(\boldsymbol{\tau}, \mathbf{v}) + \mathcal{B}(\boldsymbol{\tau}, \mathbf{v})}{\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}}} \right\}, \tag{5.4}$$

where $\mathcal{R} : \mathbf{X}_0 \to \mathbb{R}$ and $\mathcal{B} : \mathbf{X}_0 \to \mathbb{R}$ are the functional defined by

$$\mathcal{R}(\boldsymbol{\tau}, \mathbf{v}) := \mathbf{F}(\boldsymbol{\tau}, \mathbf{v}) - \mathbf{A}(\underline{\phi}_{h,0}, (\boldsymbol{\tau}, \mathbf{v})) - \mathbf{C}_{\mathbf{u}_h}(\underline{\phi}_{h,0}, (\boldsymbol{\tau}, \mathbf{v}))$$

and

$$\mathcal{B}(\boldsymbol{\tau}, \mathbf{v}) := -\mathbf{C}_{\mathbf{u}-\mathbf{u}_h}((\underline{\phi}_{h,0}, (\boldsymbol{\tau}, \mathbf{v})).$$

Let us observe that from the continuity of $\mathbf{C}$ (cf. (3.6)), we can obtain the following upper bound for $\mathcal{B}$:

$$|\mathcal{B}(\boldsymbol{\tau}, \mathbf{v})| \leq C_{\mathbf{C}} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\underline{\phi}_{h,0}\|_{\mathbf{X}} \|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}} \leq C_{\mathbf{C}} \|\underline{\phi}_0 - \underline{\phi}_{h,0}\|_{\mathbf{X}} \|\underline{\phi}_{h,0}\|_{\mathbf{X}} \|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}},$$

which together to (4.14) and assumption (5.2), implies

$$\sup_{(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}_0 \setminus \{\mathbf{0}\}} \left\{ \frac{B(\boldsymbol{\tau}, \mathbf{v})}{\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}}} \right\} \leq \frac{\alpha_{\mathbf{A}}}{4} \|\underline{\phi}_0 - \underline{\phi}_{h,0}\|_{\mathbf{X}} = \frac{\alpha_{\mathbf{A}}}{4} \|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{X}}.$$

In this way, from the latter inequality and (5.4), we easily obtain

$$\|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{X}} \leq \frac{4}{\alpha_{\mathbf{A}}} \sup_{(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}_0 \setminus \{\mathbf{0}\}} \left\{ \frac{\mathcal{R}(\boldsymbol{\tau}, \mathbf{v})}{\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{X}}} \right\}. \tag{5.5}$$

In turn, according to the definitions of the forms $\mathbf{A}, \mathbf{C}_{\mathbf{z}}$ and the functional $\mathbf{F}$ (cf. (2.14)), we find after a simple computation that for any $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}_0$, there holds

$$\mathcal{R}(\boldsymbol{\tau}, \mathbf{v}) := \mathcal{R}_1(\boldsymbol{\tau}) + \mathcal{R}_2(\mathbf{v})$$

where

$$\mathcal{R}_1(\boldsymbol{\tau}) = -\frac{\nu}{\alpha} (\mathbf{f} + g\, \mathbf{u}_h + \mathbf{div}\, \boldsymbol{\sigma}_{h,0}, \mathbf{div}\, \boldsymbol{\tau})_\Omega - \Big(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}}, \boldsymbol{\tau} \Big)_\Omega + \nu\, \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_D \rangle_\Gamma \tag{5.6}$$

and

$$\mathcal{R}_2(\mathbf{v}) = \kappa_1 \Big(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\, \nabla \mathbf{u}_h, \nabla \mathbf{v} \Big)_\Omega + \kappa_2\, (\mathbf{u}_D - \mathbf{u}_h, \mathbf{v})_\Gamma. \tag{5.7}$$

In this way, the supremum (5.5) can be bounded in terms of $\mathcal{R}_1$ and $\mathcal{R}_2$, which yields

$$\|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{X}} \leq \frac{4}{\alpha_{\mathbf{A}}} \big\{ \|\mathcal{R}_1\|_{(\mathbb{H}_0(\mathbf{div};\Omega))'} + \|\mathcal{R}_2\|_{(\mathbf{H}^1(\Omega))'} \big\}. \tag{5.8}$$

As a consequence of the above, the derivation of the upper bound in (5.3) is completed by providing suitable upper bounds for $\mathcal{R}_1$ and $\mathcal{R}_2$. We begin by establishing the corresponding estimate for $\mathcal{R}_2$.

Lemma 5.2 *There exists $C_1 > 0$ independent of h, such that*

$$\begin{aligned} ||\mathcal{R}_2||_{(\mathbf{H}^1(\Omega))'} \leq C_1 \Bigg\{ \sum_{K \in \mathcal{T}_h} \Bigg(\Big\| \frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\, \nabla \mathbf{u}_h \Big\|_{0,K}^2 \\ + \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma)} ||\mathbf{u}_D - \mathbf{u}_h||_{0,e}^2 \Bigg) \Bigg\}^{1/2} \end{aligned} \tag{5.9}$$

Proof. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. From the definition of $\mathcal{R}_2$ and the Cauchy-Schwarz inequality it readily follows that

$$\begin{aligned} |\mathcal{R}_2(\mathbf{v})| \quad &\leq \max\{\kappa_1, \kappa_2\} \left(\left\| \tfrac{\nu}{2}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\,\nabla \mathbf{u}_h \right\|_{0,\Omega}^2 \right. \\ &\left. \qquad + \|\mathbf{u}_D - \mathbf{u}_h\|_{0,\Gamma}^2 \right)^{1/2} \left(\|\nabla \mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{v}\|_{0,\Gamma}^2 \right)^{1/2}, \end{aligned}$$

which together to (3.4) clearly implies (5.9), with $C_1 = \tilde{C}_p^{1/2} \max\{\kappa_1, \kappa_2\} > 0$. □

Next, to derive the corresponding upper bound for $\mathcal{R}_1$ we first notice that, after adding and subtracting $\alpha \mathbf{u}_h$ in the first term of $\mathcal{R}_1$ (cf. (5.6)), this functional can be rewritten as

$$\mathcal{R}_1(\boldsymbol{\tau}) = \mathcal{R}_1^1(\boldsymbol{\tau}) + \mathcal{R}_1^2(\boldsymbol{\tau}), \tag{5.10}$$

with

$$\mathcal{R}_1^1(\boldsymbol{\tau}) := -\frac{\nu}{\alpha} \left(\mathbf{f} + g\,\mathbf{u}_h - \alpha \mathbf{u}_h + \mathbf{div}\, \boldsymbol{\sigma}_{h,0}, \mathbf{div}\, \boldsymbol{\tau} \right)_\Omega$$

and

$$\mathcal{R}_1^2(\boldsymbol{\tau}) := \left(-\frac{\nu}{2}\, g\,\mathrm{I} - \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} - (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}}, \boldsymbol{\tau} \right)_\Omega - \nu\,(\mathbf{div}\, \boldsymbol{\tau}, \mathbf{u}_h)_\Omega + \nu\, \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_D \rangle_\Gamma .$$

In addition, we will require two well–known approximation operators: the Raviart-Thomas interpolator (see e.g. [2, 24]) and the Clément operator onto the space of continuous piecewise linear functions [15].

