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## Centro de Investigación en Ingeniería Matemática (CI$^2$MA)

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# Dual-mixed finite element methods for the stationary Boussinesq problem

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**Abstract**

We propose and analyze two mixed approaches for numerically solving the stationary Boussinesq model describing heat driven flows. For the fluid equations, the velocity gradient and a Bernoulli stress tensor are introduced as auxiliary unknowns. For the heat equation, we consider primal and mixed-primal formulations; the latter, incorporating additionally the normal component of the temperature gradient on the Dirichlet boundary. Both dual–mixed formulations exhibit the same classical structure of the Navier–Stokes equations. We derive a priori estimates and the existence of continuous and discrete solutions for the formulations. In addition, we prove the uniqueness of solutions and optimal–order error estimates provided the data is sufficiently small. Numerical experiments are given which back up the theoretical results and illustrate the robustness and accuracy of both methods for a classic benchmark problem.

**Key words**: Mixed finite element methods, Boussinesq problem, mixed-primal formulation, dual-mixed formulation, a priori estimates.

**Mathematics subject classifications (2000)**: 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

## 1 Introduction

Natural convection, or heat driven flow problems, are described in the Boussinesq approximation framework by the Navier-Stokes and advection-diffusion equations, nonlinearly coupled via buoyancy forces and convective heat transfer. This model takes place in diverse situations arising in sciences and engineering. Typical examples include convective hydrothermal systems, environmental processes, thermal regulation of electronic devices, among others. In such applications, an accurate knowledge of the flow patterns contributes to the improvement of configuration designs and operating conditions. In light of this, several computational techniques have been proposed in order to predict the behavior of the fluid as well as to quantify the inherent physical variables (see, e.g., [1, 3, 5, 6, 4, 9, 10, 12, 19, 20] and the references therein).

One of the first finite element analyses for the Boussinesq problem is given in [1]. There, the model is considered with non-homogeneous Dirichlet and mixed boundary conditions for the velocity and the temperature, respectively. The authors propose a primal formulation and apply the topological degree theory to state existence results of solutions. Their results show that employing finite element spaces

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with the same order for the velocity and the temperature leads to optimal–order convergence. The analysis carried out in the aforementioned paper is later extended to a new mixed scheme developed in [9], in which both the velocity gradient and the temperature gradient of the fluid are incorporated as additional unknowns in the Boussinesq problem (with non–symmetric stress). There, the auxiliary variables are approximated by the lowest order Raviart–Thomas elements, and the primary unknowns are approximated by piecewise constants. Existence of solutions and convergence results are proven near a nonsingular solution, and quasi-optimal error estimates are also derived. Moreover, the data restriction to ensure uniqueness is more explicit than the primal method. However, that work does not address the physically relevant non-homogeneous Dirichlet condition case for the temperature, where more difficulties arise in the analysis (cf. [1, Section 2.5] and Section 3.2).

Primal methods for solving the generalized Boussinesq model, in which the viscosity and the thermal conductivity of the fluid depend on the temperature, have been also developed [19, 20]. In [19] divergence-conforming elements for the velocity, discontinuous elements for the pressure and Lagrange elements for the temperature are considered. Meanwhile, in [20] a conforming scheme is proposed involving the normal derivative of the temperature as an additional unknown on the boundary. Both works provide existence results of solutions under small data assumptions, uniqueness of continuous solutions under an additional regularity hypothesis, and optimal–order convergence of the discrete problems; however uniqueness of discrete solutions is left as an open question.

Recently, two new augmented mixed finite element schemes have been developed for solving the Boussinesq problem with Dirichlet boundary conditions [5, 6, 4]. The methods extend the methodology in [2], where a modified pseudostress tensor is introduced as an auxiliary unknown, and redundant stabilization terms are included in the variational formulation. These Galerkin schemes are convergent for arbitrary finite element spaces, and in particular, converge with optimal order if the auxiliary and primitive unknowns are approximated by Raviart–Thomas and Lagrange spaces, respectively. Additionally, other variables of physical interest can be computed by simple postprocessing of the discrete solution. However, the existence results are stated only under small data assumptions and for feasible stabilization parameters; numerically, we have found that the choice of stabilization parameters has a significant influence on the solvability, the stability, and the robustness of the numerical approximations.

The objective of this paper is to complement, to improve, and to contribute to the methodologies used so far to solve the Boussinesq problem. We propose two schemes based on a dual-mixed method developed in [14, 15] for the Navier-Stokes equations, in which the stress and the velocity gradient of the fluid are the primary unknowns of interest. Regarding the heat equation, we employ both primal and mixed-primal variational formulations. The latter incorporates the normal component of the temperature gradient on the Dirichlet boundary as an additional unknown. Both formulations exhibit the same classical structure of the Navier-Stokes equations. Using a suitable extension operator of the temperature Dirichlet data, we derive a priori estimates and establish existence of solutions for the continuous problem without data constraints.

Finite element methods based on the dual–mixed formulations are then described. Here, the velocity and the trace–free gradient are approximated by discontinuous piecewise polynomials, the stress is approximated by the Raviart–Thomas finite element space, and the temperature is approximated by the Lagrange finite element space. These discrete spaces are constructed over triangulations with a macroelement structure to ensure that an inf–sup condition and a discrete Korn inequality is satisfied. Similar to the continuous setting, we show that there exists a solution to the discrete problem. In addition we show that solutions are unique and that the errors converge quasi–optimally provided the data is sufficiently small.

### 1.1 Outline

The end of this section introduces some standard notations and function spaces. In Section 2 we state the model problem, the assumptions of the data, and the strong form of the dual–mixed formulation. We state the variational formulation of the continuous problem in Section 3 and derive a priori estimates and existence results. In addition we show that if the data is sufficiently small, then the solutions are unique. Section 4 gives the finite element method based on the dual–mixed approach. Similar to the continuous setting, we show that there exists a solution to the discrete scheme, and if the data is sufficiently small, solutions are unique. In Section 5 we introduce a mixed–primal formulation for the heat equation and state the convergence results. Finally, numerical experiments are presented in Section 6 which back up the theoretical results.

### 1.2 Notation

Let $\Omega \subset \mathrm{R}^n$ ($n \in \{2,3\}$) be a bounded domain with polyhedral boundary $\Gamma$ with outward unit normal $\boldsymbol{n}$, and let $\Gamma_{\mathrm{D}}, \Gamma_{\mathrm{N}} \subseteq \Gamma$ be such that $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, $|\Gamma_{\mathrm{D}}| \neq 0$ and $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$. We use $\mathrm{W}^{s,p}(\Omega)$ ($s \geq 0$) to denote the set of all $\mathrm{L}^p(\Omega)$ functions whose distributional derivatives up to order $s$ are in $\mathrm{L}^p(\Omega)$, and denote the corresponding norm and seminorm by $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$, respectively. The special case $p = 2$ is denoted by $\mathrm{H}^s(\Omega) := \mathrm{W}^{s,2}(\Omega)$, and the norm and seminorm are given by $\|\cdot\|_{s,\Omega} := \|\cdot\|_{s,2,\Omega}$ and $|\cdot|_{s,\Omega} := |\cdot|_{s,p,\Omega}$, respectively. The case $s = 1/2$ on the domain $\Gamma_{\mathrm{D}}$ is defined as

$$\|\phi\|_{1/2,\Gamma_{\mathrm{D}}} = \inf\big\{\|\psi\|_{1,\Omega} : \quad \psi \in \mathrm{H}^1(\Omega),\ \psi|_{\Gamma_{\mathrm{D}}} = \phi\big\}.$$

The pairing $(\cdot,\cdot)_D$ denotes the $\mathrm{L}^2$ inner product over a subdomain $D \subset \Omega$ for scalar, vector, and tensor functions; in the case $D = \Omega$ the subscript is omitted. For a scalar function space M, we denote by $\mathbf{M} = \mathrm{M}^n$ and $\mathbb{M} = \mathrm{M}^{n\times n}$ the corresponding vectorial and tensorial spaces, respectively. If M is a vector–valued function space, then we set $\mathbb{M} = \mathrm{M}^n$. The norm $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product function space. A generic, positive constant is denoted by $C$ which, unless labeled, is independent of any mesh parameters and data parameters.

## 2 The model problem

We consider the stationary Boussinesq problem for describing the motion of fluid of natural convection which is given by the following system of partial differential equations

$$\begin{aligned} -\,\mathbf{div}\,\mathcal{A}(\nabla\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p - \varphi\,\boldsymbol{g} &= 0 \quad \text{in } \Omega\,, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega\,, \\ -\,\kappa\,\Delta\,\varphi + \boldsymbol{u}\cdot\nabla\varphi &= 0 \quad \text{in } \Omega\,, \end{aligned} \tag{2.1a}$$

along with the boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,, \quad \varphi = \varphi_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \quad \text{and} \quad \frac{\partial\varphi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}\,. \tag{2.1b}$$

Here, $\mathcal{A}(\nabla\boldsymbol{u}) := \nu\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\big)$ is the symmetric gradient of $\boldsymbol{u}$, and the unknowns are the velocity $\boldsymbol{u}$, the pressure $p$, and the temperature $\varphi$ of a fluid occupying the region $\Omega$. The given data is the kinematic viscosity $\nu > 0$, the external force per unit mass $\boldsymbol{g} \in \mathbf{L}^2(\Omega)$, the boundary temperature

$\varphi_D \in H^{1/2}(\Gamma_D)$, and the thermal conductivity $\kappa > 0$. To simplify the presentation, it is assumed that the viscosity and thermal conductivity are constant.

The formulation we consider introduces as auxiliary unknowns the gradient of the velocity $G := \nabla \boldsymbol{u}$ and the Bernoulli stress tensor $S$ given by

$$S \,:=\, \mathcal{A}(G) \,-\, p\,\mathbb{I} \,-\, \frac{1}{2}\,(\boldsymbol{u}\otimes\boldsymbol{u}). \tag{2.2}$$

From the incompressibility condition, the first equation in (2.1a) becomes

$$\frac{1}{2}G\boldsymbol{u} \,-\, \mathbf{div}\,S \,-\, \varphi\,\boldsymbol{g} \,=\, 0.$$

Moreover, by taking the deviatoric part and trace in (2.2) we find that

$$S^{\mathrm{d}} \,=\, \mathcal{A}(G) \,-\, \frac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} \quad \text{in} \quad \Omega \quad \text{and} \quad p \,=\, -\frac{1}{2n}\,\mathrm{tr}(\,2S + \boldsymbol{u}\otimes\boldsymbol{u}\,)\,. \tag{2.3}$$

In this way, the pressure is eliminated from the formulation and can be recovered later by a simple postprocessing calculation through the second equation of (2.3). As a result, we consider the following system of equations with unknowns $G$, $S$, $\boldsymbol{u}$ and $\varphi$ :

$$\begin{gathered} G = \nabla\boldsymbol{u} \quad \text{in} \quad \Omega\,, \quad S^{\mathrm{d}} \,=\, \mathcal{A}(G) \,-\, \frac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \\ \frac{1}{2}G\boldsymbol{u} \,-\, \mathbf{div}\,S \,-\, \varphi\,\boldsymbol{g} = 0 \quad \text{in} \quad \Omega\,, \quad -\kappa\,\Delta\,\varphi \,+\, \boldsymbol{u}\cdot\nabla\varphi = 0 \quad \text{in} \quad \Omega\,, \\ \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,, \quad \varphi = \varphi_D \quad \text{on} \quad \Gamma_D\,, \quad \frac{\partial\varphi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_N \quad \text{and} \quad \int_\Omega \mathrm{tr}(2S \,+\, \boldsymbol{u}\otimes\boldsymbol{u}) = 0\,. \end{gathered} \tag{2.4}$$

Note that the incompressibility condition of the fluid is implicitly present in the new constitutive equation. The last statement in (2.4) ensures that the pressure has zero mean.

# 3 The continuous formulation

## 3.1 The dual-mixed variational problem

We now proceed to derive a variational formulation for the problem (2.4). Let $\mathbb{H}(\mathbf{div};\Omega)$ denote the space of square integrable matrix–valued functions with divergence (taken row–wise) in $\mathbf{L}^{4/3}(\Omega)$, and the corresponding norm by $\|\cdot\|^2_{\mathbf{div},\Omega} \,=\, \|\cdot\|^2_{0,\Omega} + \|\mathbf{div}\,\cdot\|^2_{0,4/3,\Omega}$ . Then set

$$\mathbb{H}_0(\mathbf{div};\Omega) \,:=\, \left\{ T \,\in\, \mathbb{H}(\mathbf{div};\Omega) \,:\, \quad \int_\Omega \mathrm{tr}(T) \,=\, 0 \right\},$$

so that the stress can be written as $S \,=\, S_0 \,+\, c\,\mathbb{I}$ where $S_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$ and

$$c \,=\, \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(S) = -\frac{1}{2n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{u}). \tag{3.1}$$

Since $S^{\mathrm{d}} \,=\, S_0^{\mathrm{d}}$ and $\mathbf{div}\,S \,=\, \mathbf{div}\,S_0$ , we rename $S_0$ by $S \in \mathbb{H}_0(\mathbf{div};\Omega)$ from now on and observe that the second and third equations of (2.4) remain unchanged. The incompressibility condition leads us to look for the unknown $G$ in the space

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \,:=\, \left\{\, H \,\in\, \mathbb{L}^2(\Omega) \,: \quad \mathrm{tr}(H) \,=\, 0 \,\right\}.$$

Multiplying the first equation of (2.4) by a test function $T \in \mathbb{H}_0(\mathbf{div};\Omega)$, integrating by parts and using the Dirichlet condition for $\boldsymbol{u}$, we obtain

$$(G,T) \,+\, (\boldsymbol{u}, \mathbf{div}\, T) \,=\, 0 \quad \forall\, T \in \mathbb{H}_0(\mathbf{div};\Omega).$$

Additionally, since $\mathbb{L}^2(\Omega) \,=\, \mathbb{L}^2_{\mathtt{tr}}(\Omega) \,\oplus\, \mathrm{R}\mathbb{I}$ (see, e.g., [11]), we observe that the constitutive equation can be written in the weak form as

$$(\mathcal{A}(G), H) \,-\, \frac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{u}, H) \,-\, (S,H) \,=\, 0 \quad \forall\, H \,\in\, \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.2}$$

In turn, the equilibrium relation given by the third equation in (2.4) is

$$\frac{1}{2}(G\boldsymbol{u}, \boldsymbol{v}) \,-\, (\mathbf{div}\, S, \boldsymbol{v}) \,-\, (\varphi\, \boldsymbol{g}, \boldsymbol{v}) \,=\, 0 \quad \forall\, \boldsymbol{v} \,\in\, \mathbf{L}^4(\Omega)\,. \tag{3.3}$$

For the temperature equation, we consider the closed subspace of $\mathrm{H}^1(\Omega)$ defined as

$$\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) \,:=\, \Big\{\psi \,\in\, \mathrm{H}^1(\Omega) \,: \quad \psi|_{\Gamma_\mathrm{D}} \,=\, 0\Big\}\,.$$

Multiplying the fourth equation of (2.4) by a function $\psi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, integrate by parts, and applying the Neumann boundary condition on $\Gamma_\mathrm{N}$ we get

$$\kappa\,(\nabla\varphi, \nabla\psi) \,+\, (\boldsymbol{u}\cdot\nabla\varphi, \psi) \,=\, 0 \quad \forall\, \psi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,.$$

The underlying formulation is then: Find $((G,\boldsymbol{u},\varphi), S) \,\in\, (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \,\times\, \mathbf{L}^4(\Omega) \,\times\, \mathrm{H}^1(\Omega)) \,\times\, \mathbb{H}_0(\mathbf{div};\Omega)$ such that $\varphi|_{\Gamma_\mathrm{D}} = \varphi_\mathrm{D}$ and

$$\begin{aligned}
(\mathcal{A}(G), H) \,-\, \frac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{u}, H) \,-\, (S,H) \;&=\; 0\\
\frac{1}{2}(G\boldsymbol{u}, \boldsymbol{v}) \,-\, (\mathbf{div}\, S, \boldsymbol{v}) \,-\, (\varphi\, \boldsymbol{g}, \boldsymbol{v}) \;&=\; 0\\
(G,T) \,+\, (\boldsymbol{u}, \mathbf{div}\, T) \;&=\; 0\\
\kappa\,(\nabla\varphi, \nabla\psi\big) \,+\, (\boldsymbol{u}\cdot\nabla\varphi, \psi) \;&=\; 0
\end{aligned} \tag{3.4}$$

for all $((H,\boldsymbol{v},\psi),T) \,\in\, (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)) \,\times\, \mathbb{H}_0(\mathbf{div};\Omega)$.

Similar to [14, 15], we now introduce the following forms to illustrate that the problem (3.4) exhibits the same structure as the usual formulation of the Navier-Stokes equations.

**Definition 3.1**

*1.* $\mathbf{a} : (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega))^2 \longrightarrow \mathrm{R}$,

$$\mathbf{a}((G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,=\, (\mathcal{A}(G), H) \,+\, \kappa\,(\nabla\varphi, \nabla\psi)\,. \tag{3.5}$$

*2.* $\mathbf{b} \,:\, \mathbb{H}_0(\mathbf{div};\Omega) \,\times\, (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times\, \mathbf{L}^4(\Omega)) \longrightarrow\, \mathrm{R}$,

$$\mathbf{b}(T,(G,\boldsymbol{u})) \,=\, (G,T) \,+\, (\boldsymbol{u}, \mathbf{div}\, T)\,. \tag{3.6}$$

*3.* $\mathbf{c} \,:\, (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega))^3 \longrightarrow \mathrm{R}$,

$$\mathbf{c}((F,\boldsymbol{w},\phi),(G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \;=\; \frac{1}{2}\big[\,(G\boldsymbol{w},\boldsymbol{v}) \,-\, (H\boldsymbol{w},\boldsymbol{u})\big] \,+\, (\boldsymbol{w}\cdot\nabla\varphi, \psi)\,.$$

The variational problem (3.4) can be written in the dual-mixed form: Find $((G,\boldsymbol{u},\varphi),S) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times \mathbf{L}^4(\Omega)\times \mathrm{H}^1(\Omega)) \times \mathbb{H}_0(\mathbf{div};\Omega)$ such that with $\varphi|_{\Gamma_\mathrm{D}} = \varphi_\mathrm{D}$ and

$$\begin{aligned} \mathbf{a}((G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,+\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,-\, \mathbf{b}(S,(H,\boldsymbol{v})) &= (\varphi\,\boldsymbol{g},\boldsymbol{v}),\\ \mathbf{b}(T,(G,\boldsymbol{u})) &= 0\,, \end{aligned} \tag{3.7}$$

for all $((H,\boldsymbol{v},\psi),T) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times \mathbf{L}^4(\Omega)\times \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega))\times \mathbb{H}_0(\mathbf{div};\Omega)$.

To simplify the presentation we define $\mathbb{H} := \mathbb{Z} \times \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, where $\mathbb{Z}$ is the kernel of $\mathbf{b}(\cdot,\cdot)$:

$$\mathbb{Z} := \Big\{ (H,\boldsymbol{v}) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\times \mathbf{L}^4(\Omega): \quad (H,T) + (\boldsymbol{v},\mathbf{div}\,T) \,=\, 0 \qquad T\in \mathbb{H}_0(\mathbf{div};\Omega)\Big\}\,. \tag{3.8}$$

Since the solution $(G,\boldsymbol{u})$ to (3.7) belongs to $\mathbb{Z}$, we deduce that

$$(G,\boldsymbol{u})\in\mathbb{Z} \quad\Longrightarrow\quad \boldsymbol{u} \,\in\, \mathbf{H}^1_0(\Omega)\,, \quad G \,=\, \nabla\boldsymbol{u} \quad\text{and}\quad \operatorname{div}\boldsymbol{u} \,=\, 0\,. \tag{3.9}$$

We summarize some key properties of the forms in the next lemma.

**Lemma 3.1** *Let $\mathbf{a}(\cdot,\cdot)$, $\mathbf{b}(\cdot,\cdot)$, $\mathbf{c}(\cdot,\cdot,\cdot)$ be the forms given in Definition* 3.1*.*

1. *$\mathbf{a}(\cdot,\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are continuous and $\mathbf{a}(\cdot,\cdot)$ is coercive on $\mathbb{H}$, i.e., there exists $C_a>0$, such that*
$$\mathbf{a}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi)) \,\geq\, C_a\,\|(G,\boldsymbol{u},\varphi)\|^2 \quad \forall\,(G,\boldsymbol{u},\varphi)\,\in\,\mathbb{H}\,.$$

2. *There exists $\beta>0$, such that*
$$\sup_{\substack{(G,\boldsymbol{u})\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{L}^4(\Omega)\\ (G,\boldsymbol{u})\neq\mathbf{0}}} \frac{\mathbf{b}(S,(G,\boldsymbol{u}))}{\|(G,\boldsymbol{u})\|} \,\geq\, \beta\,\|S\|_{\mathbf{div},\Omega} \quad \forall\,S\,\in\,\mathbb{H}_0(\mathbf{div};\Omega)\,,$$

3. *$\mathbf{c}(\cdot,\cdot,\cdot):\mathbb{H}\times\mathbb{H}\to\big(\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbf{L}^4(\Omega)\times\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\big)'$ is weakly continuous.*

*Proof.* The continuity of the bilinear forms $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ follows from Cauchy-Schwarz inequality. The coercivity of $\mathbf{a}(\cdot,\cdot)$ follows from the Korn inequality, the Poincare inequality, and the definition of $\mathbb{H}$, and the inf–sup condition is proven in [14, 15, Lemma 2.4].

