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Abstract

In this paper we introduce and analyze a hybridizable discontinuous Galerkin (HDG) method for numerically solving the coupling of fluid flow with porous media flow. Flows are governed by the Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. We consider a fully-mixed formulation in which the main unknowns in the fluid are given by the stress, the vorticity, the velocity, and the trace of the velocity, whereas the velocity, the pressure, and the trace of the pressure are the unknowns in the porous medium. In addition, a suitable enrichment of the finite dimensional subspace for the stress yields optimally convergent approximations for all unknowns, as well as a superconvergent approximation of the trace variables. To do that, similarly as in previous papers dealing with development of the *a priori* error estimates, we use the projection-based error analysis in order to simplify the corresponding study. Finally, we provide several numerical results illustrating the good performance of the proposed scheme and confirming the optimal order of convergence provided by the HDG approximation.

Key words: coupling, Stokes equations, Darcy equations, mixed finite element method, hybridized discontinuous Galerkin method

1 Introduction

The derivation of suitable numerical methods for the coupling of fluid flow with porous media flow, modelled by the Stokes and Darcy equations, has been increasing during recent years (see e.g., [2, 5, 9, 18, 19, 20, 21, 22, 25, 26, 28, 32, 33, 35, 36, 37, 38], and the references therein). The above list includes different kind of problems. In particular, porous media with cracks, and the incorporation of other linear and nonlinear equations in the coupled problem, such as Brinkman and Forchheimer. This model has applications in different areas of interest, such as chemical and petroleum engineering, hydrology, and environmental sciences, to name a few. That is the reason why it has gained relevance through the last decades, and the cause of the numerical analysis community has been putting so much effort in developing more accurate and efficient methods for solving this problem. Now, with respect to the historical perspective, we recall here that the first fully-mixed finite element method for the 2D Stokes-Darcy coupled problem has been introduced and analyzed recently in [27]. This approach allows the introduction of further unknowns of physical interest as well as the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term.

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Moreover, the fact that dual-mixed formulations are considered in both domains yields as the main unknowns the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium. The pressure and the gradient of the velocity in the fluid can then be computed through a very simple post-process of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, due to the fully-mixed approach utilized, the transmission conditions become essential, and hence they have to be imposed weakly, which leads to the incorporation of two additional unknowns to the system, namely the traces of the Darcy pressure and the Stokes velocity on the coupling interface Σ. These new unknowns are also variables of importance from a physical point of view. Then, in order to prove the unique solvability of the resulting continuous formulation, that the well known Fredholm and Babuška-Brezzi theories are applied, which also contribute to derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well-posed. Among the several different ways in which the equations and unknowns can be ordered, the one yielding a doubly mixed structure is chosen, for which the inf-sup conditions of the off-diagonal bilinear forms follow straightforwardly. Moreover, the arguments of the continuous analysis can be easily adapted to the discrete case. In particular, a feasible choice of subspaces is given by Raviart-Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the additional unknowns on the interface.

On the other hand, the hybridizable discontinuous Galerkin (HDG) method, introduced in [11] for diffusion problems, is one of the several high-order discretization schemes that benefit from the hybridization technique originally applied in [17] to the local discontinuous Galerkin (LDG) method for time dependent convection-diffusion problems. The main advantages of HDG methods include a substantial reduction of the globally coupled degrees of freedom (which was a criticism for the discontinuous Galerkin (DG) methods for elliptic problems during the last decade), and the fact that convergence is obtained even for the polynomial degree $k = 0$. Additionally, the approximate flux converges with order $k+1$ for $k \geq 0$, and an element-by-element computation of a new approximation of the scalar variable is possible, which converges with order $k+2$ for $k \geq 1$ (see e.g. [10, 14, 12]). Nevertheless, and up to our knowledge, there is still no contribution in the literature concerning HDG for fully-mixed Stokes-Darcy systems.

According to the above discussion, we are interested in this paper in applying the HDG approach to the coupled Stokes and Darcy flows problem studied in [27]. To this end, we plan to employ the same techniques given in the context of HDG schemes for the Stokes and Darcy uncoupled equations. More precisely, for Stokes problem, the hybridization for DG methods was initially introduced in [6] and then analyzed in [34, 12]. Lately, an overview of the recent work by Cockburn and co-workers on the devising of HDG methods for the Stokes equations of incompressible flow was provided in [16]. For the Darcy law, we are particularly interested in [13], where it was introduced the new projection-based technique for the study of the *a priori* error analysis of hybridizable discontinuous Galerkin methods. In addition, we follow [15], to derive a way to deal with the weak stress symmetry, and then show the optimal rate of convergence for all unknowns. The rest of this paper is organized as follows. In Section 2 we present the main aspects of the continuous problem, which includes the geometry and the coupled model. Then, in Section 3 we introduce the hybridizable discontinuous Galerkin formulation for the coupled problem. More precisely, we present the finite dimensional discontinuous subspaces and we show the unique solvability of HDG scheme. The corresponding *a priori* error estimates are derived in Section 4. In particular, we use projections whose design are inspired by the form of the numerical traces of the method, which is an innovative technique applied for the error analysis of HDG approximations. Finally, several numerical experiments validating the good performance of the method and confirming the rates of convergence derived are reported in Section 5.

We end the present section with further notations to be used below. Given a non-null space H, we set $\mathbf{H} := H^n$ and $\mathbb{H} := H^{n\times n}$. Also, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in R^{n\times n}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji}), \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}.$$

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, null tensor or null operator, and use C, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The coupled problem

In order to describe the geometry of the problem, we let Ω_S and Ω_D be bounded and simply connected polyhedral domains in R^n, $n \in \{2,3\}$, such that $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \emptyset$. Then, we let $\Gamma_S := \partial\Omega_S \setminus \bar{\Sigma}$, $\Gamma_D := \partial\Omega_D \setminus \bar{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_S \cup \Sigma \cup \Omega_D$ (and hence inward to Ω_D when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Figure 2.1 below) and by $\{\mathbf{t}_1, \mathbf{t}_2\}$, when $n = 3$.

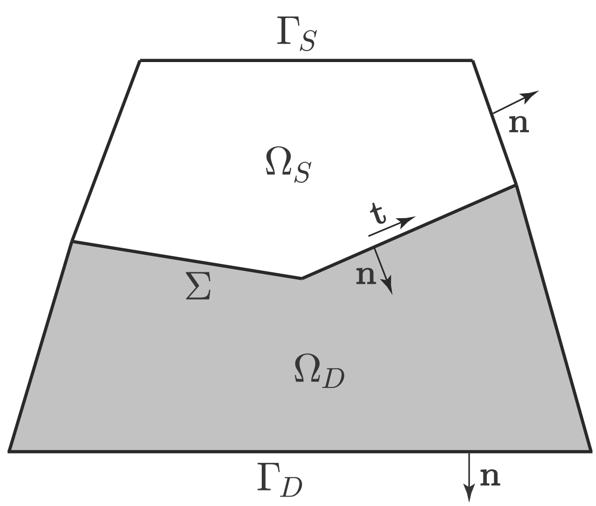

Figure 2.1: Sketch of the 2D geometry for the Stokes-Darcy coupling.

The model consists of two separate groups of equations and a set of coupling terms. In Ω_S, the governing equations are those of the Stokes problem, which are written in the following stress-velocity-pressure formulation:

$$\begin{aligned} &\boldsymbol{\sigma}_S = \nu\,\mathbf{e}(\mathbf{u}_S) - p_S\,\mathbf{I} \quad \text{in} \quad \Omega_S, \qquad \mathbf{div}(\boldsymbol{\sigma}_S) + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S, \\ &\mathrm{div}(\mathbf{u}_S) = 0 \quad \text{in} \quad \Omega_S, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S, \qquad \int_{\Omega_S} p_S = 0, \end{aligned} \tag{2.1}$$

where $\nu > 0$ is the viscosity of the fluid, $\boldsymbol{\sigma}_S$ is the stress tensor, $\mathbf{u}_S$ is the fluid velocity, p_S is the pressure, $\mathbf{I}$ is the $n\times n$ identity matrix, $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ is a known source term, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and

$$\mathbf{e}(\mathbf{u}_S) := \frac{1}{2}\left(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathtt{t}}\right)$$

is the strain tensor (or symmetric part of the velocity gradient). Now, introducing the vorticity (or skew-symmetric part of the velocity gradient) $\boldsymbol{\rho}_S := \frac{1}{2}(\nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^{\mathtt{t}})$ as a further unknown, and using that $\mathrm{tr}(\nabla \mathbf{u}_S) = \mathrm{div}(\mathbf{u}_S) = 0$ in Ω_S, and the relation $\nabla \mathbf{u}_S - \boldsymbol{\rho}_S = \mathbf{e}(\mathbf{u}_S)$ in Ω_S, we observe that the equations in (2.1) can be rewritten equivalently as

$$
\begin{gathered}
\frac{1}{\nu}\boldsymbol{\sigma}_S^{\mathtt{d}} = \nabla \mathbf{u}_S - \boldsymbol{\rho}_S \quad \text{in} \quad \Omega_S\,, \qquad \mathbf{div}(\boldsymbol{\sigma}_S) + \mathbf{f}_S = \mathbf{0} \quad \text{in} \quad \Omega_S\,,\\
\boldsymbol{\sigma}_S = \boldsymbol{\sigma}_S^{\mathtt{t}} \quad \text{in} \quad \Omega_S\,, \qquad p_S = -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}_S) \quad \text{in} \quad \Omega_S\,,\\
\mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S\,, \qquad \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\sigma}_S) = 0\,.
\end{gathered}
\tag{2.2}
$$

In turn, in Ω_D we consider the following Darcy model:

$$
\begin{gathered}
\mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in} \quad \Omega_D\,, \qquad \mathrm{div}(\mathbf{u}_D) = f_D \quad \text{in} \quad \Omega_D\,,\\
\mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\,,
\end{gathered}
\tag{2.3}
$$

where $\mathbf{u}_D$ and p_D denote the velocity and pressure, respectively, and the source term $f_D \in L^2(\Omega_D)$ is such that $\int_{\Omega_D} f_D = 0$. The tensor valued function $\mathbf{K}$, which describes the permeability of Ω_D divided by the viscosity ν, satisfies $\mathbf{K}^{\mathtt{t}} = \mathbf{K}$, and has $L^\infty(\Omega_D)$ components. Also, we assume that there exists $\alpha_{\mathbf{K}} > 0$ such that

$$
\mathbf{w} \cdot \mathbf{K}(\mathbf{x})\mathbf{w} \;\geq\; \alpha_{\mathbf{K}} \|\mathbf{w}\|^2_{R^n}\,,
$$

for almost all $\mathbf{x} \in \Omega_D$, and for all $\mathbf{w} \in R^n$. Finally, the transmission conditions on Σ are given by

$$
\begin{aligned}
\mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad \text{on} \quad \Sigma\,,\\
\boldsymbol{\sigma}_S \mathbf{n} + \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}(\mathbf{u}_S \cdot \mathbf{t}_\ell)\mathbf{t}_\ell &= -p_D \mathbf{n} \quad \text{on} \quad \Sigma\,,
\end{aligned}
\tag{2.4}
$$

where $\{\kappa_1, \ldots, \kappa_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.4) is the conservation of mass, and the second one establishes the balance of normal forces and the Beavers-Joseph-Saffman law.

3 The HDG method

3.1 Notation

We begin by introducing some preliminary notations. Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D without the presence of hanging nodes, which are formed by shape-regular n-simplex of diameter h_T and assume that they match in Σ so that $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. In addition, let $\mathcal{E}_h^*$ be the set of faces F of $\mathcal{T}_h^*$, and $\partial\mathcal{T}_h^* := \cup\{\partial T \,:\, T \in \mathcal{T}_h^*\}$ $\forall\, * \in \{S, D\}$. Next, let $(\cdot,\cdot)_U$ denote the usual L^2, $\mathbf{L}^2$ and $\mathbb{L}^2$ inner product over the domain $U \subseteq R^n$, and similarly let $\langle\cdot,\cdot\rangle_G$ be the L^2 and $\mathbf{L}^2$ inner product over the surface $G \subseteq R^{n-1}$. Then, for each $* \in \{S, D\}$ we introduce the inner products:

$$
(\cdot,\cdot)_{\mathcal{T}_h^*} := \sum_{T\in\mathcal{T}_h^*} (\cdot,\cdot)_T\,, \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h^*} := \sum_{T\in\mathcal{T}_h^*} \langle\cdot,\cdot\rangle_{\partial T}\,, \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h^*\setminus\Sigma} := \sum_{T\in\mathcal{T}_h^*} \sum_{F\in\partial T\setminus\Sigma} \langle\cdot,\cdot\rangle_F\,.
$$

Furthermore, given $r \geq 0$ and $* \in \{S, D\}$, we define

$$
H^r(\mathcal{T}_h^*) := \left\{ v \in L^2(\Omega_*) \,:\, v|_T \in H^r(T) \quad \forall\, T \in \mathcal{T}_h^* \right\}\,,
$$

whence $\mathbf{H}^r(\mathcal{T}_h^*)$ and $\mathbb{H}^r(\mathcal{T}_h^*)$ denote the vectorial and tensorial versions of $H^r(\mathcal{T}_h^*)$, respectively.

On the other hand, let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outward unit normal vectors on the boundaries of two neighboring elements T^+ and T^-, respectively. We use $\boldsymbol{\tau}^\pm$ to denote the traces of $\boldsymbol{\tau}$ on $F := \partial T^+ \cap \partial T^-$ from the interior of $T^\pm$, where $\boldsymbol{\tau}$ is a second-order tensorial function. Then, we define the jumps $[\![\cdot]\!]$ of tensor variables on each interior face as follows

$$[\![\boldsymbol{\tau}]\!] \;:=\; \boldsymbol{\tau}^+\mathbf{n}^+ \;+\; \boldsymbol{\tau}^-\mathbf{n}^- \,.$$

3.2 Subspaces

Next, given $k \geq 1$, and U a domain or either a closed or open Lipschitz curve if $n = 2$ (resp. surface if $n = 3$), we let $\mathrm{P}_k(U)$ be the space of polynomials of total degree at most k defined on U. In addition, given $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$, we let

$$\mathbb{A}(T) \;:=\; \mathbb{P}_k(T) \cap \mathbb{L}^2_{\text{skew}}(T)\,,$$

and (see [31, 15])

$$\mathbb{B}(T) \;:=\; \operatorname{curl}\left(\operatorname{curl}\left(\mathbb{A}(T)\right) b_T\right)\,,$$

where $\mathbb{L}^2_{\text{skew}}(T) := \{\boldsymbol{\eta} \in \mathbb{L}^2(T) \,:\, \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\}$ is the subspace of skew-symmetric tensors of $\mathbb{L}^2(T)$, and b_T is the scalar bubble function in $\mathrm{P}_{n+1}(T)$. Furthermore, in three space dimensions the ith row of $\operatorname{curl}(\boldsymbol{\tau})$ is nothing but $\operatorname{curl}(\cdot)$ applied to the ith row of $\boldsymbol{\tau}$. In the two-dimensional case, given vector and tensor valued fields $\mathbf{v} := (v_1, v_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\operatorname{curl}(\mathbf{v}) \;:=\; \begin{pmatrix} \frac{\partial v_1}{\partial x_2} & -\frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} & -\frac{\partial v_2}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \operatorname{curl}(\boldsymbol{\tau}) \;:=\; \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, the finite dimensional discontinuous subspaces are given by

$$\begin{aligned}
\mathbb{S}_h &:= \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_S) \,:\, \boldsymbol{\tau}|_T \in \mathbb{P}_k(T) + \mathbb{B}(T) \quad \forall\, T \in \mathcal{T}_h^S\right\}\,, \\
\mathbb{A}_h &:= \left\{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega_S) \,:\, \boldsymbol{\eta}|_T \in \mathbb{A}(T) \quad \forall\, T \in \mathcal{T}_h^S\right\}\,, \\
\mathbf{V}_h^* &:= \left\{\mathbf{v} \in \mathbf{L}^2(\Omega_*) \,:\, \mathbf{v}|_T \in \mathbf{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^*\right\} \qquad \forall\, * \in \{S, D\}\,, \\
\mathbf{M}_h &:= \left\{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h^S) \,:\, \boldsymbol{\mu}|_F \in \mathbf{P}_k(F) \quad \forall\, F \in \mathcal{E}_h^S \quad \text{and} \quad \boldsymbol{\mu}|_{\Gamma_S} \;=\; \mathbf{0}\right\}\,, \\
P_h &:= \left\{q \in L^2(\Omega_D) \,:\, q|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^D\right\}\,, \\
N_h &:= \left\{\psi \in L^2(\mathcal{E}_h^D) \,:\, \psi|_F \in \mathrm{P}_k(F) \quad \forall\, F \in \mathcal{E}_h^D\right\}\,.
\end{aligned}$$

The purpose of enriching here the space $\mathbb{S}_h$ with $\mathbb{B}(T)$ will become clear in the *a priori* error analysis given below in Section 4. Note that if $\boldsymbol{\tau}_{S,h} \in \mathbb{B}_h := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_S) \,:\, \boldsymbol{\tau}|_T \in \mathbb{B}(T) \quad \forall\, T \in \mathcal{T}_h^S\}$, we have that

$$\mathbf{div}(\boldsymbol{\tau}_{S,h}) \;=\; \mathbf{0} \quad \text{in} \quad \Omega_S \quad \text{and} \quad \boldsymbol{\tau}_{S,h}\mathbf{n} \;=\; \mathbf{0} \quad \text{in} \quad \mathcal{E}_h^S\,. \tag{3.1}$$