The Raviart-Thomas interpolation operator $\Pi_h^k : \mathbf{H}^1(\Omega) \to \mathbf{H}_h(\mathrm{div}\,;\Omega)$ (cf. (4.16)), is given by the conditions

$$\Pi_h^k \tau \in \mathbf{H}_h(\mathrm{div}\,;\Omega) \quad \text{and} \quad \int_e \Pi_h^k \tau \cdot \mathbf{n} = \int_e \tau \cdot \mathbf{n} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}. \tag{5.11}$$

As a consequence of (5.11), there holds

$$\mathrm{div}\,(\Pi_h^k \tau) = \mathcal{P}_h(\mathrm{div}\,\tau), \tag{5.12}$$

where $\mathcal{P}_h$, is the $L^2(\Omega)$-orthogonal projector onto the piecewise polynomials functions of degree $\leq k$ on Ω. In what follows we will utilize a tensor version of Π_h^k, say $\boldsymbol{\Pi}_h^k : \mathbb{H}^1(\Omega) \to \mathbb{H}_h(\Omega)$, which is defined row-wise by Π_h^k. The local approximation properties of Π_h^k (and hence of $\boldsymbol{\Pi}_h^k$) are stated as follows (see e.g. [2, 24] for details): there exist constants $\hat{c}_1$, $\hat{c}_2 > 0$, independent of h, such that for all $\tau \in \mathbf{H}^1(\Omega)$ there hold

$$\|\tau - \Pi_h^k \tau\|_{0,K} \leq \hat{c}_1\, h_K\, \|\tau\|_{1,K} \qquad \forall\, K \in \mathcal{T}_h,$$

and

$$\|\tau \cdot \mathbf{n} - \Pi_h^k \tau \cdot \mathbf{n}\|_{0,e} \leq \hat{c}_2\, h_e^{1/2}\, \|\tau\|_{1,K_e} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h,$$

where K_e is a triangle of $\mathcal{T}_h$ containing e on its boundary.

The Clément operator $I_h : H^1(\Omega) \to Y_h$ approximates optimally non–smooth functions by continuous piecewise linear functions:

$$Y_h := \{ v \in C(\bar{\Omega}) : \quad v|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}_h \}.$$

It is well known that I_h satisfies the following approximation properties (see [15] for details): there exist constants $\hat{c}_3$, $\hat{c}_4 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there hold

$$\|v - I_h v\|_{0,K} \,\leq\, \hat{c}_3\, h_K \,\|v\|_{1,\Delta(K)} \qquad \forall\, K \,\in\, \mathcal{T}_h\,,$$

and

$$\|v - I_h v\|_{0,e} \,\leq\, \hat{c}_4\, h_e^{1/2}\,\|v\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h\,,$$

where

$$\Delta(K) \,:=\, \cup\{K' \,\in\, \mathcal{T}_h :\quad K' \cap K \neq \emptyset\} \quad \text{and} \quad \Delta(e) \,:=\, \cup\{K' \,\in\, \mathcal{T}_h :\quad K' \cap e \neq \emptyset\}\,.$$

In what follows we will utilize a vector version of I_h, say $\mathbf{I}_h : \mathbf{H}^1(\Omega) \to Y_h \times Y_h$, which is defined row-wise by $\mathbf{I}_h$.

We are now in position of establishing the corresponding estimate for $\mathcal{R}_1$.

Lemma 5.3 *There exists C_2, independents of h, such that*

$$\begin{aligned}
||\mathcal{R}_1||_{(\mathbb{H}_0(\mathbf{div}\,;\Omega))'} \leq C_2 \Bigg\{ \sum_{K\in\mathcal{T}_h} \Bigg(& h_K^2 \,\left\| \frac{\nu}{2}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\, \nabla \mathbf{u}_h \right\|_{0,K}^2 \\
& + ||\mathbf{f} + g\,\mathbf{u}_h + \mathbf{div}\,\boldsymbol{\sigma}_{h,0} - \alpha\,\mathbf{u}_h||_{0,K}^2 \\
& + h_K^2 \left\| \mathbf{rot}\,\left(\frac{\nu}{2}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \right\|_{0,K}^2 \\
& + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Omega)} h_e \left\| \left[\left(\frac{\nu}{2}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} \right] \right\|_{0,e}^2 \\
& + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Gamma)} h_e \Bigg(\left\| \left(\frac{\nu}{2}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} - \nu\, \mathbf{u}_D' \right\|_{0,e}^2 \\
& + \|\mathbf{u}_D - \mathbf{u}_h\|_{0,e}^2 \Bigg)\Bigg)\Bigg\}^{1/2}
\end{aligned} \tag{5.13}$$

Proof. Let $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}\,;\Omega)$. First, by applying [32, Lemma 3.3] we know that there exist $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega)$ and $\boldsymbol{\phi} \in \mathbf{H}^1(\Omega)$ satisfying the following Helmholtz decomposition

$$\boldsymbol{\tau} = \boldsymbol{\eta} + \mathbf{curl}\boldsymbol{\phi} \quad \text{in} \quad \Omega. \tag{5.14}$$

Moreover, we know that this decomposition is stable, namely

$$\|\boldsymbol{\eta}\|_{1,\Omega} + \|\boldsymbol{\phi}\|_{1,\Omega} \leq C\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega}, \tag{5.15}$$

with $C > 0$ independent of h.

Now, let $\boldsymbol{\tau}_h = \Pi_h^k(\boldsymbol{\eta}) + \mathbf{curl}(\mathbf{I}_h(\boldsymbol{\phi})) - c\mathrm{I} \in \mathbb{H}_{h,0}$, with $c = \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}\,\left(\Pi_h^k(\boldsymbol{\eta}) + \mathbf{curl}(\mathbf{I}_h(\boldsymbol{\phi}))\right)$. It is clear from the definition of $\mathcal{R}_1$ and the compatibility condition (2.2), that

$$\mathcal{R}_1(\boldsymbol{\tau}_h) = 0 \quad \text{and} \quad \mathcal{R}_1(I) = 0,$$

which implies

$$\mathcal{R}_1(\boldsymbol{\tau}) = \mathcal{R}_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \mathcal{R}_1^1(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^k(\boldsymbol{\eta})) + \mathcal{R}_1^2(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^k(\boldsymbol{\eta})) + \mathcal{R}_1^2(\mathbf{curl}(\boldsymbol{\phi} - \mathbf{I}_h(\boldsymbol{\phi}))). \tag{5.16}$$

In this way, to obtain the desired estimate it suffices to bound each term on the right hand side of (5.16).