To show the weak continuity of $\mathbf{c}(\cdot,\cdot,\cdot)$, let $(G,\boldsymbol{u},\varphi)\in\mathbb{H}$ and $\{(G_n,\boldsymbol{u}_n,\varphi_n)\}_{n\geq1}\subset\mathbb{H}$ such that $(G_n,\boldsymbol{u}_n,\varphi_n)\rightharpoonup(G,\boldsymbol{u},\varphi)$ in $\mathbb{H}$. Then, it follows from (3.9) that

$$\boldsymbol{u},\boldsymbol{u}_n\in\mathbf{H}^1_0(\Omega)\,,\quad G_n=\nabla\boldsymbol{u}_n \quad G=\nabla\boldsymbol{u} \quad\text{and}\quad \operatorname{div}(\boldsymbol{u}_n) \,=\, \operatorname{div}(\boldsymbol{u}) \,=\, 0\,,\quad \text{for each } n\,,$$

and therefore $\boldsymbol{u}_n\to\boldsymbol{u}$ (and $\varphi_n\to\varphi$) strongly in $\mathbf{L}^4(\Omega)$ due to the Rellich-Kondrachov Theorem. Using the definition of $\mathbf{c}(\cdot,\cdot,\cdot)$, we find for all $(H,\boldsymbol{v},\psi)\in\mathbb{H}$ that

$$\begin{aligned} &\mathbf{c}((G_n,\boldsymbol{u}_n,\varphi_n),(G_n,\boldsymbol{u}_n,\varphi_n),(H,\boldsymbol{v},\psi)) \,-\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \\ &\quad= \frac{1}{2}\big[(G_n\boldsymbol{u}_n,\boldsymbol{v}) \,-\, (H\boldsymbol{u}_n,\boldsymbol{u}_n)\big] \,+\, (\boldsymbol{u}_n\cdot\nabla\varphi_n,\psi) \,-\, \frac{1}{2}\big[(G\boldsymbol{u},\boldsymbol{v}) \,-\, (H\boldsymbol{u},\boldsymbol{u})\big] \,-\, (\boldsymbol{u}\cdot\nabla\varphi,\psi) \\ &\quad= \frac{1}{2}\big[((G_n-G)\boldsymbol{u}_n,\boldsymbol{v}) + (G(\boldsymbol{u}_n-\boldsymbol{u}),\boldsymbol{v}) + (H(\boldsymbol{u}-\boldsymbol{u}_n),\boldsymbol{u}_n) + (H\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}_n)\big] \\ &\qquad - ((\boldsymbol{u}_n-\boldsymbol{u})\cdot\nabla\psi,\varphi_n) + (\boldsymbol{u}\cdot\nabla\psi,\varphi-\varphi_n) \\ &\quad\leq \frac{1}{2}\big[(G_n-G,\boldsymbol{v}\otimes\boldsymbol{u}_n) + \|\boldsymbol{u}_n-\boldsymbol{u}\|_{0,4,\Omega}\big(\|G\|_{0,\Omega}\|\boldsymbol{v}\|_{0,4,\Omega} + \|H\|_{0,\Omega}(\|\boldsymbol{u}\|_{0,4,\Omega}+\|\boldsymbol{u}_n\|_{0,4,\Omega})\big)\big] \\ &\qquad\quad + \|\boldsymbol{u}_n-\boldsymbol{u}\|_{0,4,\Omega}\|\nabla\psi\|_{0,\Omega}\|\varphi_n\|_{0,4,\Omega} + \|\boldsymbol{u}\|_{0,4,\Omega}\|\nabla\psi\|_{0,\Omega}\|\varphi-\varphi_n\|_{0,4,\Omega} \quad\longrightarrow\quad 0 \text{ as } \ n\to\infty\,, \end{aligned} \tag{3.10}$$

which follows from the fact that $\{\boldsymbol{u}_n\}_{n\geq1}$ and $\{\varphi_n\}_{n\geq1}$ are bounded sequences in their corresponding spaces. Thus, $\mathbf{c}(\cdot,\cdot,\cdot)$ is weakly continuous. $\square$

### 3.2 Well-posedness

Observe that the problem (3.7) can be equivalently written as: Find $((G,\boldsymbol{u}),\varphi) \in \mathbb{Z} \times \mathrm{H}^1(\Omega)$ with $\varphi|_{\Gamma_\mathrm{D}} = \varphi_\mathrm{D}$ and such that

$$\mathbf{a}((G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,+\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,=\, (\varphi\,\boldsymbol{g},\boldsymbol{v}) \quad \forall\,(H,\boldsymbol{v},\psi) \in \mathbb{H}\,, \tag{3.11}$$

which follows straightforwardly from the properties of the forms stated in Lemma 3.1. In this way, the solvability of our dual-mixed formulation is studied as follows. In Section 3.2.1 below, we derive a priori estimates for continuous solutions $G$, $\boldsymbol{u}$ and $\varphi$ to the restricted problem (3.11). Next, in Section 3.2.2, we employ a fixed point approach to establish existence and uniqueness results. Then, the inf-sup condition of the bilinear form $\mathbf{b}(\cdot,\cdot)$ stated in the previous lemma will be applied to show the existence of the tensor $S$.

#### 3.2.1 A priori estimates

To derive estimates for solutions of (3.11), we require the following technical result.

**Lemma 3.2** *Let $\Omega$ be a bounded domain in $\mathrm{R}^n$, $n=2$ or $n=3$, with Lipschitz continuous boundary. Then for any $\delta \in (0,1)$, there exists an extension operator $E_\delta : \mathrm{H}^{1/2}(\Gamma_\mathrm{D}) \to \mathrm{H}^1(\Omega)$ such that $\|E_\delta\psi\|_{0,3,\Omega} \le C\delta\|\psi\|_{1/2,\Gamma_\mathrm{D}}$ and $\|E_\delta\psi\|_{1,\Omega} \le C\delta^{-4}\|\psi\|_{1/2,\Gamma_\mathrm{D}}$ for all $\psi \in \mathrm{H}^{1/2}(\Gamma_\mathrm{D})$.*

*Proof.* We employ arguments similar to in [1, Lemma 2.8] and [17, Lemma 4.1].

Define the subdomain

$$\Omega_\delta \,:=\, \left\{\, \mathbf{x} \in \mathbf{R} : \quad \mathrm{dist}(\mathbf{x},\Gamma) \,<\, \delta^6 \,\right\},$$

and let $\beta_\delta \in \mathrm{W}^{1,\infty}(\Omega)$ such that

$$0 \,\le\, \beta_\delta \,\le\, 1 \;\text{ in }\; \Omega_\delta\,, \quad \beta_\delta \,\equiv\, 0 \;\text{ in }\; \mathbf{R}\setminus\Omega_\delta\,, \quad \text{and} \quad \|\nabla\beta_\delta\|_{\infty,\Omega} \le C\delta^{-6}\,.$$

Let $E : \mathrm{H}^{1/2}(\Gamma_\mathrm{D}) \to \mathrm{H}^1(\Omega)$ be an extension operator satisfying $\|E\psi\|_{1,\Omega} \le C\|\psi\|_{1/2,\Gamma_\mathrm{D}}\ \forall\,\psi \in \mathrm{H}^{1/2}(\Gamma_\mathrm{D})$, and set $E_\delta := \beta_\delta E$. We then have, by Hölder's inequality and a Sobolev embedding,

$$\|E_\delta\psi\|^3_{0,3,\Omega} \le \|E\psi\|^3_{0,3,\Omega\cap\Omega_\delta} \le |\Omega_\delta|^{1/2}\|E\psi\|^3_{0,6,\Omega} \le C\delta^3\|E\psi\|^3_{1,\Omega} \le C\delta^3\|\psi\|^3_{1/2,\Gamma_\mathrm{D}}.$$

This is the first inequality. By similar arguments we find

$$\begin{aligned}\|\nabla E_\delta\psi\|_{0,\Omega} &\le C\delta^{-6}\|E\psi\|_{0,\Omega\cap\Omega_\delta} + \|\nabla E\psi\|_{0,\Omega}\\ &\le C\delta^{-6}|\Omega_\delta|^{1/3}\|E\psi\|_{0,6,\Omega} + \|\nabla E\psi\|_{0,\Omega} \le C\delta^{-4}\|\psi\|_{1/2,\Gamma_\mathrm{D}}\,,\end{aligned}$$

which gives the desired result. □

**Theorem 3.3** *Any solution $(G,\boldsymbol{u},\varphi)$ to (3.11) satisfies the a priori estimates*

$$\|(G,\boldsymbol{u})\| \,\le\, C_1(\varphi_\mathrm{D},\boldsymbol{g}) \qquad \text{and} \qquad \|\varphi\|_{1,\Omega} \,\le\, C_2(\varphi_\mathrm{D},\boldsymbol{g})\,, \tag{3.12}$$

*where $C_1(\varphi_\mathrm{D},\boldsymbol{g}) = C\nu^{-5}\kappa^{-4}\|\varphi_\mathrm{D}\|^5_{1/2,\Gamma_\mathrm{D}}\|\boldsymbol{g}\|^5_{0,\Omega}$, and $C_2(\varphi_\mathrm{D},\boldsymbol{g}) = C\nu^{-4}\kappa^{-4}\|\varphi_\mathrm{D}\|^5_{1/2,\Gamma_\mathrm{D}}\|\boldsymbol{g}\|^4_{0,\Omega}$.*

*Proof.* Let $\varphi_1 = E_\delta\varphi_\mathrm{D} \in \mathrm{H}^1(\Omega)$ be an extension of $\varphi_\mathrm{D}$ with $\delta > 0$ to be determined (cf. Lemma 3.2), and set $\varphi_0 = \varphi - \varphi_1 \,\in\, \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$. Replacing $\varphi = \varphi_0 + \varphi_1$ into (3.11) yields

$$\begin{aligned}\mathbf{a}((G,\boldsymbol{u},\varphi_0),(H,\boldsymbol{v},\psi)) + \mathbf{c}((G,\boldsymbol{u},\varphi_0),(G,\boldsymbol{u},\varphi_0),(H,\boldsymbol{v},\psi)) \,&=\, (\varphi_0\,\boldsymbol{g},\boldsymbol{v}) \,+\, (\varphi_1\,\boldsymbol{g},\boldsymbol{v})\\ -\,\kappa\,(\nabla\varphi_1,\nabla\psi) \,-\, (\boldsymbol{u}\cdot\nabla\varphi_1,\psi) \quad &\forall\,(H,\boldsymbol{v},\psi) \in \mathbb{H}\,.\end{aligned}$$

Decoupling the equations, taking $(H, \boldsymbol{v}, \psi) = (G, \boldsymbol{u}, \varphi_0)$, and using the skew-symmetric property of $\mathbf{c}(\cdot,\cdot,\cdot)$, we obtain

$$\begin{aligned} (\mathcal{A}(G), G) &= (\varphi_0\,\boldsymbol{g}, \boldsymbol{u}) + (\varphi_1\,\boldsymbol{g}, \boldsymbol{u}) \\ \kappa\,\|\nabla\varphi_0\|_{0,\Omega}^2 &= -\kappa\,(\nabla\varphi_1, \nabla\varphi_0) - (\boldsymbol{u}\cdot\nabla\varphi_1, \varphi_0) = -\kappa\,(\nabla\varphi_1, \nabla\varphi_0) + (\boldsymbol{u}\cdot\nabla\varphi_0, \varphi_1)\,, \end{aligned} \tag{3.13}$$

where an integration–by–parts formula was used to derive the last equality. Next, applying Hölder's inequality in the first equation of (3.13) and two Sobolev embeddings, we find that

$$\big(\mathcal{A}(G), G\big) \,\le\, \|\boldsymbol{g}\|_{0,\Omega}\,\big(\,\|\varphi_0\|_{0,3,\Omega} + \|\varphi_1\|_{0,3,\Omega}\,\big)\,\|\boldsymbol{u}\|_{0,6,\Omega} \,\le\, C\|\boldsymbol{g}\|_{0,\Omega}\,\big(\,\|\varphi_0\|_{1,\Omega} + \|\varphi_1\|_{1,\Omega}\,\big)\,\|G\|_{0,\Omega}\,.$$

Therefore by Korn's inequality and the estimate $\|\boldsymbol{u}\|_{0,4,\Omega} \le C\|G\|_{0,\Omega}$,

$$\nu\,\|(G, \boldsymbol{u})\| \,\le\, C\,\|\boldsymbol{g}\|_{0,\Omega}\,\big(\,\|\varphi_0\|_{1,\Omega} + \|\varphi_1\|_{1,\Omega}\,\big)\,. \tag{3.14}$$

Likewise, from the second equation in (3.13), we bound the $\mathrm{L}^2-$norm of $\nabla\varphi_0$ by applying Hölder's inequality and a Sobolev embedding:

$$\kappa\|\nabla\varphi_0\|_{0,\Omega}^2 \,\le\, \kappa\,\|\nabla\varphi_1\|_{0,\Omega}\,\|\nabla\varphi_0\|_{0,\Omega} + C\,\|G\|_{0,\Omega}\,\|\nabla\varphi_0\|_{0,\Omega}\,\|\varphi_1\|_{0,3,\Omega}\,. \tag{3.15}$$

Therefore, simplifying and applying the Poincaré inequality and Lemma 3.2, we obtain

$$\|\varphi_0\|_{1,\Omega} \,\le\, C\big(\,\|\varphi_1\|_{1,\Omega} + \kappa^{-1}\,\delta\,\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|(G, \boldsymbol{u})\|\,\big). \tag{3.16}$$

Thus, applying this estimate in (3.14), we have

$$\nu\,\|(G, \boldsymbol{u})\| \,\le\, C\|\boldsymbol{g}\|_{0,\Omega}\,\big(\|\varphi_1\|_{1,\Omega} + \kappa^{-1}\,\delta\,\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|(G, \boldsymbol{u})\|\,\big)\,.$$

Taking $\delta > 0$ such that

$$C\,\kappa^{-1}\,\delta\,\nu^{-1}\,\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{g}\|_{0,\Omega} \,=\, \frac{1}{2} \tag{3.17}$$

then yields

$$\|(G, \boldsymbol{u})\| \,\le\, C\,\nu^{-1}\|\boldsymbol{g}\|_{0,\Omega}\,\|\varphi_1\|_{1,\Omega}\,. \tag{3.18}$$

Finally, we obtain the a priori estimate for $\varphi$ by combining (3.17)–(3.18) with (3.16):

$$\begin{aligned} \|\varphi\|_{1,\Omega} &\le \|\varphi_0\|_{1,\Omega} + \|\varphi_1\|_{1,\Omega} \\ &\le C\big(\,\|\varphi_1\|_{1,\Omega} + \kappa^{-1}\,\delta\,\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\nu^{-1}\,\|\boldsymbol{g}\|_{0,\Omega}\|\varphi_1\|_{1,\Omega}\big) \le C\|\varphi_1\|_{1,\Omega}. \end{aligned} \tag{3.19}$$

The desired estimate (3.12) now follows from (3.17)–(3.19) and Lemma 3.2. □

### 3.2.2 Existence of solutions

In this section we establish an existence result to the problem (3.11) by using the standard Leray-Schauder principle (cf. [13, Theorem 11.3], [22, Theorem 6.A], [16],[18]). To this end, for $(G, \boldsymbol{u}, \varphi_0) \in \mathbb{H}$, we define the linear functionals $\mathcal{F}_{i,(G,\boldsymbol{u},\varphi_0)} : \mathbb{H} \to \mathbb{H}'$ by

$$\begin{aligned} \mathcal{F}_{1,(G,\boldsymbol{u},\varphi_0)}((H, \boldsymbol{v}, \psi)) &= -\mathbf{c}((G, \boldsymbol{u}, \varphi_0), (G, \boldsymbol{u}, \varphi_0), (H, \boldsymbol{v}, \psi)) \\ \mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)}((H, \boldsymbol{v}, \psi)) &= -\,(\boldsymbol{u}\cdot\nabla\varphi_1, \psi) + (\varphi_0\,\boldsymbol{g}, \boldsymbol{v})\,, \\ \mathcal{F}_3((H, \boldsymbol{v}, \psi)) &= (\varphi_1\,\boldsymbol{g}, \boldsymbol{v}) - \kappa\,(\nabla\varphi_1, \nabla\psi)\,, \end{aligned} \tag{3.20}$$

for all $(H, \boldsymbol{v}, \psi) \in \mathbb{H}$, where $\varphi_1 = E_\delta\varphi_{\mathrm{D}} \in \mathrm{H}^1(\Omega)$ with $\delta > 0$ given by (3.17).

**Lemma 3.4** *The functionals $\mathcal{F}_{i,(G,\boldsymbol{u},\varphi_0)}$ satisfy*

$$\|\mathcal{F}_{1,(G,\boldsymbol{u},\varphi_0)}\|_{\mathbb{H}'} \,\leq\, C_3(\boldsymbol{u})\,\|(G,\boldsymbol{u},\varphi_0)\|\,, \quad \|\mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)}\|_{\mathbb{H}'} \,\leq\, C_4(\varphi_{\mathrm{D}},\boldsymbol{g})\,\|(G,\boldsymbol{u},\varphi_0)\| \\ \text{and} \quad \|\mathcal{F}_3\|_{\mathbb{H}'} \,\leq\, C_5(\varphi_{\mathrm{D}},\boldsymbol{g})\,, \tag{3.21}$$

*with $C_3(\boldsymbol{u}) = C\|\boldsymbol{u}\|_{0,3,\Omega}$, $C_4(\varphi_{\mathrm{D}},\boldsymbol{g}) = C\max\{\|\boldsymbol{g}\|_{0,\Omega}, \nu^{-4}\kappa^{-4}\|\varphi_{\mathrm{D}}\|^5_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{g}\|^4_{0,\Omega}\}$, and $C_5(\varphi_{\mathrm{D}},\boldsymbol{g}) = C\nu^{-4}\kappa^{-4}\|\varphi_{\mathrm{D}}\|^5_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{g}\|^4_{0,\Omega}\left(\kappa + \|\boldsymbol{g}\|_{0,\Omega}\right)$.*

The proof of Lemma 3.4 is standard, relying only on Hölder's inequality, Sobolev embeddings and Lemma 3.2; the proof is given in Appendix A.

We consider the sequence of fixed point problems: Find $(G,\boldsymbol{u},\varphi_0) \in \mathbb{H}$ such that

$$(G,\boldsymbol{u},\varphi_0) \,=\, \tau A((G,\boldsymbol{u},\varphi_0)) \quad \text{for each} \quad \tau \in [0,1]\,, \tag{3.22}$$

where the operator $\tau A : \mathbb{H} \longrightarrow \mathbb{H}$ is defined for all $(G,\boldsymbol{u},\varphi_0) \in \mathbb{H}$ as $\tau A((G,\boldsymbol{u},\varphi_0)) \,=\, (\widehat{G},\widehat{\boldsymbol{u}},\widehat{\varphi}_0)$ and $(\widehat{G},\widehat{\boldsymbol{u}},\widehat{\varphi}_0) \in \mathbb{H}$ satisfies

$$\mathbf{a}((\widehat{G},\widehat{\boldsymbol{u}},\widehat{\varphi}_0),(H,\boldsymbol{v},\psi)) \,=\, \tau\big(\mathcal{F}_{1,(G,\boldsymbol{u},\varphi_0)} \,+\, \mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)} \,+\mathcal{F}_3\big)\,(H,\boldsymbol{v},\psi) \quad \forall\,(H,\boldsymbol{v},\psi) \in \mathbb{H}\,. \tag{3.23}$$

In this way, we realize that the problems (3.11) and (3.22) (with $\tau = 1$) are equivalent.

We observe that $\tau A$ is well-defined by virtue of Lax-Milgram Theorem (see e.g. [11, Theorem 1.1]), since $\mathbf{a}(\cdot,\cdot)$ is continuous and coercive on $\mathbb{H}$ (see Lemma 3.1), and $\mathcal{F}_{1,(G,\boldsymbol{u},\varphi_0)} + \mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)} + \mathcal{F}_3 \in \mathbb{H}'$.