Finally, for convenience of further analysis, we define the subspace

$$\widetilde{\mathbb{S}}_h \;:=\; \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_S) \,:\, \boldsymbol{\tau}|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^S\}$$

and notice that

$$\mathbb{S}_h \;=\; \widetilde{\mathbb{S}}_h \;+\; \mathbb{B}_h\,.$$

3.3 Formulation

Proceeding as in [13, 12, 15], we deduce that the HDG formulation of the coupled system (2.2)-(2.3)-(2.4) reduces to: Find $(\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{S,h}, \boldsymbol{\rho}_{S,h}, \boldsymbol{\lambda}_{S,h}, \mathbf{u}_{D,h}, p_{D,h}, \varphi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times P_h \times N_h$, such that

$$
\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}^{\mathtt{d}}_{S,h}, \boldsymbol{\tau}^{\mathtt{d}}_{S,h})_{\mathcal{T}_h^S} + (\mathbf{u}_{S,h}, \mathbf{div}(\boldsymbol{\tau}_{S,h}))_{\mathcal{T}_h^S} - \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \widehat{\mathbf{u}}_{S,h}\rangle_{\partial\mathcal{T}_h^S} & \\
+ (\boldsymbol{\rho}_{S,h}, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} &= 0\,, && \text{(3.2a)}\\
(\boldsymbol{\sigma}_{S,h}, \nabla \mathbf{v}_{S,h})_{\mathcal{T}_h^S} - \langle \widehat{\boldsymbol{\sigma}_{S,h}\mathbf{n}}, \mathbf{v}_{S,h}\rangle_{\partial\mathcal{T}_h^S} &= (\mathbf{f}_S, \mathbf{v}_{S,h})_{\mathcal{T}_h^S}\,, && \text{(3.2b)}\\
(\boldsymbol{\eta}_{S,h}, \boldsymbol{\sigma}_{S,h})_{\mathcal{T}_h^S} &= 0\,, && \text{(3.2c)}\\
\langle \widehat{\boldsymbol{\sigma}_{S,h}\mathbf{n}}, \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S\setminus\Sigma} &= 0\,, && \text{(3.2d)}\\
(\mathbf{K}^{-1}\mathbf{u}_{D,h}, \mathbf{v}_{D,h})_{\mathcal{T}_h^D} - (p_{D,h}, \mathrm{div}(\mathbf{v}_{D,h}))_{\mathcal{T}_h^D} + \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \widehat{p}_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} & \\
- \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \widehat{p}_{D,h}\rangle_{\Sigma} &= 0\,, && \text{(3.2e)}\\
-(\mathbf{u}_{D,h}, \nabla q_{D,h})_{\mathcal{T}_h^D} + \langle \widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}, q_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}, q_{D,h}\rangle_{\Sigma} &= (f_D, q_{D,h})_{\mathcal{T}_h^D}\,, && \text{(3.2f)}\\
\langle \widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}, \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} &= 0\,, && \text{(3.2g)}\\
\langle \widehat{\mathbf{u}}_{S,h}\cdot\mathbf{n} - \widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}, \psi_{D,h}\rangle_{\Sigma} &= 0\,, && \text{(3.2h)}\\
\left\langle \widehat{\boldsymbol{\sigma}_{S,h}\mathbf{n}} \;+\; \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}(\widehat{\mathbf{u}}_{S,h}\cdot\mathbf{t}_\ell)\mathbf{t}_\ell \;+\; \widehat{p}_{D,h}\mathbf{n},\; \boldsymbol{\mu}_{S,h}\right\rangle_{\Sigma} &= 0\,, && \text{(3.2i)}\\
(\mathrm{tr}\,(\boldsymbol{\sigma}_{S,h})\,, 1)_{\Omega_S} &= 0\,, && \text{(3.2j)}
\end{aligned}
$$

for all $(\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}, \boldsymbol{\mu}_{S,h}, \mathbf{v}_{D,h}, q_{D,h}, \psi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times P_h \times N_h$, with the numerical fluxes $\widehat{\mathbf{u}}_{S,h}$, $\widehat{\boldsymbol{\sigma}_{S,h}\mathbf{n}}$, $\widehat{p}_{D,h}$ and $\widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}$ given by

$$
\widehat{\mathbf{u}}_{S,h} \;=\; \boldsymbol{\lambda}_{S,h} \quad \text{in} \quad \mathcal{E}_h^S\,, \qquad \widehat{\boldsymbol{\sigma}_{S,h}\mathbf{n}} \;=\; \boldsymbol{\sigma}_{S,h}\mathbf{n} - \mathbf{S}(\mathbf{u}_{S,h} - \widehat{\mathbf{u}}_{S,h}) \quad \text{in} \quad \partial\mathcal{T}_h^S\,,
$$

$$
\widehat{p}_{D,h} \;=\; \varphi_{D,h} \quad \text{in} \quad \mathcal{E}_h^D\,, \qquad \text{and} \qquad \widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}} \;=\; \begin{cases} \mathbf{u}_{D,h}\cdot\mathbf{n} + \tau(p_{D,h} - \widehat{p}_{D,h}) & \text{on} \quad \partial\mathcal{T}_h^D\setminus\Sigma\,,\\ \mathbf{u}_{D,h}\cdot\mathbf{n} - \tau(p_{D,h} - \widehat{p}_{D,h}) & \text{on} \quad \Sigma\,, \end{cases}
$$

where $\mathbf{S}$ is an stabilization tensor to be defined below, and $\tau > 0$ is a constant function in $\mathcal{E}_h^D$.

From (3.2) we observe that equations (3.2a)-(3.2b)-(3.2c) and (3.2j) arise from the application of the HDG approximation to the Stokes system (2.2), and similarly (3.2e) and (3.2f) arise from the Darcy system (2.3). In addition, expressions (3.2d) and (3.2g) are the weak imposition of the continuity of the normal component of the fluxes, as it is natural in HDG schemes. In particular, note that the Neumann condition $\mathbf{u}_D\cdot\mathbf{n} = 0$ on Γ_D is considered in (3.2g). Finally, equations (3.2h) and (3.2i) constitute the HDG setting of the transmission conditions (2.4) on Σ.

On the other hand, the definition of $\widehat{\mathbf{u}_{D,h}\cdot\mathbf{n}}$ is consistent with that given in [13], that is $\widehat{\mathbf{u}}_{D,h} := \mathbf{u}_{D,h} + \tau(p_{D,h} - \widehat{p}_{D,h})\mathbf{n}$ on $\partial\mathcal{T}_h^D$, for some non-negative penalty function τ defined on $\partial\mathcal{T}_h^D$, which we assume to be constant on each face of the triangulation. To this respect, note first that problem (3.2) can be reformulated as: Find $(\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{S,h}, \boldsymbol{\rho}_{S,h}, \boldsymbol{\lambda}_{S,h}, \mathbf{u}_{D,h}, p_{D,h}, \varphi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times$

$P_h \times N_h$, such that

$$
\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}^{\mathtt{d}}_{S,h}, \boldsymbol{\tau}^{\mathtt{d}}_{S,h})_{\mathcal{T}_h^S} + (\mathbf{u}_{S,h}, \mathbf{div}(\boldsymbol{\tau}_{S,h}))_{\mathcal{T}_h^S} - \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\lambda}_{S,h}\rangle_{\partial\mathcal{T}_h^S} \qquad & \\
+ (\boldsymbol{\rho}_{S,h}, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} \;&=\; 0\,, && (3.3a)\\
-(\mathbf{v}_{S,h}, \mathbf{div}(\boldsymbol{\sigma}_{S,h}))_{\mathcal{T}_h^S} + \langle \mathbf{S}(\mathbf{u}_{S,h} - \boldsymbol{\lambda}_{S,h}), \mathbf{v}_{S,h}\rangle_{\partial\mathcal{T}_h^S} \;&=\; (\mathbf{f}_S, \mathbf{v}_{S,h})_{\mathcal{T}_h^S}\,, && (3.3b)\\
(\boldsymbol{\eta}_{S,h}, \boldsymbol{\sigma}_{S,h})_{\mathcal{T}_h^S} \;&=\; 0\,, && (3.3c)\\
\langle \boldsymbol{\sigma}_{S,h}\mathbf{n}, \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S\setminus\Sigma} - \langle \mathbf{S}(\mathbf{u}_{S,h} - \boldsymbol{\lambda}_{S,h}), \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S\setminus\Sigma} \;&=\; 0\,, && (3.3d)\\
(\mathbf{K}^{-1}\mathbf{u}_{D,h}, \mathbf{v}_{D,h})_{\mathcal{T}_h^D} - (p_{D,h}, \mathrm{div}(\mathbf{v}_{D,h}))_{\mathcal{T}_h^D} + \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} \qquad & \\
- \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_{\Sigma} \;&=\; 0\,, && (3.3e)\\
(q_{D,h}, \mathrm{div}(\mathbf{u}_{D,h}))_{\mathcal{T}_h^D} + \langle \tau(p_{D,h} - \varphi_{D,h}), q_{D,h}\rangle_{\partial\mathcal{T}_h^D} \;&=\; (f_D, q_{D,h})_{\mathcal{T}_h^D}\,, && (3.3f)\\
\langle \mathbf{u}_{D,h}\cdot\mathbf{n}, \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} + \langle \tau(p_{D,h} - \varphi_{D,h}), \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} \;&=\; 0\,, && (3.3g)\\
\langle \boldsymbol{\lambda}_{S,h}\cdot\mathbf{n}, \psi_{D,h}\rangle_{\Sigma} - \langle \mathbf{u}_{D,h}\cdot\mathbf{n}, \psi_{D,h}\rangle_{\Sigma} + \langle \tau(p_{D,h} - \varphi_{D,h}), \psi_{D,h}\rangle_{\Sigma} \;&=\; 0\,, && (3.3h)\\
\langle \boldsymbol{\sigma}_{S,h}\mathbf{n}, \boldsymbol{\mu}_{S,h}\rangle_{\Sigma} - \langle \mathbf{S}(\mathbf{u}_{S,h} - \boldsymbol{\lambda}_{S,h}), \boldsymbol{\mu}_{S,h}\rangle_{\Sigma} + \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}\langle \boldsymbol{\lambda}_{S,h}\cdot\mathbf{t}_\ell, \boldsymbol{\mu}_{S,h}\cdot\mathbf{t}_\ell\rangle_{\Sigma} \qquad & \\
+ \langle \boldsymbol{\mu}_{S,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_{\Sigma} \;&=\; 0\,, && (3.3i)\\
(\mathrm{tr}\,(\boldsymbol{\sigma}_{S,h})\,, 1)_{\Omega_S} \;&=\; 0\,, && (3.3j)
\end{aligned}
$$

for all $(\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}, \boldsymbol{\mu}_{S,h}, \mathbf{v}_{D,h}, q_{D,h}, \psi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times P_h \times N_h$, where (3.2b) and (3.2f) has been rewritten, respectively, using that

$$
(\boldsymbol{\sigma}_{S,h}, \nabla\mathbf{v}_{S,h})_{\mathcal{T}_h^S} \;=\; -(\mathbf{v}_{S,h}, \mathbf{div}(\boldsymbol{\sigma}_{S,h}))_{\mathcal{T}_h^S} \;+\; \langle \boldsymbol{\sigma}_{S,h}\mathbf{n}, \mathbf{v}_{S,h}\rangle_{\partial\mathcal{T}_h^S}
$$

and

$$
-(\mathbf{u}_{D,h}, \nabla q_{D,h})_{\mathcal{T}_h^D} \;=\; (q_{D,h}, \mathrm{div}(\mathbf{u}_{D,h}))_{\mathcal{T}_h^D} \;-\; \langle \mathbf{u}_{D,h}\cdot\mathbf{n}, q_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} \;+\; \langle \mathbf{u}_{D,h}\cdot\mathbf{n}, q_{D,h}\rangle_{\Sigma}\,.
$$

We complete the definition of the HDG method by describing the stabilization tensor $\mathbf{S}$. We first recall that general conditions for $\mathbf{S}$ were proposed in [12]. In particular, given $F \in \mathcal{E}_h^S$, we assume that $\mathbf{S}|_F$ is a symmetric and positive definite constant tensor.

3.4 Solvability analysis

The following theorem establishes the unique solvability of the HDG scheme (3.3).

Theorem 3.1. *There exists a unique solution for the linear problem* (3.3).

Proof. We first note that the existence of the solution follows from its uniqueness. Thus, it suffices to show that when the right-hand sides of (3.3) vanish, then $\boldsymbol{\sigma}_{S,h}$, $\mathbf{u}_{S,h}$, $\boldsymbol{\rho}_{S,h}$, $\boldsymbol{\lambda}_{S,h}$, $\mathbf{u}_{D,h}$, $p_{D,h}$, and $\varphi_{D,h}$ also vanish. Indeed, assuming that $\mathbf{f}_S = \mathbf{0}$ and $f_D = 0$, and taking $\boldsymbol{\tau}_{S,h} = \boldsymbol{\sigma}_{S,h}$, $\mathbf{v}_{S,h} = \mathbf{u}_{S,h}$,

$\boldsymbol{\eta}_{S,h} = \boldsymbol{\rho}_{S,h}$ and $\boldsymbol{\mu}_{S,h} = \boldsymbol{\lambda}_{S,h}$ in (3.3a), (3.3b), (3.3c), (3.3d) and (3.3i), we easily obtain

$$\begin{aligned}\frac{1}{\nu}\|\boldsymbol{\sigma}^{\mathtt{d}}_{S,h}\|^2_{0,\Omega_S} \,+\, \langle \mathbf{S}(\mathbf{u}_{S,h} - \boldsymbol{\lambda}_{S,h}), \mathbf{u}_{S,h} - \boldsymbol{\lambda}_{S,h}\rangle_{\partial\mathcal{T}_h^S} \\ +\, \sum_{\ell=1}^{n-1} \kappa_\ell^{-1} \|\boldsymbol{\lambda}_{S,h}\cdot \mathbf{t}_\ell\|^2_{0,\Sigma} \,+\, \langle \boldsymbol{\lambda}_{S,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_\Sigma \;=\; 0\,. \end{aligned} \tag{3.4}$$

Similarly, taking $\mathbf{v}_{D,h} = \mathbf{u}_{D,h}$, $q_{D,h} = p_{D,h}$ and $\psi_{D,h} = \varphi_{D,h}$ in (3.3e), (3.3f), (3.3g), and (3.3h), we find that

$$(\mathbf{K}^{-1}\mathbf{u}_{D,h}, \mathbf{u}_{D,h})_{\mathcal{T}_h^D} + \langle \tau(p_{D,h} - \varphi_{D,h}), p_{D,h} - \varphi_{D,h}\rangle_{\partial\mathcal{T}_h^D} - \langle \boldsymbol{\lambda}_{S,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_\Sigma \;=\; 0\,. \tag{3.5}$$

Next, adding (3.4) and (3.5), and using the properties of $\mathbf{S}$, $\mathbf{K}$ and the fact that $\nu, \kappa_1, \ldots, \kappa_{n-1}, \tau > 0$, it follows that

$$\begin{gathered}\boldsymbol{\sigma}^{\mathtt{d}}_{S,h} = \mathbf{0} \quad\text{in}\quad \Omega_S\,,\quad \mathbf{u}_{S,h} = \boldsymbol{\lambda}_{S,h} \quad\text{on}\quad \mathcal{E}_h^S\,,\quad \boldsymbol{\lambda}_{S,h}\cdot\mathbf{t}_\ell = 0 \quad\text{in}\quad \Sigma \quad \forall\,\ell\in\{1,\ldots,n-1\}\,, \\ \mathbf{u}_{D,h} = \mathbf{0} \quad\text{in}\quad \Omega_D\,,\quad\text{and}\quad p_{D,h} = \varphi_{D,h} \quad\text{on}\quad \mathcal{E}_h^D\,.\end{gathered} \tag{3.6}$$

Now, using that $\mathbf{u}_{S,h} = \boldsymbol{\lambda}_{S,h}$ on $\mathcal{E}_h^S$ and (3.1), we deduce from (3.3b) and (3.3d) that $\mathbf{div}(\boldsymbol{\sigma}_{S,h}) = \mathbf{0}$ in $\mathcal{T}_h^S$ and $[\![\boldsymbol{\sigma}_{S,h}]\!] = \mathbf{0}$ on $\mathcal{E}_h^S \setminus (\Sigma\cup\Gamma_S)$, which together with $\boldsymbol{\sigma}^{\mathtt{d}}_{S,h} = \mathbf{0}$ in Ω_S implies that $\boldsymbol{\sigma}_{S,h} = c\,\mathbf{I}$ in Ω_S, where $c \in R$. Thus, applying (3.3j) we arrive to $\boldsymbol{\sigma}_{S,h} = \mathbf{0}$ in Ω_S. In turn, according to the foregoing analysis, and integrating by parts the second terms in (3.3a) and (3.3e), we see that (3.3) reduces to the system:

$$-(\nabla\mathbf{u}_{S,h}, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} + (\boldsymbol{\rho}_{S,h}, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} \;=\; 0 \quad \forall\,\boldsymbol{\tau}_{S,h}\in\mathbb{S}_h\,, \tag{3.7}$$

$$(\nabla p_{D,h}, \mathbf{v}_{D,h})_{\mathcal{T}_h^D} \;=\; 0 \quad \forall\,\mathbf{v}_{D,h}\in\mathbf{V}_h^D\,, \tag{3.8}$$

$$\langle \boldsymbol{\lambda}_{S,h}\cdot\mathbf{n}, \psi_{D,h}\rangle_\Sigma \;=\; 0 \quad \forall\,\psi_{D,h}\in N_h\,, \tag{3.9}$$

$$\langle \boldsymbol{\mu}_{S,h}\cdot\mathbf{n}, \varphi_{D,h}\rangle_\Sigma \;=\; 0 \quad \forall\,\boldsymbol{\mu}_{S,h}\in\mathbf{M}_h\,. \tag{3.10}$$

It is clear from (3.9) that $\boldsymbol{\lambda}_{S,h}\cdot\mathbf{n} = 0$ on Σ, which together with the fact that $\boldsymbol{\lambda}_{S,h}\cdot\mathbf{t}_\ell = 0$ on Σ $\forall\,\ell\in\{1,\ldots,n-1\}$ implies that $\boldsymbol{\lambda}_{S,h} = \mathbf{u}_{S,h} = \mathbf{0}$ on $\partial\Omega_S$. In addition, it follows from (3.7) that $\boldsymbol{\rho}_{S,h} = \nabla\mathbf{u}_{S,h}$ in Ω_S, which establishes that $\mathbf{e}(\mathbf{u}_{S,h}) = \mathbf{0}$ in Ω_S, and hence $\mathbf{u}_{S,h}$ belongs to the space of infinitesimal rigid motions (see [3, Exercise 11.x.2]). In this way, using that $\mathbf{u}_{S,h} = \mathbf{0}$ on $\partial\Omega_S$, it is easy to prove that $\mathbf{u}_{S,h} = \mathbf{0}$ in Ω_S, and then we conclude that $\boldsymbol{\rho}_{S,h} = \mathbf{0}$ in Ω_S and $\boldsymbol{\lambda}_{S,h} = \mathbf{0}$ on $\mathcal{E}_h^S$. Finally, from (3.8) we have that $\nabla p_{D,h} = \mathbf{0}$ in $\mathcal{T}_h^D$, which using that $p_{D,h}$ is continuous in Ω_D ($p_{D,h} = \varphi_{D,h}$ in $\mathcal{E}_h^D$), yields $p_{D,h}$ constant in Ω_D. But, recalling that $\varphi_{D,h} = 0$ on Σ (cf. (3.10)), we deduce that $p_{D,h} = 0$ in Ω_S, which gives $\varphi_{D,h} = p_{D,h} = 0$ in $\mathcal{E}_h^D$ and completes the proof. $\square$

4 A priori error analysis

We now aim to derive the *a priori* error estimates for the HDG scheme (3.3). To this end, we use the projection-based error analysis developed in [13, 12, 15].

4.1 The projections

The projected functions are denoted by

$$\begin{aligned}\Pi_S \;:\; \mathbb{H}^1(\mathcal{T}_h^S)\times\mathbf{H}^1(\mathcal{T}_h^S) \;&\longrightarrow\; \widetilde{\mathbb{S}}_h\times\mathbf{V}_h^S \\ (\boldsymbol{\Phi}_S, \boldsymbol{\varphi}_S) \;&\longmapsto\; \Pi_S(\boldsymbol{\Phi}_S, \boldsymbol{\varphi}_S) \;:=\; (\Pi\boldsymbol{\Phi}_S, \Pi\boldsymbol{\varphi}_S)\,,\end{aligned}$$

and

$$\begin{array}{rcl} \Pi_D & : & \mathbf{H}^1(\mathcal{T}_h^D) \times H^1(\mathcal{T}_h^D) \quad \longrightarrow \quad \mathbf{V}_h^D \times P_h \\ & & (\boldsymbol{\varphi}_D, \phi_D) \qquad\quad \longmapsto \quad \Pi_D(\boldsymbol{\varphi}_D, \phi_D) \ := \ (\Pi\boldsymbol{\varphi}_D, \Pi\phi_D)\,, \end{array}$$

where, as usual, we denote $\Pi\boldsymbol{\Phi}_S$ and $\Pi\boldsymbol{\varphi}_S$ only for convenience, since it is clear that $\Pi\boldsymbol{\Phi}_S$ and $\Pi\boldsymbol{\varphi}_S$ depend both on $\boldsymbol{\Phi}_S$ and $\boldsymbol{\varphi}_S$. The same convention is applied to $\Pi\boldsymbol{\varphi}_D$ and $\Pi\phi_D$.