First, employing the Cauchy-Schwarz inequality and the continuity of $\boldsymbol{\Pi}_h^k$ (see [24, Lemma 4.4]), we easily obtain

$$\begin{aligned} |\mathcal{R}_1^1(\boldsymbol{\eta} - \Pi_h^k(\boldsymbol{\eta}))| &\leq C\,\|\mathbf{f} + g\,\mathbf{u}_h + \mathbf{div}\,\boldsymbol{\sigma}_{h,0} - \alpha\,\mathbf{u}_h\|_{0,\Omega}\,\|\boldsymbol{\eta}\|_{1,\Omega} \\ &= C\left\{\sum_{K\in\mathcal{T}_h}\|\mathbf{f} + g\,\mathbf{u}_h + \mathbf{div}\,\boldsymbol{\sigma}_{h,0} - \alpha\,\mathbf{u}_h\|_{0,K}^2\right\}^{1/2}\|\boldsymbol{\eta}\|_{1,\Omega}. \end{aligned} \tag{5.17}$$

In turn, proceeding analogously as in the proof of lemmas 5.3 and 5.4 in [29], that is, integrating by parts locally and utilizing the local approximation properties of the Raviart-Thomas and Clément operators, we can easily deduce that

$$\begin{aligned} |\mathcal{R}_1^2(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^k(\boldsymbol{\eta}))| \leq\ & C\left\{\sum_{K\in\mathcal{T}_h}\left(h_K^2\,\left\|\frac{\nu}{2}\,g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h\otimes\mathbf{u}_h)^{\mathrm{d}} - \nu\,\nabla\mathbf{u}_h\right\|_{0,K}^2\right.\right. \\ & \left.\left. + h_e\,\|\mathbf{u}_D - \mathbf{u}_h\|_{0,e}^2\right)\right\}^{1/2}\|\boldsymbol{\eta}\|_{1,\Omega}, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} |\mathcal{R}_1^2(\mathbf{curl}(\boldsymbol{\phi} - \mathbf{I}_h(\boldsymbol{\phi})))| \leq\ & C\left\{\sum_{K\in\mathcal{T}_h}\left(h_K^2\left\|\mathbf{rot}\,\left(\frac{\nu}{2}g\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h\otimes\mathbf{u}_h)^{\mathrm{d}}\right)\right\|_{0,K}^2\right.\right. \\ & + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Gamma)} h_e\left\|\left(\frac{\nu}{2}g\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h\otimes\mathbf{u}_h)^{\mathrm{d}}\right)\mathbf{t} - \nu\mathbf{u}_D'\right\|_{0,e}^2 \\ & \left.\left. + \sum_{e\in\mathcal{E}(K)\cap\mathcal{E}_h(\Omega)} h_e\left\|\left[\left(\frac{\nu}{2}g\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h\otimes\mathbf{u}_h)^{\mathrm{d}}\right)\mathbf{t}\right]\right\|_{0,e}^2\right)\right\}^{1/2}\|\boldsymbol{\phi}\|_{1,\Omega}. \end{aligned} \tag{5.19}$$

In this way from the identities (5.14), (5.16) and the estimates (5.15), (5.17), (5.18) and (5.19), it readily follows that (5.13) holds, which concludes the proof. □

We end this section by noticing that the reliability estimate (5.3) follows straightforwardly from lemmas 5.2 and 5.3.

5.2 Efficiency of the a posteriori a error estimator

The main result of this section is stated next.

Theorem 5.4 *There exists $C_{eff} > 0$, independent of h, such that*

$$C_{eff}\,\Theta \leq \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div};\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \text{h.o.t.} \tag{5.20}$$

where h.o.t. *stands, eventually, for terms of higher order.*

In order to prove the efficiency of the a posteriori error estimator, in what follows we bound each term defining Θ in terms of local or global errors. We remark in advance that most of the results to be deduced in this section are derived by employing estimates already available in the literature. In particular, in the sequel we adapt the results from [8, 9, 10, 32, 33] to prove the efficiency of our estimator Θ. We begin with the following lemma providing the estimates for the zero-order terms appearing in the definition of Θ_K.

Lemma 5.5 *There exists $\mathcal{C}_1$ and $\mathcal{C}_2$, independents of h, such that:*

$$\left\| \frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu\, \nabla \mathbf{u}_h \right\|_{0,K} \le \mathcal{C}_1 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right), \tag{5.21}$$

$$\|\mathbf{f} + g\, \mathbf{u}_h + \mathbf{div}\, \boldsymbol{\sigma}_{h,0} - \alpha\, \mathbf{u}_h\|_{0,K} \le \mathcal{C}_2 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div};K} + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} \right), \tag{5.22}$$

for all $K \in \mathcal{T}_h$.

Proof. Similarly to the proof of [33, Theorem 3.12] we first observe that

$$\|\mathbf{u}_h \otimes \mathbf{u}_h - \mathbf{u} \otimes \mathbf{u}\|_{0,K}^2 \le 2 \left\{ \|\mathbf{u}_h\|_{\mathbf{L}^4(\Omega)}^2 + \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}^2 \right\} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^4(\Omega)}^2,$$

which together to the continuity of the injection $\mathbf{i}_c : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, hypothesis (5.2), the fact that $\mathbf{u} \in \mathbf{K}$ (cf. (3.11)) and $\mathbf{u}_h \in \mathbf{K}_h$ (cf. (4.8)), and the definition of $C_{\mathbf{C}}$ (cf. (3.6)), implies

$$\|\mathbf{u}_h \otimes \mathbf{u}_h - \mathbf{u} \otimes \mathbf{u}\|_{0,K}^2 \le 2\|\mathbf{i}_c\|^4 \left\{ \|\mathbf{u}_h\|_{1,\Omega}^2 + \|\mathbf{u}\|_{1,\Omega}^2 \right\} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2 \le \frac{\alpha_{\mathbf{A}}^2}{4(1+\kappa_1)} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2. \tag{5.23}$$

Then, the estimate (5.21) follows from (5.23), the triangle inequality and the first equation of (2.6). In turn, the estimate (5.22) readily follows from the second equation of (2.6) and the triangle inequality. □

Now, we provide the upper bound for the residual term involving the Dirichlet datum. Its proof is a direct consequence of the well known trace inequality. We omit further details.

Lemma 5.6 *There exists $\mathcal{C}_3 > 0$, independents of h, such that:*

$$\|\mathbf{u}_D - \mathbf{u}_h\|_{0,e} \le \mathcal{C}_3\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \quad \forall\, e \in \mathcal{E}(\Gamma). \tag{5.24}$$

Finally, since the datum $g \in L^4(\Omega)$ is not necessarily a polynomial function, for the remaining terms we proceed analogously to [10] and apply the results from [8, 9].

Lemma 5.7 *There exists $\mathcal{C}_4 > 0$, independents of h, such that:*

$$h_K \left\| \mathbf{rot} \left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \right\|_{0,K} \le \mathcal{C}_4 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right) + \text{h.o.t.}, \tag{5.25}$$

for all $K \in \mathcal{T}_h$.

Proof. By using the first equation of (2.6) and estimate (5.23), the result follows similarly to the proof of [10, Lemma 6.10]. We omit further details. □

Lemma 5.8 *There exists $\mathcal{C}_5 > 0$, independents of h, such that:*

$$h_e^{1/2} \left\| \left[\!\left[\left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} \right]\!\right] \right\|_{0,e}$$

$$\leq \mathcal{C}_5 \left(\sum_{K \subseteq \omega_e} \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right) + \text{h.o.t.}, \quad \forall\, e \in \mathcal{E}_h(\Omega), \tag{5.26}$$

where $\omega_e := \cup\{K' \in \mathcal{T}_h : e \in \mathcal{E}(K')\}$.

Proof. The proof follows from the first equation of (2.6), estimate (5.23), and a slight modification of the proof of [10, Lemma 6.10]. We omit further details. □

Lemma 5.9 *There exists $\mathcal{C}_6 > 0$, independents of h, such that:*

$$h_e^{1/2} \left\| \left(\frac{\nu}{2}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \mathbf{t} - \nu\, \mathbf{u}_D' \right\|_{0,e}$$

$$\leq \mathcal{C}_6 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K_e} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right) + \text{h.o.t.} \quad \forall\, e \in \mathcal{E}(\Gamma),$$

where K_e is a generic triangle having e as an edge.

Proof. Again, the proof follows from the first equation of (2.6), estimate (5.23), and a slight modification of the proof of [10, Lemma 6.10] (see also [30, Lemma 4.15]). We omit further details. □

We end this section by observing that the efficiency estimate (5.20) follows straightforwardly from the Lemmas 5.5–5.9.