**Lemma 3.5** *The operator $A$ given by* (3.22) *is compact. Moreover, the operator is locally Lipschitz continuous:*

$$\|A((G,\boldsymbol{u},\varphi_0)) - A((G',\boldsymbol{u}',\varphi_0'))\| \,\leq C_{\mathrm{LIP}}\|(G-G',\boldsymbol{u}-\boldsymbol{u}',\varphi_0-\varphi_0')\|, \tag{3.24}$$

*with*

$$C_{\mathrm{LIP}} = C_{\mathrm{LIP}}(G,\boldsymbol{u},\boldsymbol{u}',\varphi_0',\varphi_{\mathrm{D}},\boldsymbol{g}) = C_a^{-1}\Big\{C\Big(\|G\|_{0,\Omega} + \|\boldsymbol{u}\|_{0,4,\Omega} + \|\boldsymbol{u}'\|_{0,4,\Omega} + \|\varphi_0'\|_{0,4,\Omega}\Big) + C_4(\varphi_{\mathrm{D}},\boldsymbol{g})\Big\},$$

*and $C_a = C\min\{\nu,\kappa\}$ is the coercivity constant of the bilinear form $\mathbf{a}(\cdot,\cdot)$.*

*Proof.* To prove the compactness property, consider $(G,\boldsymbol{u},\varphi_0) \in \mathbb{H}$ and $\{(G_n,\boldsymbol{u}_n,\varphi_n)\}_{n\geq 1} \subset \mathbb{H}$ such that $(G_n,\boldsymbol{u}_n,\varphi_n) \rightharpoonup (G,\boldsymbol{u},\varphi_0)$ in $\mathbb{H}$. For clarity, we set $\Psi_n = (G_n,\boldsymbol{u}_n,\varphi_n) \in \mathbb{H}$, $\Psi = (G,\boldsymbol{u},\varphi_0) \in \mathbb{H}$, and

$$A(\Psi_n) \,=\, \widehat{\Psi}_n = (\widehat{G}_n,\widehat{\boldsymbol{u}}_n,\widehat{\varphi}_n) \quad \text{and} \quad A(\Psi) \,=\, \widehat{\Psi} = (\widehat{G},\widehat{\boldsymbol{u}},\widehat{\varphi})\,.$$

Using the coercivity and linearity of $\mathbf{a}(\cdot,\cdot)$ and the definition (3.23) of $A$, we find that

$$\begin{aligned} &\|A(\Psi_n) \,-\, A(\Psi)\|^2 \,=\, \|\widehat{\Psi}_n - \widehat{\Psi}\|^2 \\ &\leq\, {C_a}^{-1}\,\mathbf{a}(\widehat{\Psi}_n - \widehat{\Psi},\widehat{\Psi}_n - \widehat{\Psi}) = C_a^{-1}\Big\{\mathbf{a}(\widehat{\Psi}_n,\widehat{\Psi}_n - \widehat{\Psi}) - \mathbf{a}(\widehat{\Psi},\widehat{\Psi}_n - \widehat{\Psi})\Big\} \\ &=\, {C_a}^{-1}\Big\{\big(\mathcal{F}_{1,\Psi_n} - \mathcal{F}_{1,\Psi}\big)(\widehat{\Psi}_n - \widehat{\Psi}) \,+\, \big(\mathcal{F}_{2,\Psi_n} - \mathcal{F}_{2,\Psi}\big)(\widehat{\Psi}_n - \widehat{\Psi})\Big\}\,. \end{aligned} \tag{3.25}$$

Using the definition of $\mathcal{F}_1$ and the weak continuity of $\mathbf{c}(\cdot,\cdot,\cdot)$ (see Lemma 3.1)), we have that

$$\big(\mathcal{F}_{1,\Psi_n} - \mathcal{F}_{1,\Psi}\big)(\widehat{\Psi}_n - \widehat{\Psi}) \quad \longrightarrow \quad 0 \quad \text{as} \quad n \to \infty\,. \tag{3.26}$$

On the other hand, using the definition of $\mathcal{F}_2$ from (3.20), it follows that

$$
\begin{aligned}
\big(\mathcal{F}_{2,\Psi_n} - \mathcal{F}_{2,\Psi}\big)(\widehat{\Psi}_n - \widehat{\Psi}) &= \mathcal{F}_{2,\Psi_n-\Psi}(\widehat{\Psi}_n - \widehat{\Psi}) \,=\, ((\boldsymbol{u}_n - \boldsymbol{u})\cdot\nabla\varphi_1, \widehat{\varphi}_n - \widehat{\varphi}) \,-\, ((\varphi_n - \varphi_0)\boldsymbol{g}, \widehat{\boldsymbol{u}}_n - \widehat{\boldsymbol{u}}) \\
&\leq\, \|\boldsymbol{u}_n - \boldsymbol{u}\|_{0,4,\Omega}\,\|\nabla\varphi_1\|_{0,\Omega}\,\|\widehat{\varphi}_n - \widehat{\varphi}\|_{0,4,\Omega} \,+\, \|\boldsymbol{g}\|_{0,\Omega}\,\|\varphi_n - \varphi_0\|_{0,4,\Omega}\,\|\widehat{\boldsymbol{u}}_n - \widehat{\boldsymbol{u}}\|_{0,4,\Omega} \,\longrightarrow\, 0\,.
\end{aligned} \tag{3.27}
$$

and thus, according to (3.26) and (3.27) we deduce from (3.25) that

$$
\|A((G_n, \boldsymbol{u}_n, \varphi_n)) \,-\, A((G, \boldsymbol{u}, \varphi_0))\| \,\longrightarrow\, 0 \quad \text{as} \quad n \to \infty\,,
$$

and therefore $\{A((G_n, \boldsymbol{u}_n, \varphi_n))\}_{n\geq 1}$ converges strongly to $A((G, \boldsymbol{u}, \varphi_0))$ in $\mathbb{H}$; hence, $A$ is compact.

To show Lipschitz continuity, we take $\Psi = (G, \boldsymbol{u}, \varphi_0) \in \mathbb{H}$, $\Psi' = (G', \boldsymbol{u}', \varphi_0') \in \mathbb{H}$ and denote

$$
A(\Psi) \,=\, \widehat{\Psi} = (\widehat{G}, \widehat{\boldsymbol{u}}, \widehat{\varphi}_0) \quad \text{and} \quad A(\Psi') \,=\, \widehat{\Psi}' = (\widehat{G}', \widehat{\boldsymbol{u}}', \widehat{\varphi}_0')\,.
$$

Proceeding similarly as in (3.25) we get

$$
\|A(\Psi) \,-\, A(\Psi')\|^2 \,=\, \|\widehat{\Psi} - \widehat{\Psi}'\|^2 \,\leq\, {C_a}^{-1}\Big\{ \big(\mathcal{F}_{1,\Psi} \,-\, \mathcal{F}_{1,\Psi'}\big)(\widehat{\Psi} - \widehat{\Psi}') \,+\, \mathcal{F}_{2,\Psi-\Psi'}(\widehat{\Psi} - \widehat{\Psi}')\Big\}\,. \tag{3.28}
$$

From the estimate (3.10), we find

$$
\begin{aligned}
\big(\mathcal{F}_{1,\Psi} \,-\, \mathcal{F}_{1,\Psi'}\big)(\widehat{\Psi} - \widehat{\Psi}') &= -\mathbf{c}(\Psi, \Psi, \widehat{\Psi} - \widehat{\Psi}') + \mathbf{c}(\Psi', \Psi', \widehat{\Psi} - \widehat{\Psi}') \\
&\leq C\Big(\|G\|_{0,\Omega} + \|\boldsymbol{u}\|_{0,4,\Omega} + \|\boldsymbol{u}'\|_{0,4,\Omega} + \|\varphi_0'\|_{0,4,\Omega}\Big)\|\Psi - \Psi'\|\|\widehat{\Psi} - \widehat{\Psi}'\|.
\end{aligned} \tag{3.29}
$$

Next, applying the estimate (3.21) we obtain

$$
\big|\mathcal{F}_{2,\Psi-\Psi'}(\widehat{\Psi} - \widehat{\Psi}')\big| \,\leq\, C_4(\varphi_{\mathrm{D}}, \boldsymbol{g})\,\|\Psi - \Psi'\|\|\widehat{\Psi} - \widehat{\Psi}'\|\,. \tag{3.30}
$$

The Lipschitz condition (3.24) now follows from (3.28) and the estimates (3.29)–(3.30). □

Next, we show that the solutions to (3.23) are uniformly bounded with respect to $\tau \in [0, 1]$.

**Lemma 3.6** *Any solution to* (3.22)*, with* $\tau \in [0, 1]$*, satisfies the a priori estimate*

$$
\|(G, \boldsymbol{u})\| \,\leq C C_1(\varphi_{\mathrm{D}}, \boldsymbol{g}) \quad \textit{and} \quad \|\varphi_0\|_{1,\Omega} \,\leq\, C C_2(\varphi_{\mathrm{D}}, \boldsymbol{g})\,, \tag{3.31}
$$

*where* $C > 0$ *is independent of* $\tau$*, and the constants* $C_1(\varphi_{\mathrm{D}}, \boldsymbol{g})$ *and* $C_2(\varphi_{\mathrm{D}}, \boldsymbol{g})$ *are given in Theorem 3.3.*

*Proof.* We proceed similarly as in Section 3.2.1. Suppose $(G, \boldsymbol{u}, \varphi_0) = (G_\tau, \boldsymbol{u}_\tau, \varphi_\tau) \in \mathbb{H}$ satisfies (3.23) for a fixed $\tau \in [0, 1]$. Taking $(H, \boldsymbol{v}, \psi) = (G, \boldsymbol{u}, \varphi_0)$, using the skew-symmetric property of $\mathbf{c}(\cdot,\cdot,\cdot)$ and decoupling, we find that

$$
\begin{aligned}
\big(\mathcal{A}(G), G\big) \quad &= \quad -\,\tau\,(\varphi_0\boldsymbol{g}, \boldsymbol{u}) \,+\, \tau(\varphi_1\boldsymbol{g}, \boldsymbol{u}) \\
\kappa\,\|\nabla\varphi_0\|^2_{0,\Omega} \quad &= \quad -\,\tau\,\kappa\,(\nabla\varphi_1, \nabla\varphi_0) \,+\, \tau\,(\boldsymbol{u}\cdot\nabla\varphi_0, \varphi_1).
\end{aligned}
$$

Following the same arguments used in Theorem 3.3, we obtain

$$
\nu\,\|(G, u)\| \,\leq\, \tau\,C\,\|\boldsymbol{g}\|_{0,\Omega}\,\big(\,\|\varphi_0\|_{1,\Omega} \,+\, \|\varphi_1\|_{1,\Omega}\,\big) \,\leq\, C\,\|\boldsymbol{g}\|_{0,\Omega}\,\big(\,\|\varphi_0\|_{1,\Omega} \,+\, \|\varphi_1\|_{1,\Omega}\,\big)\,, \tag{3.32}
$$

as well as

$$
\|\varphi_0\|_{1,\Omega} \,\leq\, \tau\,C\big(\,\|\nabla\varphi_1\|_{0,\Omega} \,+\, \kappa^{-1}\|G\|_{0,\Omega}\,\|\varphi_1\|_{0,3,\Omega}\big) \,\leq\, C\big(\,\|\varphi_1\|_{1,\Omega} \,+\, \kappa^{-1}\,\delta\,\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|(G, u)\|\,\big)\,, \tag{3.33}
$$

where $\delta$ satisfies (3.17). Estimates (3.32)–(3.33) are the same as (3.14)–(3.16) in the proof of Theorem 3.3. Therefore by applying the arguments in the proof verbatim, we obtain the estimates (3.31). $\square$

Since solutions to (3.22) are uniformly bounded with respect to $\tau$, and since the operator $A$ is compact, the existence of solutions follows from a direct application of the Leray-Schauder Principle.

**Theorem 3.7** *There exists a solution* $(G, \boldsymbol{u}, \varphi)$ *to* (3.11).

Here, we emphasize that, in contrast to [5, 6, 4], the previous result establishes existence of a solution without a restriction on the data. Additionally, we are further able to establish conditions under which the solution is unique. Indeed, if $(G, \boldsymbol{u}, \varphi_0)$, $(G', \boldsymbol{u}', \varphi_0') \in \mathbb{H}$ are both solutions to (3.11) (equivalently, fixed points of $A$), then we have by Lemma 3.5 and Theorem 3.3,

$$\|A((G, \boldsymbol{u}, \varphi_0)) - A((G', \boldsymbol{u}', \varphi_0'))\| = \|(G - G', \boldsymbol{u} - \boldsymbol{u}', \varphi_0 - \varphi_0')\| \leq C_{\mathrm{LIP}} \|(G - G', \boldsymbol{u} - \boldsymbol{u}', \varphi_0 - \varphi_0'\|,$$

with

$$C_{\mathrm{LIP}} \leq C C_a^{-1} \Big\{ C_1(\varphi_{\mathrm{D}}, \boldsymbol{g}) + C_2(\varphi_{\mathrm{D}}, \boldsymbol{g}) + C_4(\varphi_{\mathrm{D}}, \boldsymbol{g}) \Big\}. \tag{3.34}$$

Therefore if the data is sufficiently small, we immediately deduce the following uniqueness result.

**Theorem 3.8** *Suppose that the data is small enough such that $C_{\mathrm{LIP}} < 1$ (cf. (3.34)). Then there exists a unique solution* $(G, \boldsymbol{u}, \varphi)$ *to* (3.11).

Note also that no additional regularity of the solution is required to establish our uniqueness result (e.g. Theorem 2.3 in [19] and [20]).

We close the section stating the existence of the tensor $S$ solution to problem (3.7). To this end, given a solution $(G, \boldsymbol{u}, \varphi)$ to (3.11), it follows from the inf–sup conditions (2) and the continuity of the forms (see Lemma 3.1) that there exists a unique $S \in \mathbb{H}_0(\mathbf{div}; \Omega)$ satisfying

$$\mathbf{b}(S, (H, \boldsymbol{v})) \,=\, \mathbf{a}((G, \boldsymbol{u}, \varphi), (H, \boldsymbol{v}, \psi)) + \mathbf{c}((G, \boldsymbol{u}, \varphi), (G, \boldsymbol{u}, \varphi), (H, \boldsymbol{v}, \psi)) - \,(\varphi\, \boldsymbol{g}, \boldsymbol{v})$$

for all $(H, \boldsymbol{v}, \psi) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$. Moreover,

$$\|S\|_{\mathbf{div},\Omega} \,\leq\, C\Big(\, \|\mathbf{a}\| \,+\, \|\mathbf{c}\|\, \|(G, \boldsymbol{u}, \varphi)\| \,+\, \|\boldsymbol{g}\|_{0,\Omega} \Big)\, \|(G, \boldsymbol{u}, \varphi)\|\,.$$

# 4 The Galerkin scheme

In this section we describe the discrete setting of the formulation (3.7). We present a family of spaces developed in [15] for the fluid unknowns satisfying a inf–sup/LBB compatibility condition as well as the Korn/Poincaré inequality in two and three dimensions

## 4.1 The discrete setting and finite element spaces

Let $\mathcal{T}_h$ be a shape–regular triangulation of $\Omega$, made up of simplices $K$ of diameter $h_K$, and meshsize $h := \max_{K \in \mathcal{T}_h} h_K$. For simplicity we assume that if $\partial K \cap \partial\Omega \neq \emptyset$, then either $|\partial K \cap \Gamma_{\mathrm{D}}| = 0$ or $|\partial K \cap \Gamma_{\mathrm{N}}| = 0$. We denote by $\mathcal{T}_h^r$ the corresponding barycentric refinement of a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, for each $h > 0$, and for a given integer $k \geq 0$, we set

$$\begin{aligned} \mathrm{P}_k(\mathcal{T}_h^r) \quad &= \quad \big\{ p_h \,\in\, \mathrm{C}(\Omega) : \quad p_h|_K \,\in\, \mathrm{P}_k(K) \quad \forall\, K \,\in \mathcal{T}_h^r \big\}\,, \\ \mathrm{P}_k^{disc}(\mathcal{T}_h^r) \quad &= \quad \big\{ p_h \,\in\, \mathrm{L}^2(\Omega) : \quad p_h|_K \,\in\, \mathrm{P}_k(K) \quad \forall\, K \,\in \mathcal{T}_h^r \big\}\,, \end{aligned}$$

as the spaces of continuous (Lagrange) and discontinuous piecewise polynomials of degree $k$ on $\mathcal{T}_h^r$, respectively. Similar to the notations described in the Section 1, the analogous vector spaces (resp., tensor spaces) with components in these spaces are denoted by $\mathbf{P}_k(\mathcal{T}_h^r)$ and $\mathbf{P}_k^{disc}(\mathcal{T}_h^r)$ (resp., $\mathbb{P}_k(\mathcal{T}_h^r)$ and $\mathbb{P}_k^{disc}(\mathcal{T}_h^r)$). The finite element subspaces approximating the unknowns $G$ and $\boldsymbol{u}$ are given by

$$\mathbb{H}_h^G = \mathbb{L}^2_{\mathtt{tr}}(\Omega) \cap \mathbb{P}_k^{disc}(\mathcal{T}_h^r) \quad \text{and} \quad \mathbf{H}_h^{\boldsymbol{u}} = \mathbf{P}_k^{disc}(\mathcal{T}_h^r), \tag{4.1}$$

and the finite element space approximating the tensor $S$ is the global Raviart–Thomas space of order $k$:

$$\mathbb{H}_h^S = \left\{ T_h \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \mathrm{c}^{\mathtt{t}}\, T_h\big|_K \in \mathbf{RT}_k(K) \quad \forall\, \mathrm{c} \in \mathrm{R}^n \quad \forall\, K \in \mathcal{T}_h^r \right\}, \tag{4.2}$$

where $\mathbf{RT}_k(K)$ is the local Raviart–Thomas space of order $k$, i.e.,

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_{\underline{k}}(K)\, \boldsymbol{x}\,,$$

and $\mathrm{P}_{\underline{k}}(K)$ stands for the homogeneous space of piecewise polynomials of degree $k$.

For the temperature, we let $\mathrm{H}_h^{\varphi} \subset \mathrm{H}^1(\Omega)$ denote the Lagrange space of degree $\leq k+1$ with respect to $\mathcal{T}_h^r$, and set

$$\mathrm{H}^{\varphi}_{h,\Gamma_{\mathrm{D}}} := \left\{ \psi_h \in \mathrm{H}_h^{\varphi} : \quad \psi_h\big|_{\Gamma_{\mathrm{D}}} = 0 \right\} \tag{4.3}$$

to be the analogous space with homogeneous Dirichlet boundary conditions. We define $\varphi_{\mathrm{D},h} := I_h^{\mathcal{SZ}}\varphi_{\mathrm{D}}|_{\Gamma_{\mathrm{D}}}$ to be the approximate Dirichlet boundary data, where $I_h^{\mathcal{SZ}} : \mathrm{H}^1(\Omega) \to \mathrm{H}_h^{\varphi}$ denotes the Scott–Zhang interpolant of degree $k+1$ [21]. Hence, $\varphi_{\mathrm{D},h}$ belongs to the discrete trace space on $\Gamma_{\mathrm{D}}$ given by

$$\mathrm{H}_h^{1/2}(\Gamma_{\mathrm{D}}) := \left\{ \psi_{\mathrm{D},h} \in \mathrm{C}(\Gamma_{\mathrm{D}}) : \quad \psi_{\mathrm{D},h}\big|_e \in \mathrm{P}_{k+1}(e) \quad \text{for all} \quad e \in \mathcal{E}^r_{\Gamma_{\mathrm{D}}} \right\},$$

where $\mathcal{E}^r_{\Gamma_{\mathrm{D}}}$ stands for the set of edges/faces on $\Gamma_{\mathrm{D}}$.