Next, given $(\boldsymbol{\Phi}_S, \boldsymbol{\varphi}_S) \in \mathbb{H}^1(\mathcal{T}_h^S) \times \mathbf{H}^1(\mathcal{T}_h^S)$, the values of the projection $\Pi_S(\boldsymbol{\Phi}_S, \boldsymbol{\varphi}_S)$ on any $T \in \mathcal{T}_h^S$ are fixed by requiring that the components satisfy the equations

$$(\Pi\boldsymbol{\Phi}_S, \boldsymbol{\tau}_S)_T \ = \ (\boldsymbol{\Phi}_S, \boldsymbol{\tau}_S)_T \quad \forall\, \boldsymbol{\tau}_S \in \mathbb{P}_{k-1}(T)\,, \tag{4.1a}$$

$$(\Pi\boldsymbol{\varphi}_S, \mathbf{v}_S)_T \ = \ (\boldsymbol{\varphi}_S, \mathbf{v}_S)_T \quad \forall\, \mathbf{v}_S \in \mathbf{P}_{k-1}(T)\,, \tag{4.1b}$$

$$\langle \Pi\boldsymbol{\Phi}_S\mathbf{n} - \mathbf{S}\Pi\boldsymbol{\varphi}_S, \boldsymbol{\mu}_S\rangle_F \ = \ \langle \boldsymbol{\Phi}_S\mathbf{n} - \mathbf{S}\boldsymbol{\varphi}_S, \boldsymbol{\mu}_S\rangle_F \quad \forall\, \boldsymbol{\mu}_S \in \mathbf{P}_k(F)\,, \quad \forall\, F \in \partial T\,. \tag{4.1c}$$

Similarly, given $(\boldsymbol{\varphi}_D, \phi_D) \in \mathbf{H}^1(\mathcal{T}_h^D) \times H^1(\mathcal{T}_h^D)$, the values of the projection $\Pi_D(\boldsymbol{\varphi}_D, \phi_D)$ on any $T \in \mathcal{T}_h^D$ are determined by requiring that

$$(\Pi\boldsymbol{\varphi}_D, \mathbf{v}_D)_T \ = \ (\boldsymbol{\varphi}_D, \mathbf{v}_D)_T \quad \forall\, \mathbf{v}_D \in \mathbf{P}_{k-1}(T)\,, \tag{4.2a}$$

$$(\Pi\phi_D, q_D)_T \ = \ (\phi_D, q_D)_T \quad \forall\, q_D \in \mathrm{P}_{k-1}(T)\,, \tag{4.2b}$$

$$\langle \Pi\boldsymbol{\varphi}_D \cdot \mathbf{n} + \tau\Pi\phi_D, \psi_D\rangle_F \ = \ \langle \boldsymbol{\varphi}_D \cdot \mathbf{n} + \tau\phi_D, \psi_D\rangle_F \quad \forall\, \psi_D \in \mathrm{P}_k(F)\,, \quad \forall\, F \in \partial T \setminus \Sigma\,, \tag{4.2c}$$

$$\langle \Pi\boldsymbol{\varphi}_D \cdot \mathbf{n} - \tau\Pi\phi_D, \psi_D\rangle_F \ = \ \langle \boldsymbol{\varphi}_D \cdot \mathbf{n} - \tau\phi_D, \psi_D\rangle_F \quad \forall\, \psi_D \in \mathrm{P}_k(F)\,, \quad \forall\, F \in \partial T \cap \Sigma\,. \tag{4.2d}$$

As in [13, 12], both projections are defined in order to preserve the numerical traces (cf. equations (3.3d), (3.3g), (3.3h) and (3.3i)). Also, as it is normal in this approach, the fact that Π_S and Π_D are well-defined arises from the fact that (4.1) and (4.2) are square linear systems, so that the existence of each projection follows from its uniqueness. In view of this, we develop next estimates for any $(\Pi\boldsymbol{\Phi}_S, \Pi\boldsymbol{\varphi}_S) \in \widetilde{\mathbb{S}}_h \times \mathbf{V}_h^S$ satisfying (4.1) without assuming uniqueness a priori. Then, we will use the approximation estimates below to prove unisolvency (see Theorem 4.2). The same format is applied to $(\Pi\boldsymbol{\varphi}_D, \Pi\phi_D) \in \mathbf{V}_h^D \times P_h$. Since the main ideas are given in [13, 12] for general choices of $\mathbf{S}$ and τ, in what follows we only give a summary of the proofs for the well-posedness and the approximation properties of the projections. In particular, for the special choice $\mathbf{S} := \alpha\,\mathbf{I}$, $\alpha \in R$, the ith row of $\Pi\boldsymbol{\Phi}_S$ and the ith component of $\Pi\boldsymbol{\varphi}_S$ are nothing but the two components of the projection for the diffusion case given in [13].

Now, we set $\mathrm{P}_k(T)^\perp$ be the orthogonal of $\mathrm{P}_{k-1}(T)$ within $\mathrm{P}_k(T)$, that is

$$\mathrm{P}_k(T)^\perp \ := \ \{p \in \mathrm{P}_k(T) \ : \ (p,q)_T \ = \ 0 \quad \forall\, q \in \mathrm{P}_{k-1}(T)\}\,,$$

and, according to our notation from the Introduction, we set $\mathbf{P}_k(T)^\perp := [\mathrm{P}_k(T)^\perp]^n$. Thus, the following lemma establishes a characterization of $\Pi\boldsymbol{\varphi}_S$.

Lemma 4.1. *Let* $(\Pi\boldsymbol{\Phi}_S, \Pi\boldsymbol{\varphi}_S) \in \widetilde{\mathbb{S}}_h \times \mathbf{V}_h^S$ *satisfying* (4.1)*. Then, for each* $T \in \mathcal{T}_h^S$, $\Pi\boldsymbol{\varphi}_S|_T$ *is the only element of* $\mathbf{P}_k(T)$ *such that*

$$\begin{aligned} (\Pi\boldsymbol{\varphi}_S, \mathbf{v}_S)_T \ &= \ (\boldsymbol{\varphi}_S, \mathbf{v}_S)_T \quad \forall\, \mathbf{v}_S \in \mathbf{P}_{k-1}(T)\,, \\ \langle \mathbf{S}\Pi\boldsymbol{\varphi}_S, \mathbf{v}_S\rangle_{\partial T} \ &= \ -(\mathbf{div}(\boldsymbol{\Phi}_S), \mathbf{v}_S)_T \ + \ \langle \mathbf{S}\boldsymbol{\varphi}_S, \mathbf{v}_S\rangle_{\partial T} \quad \forall\, \mathbf{v}_S \in \mathbf{P}_k(T)^\perp. \end{aligned}$$

Proof. It follows by applying the same techniques from [12, Proposition 4.2]. $\square$

We now collect estimates for $\Pi\boldsymbol{\varphi}_S - \boldsymbol{\varphi}_S$ and $\Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S$. Note that the assumed local regularity of the pair $(\boldsymbol{\Phi}_S, \boldsymbol{\varphi}_S)$ is clear from the right-hand side of each estimate.

Theorem 4.1. *Given $(\Pi\mathbf{\Phi}_S, \Pi\boldsymbol{\varphi}_S) \in \widetilde{\mathbb{S}}_h \times \mathbf{V}_h^S$ satisfying (4.1), there exists $C > 0$, depending only on $\mathbf{S}$, such that for each $T \in \mathcal{T}_h^S$ there hold*

$$\|\Pi\boldsymbol{\varphi}_S - \boldsymbol{\varphi}_S\|_{0,T} \;\leq\; C\Big\{ h_T^{\ell_{\boldsymbol{\varphi}_S}+1} |\boldsymbol{\varphi}_S|_{\ell_{\boldsymbol{\varphi}_S}+1,T} \;+\; h_T^{\ell_{\mathbf{\Phi}_S}+1} |\mathbf{div}(\mathbf{\Phi}_S)|_{\ell_{\mathbf{\Phi}_S},T} \Big\},$$

and

$$\|\Pi\mathbf{\Phi}_S - \mathbf{\Phi}_S\|_{0,T} \;\leq\; C\Big\{ h_T^{\ell_{\mathbf{\Phi}_S}+1} |\mathbf{\Phi}_S|_{\ell_{\mathbf{\Phi}_S}+1,T} \;+\; h_T^{\ell_{\boldsymbol{\varphi}_S}+1} |\boldsymbol{\varphi}_S|_{\ell_{\boldsymbol{\varphi}_S}+1,T} \;+\; h_T^{\ell_{\mathbf{\Phi}_S}+1} |\mathbf{div}(\mathbf{\Phi}_S)|_{\ell_{\mathbf{\Phi}_S},T} \Big\},$$

for $\ell_{\mathbf{\Phi}_S}, \ell_{\boldsymbol{\varphi}_S} \in [0, k]$.

Proof. Using Lemma 4.1, it follows from a slight modification of the proofs of [12, Lemma 4.5] and [12, Propositions 4.6 and 4.7]. $\square$

Now, we are ready to establish that the projection $\Pi_S\big(\mathbf{\Phi}_S, \boldsymbol{\varphi}_S\big) := \big(\Pi\mathbf{\Phi}_S, \Pi\boldsymbol{\varphi}_S\big)$ is well defined, whereas the respective approximation estimates are already given in the foregoing theorem.

Theorem 4.2. *The projection Π_S is well defined.*

Proof. Let us first observe that the number of independent equations arising from (4.1) is given by

$$\begin{aligned} n^2 \dim \mathrm{P}_{k-1}(T \qquad & \text{for (4.1a)}, \\ n \dim \mathrm{P}_{k-1}(T) \qquad & \text{for (4.1b)}, \\ n\,(n+1) \dim \mathrm{P}_{k}(F) \qquad & \text{for (4.1c)}, \end{aligned}$$

which yields a total of $n\,(n+1) \dim \mathrm{P}_k(T)$. In turn, the corresponding number of unknowns is

$$\begin{aligned} n^2 \dim \mathrm{P}_{k}(T) \qquad & \text{for } \Pi\mathbf{\Phi}_S, \\ n \dim \mathrm{P}_{k}(T) \qquad & \text{for } \Pi\boldsymbol{\varphi}_S, \end{aligned}$$

which gives $n\,(n+1) \dim \mathrm{P}_k(T)$. It follows that (4.1) is a square linear system, and hence, setting $\mathbf{\Phi}_S = \mathbf{0}$ and $\boldsymbol{\varphi}_S = \mathbf{0}$ in the approximation estimates (cf. Theorem 4.1), we find that the projection must vanish, which establishes the unisolvency of (4.1). $\square$

As previously announced, a similar approach is used to establish that the projection $\Pi_D\big(\boldsymbol{\varphi}_D, \boldsymbol{\phi}_D\big) := \big(\Pi\boldsymbol{\varphi}_D, \Pi\boldsymbol{\phi}_D\big)$ (cf. (4.2)) is well defined, and that corresponding approximation properties hold.

Theorem 4.3. *The projection Π_D is well defined. In addition, there exist $C > 0$, independent of $T \in \mathcal{T}_h^D$ and τ, such that*

$$\|\Pi\phi_D - \phi_D\|_{0,T} \;\leq\; C\Big\{ h_T^{\ell_{\phi_D}+1} |\phi_D|_{\ell_{\phi_D}+1,T} \;+\; \tau^{-1} h_T^{\ell_{\boldsymbol{\varphi}_D}+1} |\mathrm{div}(\boldsymbol{\varphi}_D)|_{\ell_{\boldsymbol{\varphi}_D},T} \Big\},$$

and

$$\|\Pi\boldsymbol{\varphi}_D - \boldsymbol{\varphi}_D\|_{0,T} \;\leq\; C\Big\{ h_T^{\ell_{\boldsymbol{\varphi}_D}+1} |\boldsymbol{\varphi}_D|_{\ell_{\boldsymbol{\varphi}_D}+1,T} \;+\; \tau\, h_T^{\ell_{\phi_D}+1} |\phi_D|_{\ell_{\phi_D}+1,T} \Big\},$$

where $\ell_{\boldsymbol{\varphi}_D}, \ell_{\phi_D} \in [0, k]$.

Proof. The proof can be carried out similarly as for Theorem 4.2. More precisely, it follows by applying the same techniques employed in the proof of [13, Propositions A.1, A.2 and A.3]. $\square$

We end this section by introducing other projections. First, consider $\boldsymbol{\mathcal{P}}_A : \mathbb{L}^2(\Omega_S) \to \mathbb{A}_h$ the $\mathbb{L}^2$ orthogonal projector, for which it is well known (see, e.g. [8, 23]) that there holds

$$\|\boldsymbol{\mathcal{P}}_A(\mathbf{s}) - \mathbf{s}\|_{0,T} \quad \leq \quad C\, h_T^{\ell_{\mathbf{s}}}\, |\mathbf{s}|_{\ell_{\mathbf{s}},T} \quad \forall\, \mathbf{s} \in \mathbb{H}^{\ell_{\mathbf{s}}}(T)\,, \quad \forall\, T \in \mathcal{T}_h^S\,, \tag{4.3}$$

where $\ell_{\mathbf{s}} \in [0, k+1]$. Also, we consider

$$\boldsymbol{P}_M \,:\, \mathbf{L}^2(\mathcal{E}_h^S) \longrightarrow \mathbf{M}_h \quad \text{and} \quad P_N \,:\, L^2(\mathcal{E}_h^D) \longrightarrow N_h\,, \tag{4.4}$$

the corresponding $\mathbf{L}^2$ and L^2 orthogonal projections, respectively.

4.2 The a priori error estimates

Similary as in [13, 12, 15], our first goal in this section is to provide upper estimates for the approximation errors, namely, $\mathrm{E}^{\boldsymbol{\sigma}_S} := \Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}$, $\mathrm{E}^{\mathbf{u}_S} := \Pi\mathbf{u}_S - \mathbf{u}_{S,h}$, $\mathrm{E}^{\boldsymbol{\rho}_S} := \boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_{S,h}$, $\mathrm{E}^{\mathbf{u}_D} := \Pi\mathbf{u}_D - \mathbf{u}_{D,h}$, $\mathrm{E}^{p_D} := \Pi p_D - p_{D,h}$, $\mathrm{E}^{\widehat{\mathbf{u}}_S} := \boldsymbol{P}_M(\mathbf{u}_S) - \boldsymbol{\lambda}_{S,h}$ and $\mathrm{E}^{\widehat{p}_D} := P_N(p_D) - \varphi_{D,h}$. In what follows we assume that the exact solution $(\boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, p_D)$ of our problem (2.2)-(2.3)-(2.4) is regular enough to apply Π_S and Π_D. That is, $(\boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, p_D)$ belongs to $\mathbb{H}^1(\mathcal{T}_h^S) \times \mathbf{H}^2(\mathcal{T}_h^S) \times \mathbf{H}^1(\mathcal{T}_h^D) \times H^2(\mathcal{T}_h^D)$ and admits the regularity estimate

$$\sum_{T\in\mathcal{T}_h^S} \Big\{ \|\boldsymbol{\sigma}_S\|_{1,T} \,+\, \|\mathbf{u}_S\|_{2,T} \Big\} + \sum_{T\in\mathcal{T}_h^D} \Big\{ \|\mathbf{u}_D\|_{1,T} \,+\, \|p_D\|_{2,T} \Big\} \;\leq\; C_{\mathrm{reg}} \Big\{ \|\mathbf{f}_S\|_{0,\Omega_S} \,+\, \|f_D\|_{0,\Omega_D} \Big\}\,. \tag{4.5}$$

The purpose of assuming that $\nabla\mathbf{u}_S \in \mathbb{H}^1(\mathcal{T}_h^S)$ and $\nabla p_D \in \mathbf{H}^1(\mathcal{T}_h^D)$ will become clear in the proof of Lemma 4.6 below.

Next, for the consistency of the HDG approximation, we note that the exact solution $\big(\boldsymbol{\sigma}_S,\ \mathbf{u}_S,\ \boldsymbol{\rho}_S := \frac{1}{2}(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}),\ \boldsymbol{\lambda}_S := \mathbf{u}_S|_{\mathcal{E}_h^S},\ \mathbf{u}_D,\ p_D,\ \varphi_D := p_D|_{\mathcal{E}_h^D}\big)$, satisfies also (3.3). Hence, after applying the definition of the projections (see (4.1) and (4.2)) Π_S, Π_D, $\boldsymbol{\mathcal{P}}_A$, $\boldsymbol{P}_M$ and P_N, together with the identities (3.1), and integrating by parts, we find from (3.3) that

$$\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}_S^{\mathtt{d}}, \boldsymbol{\tau}_{S,h}^{\mathtt{d}})_{\mathcal{T}_h^S} + (\Pi\mathbf{u}_S, \mathbf{div}(\boldsymbol{\tau}_{S,h}))_{\mathcal{T}_h^S} - \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{P}_M(\mathbf{u}_S)\rangle_{\partial\mathcal{T}_h^S} & \\
+ (\boldsymbol{\rho}_S, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} &= 0\,, \\
-(\mathbf{v}_{S,h}, \mathbf{div}(\Pi\boldsymbol{\sigma}_S))_{\mathcal{T}_h^S} + \langle \mathbf{S}(\Pi\mathbf{u}_S - \boldsymbol{P}_M(\mathbf{u}_S)), \mathbf{v}_{S,h}\rangle_{\partial\mathcal{T}_h^S} &= (\mathbf{f}_S, \mathbf{v}_{S,h})_{\mathcal{T}_h^S}\,, \\
(\boldsymbol{\eta}_{S,h}, \boldsymbol{\sigma}_S)_{\mathcal{T}_h^S} &= 0\,, \\
\langle \Pi\boldsymbol{\sigma}_S\mathbf{n}, \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S\setminus\Sigma} - \langle \mathbf{S}(\Pi\mathbf{u}_S - \boldsymbol{P}_M(\mathbf{u}_S)), \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S\setminus\Sigma} &= 0\,, \\
(\mathbf{K}^{-1}\mathbf{u}_{D,h}, \mathbf{v}_{D,h})_{\mathcal{T}_h^D} - (\Pi p_D, \mathrm{div}(\mathbf{v}_{D,h}))_{\mathcal{T}_h^D} + \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, P_N(p_D)\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} & \\
- \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, P_N(p_D)\rangle_{\Sigma} &= 0\,, \\
(q_{D,h}, \mathrm{div}(\Pi\mathbf{u}_D))_{\mathcal{T}_h^D} + \langle \tau(\Pi p_D - P_N(p_D)), q_{D,h}\rangle_{\partial\mathcal{T}_h^D} &= (f_D, q_{D,h})_{\mathcal{T}_h^D}\,, \\
\langle \Pi\mathbf{u}_D\cdot\mathbf{n}, \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} + \langle \tau(\Pi p_D - P_N(p_D)), \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} &= 0\,, \\
\langle \boldsymbol{P}_M(\mathbf{u}_S)\cdot\mathbf{n}, \psi_{D,h}\rangle_{\Sigma} - \langle \Pi\mathbf{u}_D\cdot\mathbf{n}, \psi_{D,h}\rangle_{\Sigma} + \langle \tau(\Pi p_D - P_N(p_D)), \psi_{D,h}\rangle_{\Sigma} &= 0\,, \\
\langle \Pi\boldsymbol{\sigma}_S\mathbf{n}, \boldsymbol{\mu}_{S,h}\rangle_{\Sigma} - \langle \mathbf{S}(\Pi\mathbf{u}_S - \boldsymbol{P}_M(\mathbf{u}_S)), \boldsymbol{\mu}_{S,h}\rangle_{\Sigma} + \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}\langle \boldsymbol{P}_M(\mathbf{u}_S)\cdot\mathbf{t}_\ell, \boldsymbol{\mu}_{S,h}\cdot\mathbf{t}_\ell\rangle_{\Sigma} & \\
+ \langle \boldsymbol{\mu}_{S,h}\cdot\mathbf{n}, P_N(p_D)\rangle_{\Sigma} &= 0\,, \\
(\mathrm{tr}\,(\Pi\boldsymbol{\sigma}_S)\,, 1)_{\Omega_S} &= 0\,,
\end{aligned}$$