5.3 Three dimensional case

In what follows we briefly discuss about the a posteriori error estimator in the three dimensional case. To that end we first introduce some notations.

Given $\mathbf{v}$ a sufficiently smooth vector field, we let

$$\operatorname{curl} \mathbf{v} := \nabla \times \mathbf{v},$$

and given a tensor field $\boldsymbol{\tau} = (\tau_{ij})_{3\times 3}$, we define

$$\mathbf{curl}\, \boldsymbol{\tau} := \begin{pmatrix} \operatorname{curl}(\tau_{11}, \tau_{12}, \tau_{13}) \\ \operatorname{curl}(\tau_{21}, \tau_{22}, \tau_{23}) \\ \operatorname{curl}(\tau_{31}, \tau_{32}, \tau_{33}) \end{pmatrix}.$$

On the other hand, given $K \in \mathcal{T}_h$, we let $\mathcal{E}(K)$ be the set of its faces, and let $\mathcal{E}_h$ be the set of all the faces of the triangulation $\mathcal{T}_h$. Then, we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h \,:\, e \subset \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h \,:\, e \subset \Gamma\}$. The faces of the tetrahedrons of $\mathcal{T}_h$ are denoted by e and their corresponding diameters by h_e. Also for each face $e \in \mathcal{E}_h$ we fix a unit normal $\mathbf{n}_e$ to e. In addition, if $\mathbf{v}$ is a sufficiently smooth vector field, and $e \in \mathcal{E}_h(\Omega)$, we let $[\![\mathbf{v} \times \mathbf{n}_e]\!] := (\mathbf{v}|_{K'} - \mathbf{v}|_{K''})\,|_e \times \mathbf{n}_e$, where K' and K'' are the elements of $\mathcal{T}_h$ having e as a common face. As in the previous section, from now on, when no confusion arises, we simple write $\mathbf{n}$ instead of $\mathbf{n}_e$. In addition, $\boldsymbol{\tau} \times \mathbf{n}$ stands for the 3×3 tensor given by

$$\boldsymbol{\tau} \times \mathbf{n} := \left(\begin{array}{c} (\tau_{11}, \tau_{12}, \tau_{13}) \times \mathbf{n} \\ (\tau_{21}, \tau_{22}, \tau_{23}) \times \mathbf{n} \\ (\tau_{31}, \tau_{32}, \tau_{33}) \times \mathbf{n} \end{array} \right),$$

and set

$$[\![\boldsymbol{\tau} \times \mathbf{n}]\!] := \left(\begin{array}{c} [\![(\tau_{11}, \tau_{12}, \tau_{13}) \times \mathbf{n}]\!] \\ [\![(\tau_{21}, \tau_{22}, \tau_{23}) \times \mathbf{n}]\!] \\ [\![(\tau_{31}, \tau_{32}, \tau_{33}) \times \mathbf{n}]\!] \end{array} \right).$$

Now, let $(\boldsymbol{\sigma}_0, \mathbf{u}) \in \mathbf{X}_0$ and $(\boldsymbol{\sigma}_{h,0}, \mathbf{u}_h) \in \mathbf{X}_{h,0}$ be the respective unique solutions of (3.3) and (4.4) . Then we define the global a posteriori error estimator

$$\widehat{\Theta} := \left\{ \sum_{K \in \mathcal{T}_h} \widehat{\Theta}_K^2 \right\}^{1/2},$$

where for each $K \in \mathcal{T}_h$:

$$\begin{aligned} \widehat{\Theta}_K^2 \; &= (1 + h_K^2) \left\| \frac{\nu}{3}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} - \nu \nabla \mathbf{u}_h \right\|_{0,K}^2 \\ &+ \|\mathbf{f} + g\, \mathbf{u}_h + \mathbf{div}\, \boldsymbol{\sigma}_{h,0} - \alpha\, \mathbf{u}_h\|_{0,K}^2 + h_K^2 \left\| \mathbf{curl} \left(\frac{\nu}{3}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \right\|_{0,K}^2 \\ &+ \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Omega)} h_e \left\| \left[\!\left[\left(\frac{\nu}{3}\, g\, \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \times \mathbf{n} \right]\!\right] \right\|_{0,e}^2 \\ &+ \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma)} \Bigg(h_e \left\| \left(\frac{\nu}{3} g \mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \times \mathbf{n} - \nu \nabla \mathbf{u}_D \times \mathbf{n} \right\|_{0,e}^2 \\ &+ (1 + h_e)\, \|\mathbf{u}_D - \mathbf{u}_h\|_{0,e}^2 \Bigg) \end{aligned}$$

The reliability of this estimator can be proved essentially by using the same arguments employed for the 2D case. In particular, analogously to the 2D case, here it is needed a stable Helmholtz decomposition for $\mathbb{H}(\mathbf{div}\,; \Omega)$. This result is a direct consequence of the following lemma. For its proof we refer the reader to [25, Theorem 3.1].

Lemma 5.10 *For each $\mathbf{v} \in H(\mathrm{div}\,; \Omega)$ there exist $z \in H^2(\Omega)$ and $\boldsymbol{\chi} \in [H^1(\Omega)]^3$, such that there hold $\mathbf{v} = \nabla z + \mathrm{curl}\, \boldsymbol{\chi}$ in Ω, and*

$$\|z\|_{2,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \leq C \|\mathbf{v}\|_{\mathrm{div}\,;\Omega},$$

where C is a positive constant independent of $\mathbf{v}$.

Finally, to prove the efficiency of the 3D estimator it suffices to control the new terms since the analysis of the rest of the terms is straightforward. The following lemma provides these desired estimates, where, for the sake of simplicity, we assume that $\mathbf{u}_D$ and g are piecewise polynomials.

Lemma 5.11 *There exist positive constants C_i, $i \in \{1,2,3\}$, independent of h, ν, $\boldsymbol{\eta}$ and α, such that*

a) $h_K \left\| \mathbf{curl}(\frac{\nu}{3}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}}) \right\|_{0,T} \le C_1 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right) \quad \forall K \in \mathcal{T}_h$,

b) $h_e^{1/2} \left\| \left[\left(\frac{\nu}{3}\, g\,\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \times \mathbf{n} \right] \right\|_{0,e} \le C_2 \left(\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,\omega_e} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right) \quad \forall e \in \mathcal{E}_h(\Omega)$,

c) *There exists $C_3 > 0$, independent of h, such that*

$$h_e^{1/2} \left\| \left(\frac{\nu}{3} g\mathrm{I} + \boldsymbol{\sigma}_{h,0}^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \right) \times \mathbf{n} - \nu\, \nabla \mathbf{u}_D \times \mathbf{n} \right\|_{0,e} \le C_3 (\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{0,K_e} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}),$$

where K_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. First, a) follows straightforwardly from [28, Lemma 4.9] and (5.23). Similarly, b) follows from [28, Lemma 4.10] and (5.23). Finally, the proof of c) follows from (5.23) a slight modification of the proof of [28, Lemma 4.13]. □

6 Numerical results

In this section we present two numerical examples, illustrating the performance of the mixed finite element scheme (4.5), confirming the reliability and efficiency of the a posteriori error estimator Θ derived in Section 5, and showing the behaviour of the associated adaptive algorithm. Our implementation is based on a *FreeFem++* code (see [35]), in conjunction with the direct linear solver UMFPACK (see [17]).