The discrete problem is: Find $((G_h, \boldsymbol{u}_h, \varphi_h), S_h) \in (\mathbb{H}_h^G \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi}) \times \mathbb{H}_h^S$ such that $\varphi_h|_{\Gamma_{\mathrm{D}}} = \varphi_{\mathrm{D},h}$ and

$$\begin{aligned}
&\mathbf{a}((G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) + \mathbf{c}^{\mathtt{skw}}((G_h, \boldsymbol{u}_h, \varphi_h), (G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) \\
&\qquad\qquad - \mathbf{b}(S_h, (H_h, \boldsymbol{v}_h)) = (\varphi_h\, \boldsymbol{g}, \boldsymbol{v}_h) \quad \forall\, (H_h, \boldsymbol{v}_h, \psi_h) \in \mathbb{H}_h^G \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}^{\varphi}_{h,\Gamma_{\mathrm{D}}} \\
&\mathbf{b}(T_h, (G_h, \boldsymbol{u}_h)) = 0 \quad \forall\, T_h \in \mathbb{H}_h^S,
\end{aligned} \tag{4.4}$$

where $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are the bilinear forms defined by (3.5) and (3.6), and the trilinear form $\mathbf{c}^{\mathtt{skw}}(\cdot,\cdot,\cdot)$ is given by

$$\begin{aligned}
\mathbf{c}^{\mathtt{skw}}((F_h, \boldsymbol{w}_h, \phi_h), (G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) &= \frac{1}{2}\left[ (G_h \boldsymbol{w}_h, \boldsymbol{v}_h) - (H_h \boldsymbol{w}_h, \boldsymbol{u}_h) \right] \\
&\quad + \frac{1}{2}\left[ (\boldsymbol{w}_h \cdot \nabla\varphi_h, \psi_h) - (\boldsymbol{w}_h \cdot \nabla\psi_h, \varphi_h) \right],
\end{aligned} \tag{4.5}$$

which comes from the discrete skew-symmetrization of the form $\mathbf{c}(\cdot,\cdot,\cdot)$. More precisely, note that the property $(\boldsymbol{u} \cdot \nabla\varphi, \psi) = -(\boldsymbol{u} \cdot \nabla\psi, \varphi)$ follows from integration by parts and the fact that $\boldsymbol{u}$ is divergence-free in $\Omega$. Nevertheless, elements in the discrete kernel

$$\mathbb{Z}_h := \left\{ (G_h, \boldsymbol{u}_h) \in \mathbb{H}_h^G \times \mathbf{H}_h^{\boldsymbol{u}} : \ \mathbf{b}(T_h, (G_h, \boldsymbol{u}_h)) = (G_h, T_h) + (\boldsymbol{u}_h, \mathbf{div}\, T_h) = 0\,, \ \forall\, T_h \in \mathbb{H}_h^S \right\}, \tag{4.6}$$

do not necessarily satisfy this property and hence $\mathbf{c}(\cdot,\cdot,\cdot)$ is not skew-symmetric at discrete level (c.f. (3.8)–(3.9)). We circumvent this issue by observing that the nonlinear convective term associated to the heat equation can also be written as

$$(\boldsymbol{u} \cdot \nabla\varphi, \psi) = \frac{1}{2}(\boldsymbol{u} \cdot \nabla\varphi, \psi) - \frac{1}{2}(\boldsymbol{u} \cdot \nabla\psi, \varphi)\,,$$

for all $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$ and for all $\varphi, \psi \in \mathrm{H}^1(\Omega)$. In particular, if we set $\psi = \varphi$, then the term at the right-hand side of the latter equality vanishes (regardless if $\boldsymbol{u}$ is divergence-free or not). This explains why we employ $\mathbf{c}^{\mathtt{skw}}(\cdot,\cdot,\cdot)$ in our formulation (4.4).

We end this section by stating the following useful compatibility properties of the subspaces $\mathbb{H}_h^G$, $\mathbf{H}_h^{\boldsymbol{u}}$ and $\mathbb{H}_h^S$ defined above. The proofs are found in [14, 15, Lemma 3.3, Lemma 4.12].

**Lemma 4.1** *Let* $\big\{(\mathbb{H}_h^G, \mathbf{H}_h^{\boldsymbol{u}}, \mathbb{H}_h^S)\big\}_{h>0}$ *be the family of finite element subspaces defined by* (4.1)–(4.2), *and let* $\mathbb{Z}_h$ *be the discrete kernel defined by* (4.6).

1. *If* $(G_h, \boldsymbol{u}_h) \in \mathbb{Z}_h$ *and* $G_h \rightharpoonup G$ *in* $\mathbb{L}^2(\Omega)$, *then* $\boldsymbol{u}_h \to \boldsymbol{u}$ *in* $\mathbb{L}^2(\Omega)$.
2. *There exists a constant* $C > 0$ *independent of* $h$ *such that* $\|\boldsymbol{u}_h\|_{0,6,\Omega} \leq C\|G_h^{\mathtt{sym}}\|_{0,\Omega}$ *for all* $(G_h, \boldsymbol{u}_h) \in \mathbb{Z}_h$
3. *If* $k \geq (n-1)$ $(n = 2, 3)$, *then the finite element triple* $\mathbb{H}_h^G \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^S$ *satisfies*

$$\sup_{\substack{(G_h,\boldsymbol{u}_h)\in\mathbb{H}_h^G\times\mathbf{H}_h^{\boldsymbol{u}}\\(G_h,\boldsymbol{u}_h)\neq\mathbf{0}}} \frac{\mathbf{b}(S_h,(G_h,\boldsymbol{u}_h))}{\|(G_h,\boldsymbol{u}_h)\|} \,\geq\, \beta^* \,\|S_h\|_{\mathbf{div},\Omega} \quad \forall S_h \,\in\, \mathbb{H}_h^S\,, \tag{4.7}$$

$$\|(G_h^{\mathtt{skw}},\boldsymbol{u}_h)\| \,\leq\, C^* \,\|G_h^{\mathtt{sym}}\|_{0,\Omega} \quad \forall\,(G_h,\boldsymbol{u}_h) \,\in\, \mathbb{Z}_h\,, \tag{4.8}$$

*with constants* $\beta^*, C^* > 0$ *depending only upon the aspect ratio of* $\mathcal{T}_h$.

**Remark 4.1** *Set* $\mathbb{H}_h := \mathbb{Z}_h \times \mathrm{H}_{h,\Gamma_\mathrm{D}}^{\varphi}$ *(cf.* (4.3) *and* (4.6)*) and observe from Lemma 4.1 and the Poincaré inequality that* $\mathbf{a}(\cdot,\cdot)$ *is coercive on* $\mathbb{H}_h$. *In particular, there exists* $C_a^* = C\min\{\nu,\kappa\} > 0$, *independent of* $h$, *such that*

$$\mathbf{a}((G_h,\boldsymbol{u}_h,\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h)) \,\geq\, C_a^* \,\|(G_h,\boldsymbol{u}_h,\varphi_h)\|^2 \quad \forall\,(G_h,\boldsymbol{u}_h,\varphi_h) \,\in\, \mathbb{H}_h\,.$$

**Remark 4.2** *In reference [14, 15], the estimate* $\|\boldsymbol{u}_h\|_{0,6,\Omega} \leq C\|G_h^{\mathtt{sym}}\|_{0,\Omega}$ *is proven provided the triangulation is quasi–uniform. However, Lemma 4.4 below and a discrete Sobolev inequality show that this mesh restriction is not needed.*

### 4.2 Preliminary results

Similar to the continuous case, we consider problem (4.4) restricted to the kernel $\mathbb{Z}_h$. In particular, we first study the problem: Find $((G_h,\boldsymbol{u}_h),\varphi_h) \in \mathbb{Z}_h \times \mathrm{H}_h^{\varphi}$ with $\varphi_h|_{\Gamma_\mathrm{D}} = \varphi_{\mathrm{D},h}$ such that

$$\mathbf{a}((G_h,\boldsymbol{u}_h,\varphi_h),(H_h,\boldsymbol{v}_h,\psi_h)) \,+\, \mathbf{c}^{\mathtt{skw}}((G_h,\boldsymbol{u}_h,\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h),(H_h,\boldsymbol{v}_h,\psi_h)) \,=\, (\varphi_h\,\boldsymbol{g},\boldsymbol{v}_h) \tag{4.9}$$

for all $(H_h,\boldsymbol{v}_h,\psi_h) \in \mathbb{H}_h$.

In advance, we point out that, due to the skew–symmetrization of the convective term, the solvability analysis of the discrete problem does not immediately follow from the continuous one. For example, it is easy to see that when proceeding as in Section 3.2.1, the discrete counterpart of the estimation (3.15) becomes

$$\begin{aligned}\kappa\|\nabla\varphi_{0,h}\|_{0,\Omega}^2 \,\leq\, & \kappa\|\nabla\varphi_{1,h}\|_{0,\Omega}\|\nabla\varphi_0\|_{0,\Omega}\\ & + C\,\|G_h\|_{0,\Omega}\big(\,\|\nabla\varphi_{0,h}\|_{0,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega} \,+\, \|\nabla\varphi_{1,h}\|_{0,\Omega}\,\|\varphi_{0,h}\|_{0,3,\Omega}\big)\,,\end{aligned}$$

where $\varphi_{1,h}$ is any discrete extension of $\varphi_{\mathrm{D},h}$, i.e., $\varphi_{1,h} \in \mathrm{H}_h^{\varphi}$ and $\varphi_{1,h}|_{\Gamma_\mathrm{D}} = \varphi_{\mathrm{D},h}$. Hence, it is observed that the factor multiplying the $\mathbb{L}^2-$norm of $G_h$ depends on the $\mathrm{H}^1$-norm of the discrete extension $\varphi_{1,h}$, not its $\mathrm{L}^3-$norm (as in the continuous case). This bound is due to estimating the term

$$(\boldsymbol{u}_h \cdot \nabla\varphi_{1,h}, \varphi_{0,h}) \,-\, (\boldsymbol{u}_h \cdot \nabla\varphi_{0,h}, \varphi_{1,h}) \tag{4.10}$$

which involves the gradient of $\varphi_{1,h}$. Proceeding as in the continuous case would therefore lead us to data constraints in order to derive a priori estimates and existence results for the discrete solution (e.g. [19, 20]). Thus, in order to overcome this restriction and to establish results at discrete level similar to the continuous one, we focus on the following goals:

1. To extend an analogous version of Lemma 3.2 providing some stability properties of discrete extensions.
2. To derive a suitable bound for (4.10) in terms of some $\mathrm{L}^p-$norm of $\varphi_{1,h}$.

### 4.2.1 A Discrete Extension Operator

To define an appropriate discrete extension operator, we first state a well–known property of the Scott–Zhang interpolant.

**Lemma 4.2 ([21],Theorem 3.1 [8],Lemma 1.130)** *Let $p$ and $\ell$ satisfy $1 \le p < \infty$ and $\ell \ge 1$ if $p = 1$, and $\ell > 1/p$ otherwise. Then for all $K \in \mathcal{T}_h^r$, for any non–negative integer $m$ and $1 \le q \le \infty$,*

$$\|I_h^{\mathcal{SZ}} v\|_{m,q,K} \,\le\, C \sum_{k=0}^{\ell} h_K^{k-m+\frac{n}{q}-\frac{n}{p}} |v|_{k,p,\omega_K} \quad \forall\, v \in \mathrm{W}^{\ell,p}(\omega_K)\,.$$

*Here, $\omega_K$ stands for the set of elements in $\mathcal{T}_h^r$ sharing at least one vertex with $K$.*

With Lemma 4.2 we obtain a discrete version of Lemma 3.2 that guarantees the existence of a discrete extension operator with similar properties found in the continuous setting.

**Lemma 4.3** *For any $\delta \in (0,1)$ there exists an $h_\delta > 0$ and an extension operator $E_{\delta,h} : \mathrm{H}^{1/2}(\Gamma_\mathrm{D}) \to \mathrm{H}_h^{\varphi}$ such that, for $h \le h_\delta$,*

$$\|E_{\delta,h}\psi_\mathrm{D}\|_{0,3,\Omega} \,\le\, C\delta\|\psi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}, \quad \textit{and} \quad \|E_{\delta,h}\psi_\mathrm{D}\|_{1,\Omega} \,\le\, C\delta^{-4}\|\psi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}\,, \tag{4.11}$$

*where $C > 0$ is independent of $h$. In particular,*

$$\|E_{\delta,h}\varphi_{\mathrm{D},h}\|_{0,3,\Omega} \,\le\, C\delta\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}, \quad \textit{and} \quad \|E_{\delta,h}\varphi_{\mathrm{D},h}\|_{1,\Omega} \,\le\, C\delta^{-4}\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}\,. \tag{4.12}$$

*Proof.* Let $E_{\delta,h} := I_h^{\mathcal{SZ}} E_\delta$, where $E_\delta$ is the extension operator constructed in Lemma 3.2. Then the second estimate in (4.11) follows from Lemmas 4.2 and 3.2:

$$\|E_{\delta,h}\psi\|_{1,\Omega} \le C\|E_\delta\psi\|_{1,\Omega} \le C\delta^{-4}\|\psi\|_{1/2,\Gamma_\mathrm{D}}.$$

Likewise Lemmas 4.2 and Hölder's inequality gives us

$$\|E_{\delta,h}\psi_\mathrm{D}\|_{0,3,K} \le C\big(h_K^{-\frac{n}{6}}\|E_\delta\psi_\mathrm{D}\|_{0,\omega_K} + h_K^{1-\frac{n}{6}}\|E_\delta\psi_\mathrm{D}\|_{1,\omega_K}\big) \le C\big(\|E_\delta\psi_\mathrm{D}\|_{0,3,\omega_K} + h_K^{1-\frac{n}{6}}\|E_\delta\psi_\mathrm{D}\|_{1,\omega_K}\big)\,.$$

Therefore by Lemma 3.2,

$$\|E_{\delta,h}\psi_{\mathrm{D}}\|_{0,3,\Omega} \,\le\, C\big(\delta + h^{1-\frac{n}{6}}\delta^{-4}\big)\|\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}.$$

Hence for $h$ sufficiently small we have $\|E_{\delta,h}\psi_{\mathrm{D}}\|_{0,3,\Omega} \le C\delta\|\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}$.

To prove (4.12) it suffices to show $\|I_h^{\mathcal{SZ}}\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \le C\|\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}$ for all $\psi_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$. To this end, for a fixed $\psi_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$, let $\psi_*, \widetilde{\psi}_* \in \mathrm{H}^1(\Omega)$ satisfy

$$\begin{aligned}
\|\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} &:= \inf\big\{\|\phi\|_{1,\Omega} \,:\, \phi\in\mathrm{H}^1(\Omega),\ \phi\big|_{\Gamma_{\mathrm{D}}} = \psi_{\mathrm{D}}\big\} \,=\, \|\psi_*\|_{1,\Omega}\,,\\
\|I_h^{\mathcal{SZ}}\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} &:= \inf\big\{\|\phi\|_{1,\Omega} \,:\, \phi\in\mathrm{H}^1(\Omega),\ \phi\big|_{\Gamma_{\mathrm{D}}} = I_h^{\mathcal{SZ}}\psi_{\mathrm{D}}\big\} \,=\, \|\widetilde{\psi}_*\|_{1,\Omega}\,.
\end{aligned}$$

By the stability properties stated in Lemma 4.2 we have

$$\|I_h^{\mathcal{SZ}}\psi_*\|_{1,\Omega} \,\le\, C\,\|\psi_*\|_{1,\Omega}\,.$$

Since $\widetilde{\psi}_*|_{\Gamma_{\mathrm{D}}} = I_h^{\mathcal{SZ}}\psi_*|_{\Gamma_{\mathrm{D}}}$, it follows from the definition of $\widetilde{\psi}_*$ that

$$\|\widetilde{\psi}_*\|_{1,\Omega} \,\le\, C\,\|I_h^{\mathcal{SZ}}\psi_*\|_{1,\Omega}\,.$$

Thus,

$$\|I_h^{\mathcal{SZ}}\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,=\, \|\widetilde{\psi}_*\|_{1,\Omega} \,\le\, C\,\|I_h^{\mathcal{SZ}}\psi_*\|_{1,\Omega} \,\le\, C\,\|\psi_*\|_{1,\Omega} \,=\, C\,\|\psi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,.$$

□

#### 4.2.2 A weak continuity property of the discrete kernel

Recall that in the continuous setting, an element in the kernel $(G,\boldsymbol{u}) \in \mathbb{Z}$ satisfies $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$. A discrete analogue of this property is now shown in the following lemma.

**Lemma 4.4** *There exists a positive constant $C$, independent of $h$, such that*

$$\sum_{K\in\mathcal{T}_h^r}\|\nabla\boldsymbol{u}_h\|_{0,K}^2 \,+\, \sum_{e\in\mathcal{E}_h^r} h_e^{-1}\|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2 \,\le\, C\,\|G_h\|_{0,\Omega}^2 \quad \forall\,(G_h,\boldsymbol{u}_h)\,\in\,\mathbb{Z}_h\,, \tag{4.13}$$

*where $\mathcal{E}_h^r$ denotes the set of edges/faces of $\mathcal{T}_h^r$. Here, $[\![\cdot]\!]$ is the jump operator given by*

$$\begin{aligned}
[\![\boldsymbol{v}]\!]|_e &= \boldsymbol{v}^+\big|_e - \boldsymbol{v}^-\big|_e, \qquad && e = \partial K_+\cap\partial K_-,\\
[\![\boldsymbol{v}]\!]|_e &= \boldsymbol{v}^+\big|_e, && e = \partial K_+\cap\partial\Omega,
\end{aligned}$$

*where $\boldsymbol{v}^\pm = \boldsymbol{v}|_{K_\pm}$, and $K_+$ has a global labeling number smaller than $K_-$.*

*Proof.* Recall that any function $T_h$ in the global Raviart-Thomas space is uniquely determined on each $K\in\mathcal{T}_h^r$ by the conditions

$$\int_K T_h : S \quad \forall\,S\,\in\,\mathbb{P}_{k-1}(K) \qquad \text{and} \qquad \int_e T_h\boldsymbol{n}\cdot\boldsymbol{v} \quad \forall\,\boldsymbol{v}\,\in\,\mathbf{P}_k(e)\,,\ e\subset\partial K\,.$$

Moreover, a simple scaling argument shows that

$$\|T_h\|_{0,K}^2 \,\le\, C\big(\,\|\Pi_{k-1,K}(T_h)\|_{0,K}^2 \,+\, \sum_{e\subset\partial K} h_e\,\|T_h\,\boldsymbol{n}_e\|_{0,e}^2\,\big) \qquad \forall\,K\in\mathcal{T}_h^r\,,$$

where $\Pi_{k-1,K}$ is the $\mathbb{L}^2-$projection onto $\mathbb{P}_{k-1}(K)$. Now, recall from (4.6) that $(G_h, \boldsymbol{u}_h) \in \mathbb{Z}_h$ if and only if

$$(G_h, T_h) + (\boldsymbol{u}_h, \mathbf{div}\, T_h) = 0 \quad \forall\, T_h \in \mathbb{H}_h^S,$$

or after integrating by parts,

$$(G_h, T_h) - \sum_{K \in \mathcal{T}_h^r} (\nabla \boldsymbol{u}_h, T_h)_{0,K} + \sum_{e \in \mathcal{E}_h^r} \int_e T_h \boldsymbol{n} \cdot [\![ \boldsymbol{u}_h ]\!] = 0. \tag{4.14}$$

Letting $T_h$ satisfy

$$T_h\, \boldsymbol{n}_e|_e = h_e^{-1} [\![ \boldsymbol{u}_h ]\!]|_e \in \mathbf{P}_k(e) \quad \forall\, e \in \mathcal{E}_h^r \qquad \text{and} \qquad \Pi_{k-1,K}(T_h) = 0 \quad \forall\, K \in \mathcal{T}_h^r,$$

we find from (4.14) and Cauchy-Schwarz inequality that

$$\sum_{e \in \mathcal{E}_h^r} h_e^{-1} \| [\![ \boldsymbol{u}_h ]\!] \|_{0,e}^2 = (G_h, T_h) \le \|G_h\|_{0,\Omega} \|T_h\|_{0,\Omega} \le C \|G_h\|_{0,\Omega} \Big( \sum_{e \in \mathcal{E}_h^r} h_e^{-1} \| [\![ \boldsymbol{u}_h ]\!] \|_{0,e}^2 \Big)^{1/2}.$$

Thus,

$$\sum_{e \in \mathcal{E}_h^r} h_e^{-1} \| [\![ \boldsymbol{u}_h ]\!] \|_{0,e}^2 \le C \|G_h\|_{0,\Omega}^2. \tag{4.15}$$

Likewise, taking now $T_h$ such that

$$T_h\, \boldsymbol{n}_e|_e = 0 \quad \forall\, e \in \mathcal{E}_h^r \qquad \text{and} \qquad \Pi_{k-1,K}(T_h) = \Pi_{k-1,K}(\nabla \boldsymbol{u}_h|_K) \quad \forall\, K \in \mathcal{T}_h^r,$$

yields

$$\sum_{K \in \mathcal{T}_h^r} \|\nabla \boldsymbol{u}_h\|_{0,K}^2 = (G_h, T_h) \le \|G_h\|_{0,\Omega} \|T_h\|_{0,\Omega} \le C \|G_h\|_{0,\Omega} \Big( \sum_{K \in \mathcal{T}_h^r} \|\nabla \boldsymbol{u}_h\|_{0,K}^2 \Big)^{1/2},$$

and therefore

$$\sum_{K \in \mathcal{T}_h^r} \|\nabla \boldsymbol{u}_h\|_{0,K}^2 \le C \|G_h\|_{0,\Omega}^2. \tag{4.16}$$

The estimate (4.13) follows by combining (4.15) and (4.16). $\square$

With the help of Lemma 4.4, we now provide a suitable upper bound for the nonlinear convective expression (4.10) in terms of the $\mathrm{L}^3-$norm of $\varphi_{1,h}$.