for all $(\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}, \boldsymbol{\mu}_{S,h}, \mathbf{v}_{D,h}, q_{D,h}, \psi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times P_h \times N_h$. Now, subtracting (3.3) from the above set of equations, and performing simple algebraic manipulations (see [13, 12, 15] for details), we obtain the error equations:

$$\begin{aligned}
\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \boldsymbol{\tau}_{S,h}^{\mathtt{d}})_{\mathcal{T}_h^S} + (\mathrm{E}^{\mathbf{u}_S}, \mathbf{div}(\boldsymbol{\tau}_{S,h}))_{\mathcal{T}_h^S} - \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S} + (\mathrm{E}^{\boldsymbol{\rho}_S}, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S} & \\
= \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \boldsymbol{\tau}_{S,h}^{\mathtt{d}})_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \boldsymbol{\tau}_{S,h})_{\mathcal{T}_h^S}\,, & \qquad (4.6a)
\end{aligned}$$

$$-(\mathbf{v}_{S,h}, \mathbf{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}))_{\mathcal{T}_h^S} + \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \mathbf{v}_{S,h}\rangle_{\partial\mathcal{T}_h^S} = 0\,, \qquad (4.6b)$$

$$(\boldsymbol{\eta}_{S,h}, \mathrm{E}^{\boldsymbol{\sigma}_S})_{\mathcal{T}_h^S} = (\boldsymbol{\eta}_{S,h}, \Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S)_{\mathcal{T}_h^S}\,, \qquad (4.6c)$$

$$\begin{aligned}
\langle \mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n}, \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S} - \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \boldsymbol{\mu}_{S,h}\rangle_{\partial\mathcal{T}_h^S} + \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}\langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{t}_\ell, \boldsymbol{\mu}_{S,h}\cdot\mathbf{t}_\ell\rangle_\Sigma & \\
+ \langle \boldsymbol{\mu}_{S,h}\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_\Sigma = 0\,, & \qquad (4.6d)
\end{aligned}$$

$$\begin{aligned}
(\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D}, \mathbf{v}_{D,h})_{\mathcal{T}_h^D} - (\mathrm{E}^{p_D}, \mathrm{div}(\mathbf{v}_{D,h}))_{\mathcal{T}_h^D} + \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_\Sigma & \\
= (\mathbf{K}^{-1}(\Pi\mathbf{u}_D - \mathbf{u}_D), \mathbf{v}_{D,h})_{\mathcal{T}_h^D}\,, & \qquad (4.6e)
\end{aligned}$$

$$(q_{D,h}, \mathrm{div}(\mathrm{E}^{\mathbf{u}_D}))_{\mathcal{T}_h^D} + \langle \tau(\mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}), q_{D,h}\rangle_{\partial\mathcal{T}_h^D} = 0\,, \qquad (4.6f)$$

$$\langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{n}, \psi_{D,h}\rangle_\Sigma + \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \psi_{D,h}\rangle_\Sigma + \langle \tau(\mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}), \psi_{D,h}\rangle_{\partial\mathcal{T}_h^D} = 0\,, \qquad (4.6g)$$

$$(\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,, 1)_{\Omega_S} = 0\,, \qquad (4.6h)$$

for all $(\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}, \boldsymbol{\mu}_{S,h}, \mathbf{v}_{D,h}, q_{D,h}, \psi_{D,h}) \in \mathbb{S}_h \times \mathbf{V}_h^S \times \mathbb{A}_h \times \mathbf{M}_h \times \mathbf{V}_h^D \times P_h \times N_h$.

4.2.1 Estimating $\mathrm{E}^{\boldsymbol{\rho}_S}$

We begin by determining an estimate for $\mathrm{E}^{\boldsymbol{\rho}_S}$. To do this, we follow [15] and consider the orthogonal decomposition

$$\mathrm{E}^{\boldsymbol{\rho}_S} = \mathrm{E}_0^{\boldsymbol{\rho}_S} + \mathrm{E}_c^{\boldsymbol{\rho}_S}\,,$$

where $\int_T \mathrm{E}_0^{\boldsymbol{\rho}_S}|_T = \mathbf{0}$ and $\mathrm{E}_c^{\boldsymbol{\rho}_S}|_T \in \mathbb{P}_0(T)$ for each $T \in \mathcal{T}_h^S$. This means that for each $T \in \mathcal{T}_h^S$ and for all $i, j \in \{1, \ldots, n\}$ there exist unique $(\mathrm{E}_0^{\boldsymbol{\rho}_S}|_T)_{ij} \in L_0^2(T)$ and $(\mathrm{E}_c^{\boldsymbol{\rho}_S}|_T)_{ij} := \frac{1}{|T|}\int_T (\mathrm{E}^{\boldsymbol{\rho}_S}|_T)_{ij} \in R$ such that $(\mathrm{E}^{\boldsymbol{\rho}_S}|_T)_{ij} = (\mathrm{E}_0^{\boldsymbol{\rho}_S}|_T)_{ij} + (\mathrm{E}_c^{\boldsymbol{\rho}_S}|_T)_{ij}$.

Next, in order to bound these two terms separately, we denote by $\mathbb{A}_h^0$ the subspace to which $\mathrm{E}_0^{\boldsymbol{\rho}_S}$ belongs, that is

$$\mathbb{A}_h^0 := \left\{\boldsymbol{\eta}_{S,h} \in \mathbb{A}_h \;:\; (\boldsymbol{\eta}_{S,h}, \boldsymbol{\tau})_T = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{P}_0(T)\,, \quad \forall\, T \in \mathcal{T}_h^S\right\}\,.$$

The following two lemmas provide the upper bounds for $\|\mathrm{E}_0^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}$ and $\|\mathrm{E}_c^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}$.

Lemma 4.2. *There exists $C > 0$, independent of the meshsize, such that*

$$\|\mathrm{E}_0^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} \leq C\Big\{\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}\Big\}\,.$$

Proof. We follow similarly as in the proof of [15, Theorem 3.6] for the three-dimensional case, keeping in mind that the proof for two dimensions should be even simpler. Indeed, we know from the discrete surjectivity result provided by [31, Lemma 2.9] (see also [15, Lemma 3.7]) that, given $\underline{\boldsymbol{\eta}} := \mathrm{E}_0^{\boldsymbol{\rho}_S} \in \mathbb{A}_h^0$, there exists $\widetilde{\boldsymbol{\tau}}_{S,h} \in \mathbb{B}_h$ such that

$$(\widetilde{\boldsymbol{\tau}}_{S,h}, \boldsymbol{\eta}_{S,h})_{\mathcal{T}_h^S} \;=\; (\mathrm{E}_0^{\boldsymbol{\rho}_S}, \boldsymbol{\eta}_{S,h})_{\mathcal{T}_h^S} \quad \forall\, \boldsymbol{\eta}_{S,h} \in \mathbb{A}_h \tag{4.7}$$

and

$$\|\widetilde{\boldsymbol{\tau}}_{S,h}\|_{0,\Omega_S} \;\leq\; C\,\|\mathrm{E}_0^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}\,, \tag{4.8}$$

where $C > 0$ is independent of $\underline{\boldsymbol{\eta}}$ and the meshsize. Next, we take $\boldsymbol{\tau}_{S,h} = \widetilde{\boldsymbol{\tau}}_{S,h}$ in the error equation (4.6a), and then apply the identities (3.1) to obtain

$$\begin{aligned}\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, (\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} + (\mathrm{E}_0^{\boldsymbol{\rho}_S}, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S} + (\mathrm{E}_c^{\boldsymbol{\rho}_S}, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S} \;&=\; \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, (\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S}\\ &\quad + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S}\,.\end{aligned} \tag{4.9}$$

In turn, it follows from (4.7) that

$$(\widetilde{\boldsymbol{\tau}}_{S,h}, \mathrm{E}_0^{\boldsymbol{\rho}_S})_{\mathcal{T}_h^S} \;=\; (\mathrm{E}_0^{\boldsymbol{\rho}_S}, \mathrm{E}_0^{\boldsymbol{\rho}_S})_{\mathcal{T}_h^S} \;=\; \|\mathrm{E}_0^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}^2$$

and

$$(\widetilde{\boldsymbol{\tau}}_{S,h}, \mathrm{E}_c^{\boldsymbol{\rho}_S})_{\mathcal{T}_h^S} \;=\; (\mathrm{E}_0^{\boldsymbol{\rho}_S}, \mathrm{E}_c^{\boldsymbol{\rho}_S})_{\mathcal{T}_h^S} \;=\; 0\,,$$

which implies together with (4.9), that

$$\|\mathrm{E}_0^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}^2 \;=\; -\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, (\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} + \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, (\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S}\,.$$

In this way, applying the Cauchy-Schwarz inequality and estimate (4.8) we conclude the proof. □

Lemma 4.3. *There exists $C > 0$, independent of the meshsize, such that*

$$\|\mathrm{E}_c^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} \;\leq\; C\Big\{\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} \;+\; \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} \;+\; \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}\Big\}\,.$$

Proof. Given $\mathrm{E}_c^{\boldsymbol{\rho}_S} \in \mathbb{A}_h^c := \{\boldsymbol{\eta} \in \mathbb{A}_h : \boldsymbol{\eta}|_T \in \mathbb{P}_0(T)\ \ \forall\, T \in \mathcal{T}_h^S\}$, we know from [1, Section 11.7, Theorem 11.9] (see also [24, Lemma 5.2]) that there exists $\widetilde{\boldsymbol{\tau}}_{S,h} \in \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_S) : \boldsymbol{\tau}|_T \in \mathbb{P}_1(T)\ \ \forall\, T \in \mathcal{T}_h^S\}$ such that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}_{S,h}) = \mathbf{0}$ in Ω_S, $\widetilde{\boldsymbol{\tau}}_{S,h}\mathbf{n} = \mathbf{0}$ on $\partial\Omega_S$,

$$(\widetilde{\boldsymbol{\tau}}_{S,h}, \boldsymbol{\eta}_{S,h})_{\mathcal{T}_h^S} \;=\; (\mathrm{E}_c^{\boldsymbol{\rho}_S}, \boldsymbol{\eta}_{S,h})_{\mathcal{T}_h^S} \quad \forall\, \boldsymbol{\eta}_{S,h} \in \mathbb{A}_h^c\,, \tag{4.10}$$

and

$$\|\widetilde{\boldsymbol{\tau}}_{S,h}\|_{\mathbf{div},\Omega_S} \;\leq\; C\,\|\mathrm{E}_c^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}\,, \tag{4.11}$$

where $C > 0$ is independent of the meshsize. Then, replacing $\boldsymbol{\tau}_{S,h} = \widetilde{\boldsymbol{\tau}}_{S,h}$ in the error equation (4.6a), we obtain that

$$\begin{aligned}&\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, (\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} - \langle \widetilde{\boldsymbol{\tau}}_{S,h}\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S} + (\mathrm{E}_0^{\boldsymbol{\rho}_S}, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S} + (\mathrm{E}_c^{\boldsymbol{\rho}_S}, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S}\\ &\quad = \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \widetilde{\boldsymbol{\tau}}_{S,h}^{\mathtt{d}})_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S}\,.\end{aligned}$$

Now, from (4.10) we have $(\mathrm{E}_c^{\boldsymbol{\rho}_S}, \widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S} = (\mathrm{E}_c^{\boldsymbol{\rho}_S}, \mathrm{E}_c^{\boldsymbol{\rho}_S})_{\mathcal{T}_h^S} = \|\mathrm{E}_c^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}^2$, and using that $\widetilde{\boldsymbol{\tau}}_{S,h}\mathbf{n} = \mathbf{0}$ on $\partial\Omega_S$, we see that

$$\langle \widetilde{\boldsymbol{\tau}}_{S,h}\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S} \;=\; \langle \widetilde{\boldsymbol{\tau}}_{S,h}\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\Omega_S} \;=\; 0\,.$$

Thus, from the foregoing identities we deduce that

$$\begin{aligned}\|\mathrm{E}_c^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}^2 &= -\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},(\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} - (\mathrm{E}_0^{\boldsymbol{\rho}_S},\widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S} + \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S,(\widetilde{\boldsymbol{\tau}}_{S,h})^{\mathtt{d}})_{\mathcal{T}_h^S} \\ &\quad + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S,\widetilde{\boldsymbol{\tau}}_{S,h})_{\mathcal{T}_h^S}\,,\end{aligned}$$

from which, applying the Cauchy-Schwarz inequality, estimate (4.11) and Lemma 4.2, the proof is completed. □

As a consequence of Lemmas 4.2 and 4.3 we conclude the estimate for $\mathrm{E}^{\boldsymbol{\rho}_S}$ given by

$$\|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} \;\le\; C\Big\{\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}\Big\}. \tag{4.12}$$

4.2.2 Estimating $\mathrm{E}^{\boldsymbol{\sigma}_S}$ and $\mathrm{E}^{\mathbf{u}_D}$

The following two lemmas show how to use the previous results to obtain estimates for $\|\mathrm{E}^{\boldsymbol{\sigma}_S}\|_{0,\Omega_S}$ and $\|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D}$. To do that, in what follows we denote

$$\|\boldsymbol{\mu}\|_{\mathbf{S}} \;:=\; \langle \mathbf{S}\boldsymbol{\mu},\boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h^S}^{1/2} \qquad \forall\ \boldsymbol{\mu}\in\mathbf{L}^2(\mathcal{E}_h^S)\,.$$

Lemma 4.4. *There exists $C>0$, independent of the meshsize, such that*

$$\begin{aligned}\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}} + \|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D} \;&\le\; C\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} \\ &\qquad + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}.\end{aligned}$$

Proof. Taking $\boldsymbol{\tau}_{S,h} := \mathrm{E}^{\boldsymbol{\sigma}_S}$, $\mathbf{v}_{S,h} := \mathrm{E}^{\mathbf{u}_S}$, $\boldsymbol{\eta}_{S,h} := -\mathrm{E}^{\boldsymbol{\rho}_S}$, $\boldsymbol{\mu}_{S,h} := \mathrm{E}^{\widehat{\mathbf{u}}_S}$, $\mathbf{v}_{D,h} := \mathrm{E}^{\mathbf{u}_D}$, $q_{D,h} := \mathrm{E}^{p_D}$, and $\psi_{D,h} := -\mathrm{E}^{p_D}$ in the error equations (4.6), and summing all them, we arrive at

$$\begin{aligned}&\frac{1}{\nu}\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S}^2 + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}}^2 + \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}\|\mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{t}_\ell\|_{0,\Sigma}^2 \\ &\quad + (\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D},\mathrm{E}^{\mathbf{u}_D})_{\mathcal{T}_h^D} + \langle\tau(\mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}),\mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}\rangle_{\partial\mathcal{T}_h^D} \\ &= \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S,(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}})_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S,(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}})_{\mathcal{T}_h^S} \\ &\quad - (\mathrm{E}^{\boldsymbol{\rho}_S},\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S)_{\mathcal{T}_h^S} + (\mathbf{K}^{-1}(\Pi\mathbf{u}_D - \mathbf{u}_D),\mathrm{E}^{\mathbf{u}_D})_{\mathcal{T}_h^D}\,.\end{aligned} \tag{4.13}$$

In particular, according to the properties of $\mathbf{K}^{-1}$, $\mathbf{S}$ and τ, and applying the Cauchy-Schwarz and Young inequality, we deduce from (4.13) that

$$\begin{aligned}&\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S}^2 + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}}^2 + \|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D}^2 \\ &\le \widetilde{C}\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S}\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} \\ &\quad + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S}\|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D}\Big\} \\ &\le \frac{1}{2}\widetilde{C}\Big\{\frac{1}{\delta_1}\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S}^2 + \delta_1\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S}^2 + \frac{1}{\delta_2}\|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}^2 + \delta_2\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S}^2 \\ &\quad + \frac{1}{\delta_3}\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S}^2 + \delta_3\|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S}^2 + \frac{1}{\delta_4}\|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}^2 + \delta_4\|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D}^2\Big\},\end{aligned}$$

for all $\delta_i > 0$, $i \in \{1,2,3,4\}$. Next, utilizing (4.12) we find that the previous inequality becomes

$$
\begin{aligned}
&\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|^2_{0,\Omega_S} \;+\; \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|^2_{\mathbf{S}} \;+\; \|\mathrm{E}^{\mathbf{u}_D}\|^2_{0,\Omega_D} \\
&\quad \le\; \widehat{C}\left\{ \left(\frac{1}{\delta_1} + \frac{1}{\delta_3} + \delta_3\right) \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|^2_{0,\Omega_S} + \left(\frac{1}{\delta_2} + \delta_3\right) \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|^2_{0,\Omega_S} \right. \\
&\qquad \left. +\; \frac{1}{\delta_4}\|\Pi\mathbf{u}_D - \mathbf{u}_D\|^2_{0,\Omega_D} + (\delta_1 + \delta_2 + \delta_3)\,\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|^2_{0,\Omega_S} + \delta_4\|\mathrm{E}^{\mathbf{u}_D}\|^2_{0,\Omega_D} \right\},
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\Big\{1 - \widehat{C}(\delta_1 + \delta_2 + \delta_3)\Big\}\,\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|^2_{0,\Omega_S} \;+\; \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|^2_{\mathbf{S}} \;+\; (1 - \widehat{C}\delta_4)\|\mathrm{E}^{\mathbf{u}_D}\|^2_{0,\Omega_D} \\
&\quad \le\; \widehat{C}\left\{ \left(\frac{1}{\delta_1} + \frac{1}{\delta_3} + \delta_3\right) \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|^2_{0,\Omega_S} + \left(\frac{1}{\delta_2} + \delta_3\right) \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|^2_{0,\Omega_S} \right. \\
&\qquad \left. +\; \frac{1}{\delta_4}\|\Pi\mathbf{u}_D - \mathbf{u}_D\|^2_{0,\Omega_D} \right\} \qquad \forall\, \delta_i > 0, \quad i \in \{1,2,3,4\}.
\end{aligned}
$$

Finally, suitable choices of $\delta_i > 0$, $i \in \{1,2,3,4\}$, complete the proof. □

Lemma 4.5. *There exists $C > 0$, independent of the meshsize, such that*

$$
\|\mathrm{E}^{\boldsymbol{\sigma}_S}\|_{0,\Omega_S} \;\;\le\;\; C\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}.
$$

Proof. From the identity $\|\mathrm{E}^{\boldsymbol{\sigma}_S}\|^2_{0,\Omega_S} = \|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|^2_{0,\Omega_S} + \frac{1}{n}\|\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,\|^2_{0,\Omega_S}$ and Lemma 4.4, it is clear that we only need to bound $\|\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,\|_{0,\Omega_S}$, for which we proceed in what follows as in [12, Proposition 3.4]. In fact, we first recall here a well-known result (see, e.g. [30, Corollary 2.4 in Chapter I]), which establishes that there exists $\beta > 0$ such that

$$
\beta\,\|q\|_{0,\Omega_S} \;\le\; \sup_{\substack{\mathbf{w}\in\mathbf{H}^1_0(\Omega_S)\\ \mathbf{w}\neq\mathbf{0}}} \frac{(q,\mathrm{div}(\mathbf{w}))_{\mathcal{T}_h^S}}{\|\mathbf{w}\|_{1,\Omega_S}} \qquad \forall\; q \in L^2_0(\Omega_S)\,.
$$

Then, applying this inequality to $q := \mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S}) \in L^2_0(\Omega_S)$ (cf. (4.6h)), we readily have

$$
\|\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,\|_{0,\Omega_S} \;\;\le\;\; \frac{1}{\beta} \sup_{\substack{\mathbf{w}\in\mathbf{H}^1_0(\Omega_S)\\ \mathbf{w}\neq\mathbf{0}}} \frac{(\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,,\mathrm{div}(\mathbf{w}))_{\mathcal{T}_h^S}}{\|\mathbf{w}\|_{1,\Omega_S}}\,. \tag{4.14}
$$

Next, let $\mathbf{P} : \mathbf{H}^1(\mathcal{T}_h^S) \to \mathbf{V}_h^S$ be any projection such that, given $\mathbf{w} \in \mathbf{H}^1(\mathcal{T}_h^S)$, there holds $(\mathbf{P}(\mathbf{w}) - \mathbf{w},\mathbf{v})_T = 0 \quad \forall\,\mathbf{v} \in \mathbf{P}_{k-1}(T), \quad \forall\, T \in \mathcal{T}_h^S$. In particular, it suffices to take $\mathbf{P} : \mathbf{H}^1(\mathcal{T}_h^S) \to \mathbf{V}_h^S$ as the orthogonal projector with respect to the $\mathbf{L}^2(\Omega_S)$ inner product, which verifies $(\mathbf{P}(\mathbf{w}) - \mathbf{w},\mathbf{v})_T = 0$ $\forall\,\mathbf{v} \in \mathbf{P}_k(T), \forall\, T \in \mathcal{T}_h^S$. It follows, integrating by parts on each $T \in \mathcal{T}_h^S$, and at the end incorporating the projectors $\boldsymbol{P}_M$ (cf. (4.4)) and $\mathbf{P}$, that for each $\mathbf{w} \in \mathbf{H}^1_0(\Omega_S)$ there holds

$$
\begin{aligned}
&\frac{1}{n}(\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,,\mathrm{div}(\mathbf{w}))_{\mathcal{T}_h^S} \;=\; -\frac{1}{n}(\nabla\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,,\mathbf{w})_{\mathcal{T}_h^S} + \frac{1}{n}\langle\mathbf{w}\cdot\mathbf{n},\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\rangle_{\partial\mathcal{T}_h^S} \\
&\;=\; \left(\mathbf{div}\left(-\frac{1}{n}\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,\mathbf{I}\right),\mathbf{w}\right)_{\mathcal{T}_h^S} + \frac{1}{n}\langle\mathbf{w}\cdot\mathbf{n},\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\rangle_{\partial\mathcal{T}_h^S} \\
&\;=\; \big(\mathbf{div}\,\big((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\big)\,,\mathbf{w}\big)_{\mathcal{T}_h^S} - (\mathbf{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}),\mathbf{w})_{\mathcal{T}_h^S} + \frac{1}{n}\langle\mathbf{w}\cdot\mathbf{n},\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\rangle_{\partial\mathcal{T}_h^S} \\
&\;=\; -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\nabla\mathbf{w})_{\mathcal{T}_h^S} + \langle(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\mathbf{n},\mathbf{w}\rangle_{\partial\mathcal{T}_h^S} - (\mathbf{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}),\mathbf{w})_{\mathcal{T}_h^S} + \left\langle\left(\frac{1}{n}\mathrm{tr}\,(\mathrm{E}^{\boldsymbol{\sigma}_S})\,\mathbf{I}\right)\mathbf{n},\mathbf{w}\right\rangle_{\partial\mathcal{T}_h^S} \\
&\;=\; -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\nabla\mathbf{w})_{\mathcal{T}_h^S} - (\mathbf{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}),\mathbf{w})_{\mathcal{T}_h^S} + \langle\mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n},\mathbf{w}\rangle_{\partial\mathcal{T}_h^S} \\
&\;=\; -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\nabla\mathbf{w})_{\mathcal{T}_h^S} - (\mathbf{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}),\mathbf{P}(\mathbf{w}))_{\mathcal{T}_h^S} + \langle\mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n},\boldsymbol{P}_M(\mathbf{w})\rangle_{\partial\mathcal{T}_h^S} \qquad \forall\;\mathbf{w} \in \mathbf{H}^1_0(\Omega_S)\,.
\end{aligned}
$$

Now, employing the error equation (4.6b) and (4.6d) with $\mathbf{v}_{S,h} := \mathbf{P}(\mathbf{w})$ and $\boldsymbol{\mu}_{S,h} := \boldsymbol{P}_M(\mathbf{w})$, respectively, we deduce together with the foregoing equation that

$$\begin{aligned}
\frac{1}{n}(\operatorname{tr}(\mathrm{E}^{\boldsymbol{\sigma}_S}), \operatorname{div}(\mathbf{w}))_{\mathcal{T}_h^S} &= -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \nabla \mathbf{w})_{\mathcal{T}_h^S} - \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \mathbf{P}(\mathbf{w})\rangle_{\partial\mathcal{T}_h^S} \\
&\quad + \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \boldsymbol{P}_M(\mathbf{w})\rangle_{\partial\mathcal{T}_h^S} - \sum_{\ell=1}^{n-1} \kappa_\ell^{-1} \langle \mathrm{E}^{\widehat{\mathbf{u}}_S} \cdot \mathbf{t}_\ell, \boldsymbol{P}_M(\mathbf{w}) \cdot \mathbf{t}_\ell \rangle_\Sigma - \langle \boldsymbol{P}_M(\mathbf{w}) \cdot \mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_\Sigma \\
&= -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \nabla \mathbf{w})_{\mathcal{T}_h^S} - \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \mathbf{P}(\mathbf{w}) - \boldsymbol{P}_M(\mathbf{w})\rangle_{\partial\mathcal{T}_h^S} \\
&\quad - \sum_{\ell=1}^{n-1} \kappa_\ell^{-1} \langle \mathrm{E}^{\widehat{\mathbf{u}}_S} \cdot \mathbf{t}_\ell, \mathbf{w} \cdot \mathbf{t}_\ell \rangle_\Sigma - \langle \mathbf{w} \cdot \mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_\Sigma \\
&= -((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \nabla \mathbf{w})_{\mathcal{T}_h^S} - \langle \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \mathbf{P}(\mathbf{w}) - \boldsymbol{P}_M(\mathbf{w})\rangle_{\partial\mathcal{T}_h^S},
\end{aligned}$$

where the terms on Σ vanish because $\mathbf{w} = \mathbf{0}$ on $\partial\Omega_S$. In this way, we find that

$$\begin{aligned}
(\operatorname{tr}(\mathrm{E}^{\boldsymbol{\sigma}_S}), \operatorname{div}(\mathbf{w}))_{\mathcal{T}_h^S} &\leq n \left\{ \|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} |\mathbf{w}|_{1,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}} \|\mathbf{P}(\mathbf{w}) - \boldsymbol{P}_M(\mathbf{w})\|_{\mathbf{S}} \right\} \\
&\leq n \left\{ \|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}} \frac{\|\mathbf{P}(\mathbf{w}) - \boldsymbol{P}_M(\mathbf{w})\|_{\mathbf{S}}}{\|\mathbf{w}\|_{1,\Omega_S}} \right\} \|\mathbf{w}\|_{1,\Omega_S} \quad \forall\, \mathbf{w} \in \mathbf{H}_0^1(\Omega_S),
\end{aligned}$$

which, replaced back into (4.14), yields

$$\|\operatorname{tr}(\mathrm{E}^{\boldsymbol{\sigma}_S})\|_{0,\Omega_S} \leq \frac{n}{\beta}\, \Psi(\mathbf{S}) \left\{ \|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}\|_{\mathbf{S}} \right\}, \tag{4.15}$$

where

$$\Psi(\mathbf{S}) := \max\left\{ 1, \sup_{\substack{\mathbf{w} \in \mathbf{H}_0^1(\Omega_S) \\ \mathbf{w} \neq \mathbf{0}}} \frac{\|\mathbf{P}(\mathbf{w}) - \boldsymbol{P}_M(\mathbf{w})\|_{\mathbf{S}}}{\|\mathbf{w}\|_{1,\Omega_S}} \right\}.$$

The above expression is bounded by a constant depending on $\mathbf{S}$ (see [12, Proposition 3.9] for details), and hence Lemma 4.4 and (4.15) complete the proof. □

4.2.3 Estimating $\mathrm{E}^{\mathbf{u}_S}$ and E^{p_D}

In order to estimate $\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}$ and $\|\mathrm{E}^{p_D}\|_{0,\Omega_D}$, we now proceed as in [13, 12, 15] and incorporate a suitable auxiliary problem. More precisely, in what follows we consider the continuous problem (2.2)-(2.3)-(2.4) with sources given by $\mathbf{f}_S := -\mathrm{E}^{\mathbf{u}_S} \in \mathbf{L}^2(\Omega_S)$ and $f_D := \mathrm{E}^{p_D} \in L^2(\Omega_D)$, that is:

$$\begin{aligned}
\frac{1}{\nu}\boldsymbol{\Phi}_S^{\mathtt{d}} - \nabla \boldsymbol{\varphi}_S + \boldsymbol{\gamma}_S &= \mathbf{0} && \text{in } \Omega_S, && (4.16a)\\
\mathbf{div}(\boldsymbol{\Phi}_S) &= \mathrm{E}^{\mathbf{u}_S} && \text{in } \Omega_S, && (4.16b)\\
\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S^{\mathtt{t}} &= \mathbf{0} && \text{in } \Omega_S, && (4.16c)\\
\boldsymbol{\varphi}_S &= \mathbf{0} && \text{on } \Gamma_S, && (4.16d)\\
\mathbf{K}^{-1}\boldsymbol{\psi}_D + \nabla \phi_D &= \mathbf{0} && \text{in } \Omega_D, && (4.16e)\\
\operatorname{div}(\boldsymbol{\psi}_D) &= \mathrm{E}^{p_D} && \text{in } \Omega_D, && (4.16f)\\
\boldsymbol{\psi}_D \cdot \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_D, && (4.16g)\\
\boldsymbol{\varphi}_S \cdot \mathbf{n} - \boldsymbol{\psi}_D \cdot \mathbf{n} &= \mathbf{0} && \text{on } \Sigma, && (4.16h)\\
\boldsymbol{\Phi}_S\mathbf{n} + \sum_{\ell=1}^{n-1} \kappa_\ell^{-1}(\boldsymbol{\varphi}_S \cdot \mathbf{t}_\ell)\mathbf{t}_\ell + \phi_D\mathbf{n} &= \mathbf{0} && \text{on } \Sigma, && (4.16i)
\end{aligned}$$

where $\boldsymbol{\gamma}_S := \frac{1}{2}\left(\nabla\boldsymbol{\varphi}_S - (\nabla\boldsymbol{\varphi}_S)^{\mathtt{t}}\right)$. According to (4.5), we know that there holds

$$\sum_{T\in\mathcal{T}_h^S}\Big\{\|\boldsymbol{\Phi}_S\|_{1,T}+\|\boldsymbol{\varphi}_S\|_{2,T}\Big\} + \sum_{T\in\mathcal{T}_h^D}\Big\{\|\boldsymbol{\psi}_D\|_{1,T}+\|\phi_D\|_{2,T}\Big\} \;\le\; C_{\mathrm{reg}}\Big\{\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}+\|\mathrm{E}^{p_D}\|_{0,\Omega_D}\Big\}, \quad (4.17)$$

and certainly $\mathbf{div}(\boldsymbol{\Phi}_S)\in\mathbf{L}^2(\Omega_S)$ and $\mathrm{div}(\boldsymbol{\psi}_D)\in L^2(\Omega_D)$.

Lemma 4.6. *There holds*

$$\begin{aligned}\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S} + \;\|\mathrm{E}^{p_D}\|_{0,\Omega_D} \;\le\; C\,h\Big\{&\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D}\\ &+ \|\Pi\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S)-\boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D-\mathbf{u}_D\|_{0,\Omega_D}\Big\}.\end{aligned}$$

Proof. First, note from (4.17) that we can apply Π_S and Π_D to the solution of (4.16), and hence we can set $\Pi_S(\boldsymbol{\Phi}_S,\boldsymbol{\varphi}_S) := (\Pi\boldsymbol{\Phi}_S,\Pi\boldsymbol{\varphi}_S)$ and $\Pi_D(\boldsymbol{\psi}_D,\phi_D) := (\Pi\boldsymbol{\psi}_D,\Pi\phi_D)$. Then, from (4.16a), (4.16b), (4.16e) and (4.16f) we have

$$\begin{aligned}\|\mathrm{E}^{\mathbf{u}_S}\|^2_{0,\Omega_S} + \|\mathrm{E}^{p_D}\|^2_{0,\Omega_D} \;&=\; (\mathrm{E}^{\mathbf{u}_S},\mathrm{E}^{\mathbf{u}_S})_{\mathcal{T}_h^S} + (\mathrm{E}^{p_D},\mathrm{E}^{p_D})_{\mathcal{T}_h^D} \;=\; (\mathrm{E}^{\mathbf{u}_S},\mathbf{div}(\boldsymbol{\Phi}_S))_{\mathcal{T}_h^S}\\ &+ (\mathrm{E}^{\boldsymbol{\sigma}_S},\frac{1}{\nu}\boldsymbol{\Phi}_S^{\mathtt{d}} - \nabla\boldsymbol{\varphi}_S + \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} + (\mathrm{E}^{p_D},\mathrm{div}(\boldsymbol{\psi}_D))_{\mathcal{T}_h^D} - (\mathrm{E}^{\mathbf{u}_D},\mathbf{K}^{-1}\boldsymbol{\psi}_D+\nabla\phi_D)_{\mathcal{T}_h^D}\\ =\; &(\mathrm{E}^{\mathbf{u}_S},\mathbf{div}(\Pi\boldsymbol{\Phi}_S))_{\mathcal{T}_h^S} + (\Pi\boldsymbol{\varphi}_S,\mathrm{div}(\mathrm{E}^{\boldsymbol{\sigma}_S}))_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S),\mathrm{E}^{\boldsymbol{\sigma}_S})_{\mathcal{T}_h^S} + (\mathrm{E}^{p_D},\mathrm{div}(\Pi\boldsymbol{\psi}_D))_{\mathcal{T}_h^D}\\ &+ (\Pi\phi_D,\mathrm{div}(\mathrm{E}^{\mathbf{u}_D}))_{\mathcal{T}_h^D} + \frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - ((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S)-\boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} - (\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D},\boldsymbol{\psi}_D)_{\mathcal{T}_h^D}\\ &- \langle(\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)\mathbf{n},\mathrm{E}^{\mathbf{u}_S}\rangle_{\partial\mathcal{T}_h^S} - \langle\mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n},\boldsymbol{\varphi}_S\rangle_{\partial\mathcal{T}_h^S} - \langle(\Pi\boldsymbol{\psi}_D-\boldsymbol{\psi}_D)\cdot\mathbf{n},\mathrm{E}^{p_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma}\\ &+ \langle(\Pi\boldsymbol{\psi}_D-\boldsymbol{\psi}_D)\cdot\mathbf{n},\mathrm{E}^{p_D}\rangle_{\Sigma} - \langle\mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n},\phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} + \langle\mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n},\phi_D\rangle_{\Sigma}\,.\end{aligned}$$

Now, using the error equations (4.6a), (4.6b), (4.6c), (4.6e) and (4.6f) in the first five terms of the above identity, and employing that $\boldsymbol{\Phi}_S$ is a symmetric tensor (cf. (4.16c)), we deduce that

$$\|\mathrm{E}^{\mathbf{u}_S}\|^2_{0,\Omega_S} + \|\mathrm{E}^{p_D}\|^2_{0,\Omega_D} \;=\; S_1 + S_2 + S_3 - \langle\mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n},\phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} + \langle\mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n},\phi_D\rangle_{\Sigma}\,, \qquad (4.18)$$

where the intermediate terms S_i, $i\in\{1,2,3\}$, are given by

$$\begin{aligned}S_1 \;:=\; &-\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - ((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}},\boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S)-\boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} - (\mathrm{E}^{\boldsymbol{\rho}_S},\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)_{\mathcal{T}_h^S}\\ &+ (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S)-\boldsymbol{\rho}_S,\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} + \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_S,(\Pi\boldsymbol{\Phi}_S)^{\mathtt{d}})_{\mathcal{T}_h^S} + (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S),\Pi\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_S)_{\mathcal{T}_h^S}\\ &+ (\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D},\Pi\boldsymbol{\psi}_D-\boldsymbol{\psi}_D)_{\mathcal{T}_h^D} - (\mathbf{K}^{-1}(\Pi\mathbf{u}_D-\mathbf{u}_D),\Pi\boldsymbol{\psi}_D)_{\mathcal{T}_h^D}\,,\end{aligned}$$

$$S_2 \;:=\; \langle\Pi\boldsymbol{\Phi}_S\mathbf{n},\mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S} - \langle(\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)\mathbf{n},\mathrm{E}^{\mathbf{u}_S}\rangle_{\partial\mathcal{T}_h^S} + \langle\mathbf{S}(\mathrm{E}^{\mathbf{u}_S}-\mathrm{E}^{\widehat{\mathbf{u}}_S}),\Pi\boldsymbol{\varphi}_S\rangle_{\partial\mathcal{T}_h^S} - \langle\mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n},\boldsymbol{\varphi}_S\rangle_{\partial\mathcal{T}_h^S}\,,$$

and

$$\begin{aligned}S_3 \;:=\; &\langle\Pi\boldsymbol{\psi}_D\cdot\mathbf{n},\mathrm{E}^{\widehat{p}_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle\Pi\boldsymbol{\psi}_D\cdot\mathbf{n},\mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} - \langle(\Pi\boldsymbol{\psi}_D-\boldsymbol{\psi}_D)\cdot\mathbf{n},\mathrm{E}^{p_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma}\\ &+ \langle(\Pi\boldsymbol{\psi}_D-\boldsymbol{\psi}_D)\cdot\mathbf{n},\mathrm{E}^{p_D}\rangle_{\Sigma} - \langle\tau(\mathrm{E}^{p_D}-\mathrm{E}^{\widehat{p}_D}),\Pi\phi_D\rangle_{\partial\mathcal{T}_h^D}\,.\end{aligned}$$

Next, performing some simple algebraic manipulations, we find that

$$\begin{aligned}S_2 \;=\; &-\langle\mathrm{E}^{\mathbf{u}_S}-\mathrm{E}^{\widehat{\mathbf{u}}_S},(\Pi\boldsymbol{\Phi}_S-\boldsymbol{\Phi}_S)\mathbf{n}-\mathbf{S}(\Pi\boldsymbol{\varphi}_S-\boldsymbol{\varphi}_S)\rangle_{\partial\mathcal{T}_h^S} + \langle\boldsymbol{\Phi}_S\mathbf{n},\mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S}\\ &- \langle\mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n}-\mathbf{S}(\mathrm{E}^{\mathbf{u}_S}-\mathrm{E}^{\widehat{\mathbf{u}}_S}),\boldsymbol{\varphi}_S\rangle_{\partial\mathcal{T}_h^S} \qquad (4.19)\end{aligned}$$

and

$$\begin{aligned} S_3 &= -\langle \mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}, (\Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)\cdot\mathbf{n} + \tau(\Pi\phi_D - \phi_D)\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} \\ &\quad + \langle \mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}, (\Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)\cdot\mathbf{n} - \tau(\Pi\phi_D - \phi_D)\rangle_{\Sigma} \\ &\quad + \langle \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} - \langle \tau(\mathrm{E}^{p_D} - \mathrm{E}^{\widehat{p}_D}), \phi_D\rangle_{\partial\mathcal{T}_h^D}\,. \end{aligned} \tag{4.20}$$

Then, from (4.1c) with $\boldsymbol{\mu}_S := \mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}$, we note that (4.19) reduces to

$$S_2 = \langle \boldsymbol{\Phi}_S\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\partial\mathcal{T}_h^S} - \langle \mathrm{E}^{\boldsymbol{\sigma}_S}\mathbf{n} - \mathbf{S}(\mathrm{E}^{\mathbf{u}_S} - \mathrm{E}^{\widehat{\mathbf{u}}_S}), \boldsymbol{P}_M(\boldsymbol{\varphi}_S)\rangle_{\partial\mathcal{T}_h^S}\,,$$

from which, applying (4.6d) with $\boldsymbol{\mu}_{S,h} := \boldsymbol{P}_M(\boldsymbol{\varphi}_S)$, the continuity of $\boldsymbol{\Phi}_S\mathbf{n}$, the fact that $\mathrm{E}^{\widehat{\mathbf{u}}_S} = \mathbf{0}$ on Γ_S, and (4.16i), we deduce that

$$\begin{aligned} S_2 &= \langle \boldsymbol{\Phi}_S\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\Sigma} + \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}\langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{t}_\ell, \boldsymbol{P}_M(\boldsymbol{\varphi}_S)\cdot\mathbf{t}_\ell\rangle_{\Sigma} + \langle \boldsymbol{P}_M(\boldsymbol{\varphi}_S)\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} \\ &= \left\langle \boldsymbol{\Phi}_S\mathbf{n} + \sum_{\ell=1}^{n-1}\kappa_\ell^{-1}(\boldsymbol{\varphi}_S\cdot\mathbf{t}_\ell)\mathbf{t}_\ell,\ \mathrm{E}^{\widehat{\mathbf{u}}_S}\right\rangle_{\Sigma} + \langle \boldsymbol{\varphi}_S\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} \\ &= -\langle \phi_D\mathbf{n}, \mathrm{E}^{\widehat{\mathbf{u}}_S}\rangle_{\Sigma} + \langle \boldsymbol{\varphi}_S\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} \\ &= -\langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma} + \langle \boldsymbol{\varphi}_S\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma}\,. \end{aligned} \tag{4.21}$$

On the other hand, from (4.2c), (4.2d), and (4.6g) with $\psi_{D,h} := P_N(\phi_D)$, we have

$$S_3 = \langle \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} + \langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma} + \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma}\,,$$

which, using the continuity of $\boldsymbol{\psi}_D\cdot\mathbf{n}$ and (4.16g), becomes

$$S_3 = -\langle \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} + \langle \mathrm{E}^{\widehat{\mathbf{u}}_S}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma} + \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma}\,.$$

In this way, from (4.21) and (4.16h), we find that

$$\begin{aligned} S_2 + S_3 &= \langle \boldsymbol{\varphi}_S\cdot\mathbf{n} - \boldsymbol{\psi}_D\cdot\mathbf{n}, \mathrm{E}^{\widehat{p}_D}\rangle_{\Sigma} + \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma} \\ &= \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\partial\mathcal{T}_h^D\setminus\Sigma} - \langle \mathrm{E}^{\mathbf{u}_D}\cdot\mathbf{n}, \phi_D\rangle_{\Sigma}\,, \end{aligned}$$

and replacing the above expression back into (4.18), we obtain

$$\begin{aligned} &\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}^2 + \|\mathrm{E}^{p_D}\|_{0,\Omega_D}^2 = S_1 \\ &= -\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - ((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S) - \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} \\ &\quad - (\mathrm{E}^{\boldsymbol{\rho}_S}, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} + \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, (\Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)^{\mathtt{d}})_{\mathcal{T}_h^S} \\ &\quad + (\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S) - \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} + (\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D}, \Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)_{\mathcal{T}_h^D} \\ &\quad - (\mathbf{K}^{-1}(\Pi\mathbf{u}_D - \mathbf{u}_D), \Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)_{\mathcal{T}_h^D} + (\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \frac{1}{\nu}\boldsymbol{\Phi}_S^{\mathtt{d}} + \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} - (\Pi\mathbf{u}_D - \mathbf{u}_D, \mathbf{K}^{-1}\boldsymbol{\psi}_D)_{\mathcal{T}_h^D} \\ &= -\frac{1}{\nu}((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - ((\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathtt{d}}, \boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S) - \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} \\ &\quad - (\mathrm{E}^{\boldsymbol{\rho}_S}, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} - (\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S, \Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)_{\mathcal{T}_h^S} + \frac{1}{\nu}(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, (\Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S)^{\mathtt{d}})_{\mathcal{T}_h^S} \\ &\quad + (\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S) - \boldsymbol{\gamma}_S)_{\mathcal{T}_h^S} + (\mathbf{K}^{-1}\mathrm{E}^{\mathbf{u}_D}, \Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)_{\mathcal{T}_h^D} \\ &\quad - (\mathbf{K}^{-1}(\Pi\mathbf{u}_D - \mathbf{u}_D), \Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D)_{\mathcal{T}_h^D} + (\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \nabla\boldsymbol{\varphi}_S)_{\mathcal{T}_h^S} + (\Pi\mathbf{u}_D - \mathbf{u}_D, \nabla\phi_D)_{\mathcal{T}_h^D}\,, \end{aligned}$$

where in the last identity we have applied (4.16a) and (4.16e). Now, from (4.1a) and (4.2a) we note that

$$(\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \nabla\boldsymbol{\varphi}_S)_{\mathcal{T}_h^S} \;=\; (\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S, \nabla\boldsymbol{\varphi}_S - \boldsymbol{P}_0(\nabla\boldsymbol{\varphi}_S))_{\mathcal{T}_h^S}$$

and

$$(\Pi\mathbf{u}_D - \mathbf{u}_D, \nabla\phi_D)_{\mathcal{T}_h^D} \;=\; (\Pi\mathbf{u}_D - \mathbf{u}_D, \nabla\phi_D - P_0(\nabla\phi_D))_{\mathcal{T}_h^D}$$

where $\boldsymbol{P}_0|_T$ and $P_0|_K$ are the $\mathbf{L}^2(T)$ and $L^2(K)$ projections onto $\mathbf{P}_0(T)$ and $\mathrm{P}_0(K)$, respectively, for each $T \in \mathcal{T}_h^S$ and $K \in \mathcal{T}_h^D$. Hence, applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}^2 + \|\mathrm{E}^{p_D}\|_{0,\Omega_D}^2 \;\le\; & C\Big\{\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathrm{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D} + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} \\
& + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}\Big\{\|\Pi\boldsymbol{\Phi}_S - \boldsymbol{\Phi}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\gamma}_S) - \boldsymbol{\gamma}_S\|_{0,\Omega_S} \\
& + \|\Pi\boldsymbol{\psi}_D - \boldsymbol{\psi}_D\|_{0,\Omega_D} + \|\boldsymbol{P}_0(\nabla\boldsymbol{\varphi}_S) - \nabla\boldsymbol{\varphi}_S\|_{0,\Omega_S} + \|P_0(\nabla\phi_D) - \nabla\phi_D\|_{0,\Omega_D}\Big\},
\end{aligned}$$

from which, using the approximation properties of Π_S and Π_D (cf. Theorems 4.1 and 4.3), and those of $\boldsymbol{\mathcal{P}}_A$ (cf. (4.3)), $\boldsymbol{P}_0$ and P_0 (see, e.g. [8]), we deduce that

$$\begin{aligned}
\|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}^2 + \|\mathrm{E}^{p_D}\|_{0,\Omega_D}^2 \;\le\; & C\,h\Big\{\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathrm{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} + \|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D} + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} \\
& + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}\Bigg\{\sum_{T\in\mathcal{T}_h^S}\Big(|\boldsymbol{\Phi}_S|_{1,T} + |\boldsymbol{\varphi}_S|_{1,\Omega_S} + \|\mathbf{div}(\boldsymbol{\Phi}_S)\|_{0,T} \\
& + |\boldsymbol{\gamma}_S|_{1,T} + |\nabla\boldsymbol{\varphi}_S|_{1,T}\Big) + \sum_{T\in\mathcal{T}_h^D}\Big(|\boldsymbol{\psi}_D|_{1,T} + |\phi_D|_{1,T} + |\nabla\phi_D|_{1,T}\Big)\Bigg\}.
\end{aligned}$$

Finally, the regularity estimate (4.17) and (4.16b) finish the proof. □

4.2.4 Estimating $\mathrm{E}^{\widehat{\mathbf{u}}_S}$ and $\mathrm{E}^{\widehat{p}_D}$

Our next goal is to derive estimates for the trace variables. To this end, as in [12, 13], we measure the errors of quantities defined on $\partial\mathcal{T}_h^S$ and $\partial\mathcal{T}_h^D$ with the seminorms:

$$\|\boldsymbol{\mu}_{S,h}\|_h \;:=\; \left\{\sum_{T\in\mathcal{T}_h^S} h_T\|\boldsymbol{\mu}_{S,h}\|_{0,\partial T}^2\right\}^{1/2} \quad\text{and}\quad \|\psi_{D,h}\|_h \;:=\; \left\{\sum_{T\in\mathcal{T}_h^D} h_T\|\psi_{D,h}\|_{0,\partial T}^2\right\}^{1/2},$$

respectively. In this way, the following lemma uses ideas from [12, Lemma 3.7] and [13, Theorem 4.1] to obtain estimates for $\|\mathrm{E}^{\widehat{\mathbf{u}}_S}\|_h$ and $\|\mathrm{E}^{\widehat{p}_D}\|_h$.

Lemma 4.7. *There hold*

$$\|\mathrm{E}^{\widehat{\mathbf{u}}_S}\|_h \le C\Big\{h\Big(\|(\mathrm{E}^{\boldsymbol{\sigma}_S})^{\mathrm{d}}\|_{0,\Omega_S} + \|\mathrm{E}^{\boldsymbol{\rho}_S}\|_{0,\Omega_S} + \|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S}\Big) + \|\mathrm{E}^{\mathbf{u}_S}\|_{0,\Omega_S}\Big\}$$

and

$$\|\mathrm{E}^{\widehat{p}_D}\|_h \;\le\; C\Big\{h\Big(\|\mathrm{E}^{\mathbf{u}_D}\|_{0,\Omega_D} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big) \;+\; \|\mathrm{E}^{p_D}\|_{0,\Omega_D}\Big\}.$$

Proof. The proof follows from a straightforward adaptation of the proofs in [12, Lemma 3.7] and [13, Theorem 4.1]. The main tools employed are the error equations (4.6a) and (4.6e), a standard scaling argument (see [4]), the Cauchy-Schwarz inequality, and an inverse inequality. □

4.2.5 The main result

As a consequence of the lemmas provided in the previous sections, now we are able to establish the *a priori* error estimates for the HDG scheme (3.3).

Theorem 4.4. *There exists $C > 0$, independent of h and the polynomial approximation degree k, such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} \;+\; \|\boldsymbol{\rho}_S - \boldsymbol{\rho}_{S,h}\|_{0,\Omega_S} \;+\; \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{0,\Omega_D} \\
&\quad \leq\; C\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
&\|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,\Omega_S} \\
&\quad \leq\; C\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_S - \mathbf{u}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
&\|p_D - p_{D,h}\|_{0,\Omega_D} \\
&\quad \leq\; C\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D} + \|\Pi p_D - p_D\|_{0,\Omega_D}\Big\}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathrm{E}^{\widehat{\mathbf{u}}_S}\|_h \;+\; \|\mathrm{E}^{\widehat{p}_D}\|_h \\
&\quad \leq\; C\,h\Big\{\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} + \|\boldsymbol{\mathcal{P}}_A(\boldsymbol{\rho}_S) - \boldsymbol{\rho}_S\|_{0,\Omega_S} + \|\Pi\mathbf{u}_D - \mathbf{u}_D\|_{0,\Omega_D}\Big\}\,.
\end{aligned}$$

Moreover, the following theorem provides the corresponding theoretical rates of convergence.

Theorem 4.5. *There exists $C > 0$, independent of h and the polynomial approximation degree k, such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S} \;+\; \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,\Omega_S} \;+\; \|\boldsymbol{\rho}_S - \boldsymbol{\rho}_{S,h}\|_{0,\Omega_S} \;+\; \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{0,\Omega_D} \\
&\quad \leq\; C\,h^{\min\{\ell_{\boldsymbol{\sigma}_S}+1,\ell_{\mathbf{u}_S}+1,\ell_{\boldsymbol{\rho}_S}\}} \sum_{T\in\mathcal{T}_h^S} \Big\{|\boldsymbol{\sigma}_S|_{\ell_{\boldsymbol{\sigma}_S}+1,T} + |\mathbf{u}_S|_{\ell_{\mathbf{u}_S}+1,T} + |\boldsymbol{\rho}_S|_{\ell_{\boldsymbol{\rho}_S},T} + |\mathbf{div}(\boldsymbol{\sigma}_S)|_{\ell_{\boldsymbol{\sigma}_S},T}\Big\} \\
&\qquad +\; C\,h^{\min\{\ell_{\mathbf{u}_D}+1,\ell_{p_D}+1\}} \sum_{T\in\mathcal{T}_h^D} \Big\{|\mathbf{u}_D|_{\ell_{\mathbf{u}_D}+1,T} + |p_D|_{\ell_{p_D}+1,T}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
&\|p_D - p_{D,h}\|_{0,\Omega_D} \\
&\quad \leq\; C\,h^{\min\{\ell_{\boldsymbol{\sigma}_S}+1,\ell_{\mathbf{u}_S}+1,\ell_{\boldsymbol{\rho}_S}\}} \sum_{T\in\mathcal{T}_h^S} \Big\{|\boldsymbol{\sigma}_S|_{\ell_{\boldsymbol{\sigma}_S}+1,T} + |\mathbf{u}_S|_{\ell_{\mathbf{u}_S}+1,T} + |\boldsymbol{\rho}_S|_{\ell_{\boldsymbol{\rho}_S},T} + |\mathbf{div}(\boldsymbol{\sigma}_S)|_{\ell_{\boldsymbol{\sigma}_S},T}\Big\} \\
&\qquad +\; C\,h^{\min\{\ell_{\mathbf{u}_D}+1,\ell_{p_D}+1\}} \sum_{T\in\mathcal{T}_h^D} \Big\{|\mathbf{u}_D|_{\ell_{\mathbf{u}_D}+1,T} + |p_D|_{\ell_{p_D}+1,T} + |\mathrm{div}(\mathbf{u}_D)|_{\ell_{\mathbf{u}_D},T}\Big\}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\|\boldsymbol{P}_M(\mathbf{u}_S) - \boldsymbol{\lambda}_{S,h}\|_h \;+\; \|P_N(p_D) - \varphi_{D,h}\|_h \\
&\quad \leq\; C\,h^{1+\min\{\ell_{\boldsymbol{\sigma}_S}+1,\ell_{\mathbf{u}_S}+1,\ell_{\boldsymbol{\rho}_S}\}} \sum_{T\in\mathcal{T}_h^S} \Big\{|\boldsymbol{\sigma}_S|_{\ell_{\boldsymbol{\sigma}_S}+1,T} + |\mathbf{u}_S|_{\ell_{\mathbf{u}_S}+1,T} + |\boldsymbol{\rho}_S|_{\ell_{\boldsymbol{\rho}_S},T} + |\mathbf{div}(\boldsymbol{\sigma}_S)|_{\ell_{\boldsymbol{\sigma}_S},T}\Big\} \\
&\qquad +\; C\,h^{1+\min\{\ell_{\mathbf{u}_D}+1,\ell_{p_D}+1\}} \sum_{T\in\mathcal{T}_h^D} \Big\{|\mathbf{u}_D|_{\ell_{\mathbf{u}_D}+1,T} + |p_D|_{\ell_{p_D}+1,T}\Big\}\,,
\end{aligned}$$

for $\ell_{\boldsymbol{\sigma}_S}, \ell_{\mathbf{u}_S}, \ell_{\mathbf{u}_D}, \ell_{p_D} \in [0,k]$ and $\ell_{\boldsymbol{\rho}_S} \in [0,k+1]$.

Proof. It follows from Theorem 4.4 and the approximation properties of Π_S and Π_D (cf. Theorems 4.1 and 4.3), and those of $\boldsymbol{\mathcal{P}}_A$ (cf. (4.3)). $\square$

In addition, we know from (2.2) that $p_S = -\frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\sigma}_S)$, which suggests to define the following postprocessed approximation of the pressure:

$$p_{S,h} \;:=\; -\frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\sigma}_{S,h}) \quad \text{in} \quad \Omega_S\,, \tag{4.22}$$

and therefore

$$\|p_S - p_{S,h}\|_{0,\Omega_S} \;=\; \frac{1}{n}\,\|\mathrm{tr}\,(\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h})\,\|_{0,\Omega_S} \;\le\; \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}\,, \tag{4.23}$$

which, thanks to Theorem 4.5, gives the *a priori* error estimate for the pressure in the fluid as well.

5 Numerical results

In this section we present three numerical experiments illustrating the performance of the HDG method (3.3) introduced and analyzed in Section 3. We let N_{total} be the total number of degrees of freedom, and N_{comp} be the number of degrees of freedom effectively employed in the computations (involved in the resolution of the corresponding linear system). In other words, N_{total} is the total number of unknowns defining $\boldsymbol{\sigma}_{S,h}$, $\mathbf{u}_{S,h}$, $\boldsymbol{\rho}_{S,h}$, $\boldsymbol{\lambda}_{S,h}$, $\mathbf{u}_{D,h}$, $p_{D,h}$ and $\varphi_{D,h}$, whereas N_{comp} is the total number of unknowns defining $\boldsymbol{\lambda}_{S,h}$ and $\varphi_{D,h}$ plus one constant for each $T \in \mathcal{T}_h^S$, which take care of the condition $\int_{\Omega_S} \mathrm{tr}\,(\boldsymbol{\sigma}_S) = 0$ (see [29, Section 5] for details). Also, the individual errors are defined by

$$\begin{gathered}
\mathtt{e}(\boldsymbol{\sigma}_S) \;:=\; \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{0,\Omega_S}\,, \quad \mathtt{e}(\mathbf{u}_S) \;:=\; \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,\Omega_S}\,, \quad \mathtt{e}(\boldsymbol{\rho}_S) \;:=\; \|\boldsymbol{\rho}_S - \boldsymbol{\rho}_{S,h}\|_{0,\Omega_S}\,,\\
\mathtt{e}(\boldsymbol{\lambda}_S) \;:=\; \|\boldsymbol{P}_M(\mathbf{u}_S) - \boldsymbol{\lambda}_{S,h}\|_h\,, \quad \mathtt{e}(p_S) \;:=\; \|p_S - p_{S,h}\|_{0,\Omega_S}\,, \quad \mathtt{e}(\mathbf{u}_D) \;:=\; \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{0,\Omega_D}\,,\\
\mathtt{e}(p_D) \;:=\; \|p_D - p_{D,h}\|_{0,\Omega_D}\,, \quad \text{and} \quad \mathtt{e}(\varphi_D) \;:=\; \|P_N(p_D) - \varphi_{D,h}\|_h\,,
\end{gathered}$$

where $p_{S,h}$ is computed by the postprocessing formulae (4.22). Then, we define the experimental rates of convergence as

$$\mathtt{r}(\cdot) \;:=\; \frac{\log\big(\mathtt{e}(\cdot)\,/\,\mathtt{e}'(\cdot)\big)}{\log(h\,/\,h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the corresponding errors for two consecutive triangulations with mesh sizes h and h', respectively.

The examples to be considered in this section are described next. In all of them we choose $\nu = 1$, $\kappa_1 = \ldots = \kappa_{n-1} = 1$, $\mathbf{S}|_F = \mathbf{I}$ for all $F \in \mathcal{E}_h^S$, and $\tau|_F = 1$ in each $F \in \mathcal{E}_h^D$. Example 1 ($n = 2$) and 2 ($n = 3$) are used to illustrate the performance of the HDG scheme (3.3) and to corroborate the rates of convergence given in Theorem 4.5, when the solution is regular enough and the domains are convex. Example 3 ($n = 2$) is utilized to illustrate the behaviour of the same estimate for non convex domains and solutions with low regularity. We use $k \in \{1,2,3\}$ and $k \in \{1,2\}$ for the 2D and 3D numerical experiments, respectively. The numerical results presented below were obtained using a C^{++} code, which was developed following the same techniques from [7] (see also [29]).

In Example 1 we consider the regions $\Omega_S := (0,1)\times(0,1)$ and $\Omega_D := (0,1)\times(-1,0)$, $\mathbf{K} = \mathbf{I}$, and the data $\mathbf{f}_S$ and f_D are chosen so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}_S(\mathbf{x}) \;&=\; \mathbf{curl}\Big(x_1 x_2^2 (x_1 - 1)(x_2 - 1)\sin(\pi x_1)\sin(\pi x_2)\Big)\,,\\
p_S(\mathbf{x}) \;&=\; \cos(\pi x_1)\cos(\pi x_2)\,,
\end{aligned}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_S$, and

$$p_D(\mathbf{x}) \quad = \quad \cos(\pi x_1)\cos(\pi x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D\,,$$

where $\mathbf{curl}(v) := \left(\frac{\partial v}{\partial x_2},\ -\frac{\partial v}{\partial x_1}\right)^{\mathtt{t}}$. Concerning the triangulations employed in our computations, we first consider seven meshes that are Cartesian refinements of a domain defined in terms of squares, and then we split each square into four congruent triangles.

In Example 2 we consider $\Omega_S := (0,1)^2 \times \left(\frac{1}{2}, 1\right)$ and $\Omega_D := (0,1)^2 \times \left(0, \frac{1}{2}\right)$, $\mathbf{K} = \left(\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{smallmatrix}\right)$, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\begin{aligned} \mathbf{u}_S(\mathbf{x}) \quad &= \quad \mathrm{curl}\begin{pmatrix} x_1^2(1-x_1)^2 x_2^2(1-x_2)^2(1-x_3)^2(1-2x_3)^3\sin(\pi x_1) \\ x_1^2(1-x_1)^2 x_2^2(1-x_2)^2(1-x_3)^2(1-2x_3)^3\sin(\pi x_2) \\ x_1^2(1-x_1)^2 x_2^2(1-x_2)^2(1-x_3)^2(1-2x_3)^3\sin(\pi x_3) \end{pmatrix}, \\ p_S(\mathbf{x}) \quad &= \quad \cos(\pi x_2)\cos(\pi x_3)\exp(x_1)\,, \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2, x_3) \in \Omega_S$, and

$$p_D(\mathbf{x}) \quad = \quad x_1 x_2 x_3 (1-x_1)(1-x_2)(1-2x_3)^2 \sin(2\pi x_1)\sin(2\pi x_2)\sin(\pi x_3)\,,$$

for all $\mathbf{x} := (x_1, x_2, x_3) \in \Omega_D$.

Finally, in Example 3 we consider $\mathbf{K} = 5\,\mathbf{I}$, $\Omega_D := (-1,1) \times (-2,-1)$, and let Ω_S be the L-shaped domain given by $(-1,1)^2 \setminus [0,1]^2$. Then we choose $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\begin{aligned} \mathbf{u}_S(\mathbf{x}) \quad &= \quad \mathbf{curl}\left(3(x_1^2 + x_2^2)^{5/6}(x_2+1)^3\right)\,, \\ p_S(\mathbf{x}) \quad &= \quad \frac{1}{5}(x_1^3 - 3x_1)\cos(\pi x_2)\,, \end{aligned}$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_S$, and

$$p_D(\mathbf{x}) \quad = \quad \frac{1}{5}(x_1^3 - 3x_1)\cos(\pi x_2) \quad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_D\,.$$

Note that $\mathbf{u}_S$ is divergence free, $\int_{\Omega_S} p_S = 0$, and $\nabla \mathbf{u}_S$ has a singularity at the origin. In addition, it is easy to check that this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ, and the boundary condition $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D. However, the Dirichlet condition for the Stokes velocity on Γ_S is non-homogeneous ($\mathbf{u}_S = \mathbf{g} \neq \mathbf{0}$ on Γ_S). For that reason, we have modified our implementation to allow non-zero Dirichlet conditions. It is important to remark here that all the analysis in the previous sections can be extended straightforwardly to this case by eliminating the condition $\boldsymbol{\mu}|_{\Gamma_S} = \mathbf{0}$ in the definition of the subspace $\mathbf{M}_h$, and then considering in (3.3) the new equation $\langle \widehat{\mathbf{u}}_{S,h}, \boldsymbol{\mu}_{S,h} \rangle_{\Gamma_S} = \langle \mathbf{g}, \boldsymbol{\mu}_{S,h} \rangle_{\Gamma_S}\ \forall\ \boldsymbol{\mu}_{S,h} \in \mathbf{M}_h$.

In Tables 5.1$-$5.4 we summarize the convergence history of the HDG method (3.3) as applied to Examples 1 and 2. We observe there, looking at the experimental rates of convergence, that the orders predicted for each k by Theorem 4.5, are attained in all the unknowns for these smooth examples. In particular, $\|\boldsymbol{P}_M(\mathbf{u}_S) - \boldsymbol{\lambda}_{S,h}\|_h$ and $\|P_N(p_D) - \varphi_{D,h}\|_h$ present a superconvergence with an additional powers of h, also as predicted in Theorem 4.5.

On the other hand, in Tables 5.5$-$5.6 we summarize the convergence history of the HDG method (3.3) as applied to Example 3 for the polynomial degrees $k \in \{1,2,3\}$. In this case, and because of the singularity at the origin of the exact solution, the theoretical orders of convergence are far to be attained. In fact, it is easy to show that $\mathbf{u}_S$ belong to $\mathbf{H}^{4/3}(\Omega_S)$, whence $\boldsymbol{\sigma}_S \in \mathbb{H}^{2/3}(\Omega_S)$, which

k	h	N_{total}	N_{comp}	$\mathtt{e}(\boldsymbol{\sigma}_S)$	$\mathtt{r}(\boldsymbol{\sigma}_S)$	$\mathtt{e}(\mathbf{u}_S)$	$\mathtt{r}(\mathbf{u}_S)$	$\mathtt{e}(\boldsymbol{\rho}_S)$	$\mathtt{r}(\boldsymbol{\rho}_S)$	$\mathtt{e}(\boldsymbol{\lambda}_S)$	$\mathtt{r}(\boldsymbol{\lambda}_S)$	$\mathtt{e}(p_S)$	$\mathtt{r}(p_S)$
1	0.2000	8290	2091	2.00e-2	−−	4.15e-3	−−	1.70e-2	−−	1.42e-3	−−	8.56e-3	−−
	0.1000	32980	8181	5.42e-3	1.89	1.06e-3	1.97	4.83e-3	1.82	2.01e-4	2.81	2.39e-3	1.84
	0.0667	74070	18271	2.47e-3	1.94	4.74e-4	1.99	2.23e-3	1.90	6.23e-5	2.90	1.10e-3	1.91
	0.0500	131560	32361	1.41e-3	1.96	2.67e-4	1.99	1.28e-3	1.93	2.68e-5	2.93	6.30e-4	1.94
	0.0400	205450	50451	9.06e-4	1.97	1.71e-4	2.00	8.28e-4	1.95	1.39e-5	2.94	4.07e-4	1.95
	0.0333	295740	72541	6.32e-4	1.97	1.19e-4	2.00	5.79e-4	1.96	8.11e-6	2.96	2.85e-4	1.96
	0.0286	402430	98631	4.66e-4	1.98	8.74e-5	2.00	4.28e-4	1.97	5.14e-6	2.96	2.10e-4	1.97
2	0.2000	15435	3036	1.65e-3	−−	3.11e-4	−−	1.55e-3	−−	8.22e-5	−−	6.99e-4	−−
	0.1000	61470	11871	2.05e-4	3.01	3.99e-5	2.96	1.92e-4	3.01	5.00e-6	4.04	8.58e-5	3.03
	0.0667	138105	26506	6.04e-5	3.01	1.19e-5	2.98	5.66e-5	3.01	9.80e-7	4.02	2.53e-5	3.01
	0.0500	245340	46941	2.55e-5	3.00	5.04e-6	2.99	2.39e-5	3.00	3.09e-7	4.01	1.06e-5	3.01
	0.0400	383175	73176	1.30e-5	3.00	2.58e-6	2.99	1.22e-5	3.00	1.27e-7	4.00	5.43e-6	3.01
	0.0333	551610	105211	7.54e-6	3.00	1.50e-6	2.99	7.07e-6	3.00	6.10e-8	4.00	3.14e-6	3.00
	0.0286	750645	143046	4.75e-6	3.00	9.43e-7	3.00	4.45e-6	3.00	3.29e-8	4.00	1.98e-6	3.00
3	0.2000	24580	3981	1.08e-4	−−	2.10e-5	−−	9.98e-5	−−	4.05e-6	−−	4.27e-5	−−
	0.1000	97960	15561	7.14e-6	3.93	1.34e-6	3.97	6.63e-6	3.91	1.31e-7	4.95	2.79e-6	3.93
	0.0667	220140	34741	1.43e-6	3.96	2.66e-7	3.99	1.33e-6	3.95	1.75e-8	4.97	5.59e-7	3.97
	0.0500	391120	61521	4.58e-7	3.96	8.45e-8	3.99	4.27e-7	3.96	4.20e-9	4.97	1.78e-7	3.97
	0.0400	610900	95901	1.89e-7	3.96	3.47e-8	3.98	1.76e-7	3.96	1.39e-9	4.97	7.37e-8	3.96
	0.0333	879480	137881	9.18e-8	3.97	1.68e-8	3.98	8.56e-8	3.96	5.61e-10	4.96	3.58e-8	3.96
	0.0286	1196860	187461	4.99e-8	3.96	9.10e-9	3.98	4.65e-8	3.96	2.69e-10	4.77	1.94e-8	3.96

Table 5.1: History of convergence for Example 1 (Stokes variables).

implies that we can expect $\|\Pi\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S\|_{0,\Omega_S} = \mathcal{O}(h^{2/3})$. We use here that Π_S can also be defined for $\boldsymbol{\sigma}_S \in \mathbb{H}^\delta(\mathcal{T}_h^S)$ with $\delta > 1/2$. Thus, thanks to Theorem 4.5 and (4.23), we can explain the a priori estimates in Tables 5.5−5.6 for $\boldsymbol{\sigma}_S$, $\mathbf{u}_S$, $\boldsymbol{\rho}_S$, p_S, and also for $\|\boldsymbol{P}_M(\mathbf{u}_S) - \boldsymbol{\lambda}_{S,h}\|_h$ and $\|P_N(p_D) - \varphi_{D,h}\|_h$, which must converge with $\mathcal{O}(h^{1+2/3})$. In addition, the convergence of $\mathbf{u}_D$ and p_D is a bit faster than expected, which could correspond to a special feature of this example.

Finally, some components of the approximate solutions for the three examples are displayed in Figures 5.2, 5.3 and 5.4. They all correspond to those obtained with the fourth mesh and for the polynomial degree $k = 2$. Here we use the notations $\boldsymbol{\sigma}_{S,h} = (\,[\boldsymbol{\sigma}_{S,h}]_{ij}\,)_{i,j=1,\dots,n}$, $\boldsymbol{\rho}_{S,h} = (\,[\boldsymbol{\rho}_{S,h}]_{ij}\,)_{i,j=1,\dots,n}$, and $\mathbf{u}_{*,h} = (\,[\mathbf{u}_{*,h}]_i\,)_{i=1,\dots,n}$ for $* \in \{S, D\}$.

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k	h	N_{total}	N_{comp}	$\mathtt{e}(\mathbf{u}_D)$	$\mathtt{r}(\mathbf{u}_D)$	$\mathtt{e}(p_D)$	$\mathtt{r}(p_D)$	$\mathtt{e}(\varphi_D)$	$\mathtt{r}(\varphi_D)$
	0.2000	8290	2091	2.50e-2	−−	8.45e-3	−−	1.40e-3	−−
	0.1000	32980	8181	6.33e-3	1.98	2.12e-3	1.99	1.82e-4	2.94
	0.0667	74070	18271	2.82e-3	1.99	9.45e-4	2.00	5.46e-5	2.97
1	0.0500	131560	32361	1.59e-3	1.99	5.32e-4	2.00	2.32e-5	2.98
	0.0400	205450	50451	1.02e-3	2.00	3.40e-4	2.00	1.19e-5	2.99
	0.0333	295740	72541	7.08e-4	2.00	2.36e-4	2.00	6.90e-6	2.99
	0.0286	402430	98631	5.21e-4	2.00	1.74e-4	2.00	4.34e-6	3.00
	0.2000	15435	3036	1.08e-3	−−	3.51e-4	−−	4.18e-5	−−
	0.1000	61470	11871	1.36e-4	2.99	4.42e-5	2.99	2.62e-6	4.00
	0.0667	138105	26506	4.04e-5	3.00	1.31e-5	3.00	5.19e-7	4.00
2	0.0500	245340	46941	1.70e-5	3.00	5.53e-6	3.00	1.65e-7	3.98
	0.0400	383175	73176	8.73e-6	3.00	2.83e-6	3.00	6.79e-8	3.98
	0.0333	551610	105211	5.05e-6	3.00	1.64e-6	3.00	3.29e-8	3.98
	0.0286	750645	143046	3.19e-6	2.99	1.03e-6	3.00	1.78e-8	3.98
	0.2000	24580	3981	3.83e-5	−−	1.17e-5	−−	1.24e-6	−−
	0.1000	97960	15561	2.39e-6	4.00	7.34e-7	3.99	3.98e-8	4.96
	0.0667	220140	34741	4.75e-7	3.99	1.46e-7	3.99	5.43e-9	4.91
3	0.0500	391120	61521	1.50e-7	3.99	4.64e-8	3.98	1.32e-9	4.92
	0.0400	610900	95901	6.22e-8	3.96	1.91e-8	3.98	4.53e-10	4.79
	0.0333	879480	137881	3.02e-8	3.97	9.23e-9	3.98	1.93e-10	4.68
	0.0286	1196860	187461	1.65e-8	3.92	5.01e-9	3.96	9.53e-11	4.58

Table 5.2: History of convergence for Example 1 (Darcy variables).

k	h	N_{total}	N_{comp}	$\mathtt{e}(\boldsymbol{\sigma}_S)$	$\mathtt{r}(\boldsymbol{\sigma}_S)$	$\mathtt{e}(\mathbf{u}_S)$	$\mathtt{r}(\mathbf{u}_S)$	$\mathtt{e}(\boldsymbol{\rho}_S)$	$\mathtt{r}(\boldsymbol{\rho}_S)$	$\mathtt{e}(\boldsymbol{\lambda}_S)$	$\mathtt{r}(\boldsymbol{\lambda}_S)$	$\mathtt{e}(p_S)$	$\mathtt{r}(p_S)$
	0.4330	21696	5569	5.69e-2	−−	6.43e-3	−−	3.30e-2	−−	6.79e-3	−−	2.32e-2	−−
	0.2887	72360	17929	3.16e-2	1.45	2.88e-3	1.98	1.81e-2	1.49	2.41e-3	2.56	1.01e-2	2.05
	0.2165	170496	41473	1.82e-2	1.92	1.62e-3	1.99	1.13e-2	1.63	1.12e-3	2.67	5.66e-3	2.02
1	0.1732	331800	79801	1.18e-2	1.94	1.04e-3	1.99	7.70e-3	1.71	6.06e-4	2.74	3.60e-3	2.02
	0.1443	571968	136513	8.30e-3	1.94	7.25e-4	1.99	5.58e-3	1.76	3.65e-4	2.78	2.47e-3	2.08
	0.1237	906696	215209	6.13e-3	1.96	5.33e-4	1.99	4.23e-3	1.80	2.37e-4	2.81	1.81e-3	2.00
	0.1083	1351680	319489	4.71e-3	1.98	4.09e-4	1.99	3.32e-3	1.83	1.62e-4	2.83	1.40e-3	1.94
	0.4330	50688	10945	5.19e-3	−−	6.20e-4	−−	3.45e-3	−−	5.04e-4	−−	2.16e-3	−−
	0.2887	169344	35209	1.89e-3	2.49	1.86e-4	2.97	1.17e-3	2.67	1.27e-4	3.40	6.49e-4	2.97
	0.2165	399360	81409	8.81e-4	2.66	7.99e-5	2.93	5.31e-4	2.74	4.33e-5	3.75	2.86e-4	2.85
2	0.1732	777600	156601	4.71e-4	2.81	4.12e-5	2.97	2.86e-4	2.78	1.87e-5	3.76	1.48e-4	2.94
	0.1443	1340928	267841	2.77e-4	2.91	2.40e-5	2.97	1.72e-4	2.80	9.35e-6	3.80	8.65e-5	2.96
	0.1237	2126208	422185	1.76e-4	2.94	1.52e-5	2.97	1.12e-4	2.78	5.14e-6	3.89	5.46e-5	2.99
	0.1083	3170304	626689	1.18e-4	2.98	1.02e-5	2.99	7.79e-5	2.72	3.06e-6	3.89	3.68e-5	2.95

Table 5.3: History of convergence for Example 2 (Stokes variables).

k	h	N_{total}	N_{comp}	$\mathtt{e}(\mathbf{u}_D)$	$\mathtt{r}(\mathbf{u}_D)$	$\mathtt{e}(p_D)$	$\mathtt{r}(p_D)$	$\mathtt{e}(\varphi_D)$	$\mathtt{r}(\varphi_D)$
	0.4330	21696	5569	3.47e-3	−−	8.44e-4	−−	3.80e-4	−−
	0.2887	72360	17929	1.72e-3	1.73	4.27e-4	1.68	1.46e-4	2.36
	0.2165	170496	41473	1.01e-3	1.84	2.53e-4	1.82	7.35e-5	2.39
1	0.1732	331800	79801	6.62e-4	1.90	1.65e-4	1.89	4.13e-5	2.58
	0.1443	571968	136513	4.66e-4	1.93	1.16e-4	1.93	2.53e-5	2.70
	0.1237	906696	215209	3.45e-4	1.95	8.61e-5	1.95	1.60e-5	2.97
	0.1083	1351680	319489	2.66e-4	1.96	6.63e-5	1.96	1.07e-5	2.99
	0.4330	50688	10945	1.21e-3	−−	2.86e-4	−−	6.78e-5	−−
	0.2887	169344	35209	4.09e-4	2.68	9.57e-5	2.70	1.45e-5	3.81
	0.2165	399360	81409	1.82e-4	2.81	4.25e-5	2.82	4.78e-6	3.85
2	0.1732	777600	156601	9.57e-5	2.88	2.23e-5	2.89	2.01e-6	3.89
	0.1443	1340928	267841	5.62e-5	2.92	1.31e-5	2.92	9.76e-7	3.95
	0.1237	2126208	422185	3.56e-5	2.95	8.30e-6	2.95	5.31e-7	3.95
	0.1083	3170304	626689	2.40e-5	2.96	5.59e-6	2.96	3.14e-7	3.94

Table 5.4: History of convergence for Example 2 (Darcy variables).

k	h	N_{total}	N_{comp}	$\mathtt{e}(\boldsymbol{\sigma}_S)$	$\mathtt{r}(\boldsymbol{\sigma}_S)$	$\mathtt{e}(\mathbf{u}_S)$	$\mathtt{r}(\mathbf{u}_S)$	$\mathtt{e}(\boldsymbol{\rho}_S)$	$\mathtt{r}(\boldsymbol{\rho}_S)$	$\mathtt{e}(\boldsymbol{\lambda}_S)$	$\mathtt{r}(\boldsymbol{\lambda}_S)$	$\mathtt{e}(p_S)$	$\mathtt{r}(p_S)$
	0.0500	178040	43641	1.19e-1	−−	5.00e-2	−−	1.48e-1	−−	3.80e-3	−−	4.01e-2	−−
	0.0400	278050	68051	1.02e-1	0.68	4.27e-2	0.71	1.27e-1	0.69	2.79e-3	1.39	3.44e-2	0.69
	0.0333	400260	97861	9.01e-2	0.67	3.76e-2	0.69	1.12e-1	0.68	2.18e-3	1.36	3.04e-2	0.68
1	0.0286	544670	133071	8.13e-2	0.67	3.39e-2	0.68	1.01e-1	0.67	1.77e-3	1.34	2.74e-2	0.67
	0.0250	711280	173681	7.44e-2	0.67	3.09e-2	0.68	9.22e-2	0.67	1.48e-3	1.33	2.51e-2	0.67
	0.0222	900090	219691	6.88e-2	0.67	2.86e-2	0.68	8.52e-2	0.67	1.27e-3	1.32	2.32e-2	0.67
	0.0200	1111100	271101	6.41e-2	0.66	2.66e-2	0.67	7.94e-2	0.67	1.10e-3	1.31	2.16e-2	0.67
	0.0500	331860	63061	6.94e-2	−−	3.22e-2	−−	8.82e-2	−−	9.79e-4	−−	1.83e-2	−−
	0.0400	518325	98326	5.99e-2	0.66	2.78e-2	0.67	7.60e-2	0.67	7.00e-4	1.50	1.58e-2	0.66
	0.0333	746190	141391	5.31e-2	0.66	2.46e-2	0.67	6.73e-2	0.67	5.33e-4	1.49	1.40e-2	0.66
2	0.0286	1015455	192256	4.80e-2	0.66	2.22e-2	0.67	6.07e-2	0.67	4.25e-4	1.47	1.27e-2	0.66
	0.0250	1326120	250921	4.39e-2	0.66	2.03e-2	0.67	5.56e-2	0.67	3.50e-4	1.46	1.16e-2	0.66
	0.0222	1678185	317386	4.06e-2	0.66	1.88e-2	0.67	5.14e-2	0.67	2.95e-4	1.45	1.07e-2	0.66
	0.0200	2071650	391651	3.79e-2	0.66	1.75e-2	0.67	4.79e-2	0.67	2.53e-4	1.44	9.99e-3	0.66
	0.0500	528880	82481	5.01e-2	−−	2.31e-2	−−	5.99e-2	−−	4.27e-4	−−	1.26e-2	−−
	0.0400	826100	128601	4.32e-2	0.66	1.99e-2	0.66	5.16e-2	0.67	2.95e-4	1.65	1.08e-2	0.66
	0.0333	1189320	184921	3.83e-2	0.66	1.77e-2	0.66	4.57e-2	0.67	2.18e-4	1.65	9.59e-3	0.67
3	0.0286	1618540	251441	3.46e-2	0.66	1.59e-2	0.67	4.13e-2	0.67	1.69e-4	1.65	8.66e-3	0.67
	0.0250	2113760	328161	3.16e-2	0.66	1.46e-2	0.67	3.78e-2	0.67	1.36e-4	1.65	7.92e-3	0.67
	0.0222	2674980	415081	2.93e-2	0.66	1.35e-2	0.67	3.49e-2	0.67	1.12e-4	1.65	7.33e-3	0.67
	0.0200	3302200	512201	2.73e-2	0.66	1.26e-2	0.67	3.25e-2	0.67	9.40e-5	1.65	6.83e-3	0.67

Table 5.5: History of convergence for Example 3 (Stokes variables).

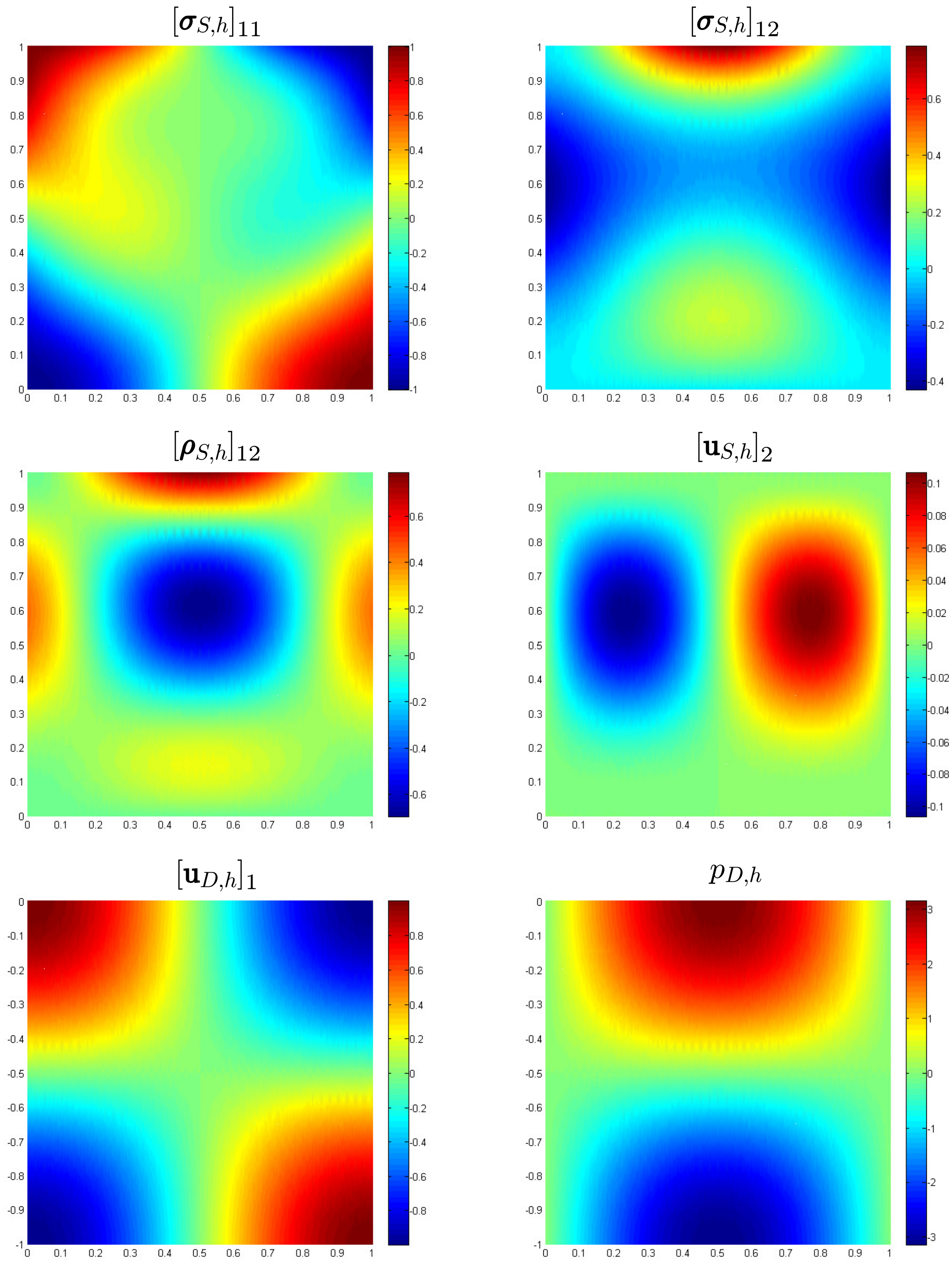

Figure 5.2: Example 1, some components of the approximate solutions.

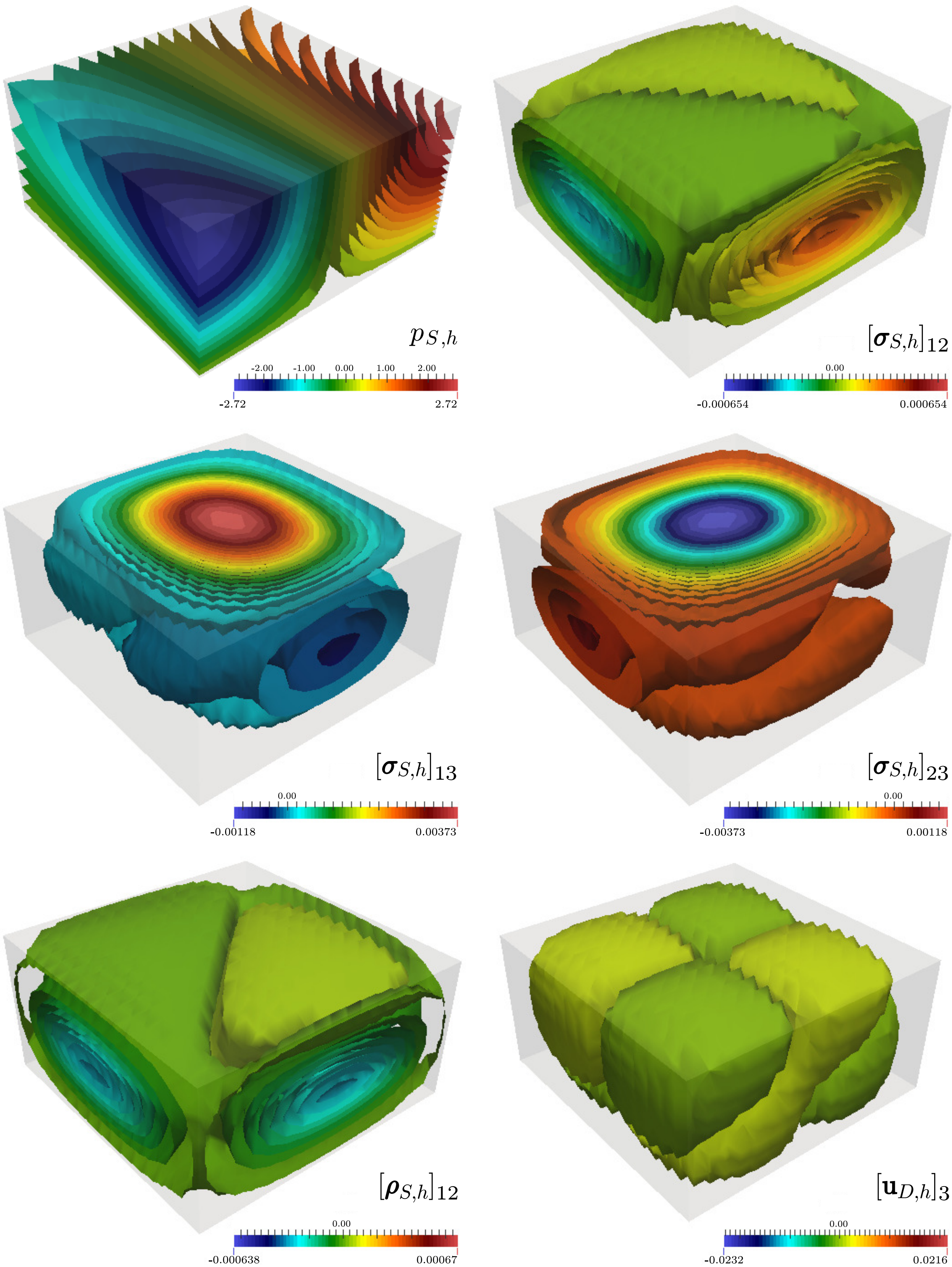

Figure 5.3: Example 2, iso-surfaces of some components of the approximate solutions.

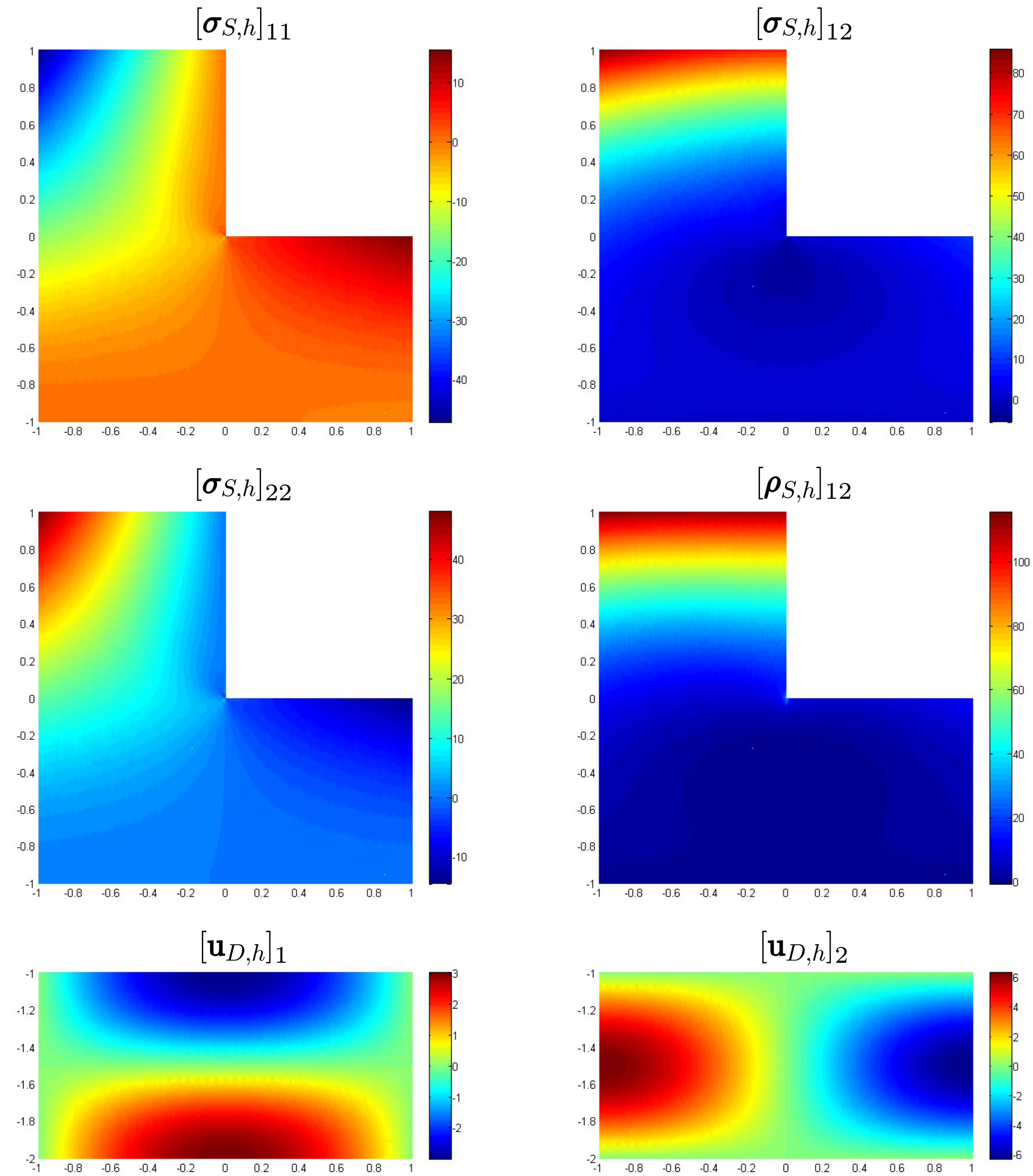

Figure 5.4: Example 3, some components of the approximate solutions.

k	h	N_{total}	N_{comp}	$\mathtt{e}(\mathbf{u}_D)$	$\mathtt{r}(\mathbf{u}_D)$	$\mathtt{e}(p_D)$	$\mathtt{r}(p_D)$	$\mathtt{e}(\varphi_D)$	$\mathtt{r}(\varphi_D)$
	0.0500	178040	43641	2.84e-3	$--$	2.98e-3	$--$	8.30e-3	$--$
	0.0400	278050	68051	1.82e-3	1.99	2.21e-3	1.32	6.37e-3	1.19
	0.0333	400260	97861	1.27e-3	1.98	1.75e-3	1.29	5.13e-3	1.19
1	0.0286	544670	133071	9.36e-4	1.98	1.44e-3	1.27	4.27e-3	1.19
	0.0250	711280	173681	7.19e-4	1.97	1.22e-3	1.26	3.64e-3	1.19
	0.0222	900090	219691	5.70e-4	1.97	1.05e-3	1.25	3.16e-3	1.19
	0.0200	1111100	271101	4.64e-4	1.96	9.22e-4	1.24	2.79e-3	1.19
	0.0500	331860	63061	6.38e-5	$--$	4.82e-4	$--$	1.50e-3	$--$
	0.0400	518325	98326	4.65e-5	1.42	3.72e-4	1.16	1.16e-3	1.15
	0.0333	746190	141391	3.68e-5	1.28	3.01e-4	1.17	9.36e-4	1.16
2	0.0286	1015455	192256	3.05e-5	1.22	2.51e-4	1.17	7.81e-4	1.17
	0.0250	1326120	250921	2.60e-5	1.18	2.15e-4	1.18	6.67e-4	1.18
	0.0222	1678185	317386	2.27e-5	1.16	1.87e-4	1.18	5.81e-4	1.18
	0.0200	2071650	391651	2.01e-5	1.15	1.65e-4	1.18	5.13e-4	1.18
	0.0500	528880	82481	5.12e-6	$--$	3.83e-5	$--$	1.19e-4	$--$
	0.0400	826100	128601	4.32e-6	0.76	3.08e-5	0.98	9.56e-5	0.98
	0.0333	1189320	184921	3.76e-6	0.77	2.55e-5	1.03	7.92e-5	1.03
3	0.0286	1618540	251441	3.34e-6	0.76	2.16e-5	1.06	6.73e-5	1.06
	0.0250	2113760	328161	3.03e-6	0.74	1.87e-5	1.08	5.82e-5	1.08
	0.0222	2674980	415081	2.78e-6	0.72	1.65e-5	1.10	5.11e-5	1.10
	0.0200	3302200	512201	2.58e-6	0.69	1.46e-5	1.11	4.55e-5	1.11

Table 5.6: History of convergence for Example 3 (Darcy variables).

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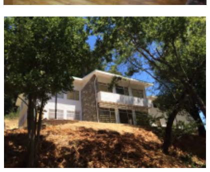

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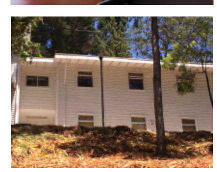

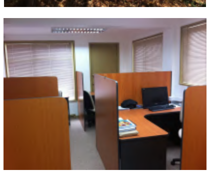

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