In what follows, N stands for the total number of degrees of freedom defining $\mathbf{X}_{h,0}$. Denoting by $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}_0$ and $(\boldsymbol{\sigma}_{h,0}, \mathbf{u}) \in \mathbf{X}_{h,0}$, the respective solutions of (3.3) and (4.4), and by p_h the post-processed discrete pressure given by (4.25), we define the individual errors as

$$\mathrm{e}(\boldsymbol{\sigma}_0) := \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}\|_{\mathbf{div};\Omega}, \quad \mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \quad \text{and} \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega}.$$

In addition, denoting by $\mathrm{e}(\boldsymbol{\sigma}_0, u) := \{\mathbf{e}(\boldsymbol{\sigma}_0)^2 + (\mathbf{e}(\mathbf{u}))\}^{1/2}$ the total error, the effectivity index with respect to Θ (cf. (5.1)) is given by

$$\mathrm{eff}(\Theta) := \mathrm{e}(\boldsymbol{\sigma}_0, u)/\Theta.$$

Furthermore, we define the experimental rates of convergence as

$$r(\boldsymbol{\sigma}_0) := \frac{\log(\mathrm{e}(\boldsymbol{\sigma}_0)/\mathrm{e}'(\boldsymbol{\sigma}_0))}{\log(h/h')}, \quad r(\mathbf{u}) := \frac{\log(\mathrm{e}(\mathbf{u})/\mathrm{e}'(\mathbf{u}))}{\log(h/h')},$$
$$r(p) := \frac{\log(\mathrm{e}(p)/\mathrm{e}'(p))}{\log(h/h')}, \qquad r(\boldsymbol{\sigma}_0, \mathbf{u}) := \frac{\log(\mathrm{e}(\boldsymbol{\sigma}_0, \mathbf{u})/\mathrm{e}'(\boldsymbol{\sigma}, \mathbf{u}))}{\log(h/h')},$$

where h and h' are two consecutive meshsizes with errors e and e$'$. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all the cases the nonlinear system is solved by applying a Newton method (stopped when the L^2–norm of the total residual attains the tolerance 1E-6). Example 1 is used to illustrate the performance of the two dimensional mixed finite element scheme under a quasi-uniform refinement, and to test the behaviour of the iterative method when $\|g\|_{L^4(\Omega)}$ increases, whereas Example 2 is utilized to assess the accuracy of the proposed scheme along with the properties of the adaptive error estimator Θ defined in (5.1). For the later we apply the following adaptive procedure from [39]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (4.5) for the current mesh $\mathcal{T}_h$.

3) Compute Θ_T for each triangle $T \in \mathcal{T}_h$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2} \max_{T \in \mathcal{T}_h} \{\Theta_T : T \in \mathcal{T}_h\} .$$

6) Define the resulting mesh as the current meshe $\mathcal{T}_h$, and go to step 2.

In Example 1 we choose the domain $\Omega := (0,1)^2$, the parameters $\nu = 1$, $\alpha = 1$, $(\kappa_1, \kappa_2) = (\nu, \nu^2/2)$ (according to the assumptions of Lemma 3.2), and take $\mathbf{f}$, g and $\mathbf{u}_D$ so that the exact solution is given by the smooth functions

$$\mathbf{u}(x_1, x_2) := \begin{pmatrix} \pi\, e^{x_1} \cos(\pi x_2) + e^{x_2} \sin(\pi x_1) + \frac{\lambda x_1}{2} \\ \\ -\pi\, e^{x_2} \cos(\pi x_1) - e^{x_1} \sin(\pi x_2) + \frac{\lambda x_2}{2} \end{pmatrix},$$

$$p(x_1, x_2) := e^{x_1 + x_2} \cos(\pi x_1) - \frac{1 - e^2}{\pi^2 + 1},$$

where λ is a real parameter. It is not difficult to see that $\|g\|_{L^4(\Omega)} = |\lambda|$, so that condition (3.9) becomes

$$|\lambda|^2 \leq \frac{C_p}{2\|\mathbf{i}_c\|^2}.$$

In Table 6.1 below we summarize the convergence history obtained for this example, considering $\lambda = 10$, a sequence of quasi-uniform triangulations and $\mathrm{RT}_0 - \mathbf{P}_1$ (table at the top) and $\mathrm{RT}_1 - \mathbf{P}_2$ (table at the bottom) approximations. We observe there that the rates of convergence $O(h)$ and $O(h^2)$ predicted by Theorem 4.7 (when $s = 1$ and $s = 2$, respectively) are attained in both cases for all the unknowns. Next, in order to test the real influence of hypothesis (3.9) on our method, in Table 6.2 we illustrate the behaviour of the iterative method for different values of λ. We observe there that when $\lambda = 500$, for the first three meshes the iterative method takes more than 200 iterations to converge, reason why this information is not reported in those cases. This behaviour shows that the performance of the method is clearly influenced by hypothesis

(3.9). However, it is also important to remark that for values of λ not greater than 10, and for those meshes where the iterative method converges, the number of iterations remains reasonably bounded. Exact and computed solutions, considering the $\mathrm{RT}_0 - \mathbf{P}_1$ approximation, are shown in Figure 6.1 for $\lambda = 10$ and $N = 6214575$.

In our second example we assess the capability of our adaptive algorithm to capturing the presence of high gradients and singularities of the solution. To that end we consider the parameters $\nu = \alpha = 1$, $(\kappa_1, \kappa_2) = (1, 1/2)$, the domain as the non-convex L–shaped region $\Omega = (-1,1)^2 \backslash [0,1]^2$, and take $\mathbf{f}$, g and $\mathbf{u}_D$ so that the exact solution is given by the functions

$$\mathbf{u}(x_1,x_2) := \begin{pmatrix} -\cos(\pi x_1)\sin(\pi x_2) + \frac{x_1}{2} \\ \\ \cos(\pi x_2)\sin(\pi x_1) + \frac{x_2}{2} \end{pmatrix},$$

$$p(x_1,x_2) := \frac{1 - x_1}{(0.01 - x_1)^2 + (0.01 - x_2)^2} - l,$$

where $l \in \mathbb{R}$ is a constant chosen in such a way $(p,1)_\Omega = 0$. Notice that p has high gradients around the origin and $\operatorname{div} \mathbf{u} = 1$. In Table 6.3 we present the convergence history of the method (in its lowest-order configuration), considering firstly a quasi-uniform refinement (table at the top) and secondly an adaptive refinement (table at the bottom). We observe there that the errors of the adaptive procedure decrease faster than those obtained by the quasi-uniform one, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figure 6.2 where we display the total errors $\mathrm{e}(\boldsymbol{\sigma}_0, \mathbf{u})$ vs. the degrees of freedom N for both refinements. As shown by the values of $r(\boldsymbol{\sigma}_0, \mathbf{u})$, the adaptive method is able to keep the quasi-optimal rate of convergence $O(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ. Intermediate meshes obtained with the adaptive refinements are displayed in Figure 6.3. Note that the method is able to recognize the region with high gradients.