**Lemma 4.5** *Set $\varphi_h = \varphi_{0,h} + \varphi_{1,h}$, where $\varphi_{0,h} \in \mathrm{H}^{\varphi}_{h,\Gamma_{\mathrm{D}}}$ and $\varphi_{1,h}$ is a discrete extension of $\varphi_{\mathrm{D},h}$. Then for any $(G_h, \boldsymbol{u}_h) \in \mathbb{Z}_h$, there exists a positive constant $C$, independent of $h$, such that*

$$\big| (\boldsymbol{u}_h \cdot \nabla \varphi_{1,h}, \varphi_{0,h}) - (\boldsymbol{u}_h \cdot \nabla \varphi_{0,h}, \varphi_{1,h}) \big| \le C \|G_h\|_{0,\Omega} \|\varphi_{0,h}\|_{1,\Omega} \|\varphi_{1,h}\|_{0,3,\Omega}. \tag{4.17}$$

*Proof.* Integrating by parts we find

$$\begin{aligned}(\boldsymbol{u}_h \cdot \nabla \varphi_{1,h}, \varphi_{0,h}) &= - \sum_{K \in \mathcal{T}_h^r} (\mathrm{div}(\varphi_{0,h}\, \boldsymbol{u}_h), \nabla \varphi_{1,h})_K + \sum_{e \in \mathcal{E}_h^r} ([\![ \boldsymbol{u}_h \cdot \boldsymbol{n} ]\!] \varphi_{0,h}, \varphi_{1,h})_e \\ &= -(\boldsymbol{u}_h \cdot \nabla \varphi_{0,h}, \varphi_{1,h}) - \sum_{K \in \mathcal{T}_h^r} (\mathrm{div}(\boldsymbol{u}_h)\, \varphi_{0,h}, \varphi_{1,h})_K + \sum_{e \in \mathcal{E}_h^r} ([\![ \boldsymbol{u}_h \cdot \boldsymbol{n} ]\!] \varphi_{0,h}, \varphi_{1,h})_e,\end{aligned}$$

and therefore,

$$
\begin{aligned}
&(\boldsymbol{u}_h \cdot \nabla\varphi_{1,h}, \varphi_{0,h}) \,-\, (\boldsymbol{u}_h \cdot \nabla\varphi_{0,h}, \varphi_{1,h}) \\
&\qquad = -2\,(\boldsymbol{u}_h \cdot \nabla\varphi_{0,h}, \varphi_{1,h}) \,-\, \sum_{K\in\mathcal{T}_h^r} (\mathrm{div}(\boldsymbol{u}_h)\,\varphi_{0,h}, \varphi_{1,h})_K \,+\, \sum_{e\in\mathcal{E}_h^r} ([\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!]\varphi_{0,h}, \varphi_{1,h})_e \\
&\qquad =: \; J_1 \,+\, J_2 \,+\, J_3\,.
\end{aligned}
$$

Next, we proceed to estimate each term $J_i$ by applying Hölder's inequality, Sobolev embeddings, and the Lemmas 4.1–4.4. Thus,

$$
|J_1| \,\le\, 2\,\|\boldsymbol{u}_h\|_{0,6,\Omega}\,\|\nabla\varphi_{0,h}\|_{0,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega} \,\le\, C\,\|G_h\|_{0,\Omega}\,\|\varphi_{0,h}\|_{1,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega}\,,
$$

and likewise

$$
|J_2| \,\le\, C\Big(\sum_{K\in\mathcal{T}_h^r} \|\nabla\boldsymbol{u}_h\|_{0,\Omega}^2\Big)^{1/2}\|\varphi_{0,h}\|_{0,6,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega} \,\le\, C\,\|G_h\|_{0,\Omega}\,\|\varphi_{0,h}\|_{1,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega}\,.
$$

Finally, we further use an inverse inequality to get

$$
\begin{aligned}
|J_3| \,&\le\, \Big(\sum_{e\in\mathcal{E}_h^r} h_e^{-1}\|[\![\boldsymbol{u}_h]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^r} h_e\|\varphi_{0,h}\|_{0,6,e}^6\Big)^{1/6}\Big(\sum_{e\in\mathcal{E}_h^r} h_e\|\varphi_{1,h}\|_{0,3,e}^3\Big)^{1/3} \\
&\le\, C\,\|G_h\|_{0,\Omega}\,\|\varphi_{0,h}\|_{1,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega}\,.
\end{aligned}
$$

Combining these upper bounds yields the estimate (4.17). □

### 4.3 A priori estimates

We now derive a priori estimates of solutions of (4.9).

**Theorem 4.6** *There exists an $h_\delta > 0$ such that for $h \le h_\delta$, any solution $(G_h, \boldsymbol{u}_h, \varphi_h)$ to (4.9) satisfies*

$$
\|(G_h, \boldsymbol{u}_h)\| \le C_1^*(\varphi_{\mathrm{D}}, \boldsymbol{g}) \quad \text{and} \quad \|\varphi_h\|_{1,\Omega} \le C_2^*(\varphi_{\mathrm{D}}, \boldsymbol{g}),
$$

*where $C_1^*(\varphi_{\mathrm{D}}, \boldsymbol{g}) = CC_1(\varphi_{\mathrm{D}}, \boldsymbol{g}) > 0$, $C_2^*(\varphi_{\mathrm{D}}, \boldsymbol{g}) = CC_2(\varphi_{\mathrm{D}}, \boldsymbol{g})$, $C > 0$ is independent of $h$, and $C_1(\varphi_{\mathrm{D}}, \boldsymbol{g})$ and $C_2(\varphi_{\mathrm{D}}, \boldsymbol{g})$ are given in Theorem 3.3.*

*Proof.* Let $\varphi_{1,h} = E_{\delta,h}\varphi_{\mathrm{D},h} \in \mathrm{H}_h^\varphi$ be the discrete extension of $\varphi_{\mathrm{D},h}$ satisfying (4.11), and let $\varphi_{0,h} = \varphi_h - \varphi_{1,h} \in \mathrm{H}_{h,\Gamma_{\mathrm{D}}}^\varphi$. Then problem (4.9) takes the equivalent form: Find $(G_h, \boldsymbol{u}_h, \varphi_{0,h}) \in \mathbb{H}_h$ such that

$$
\begin{aligned}
&\mathbf{a}((G_h, \boldsymbol{u}_h, \varphi_{0,h}), (H_h, \boldsymbol{v}_h, \psi_h)) + \mathbf{c}^{\mathtt{skw}}((G_h, \boldsymbol{u}_h, \varphi_{0,h}), (G_h, \boldsymbol{u}_h, \varphi_{0,h}), (H_h, \boldsymbol{v}_h, \psi_h)) \,=\, (\varphi_{0,h}\,\boldsymbol{g}, \boldsymbol{v}_h) \\
&\quad + (\varphi_{1,h}\,\boldsymbol{g}, \boldsymbol{v}_h) \,+\, \kappa\,(\nabla\varphi_{1,h}, \nabla\psi_h) \,-\, \frac{1}{2}\big[(\boldsymbol{u}_h \cdot \nabla\varphi_{1,h}, \psi_h) - (\boldsymbol{u}_h \cdot \nabla\psi_h, \varphi_{1,h})\big] \;\; \forall\,(H_h, \boldsymbol{v}_h, \psi_h) \in \mathbb{H}_h\,.
\end{aligned}
$$

Similarly to the continuous case, to derive a priori estimates, we take $(H_h, \boldsymbol{v}_h, \psi_h) = (G_h, \boldsymbol{u}_h, \varphi_{0,h})$ decouple the equations and use the skew-symmetric property of the trilinear form to obtain

$$
\begin{aligned}
(\mathcal{A}(G_h), G_h) \;&=\; (\varphi_{0,h}\,\boldsymbol{g}, \boldsymbol{u}_h) \,+\, (\varphi_{1,h}\,\boldsymbol{g}, \boldsymbol{u}_h) \\
\kappa\,\|\nabla\varphi_{0,h}\|_{0,\Omega}^2 \;&=\; -\kappa\,(\nabla\varphi_{1,h}, \nabla\varphi_{0,h}) \,-\, \frac{1}{2}\,\big[(\boldsymbol{u}_h \cdot \nabla\varphi_{1,h}, \varphi_{0,h}) - (\boldsymbol{u}_h \cdot \nabla\varphi_{0,h}, \varphi_{1,h})\big]\,.
\end{aligned}
\tag{4.18}
$$

In light of the discrete Korn inequality stated in Lemma 4.1, we can apply the same arguments in the proof of Theorem 3.3 to obtain

$$\nu\|(G_h,\boldsymbol{u}_h)\| \,\le\, C\|\boldsymbol{g}\|_{0,\Omega}\big(\,\|\varphi_{0,h}\|_{1,\Omega} \,+\, \|\varphi_{1,h}\|_{1,\Omega}\,\big)\,. \tag{4.19}$$

For the second equation in (4.18), we employ the estimate (4.17) for the nonlinear convective term provided by the Lemma 4.5 to get

$$\kappa\|\nabla\varphi_{0,h}\|^2_{0,\Omega} \,\le\, \kappa\,\|\varphi_{1,h}\|_{1,\Omega}\,\|\varphi_{0,h}\|_{1,\Omega} \,+\, C\,\|G_h\|_{0,\Omega}\,\|\varphi_{0,h}\|_{1,\Omega}\,\|\varphi_{1,h}\|_{0,3,\Omega}\,.$$

Applying Poincaré inequality on the left–hand side and Lemma 4.3 on the right–hand side and simplifying, we obtain

$$\|\varphi_{0,h}\|_{1,\Omega} \,\le\, C\big(\,\|\varphi_{1,h}\|_{1,\Omega} \,+\, \kappa^{-1}\,\delta\|\varphi_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|(G_h,\boldsymbol{u}_h)\|\,\big)\,. \tag{4.20}$$

Note that the estimates (4.19)–(4.20) are the same as (3.14)–(3.16) in Theorem 3.3 (up to an $h$–independent multiplicative factor). Therefore by applying the same arguments in the proof of Theorem 3.3 we obtain the desired estimates. □

### 4.4 Well-posedness

Analogous to the continuous analysis, we observe that a solution $(G_h,\boldsymbol{u}_h,\varphi_{0,h})\in\mathbb{H}_h$ to the problem (4.9) equivalently satisfies the discrete fixed point equation

$$(G_h,\boldsymbol{u}_h,\varphi_{0,h}) \,=\, A_h((G_h,\boldsymbol{u}_h,\varphi_{0,h}))\,,$$

where $\varphi_h=\varphi_{0,h}+\varphi_{1,h}$, $\varphi_{1,h}=E_{\delta,h}\varphi_{\mathrm{D},h}$ is the discrete extension of $\varphi_{\mathrm{D}}$ satisfying the conditions in Theorem 4.6, and $A_h((G_h,\boldsymbol{u}_h,\varphi_{0,h}))=(\widehat{G}_h,\widehat{\boldsymbol{u}}_h,\widehat{\varphi}_{0,h})$ is uniquely defined by the variational problem

$$\mathbf{a}((\widehat{G}_h,\widehat{\boldsymbol{u}}_h,\widehat{\varphi}_{0,h}),(H_h,\boldsymbol{v}_h,\psi_h)) \,=\, \big(\mathcal{F}^h_{1,(G_h,\boldsymbol{u}_h,\varphi_{0,h})} \,+\, \mathcal{F}^h_{2,(G_h,\boldsymbol{u}_h,\varphi_{0,h})} \,+\mathcal{F}^h_3\big)\,(H_h,\boldsymbol{v}_h,\psi_h)\,,$$

for all $(H_h,\boldsymbol{v}_h,\psi_h)\in\mathbb{H}_h$. Here, $\mathcal{F}^h_{1,(G,\boldsymbol{u},\varphi_0)}$, $\mathcal{F}^h_{2,(G,\boldsymbol{u},\varphi_0)}$ and $\mathcal{F}^h_3$ are the linear functionals defined by

$$\begin{aligned}
\mathcal{F}^h_{1,(G_h,\boldsymbol{u}_h,\varphi_{0,h})}((H_h,\boldsymbol{v}_h,\psi_h)) &= \mathbf{c}^{\mathbf{skw}}((G_h,\boldsymbol{u}_h,\varphi_{0,h}),(G_h,\boldsymbol{u}_h,\varphi_{0,h}),(H_h,\boldsymbol{v}_h,\psi_h))\\
\mathcal{F}^h_{2,(G_h,\boldsymbol{u}_h,\varphi_{0,h})}((H_h,\boldsymbol{v}_h,\psi_h)) &= -\frac{1}{2}(\boldsymbol{u}_h\cdot\nabla\varphi_{1,h},\psi_h) \,+\, \frac{1}{2}(\boldsymbol{u}_h\cdot\nabla\psi_h,\varphi_{1,h}) \,+\, (\varphi_{0,h}\,\boldsymbol{g},\boldsymbol{v}_h)\,,\\
\mathcal{F}^h_3((H_h,\boldsymbol{v}_h,\psi_h)) &= (\varphi_{1,h}\,\boldsymbol{g},\boldsymbol{v}_h) \,-\, \kappa\,(\nabla\varphi_{1,h},\nabla\psi_h)\,,
\end{aligned}$$

for all $(H_h,\boldsymbol{v}_h,\psi_h)\in\mathbb{H}_h$. From the Hölder Cauchy-Schwarz inequalities and Lemma 4.1 there holds

$$\begin{aligned}
|\mathcal{F}^h_{1,(G_h,\boldsymbol{u}_h,\varphi_{0,h})}(H_h,\boldsymbol{v}_h,\psi_h)| &\le C^*_3(\boldsymbol{u}_h)\,\|(G_h,\boldsymbol{u}_h,\varphi_{0,h})\|\,\|(H_h,\boldsymbol{v}_h,\psi_h)\|\,,\\
|\mathcal{F}^h_{2,(G_h,\boldsymbol{u}_h,\varphi_{0,h})}(H_h,\boldsymbol{v}_h,\psi_h)| &\le C^*_4(\varphi_{\mathrm{D}},\boldsymbol{g})\,\|(G_h,\boldsymbol{u}_h,\varphi_{0,h})\|\,\|(H_h,\boldsymbol{v}_h,\psi_h)\|\,,\,,\\
|\mathcal{F}^h_3(H_h,\boldsymbol{v}_h,\psi_h)| &\le C^*_5(\varphi_{\mathrm{D}},\boldsymbol{g})\,\|(H_h,\boldsymbol{v}_h,\psi_h)\|\,,
\end{aligned}$$

where $C^*_3(\boldsymbol{u}_h)=C\|\boldsymbol{u}_h\|_{0,3,\Omega}$, $C^*_4(\varphi_{\mathrm{D}},\boldsymbol{g})=CC_4(\varphi_{\mathrm{D}},\boldsymbol{g})$, and $C^*_5(\varphi_{\mathrm{D}},\boldsymbol{g})=C_5(\varphi_{\mathrm{D}},\boldsymbol{g})$. Since the bilinear form $\mathbf{a}(\cdot,\cdot)$ is uniformly continuous and coercive in $\mathbb{H}_h$, $A_h$ is well-defined thanks to the Lax-Milgram Theorem. Since $A_h$ is a compact operator, we trivially have the following existence result. Its proof is identical to the proof of Theorem 3.7.

**Theorem 4.7** *There exists a solution $(G_h,\boldsymbol{u}_h,\varphi_h)$ satisfying (4.9) provided $h\le h_\delta$.*

Next, to establish a uniqueness result we study the continuity of $A_h$ by proceeding as in the continuous case. Take $\Psi_h := (G_h, \boldsymbol{u}_h, \varphi_{0,h}),\ \Psi'_h := (G'_h, \boldsymbol{u}'_h, \varphi'_{0,h}) \in \mathbb{H}_h$ and denote

$$A_h((G_h, \boldsymbol{u}_h, \varphi_{0,h})) \,=\, (\widehat{G}_h, \widehat{\boldsymbol{u}}_h, \widehat{\varphi}_{0,h}) =: \widehat{\Psi}_h \quad \text{and} \quad A_h((G'_h, \boldsymbol{u}'_h, \varphi'_{0,h})) \,=\, (\widehat{G}'_h, \widehat{\boldsymbol{u}}'_h, \widehat{\varphi}'_{0,h}) := \widehat{\Psi}'_h\,.$$

It follows, similarly to (3.28), by employing the definition of $A_h$, and the coercivity of $\mathbf{a}(\cdot,\cdot)$ that

$$\begin{aligned}
&\|A_h(\Psi_h) \,-\, A(\Psi'_h)\|^2 \,=\, \|\widehat{\Psi}_h - \widehat{\Psi}'_h\|^2 \\
&\leq \frac{1}{C_a^*}\Big\{\big(\mathcal{F}^h_{1,\Psi_h} \,-\, \mathcal{F}^h_{1,\Psi'_h}\big)(\widehat{\Psi}_h - \widehat{\Psi}'_h) \,+\, \mathcal{F}^h_{2,\Psi_h-\Psi'_h}(\widehat{\Psi}_h - \widehat{\Psi}'_h)\Big\}\,.
\end{aligned}$$

Applying the same arguments to derive (3.29) and (3.30) we then obtain

$$\begin{aligned}
&\|A_h(\Psi_h) \,-\, A(\Psi'_h)\|^2 \,=\, \|\widehat{\Psi}_h - \widehat{\Psi}'_h\|^2 \\
&\leq \frac{1}{C_a^*}\Big\{C\Big(\|G_h\|_{0,\Omega} + \|\boldsymbol{u}_h\|_{0,4,\Omega} + \|\boldsymbol{u}'_h\|_{0,4,\Omega} + \|\varphi'_{0,h}\|_{0,4,\Omega}\Big) + C_4^*(\varphi_{\mathrm{D}},\boldsymbol{g})\Big\}\|\Psi_h - \Psi'_h\|\|\widehat{\Psi} - \widehat{\Psi}'\|.
\end{aligned}$$

Therefore

$$\|A_h(\Psi_h) - A(\Psi'_h)\| \leq C^*_{\mathrm{LIP}}\|(\Psi_h - \Psi'_h)\|,$$

with $C^*_{\mathrm{LIP}} \,=\, C^*_{\mathrm{LIP}}(G_h, \boldsymbol{u}_h, \boldsymbol{u}'_h, \varphi'_0, \varphi_{\mathrm{D}}, \boldsymbol{g}) \,=\, \frac{1}{C_a^*}\Big\{\,C\Big(\|G_h\|_{0,\Omega} + \|\boldsymbol{u}_h\|_{0,4,\Omega} + \|\boldsymbol{u}'_h\|_{0,4,\Omega} + \|\varphi'_{0,h}\|_{0,4,\Omega}\Big) + C_4^*(\varphi_{\mathrm{D}},\boldsymbol{g})\Big\}$. Now if $(G_h, \boldsymbol{u}_h, \varphi_h), (G'_h, \boldsymbol{u}'_h, \varphi'_h) \in \mathbb{H}_h$ are two solutions to (4.9) then

$$\|(G_h - G'_h, \boldsymbol{u}_h - \boldsymbol{u}'_h, \varphi_{0,h} - \varphi'_{0,h})\| \leq C^*_{\mathrm{LIP}}\|(G_h - G'_h, \boldsymbol{u}_h - \boldsymbol{u}'_h, \varphi_{0,h} - \varphi'_{0,h})\|,$$

and by Theorem 4.6

$$C^*_{\mathrm{LIP}} \leq \frac{C}{C_a^*}\Big\{C_1^*(\varphi_{\mathrm{D}},\boldsymbol{g}) + C_2^*(\varphi_{\mathrm{D}},\boldsymbol{g}) + C_4^*(\varphi_{\mathrm{D}},\boldsymbol{g})\Big\}\,. \tag{4.21}$$

Thus, we arrive at the following uniqueness result.

**Theorem 4.8** *If the data is sufficiently small so that the constant $C^*_{\mathrm{LIP}}$ satisfies $C^*_{\mathrm{LIP}} < 1$, then solutions to* (4.9) *are unique.*

Finally, such as in the continuous case, the existence of the discrete tensor $S_h$ follows from the inf-sup condition given in Lemma 4.1. Furthermore, we have that

$$\|S_h\|_{\mathbf{div},\Omega} \,\leq\, C\Big(\,\|\mathbf{a}\| \,+\, \|\mathbf{c}^{\mathbf{skw}}\|\,\|(G_h, \boldsymbol{u}_h, \varphi_h)\| \,+\, \|\boldsymbol{g}\|_{0,\Omega}\Big)\,\|(G_h, \boldsymbol{u}_h, \varphi_h)\|\,. \tag{4.22}$$

### 4.5 A priori error analysis

In this section we proceed to derive error estimates for our numerical scheme. To this end, we recall from Theorems 3.3 and 4.6 that the following a priori estimates hold

$$\begin{aligned}
\|(G, \boldsymbol{u})\| \,&\leq\, C_1(\varphi_{\mathrm{D}},\boldsymbol{g}) \qquad \text{and} \qquad \|\varphi\|_{1,\Omega} \,\leq\, C_2(\varphi_{\mathrm{D}},\boldsymbol{g})\,, \\
\|(G_h, \boldsymbol{u}_h)\| \,&\leq\, C_1^*(\varphi_{\mathrm{D}},\boldsymbol{g}) \qquad \text{and} \qquad \|\varphi_h\|_{1,\Omega} \,\leq\, C_2^*(\varphi_{\mathrm{D}},\boldsymbol{g})\,,
\end{aligned}$$

Moreover, from the Theorems 3.8 and 4.8, we have that if the data is sufficiently small so that if $C_{\rm LIP} < 1$ and $C^*_{\rm LIP} < 1$ (cf. (3.34) and (4.21)), then the solutions are unique. Therefore by setting

$$R := \max\left\{ C_1(\varphi_{\rm D}, \boldsymbol{g}), C_2(\varphi_{\rm D}, \boldsymbol{g}) \right\}, \qquad \text{and} \qquad R^* := \max\left\{ C_1^*(\varphi_{\rm D}, \boldsymbol{g}), C_2^*(\varphi_{\rm D}, \boldsymbol{g}) \right\}, \tag{4.23}$$

it follows that

$$\|(G, \boldsymbol{u}, \varphi)\| \le R, \qquad \text{and} \qquad \|(G_h, \boldsymbol{u}_h, \varphi_h)\| \le R^*. \tag{4.24}$$

We state the convergence of our Galerkin scheme through the next result.

**Theorem 4.9** *Assume that the hypotheses of the Theorems* 3.8 *and* 4.8 *hold, and the data is sufficiently small so that*

$$\frac{1}{C_a^*}\Big( \|\boldsymbol{g}\|_{0,\Omega} + R^* \|\mathbf{c}^{\tt skw}\| \Big) \le \frac{1}{2}, \tag{4.25}$$

*where $C_a^*$ is the coercivity constant of the bilinear form $\mathbf{a}(\cdot,\cdot)$ on $\mathbb{H}_h \times \mathbb{H}_h$ and $R^*$ is defined as in* (4.23)*. Suppose further that the solution satisfies $((G, \boldsymbol{u}, \varphi), S) \in \big(\mathbb{H}^s(\Omega) \times \mathbf{H}^s(\Omega) \times \mathrm{H}^{s+1}(\Omega)\big) \times \mathbb{H}^s(\Omega)$ with $\mathbf{div}\, S \in \mathbf{H}^s(\Omega)$ for some $s \in (0, k+1]$. Then, the errors satisfy*

$$\|((G, \boldsymbol{u}, \varphi), S) - ((G_h, \boldsymbol{u}_h, \varphi_h), S_h)\| \le C h^s, \tag{4.26}$$

*where the constant $C > 0$ depends on the data and high-order norms of the solution, but is independent of $h$.*

*Proof.* We extend in detail the proof of the a priori error estimate result for the dual-mixed formulation of the Navier-Stokes equations given in [14, 15, Theorem 3.4]. In this way, by subtracting (4.4) from (3.7) we obtain the following nonlinear error equation:

$$\begin{aligned} \mathbf{a}((G - G_h, \boldsymbol{u} - \boldsymbol{u}_h, \varphi - \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) - \mathbf{b}(S - S_h, (H_h, \boldsymbol{v}_h, \psi_h)) \;=\; ((\varphi - \varphi_h)\, \boldsymbol{g}, \boldsymbol{v}_h) \\ \mathbf{c}^{\tt skw}((G_h, \boldsymbol{u}_h, \varphi_h), (G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) - \mathbf{c}((G, \boldsymbol{u}, \varphi), (G, \boldsymbol{u}, \varphi), (H_h, \boldsymbol{v}_h, \psi_h)). \end{aligned} \tag{4.27}$$

Let $(G_p, \boldsymbol{u}_p, \varphi_p) \in \mathbb{Z} \times \mathrm{H}_h^{\varphi}$ be arbitrary, where $\varphi_p|_{\Gamma_{\rm D}} = \varphi_{{\rm D},h}$ and write

$$\begin{aligned} (E, \mathbf{e}, e) := (G - G_h, \boldsymbol{u} - \boldsymbol{u}_h, \varphi - \varphi_h) \;&=\; (G - G_p, \boldsymbol{u} - \boldsymbol{u}_p, \varphi - \varphi_p) + (G_p - G_h, \boldsymbol{u}_p - \boldsymbol{u}_h, \varphi_p - \varphi_h) \\ &=: \; (E_p, \mathbf{e}_p, e_p) + (E_h, \mathbf{e}_h, e_h). \end{aligned} \tag{4.28}$$

Note that $e_h = \varphi_p - \varphi_h \in \mathrm{H}^{\varphi}_{h,\Gamma_{\rm D}}$, and so $(E_h, \mathbf{e}_h, e_h) \in \mathbb{H}$. Hence, using the coercivity of $\mathbf{a}(\cdot,\cdot)$ in $\mathbb{H}$ and the equation (4.27) with $(H_h, \boldsymbol{v}_h, \psi_h) = (E_h, \mathbf{e}_h, e_h)$, we find that

$$\begin{aligned} C_a^* \|(E_h, \mathbf{e}_h, e_h)\|^2 &\le \mathbf{a}((E_h, \mathbf{e}_h, e_h), (E_h, \mathbf{e}_h, e_h)) \\ &= \mathbf{a}((E_p, \mathbf{e}_p, e_p), (E_h, \mathbf{e}_h, e_h)) + \mathbf{a}((E, \mathbf{e}, e), (E_h, \mathbf{e}_h, e_h)) \\ &= \mathbf{a}((E_p, \mathbf{e}_p, e_p), (E_h, \mathbf{e}_h, e_h)) + \mathbf{b}(S - S_h, (E_h, \mathbf{e}_h)) + ((\varphi - \varphi_h)\, \boldsymbol{g}, \mathbf{e}_h) \\ &\quad + \mathbf{c}^{\tt skw}((G_h, \boldsymbol{u}_h, \varphi_h), (G_h, \boldsymbol{u}_h, \varphi_h), (E_h, \mathbf{e}_h, e_h)) - \mathbf{c}((G, \boldsymbol{u}, \varphi), (G, \boldsymbol{u}, \varphi), (E_h, \mathbf{e}_h, e_h)). \end{aligned} \tag{4.29}$$

Now, we proceed to bound each term of the right-hand side in (4.29).

First, since $(E_h, \mathbf{e}_h) \in \mathbb{Z}_h$, we have for any $T_h \in \mathbb{H}_h^S$ that

$$\begin{aligned} \mathbf{b}(S - S_h, (E_h, \mathbf{e}_h)) &= \mathbf{b}(S - T_h, (E_h, \mathbf{e}_h)) + \mathbf{b}(T_h - S_h, (E_h, \mathbf{e}_h)) \\ &\le \|\mathbf{b}\|\, \|S - T_h\|_{\mathbf{div},\Omega}\, \|(E_h, \mathbf{e}_h)\|. \end{aligned} \tag{4.30}$$

For the trilinear forms, observe that by adding and subtracting $(G_h, \boldsymbol{u}_h, \varphi_h)$ in the second component of $\mathbf{c}(\cdot,\cdot,\cdot)$ and that this form is consistent with $\mathbf{c}^{\mathtt{skw}}(\cdot,\cdot,\cdot)$ on $\mathbb{Z}$; thus,

$$\begin{aligned}
&\mathbf{c}^{\mathtt{skw}}((G_h,\boldsymbol{u}_h,\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h),(E_h,\mathbf{e}_h,e_h)) \,-\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(E_h,\mathbf{e}_h,e_h)) \\
&\quad = \mathbf{c}^{\mathtt{skw}}((G,\boldsymbol{u},\varphi),(G-G_h,\boldsymbol{u}-\boldsymbol{u}_h,\varphi-\varphi_h),(E_h,\mathbf{e}_h,e_h)) \\
&\qquad - \mathbf{c}^{\mathtt{skw}}((G-G_h,\boldsymbol{u}-\boldsymbol{u}_h,\varphi-\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h),(E_h,\mathbf{e}_h,e_h))\,.
\end{aligned} \tag{4.31}$$

Therefore by adding and subtracting $(G_p, \boldsymbol{u}_p, \varphi_p)$ in the second component of the first term at the right of the latter expression, and employing the skew-symmetric property of the trilinear form we deduce that

$$\begin{aligned}
&\mathbf{c}^{\mathtt{skw}}((G_h,\boldsymbol{u}_h,\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h),(E_h,\mathbf{e}_h,e_h)) \,-\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(E_h,\mathbf{e}_h,e_h)) \\
&\quad = \mathbf{c}^{\mathtt{skw}}((G,\boldsymbol{u},\varphi),(E_p,\mathbf{e}_p,e_p),(E_h,\mathbf{e}_h,e_h)) \,+\, \mathbf{c}^{\mathtt{skw}}((E_p,\mathbf{e}_p,e_p),(G_h,\boldsymbol{u}_h,\varphi_h),(E_h,\mathbf{e}_h,e_h)) \\
&\qquad + \mathbf{c}^{\mathtt{skw}}((E_h,\mathbf{e}_h,e_h),(G_h,\boldsymbol{u}_h,\varphi_h),(E_h,\mathbf{e}_h,e_h))\,.
\end{aligned} \tag{4.32}$$

Thus, applying (4.30)–(4.32) to (4.29), bounding the resulting terms and simplifying yields

$$\begin{aligned}
C_a^* \,\|(E_h,\mathbf{e}_h,e_h)\| \,&\leq\, \|\mathbf{a}\|\,\|(E_p,\mathbf{e}_p,e_p)\| \,+\, \|\mathbf{b}\|\,\|S-T_h\|_{\mathbf{div},\Omega} \,+\, \|\boldsymbol{g}\|_{0,\Omega}(\|e_p\|_{1,\Omega}+\|e_h\|_{1,\Omega}) \\
&+ \|\mathbf{c}^{\mathtt{skw}}\|\Big( \big(\|(G,\boldsymbol{u},\varphi)\| \,+\, \|(G_h,\boldsymbol{u}_h,\varphi_h)\|\big)\,\|(E_p,\mathbf{e}_p,e_p)\| \,+\, \|(G_h,\boldsymbol{u}_h,\varphi_h)\|\,\|(E_h,\mathbf{e}_h,e_h)\|\Big)\,.
\end{aligned}$$

Hence, by manipulating terms, and using the bounds (4.24) we get

$$\begin{aligned}
\|(E_h,\mathbf{e}_h,e_h)\| \,&\leq\, C_a^{-1}\Big(\|\mathbf{a}\| \,+\, \|\boldsymbol{g}\|_{0,\Omega} \,+\, \big(R \,+\, R^*\big)\,\|\mathbf{c}^{\mathtt{skw}}\|\Big)\,\|(E_p,\mathbf{e}_p,e_p)\| \,+ C_a^{-1}\,\|\mathbf{b}\|\,\|S-T_h\|_{\mathbf{div},\Omega} \\
&\quad + C_a^{-1}\Big(\|\boldsymbol{g}\|_{0,\Omega} \,+\, R^*\,\|\mathbf{c}^{\mathtt{skw}}\|\Big)\,\|(E_h,\mathbf{e}_h,e_h)\|\,.
\end{aligned}$$

In this way, if the data is sufficiently small so that the hypothesis (4.25) holds, then the last term on the right can be absorbed into the left:

$$\|(E_h,\mathbf{e}_h,e_h)\| \,\leq\, \frac{2}{C_a^*}\Big\{\Big(\|\mathbf{a}\| \,+\, \|\boldsymbol{g}\|_{0,\Omega} \,+\, \big(R \,+\, R^*\big)\,\|\mathbf{c}^{\mathtt{skw}}\|\Big)\,\|(E_p,\mathbf{e}_p,e_p)\| \,+\, \|\mathbf{b}\|\,\|S-T_h\|_{\mathbf{div},\Omega}\Big\}\,.$$

It then follows from (4.28) that

$$\begin{aligned}
\|(E,\mathbf{e},e)\| \,&\leq\, C\Big(\|(E_p,\mathbf{e}_p,e_p)\| \,+\, \|S-T_h\|_{\mathbf{div},\Omega}\Big) \\
&\leq\, C\Big\{\inf_{(G_p,\boldsymbol{u}_p,\varphi_p)\in\mathbb{Z}_h\times\mathrm{H}_h^{\varphi}} \|(G-G_p,\boldsymbol{u}-\boldsymbol{u}_p,\varphi-\varphi_p)\| \,+\, \inf_{T_h\in\mathbb{H}_h^S}\|S-T_h\|_{\mathbf{div},\Omega}\Big\} \\
&\leq\, C\Big\{\inf_{H_h\in\mathbb{H}_h^G}\|G-H_h\|_{0,\Omega} + \inf_{\boldsymbol{v}_h\in\mathbf{H}_h^{\boldsymbol{u}}}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{0,4,\Omega} + \inf_{\psi_h\in\mathrm{H}_h^{\varphi}}\|\varphi-\psi_h\|_{1,\Omega} + \inf_{T_h\in\mathbb{H}_h^S}\|S-T_h\|_{\mathbf{div},\Omega}\Big\}\,,
\end{aligned} \tag{4.33}$$

where the last statement follows from the inf–sup condition.

Finally, we estimate the error for the stress tensor. To this end we have by the discrete inf-sup condition (4.7), for arbitrary $T_h \in \mathbb{H}_h^S$,

$$
\begin{aligned}
\beta^* \, \|T_h - S_h\|_{\mathbf{div},\Omega} \, &\le \sup_{\substack{(H_h,\boldsymbol{v}_h)\in\mathbb{H}_h^G\times\mathbb{H}_h^{\boldsymbol{u}}\\ (H_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \frac{\mathbf{b}(T_h - S_h, (H_h,\boldsymbol{v}_h))}{\|(H_h,\boldsymbol{v}_h)\|} \\
&\le \sup_{\substack{(H_h,\boldsymbol{v}_h)\in\mathbb{H}_h^G\times\mathbb{H}_h^{\boldsymbol{u}}\\ (H_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \frac{\mathbf{b}(T_h - S, (H_h,\boldsymbol{v}_h))}{\|(H_h,\boldsymbol{v}_h)\|} + \sup_{\substack{(H_h,\boldsymbol{v}_h)\in\mathbb{H}_h^G\times\mathbb{H}_h^{\boldsymbol{u}}\\ (H_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \frac{\mathbf{b}(S - S_h, (H_h,\boldsymbol{v}_h))}{\|(H_h,\boldsymbol{v}_h)\|} \\
&\le \|\mathbf{b}\| \, \|S - T_h\|_{\mathbf{div},\Omega} + \sup_{\substack{(H_h,\boldsymbol{v}_h)\in\mathbb{H}_h^G\times\mathbb{H}_h^{\boldsymbol{u}}\\ (H_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \frac{\mathbf{b}(S - S_h, (H_h,\boldsymbol{v}_h))}{\|(H_h,\boldsymbol{v}_h)\|} \,.
\end{aligned} \tag{4.34}
$$

Using the error equation (4.27) and the identity (4.31) we have

$$
\begin{aligned}
&\mathbf{b}(S - S_h, (H_h,\boldsymbol{v}_h)) \, = \, \mathbf{a}((G - G_h, \boldsymbol{u} - \boldsymbol{u}_h, \varphi - \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) \, - \, ((\varphi - \varphi_h)\, \boldsymbol{g}, \boldsymbol{v}_h) \\
&\qquad - \mathbf{c}^{\mathbf{skw}}((G_h,\boldsymbol{u}_h,\varphi_h),(G_h,\boldsymbol{u}_h,\varphi_h),(H_h,\boldsymbol{v}_h,\psi_h)) \, + \, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(H_h,\boldsymbol{v}_h,\psi_h)) \\
&\le \|\mathbf{a}\| \, \|(G - G_h, \boldsymbol{u} - \boldsymbol{u}_h, \varphi - \varphi_h)\| \, \|(H_h,\boldsymbol{v}_h,\psi_h)\| \, + \, \|\varphi - \varphi_h\|_{1,\Omega} \, \|\boldsymbol{g}\|_{0,\Omega} \, \|\boldsymbol{v}_h\|_{0,\Omega} \\
&\qquad + (R + R^*) \, \|(G - G_h, \boldsymbol{u} - \boldsymbol{u}_h, \varphi - \varphi_h)\| \, \|\mathbf{c}^{\mathbf{skw}}\| \, \|(H_h,\boldsymbol{v}_h)\| \,.
\end{aligned} \tag{4.35}
$$

Applying (4.35) to bound the last term in (4.34) and then using the triangle inequality yields

$$
\|S - S_h\|_{\mathbf{div},\Omega} \, \le \, C \Big( \|S - T_h\|_{\mathbf{div},\Omega} + \|(E,\mathbf{e},e)\| \Big) \, \le \, C \Big\{ \inf_{T_h \in \mathbb{H}_h^S} \|S - T_h\|_{\mathbf{div},\Omega} + \|(E,\mathbf{e},e)\| \Big\} \,. \tag{4.36}
$$

Thus, (4.26) follows by combining (4.33), (4.36) and standard approximation properties of the finite element spaces. $\square$

# 5 An alternative formulation

In this section we introduce and analyze an alternative formulation for the problem (2.4) which differs from (3.4) on the treatment of the mixed boundary conditions for the temperature. More precisely, along with the set of equations (3.2) and (3.3) associated to the fluid, we consider a primal-mixed formulation for the heat equation [5, 6].

## 5.1 The continuous problem and its well-posedness

Multiplying the fourth equation of (2.4) by a function $\psi \in \mathrm{H}^1(\Omega)$, and after integrating by parts and employing the Neumann boundary condition, we introduce the normal derivative of the temperature $\lambda := -\kappa \nabla\varphi \cdot \boldsymbol{n} \in \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})$ as a new unknown on $\Gamma_{\mathrm{D}}$, namely,

$$
\kappa \, (\nabla\varphi, \nabla\psi) \, + \, \langle \lambda, \psi \rangle_{\Gamma_{\mathrm{D}}} \, + \, (\boldsymbol{u} \cdot \nabla\varphi, \psi) \, = \, 0 \quad \forall\, \psi \in \mathrm{H}^1(\Omega) \,,
$$

where $\langle \cdot, \cdot \rangle_{\Gamma_{\mathrm{D}}} := \langle \,\cdot\,, \gamma_0(\cdot)|_{\Gamma_{\mathrm{D}}} \rangle_{\Gamma_{\mathrm{D}}}$ stands for the dual product between $\mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})$ and $\mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$, and $\gamma_0|_{\Gamma_{\mathrm{D}}} : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ is the trace operator $\gamma_0$ in $\mathrm{H}^1(\Omega)$ restricted to $\Gamma_{\mathrm{D}}$. The Dirichlet condition is then weakly imposed as

$$
\langle \xi, \varphi \rangle_{\Gamma_{\mathrm{D}}} \, = \, \langle \xi, \varphi_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} \quad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}) \,. \tag{5.1}
$$

Hence, the underlying formulation is: Find $((G,\boldsymbol{u}),S,(\varphi,\lambda)) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega)) \times \mathbb{H}_0(\mathbf{div};\Omega) \times (\mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}))$ such that

$$\begin{array}{rcl} (\mathcal{A}(G),H) \,-\, \dfrac{1}{2}(\boldsymbol{u}\otimes\boldsymbol{u},H) \,-\, (S,H) & = & 0 \\ \dfrac{1}{2}(G\boldsymbol{u},\boldsymbol{v}) \,-\, (\mathbf{div}\,S,\boldsymbol{v}) \,-\, (\varphi\,\boldsymbol{g},\boldsymbol{v}) & = & 0 \\ (G,T) \,+\, (\boldsymbol{u},\mathbf{div}\,T) & = & 0 \\ \kappa\,(\nabla\varphi,\nabla\psi) \,+\, \langle\lambda,\psi\rangle_{\Gamma_{\mathrm{D}}} \,+\, (\boldsymbol{u}\cdot\nabla\varphi,\psi) & = & 0 \\ \langle\xi,\varphi\rangle_{\Gamma_{\mathrm{D}}} & = & \langle\xi,\varphi_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}}\,. \end{array} \tag{5.2}$$

for all $((H,\boldsymbol{v}),T,(\psi,\xi)) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega)) \times \mathbb{H}_0(\mathbf{div};\Omega) \times (\mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}))$.

Define the bilinear form $\widetilde{\mathbf{b}} : (\mathbb{H}_0(\mathbf{div};\Omega) \times \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})) \times (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega)) \longrightarrow \mathrm{R}$,

$$\widetilde{\mathbf{b}}((T,\xi),(H,\boldsymbol{v},\psi)) \,=\, (H,T) \,+\, (\boldsymbol{v},\mathbf{div}\,T) \,-\, \langle\xi,\psi\rangle_{\Gamma_{\mathrm{D}}}\,, \tag{5.3}$$

whose kernel is $\mathbb{H} = \mathbb{Z} \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$, where $\mathbb{Z}$ is given by (3.8). With the same forms $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{c}(\cdot,\cdot,\cdot)$ from Definition 3.1, we see that problem (5.2) is equivalent to: Find $((G,\boldsymbol{u},\varphi),(S,\lambda)) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega)) \times (\mathbb{H}_0(\mathbf{div};\Omega) \times \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}))$ such that:

$$\begin{array}{rcl} \mathbf{a}((G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,+\, \mathbf{c}((G,\boldsymbol{u},\varphi),(G,\boldsymbol{u},\varphi),(H,\boldsymbol{v},\psi)) \,-\, \widetilde{\mathbf{b}}((S,\lambda),(H,\boldsymbol{v},\psi)) & = & (\varphi\,\boldsymbol{g},\boldsymbol{v}) \\ \widetilde{\mathbf{b}}((T,\xi),(G,\boldsymbol{u},\varphi)) & = & \langle\xi,\varphi_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}} \end{array} \tag{5.4}$$

for all $((H,\boldsymbol{v},\psi),(T,\xi)) \in (\mathbb{L}^2_{\mathtt{tr}}(\Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega)) \times (\mathbb{H}_0(\mathbf{div};\Omega) \times \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}))$.

Observe that the properties relative to the forms $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{c}(\cdot,\cdot,\cdot)$ stated in Lemma 3.1 hold. Regarding the bilinear form $\widetilde{\mathbf{b}}(\cdot,\cdot)$, note that it involves additionally the term $\langle\xi,\psi\rangle_{\Gamma_{\mathrm{D}}}$ associated to the Lagrange multiplier. Denote by $\mathcal{R}_{-1/2,\Gamma_{\mathrm{D}}} : \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}}) \longrightarrow \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ the usual Riesz operator and by $\mathcal{R}^*_{-1/2,\Gamma_{\mathrm{D}}}$ its adjoint (which are bijective). Since

$$\langle\xi,\psi\rangle_{\Gamma_{\mathrm{D}}} \,=\, \langle\xi,\gamma_0(\psi)|_{\Gamma_{\mathrm{D}}}\rangle_{\Gamma_{\mathrm{D}}} \,=\, \langle\xi,\big(\mathcal{R}^*_{-1/2,\Gamma_{\mathrm{D}}}\circ\gamma_0|_{\Gamma_{\mathrm{D}}}\big)(\psi)\rangle_{-1/2,\Gamma_{\mathrm{D}}},$$

and since the operator $\mathcal{R}^*_{-1/2,\Gamma_{\mathrm{D}}}\circ\gamma_0|_{\Gamma_{\mathrm{D}}} : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})$ is surjective, Lemma 3.1 implies that $\widetilde{\mathbf{b}}(\cdot,\cdot)$ satisfies the inf-sup condition. Thus, there exists a positive constant $\widetilde{\beta}$ such that

$$\sup_{\substack{(H,\boldsymbol{v},\psi)\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbf{L}^4(\Omega)\times\mathrm{H}^1(\Omega)\\(H,\boldsymbol{v},\psi)\neq\mathbf{0}}} \frac{\widetilde{\mathbf{b}}((T,\xi),(H,\boldsymbol{v},\psi))}{\|(H,\boldsymbol{v},\psi)\|} \,\geq\, \widetilde{\beta}\,\|(T,\xi)\| \quad \forall\,(T,\xi) \,\in\, \mathbb{H}_0(\mathbf{div};\Omega)\times\mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})\,. \tag{5.5}$$

Note that the variational problem (5.4) restricted to the kernel $\mathbb{H}$ reduces to problem (3.11). Hence the corresponding solvability analysis follows from Section 3.2. In particular, from the Theorem 3.3 we have the same a priori estimates stated there for $G$, $\boldsymbol{u}$ and $\varphi$, and from Theorems 3.7 and 3.8, existence of continuous solution is guaranteed with no constraint on data and the uniqueness follows for small data assumption. In turn, the existence of the stress tensor $S$ and the Lagrange multiplier $\lambda$ is a consequence of the inf-sup condition (5.5), and

$$\|(S,\lambda)\| \,\leq\, C\Big(\,\|\mathbf{a}\| \,+\, \|\mathbf{c}\|\,\|(G,\boldsymbol{u},\varphi)\| \,+\, \|\boldsymbol{g}\|_{0,\Omega}\Big)\,\|(G,\boldsymbol{u},\varphi)\|\,.$$

### 5.2 The discrete scheme

To discretize the primal–mixed formulation, we adopt the notations introduced in Section 4.1, and in addition, consider an independent triangulation $\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_m\}$ of $\Gamma_{\rm D}$ (consisting of straight segments in $\mathrm{R}^2$ or triangles in $\mathrm{R}^3$) and define $\widetilde{h} := \max\limits_{j\in\{1,\ldots,m\}} |\widetilde{\Gamma}_j|$. Then, with the same integer $k \geq 0$ employed in the definitions (4.1)–(4.2), we introduce the finite element subspace

$$\mathrm{H}^{\lambda}_{\widetilde{h}} := \Big\{ \xi_{\widetilde{h}} \in \mathrm{L}^2(\Gamma_{\rm D}) : \quad \xi_{\widetilde{h}}\Big|_{\widetilde{\Gamma}_j} \in \mathrm{P}_k(\widetilde{\Gamma}_j) \quad \forall\, j \in \{1, 2, \cdots, m\} \Big\}. \tag{5.6}$$

The discrete problem based on (5.4) is then: Find $((G_h, \boldsymbol{u}_h, \varphi_h), (S_h, \lambda_{\widetilde{h}})) \in (\mathbb{H}^G_h \times \mathbf{H}^{\boldsymbol{u}}_h \times \mathrm{H}^{\varphi}_h) \times (\mathbb{H}^S_h \times \mathrm{H}^{\lambda}_{\widetilde{h}})$ such that:

$$\begin{aligned}
&\mathbf{a}((G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) \,+\, \mathbf{c}^{\mathtt{skw}}((G_h, \boldsymbol{u}_h, \varphi_h), (G_h, \boldsymbol{u}_h, \varphi_h), (H_h, \boldsymbol{v}_h, \psi_h)) \\
&\qquad\qquad - \widetilde{\mathbf{b}}((S_h, \lambda_{\widetilde{h}}), (H_h, \boldsymbol{v}_h, \psi_h)) = (\varphi_h\, \boldsymbol{g}, \boldsymbol{v}_h) \quad \forall\, (H_h, \boldsymbol{v}_h, \psi_h) \in \mathbb{H}^G_h \times \mathbf{H}^{\boldsymbol{u}}_h \times \mathrm{H}^{\varphi}_h \\
&\widetilde{\mathbf{b}}((T_h, \xi_{\widetilde{h}}), (G_h, \boldsymbol{u}_h, \varphi_h)) = \langle \xi_{\widetilde{h}}, \varphi_{\rm D}\rangle_{\Gamma_{\rm D}} \quad \forall\, (T_h, \xi_{\widetilde{h}}) \in \mathbb{H}^S_h \times \mathrm{H}^{\lambda}_{\widetilde{h}}\,,
\end{aligned} \tag{5.7}$$

where $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{c}^{\mathtt{skw}}(\cdot,\cdot)$ are the forms defined by (3.5) and (4.5), and $\widetilde{\mathbf{b}}(\cdot,\cdot)$ is defined by (5.3).

The first step to show that problem (5.7) is well–posed is to verify that the finite element spaces are compatible. This issue is addressed in the next result. The proof essentially follows from [11, Lemma 4.7] and the inf-sup property (4.7) in Lemma 4.1.

**Lemma 5.1** *There exist $C_0 > 0$ and $\widehat{\beta}^* > 0$, independent of $h$ and $\widetilde{h}$, such that for all $h \leq C_0\, \widetilde{h}$, there holds*

$$\sup_{\substack{\psi_h\in \mathrm{H}^{\varphi}_h\\ \psi_h\neq 0}} \frac{\langle \xi_{\widetilde{h}}, \psi_h\rangle_{\Gamma_{\rm D}}}{\|\psi_h\|_{1,\Omega}} \,\geq\, \widehat{\beta}^*\, \|\xi_{\widetilde{h}}\|_{-1/2,\Gamma_{\rm D}} \quad \forall\, \xi_{\widetilde{h}} \in \mathrm{H}^{\lambda}_{\widetilde{h}}. \tag{5.8}$$

*Consequently,*

$$\sup_{\substack{(H_h,\boldsymbol{v}_h,\psi_h)\in \mathbb{H}^G_h\times \mathbf{H}^{\boldsymbol{u}}_h\times \mathrm{H}^{\varphi}_h\\ (H_h,\boldsymbol{v}_h,\psi_h)\neq \mathbf{0}}} \frac{\widetilde{\mathbf{b}}((T_h, \xi_{\widetilde{h}}), (H_h, \boldsymbol{v}_h, \psi_h))}{\|(H_h, \boldsymbol{v}_h, \psi_h)\|} \,\geq\, \widetilde{\beta}^*\, \|(T_h, \xi_{\widetilde{h}})\| \quad \forall\, (H_h, \xi_{\widetilde{h}}) \in \mathbb{H}^S_h \times \mathrm{H}^{\lambda}_{\widetilde{h}}\,, \tag{5.9}$$

*with $\widetilde{\beta}^* := \min\{\beta^*, \widehat{\beta}^*\}$.*

We introduce the discrete kernel $\mathrm{Z}_h$ given by

$$\mathrm{Z}_h := \Big\{ \psi_h \in \mathrm{H}^{\varphi}_h : \quad \langle\, \xi_{\widetilde{h}}\,, \psi_h\,\rangle_{\Gamma_{\rm D}} \,=\, 0 \quad \forall\, \xi_{\widetilde{h}} \in \mathrm{H}^{\lambda}_{\widetilde{h}} \Big\}.$$

Such as in [11, Section 4.3], observe that $\xi_{\widetilde{h}} \equiv 1$ belongs to $\mathrm{H}^{\lambda}_{\widetilde{h}}$ and then

$$\mathrm{Z}_h \subseteq \Big\{ \psi \in \mathrm{H}^1(\Omega) : \quad \langle\, 1\,, \psi\,\rangle_{\Gamma_{\rm D}} \,=\, 0 \Big\} \,=\, \Big\{ \psi \in \mathrm{H}^1(\Omega) : \quad \int_{\Gamma_{\rm D}} \psi \,=\, 0 \Big\}.$$

Therefore, from the Poincaré inequality, we have that $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ are equivalent in $\mathrm{Z}_h$. In this way, setting $\widetilde{\mathbb{H}}_h = \mathbb{Z}_h \times \mathrm{Z}_h$, it is easy to see that this property along with Lemma 4.1 implies that the bilinear form $\mathbf{a}(\cdot,\cdot)$ is coercive, that is,

$$\mathbf{a}((G_h, \boldsymbol{u}_h, \varphi_h), (G_h, \boldsymbol{u}_h, \varphi_h)) \,\geq\, \widetilde{C}^*_a\, \|(G_h, \boldsymbol{u}_h, \varphi_h)\|^2 \quad \forall\, (G_h, \boldsymbol{u}_h, \varphi_h) \in \widetilde{\mathbb{H}}_h\,. \tag{5.10}$$

**Remark 5.1** *The formulation* (4.4) *involves an approximation of the boundary temperature whereas problem* (5.7) *incorporates it via the discrete form of the corresponding weak imposition* (5.1)*. Because of this difference, the analogous extension* $\varphi_{1,h}$ *to be used in the discrete analysis must be defined differently (cf. Sections 4.3–4.4). To this end, denote by* $\Pi_{\mathrm{Z}_h^\perp}$ *the orthogonal projection from* $\mathrm{H}_h^\varphi$ *onto the kernel complement* $\mathrm{Z}_h^\perp$*, and observe that the inf–sup condition* (5.8) *is equivalent to (see [11, Lemma 2.1])*

$$\sup_{\substack{\xi_{\widetilde{h}}\in \mathrm{H}^\lambda_{\widetilde{h}}\\ \xi_{\widetilde{h}}\neq 0}} \frac{\langle \xi_{\widetilde{h}}, \Pi_{\mathrm{Z}_h^\perp}\psi_h\rangle_{\Gamma_\mathrm{D}}}{\|\xi_{\widetilde{h}}\|_{-1/2,\Gamma_\mathrm{D}}} \,\geq\, \widehat{\beta}^*\,\|\Pi_{\mathrm{Z}_h^\perp}\psi_h\|_{1,\Omega} \quad \forall\,\psi_h\in \mathrm{H}_h^\varphi \quad \textit{and} \quad \forall\, h\leq C_0\widetilde{h}\,.$$

*In particular, since* $\langle \xi_{\widetilde{h}}, \Pi_{\mathrm{Z}_h^\perp}\varphi_h\rangle_{\Gamma_\mathrm{D}} = \langle \xi_{\widetilde{h}}, \varphi_\mathrm{D}\rangle_{\Gamma_\mathrm{D}} \leq \|\xi_{\widetilde{h}}\|_{-1/2,\Gamma_\mathrm{D}}\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}$ $\forall\,\xi_{\widetilde{h}}\in \mathrm{H}^\lambda_{\widetilde{h}}$*, there holds*

$$\|\Pi_{\mathrm{Z}_h^\perp}\varphi_h\|_{1,\Omega} \,\leq\, (1/\widehat{\beta}^*)\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \quad \forall\, h\leq C_0\widetilde{h}\,.$$

*As a result, applying Lemma 4.3 to* $\Pi_{\mathrm{Z}_h^\perp}\varphi_h|_{\Gamma_\mathrm{D}}\in \mathrm{H}^{1/2}(\Gamma_\mathrm{D})$*, and a trace inequality, we conclude that for any* $\delta\in(0,1)$ *there exists an* $h_\delta>0$ *such that*

$$\|E_{\delta,h}\big(\Pi_{\mathrm{Z}_h^\perp}\varphi_h|_{\Gamma_\mathrm{D}}\big)\|_{0,3,\Omega} \,\leq\, C\delta\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \quad \textit{and} \quad \|E_{\delta,h}\big(\Pi_{\mathrm{Z}_h^\perp}\varphi_h|_{\Gamma_\mathrm{D}}\big)\|_{1,\Omega} \,\leq\, C\delta^{-4}\|\varphi_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}\,, \tag{5.11}$$

*for all* $h\leq\{h_\delta, C_0\widetilde{h}\}$*. The discrete extension is then defined as* $\varphi_{1,h} = E_{\delta,h}\big(\Pi_{\mathrm{Z}_h^\perp}\varphi_h\big|_{\Gamma_\mathrm{D}}\big)$*.*

We are in position to state the main result of this section.

**Theorem 5.2** *Let the discrete spaces* $\mathbb{H}_h^G$*,* $\mathbf{H}_h^{\boldsymbol{u}}$*,* $\mathbb{H}_h^S$*, and* $\mathrm{H}_h^\varphi$ *be defined as in Section* 4.1*, and* $\mathrm{H}^\lambda_{\widetilde{h}}$ *be defined by* (5.6)*. Then, there exist an* $h_\delta>0$ *and at least one solution* $((G_h,\boldsymbol{u}_h,\varphi_h),(S_h,\lambda_{\widetilde{h}}))$ *to* (5.7) *for all* $h\leq\{h_\delta, C_0\widetilde{h}\}$*, satisfying*

$$\begin{gathered}\|(G_h,\boldsymbol{u}_h)\|\leq \widetilde{C}_1^*(\varphi_\mathrm{D},\boldsymbol{g})\,, \quad \|\varphi_h\|_{1,\Omega}\leq \widetilde{C}_2^*(\varphi_\mathrm{D},\boldsymbol{g})\,, \quad \textit{and}\\ \|(S_h,\lambda_{\widetilde{h}})\| \,\leq\, C\Big(\,\|\mathbf{a}\| \,+\, \|\mathbf{c}^{\mathbf{skw}}\|\,\|(G_h,\boldsymbol{u}_h,\varphi_h)\| \,+\, \|\boldsymbol{g}\|_{0,\Omega}\Big)\,\|(G_h,\boldsymbol{u}_h,\varphi_h)\|\,,\end{gathered} \tag{5.12}$$

*where* $\widetilde{C}_1^*(\varphi_\mathrm{D},\boldsymbol{g}) = CC_1(\varphi_\mathrm{D},\boldsymbol{g})>0$*,* $\widetilde{C}_2^*(\varphi_\mathrm{D},\boldsymbol{g}) = CC_2(\varphi_\mathrm{D},\boldsymbol{g})$*,* $C>0$ *is independent of* $h$ *and* $\widetilde{h}$*, and* $C_1(\varphi_\mathrm{D},\boldsymbol{g})$ *and* $C_2(\varphi_\mathrm{D},\boldsymbol{g})$ *are given in Theorem 3.3. Moreover, provided the data is small enough (cf.* (5.14)–(5.15) *below) and* $((G,\boldsymbol{u},\varphi),(S,\lambda))\in\big(\mathbb{H}^s(\Omega)\times\mathbf{H}^s(\Omega)\times\mathrm{H}^{s+1}(\Omega)\big)\times\big(\mathbb{H}^s(\Omega)\times\mathrm{H}^{-1/2+s}(\Gamma_\mathrm{D})\big)$ *with* $\mathbf{div}\,S\in\mathbf{H}^s(\Omega)$ *for some* $s\in(0,k+1]$*, the errors satisfy*

$$\|((G,\boldsymbol{u},\varphi),(S,\lambda)) \,-\, ((G_h,\boldsymbol{u}_h,\varphi_h),(S_h,\lambda_{\widetilde{h}}))\| \,\leq\, C\,h^s \,+\, C\,\widetilde{h}^s \tag{5.13}$$

*where* $C>0$ *depends on the data and high-order norms of the solution, but is independent of* $h$ *and* $\widetilde{h}$*.*

*Proof.* Observe that, thanks to the inf–sup condition (5.9), the coercivity result (5.10) and Remark 5.1, the same arguments used in Sections 4.2–4.5 hold by replacing $\mathrm{H}^\varphi_{h,\Gamma_\mathrm{D}}$, $\mathbb{H}_h$ and $\mathbf{b}(\cdot,\cdot)$ by $\mathrm{Z}_h$, $\widetilde{\mathbb{H}}_h$, and $\widetilde{\mathbf{b}}(\cdot,\cdot)$, respectively, and defining $\varphi_{1,h} = E_{\delta,h}\big(\Pi_{\mathrm{Z}_h^\perp}\varphi_h|_{\Gamma_\mathrm{D}}\big)$ which satisfies the estimates (5.11).

Next, the same fixed-point approach in Section 4.4 shows the existence of solutions. The arguments in this section (cf. (4.21) and Theorem 4.8) also show the uniqueness of solutions provided the data is sufficiently small so that the resulting Lipschitz continuity constant, denoted by $\widetilde{C}^*_{\mathrm{LIP}}$, satisfies

$$\widetilde{C}^*_{\mathrm{LIP}} \leq \frac{C}{\widetilde{C}^*_a}\Big\{\widetilde{C}_1^*(\varphi_\mathrm{D},\boldsymbol{g}) + \widetilde{C}_2^*(\varphi_\mathrm{D},\boldsymbol{g}) + \widetilde{C}_4^*(\varphi_\mathrm{D},\boldsymbol{g})\Big\} \,<\, 1\,. \tag{5.14}$$

Likewise, the a priori estimate (5.12) for the tensor and the Lagrange multiplier as well as the corresponding existence result are a consequence of the inf–sup condition (5.9) (cf. (4.22)). Finally, the error estimate (5.13) is obtained by slightly modifying the proof of Theorem (4.9), with $\widetilde{\mathbf{b}}(\cdot,\cdot)$ in place of $\mathbf{b}(\cdot,\cdot)$, and noting that the small data constraint (4.25) takes the form

$$\frac{1}{\widetilde{C}_a^*}\Big(\|\boldsymbol{g}\|_{0,\Omega} + \widetilde{R}^* \,\|\mathbf{c}^{\mathtt{skw}}\|\Big) \leq \frac{1}{2} \tag{5.15}$$

with $\widetilde{R}^* = \max\{\widetilde{C}_1^*(\varphi_{\mathrm{D}},\boldsymbol{g}), \widetilde{C}_2^*(\varphi_{\mathrm{D}},\boldsymbol{g})\}$.

□

# 6 Numerical results

In this section we present a two examples to support the theoretical results and to illustrate the performance of our dual-mixed finite element schemes. The computations are performed on a set of meshes $\mathcal{T}_h^r$ created as a barycenter refinement of uniform triangular meshes $\mathcal{T}_h$ (cf. Figure 1) which satisfy the macro–element structure required for the inf–sup/LBB compatibility condition at discrete level (see Section 4.1). We consider $n = 2$ and order of approximation $k = 1$, and thus the finite element spaces for the fluid unknowns in both formulations are given explicitly as

$$\mathbb{H}_h^G = \mathbb{L}^2_{\mathtt{tr}}(\Omega) \cap \mathbb{P}_1^{disc}(\mathcal{T}_h^r), \qquad \mathbf{H}_h^{\boldsymbol{u}} = \mathbf{P}_1^{disc}(\mathcal{T}_h^r), \qquad \mathbb{H}_h^S = \mathbb{H}_0(\mathbf{div};\Omega) \cap \mathbb{RT}_1(\mathcal{T}_h^r).$$

For the heat equation unknowns, we consider the subspaces

$$\mathrm{H}_h^{\varphi} = \mathrm{P}_2(\mathcal{T}_h^r), \qquad \text{and} \qquad \mathrm{H}_{\widetilde{h}}^{\lambda} = \mathrm{P}_1^{disc}(\mathcal{T}_{\widetilde{h}}^r \cap \Gamma_{\mathrm{D}}),$$

where $\mathrm{H}_{\widetilde{h}}^{\lambda}$ is only employed for the formulation involving the Lagrange multiplier. Similar to [5], we take $\widetilde{h}$ as two times $h$, which comes from the restriction on the mesh sizes $h \leq C\widetilde{h}$ when considering the constant $C = 1/2$. The numeric results confirm that this choice is suitable.

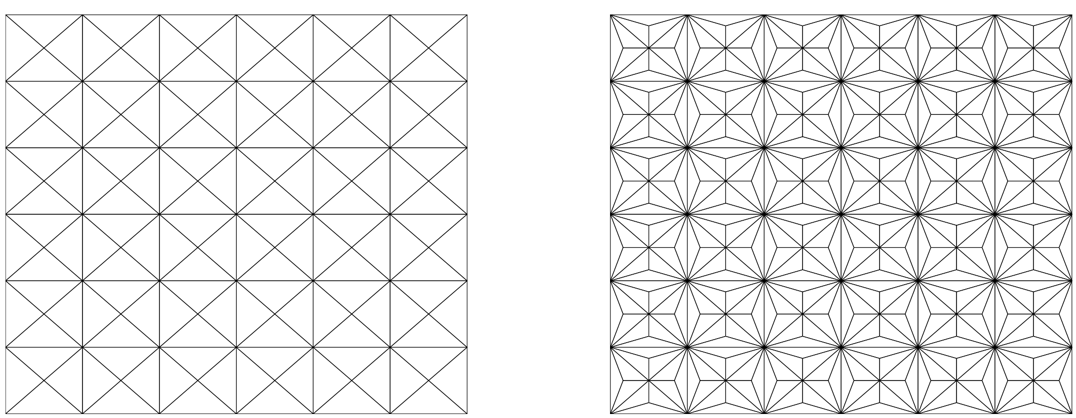

Figure 1: Uniform mesh and its barycenter refinement with meshsize $h = 1/3$ of the square $[-1, 1]^2$.

The individual errors are denoted by:

$$\mathsf{e}(G) := \|G - G_h\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}, \quad \mathsf{e}(S) := \|S - S_h\|_{\mathbf{div},\Omega},$$

$$\mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega}, \qquad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{0,\Gamma}, \qquad \text{and} \qquad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega},$$

where $\|\cdot\|^2_{\mathbf{div},\Omega} = \|\cdot\|^2_{0,\Omega} + \|\mathbf{div}\,\cdot\|^2_{0,\Omega}$, $p$ is the exact pressure of the fluid, and $p_h$ is the recovered discrete pressure suggested by the formulas given in the second equation of (2.3) and (3.1), namely,

$$p_h = -\frac{1}{2n}\operatorname{tr}\Big\{2S_h + c_h\mathbb{I} + (\boldsymbol{u}_h \otimes \boldsymbol{u}_h)\Big\}, \quad \text{with} \quad c_h := -\frac{1}{2n|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{u}_h \otimes \boldsymbol{u}_h).$$

Moreover, it is easy to see that there exists $C > 0$, independents of $h$, such that

$$\|p - p_h\|_{0,\Omega} \le C\Big\{\|S - S_h\|_{0,\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}\Big\}$$

which says that the rate of convergence of the postprocessed discrete pressure is the same of $S$ and $\boldsymbol{u}$. In turn, we let $r(\cdot)$ be the experimental rate of convergence given by

$$r(\cdot) := \frac{\log(\mathsf{e}(\cdot)/\mathsf{e}'(\cdot))}{\log(h/h')}$$

where $h$ and $h'$ (resp. $\widetilde{h}$ and $\widetilde{h}'$ for $\lambda$) denote two consecutive mesh sizes with errors $\mathsf{e}$ and $\mathsf{e}'$.

**Example 1.** In our first example we illustrate the accuracy of our methods considering manufactured non-homogeneous exact solutions. For the dual-mixed formulation we set $\Omega = (0,1)^2$, and

$$\boldsymbol{u}(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2)e^{x_1^2+x_2}\begin{pmatrix} 2\pi\sin(\pi x_1)\cos(\pi x_2) + \sin(\pi x_1)\sin(\pi x_2) \\ -2\pi\sin(\pi x_2)\cos(\pi x_1) - 2x_1\sin(\pi x_1)\sin(\pi x_2)\end{pmatrix},$$
$$p(x_1,x_2) = x_2 x_1^4 - 0.1 \qquad \text{and} \qquad \varphi(x_1,x_2) = (x_1-1)^2\sin^2(\pi(x_2-1)),$$

and for testing the alternative scheme we take $\Omega = (-1,1)^2$ and

$$\boldsymbol{u}(x_1,x_2) = \begin{pmatrix} 2\pi\cos(\pi x_2)\sin^2(\pi x_1)\sin(\pi x_2) \\ -2\pi\cos(\pi x_1)\sin(\pi x_1)\sin^2(\pi x_2)\end{pmatrix},$$
$$p(x_1,x_2) = 5x_1\sin(x_2) \qquad \text{and} \qquad \varphi(x_1,x_2) = e^{sin(x_1)+sin(x_2)}.$$

In both cases, the Dirichlet data for the temperature $\varphi_{\mathrm{D}}$, and the right-hand sides are constructed with the corresponding manufactured exact solutions on the respective domains, and consider $\nu = 1$, $\kappa = 1$, $\mathbf{g} = (1,0)^{\mathtt{t}}$. In Table 1 we present the convergence history of the computed solutions for both schemes, and observe that the convergence rates are quadratic with respect to $h$ and $\widetilde{h}$; these results are in agreement with Theorems 4.9 and 5.2 with $k = 1$.

**Example 2. The natural convection problem in a differentially heated cavity.** In this example we study the robustness of our dual-mixed method by solving a benchmark problem in natural convection flows (see [7] and [12]). We consider $\Omega = (0,1)^2$ and boundary conditions corresponding to internal flow (no slip for the velocity) with the top and bottom insulated, and heating/cooling applied to the left and right side. The external force field corresponding to the buoyancy term is given as $\mathrm{Ra}\,\varphi\boldsymbol{g}$, where Ra is the Rayleigh number and the gravity $\boldsymbol{g}$ is assumed to act upward vertically, and we take the physical parameters $\nu = \kappa = 1$.

In figure 2, we display the approximations of the velocity (its magnitude and streamlines), the temperature and pressure for several values of $\mathrm{Ra} \in [1000, 1000000]$, and in Figure 3 we present the velocity vector field, streamlines and components for the highest values of Ra. It is observed that the flow substantially changes as a result of the convective effects when Ra increases. In particular, the fluid rises along the hot side and comes down along the cold wall, a secondary flow arises at a Rayleigh number between $10^4$ and $10^5$, and boundary layers appears near the vertical walls due to the isothermal deformation.

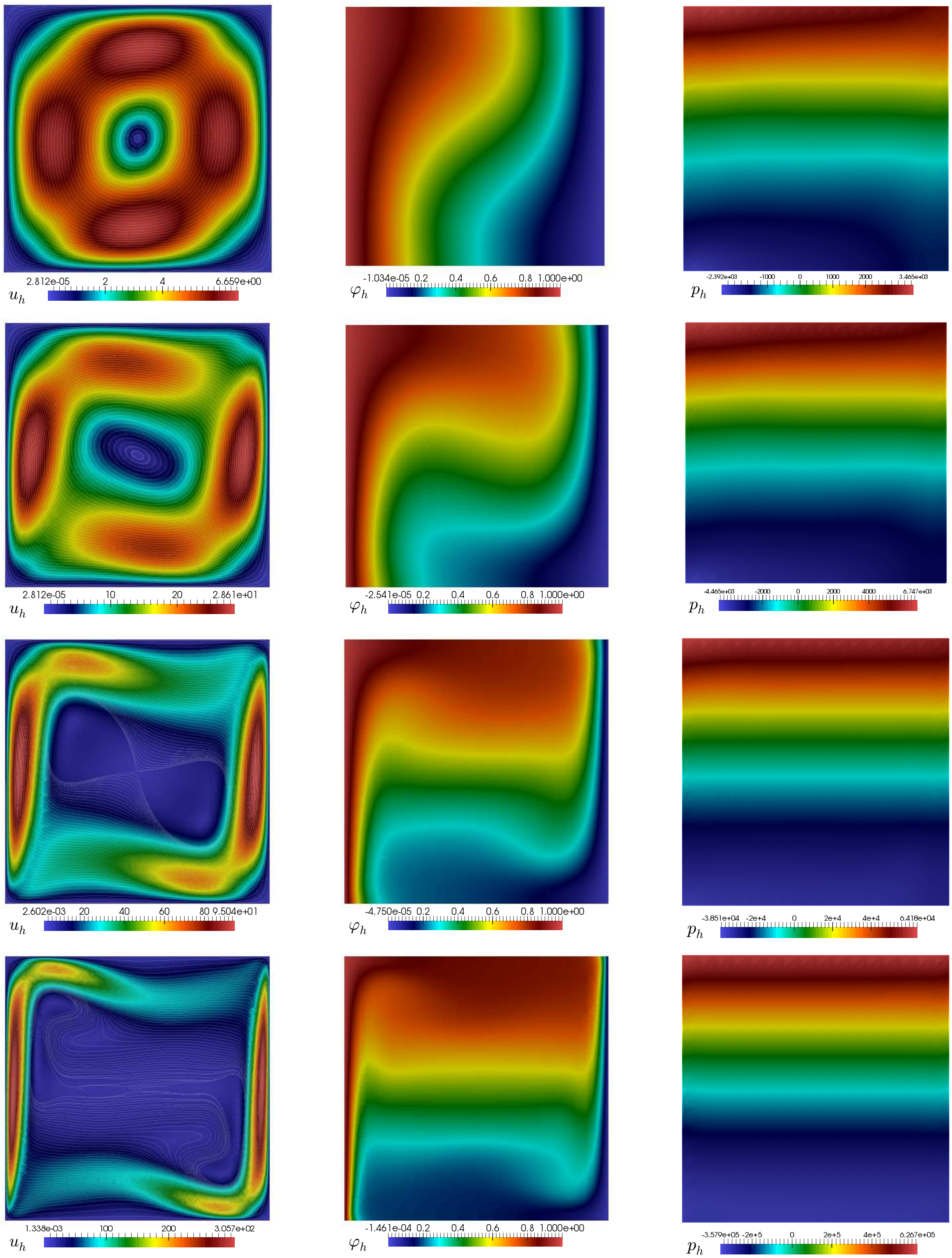

Figure 2: Example 2: Velocity streamlines (left), temperature (center) and pressure (right) profiles of the natural convection problem with $\mathrm{Ra} = 100 \times 10^n$ (n-th row).

| $h$ | $e(G)$ | $r(G)$ | $e(\boldsymbol{u})$ | $r(\boldsymbol{u})$ | $e(S)$ | $r(S)$ | $e(p)$ | $r(p)$ | $e(\varphi)$ | $r(\varphi)$ | $e(\lambda)$ | $r(\lambda)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Dual-mixed scheme | | | | | | | | | | | | |
| 0.5000 | 4.8642 | – | 0.3243 | – | 12.872 | – | 2.4383 | – | 0.0973 | – | – | – |
| 0.2500 | 1.9831 | 1.2944 | 0.1134 | 1.5166 | 4.9201 | 1.3875 | 0.8999 | 1.4380 | 0.0357 | 1.4449 | – | – |
| 0.1250 | 0.7171 | 1.4675 | 0.0342 | 1.7294 | 1.6859 | 1.5452 | 0.3210 | 1.4870 | 0.0112 | 1.6762 | – | – |
| 0.0625 | 0.2173 | 1.7224 | 0.0094 | 1.8585 | 0.4934 | 1.7726 | 0.0965 | 1.7337 | 0.0032 | 1.8263 | – | – |
| 0.03125 | 0.0598 | 1.8625 | 0.0025 | 1.9232 | 0.1328 | 1.8938 | 0.0264 | 1.8685 | 0.0008 | 1.9109 | – | – |
| Scheme with Lagrange multiplier | | | | | | | | | | | | |
| 0.5000 | 2.6116 | – | 0.6632 | – | 44.1841 | – | 2.1437 | – | 0.3007 | – | 0.7252 | – |
| 0.2500 | 1.8680 | 0.4834 | 0.1325 | 2.3239 | 9.0464 | 2.2881 | 1.1550 | 0.8921 | 0.0577 | 2.3818 | 0.1093 | 2.7304 |
| 0.1250 | 0.5302 | 1.8168 | 0.0326 | 2.0238 | 2.3195 | 1.9635 | 0.3414 | 1.7583 | 0.0139 | 2.0504 | 0.0279 | 1.9708 |
| 0.0833 | 0.2406 | 1.9487 | 0.0144 | 2.0152 | 1.0367 | 1.9862 | 0.1575 | 1.9085 | 0.0061 | 2.0288 | 0.0127 | 1.9341 |
| 0.0625 | 0.1363 | 1.9752 | 0.0081 | 2.0084 | 0.5843 | 1.9929 | 0.0898 | 1.9509 | 0.0034 | 2.0066 | 0.0073 | 1.9823 |
| 0.0417 | 0.0609 | 1.9866 | 0.0036 | 2.0037 | 0.2601 | 1.9955 | 0.0404 | 1.9732 | 0.0015 | 2.0057 | 0.0033 | 1.9934 |

Table 1: Example 1: mesh sizes, errors and rates of convergence for the dual-mixed approximations of the Boussinesq equations.

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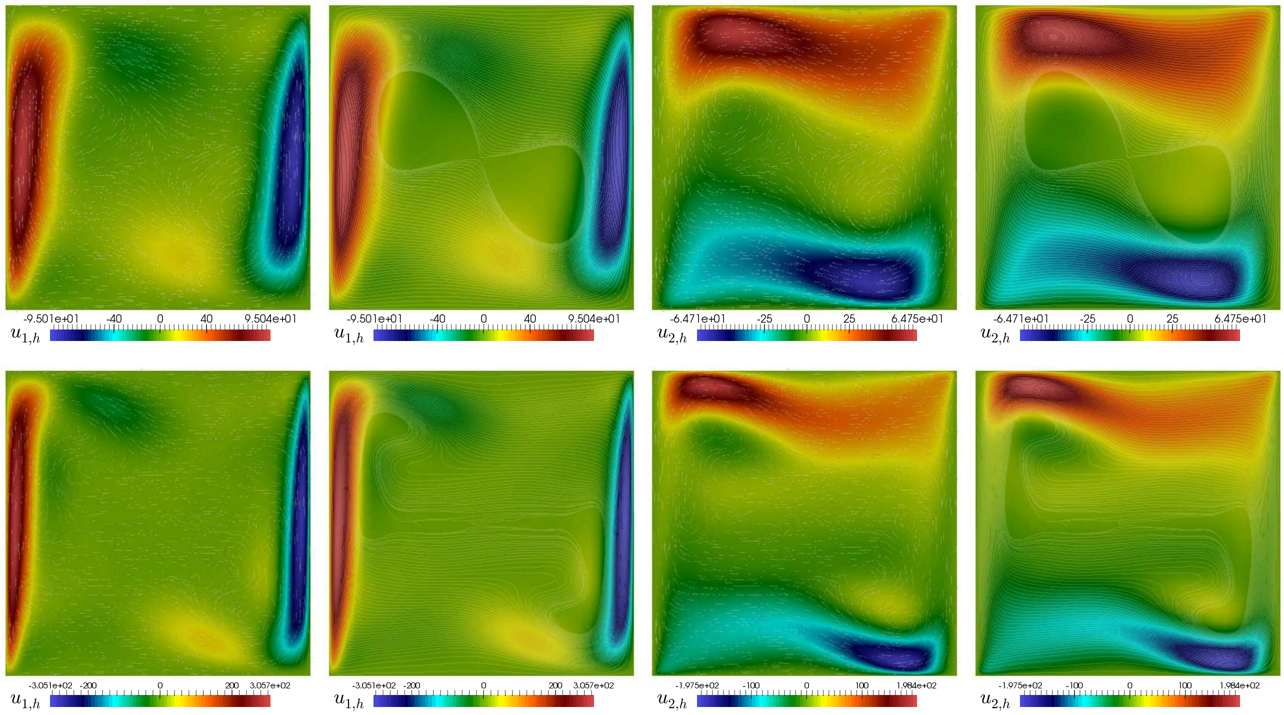


Figure 3: Example 2: Velocity vector field, streamlines and components for $\mathrm{Ra} = 10^5$ and $\mathrm{Ra} = 10^6$ (top and bottom, respectively).


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# A Proof of Lemma 3.4

From the definition of $\mathbf{c}(\cdot,\cdot,\cdot)$, Hölder's inequality, Sobolev embeddings and the Poincaré and Cauchy-Schwarz inequalities we have that

$$\begin{aligned}\mathcal{F}_{1,(G,\boldsymbol{u},\varphi_0)}((H,\boldsymbol{v},\psi)) \,&=\, \frac{1}{2}\big[\,(G,\boldsymbol{v}\otimes\boldsymbol{u}) \,-\, (\boldsymbol{u}\otimes\boldsymbol{u},H)\,\big] \,+\, (\boldsymbol{u}\cdot\nabla\varphi_0,\psi)\\ &\leq\, \|\boldsymbol{u}\|_{0,3,\Omega}\Big(\,\|G\|_{0,\Omega}\,\|\boldsymbol{v}\|_{0,6,\Omega} \,+\, \|\boldsymbol{u}\|_{0,6,\Omega}\,\|H\|_{0,\Omega} \,+\, \|\nabla\varphi_0\|_{0,\Omega}\,\|\psi\|_{0,6,\Omega}\,\Big)\\ &\leq\, C\,\|\boldsymbol{u}\|_{0,3,\Omega}\,\|(G,\boldsymbol{u},\varphi_0)\|\,\|(H,\boldsymbol{v},\psi)\|\,.\end{aligned}$$

Similarly, we find that

$$\begin{aligned}\mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)}((H,\boldsymbol{v},\psi)) \,&\leq\, C\,\Big(\,\|\boldsymbol{u}\|_{0,3,\Omega}\,\|\nabla\varphi_1\|_{0,\Omega}\,\|\psi\|_{0,6,\Omega} \,+\, \|\boldsymbol{g}\|_{0,\Omega}\,\|\varphi_0\|_{0,4,\Omega}\,\|\boldsymbol{v}\|_{0,4,\Omega}\Big)\,,\\ &\leq C\max\{\|\boldsymbol{g}\|_{0,\Omega},\|\varphi_1\|_{1,\Omega}\}\,\|(G,\boldsymbol{u},\varphi_0)\|\|(H,\boldsymbol{v},\psi)\|\,,\end{aligned}$$

then applying Lemma 3.2 to bound the $\mathrm{H}^1$-norm of the extension $\varphi_1$ with $\delta$ given by (3.17), and defining $C_4(\varphi_{\mathrm{D}},\boldsymbol{g}) := C\max\{\|\boldsymbol{g}\|_{0,\Omega},\nu^{-4}\kappa^{-4}\|\varphi_{\mathrm{D}}\|^5_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{g}\|^4_{0,\Omega}\}$, we get

$$\big|\mathcal{F}_{2,(G,\boldsymbol{u},\varphi_0)}((H,\boldsymbol{v},\psi))\big| \,\leq\, C_4(\varphi_{\mathrm{D}},\boldsymbol{g})\,\|(G,\boldsymbol{u},\varphi_0)\|\,\|(H,\boldsymbol{v},\psi)\|\,.$$

Likewise, with $C_5(\varphi_{\mathrm{D}},\boldsymbol{g}) := C\nu^{-4}\kappa^{-4}\|\varphi_{\mathrm{D}}\|^5_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{g}\|^4_{0,\Omega}\big(\,\kappa \,+\, \|\boldsymbol{g}\|_{0,\Omega}\,\big)$, we observe that

$$\big|\mathcal{F}_3((H,\boldsymbol{v},\psi))\big| \leq C\,\Big(\,\|\boldsymbol{g}\|_{0,\Omega}\,\|\varphi_1\|_{1,\Omega}\|\boldsymbol{v}\|_{0,6,\Omega} \,+\, \kappa\,\|\nabla\varphi_1\|_{0,\Omega}\,\|\nabla\psi\|_{0,\Omega}\,\Big) \,\leq\, C_5(\varphi_{\mathrm{D}},\boldsymbol{g})\,\|\,(H,\boldsymbol{v},\psi)\,\|\,.$$

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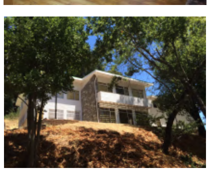

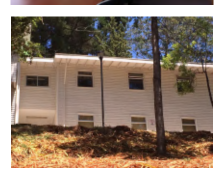


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