$\mathrm{RT}_0 - \mathbf{P}_1$ SCHEME WITH QUASI-UNIFORM REFINEMENT

N	h	$\mathrm{e}(\boldsymbol{\sigma}_0)$	$r(\boldsymbol{\sigma}_0)$	$\mathrm{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathrm{e}(p)$	$r(p)$
683	0.1901	19.3727	–	2.7597	–	3.3330	–
2539	0.1025	9.8414	1.0316	1.2952	1.1523	1.5895	1.1278
9883	0.0490	4.8180	1.0511	0.6137	1.0990	0.7609	1.0841
39059	0.0268	2.4470	0.9860	0.3090	0.9988	0.3852	0.9906
157043	0.0140	1.2077	1.0150	0.1523	1.0166	0.1870	1.0385
621475	0.0078	0.6079	0.9980	0.0744	1.0415	0.0944	0.9946

$\mathrm{RT}_1 - \mathbf{P}_2$ SCHEME WITH QUASI-UNIFORM REFINEMENT

N	h	$\mathrm{e}(\boldsymbol{\sigma}_0)$	$r(\boldsymbol{\sigma}_0)$	$\mathrm{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathrm{e}(p)$	$r(p)$
571	0.3727	5.7921	–	0.7352	–	2.4682	–
2287	0.1901	1.3191	2.1977	0.1895	2.0136	0.5277	2.2915
8687	0.1025	0.3284	2.2532	0.0377	2.6153	0.1194	2.4085
34199	0.0490	0.0783	1.9418	0.0087	1.9838	0.0290	1.9168
135931	0.0268	0.0206	2.0595	0.0022	2.1167	0.0076	2.0682
548107	0.0140	0.0050	2.3300	0.0005	2.3463	0.0019	2.3203

Table 6.1: Example 1: convergence history for the $\mathrm{RT}_0 - \mathbf{P}_1$ (table at the top) and $\mathrm{RT}_1 - \mathbf{P}_2$ approximations of the two dimensional version of the Navier-Stokes-Brinkman problem (2.13) under quasi-uniform refinement

λ	$h = 0.1901$	$h = 0.1025$	$h = 0.0490$	$h = 0.0256$	$h = 0.0140$
1	4	4	4	4	4
10	5	4	4	4	4
100	–	6	5	5	5
250	–	–	6	6	5
500	–	–	–	5	5

Table 6.2: Example 1: convergence behaviour of the iterative method with respect to the parameter λ

References

[1] D.N. Arnold, J. Douglas and Ch.P. Gupta, *A family of higher order mixed finite element methods for plane elasticity.* Numerische Mathematik, vol. 45, pp. 1–22, (1984).

[2] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[3] H. Brinkman *A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles* Applied Scientific Research, A1, pp. 27–34, (1947).

[4] H. Brinkman *On the permeability of media consisting of closely packed porous particles* Applied Scientific Research, A1, pp. 81–86, (1947).

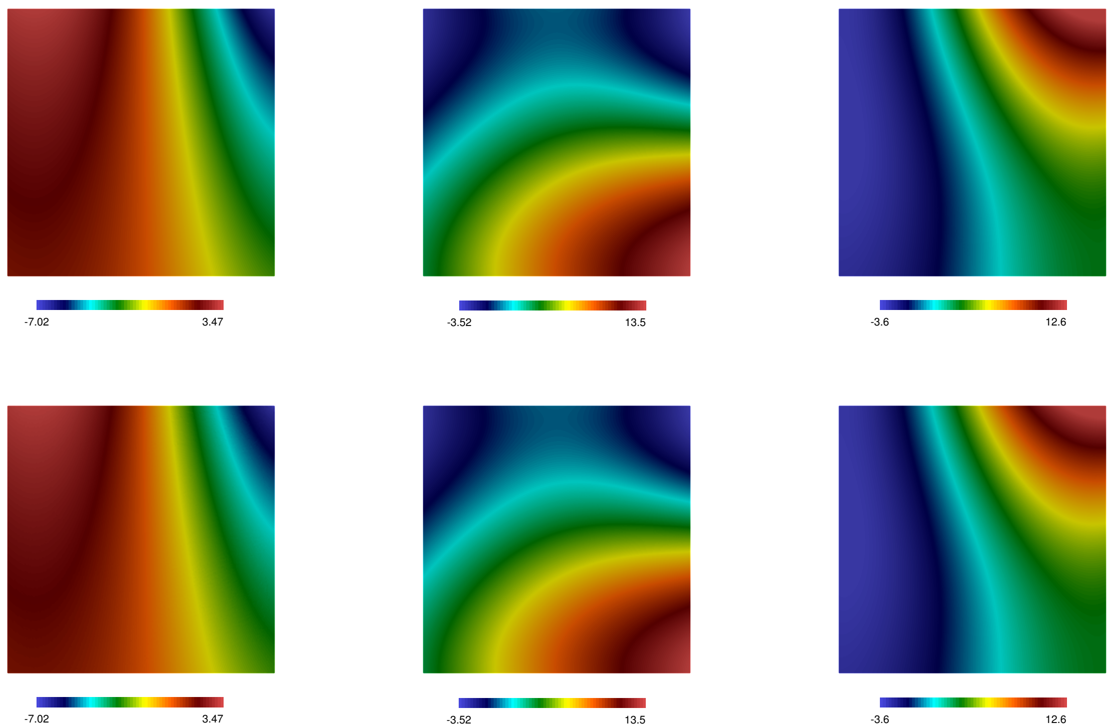


Figure 6.1: Example 1: pressure (left), first component of $\mathbf{u}$ (center) and second component of $\mathbf{u}$ (right). Exact solutions at the top and computed solutions at the bottom with $N = 621475$ and $\lambda = 10$

[5] Z. Cai and Y. Wang, *Pseudostress-velocity formulation for incompressible Navier-Stokes equations.* International Journal for Numerical Methods in Fluids, 63, no. 3, 341–356, (2010).

[6] J. Camaño, G. N. Gatica, R. Oyarzúa and G. Tierra, *An augmented mixed finite element method for the Navier-Stokes equations with variable viscosity*, SIAM Journal on Numerical Analysis, vol. 54, 2, pp. 1069-1092, (2016).

[7] J. Camaño, R. Oyarzúa and G. Tierra, *Analysis of an augmented mixed-FEM for the Navier-Stokes problem.* Mathematics of Computation, to appear.

[8] C. Cartensen, *A posteriori error estimate for the mixed finite element method*, Mathematics of Computation, vol. 66, 218, pp. 465-478, (1997).

[9] C. Cartensen and G. Dolzmann, *A posteriori error estimates for mixed FEM in elasticity*, Numerische Mathematik, vol 81, 2, pp. 187-209, (1998).

[10] S. Caucao, D. Mora and R. Oyarzúa. *A priori and a posteriori error analysis of a pseudostress-based mixed formulation of the Stokes problem with varying density.* IMA Journal of Numerical Analysis, vol. 36, 2, pp. 947-983, (2016).

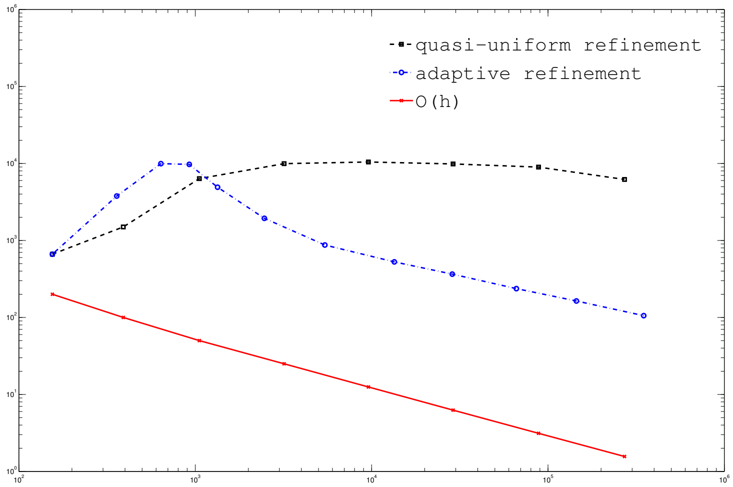


Figure 6.2: Example 2: log-log plot of the total errors vs. degrees of freedom associated to uniform and adaptive mesh refinements

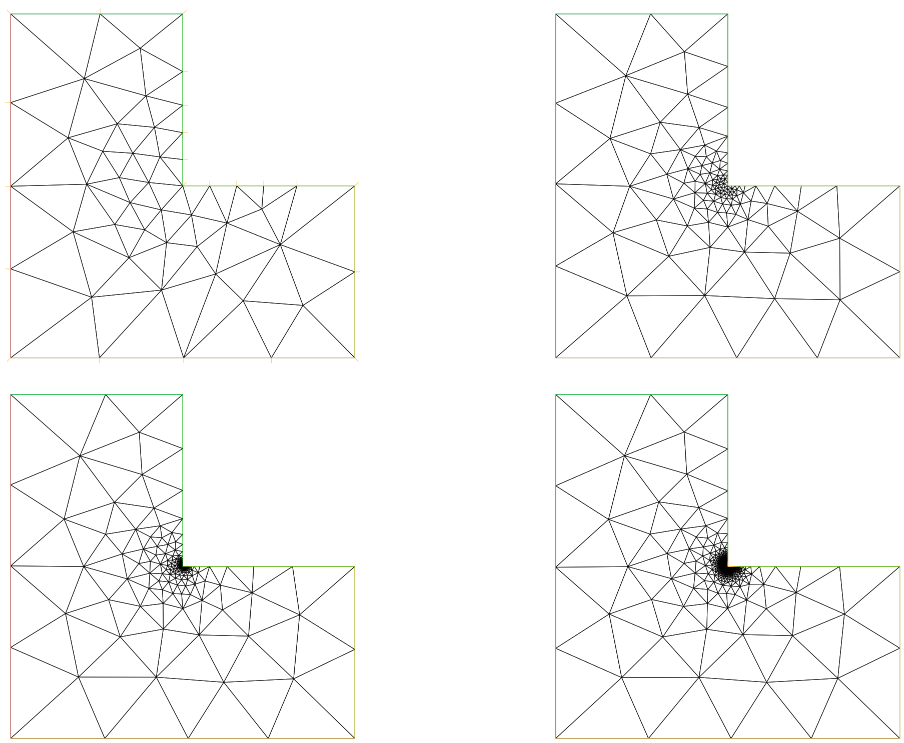


Figure 6.3: Example 2: four snapshots of successively refined meshes according to the indicator Θ

$\mathrm{RT}_0 - \mathbf{P}_1$ scheme with quasi-uniform refinement

N	$\mathrm{e}(\boldsymbol{\sigma}_0)$	$\mathrm{e}(\mathbf{u})$	$\mathrm{e}(\boldsymbol{\sigma}_0, \mathbf{u})$	$r(\boldsymbol{\sigma}_0, \mathbf{u})$	Θ	eff
155	0.6625e+03	12.5812	0.6626e+03	–	0.6734e+03	0.9839
391	1.4988e+03	15.5055	1.4988e+03	–	1.5063e+03	0.9951
1055	6.3533e+03	30.0775	6.3534e+03	–	6.3663e+03	0.9980
3191	9.9467e+03	26.0262	9.9468e+03	–	9.9507e+03	0.9996
9593	1.0444e+04	27.4547	1.0444e+04	–	1.0447e+04	0.9997
28977	9.8486e+03	19.6191	9.8486e+03	0.1063	9.8516e+03	0.9997
88435	8.9751e+03	17.5940	8.9751e+03	0.1665	8.9767e+03	0.9998
271769	6.2020e+03	10.1434	6.2020e+03	0.6584	6.2030e+03	0.9998

$\mathrm{RT}_0 - \mathbf{P}_1$ scheme with adaptive refinement

N	$\mathrm{e}(\boldsymbol{\sigma}_0)$	$\mathrm{e}(\mathbf{u})$	$\mathrm{e}(\boldsymbol{\sigma}_0, \mathbf{u})$	$r(\boldsymbol{\sigma}_0, \mathbf{u})$	Θ	eff
155	0.6625e+03	12.5812	0.6626e+03	–	0.6734e+03	0.9839
359	3.7736e+03	20.7460	3.7736e+03	–	3.7790e+03	0.9986
639	9.9393e+03	26.4936	9.9393e+03	–	9.9418e+03	0.9997
925	9.7413e+03	19.4505	9.7414e+03	0.1088	9.7434e+03	0.9998
1339	4.9119e+03	7.5203	4.9119e+03	3.7023	4.9128e+03	0.9998
2467	1.9373e+03	2.8921	1.9373e+03	3.0450	1.9380e+03	0.9997
5431	0.8723e+03	2.4567	0.8723e+03	2.0222	0.8731e+03	0.9991
13455	0.5268e+03	2.3050	0.5268e+03	1.1118	0.5274e+03	0.9989
28677	0.3656e+03	2.2725	0.3656e+03	0.9656	0.3661e+03	0.9987
66195	0.2369e+03	2.2400	0.2369e+03	1.0373	0.2373e+03	0.9984
144965	0.1637e+03	1.5330	0.1637e+03	0.9425	0.1640e+03	0.9983
349215	0.1055e+03	1.0279	0.1055e+03	1.0005	0.1057e+03	0.9981

Table 6.3: Example 2: convergence history and effectivity index for the $\mathrm{RT}_0 - \mathbf{P}_1$ approximation of the two dimensional version of the Navier-Stokes-Brinkman problem (2.13) under quasi-uniform and adaptive refinements

[11] S. Caucao, G. N. Gatica, R. Oyarzúa and I. Šebestová, *A fully-mixed finite element method for the Navier-Stokes/Darcy coupled problem with nonlinear viscosity*, Journal of Numerical Mathematics, to appear.

[12] E. Colmenares, G. N. Gatica and R. Oyarzúa *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem*, Numerical Methods for Partial Differential Equations, vol. 32, 2, pp. 445-478, (2016).

[13] D. Chen, D. Y. H. Pui, and B. Y. H. Liu. *Optimization of Pleated Filter Designs Using a Finite-Element Numerical Model.* Aerosol Science and Technology, vol. 23, 4, pp. 579–590, (1995).

[14] P.G. Ciarlet, The finite Element Method for Elliptic Problems. Studies in Mathematics and its Applications, Vol. 4. North-Holland Publishing Co., Amsterdam-New York-Oxford, (1978).

[15] P. Clément, *Approximation by finite element functions using local regularisation* RAIRO Modélisation Mathématique et Analyse Numérique, 9, 77–84, (1975).

[16] M. Discacciati and R. Oyarzúa, *A conforming mixed finite element method for the Navier-Stokes/Darcy coupled problem*, Numerische Mathematik, to appear.

[17] T. Davis, *Algorithm 832: UMFPACK V4.3 - an unsymmetric-pattern multifrontal method.* ACM Transactions on Mathematical Software, vol. 30, pp. 196–199, (2004).

[18] M. Farhloul, S. Nicaise and L. Paquet, *A priori and a posteriori error estimations for the dual mixed finite element method of the Navier-Stokes problem* Numerical Methods for Partial Differential Equations, vol. 25, no. 4, 843–869, (2009).

[19] L. Figueroa, G.N. Gatica and A. Márquez, *Augmented mixed finite element methods for the stationary Stokes Equations.* SIAM Journal on Scientific Computing, vol. 31, 2, pp. 1082–1119, (2008).

[20] L.P. Franca and T. J. Hughes, *Two classes of mixed finite element methods.* Computer Methods in Applied Mechanics and Engineering, vol. 69, 1, pp. 89–129, (1988).

[21] L.P. Franca and R. Stenberg, *Error analysis of Galerkin least squares methods for the elasticity equations*, SIAM Journal on Numerical Analysis, vol. 28, 6, pp. 1680–1697, (1991).

[22] V. Girault and P.-A. Raviart, Finite Element Approximation of the Navier–Stokes Equations. Lecture Notes in Mathematics, 749. Springer-Verlag, Berlin-New York, (1979).

[23] G.N. Gatica, *Analysis of a new augmented mixed finite element method for linear elasticity allowing* RT_0–P_1–P_0 *approximations.* ESAIM: Mathematical Modelling and Numerical Analysis, vol. 40, 1, pp. 1–28, (2006).

[24] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method. Theory and Applications. Springer Briefs in Mathematics, Springer, Cham Heidelberg New York Dordrecht London, (2014).

[25] G.N. Gatica, *A note on stable Helmholtz decompositions in 3D.* Preprint 2016-03, Centro de Investigación en Ingeniería Matemática ($\mathrm{CI^2MA}$), Universidad de Concepción, Chile.

[26] G.N. Gatica, L.F. Gatica and A. Márquez, *Analysis of a pseudostress-based mixed finite element method for the Brinkman model of porous media flow.* Numerische Mathematik, vol 126, 4, pp. 635-677, (2014).

[27] G.N. Gatica, L.F. Gatica and F. Sequeira, *Analysis of an augmented pseudostress-based mixed formulation for a nonlinear Brinkman model of porous media flow*, Computer Methods in Applied Mechanics and Engineering, vol. 289, 1, pp. 104-130, (2015).

[28] G.N. Gatica, L.F. Gatica and F. Sequeira, *A priori and a posteriori error analyses of a pseudostress-based mixed formulation for linear elasticity.* Comput. Math. Appl., vol. 71, no. 2, pp. 585-614, (2016).

[29] G.N. Gatica, A. Márquez and W. Rudolph, *A priori and a posteriori error analyses of augmented twofold saddle point formulations for nonlinear elasticity problems.* Computer Methods in Applied Mechanics and Engineering, vol. 264, 1, pp. 23-48, (2013).

[30] G.N. Gatica, A. Márquez and M.A. Sánchez, *Analysis of a velocity-pressure–pseudostress formulation for the stationary Stokes equations.* Computer Methods in Applied Mechanics and Engineering, 199, no. 17-20, pp. 1064–1079, (2010).

[31] G.N. Gatica, S. Meddahi and R. Oyarzúa, *A conforming mixed finite-element method for the coupling of fluid flow with porous media flow*, IMA Journal of Numerical Analysis, vol. 29, 1, pp. 86–108, (2009).

[32] G.N. Gatica, R. Oyarzúa and F.J. Sayas, *A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem*, Computer Methods in Applied Mechanics and Engineering, vol. 200, pp. 1877-1891, (2011).

[33] G. N. Gatica, R. Ruiz-Baier and G. Tierra, *A posteriori error analysis of an augmented mixed method for the Navier-Stokes equations with nonlinear viscosity.* Preprint 2016-11, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile.

[34] L. Guta and S. Sundar, *Navier-Stokes-Brinkman system for interaction of viscous waves with a submerged porous structure.* Tamkang journal of mathematics, vol. 41, 3, pp. 217–243, (2010).

[35] F. Hecht, *New development in FreeFem++.* Journal of Numerical Mathematics, vol. 20, 3-4, pp. 251–265, (2012).

[36] J.S. Howell and N. Walkington, *Dual Mixed Finite Element Methods for the Navier Stokes Equations.* ESAIM: Mathematical Modelling and Numerical Analysis, vol. 47, 789–805, (2013).

[37] O. Iliev and V. Laptev *On numerical simulation of flow through oil filters* Computing and Visualization in Science, vol. 6, 2, pp. 139–146, (2004).

[38] R. Ingram, *Finite element approximation of nonselenoidal, viscous flows around porous around porous media and solid obstacles.* SIAM Journal on Numerical Analysis, vol 49, no. 2, pp. 491–520, (2011).

[39] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive-Mesh-Refinement Techniques. Wiley-Teubner (Chichester), (1996).

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

PRE-PUBLICACIONES 2016

2016-16 Luis M. Castro, José González, Víctor H. Lachos, Alexandre Patriota: *A confidence set analysis for observed samples: A fuzzy set approach*

2016-17 Raimund Bürger, Stefan Diehl, Camilo Mejías: *A model of continuous sedimentation with compression and reactions*

2016-18 Carlos Garcia, Gabriel N. Gatica, Salim Meddahi: *Finite element analysis of a pressure-stress formulation for the time-domain fluid-structure interaction problem*

2016-19 Anahi Gajardo, Benjamin Hellouin, Diego Maldonado, Andres Moreira: *Nontrivial turmites are Turing-universal*

2016-20 Leonardo E. Figueroa: *Error in Sobolev norms of orthogonal projection onto polynomials in the unit ball*

2016-21 Raimund Bürger, Julio Careaga, Stefan Diehl: *Entropy solutions of a scalar conservation law modelling sedimentation in vessels with varying cross-sectional area*

2016-22 Akhlesh Lakhtakia, Peter Monk, Cinthya Rivas, Rodolfo Rodríguez, Manuel Solano: *Asymptotic model for finite-element calculations of diffraction by shallow metallic surface-relief gratings*

2016-23 Sergio Caucao, Gabriel N. Gatica, Ricardo Oyarzúa: *A posteriori error analysis of a fully-mixed formulation for the Navier-Stokes/Darcy coupled problem with nonlinear viscosity*

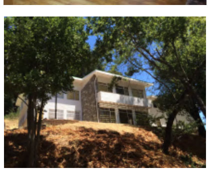
2016-24 Jessika Camaño, Gabriel N. Gatica, Ricardo Oyarzúa, Ricardo Ruiz-Baier: *An augmented stress-based mixed finite element method for the Navier-Stokes equations with nonlinear viscosity*

2016-25 Jessika Camaño, Luis F. Gatica, Ricardo Oyarzúa: *A priori and a posteriori error analyses of a flux-based mixed-FEM for convection-diffusion-reaction problems*

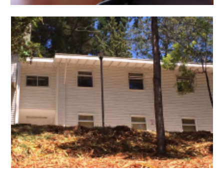
2016-26 Mario Álvarez, Gabriel N. Gatica, Ricardo Ruiz-Baier: *A posteriori error analysis for a sedimentation-consolidation system*

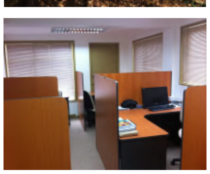
2016-27 Luis F. Gatica, Ricardo Oyarzúa, Nestor Sánchez: *A priori and a posteriori error analysis of an augmented mixed-FEM for the Navier-Stokes-Brinkman problem*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: `http://www.ci2ma.udec.cl`



CENTRO DE INVESTIGACIÓN EN
INGENIERÍA MATEMÁTICA (CI^{2}MA)
Universidad de Concepción





Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl