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Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem

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Analysis of an augmented mixed–primal formulation for the stationary Boussinesq problem*

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Abstract

In this paper we propose and analyze a new mixed variational formulation for the stationary Boussinesq problem. Our method, which employs a technique previously applied to the Navier-Stokes equations, is based first on the introduction of a modified pseudostress tensor depending nonlinearly on the velocity through the respective convective term. Next, the pressure is eliminated, and an augmented approach for the fluid flow, which incorporates Galerkin type terms arising from the constitutive and equilibrium equations, and from the Dirichlet boundary condition, is coupled with a primal-mixed scheme for the main equation modeling the temperature. In this way, the only unknowns of the resulting formulation are given by the aforementioned nonlinear pseudostress, the velocity, the temperature, and the normal derivative of the latter on the boundary. An equivalent fixed-point setting is then introduced and the corresponding classical Banach Theorem, combined with the Lax-Milgram Theorem and the Babuška-Brezzi theory, are applied to prove the unique solvability of the continuous problem. In turn, the Brouwer and the Banach fixed point theorems are utilized to establish existence and uniqueness of solution, respectively, of the associated Galerkin scheme. In particular, Raviart-Thomas spaces of order k for the pseudostress, continuous piecewise polynomials of degree $\leq k+1$ for the velocity and the temperature, and piecewise polynomials of degree $\leq k$ for the boundary unknown become feasible choices. Finally, we derive optimal a priori error estimates, and provide several numerical results illustrating the good performance of the augmented mixed-primal finite element method and confirming the theoretical rates of convergence.

Key words: Boussinesq equations, augmented mixed–primal formulation, fixed point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

The devising of suitable numerical methods for solving the Boussinesq equations and its generalizations, such as temperature-dependent coefficient problems, has become a very active research area in recent years (see, e.g. [2, 4, 13, 15, 16, 26, 28, 31], and the references therein). This fact has

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been motivated by its diverse applications in industry (fume cupboard ventilation, heat exchangers, cooling of electronic equipments, cooling of nuclear reactors, etc.), and in geophysics or oceanography (climate predictions, oceanic flows, etc.), to name a few. The Bousisnesq model, also known as natural-convection flow model, is a system of equations modelling incompressible non-isothermal fluid flows. They couple the stationary incompressible Navier-Stokes equations for the fluid variables (velocity and pressure) with a convection-diffusion equation for the temperature variable. The coupling is through a buoyancy term typically acting in direction opposite to gravity and through the convective term in the convection-diffusion equation.

Up to the authors' knowledge, one of the first works in analyzing a finite element discretization for the Boussinesq equations is [4]. In that work, the authors provide a complete analysis of a primal formulation for the coupled problem, in which the main unknowns are the velocity, the pressure and the temperature of the fluid. At the continuous level it is proved existence of at least one weak solution, and uniqueness of solution under a smallness assumption on the data. On the other hand, at the discrete level it is established suitable assumptions on the finite elements subspaces ensuring that the associated Galerkin scheme is well posed and convergent. In particular, the use of any pair of stable Stokes elements for the fluid variables and Lagrange elements for the temperature leads to a convergent scheme. Later on, a new mixed formulation for the two-dimensional Boussinesq equations has been introduced and analyzed in [15], where the gradient of the velocity and the gradient of the temperature are set as further unknowns (besides the velocity, pressure and temperature). The corresponding mixed finite element scheme employs Raviart-Thomas elements of lowest order for the gradients and piecewise constants for the velocity, temperature and pressure. Existence of solution and convergence of the numerical scheme are proved near a nonsingular solution and quasi-optimal error estimates are provided.

The purpose of the present work is to contribute in the development of new numerical methods for the Boussinesq problem allowing, on the one hand, optimal convergence, and on the other hand, the possibility of computing further variables of interest, such as the vorticity and the gradient of the velocity, as a postprocess of the discrete solutions. In this direction, a new optimally convergent augmented-mixed finite element method for the Navier-Stokes equation has been recently developed in [10] (see also [6], [7], [8], [25] for related works). This method, which extends recent results on pseudostress-based formulations for the Stokes problem (see e.g. [9], [17], [21], [22], [24], and the references therein), consists in a new formulation for the Navier-Stokes problem with Dirichlet boundary conditions, where the main unknowns are the velocity and the so called nonlinear pseudostress tensor depending nonlinearly on the velocity through the respective convective term. The pressure is eliminated by using the incompressibility condition, and can be recovered as a simple postprocess of the nonlinear pseudostress tensor, as well as the vorticity and the gradient of the fluid. Due to the presence of the convective term in the system, the velocity is kept in H^1, which leads to the incorporation of Galerkin type terms arising from the constitutive and equilibrium equations, and from the Dirichlet boundary condition, into the variational formulation. The introduction of these terms allows to circumvent the necessity of proving inf-sup conditions, and as a result, to relax the hypotheses on the corresponding discrete subspaces (see for instance [5], [18] and [19] for the foundations of this procedure). In this way, the classical Banach's fixed point Theorem and Lax-Milgram's Lemma can be applied to prove existence and uniqueness of solution of the continuous and discrete problems.

According to the above discussion, in the present paper we employ the augmented-mixed formulation introduced in [10] for the Navier-Stokes equations, which is coupled with a primal-mixed scheme for the convection-diffusion equation modelling the temperature, and introduce a new augmented mixed-primal variational formulation for the Boussinesq equations, which yields the aforementioned nonlinear pseudostress, the velocity, the temperature, and the normal derivative of the latter on the boundary as the main unknowns of the resulting formulation. Next, following basically the approach from [3] for a related coupled flow-transport problem, we introduce an equivalent fixed-point setting, and then apply the classical Banach Theorem combined with the Lax-Milgram Theorem and the Babuška-Brezzi theory, to prove the unique solvability of the continuous problem for sufficiently small data. Analogously, we apply a fixed-point argument and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. In particular, Raviart-Thomas spaces of order k for the nonlinear pseudostress, continuous piecewise polynomials of degree $\leq k+1$ for the velocity and the temperature, and piecewise polynomials of degree $\leq k$ for the boundary unknown become feasible choices. These choices of finite elements subspaces yield optimally convergent Galerkin schemes.

Outline

We have organized the contents of this paper as follows. The remainder of this section introduces some standard notation and functional spaces. In Section 2 we introduce the model problem written in terms of the velocity, pressure and temperature. Then, utilizing the incompressibility condition, we eliminate the pressure and rewrite the equations equivalently in terms of the nonlinear pseudostress, velocity and temperature. In Section 3 we derive the augmented mixed-primal variational formulation, clearly justifying the necessity of augmentation, and analyze its well-posedness under a smallness assumption on the data. Next, in Section 4 we define the Galerkin scheme, and derive general hypotheses on the finite element subspaces ensuring that the discrete scheme becomes well posed. Here we apply the Brouwer theorem to prove existence of solution whereas the Banach fixed point theorem is utilized to prove uniqueness of solution. In addition, suitable choices of finite element subspaces satisfying these assumptions are introduced in Section 4.3. In Section 5 we provide the corresponding Cea's estimate and establish the rate of convergence associated to the finite element subspaces defined in Section 4.3. Finally, in Section 6 we provide several numerical results illustrating the performance of the augmented mixed-primal finite element method and confirming the theoretical rates of convergence.

Preliminaries

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, a given bounded domain with polyhedral boundary Γ, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space.

We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\right\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathrm{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega},$$

is a Hilbert space. As usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. Also, for any vector fields $\boldsymbol{v} = (v_i)_{i=1,n}$ and $\boldsymbol{w} = (w_i)_{i=1,n}$ we set the gradient, divergence, and tensor product operators, as

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \boldsymbol{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \boldsymbol{v} \otimes \boldsymbol{w} := (v_i\, w_j)_{i,j=1,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\, \boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau})\, \mathbb{I}.$$

2 The model problem

We consider the stationary Boussinesq problem given by

$$\begin{aligned} -\mu\, \Delta \boldsymbol{u} + (\nabla \boldsymbol{u})\, \boldsymbol{u} + \nabla p - \boldsymbol{g}\, \varphi &= 0 && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in} \quad \Omega, \\ -\operatorname{div}(\mathbb{K}\, \nabla \varphi) + \boldsymbol{u} \cdot \nabla \varphi &= 0 && \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on} \quad \Gamma, \\ \varphi &= \varphi_D && \text{on} \quad \Gamma, \end{aligned} \tag{2.1}$$

where the unknowns are the velocity $\boldsymbol{u}$, the pressure p, and the temperature φ of a fluid occupying the region Ω. The given data are the fluid viscosity $\mu > 0$, the external force per unit mass $\boldsymbol{g} \in \mathbf{L}^\infty(\Omega)$, the boundary velocity $\boldsymbol{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, the boundary temperature $\varphi_D \in \mathrm{H}^{1/2}(\Gamma)$, and a uniformly positive definite tensor $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ describing the thermal conductivity. Note that $\boldsymbol{u}_D$ must satisfy the compatibility condition

$$\int_\Gamma \boldsymbol{u}_D \cdot \boldsymbol{\nu} = 0, \tag{2.2}$$

which comes from the incompressibility condition of the fluid. Uniqueness of a pressure solution of (2.1), (see e.g. [28]), is ensured in the space $\mathrm{L}_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$. We now introduce the auxiliary tensor unknown

$$\boldsymbol{\sigma} := \mu\, \nabla \boldsymbol{u} - (\boldsymbol{u} \otimes \boldsymbol{u}) - p\, \mathbb{I} \quad \text{in} \quad \Omega, \tag{2.3}$$

and realize that the first equation in (2.1) can be rewritten as

$$-\, \mathbf{div}\, \boldsymbol{\sigma} - \boldsymbol{g}\, \varphi = 0 \quad \text{in} \quad \Omega. \tag{2.4}$$

Moreover, it is easy to see that (2.3) together with the incompressibility condition given by the second equation in (2.1) are equivalent to the pair of equations

$$\begin{aligned} \mu\, \nabla \boldsymbol{u} - (\boldsymbol{u} \otimes \boldsymbol{u})^{\mathtt{d}} &= \boldsymbol{\sigma}^{\mathtt{d}} \quad \text{in} \quad \Omega, \\ p &= -\frac{1}{n} \operatorname{tr}(\boldsymbol{\sigma} + \boldsymbol{u} \otimes \boldsymbol{u}) \quad \text{in} \quad \Omega. \end{aligned} \tag{2.5}$$

Consequently, we can eliminate the pressure unknown (which can be approximated later on by the postprocessed formula suggested by the second equation of (2.5)), and arrive at the following system of equations with unknowns $\boldsymbol{u}$, $\boldsymbol{\sigma}$, and φ

$$\begin{aligned} \mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathtt{d}} &= \boldsymbol{\sigma}^{\mathtt{d}} && \text{in} \quad \Omega\,,\\ -\,\mathbf{div}\,\boldsymbol{\sigma} - \boldsymbol{g}\,\varphi &= 0 && \text{in} \quad \Omega\,,\\ -\,\mathrm{div}(\mathbb{K}\,\nabla\varphi) + \boldsymbol{u}\cdot\nabla\varphi &= 0 && \text{in} \quad \Omega\,,\\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on} \quad \Gamma\,,\\ \varphi &= \varphi_D && \text{on} \quad \Gamma\,,\\ \int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u}) &= 0\,. \end{aligned} \tag{2.6}$$

Note that the incompressibility of the fluid is implicitly present in the new constitutive equation relating $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ (first equation of (2.6)). In turn, the fact that the pressure p must belong to $\mathrm{L}^2_0(\Omega)$ (for uniqueness reasons) is guaranteed by the equivalent statement given by the last equation of (2.6).

3 The continuous formulation

3.1 The augmented mixed–primal formulation

In what follows, we derive a weak formulation of problem (2.6). We start by recalling (see e.g. [5], [20]) that there holds

$$\mathbb{H}(\mathbf{div};\Omega) \,=\, \mathbb{H}_0(\mathbf{div};\Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}\,, \tag{3.1}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega) \,:=\, \left\{ \zeta \,\in\, \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\zeta) \,=\, 0 \right\}.$$

More precisely, for each $\zeta \in \mathbb{H}(\mathbf{div};\Omega)$ there exists a unique $\zeta_0 \,:=\, \zeta - \left(\dfrac{1}{n\,|\Omega|} \displaystyle\int_\Omega \mathrm{tr}(\zeta) \right) \mathbb{I} \,\in\, \mathbb{H}_0(\mathbf{div};\Omega)$ and $c \,:=\, \dfrac{1}{n\,|\Omega|} \displaystyle\int_\Omega \mathrm{tr}(\zeta) \,\in\, \mathrm{R}$, such that

$$\zeta \,=\, \zeta_0 \,+\, c\,\mathbb{I}\,. \tag{3.2}$$

In particular, the eventual solution $\boldsymbol{\sigma}$ in (2.6) can be descomposed as $\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 \,+\, c\,\mathbb{I}$ where $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$ and, according to the last equation in (2.6), c is given explicity en terms of $\boldsymbol{u}$ as

$$c \,=\, -\frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{u})\,. \tag{3.3}$$

Hence, since $\boldsymbol{\sigma}^{\mathtt{d}} \,=\, \boldsymbol{\sigma}_0^{\mathtt{d}}$ and $\mathbf{div}\,\boldsymbol{\sigma} \,=\, \mathbf{div}\,\boldsymbol{\sigma}_0$, throughout the rest of the paper we rename $\boldsymbol{\sigma}_0$ as $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$ and observe that the first and second equations of (2.6) remain unchanged. In this way, multiplying the constitutive equation by a test function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$ and using the Dirichlet condition for $\boldsymbol{u}$, we get

$$\int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \mu \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} \,+\, \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \,=\, \mu\,\langle\,\boldsymbol{\tau}\boldsymbol{\nu}\,,\,\boldsymbol{u}_D\,\rangle_\Gamma \quad \forall\,\boldsymbol{\tau} \,\in\, \mathbb{H}(\mathbf{div};\Omega)\,, \tag{3.4}$$

where $\langle \cdot , \cdot \rangle_\Gamma$ stands for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Note, however, that (3.4) is actually satisfied in advance for $\boldsymbol{\tau} = d\,\mathbb{I}$ with $d \in \mathrm{R}$, since in this case all the terms appearing there vanish. In particular, the compatibility condition (2.2) explains this fact for the term on the right hand side of (3.4). According to this and the decomposition (3.1), we realize that (3.4), which is the weak form of the constitutive equation, reduces, equivalently, to

$$\int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \mu \int_\Omega \boldsymbol{u} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u})^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} = \mu \langle \boldsymbol{\tau\nu} , \boldsymbol{u}_D \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) . \tag{3.5}$$

In turn, the equilibrium equation given by the second equation of (2.6) can be rewritten as

$$- \mu \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\, \boldsymbol{\sigma} - \mu \int_\Omega \varphi\, \boldsymbol{g} \cdot \boldsymbol{v} = 0 \quad \forall\, \boldsymbol{v} \in \mathbf{L}^2(\Omega) . \tag{3.6}$$

On the other hand, regarding the heat equation modelling φ, we multiply the third equation of (2.6) by $\psi \in \mathrm{H}^1(\Omega)$, integrate by parts and introduce, as a new unknown, the normal component of the temperature flux, that is $\lambda := -\mathbb{K}\, \nabla\varphi \cdot \boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$, so that we get

$$\int_\Omega \mathbb{K}\, \nabla\varphi \cdot \nabla\psi + \langle \lambda, \psi \rangle_\Gamma + \int_\Omega (\boldsymbol{u} \cdot \nabla\varphi)\, \psi = 0 \quad \forall\, \psi \in \mathrm{H}^1(\Omega) . \tag{3.7}$$

Finally, the Dirichlet condition $\varphi = \varphi_D$ on Γ is imposed weakly as

$$\langle \xi, \varphi \rangle_\Gamma = \langle \xi, \varphi_D \rangle_\Gamma \quad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma) . \tag{3.8}$$

Before continuing we observe that the third terms on the left hand sides of (3.4) and (3.7) require the unknown $\boldsymbol{u}$ to live in a smaller space than $\mathbf{L}^2(\Omega)$. Indeed, by applying Cauchy-Schwarz and Hölder inequalities, and then the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ (cf. [1, Theorem 4.12], [29, Theorem 1.3.4]), we find that there exist positive constants $c_1(\Omega)$ and $c_2(\Omega)$, such that

$$\left| \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{w})^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \right| \le c_1(\Omega)\, \|\boldsymbol{u}\|_{1,\Omega}\, \|\boldsymbol{w}\|_{1,\Omega}\, \|\boldsymbol{\tau}\|_{0,\Omega} \quad \forall\, \boldsymbol{u}, \boldsymbol{w} \in \mathbf{H}^1(\Omega) , \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) , \tag{3.9}$$

and

$$\left| \int_\Omega (\boldsymbol{u} \cdot \nabla\varphi)\, \psi \right| \le c_2(\Omega)\, \|\boldsymbol{u}\|_{1,\Omega}\, \|\psi\|_{1,\Omega}\, |\varphi|_{1,\Omega} \quad \forall\, \boldsymbol{u} \in \mathbf{H}^1(\Omega), \quad \forall\, \varphi, \psi \in \mathrm{H}^1(\Omega) . \tag{3.10}$$

According to the above, and in order to be able to analyze the present variational formulation of (2.6), we now augment (3.5) - (3.8) through the incorporation of the following redundant Galerkin terms

$$\begin{aligned} \kappa_1 \int_\Omega \Big(\mu \nabla \boldsymbol{u} - (\boldsymbol{u} \otimes \boldsymbol{u})^{\mathtt{d}} - \boldsymbol{\sigma}^{\mathtt{d}} \Big) : \nabla \boldsymbol{v} &= 0 && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega) , \\ \kappa_2 \int_\Omega \mathbf{div}\, \boldsymbol{\sigma} \cdot \mathbf{div}\, \boldsymbol{\tau} + \kappa_2 \int_\Omega \varphi\, \boldsymbol{g} \cdot \mathbf{div}\, \boldsymbol{\tau} &= 0 && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) , \\ \kappa_3 \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v} &= \kappa_3 \int_\Gamma \boldsymbol{u}_D \cdot \boldsymbol{v} && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega) , \end{aligned} \tag{3.11}$$

where κ_1, κ_2 and κ_3 are positive parameters to be specified later. Note that the identities required in (3.11) are nothing but the constitutive and the equilibrium equations along with the Dirichlet condition for the velocity, but all them tested differently from (3.5) - (3.6). In this way, we arrive at the following augmented mixed-primal formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}, \varphi, \lambda) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) + \mathbf{B}_{\boldsymbol{u}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) &= F_\varphi(\boldsymbol{\tau},\boldsymbol{v}) + F_D(\boldsymbol{\tau},\boldsymbol{v}), \\ \mathbf{a}(\varphi,\psi) + \mathbf{b}(\psi,\lambda) &= F_{\boldsymbol{u},\varphi}(\psi), \\ \mathbf{b}(\varphi,\xi) &= G(\xi), \end{aligned} \tag{3.12}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, \psi, \xi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, where the forms $\mathbf{A}$, $\mathbf{B}_{\boldsymbol{w}}$, $\mathbf{a}$, and $\mathbf{b}$ are defined, respectively, as

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) := \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : (\boldsymbol{\tau}^{\mathrm{d}} - \kappa_1 \nabla \boldsymbol{v}) + \int_\Omega (\mu \boldsymbol{u} + \kappa_2 \,\mathbf{div}\,\boldsymbol{\sigma}) \cdot \mathbf{div}\,\boldsymbol{\tau} \\ - \mu \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\sigma} + \mu \kappa_1 \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + \kappa_3 \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v}, \end{aligned} \tag{3.13}$$

$$\mathbf{B}_{\boldsymbol{w}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) := -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{w})^{\mathrm{d}} : \left(\kappa_1 \nabla \boldsymbol{v} - \boldsymbol{\tau}^{\mathrm{d}}\right), \tag{3.14}$$

$$\mathbf{a}(\varphi,\psi) := \int_\Omega \mathbb{K} \nabla \varphi \cdot \nabla \psi, \tag{3.15}$$

and

$$\mathbf{b}(\psi,\xi) := \langle \xi, \psi \rangle_\Gamma, \tag{3.16}$$

for all $(\boldsymbol{\sigma},\boldsymbol{u}), (\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$, for all $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$, for all $\varphi, \psi \in \mathrm{H}^1(\Omega)$, and for all $\xi \in \mathrm{H}^{-1/2}(\Gamma)$. Note that $\mathbf{A}$, $\mathbf{B}_{\boldsymbol{w}}$ (with a given $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$), $\mathbf{a}$, and $\mathbf{b}$ are bilinear. In turn, F_φ (with a given $\varphi \in \mathrm{H}^1(\Omega)$), F_D, $F_{\boldsymbol{u},\varphi}$ (with a given $(\boldsymbol{u},\varphi) \in \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$), and G are the bounded linear functionals defined by

$$F_\varphi(\boldsymbol{\tau},\boldsymbol{v}) := \int_\Omega \varphi\, \mathbf{g} \cdot \left(\mu \boldsymbol{v} - \kappa_2 \,\mathbf{div}\,\boldsymbol{\tau}\right) \quad \forall (\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega), \tag{3.17}$$

$$F_D(\boldsymbol{\tau},\boldsymbol{v})) := \kappa_3 \int_\Gamma \boldsymbol{u}_D \cdot \boldsymbol{v} + \mu \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_D \rangle_\Gamma \quad \forall (\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega), \tag{3.18}$$

$$F_{\boldsymbol{u},\varphi}(\psi) := -\int_\Omega (\boldsymbol{u} \cdot \nabla \varphi)\, \psi \quad \forall \psi \in \mathrm{H}^1(\Omega), \tag{3.19}$$

and

$$G(\xi) := \langle \xi, \varphi_D \rangle_\Gamma \quad \forall \xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{3.20}$$

The well-posedness of (3.12) is addressed below in Sections 3.2, 3.3, and 3.4 by applying the fixed point approach that is explained next. We only remark in advance that it aims to decouple the primal unknowns given by the velocity $\boldsymbol{u}$ and the temperature φ, through the introduction of two uncoupled linear problems.

3.2 A fixed point approach

We now describe our fixed-point strategy to solve (3.12). We start by denoting $\mathbf{H} := \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and defining the operator $\mathbf{S} : \mathbf{H} \longrightarrow \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ by

$$\mathbf{S}(\boldsymbol{w},\phi) := (\mathbf{S}_1(\boldsymbol{w},\phi), \mathbf{S}_2(\boldsymbol{w},\phi)) = (\boldsymbol{\sigma},\boldsymbol{u}) \quad \forall (\boldsymbol{w},\phi) \in \mathbf{H}, \tag{3.21}$$

where $(\boldsymbol{\sigma}, \boldsymbol{u})$ is the unique solution of the problem: Find $(\boldsymbol{\sigma},\, \boldsymbol{u}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ such that

$$\mathbf{A}(\,(\boldsymbol{\sigma},\boldsymbol{u})\,,\,(\boldsymbol{\tau},\boldsymbol{v})\,) \,+\, \mathbf{B}_{\boldsymbol{w}}(\,(\boldsymbol{\sigma},\boldsymbol{u})\,,\,(\boldsymbol{\tau},\boldsymbol{v})\,) \,=\, \big(F_\phi + F_D\big)(\boldsymbol{\tau},\boldsymbol{v})\,, \tag{3.22}$$

for all $(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$. Here, the form $\mathbf{A}$ and the functional F_D are defined exactly as in (3.13) and (3.18), respectively. In turn, the bilinear form $\mathbf{B}_{\boldsymbol{w}}(\,\cdot,\cdot\,)$ and the linear functional F_ϕ are given by (3.14) and (3.17) (with ϕ instead of φ), respectively.

In addition, we also introduce the operator $\widetilde{\mathbf{S}} : \mathbf{H} \longrightarrow \mathrm{H}^1(\Omega)$ defined as

$$\widetilde{\mathbf{S}}(\boldsymbol{w},\phi) \,:=\, \varphi \quad \forall\,(\boldsymbol{w},\phi) \in \mathbf{H}\,, \tag{3.23}$$

where $\varphi \in \mathrm{H}^1(\Omega)$ is the first component of the unique solution of the problem: Find $(\varphi,\lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \mathbf{a}(\varphi,\,\psi) + \mathbf{b}(\psi,\,\lambda) &= F_{\boldsymbol{w},\phi}(\psi) \quad \forall\,\psi \in \mathrm{H}^1(\Omega) \\ \mathbf{b}(\varphi\,,\xi) &= G(\xi) \quad \forall\,\xi \in \mathrm{H}^{-1/2}(\Gamma)\,, \end{aligned} \tag{3.24}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the forms introduced in (3.15) - (3.16) and $F_{\boldsymbol{w},\phi}$ is defined by (3.19).

In this way, by introducing the operator $\mathbf{T} : \mathbf{H} \longrightarrow \mathbf{H}$ as

$$\mathbf{T}(\boldsymbol{w},\phi) \,:=\, (\mathbf{S}_2(\boldsymbol{w},\phi), \widetilde{\mathbf{S}}(\mathbf{S}_2(\boldsymbol{w},\phi),\phi)) \quad \forall\,(\boldsymbol{w},\phi) \in \mathbf{H}\,, \tag{3.25}$$

we realize that (3.12) can be rewritten as the fixed-point problem: Find $(\boldsymbol{u},\varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\boldsymbol{u},\varphi) \,=\, (\boldsymbol{u},\varphi)\,. \tag{3.26}$$

This fact certainly requires that both operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ be well defined. In other words, we first need to analyze the well-posedness of the uncoupled problems (3.22) and (3.24), which is precisely what we carry out in the following section.

3.3 Well-posedness of the uncoupled problems

We begin by recalling the following lemmas which are useful to prove ellipticity properties.

Lemma 3.1 *There exists $c_3(\Omega) > 0$ such that*

$$c_3(\Omega)\,\|\boldsymbol{\tau}_0\|_{0,\Omega}^2 \,\le\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall \boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 \,+\, c\mathbb{I} \,\in\, \mathbb{H}(\mathbf{div};\Omega),$$

Proof. See [5, Proposition 3.1]. □

Lemma 3.2 *There exists $c_4(\Omega) > 0$ such that*

$$|\boldsymbol{v}|_{1,\Omega}^2 \,+\, \|\boldsymbol{v}\|_{0,\Gamma}^2 \,\ge\, c_4(\Omega)\,\|\boldsymbol{v}\|_{1,\Omega}^2 \quad \forall\,\boldsymbol{v} \,\in\, \mathbf{H}^1(\Omega).$$

Proof. See [17, Lemma 3.3]. □

The next result provides conditions under which the operator $\mathbf{S}$ in (3.21) is well-defined, or equivalently, the problem (3.22) is well-posed.

Lemma 3.3 *Assume that $\kappa_1 \in (0, 2\delta)$ with $\delta \in (0, 2\mu)$, and $\kappa_2, \kappa_3 > 0$. Then, there exists $r_0 > 0$ such that for each $r \in (0, r_0)$, the problem (3.22) has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}) := \mathbf{S}(\boldsymbol{w}, \phi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ for each $(\boldsymbol{w}, \phi) \in \mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega} \le r$. Moreover, there exists a constant $c_{\mathbf{S}} > 0$, independent of $(\boldsymbol{w}, \phi)$, such that there holds*

$$\|\mathbf{S}(\boldsymbol{w}, \phi)\| = \|(\boldsymbol{\sigma}, \boldsymbol{u})\| \le c_{\mathbf{S}} \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} \|\phi\|_{0,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} . \tag{3.27}$$

Proof. For a given $\boldsymbol{w}$ in $\mathbf{H}^1(\Omega)$, we observe from (3.14) that $\mathbf{B}_{\boldsymbol{w}}$ is clearly a bilinear form. Also, from Cauchy-Schwarz's inequality and the trace theorem with constant $c_0(\Omega)$, we get

$$\begin{aligned}
|\mathbf{A}(\,(\boldsymbol{\sigma}, \boldsymbol{u})\,,\,(\boldsymbol{\tau}, \boldsymbol{v})\,)| \,\le\, & \|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \kappa_1 \|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} |\boldsymbol{v}|_{1,\Omega} + \mu \|\boldsymbol{u}\|_{0,\Omega} \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \\
+ \;& \kappa_2 \|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega} \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} + \mu \|\boldsymbol{v}\|_{0,\Omega} \|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega} \\
+ \;& \mu\kappa_1 |\boldsymbol{u}|_{1,\Omega} |\boldsymbol{v}|_{1,\Omega} + c_0(\Omega)\,\kappa_3 \|\boldsymbol{u}\|_{0,\Gamma} \|\boldsymbol{v}\|_{0,\Omega} ,
\end{aligned}$$

whereas, utilizing the estimation (3.9), we deduce that for all $(\boldsymbol{\sigma}, \boldsymbol{u})$, $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ there holds

$$|\mathbf{B}_{\boldsymbol{w}}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v}))| \le c_1(\Omega)\,(\kappa_1^2 + 1)^{1/2} \|\boldsymbol{w}\|_{1,\Omega} \|\boldsymbol{u}\|_{1,\Omega} \|(\boldsymbol{\tau}, \boldsymbol{v})\| . \tag{3.28}$$

It follows from the foregoing inequalities that there exists a positive constant, denoted by $\|\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}\|$, and depending on μ, κ_1, κ_2, $\kappa_3, c_0(\Omega), c_1(\Omega)$, and $\|\boldsymbol{w}\|_{1,\Omega}$, such that

$$|\mathbf{A}(\,(\boldsymbol{\sigma}, \boldsymbol{u})\,,\,(\boldsymbol{\tau}, \boldsymbol{v})\,) + \mathbf{B}_{\boldsymbol{w}}(\,(\boldsymbol{\sigma}, \boldsymbol{u})\,,\,(\boldsymbol{\tau}, \boldsymbol{v})\,)| \le \|\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}\| \, \|(\boldsymbol{\sigma}, \boldsymbol{u})\| \, \|(\boldsymbol{\tau}, \boldsymbol{v})\| \tag{3.29}$$

for all $(\boldsymbol{\sigma}, \boldsymbol{u})\,,\,(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$. In turn, we have from (3.13) that

$$\mathbf{A}(\,(\boldsymbol{\tau}, \boldsymbol{v})\,,\,(\boldsymbol{\tau}, \boldsymbol{v})\,) = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 - \kappa_1 \int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla \boldsymbol{v} + \kappa_2 \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 + \mu\,\kappa_1 |\boldsymbol{v}|_{1,\Omega}^2 + \kappa_3 \|\boldsymbol{v}\|_{0,\Gamma}^2 ,$$

which, using the Cauchy-Schwarz and Young inequalities, and then Lemmas 3.1 and 3.2, yields for any $\delta > 0$ and for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$,

$$\begin{aligned}
\mathbf{A}(\,(\boldsymbol{\tau}, \boldsymbol{v})\,,\,(\boldsymbol{\tau}, \boldsymbol{v})\,) &\ge \Big(1 - \frac{\kappa_1}{2\,\delta}\Big) \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \kappa_2 \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 + \kappa_1 \Big(\mu - \frac{\delta}{2}\Big) |\boldsymbol{v}|_{1,\Omega}^2 + \kappa_3 \|\boldsymbol{v}\|_{0,\Gamma}^2 \\
&\ge \alpha_3 \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}^2 + c_4(\Omega)\,\alpha_2 \|\boldsymbol{v}\|_{1,\Omega}^2 \ge \alpha(\Omega)\, \|(\boldsymbol{\tau}, \boldsymbol{v})\|^2 ,
\end{aligned} \tag{3.30}$$

where, assuming the stipulated hypotheses for δ and κ_1,

$$\alpha_1 := \min\Big\{1 - \frac{\kappa_1}{2\,\delta}, \frac{\kappa_2}{2}\Big\}, \quad \alpha_2 := \min\Big\{\kappa_1\Big(\mu - \frac{\delta}{2}\Big), \kappa_3\Big\}$$
$$\alpha_3 := \min\Big\{\alpha_1\, c_3(\Omega), \frac{\kappa_2}{2}\Big\}, \quad \text{and} \quad \alpha(\Omega) := \min\big\{\alpha_3, c_4(\Omega)\,\alpha_2\big\} .$$

The above shows that $\mathbf{A}$ is elliptic with constant $\alpha(\Omega)$, and hence, employing (3.28), we deduce that for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ there holds

$$\big(\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}\big)\big((\boldsymbol{\tau}, \boldsymbol{v}), (\boldsymbol{\tau}, \boldsymbol{v})\big) \ge \Big(\alpha(\Omega) - (\kappa_1^2 + 1)^{1/2} c_1(\Omega) \|\boldsymbol{w}\|_{1,\Omega}\Big) \|(\boldsymbol{\tau}, \boldsymbol{v})\|^2 \ge \frac{\alpha(\Omega)}{2} \|(\boldsymbol{\tau}, \boldsymbol{v})\|^2 , \tag{3.31}$$

provided $(\kappa_1^2 + 1\,)^{1/2} c_1(\Omega) \|\boldsymbol{w}\|_{1,\Omega} \le \dfrac{\alpha(\Omega)}{2}$. Therefore, the ellipticity of the form $\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}$ is ensured with the constant $\dfrac{\alpha(\Omega)}{2}$, independent of $\boldsymbol{w}$, by requiring $\|\boldsymbol{w}\|_{1,\Omega} \le r_0$, with

$$r_0 := \frac{\alpha(\Omega)}{2\,(\kappa_1^2 + 1)^{1/2}\, c_1(\Omega)} . \tag{3.32}$$

Next, concerning the functionals F_ϕ and F_D, we first see that, for a given $\phi \in \mathrm{H}^1(\Omega)$, F_ϕ is clearly linear in $\mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$, and by using Cauchy-Schwarz's inequality and the trace theorems in $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{H}^1(\Omega)$ with constants 1 and $c_0(\Omega)$, respectively, we find that

$$\| F_\phi \| \,\le\, (\mu^2 \,+\, \kappa_2^2)^{1/2} \, \|\boldsymbol{g}\|_{\infty,\Omega} \, \|\phi\|_{0,\Omega} \,. \tag{3.33}$$

and

$$\| F_D \| \,\le\, \kappa_3 \, c_0(\Omega) \, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \mu \, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \,. \tag{3.34}$$

In this way, denoting $M_{\mathbf{S}} := \max\left\{(\mu^2+\kappa_2^2)^{1/2}, \kappa_3\, c_0(\Omega)\right\}$, we deduce from (3.33) and (3.34) that

$$\|F_\phi \,+\, F_D\| \,\le\, M_{\mathbf{S}} \left\{ \|\boldsymbol{g}\|_{\infty,\Omega} \, \|\phi\|_{0,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \right\}. \tag{3.35}$$

We conclude by Lax-Milgram Theorem (see e.g. [20], Theorem 1.1) that there is a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}) := \mathbf{S}(\boldsymbol{w}, \phi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ of (3.22), and the corresponding continuous dependence result together with the constant of ellipticity $\alpha(\Omega)/2$ and the estimate (3.35) imply (3.27) with the positive constant $c_{\mathbf{S}} := \dfrac{2M_{\mathbf{S}}}{\alpha(\Omega)}$, which is clearly independent of $\boldsymbol{w}$ and ϕ. □

On the other hand, a straightforward application of the Babŭska-Brezzi theory provides the well-posedness of (3.24). In fact, we have the following result.

Lemma 3.4 *For each* $(\boldsymbol{w}, \phi) \in \mathbf{H} := \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ *there exists a unique pair* $(\varphi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ *solution of problem* (3.24)*, and there holds*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi)\| \,\le\, \|(\varphi, \lambda)\| \,\le\, c_{\widetilde{\mathbf{S}}} \left\{ \, \|\boldsymbol{w}\|_{1,\Omega} \, |\phi|_{1,\Omega} \,+\, \|\varphi_D\|_{1/2,\Gamma} \right\}, \tag{3.36}$$

where $c_{\widetilde{\mathbf{S}}}$ *is a positive constant independent of* $(\boldsymbol{w}, \phi)$*.*

Proof. It is clear from (3.15) and (3.16) that $\mathbf{a}$ and $\mathbf{b}$ are bounded bilinear forms in $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and $\mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, respectively, with constants $\|\mathbf{a}\| := \|\mathbb{K}\|_{\infty,\Omega}$ and $\|\mathbf{b}\| := 1$. In addition, it is easy to see that the bilinear form $\mathbf{b}$ satisfies the inf-sup condition since its induced operator is given by $\mathcal{R}^*_{-1/2} \circ \gamma_0 : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{-1/2}(\Gamma)$, where $\gamma_0 : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{1/2}(\Gamma)$ is the trace operator, which is surjective, and $\mathcal{R}_{-1/2} : \mathrm{H}^{-1/2}(\Gamma) \longrightarrow \mathrm{H}^{1/2}(\Gamma)$ is the usual Riesz operator, which is bijective. Moreover, it is clear that the kernel of the aforementioned induced operator is $V := \mathrm{H}^1_0(\Omega)$, and hence, recalling that $\mathbb{K}$ is a uniformly positive definite tensor, and using the Friedrichs-Poincaré inequality, we deduce that $\mathbf{a}$ is $V-$elliptic with a constant $\alpha_{\mathbf{a}}(\Omega)$ depending only on Ω. In turn, it is quite clear that for each $(\boldsymbol{w}, \phi) \in \mathbf{H}$ the functionals $F_{\boldsymbol{w},\phi}$ and G are linear and bounded in $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{1/2}(\Gamma)$, respectively. In particular, according to the duality pairing of $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, and the estimate (3.10), it follows from (3.19) and (3.20) that

$$\|F_{\boldsymbol{w},\phi}\|_{(H^1(\Omega))'} \,\le\, c_2(\Omega) \, \|\boldsymbol{w}\|_{1,\Omega} \, |\phi|_{1,\Omega} \tag{3.37}$$

and

$$\|G\|_{-1/2,\Gamma} \,\le\, \|\varphi_D\|_{1/2,\Gamma} \,. \tag{3.38}$$

In this way, the Babŭska-Brezzi theory (see e.g. [20, Theorem 2.3]) ensures the existence of a unique $(\varphi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ solution of (3.24) and a positive constant $c_{\widetilde{\mathbf{S}}}$ depending on $\|\mathbf{a}\|$, $\alpha_{\mathbf{a}}(\Omega)$, $c_2(\Omega)$ and the inf-sup constant of $\mathbf{b}$, such that the estimate (3.36) holds. □

3.4 Solvability analysis of the fixed point equation

Having proved the well-posedness of the uncoupled problems (3.22) and (3.24), which ensures that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\mathbf{T}$ (cf. Section 3.2) are well defined, we now aim to establish the existence of a unique fixed point of the operator $\mathbf{T}$. For this purpose, in what follows we verify the hypothesis of the Banach fixed point Theorem. We begin with the following result.

Lemma 3.5 *Let $r \in \big(0, r_0\big)$, with r_0 given by (3.32) (cf. proof of Lemma 3.3), let W be the closed ball in $\mathbf{H}$ defined by $W := \Big\{ (\boldsymbol{w}, \phi) \in \mathbf{H} : \quad \|(\boldsymbol{w}, \phi)\| \le r \Big\}$, and assume that the data satisfy*

$$c(r) \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} + c_{\widetilde{\mathbf{S}}} \, \|\varphi_D\|_{1/2,\Gamma} \le r \,, \tag{3.39}$$

where

$$c(r) := \max\big\{r, 1\big\} \big(1 + c_{\widetilde{\mathbf{S}}} r\big) \, c_{\mathbf{S}} \,,$$

with $c_{\mathbf{S}}$ and $c_{\widetilde{\mathbf{S}}}$ as in (3.27) and (3.36), respectively. Then there holds $\mathbf{T}(W) \subseteq W$.

Proof. Given $(\boldsymbol{w}, \phi)$ in the ball W of radius $r \in \big(0, r_0\big)$, it follows that $(\boldsymbol{u}, \varphi) := \mathbf{T}(\boldsymbol{w}, \phi)$ is well defined since $\|\boldsymbol{w}\|_{1,\Omega} \le r$. Then, according to the definition of the operator $\mathbf{T}$ (cf. (3.25)), and employing the continuous dependence estimates (3.36) and (3.27), it follows that

$$\begin{aligned}
\|(\boldsymbol{u}, \varphi)\| \;&\le\; \|\mathbf{S}_2(\boldsymbol{w}, \phi)\|_{1,\Omega} + c_{\widetilde{\mathbf{S}}} \Big\{ r \, \|\mathbf{S}_2(\boldsymbol{w}, \phi)\|_{1,\Omega} + \|\varphi_D\|_{1/2,\Gamma} \Big\} \\
&\le\; \Big(1 + c_{\widetilde{\mathbf{S}}} r \Big) \|\mathbf{S}_2(\boldsymbol{w}, \phi)\|_{1,\Omega} + c_{\widetilde{\mathbf{S}}} \, \|\varphi_D\|_{1/2,\Gamma} \\
&\le\; \Big(1 + c_{\widetilde{\mathbf{S}}} r \Big) c_{\mathbf{S}} \Big\{ r \, \|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} + c_{\widetilde{\mathbf{S}}} \, \|\varphi_D\|_{1/2,\Gamma} \\
&\le\; c(r) \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} + c_{\widetilde{\mathbf{S}}} \, \|\varphi_D\|_{1/2,\Gamma} \,,
\end{aligned}$$

and hence the result follows from the assumption (3.39). □

Next, we establish two lemmas that will be useful to derive conditions under which the operator $\mathbf{T}$ is continuous. We start with the following estimate regarding the operator $\mathbf{S}$.

Lemma 3.6 *Let $r \in \big(0, r_0\big)$, with r_0 given by (3.32). Then there exists a positive constant $C_{\mathbf{S}}$, depending on the viscosity μ, the stabilization parameters κ_1 and κ_2, the constant $c_1(\Omega)$ (cf. (3.9)), and the ellipticity constant $\alpha(\Omega)$ of the bilinear form $\mathbf{A}$ (cf. (3.30) in the proof of Lemma 3.3), such that*

$$\|\mathbf{S}(\boldsymbol{w}, \phi) - \mathbf{S}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \le C_{\mathbf{S}} \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} \, \|\phi - \widetilde{\phi}\|_{0,\Omega} + \|\mathbf{S}_2(\boldsymbol{w}, \phi)\|_{1,\Omega} \, \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} \Big\} , \tag{3.40}$$

for all $(\boldsymbol{w}, \phi)\,, (\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega}\,, \|\widetilde{\boldsymbol{w}}\|_{1,\Omega} \le r$.

Proof. Given r and $(\boldsymbol{w}, \phi)$, $(\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \mathbf{H}$ as indicated, we let $(\boldsymbol{\sigma}, \boldsymbol{u}) := \mathbf{S}(\boldsymbol{w}, \phi)$ and $(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}) := \mathbf{S}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})$ be the corresponding solutions of problem (3.22). Then, using the bilinearity of $\mathbf{A}$ and $\mathbf{B}_{\boldsymbol{w}}$ for any $\boldsymbol{w}$, it follows easily from (3.22) that

$$\big(\mathbf{A} + \mathbf{B}_{\widetilde{\boldsymbol{w}}}\big)((\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}), (\boldsymbol{\tau}, \boldsymbol{v})) \;=\; F_{\phi - \widetilde{\phi}}(\boldsymbol{\tau}, \boldsymbol{v}) \;-\; \mathbf{B}_{\boldsymbol{w} - \widetilde{\boldsymbol{w}}}((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v}))$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$. Hence, applying the ellipticity of $\mathbf{A} + \mathbf{B}_{\widetilde{\boldsymbol{w}}}$ (cf. (3.31)), and employing the bounds (3.33) and (3.28) for $F_{\phi-\widetilde{\phi}}$ and $\mathbf{B}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}$, respectively, we find that

$$
\begin{aligned}
&\frac{\alpha(\Omega)}{2} \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}})\|^2 \le \big(\mathbf{A} + \mathbf{B}_{\widetilde{\boldsymbol{w}}}\big)((\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}), (\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}}))\\
&\quad = F_{\phi-\widetilde{\phi}}\big((\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}})\big) - \mathbf{B}_{\boldsymbol{w}-\widetilde{\boldsymbol{w}}}\big((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}})\big)\\
&\quad \le \Big\{(\mu^2 + \kappa_2^2)^{1/2} \|\boldsymbol{g}\|_{\infty,\Omega} \|\phi - \widetilde{\phi}\|_{0,\Omega} + (\kappa_1^2 + 1)^{1/2} c_1(\Omega) \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega}\Big\} \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{u}})\|,
\end{aligned}
$$

which, denoting $C_{\mathbf{S}} := \dfrac{2}{\alpha(\Omega)} \max\Big\{(\mu^2 + \kappa_2^2)^{1/2}, (\kappa_1^2 + 1)^{1/2} c_1(\Omega)\Big\}$ and recalling that $\boldsymbol{u} = \mathbf{S}_2(\boldsymbol{w}, \phi)$, yields (3.40) and concludes the proof. □

In turn, the following result establishes the Lipschitz-continuity of the operator $\widetilde{\mathbf{S}}$.

Lemma 3.7 *There exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending on $c_2(\Omega)$ (cf. (3.10)) and the ellipticity constant $\alpha_{\mathbf{a}}(\Omega)$ of the bilinear form $\mathbf{a}$ in the kernel of $\mathbf{b}$, such that*

$$
\|\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi) - \widetilde{\mathbf{S}}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \le C_{\widetilde{\mathbf{S}}} \Big\{ \|\boldsymbol{w}\|_{1,\Omega} |\phi - \widetilde{\phi}|_{1,\Omega} + \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} |\widetilde{\phi}|_{1,\Omega} \Big\} \tag{3.41}
$$

for all $(\boldsymbol{w}, \phi)$, $(\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \mathbf{H}$.

Proof. Given $(\boldsymbol{w}, \phi)$, $(\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in \mathbf{H}$, we let (φ, λ), $(\widetilde{\varphi}, \widetilde{\lambda}) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ be the corresponding solutions of (3.24), so that $\varphi := \widetilde{\mathbf{S}}(\boldsymbol{w}, \phi)$ and $\widetilde{\varphi} := \widetilde{\mathbf{S}}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})$. Then, using the linearity of the forms $\mathbf{a}$ and $\mathbf{b}$, we deduce from both formulations (3.24) that

$$
\begin{aligned}
\mathbf{a}(\varphi - \widetilde{\varphi}, \psi) + \mathbf{b}(\psi, \lambda - \widetilde{\lambda}) &= F_{\boldsymbol{w},\phi-\widetilde{\phi}}(\psi) + F_{\boldsymbol{w}-\widetilde{\boldsymbol{w}},\widetilde{\phi}}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\\
\mathbf{b}(\varphi - \widetilde{\varphi}, \xi) &= 0 \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma).
\end{aligned} \tag{3.42}
$$

Next, noting from the second equation of (3.42) that $\varphi - \widetilde{\varphi}$ belongs to the kernel V of $\mathbf{b}$, taking $\psi = \varphi - \widetilde{\varphi}$ and $\xi = \lambda - \widetilde{\lambda}$ in (3.42), using the ellipticity of $\mathbf{a}$ in V, and employing the bound (3.37) for $F_{\boldsymbol{w},\phi-\widetilde{\phi}}$ and $F_{\boldsymbol{w}-\widetilde{\boldsymbol{w}},\widetilde{\phi}}$, we deduce starting from the first equation of (3.42) that

$$
\begin{aligned}
&\alpha_{\mathbf{a}}(\Omega) \|\varphi - \widetilde{\varphi}\|_{1,\Omega}^2 \le \mathbf{a}(\varphi - \widetilde{\varphi}, \varphi - \widetilde{\varphi}) = \big|F_{\boldsymbol{w},\phi-\widetilde{\phi}}(\varphi - \widetilde{\varphi}) + F_{\boldsymbol{w}-\widetilde{\boldsymbol{w}},\widetilde{\phi}}(\varphi - \widetilde{\varphi})\big|\\
&\quad \le c_2(\Omega)\Big\{\|\boldsymbol{w}\|_{1,\Omega} |\phi - \widetilde{\phi}|_{1,\Omega} + \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} |\widetilde{\phi}|_{1,\Omega}\Big\} \|\varphi - \widetilde{\varphi}\|_{1,\Omega},
\end{aligned}
$$

which gives (3.41) with $C_{\widetilde{\mathbf{S}}} := \dfrac{c_2(\Omega)}{\alpha_{\mathbf{a}}(\Omega)}$. □

As a consequence of the previous lemmas, we have the following result.

Lemma 3.8 *Let $r \in \big(0, r_0\big)$, with r_0 given by (3.32), and let $W := \Big\{ (\boldsymbol{w}, \phi) \in \mathbf{H} : \quad \|(\boldsymbol{w}, \phi)\| \le r \Big\}$. Then, there exists $C_{\mathbf{T}} > 0$, depending on r and the constants $c_{\mathbf{S}}$, $C_{\mathbf{S}}$, and $C_{\widetilde{\mathbf{S}}}$ (cf. (3.27), (3.40), and (3.41), respectively), such that*

$$
\|\mathbf{T}(\boldsymbol{w}, \phi) - \mathbf{T}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \le C_{\mathbf{T}} \Big\{\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\Big\} \|(\boldsymbol{w}, \phi) - (\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \tag{3.43}
$$

for all $(\boldsymbol{w}, \phi)$, $(\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in W$.

Proof. Given $r \in \big(0, r_0\big)$ and $(\boldsymbol{w},\phi)$, $(\widetilde{\boldsymbol{w}},\widetilde{\phi}) \in W$, we first observe, according to the definition of $\mathbf{T}$ (cf. (3.25)), the Lipschitz-continuity of $\widetilde{\mathbf{S}}$ (cf. (3.41)), and the fact that $\|\widetilde{\phi}\|_{1,\Omega} \le r$, that

$$
\begin{aligned}
\|\mathbf{T}(\boldsymbol{w},\phi) \,-\, \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,&\le\, \|\mathbf{S}_2(\boldsymbol{w},\phi) - \mathbf{S}_2(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,+\, \|\widetilde{\mathbf{S}}(\mathbf{S}_2(\boldsymbol{w},\phi),\phi) - \widetilde{\mathbf{S}}(\mathbf{S}_2(\widetilde{\boldsymbol{w}},\widetilde{\phi}),\widetilde{\phi})\| \\
&\le\, \big(1 + C_{\widetilde{\mathbf{S}}}\, r\big)\, \|\mathbf{S}_2(\boldsymbol{w},\phi) - \mathbf{S}_2(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,+\, C_{\widetilde{\mathbf{S}}}\, \|\mathbf{S}_2(\boldsymbol{w},\phi)\|_{1,\Omega}\, |\phi - \widetilde{\phi}|_{1,\Omega}\,,
\end{aligned}
$$

which, employing the Lipschitz-continuity of $\mathbf{S}$ (cf. (3.40)), yields

$$
\begin{aligned}
\|\mathbf{T}(\boldsymbol{w},\phi) \,-\, \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,&\le\, \big(1 + C_{\widetilde{\mathbf{S}}}\, r\big)\, C_{\mathbf{S}}\, \|\boldsymbol{g}\|_{\infty,\Omega}\, \|\phi - \widetilde{\phi}\|_{0,\Omega} \\
&+ \Big\{ \big(1 + C_{\widetilde{\mathbf{S}}}\, r\big)\, C_{\mathbf{S}}\, \|\boldsymbol{w} - \widetilde{\boldsymbol{w}}\|_{1,\Omega} \,+\, C_{\widetilde{\mathbf{S}}}\, |\phi - \widetilde{\phi}|_{1,\Omega} \Big\}\, \|\mathbf{S}_2(\boldsymbol{w},\phi)\|\,.
\end{aligned}
\tag{3.44}
$$

Then, applying the a priori estimate for $\mathbf{S}$ (cf. (3.27)), noting now that $\|\phi\|_{1,\Omega} \le r$, and performing some algebraic manipulations, we deduce from (3.44) that

$$
\|\mathbf{T}(\boldsymbol{w},\phi) \,-\, \mathbf{T}(\widetilde{\boldsymbol{w}},\widetilde{\phi})\| \,\le\, \bigg\{ C_{\mathbf{T},1}\, \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, C_{\mathbf{T},2} \Big\{ \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} \bigg\}\, \|(\boldsymbol{w},\phi) - (\widetilde{\boldsymbol{w}},\tilde{\phi})\|\,,
$$

where

$$
C_{\mathbf{T},1} \,:=\, \big(1 + C_{\widetilde{\mathbf{S}}}\, r\big)\, C_{\mathbf{S}}\, (1 + c_{\mathbf{S}}\, r) \,+\, C_{\widetilde{\mathbf{S}}}\, c_{\mathbf{S}}\, r \quad \text{and} \quad C_{\mathbf{T},2} \,:=\, \Big\{ \big(1 + C_{\widetilde{\mathbf{S}}}\, r\big)\, C_{\mathbf{S}} \,+\, C_{\widetilde{\mathbf{S}}} \Big\}\, c_{\mathbf{S}}\,.
$$

In this way, (3.43) follows from the foregoing inequality by defining $C_{\mathbf{T}} \,:=\, \max\big\{C_{\mathbf{T},1}, C_{\mathbf{T},2}\big\}$. □

We are ready now to prove that our fixed-point scheme (3.26) is well-posed. Indeed, we know from Lemmas 3.3 and 3.4 that the operator $\mathbf{T}$ is well-defined. Furthermore, the assumption on the data given by (3.39) (cf. Lemma 3.5) guarantees that $\mathbf{T}$ maps W into itself for any ball W in $\mathbf{H}$ with radius $r \in (0, r_0)$. In turn, it is clear from Lemma 3.8 that $\mathbf{T}$ is Lipschitz-continuous. In addition, assuming additionally that $\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}$ is sufficiently small, $\mathbf{T}$ becomes a contraction, and hence the Banach fixed point Theorem can be applied. More precisely, we have the following result.

Theorem 3.9 *Let $\kappa_1 \in (0, 2\,\delta)$, with $\delta \in (0, 2\,\mu)$, and $\kappa_2, \kappa_3 > 0$, and given $r \in (0, r_0)$, let $W \,:=\, \Big\{ (\boldsymbol{w},\phi) \in \mathbf{H} : \quad \|(\boldsymbol{w},\phi)\| \,\le\, r \Big\}$. Assume that the data satisfy*

$$
c(r) \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} \,+\, c_{\widetilde{\mathbf{S}}}\, \|\varphi_D\|_{1/2,\Gamma} \,\le\, r
$$

and

$$
C_{\mathbf{T}} \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} \,<\, 1\,.
$$

Then, problem (3.12) has a unique solution $(\boldsymbol{\sigma},\boldsymbol{u},\varphi,\lambda) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{1/2}(\Gamma)$, with $(\boldsymbol{u},\varphi) \in W$. Moreover, there hold

$$
\|(\boldsymbol{\sigma},\boldsymbol{u})\| \,\le\, c_{\mathbf{S}} \Big\{ r\, \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\}
$$

and

$$
\|(\varphi,\lambda)\| \,\le\, c_{\widetilde{\mathbf{S}}} \Big\{ r\, \|\boldsymbol{u}\|_{1,\Omega} \,+\, \|\varphi_D\|_{1/2,\Gamma} \Big\}\,.
$$

Proof. It follows from Lemmas 3.5 and 3.8, the Banach fixed point theorem, and the a priori estimates (3.27) and (3.36). We omit further details. □

4 The Galerkin scheme

In this section we introduce and analyze the Galerkin scheme of the augmented mixed-primal formulation (3.12). To this end, we adopt the discrete analogue of the fixed-point strategy introduced in Section 3.2.

4.1 Preliminaries

We begin by considering arbitrary finite dimensional subspaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div};\Omega)\,, \quad \mathbf{H}_h^{\boldsymbol{u}} \subseteq \mathbf{H}^1(\Omega)\,, \quad \mathrm{H}_h^{\varphi} \subseteq \mathrm{H}^1(\Omega)\,, \quad \text{and} \quad \mathrm{H}_h^{\lambda} \subseteq \mathrm{H}^{-1/2}(\Gamma)\,, \tag{4.1}$$

whose specific choices will be described later on in Section 4.3. Hereafter, h stands for the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of triangles K (when $d = 2$) or tetrahedra K (when $d = 3$) of diameter h_K, that is $h := \max\Big\{ h_K : \quad K \in \mathcal{T}_h \Big\}$. According to the above, the corresponding Galerkin scheme of problem (3.12) reads: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h, \lambda_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}$ such that

$$\begin{aligned} \mathbf{A}(\,(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\,,(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\,) + \mathbf{B}_{\boldsymbol{u}_h}(\,(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\,,(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\,) &= F_{\varphi_h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h) + F_D(\boldsymbol{\tau}_h,\boldsymbol{v}_h) \\ \mathbf{a}(\varphi_h,\psi_h) + \mathbf{b}(\psi_h\,,\lambda_h) &= F_{\boldsymbol{u}_h,\varphi_h}(\,\psi_h\,) \\ \mathbf{b}(\varphi_h\,,\xi_h) &= G(\xi_h)\,, \end{aligned} \tag{4.2}$$

for all $(\boldsymbol{\tau}_h, \boldsymbol{v}_h, \psi_h, \xi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}$.

In order to address the well-posedness of (4.2), we proceed in what follows analogously as in Section 3.2. Indeed, we first set $\mathbf{H}_h := \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi}$ and define the operator $\mathbf{S}_h : \mathbf{H}_h \longrightarrow \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ by

$$\mathbf{S}_h(\boldsymbol{w}_h,\phi_h) := (\mathbf{S}_{1,h}(\boldsymbol{w}_h,\phi_h),\mathbf{S}_{2,h}(\boldsymbol{w}_h,\phi_h)) = (\boldsymbol{\sigma}_h,\boldsymbol{u}_h) \qquad \forall\,(\boldsymbol{w}_h,\phi_h)\in\mathbf{H}_h\,,$$

where $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ is the unique solution of

$$\mathbf{A}(\,(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\,,(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\,) + \mathbf{B}_{\boldsymbol{w}_h}(\,(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\,,(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\,) = F_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h) + F_D(\boldsymbol{\tau}_h,\boldsymbol{v}_h) \tag{4.3}$$

for all $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$. Just for sake of completeness we recall here that the form $\mathbf{A}$ and the functional F_D are defined in (3.13) and (3.18), respectively. In turn, with $\boldsymbol{w}_h$ and ϕ_h given, the bilinear form $\mathbf{B}_{\boldsymbol{w}_h}(\cdot,\cdot)$ and the linear functional F_{ϕ_h} are those corresponding to (3.14) and (3.17), respectively, with $\boldsymbol{w} = \boldsymbol{w}_h$ and $\varphi = \phi_h$.

Furthermore, we introduce the operator $\widetilde{\mathbf{S}}_h : \mathbf{H}_h \longrightarrow \mathrm{H}_h^{\varphi}$ defined as

$$\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h,\phi_h) := \varphi_h \qquad \forall\,(\boldsymbol{w}_h,\phi_h) \in \mathbf{H}_h\,,$$

where $\varphi_h \in \mathrm{H}_h^{\varphi}$ is the first component of the unique solution of the problem: Find $(\varphi_h, \lambda_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}$ such that

$$\begin{aligned} \mathbf{a}(\varphi_h,\psi_h) + \mathbf{b}(\psi_h,\lambda_h) &= F_{\boldsymbol{w}_h,\phi_h}(\psi_h) \quad \forall\,\psi_h\in\mathrm{H}_h^{\varphi} \\ \mathbf{b}(\varphi_h\,,\xi_h) &= G(\xi_h) \quad \forall\,\xi_h\in\mathrm{H}_h^{\lambda}\,. \end{aligned} \tag{4.4}$$

Certainly, $\mathbf{a}$ and $\mathbf{b}$ are the forms introduced in (3.15) - (3.16), and $F_{\boldsymbol{w}_h,\phi_h}$ is defined as in (3.19) with $\boldsymbol{u} = \boldsymbol{w}_h$ and $\varphi = \phi_h$.

Therefore, by introducing the operator $\mathbf{T}_h : \mathbf{H}_h \longrightarrow \mathbf{H}_h$ as

$$\mathbf{T}_h(\boldsymbol{w}_h,\phi_h) =: (\mathbf{S}_{2,h}(\boldsymbol{w}_h,\phi_h),\widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\boldsymbol{w}_h,\phi_h),\phi_h)) \qquad \forall\,(\boldsymbol{w}_h,\phi_h) \in \mathbf{H}_h\,, \tag{4.5}$$

we see that solving (4.2) is equivalent to finding a fixed point of $\mathbf{T}_h$, that is $(\boldsymbol{u}_h, \varphi_h) \in \mathbf{H}_h$ such that

$$\mathbf{T}_h(\boldsymbol{u}_h, \varphi_h) \;=\; (\boldsymbol{u}_h, \varphi_h)\,. \tag{4.6}$$

In the following section we first establish the well-posedness of both (4.3) and (4.4), thus confirming that $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, and hence $\mathbf{T}_h$, are all well defined, and then address the solvability of the discrete fixed point equation (4.6).

4.2 Solvability analysis

We begin by remarking that the same tools utilized in the proof of Lemma 3.3 can be employed now to prove the unique solvability of the discrete problem (4.3). In fact, it is quite straightforward to see that for each $\boldsymbol{w}_h \in \mathbf{H}_h^{\boldsymbol{u}}$ the bilinear form $\mathbf{A} + \mathbf{B}_{\boldsymbol{w}_h}$ is bounded as in (3.29) with a constant depending on μ, κ_1, κ_2, κ_3, $c_0(\Omega)$, and $\|\boldsymbol{w}_h\|_{1,\Omega}$. In addition, under the same assumptions from Lemma 3.3 on the stabilization parameters and the given $\boldsymbol{w}_h \in \mathbf{H}_h^{\boldsymbol{u}}$ (instead of $\boldsymbol{w}$), $\mathbf{A} + \mathbf{B}_{\boldsymbol{w}_h}$ becomes elliptic in $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ with the same constant obtained in (3.31). On the other hand, it is clear that for each $\phi_h \in \mathrm{H}_h^{\varphi}$ the functional F_{ϕ_h} is linear and bounded as in (3.33). The foregoing discussion and the Lax-Milgram theorem allow to conclude the following result.

Lemma 4.1 *Assume that $\kappa_1 \in (0, 2\,\delta)$ with $\delta \in (0, 2\,\mu)$, and $\kappa_2, \kappa_3 > 0$. Then, for each $r \in (0, r_0)$ and for each $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega} \le r$, the problem (4.3) has a unique solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) =: \mathbf{S}_h(\boldsymbol{w}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$. Moreover, with the same constant $c_{\mathbf{S}} > 0$ from Lemma 3.3, which is independent of $(\boldsymbol{w}_h, \phi_h)$, there holds*

$$\|\mathbf{S}_h(\boldsymbol{w}_h, \phi_h)\| \;=\; \|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\| \;\le\; c_{\mathbf{S}} \left\{ \|\boldsymbol{g}\|_{\infty,\Omega}\, \|\phi_h\|_{0,\Omega} \;+\; \|\boldsymbol{u}_D\|_{0,\Gamma} \;+\; \|\boldsymbol{u}_D\|_{1/2,\Gamma} \right\}. \tag{4.7}$$

On the other hand, in order to analyze problem (4.4), we need to incorporate further hypotheses on the discrete spaces H_h^{φ} and H_h^{λ}. For this purpose, we now let V_h be the discrete kernel of $\mathbf{b}$, that is

$$V_h \;:=\; \left\{ \psi_h \,\in\, H_h^{\varphi} : \quad \mathbf{b}(\psi_h, \xi_h) = 0 \qquad \forall\, \xi_h \,\in\, \mathrm{H}_h^{\lambda} \right\}.$$

Then, we assume that the following discrete inf-sup conditions hold:

(H.1) There exists a constant $\widehat{\alpha} > 0$, independent of h, such that

$$\sup_{\substack{\psi_h \in V_h \\ \psi_h \neq 0}} \frac{\mathbf{a}(\psi_h,\, \psi_h)}{\|\psi_h\|_{1,\Omega}} \;\ge\; \widehat{\alpha}\, \|\psi_h\|_{1,\Omega} \qquad \forall \psi_h \,\in\, V_h\,. \tag{4.8}$$

(H.2) There exists a constant $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\psi_h \in \mathrm{H}_h^{\varphi} \\ \psi_h \neq 0}} \frac{\mathbf{b}(\psi_h,\, \xi_h)}{\|\psi_h\|_{1,\Omega}} \;\ge\; \widehat{\beta}\, \|\xi_h\|_{-1/2,\Gamma} \qquad \forall \xi_h \,\in\, \mathrm{H}_h^{\lambda}\,. \tag{4.9}$$

Specific examples of spaces verifying **(H.1)** and **(H.2)** are described later on in Section 4.3.

We are now in a position to establish the following result.

Lemma 4.2 *For each* $(\boldsymbol{w}_h, \phi_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi}$ *there exists a unique pair* $(\varphi_h, \lambda_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}$ *solution of problem* (4.4)*, and there holds*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h, \phi_h)\| \,\le\, \|(\varphi_h, \lambda_h)\| \,\le\, \widetilde{c}_{\widetilde{\mathbf{S}}} \Big\{ \|\boldsymbol{w}_h\|_{1,\Omega} \, |\phi_h|_{1,\Omega} \,+\, \|\varphi_D\|_{1/2,\Gamma} \Big\}, \tag{4.10}$$

where $\widetilde{c}_{\widetilde{\mathbf{S}}}$ *is a positive constant depending on* $\|\mathbf{a}\|$, $\widehat{\alpha}$ *(cf.* (4.8)*)*, $\widehat{\beta}$ *(cf.* (4.9)*), and* $c_2(\Omega)$.

Proof. It follows from a straightforward application of the discrete Babuška-Brezzi theory (see e.g. [20, Theorem 2.4]). In fact, we first notice that the bilinear forms $\mathbf{a}$ and $\mathbf{b}$ are certainly bounded on any pair of subspaces of the corresponding continuous spaces. In turn, the linear functional $F_{\boldsymbol{w}_h,\phi_h}$ is bounded on H_h^{φ} exactly as stated in (3.37) but replacing there $\boldsymbol{w}$ and ϕ by $\boldsymbol{w}_h$ and ϕ_h, respectively, whereas the restricction of G to H_h^{λ} is clearly bounded as indicated in (3.38). The other hypotheses required by the theory are exactly those described in **(H.1)** and **(H.2)**, and hence we omit further details. □

We now aim to show the solvability of (4.2) by analyzing the equivalent fixed point equation (4.6). To this end, in what follows we verify the hypotheses of the Brouwer fixed point theorem, which reads as follows (see, e.g. [12], Theorem 9.9-2).

Theorem 4.3 *Let* W *be a compact and convex subset of a finite dimensional Banach space* X*, and let* $T : W \longrightarrow W$ *be a continuous mapping. Then* T *has at least one fixed point.*

The discrete version of Lemma 3.5 is given as follows.

Lemma 4.4 *Let* $r \in \big(0, r_0\big)$*, with* r_0 *given by* (3.32) *(cf. proof of Lemma* 3.3*), let*

$$W_h \,:=\, \Big\{ (\boldsymbol{w}_h, \phi_h) \,\in\, \mathbf{H}_h : \quad \|(\boldsymbol{w}_h, \phi_h)\| \,\le\, r \Big\},$$

and assume that the data satisfy

$$\widetilde{c}(r) \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} \,+\, \widetilde{c}_{\widetilde{\mathbf{S}}} \, \|\varphi_D\|_{1/2,\Gamma} \,\le\, r\,, \tag{4.11}$$

where

$$\widetilde{c}(r) \,:=\, \max\big\{r, 1\big\} \, \big(1 \,+\, \widetilde{c}_{\widetilde{\mathbf{S}}} r\big) \, c_{\mathbf{S}}\,,$$

with $c_{\mathbf{S}}$ *and* $\widetilde{c}_{\widetilde{\mathbf{S}}}$ *as in* (3.27) *(or* (4.7)*), and* (4.10)*, respectively. Then there holds* $\mathbf{T}_h(W_h) \subseteq W_h$.

Proof. It follows by similar arguments to those employed in the proof of Lemma 3.5 by using now the discrete stability estimates given by (4.7) and (4.10). □

Next, we provide the discrete analogues of Lemmas 3.6 and 3.7, whose proofs, being either analogous or similar to the corresponding continuous ones, are omitted. We just remark that Lemma 4.5 below is proved almost verbatim as the proof of Lemma 3.6, whereas Lemma 4.6 is derived by using the discrete inf-sup condition (4.8) instead of the V_h-ellipticity of $\mathbf{a}$ (analogously as it was for Lemma 3.7), where V_h is the discrete kernel of $\mathbf{b}$. To this respect, note that (4.8) is more general, and hence less restrictive, than assuming that the bilinear form $\mathbf{a}$ is elliptic in V_h. In other words, the latter is not necessary but only sufficient condition for (4.8), which is precisely what we apply below in Section 4.3 for a particular choice of subspaces. In turn, unless V_h is contained in V, which occurs in many cases but not always, the V_h-ellipticity of $\mathbf{a}$ does not follow from its eventual V-ellipticity.

Lemma 4.5 *Let $r \in \big(0, r_0\big)$, with r_0 given by* (3.32)*. Then there holds*

$$\|\mathbf{S}_h(\boldsymbol{w}_h,\phi_h) - \mathbf{S}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \,\le\, C_{\mathbf{S}}\left\{ \|\boldsymbol{g}\|_{\infty,\Omega}\,\|\phi_h - \widetilde{\phi}_h\|_{0,\Omega} + \|\mathbf{S}_{2,h}(\boldsymbol{w}_h,\phi_h)\|_{1,\Omega}\,\|\boldsymbol{w}_h - \widetilde{\boldsymbol{w}}_h\|_{1,\Omega}\right\} \quad (4.12)$$

for all $(\boldsymbol{w}_h,\phi_h)\,,(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h) \in \mathbf{H}_h$ such that $\|\boldsymbol{w}_h\|_{1,\Omega}\,,\|\widetilde{\boldsymbol{w}}_h\|_{1,\Omega} \le r$, where $C_{\mathbf{S}}$ is the same positive constant from Lemma 3.6*.*

Lemma 4.6 *There exists a positive constant $\widetilde{C}_{\widetilde{\mathbf{S}}}$, depending on $c_2(\Omega)$ (cf.* (3.10)*) and the discrete inf-sup constant $\widehat{\alpha}$ (cf.* (4.8)*), such that*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{w}_h,\phi_h) \,-\, \widetilde{\mathbf{S}}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \,\le\, \widetilde{C}_{\widetilde{\mathbf{S}}}\left\{ \|\boldsymbol{w}_h\|_{1,\Omega}\,|\phi_h - \widetilde{\phi}_h\|_{1,\Omega} \,+\, \|\boldsymbol{w}_h - \widetilde{\boldsymbol{w}}_h\|_{1,\Omega}\,|\widetilde{\phi}_h|_{1,\Omega}\right\} \qquad (4.13)$$

for all $(\boldsymbol{w}_h,\phi_h)$, $(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h) \in \mathbf{H}_h$.

As a consequence of the foregoing lemmas, we are able to establish next the continuity of the operator $\mathbf{T}_h$.

Lemma 4.7 *Let $r \in \big(0, r_0\big)$, with r_0 given by* (3.32)*, and let*

$$W_h \,:=\, \Big\{ (\boldsymbol{w}_h,\phi_h) \,\in\, \mathbf{H}_h : \quad \|(\boldsymbol{w}_h,\phi_h)\| \,\le\, r \Big\}\,.$$

Then, there exists $\widetilde{C}_{\mathbf{T}} > 0$, depending on r and the constants $c_{\mathbf{S}}$, $C_{\mathbf{S}}$, and $\widetilde{C}_{\widetilde{\mathbf{S}}}$ (cf. (4.7)*,* (4.12)*, and* (4.13)*, respectively), such that*

$$\|\mathbf{T}_h(\boldsymbol{w}_h,\phi_h) \,-\, \mathbf{T}_h(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \,\le\, \widetilde{C}_{\mathbf{T}}\Big\{\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma}\Big\}\,\|(\boldsymbol{w}_h,\phi_h) - (\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h)\| \qquad (4.14)$$

for all $(\boldsymbol{w}_h,\phi_h)$, $(\widetilde{\boldsymbol{w}}_h,\widetilde{\phi}_h) \in W_h$.

Proof. It follows analogously to the proof of Lemma 3.8 by using now the estimates (4.7), (4.12), and (4.13), instead of (3.27), (3.40), and (3.41), respectively. Consequently, the resulting constant $\widetilde{C}_{\mathbf{T}}$ is given by $\max\big\{\widetilde{C}_{\mathbf{T},1},\widetilde{C}_{\mathbf{T},2}\big\}$, where

$$\widetilde{C}_{\mathbf{T},1} \,:=\, \big(1 + \widetilde{C}_{\widetilde{\mathbf{S}}}\,r\big)\,C_{\mathbf{S}}\,(1 + c_{\mathbf{S}}\,r) \,+\, \widetilde{C}_{\widetilde{\mathbf{S}}}\,c_{\mathbf{S}}\,r \quad \text{and} \quad \widetilde{C}_{\mathbf{T},2} \,:=\, \Big\{\big(1 + \widetilde{C}_{\widetilde{\mathbf{S}}}\,r\big)\,C_{\mathbf{S}} \,+\, \widetilde{C}_{\widetilde{\mathbf{S}}}\Big\}\,c_{\mathbf{S}}\,.$$

□

We are able now to establish the existence of a fixed-point of the operator $\mathbf{T}_h$.

Theorem 4.8 *Let $\kappa_1 \,\in\, (0, 2\,\delta)$, with $\delta \,\in\, (0, 2\,\mu)$, and κ_2, $\kappa_3 \,>\, 0$, and given $r \,\in\, (0, r_0)$, let $W_h \,:=\, \Big\{ (\boldsymbol{w}_h,\phi_h) \,\in\, \mathbf{H}_h : \quad \|(\boldsymbol{w}_h,\phi_h)\| \,\le\, r \Big\}$. Assume that the data satisfy*

$$\widetilde{c}(r)\Big\{\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma}\Big\} \,+\, \widetilde{c}_{\widetilde{\mathbf{S}}}\,\|\varphi_D\|_{1/2,\Gamma} \,\le\, r\,,$$

where the constant $\widetilde{c}(r)$ is defined in Lemma 4.4*. Then, problem* (4.2) *has at least one solution $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\varphi_h,\lambda_h) \,\in\, \mathbb{H}_h^{\boldsymbol{\sigma}} \,\times\, \mathbf{H}_h^{\boldsymbol{u}} \,\times\, \mathrm{H}_h^{\varphi} \,\times\, \mathrm{H}_h^{\lambda}$, with $(\boldsymbol{u}_h,\varphi_h) \in W_h$. Moreover, there hold*

$$\|(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,\le\, c_{\mathbf{S}}\left\{ r\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|\boldsymbol{u}_D\|_{0,\Gamma} \,+\, \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\}$$

and

$$\|(\varphi_h,\lambda_h)\| \,\le\, \widetilde{c}_{\widetilde{\mathbf{S}}}\left\{ r\,\|\boldsymbol{u}_h\|_{1,\Omega} \,+\, \|\varphi_D\|_{1/2,\Gamma}\right\}.$$

Proof. Thanks to Lemmas 4.4 and 4.7, it follows from a straightforward application of the Brouwer fixed point theorem (cf. Theorem 4.3). □

Furthermore, by requiring a stronger assumption on the data so that the operator $\mathbf{T}_h$ becomes a contraction, we obtain the following existence and uniqueness result for (4.2).

Theorem 4.9 *In addition to the hypotheses of Theorem 4.8, assume that the data satisfy*

$$\widetilde{C}_{\mathbf{T}} \Big\{ \|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \Big\} < 1 ,$$

where $\widetilde{C}_{\mathbf{T}}$ is the constant from Lemma 4.7. Then, problem (4.2) has a unique solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h, \lambda_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi} \times \mathrm{H}_{\widetilde{h}}^{\lambda}$, with $(\boldsymbol{u}_h, \varphi_h) \in W_h$, and the same a priori estimates from Theorem 4.8 hold.

Proof. It follows from (4.14) and a direct application of the Banach fixed point theorem. □

4.3 Specific finite element subspaces

In this section we introduce specific finite element subspaces satisfying (4.1), and the discrete inf-sup conditions given by the hypotheses **(H.1)** and **(H.2)**. In what follows, given an integer $k \geq 0$ and a set $\mathrm{S} \subseteq \mathrm{R}^n$, $\mathrm{P}_k(\mathrm{S})$ denotes the space of polynomial functions on S of degree $\leq k$. Then, with the same notations from Section 4.1, we define for each $K \in \mathcal{T}_h$ the local Raviart–Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\,\boldsymbol{x} ,$$

where, according to the terminology described in Section 1, $\mathbf{P}_k(K) := [\,\mathrm{P}_k(K)\,]^n$, and $\boldsymbol{x}$ is a generic vector in R^n. Similarly, $\mathbf{C}(\overline{\Omega}) = [C(\overline{\Omega})]^n$. Then, we introduce the finite element subspaces approximating the unknowns $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ as the global Raviart–Thomas space of order k, and the Lagrange space given by the continuous piecewise polynomial vectors of degree $\leq k+1$, respectively, that is

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \boldsymbol{c}^t\,\boldsymbol{\tau}\Big|_K \in \mathbf{RT}_k(K)\,, \quad \forall\, \boldsymbol{c} \in \mathrm{R}^n \quad \forall\, K \in \mathcal{T}_h \Big\} \tag{4.15}$$

and

$$\mathbf{H}_h^{\boldsymbol{u}} := \Big\{ \boldsymbol{v}_h \in \mathbf{C}(\overline{\Omega}) : \quad \boldsymbol{v}_h\Big|_K \in \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\} . \tag{4.16}$$

Also, the approximating space for the temperature φ is given by the continuous piecewise polynomials of degree $\leq k+1$, that is

$$\mathrm{H}_h^{\varphi} := \Big\{ \psi_h \in \mathrm{C}(\overline{\Omega}) : \quad \psi_h\Big|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\} . \tag{4.17}$$

Next, for reasons that become clear below in Lemma 4.10, we let $\big\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_m\big\}$ be an independent triangulation of Γ (made of triangles in R^3 or straight segments in R^2), and define $\widetilde{h} := \max\limits_{j\in\{1,\ldots,m\}} |\widetilde{\Gamma}_j|$. Then, with the same integer $k \geq 0$ employed in the definitions (4.15), (4.16), and (4.17), we set

$$\mathrm{H}_{\widetilde{h}}^{\lambda} := \Big\{ \xi_{\widetilde{h}} \in \mathrm{L}^2(\Gamma) : \quad \xi_{\widetilde{h}}\Big|_{\widetilde{\Gamma}_j} \in \mathrm{P}_k(\widetilde{\Gamma}_j) \quad \forall\, j \in \{1, 2, \cdots, m\} \Big\}. \tag{4.18}$$

On the other hand, in order to check that H_h^{φ} and $\mathrm{H}_{\widetilde{h}}^{\lambda}$ do satisfy the assumptions **(H1)** and **(H2)** of the previous section, we first observe that the discrete kernel of $\mathbf{b}$ is given by

$$V_h := \Big\{ \psi_h \in \mathrm{H}_h^{\varphi} : \quad \langle \xi_{\widetilde{h}}, \psi_h \rangle_\Gamma = 0 \quad \forall\, \xi_{\widetilde{h}} \in \mathrm{H}_{\widetilde{h}}^{\lambda} \Big\} .$$

In particular, $\xi_{\widetilde{h}} \equiv 1$ belongs to $\mathrm{H}^{\lambda}_{\widetilde{h}}$, and hence V_h is contained in the space

$$\widehat{V} := \left\{ \psi \in \mathrm{H}^1(\Omega) : \quad \int_{\Gamma} \psi = 0 \right\},$$

where, thanks to the generalized Poincaré inequality, $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ become equivalent. This fact together with the uniform positiveness of $\mathbb{K}$ imply that the bilinear form $\mathbf{a}$ is V_h-elliptic, and thus the assumption **(H.1)** is trivially satisfied.

In turn, concerning the discrete inf-sup condition for the bilinear form $\mathbf{b}$, we recall the following result from [20].

Lemma 4.10 *There exist $C_0 > 0$ and $\beta > 0$, independent of h and $\widetilde{h}$, such that for all $h \le C_0 \widetilde{h}$, there holds*

$$\sup_{\substack{\psi_h \in \mathrm{H}^{\varphi}_h \\ \psi_h \neq 0}} \frac{\mathbf{b}(\psi_h, \xi_{\widetilde{h}})}{\|\psi_h\|_{1,\Omega}} \ge \widehat{\beta}\, \|\xi_{\widetilde{h}}\|_{-1/2,\Gamma} \quad \forall \xi_{\widetilde{h}} \in \mathrm{H}^{\lambda}_{\widetilde{h}}. \tag{4.19}$$

Proof. It follows basically from the same arguments from [20, Lemma 4.7], where the approximating spaces for φ and λ are defined as above but with $k = 0$. In fact, it suffices to replace the orthogonal projector from $\mathrm{H}^1(\Omega)$ onto the continuous piecewise polynomials of degree ≤ 1 (employed there), by the one onto the continuous piecewise polynomials of degree $\le k+1$ (required here). Further details are omitted. □

It is important to remark here that, under the present choices of finite element subspaces, the restriction on the meshsizes required by Lemma 4.10 must be incorporated in the statements of Theorems 4.8 and 4.9, as well as henceforth in the subsequent results in which these specific spaces are involved. We end this section by recalling from [20] the approximation properties of the specific finite element subspaces introduced here.

($\mathbf{AP}^{\boldsymbol{\sigma}}_h$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div};\Omega)$ with $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\sigma}, \mathbb{H}^{\boldsymbol{\sigma}}_h) \le C\, h^s \left\{ \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \right\}. \tag{4.20}$$

($\mathbf{AP}^{\boldsymbol{u}}_h$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{u} \in \mathbf{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{u}, \mathbf{H}^{\boldsymbol{u}}_h) \le C\, h^s\, \|\boldsymbol{u}\|_{s+1,\Omega}. \tag{4.21}$$

($\mathbf{AP}^{\varphi}_h$)] there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\varphi \in \mathrm{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}(\varphi, \mathrm{H}^{\varphi}_h) \le C\, h^s\, \|\varphi\|_{s+1,\Omega}. \tag{4.22}$$

($\mathbf{AP}^{\lambda}_{\widetilde{h}}$) there exists $C > 0$, independent of $\widetilde{h}$, such that for each $s \in (0, k+1]$, and for each $\lambda \in \mathrm{H}^{-1/2+s}(\Gamma)$, there holds

$$\mathrm{dist}(\lambda, \mathrm{H}^{\lambda}_{\widetilde{h}}) \le C\, \widetilde{h}^s\, \|\lambda\|_{-1/2+s,\Gamma}. \tag{4.23}$$

5 A priori error analysis

In this section we derive an a priori error estimate for our Galerkin scheme with arbitrary finite element subspaces satisfying the hypotheses stated in Section 4.2. More precisely, given $(\boldsymbol{\sigma}, \boldsymbol{u}, \varphi, \lambda) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{1/2}(\Gamma)$, with $(\boldsymbol{u}, \varphi) \in W$, and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h, \lambda_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}$, with $(\boldsymbol{u}_h, \varphi_h) \in W_h$, solutions of problems (3.12) and (4.2), respectively, we are interested in obtaining an upper bound for

$$\|(\boldsymbol{\sigma}, \boldsymbol{u}, \varphi, \lambda) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \varphi_h, \lambda_h)\| .$$

For this purpose, we first rearrange (3.12) and (4.2) as the following pairs of continuous and discrete formulations

$$\begin{aligned} \big(\mathbf{A} + \mathbf{B}_{\boldsymbol{u}}\big)((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) &= \big(F_{\varphi} + F_D\big)(\boldsymbol{\tau}, \boldsymbol{v}) \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)\,, \\ \big(\mathbf{A} + \mathbf{B}_{\boldsymbol{u}_h}\big)((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) &= \big(F_{\varphi_h} + F_D\big)(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\,, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) &= F_{\boldsymbol{u},\varphi}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ \mathbf{b}(\varphi, \xi) &= G(\xi) \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,, \\ \mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) &= F_{\boldsymbol{u}_h,\varphi_h}(\psi_h) \qquad \forall\, \psi_h \in \mathrm{H}_h^{\varphi}\,, \\ \mathbf{b}(\varphi_h, \xi_h) &= G(\xi_h) \qquad \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,. \end{aligned} \tag{5.2}$$

Next, we recall from [30, Theorems 11.1 and 11.2] two abstract results that will be employed in our subsequent analysis. The first one is the standard Strang Lemma for elliptic variational problems, which will be straightforwardly applied to the pair (5.1). In turn, the second result is a generalized Strang-type estimate for saddle point problems whose continuous and discrete schemes differ only in the functionals involved, as it is the case of (5.2).

Lemma 5.1 *Let V be a Hilbert space, $F \in V'$, and $A : V \times V \to \mathrm{R}$ be a bounded and $V-$elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite dimensional subspaces of V, and for each $h > 0$ consider a bounded bilinear form $A_h : V_h \times V_h \to \mathrm{R}$ and a functional $F_h \in V_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic, that is, there exists a constant $\widetilde{\alpha} > 0$, independent of h, such that*

$$A_h(v_h, v_h) \,\geq\, \widetilde{\alpha}\, \|v_h\|_V^2 \quad \forall\, v_h \in V_h\,, \quad \forall h > 0\,.$$

In turn, let $u \in V$ and $u_h \in V_h$ such that

$$A(u, v) = F(v) \quad \forall\, v \in V \quad \text{and} \quad A_h(u_h, v_h) = F_h(v_h) \quad \forall\, v_h \in V_h\,.$$

Then, for each $h > 0$ there holds

$$\begin{aligned} \|u - u_h\|_V \,\leq\, C_{\mathrm{ST}} \Bigg\{ & \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_V} \\ & + \inf_{\substack{v_h \in V_h \\ v_h \neq 0}} \left(\|u - v_h\|_V + \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|A(v_h, w_h) - A_h(v_h, w_h)|}{\|w_h\|_V} \right) \Bigg\}\,, \end{aligned} \tag{5.3}$$

where $C_{\mathrm{ST}} := \widetilde{\alpha}^{-1} \max\{\,1, \|A\|\,\}$.

Lemma 5.2 *Let H and Q be Hilbert spaces, $F \in H'$, $G \in Q'$, and let $a : H \times H \to \mathrm{R}$ and $b : H \times Q \to \mathrm{R}$ be bounded bilinear forms satisfying the hypotheses of the Babuška-Brezzi theory. Furthermore, let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be sequences of finite dimensional subspaces of H and Q, respectively, and for each $h > 0$ consider functionals $F_h \in H'_h$ and $G_h \in Q'_h$. In addition, assume that a and b satisfy the hypotheses of the discrete Babuška-Brezzi theory uniformly on H_h and Q_h, that is, there exist positive constants $\bar{\alpha}$ and $\bar{\beta}$, independent of h, such that, denoting by V_h the discrete kernel of b, there holds*

$$\sup_{\substack{\psi_h\in V_h\\ \psi_h\neq 0}} \frac{a(\psi_h,\,\psi_h)}{\|\psi_h\|_{1,\Omega}} \geq \bar{\alpha}\,\|\psi_h\|_{1,\Omega} \quad \forall \psi_h \in V_h \quad \text{and} \quad \sup_{\substack{\psi_h\in H_h\\ \psi_h\neq 0}} \frac{b(\psi_h,\xi_h)}{\|\psi_h\|_H} \geq \bar{\beta}\,\|\xi_h\|_Q \quad \forall \xi_h \in Q_h. \tag{5.4}$$

In turn, let $(\varphi,\lambda) \in H \times Q$ and $(\varphi_h,\lambda_h) \in H_h \times Q_h$, such that

$$\begin{aligned} a(\varphi,\psi) + b(\psi,\lambda) &= F(\psi) \qquad \forall\, \psi \in H \\ b(\varphi,\xi) &= G(\xi) \qquad \forall\, \xi \in Q\,, \end{aligned}$$

and

$$\begin{aligned} a(\varphi_h,\psi_h) + b(\psi_h,\lambda_h) &= F_h(\psi_h) \qquad \forall\, \psi_h \in H_h \\ b(\varphi_h,\xi_h) &= G_h(\xi_h) \qquad \forall\, \xi_h \in Q_h\,. \end{aligned}$$

Then, for each $h > 0$ there holds

$$\begin{aligned} \|\varphi - \varphi_h\|_H + \|\lambda - \lambda_h\|_Q \leq \bar{C}_{\mathrm{ST}} \Bigg\{ &\inf_{\substack{\psi_h\in H_h\\ \psi_h\neq 0}} \|\varphi - \psi_h\|_H + \inf_{\substack{\xi_h\in Q_h\\ \xi_h\neq 0}} \|\lambda - \xi_h\|_Q \\ &+ \sup_{\substack{\phi_h\in H_h\\ \phi_h\neq 0}} \frac{|F(\phi_h) - F_h(\phi_h)|}{\|\phi_h\|_H} + \sup_{\substack{\eta_h\in Q_h\\ \eta_h\neq 0}} \frac{|G(\eta_h) - G_h(\eta_h)|}{\|\eta_h\|_H} \Bigg\} \end{aligned} \tag{5.5}$$

where $\bar{C}_{\mathrm{ST}}$ is a positive constant depending only on $\|a\|$, $\|b\|$, $\bar{\alpha}$ and $\bar{\beta}$.

In what follows, we denote as usual

$$\mathrm{dist}\Big((\boldsymbol{\sigma},\boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\Big) = \inf_{(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in \mathbb{H}_h^{\boldsymbol{\sigma}}\times \mathbf{H}_h^{\boldsymbol{u}}} \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|$$

and

$$\mathrm{dist}\Big((\varphi,\lambda), \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\lambda}\Big) = \inf_{(\psi_h,\xi_h)\in \mathrm{H}_h^{\varphi}\times \mathrm{H}_h^{\lambda}} \|(\varphi,\lambda) - (\psi_h,\xi_h)\|$$

Then, we have the following lemma establishing a preliminary estimate for $\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|$.

Lemma 5.3 *Let $C_{\mathrm{ST}} := \dfrac{2}{\alpha(\Omega)} \max\{1, \|\,\mathbf{A} + \mathbf{B}_{\boldsymbol{u}}\,\|\}$, where $\alpha(\Omega)$ is the constant yielding the ellipticity of both $\mathbf{A}$ and $\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}$ for any $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$ (cf. (3.30) and (3.31) in the proof of Lemma 3.3). Then, there holds*

$$\begin{aligned} \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \leq C_{\mathrm{ST}} \Bigg\{ &\Big(1 + c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big)\, \mathrm{dist}\Big((\boldsymbol{\sigma},\boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\Big) \\ &+ c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\,\|\boldsymbol{u}\|_{1,\Omega} + (\mu^2+\kappa_2^2)^{1/2}\,\|\boldsymbol{g}\|_{\infty,\Omega}\,\|\varphi-\varphi_h\|_{0,\Omega} \Bigg\}. \end{aligned} \tag{5.6}$$

Proof. From Lemma 3.3 we have that the bilinear forms $\mathbf{A} + \mathbf{B}_{\boldsymbol{u}}$ and $\mathbf{A} + \mathbf{B}_{\boldsymbol{u}_h}$ are both bounded and elliptic with the same constant $\frac{2}{\alpha(\Omega)}$. Also, $F_\varphi + F_D$ and $F_{\varphi_h} + F_D$ are bounded linear functionals in $\mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ and $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$, respectively. Then, a straightforward application of Lemma 5.1 to the context (5.1) gives

$$\begin{aligned}\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,\le\, C_{\mathrm{ST}} \Bigg\{ \Big\| F_{\varphi-\varphi_h}\Big|_{\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}}\Big\| \\ + \inf_{\substack{(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\\ (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\neq\mathbf{0}}} \left(\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\| + \sup_{\substack{(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\\ (\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\neq\mathbf{0}}} \frac{|\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}((\boldsymbol{\tau}_h,\boldsymbol{v}_h),(\boldsymbol{\zeta}_h,\boldsymbol{w}_h))|}{\|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\|}\right)\Bigg\}. \end{aligned} \tag{5.7}$$

where $C_{\mathrm{ST}} := \frac{2}{\alpha(\Omega)} \max\{1, \|\mathbf{A} + \mathbf{B}_{\boldsymbol{u}}\|\}$. We now proceed to estimate each term appearing at the right-hand side of the foregoing inequality. Firstly, employing (3.33) (cf. proof of Lemma 3.3) with $\phi = \varphi - \varphi_h$, we readily obtain

$$\Big\| F_{\varphi-\varphi_h}\Big|_{\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}}\Big\| \,\le\, (\mu^2+\kappa^2)^{1/2}\, \|\boldsymbol{g}\|_{\infty,\Omega}\, \|\varphi - \varphi_h\|_{0,\Omega}\,. \tag{5.8}$$

In turn, by applying (3.28) with $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_h$, adding and substracting $\boldsymbol{u}$, and then bounding $\|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,\Omega}$ by $\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|$, we find that

$$\begin{aligned} |\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}((\boldsymbol{\tau}_h,\boldsymbol{v}_h),(\boldsymbol{\zeta}_h,\boldsymbol{w}_h))| &\le c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, \|\boldsymbol{v}_h\|_{1,\Omega}\, \|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\| \\ &\le c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|\, \|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\| \\ &+ c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, \|\boldsymbol{u}\|_{1,\Omega}\, \|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\|\,, \end{aligned}$$

which yields

$$\begin{aligned} \sup_{\substack{(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\\ (\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\neq\mathbf{0}}} \frac{|\mathbf{B}_{\boldsymbol{u}-\boldsymbol{u}_h}((\boldsymbol{\tau}_h,\boldsymbol{v}_h),(\boldsymbol{\zeta}_h,\boldsymbol{w}_h))|}{\|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h)\|} &\le c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, \|\boldsymbol{u}\|_{1,\Omega} \\ &+ c_1(\Omega)\,(\kappa_1^2+1)^{1/2} \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|\,. \end{aligned} \tag{5.9}$$

In this way, by replacing (5.8) and (5.9) back into (5.7), and applying the infimum to the resulting term having $\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|$ as a factor, we get (5.6) and conclude the proof. $\square$

Next, as for the error $\|(\varphi,\lambda) - (\varphi_h,\lambda_h)\|$ arising from (5.2), we have the following result.

Lemma 5.4 *There exists a constant $\widehat{C}_{\mathrm{ST}} > 0$, depending only on $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\widehat{\alpha}$ (cf. (4.8)) and $\widehat{\beta}$ (cf. (4.9)), such that*

$$\begin{aligned} \|(\varphi,\lambda) - (\varphi_h,\lambda_h)\| \,\le\, \widehat{C}_{\mathrm{ST}} \Big\{ c_2(\Omega)\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\, |\varphi|_{1,\Omega} \\ + c_2(\Omega)\, \|\boldsymbol{u}_h\|_{1,\Omega}\, |\varphi-\varphi_h|_{1,\Omega} + \mathrm{dist}\Big((\varphi,\lambda), \mathbb{H}_h^{\varphi}\times\mathbf{H}_h^{\lambda}\Big)\Big\}\,. \end{aligned} \tag{5.10}$$

Proof. We first observe that (**H.1**) and (**H.2**) from Section 4.2 guarantee that the hypothesis (5.4) in Lemma 5.2 is satisfied. Hence, by applying this lemma to the context given by (5.2), we find that the corresponding estimate (5.5) becomes

$$\|(\varphi,\lambda) - (\varphi_h,\lambda_h)\| \,\le\, \widehat{C}_{\mathrm{ST}} \left\{ \Big\| \Big(F_{\boldsymbol{u},\varphi} - F_{\boldsymbol{u}_h,\varphi_h}\Big)\Big|_{\mathbb{H}_h^{\varphi}} \Big\| + \mathrm{dist}\Big((\varphi,\lambda), \mathbb{H}_h^{\varphi}\times\mathbf{H}_h^{\lambda}\Big)\right\}, \tag{5.11}$$

where $\widehat{C}_{\rm ST}$ is a positive constant depending only on $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\widehat{\alpha}$, and $\widehat{\beta}$. Next, by rewriting

$$F_{\boldsymbol{u},\varphi} - F_{\boldsymbol{u}_h,\varphi_h} = F_{\boldsymbol{u}-\boldsymbol{u}_h,\varphi} + F_{\boldsymbol{u}_h,\varphi-\varphi_h}\,,$$

and using the bound (3.37), we deduce that

$$\left\|\Big(F_{\boldsymbol{u}-\boldsymbol{u}_h,\varphi} + F_{\boldsymbol{u}_h,\varphi_h-\varphi_h}\Big)\Big|_{\mathbb{H}_h^\varphi}\right\| \,\le\, c_2(\Omega)\Big\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\,|\varphi|_{1,\Omega} + \|\boldsymbol{u}_h\|_{1,\Omega}\,|\varphi-\varphi_h|_{1,\Omega}\Big\}\,.$$

Finally, the required estimate (5.10) follows by replacing the foregoing inequality in (5.11). □

We are now in a position to derive the Céa estimate for the global error

$$\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,+\, \|(\varphi,\lambda) - (\varphi_h,\lambda_h)\|\,.$$

Indeed, by adding the estimates (5.6) and (5.10) from Lemmas 5.3 and 5.4, respectively, we find that

$$\begin{aligned}
\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,+\, \|(\varphi,\lambda) - (\varphi_h,\lambda_h)\| \,&\le\, \widehat{C}_{\rm ST}\,\mathrm{dist}\Big((\varphi,\lambda),\mathbb{H}_h^\varphi\times\mathbf{H}_h^\lambda\Big)\\
+\quad & C_{\rm ST}\,\Big(1 + c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\Big((\boldsymbol{\sigma},\boldsymbol{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\Big)\\
+\quad & \Big(\widehat{C}_{\rm ST}\,c_2(\Omega)\,\|\boldsymbol{u}_h\|_{1,\Omega} + C_{\rm ST}\,(\mu^2+\kappa_2^2)^{1/2}\,\|\boldsymbol{g}\|_{\infty,\Omega}\Big)\,\|\varphi-\varphi_h\|_{1,\Omega}\\
+\quad & \Big(\widehat{C}_{\rm ST}\,c_2(\Omega)\,|\varphi|_{1,\Omega} + C_{\rm ST}\,c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\,\|\boldsymbol{u}\|_{1,\Omega}\Big)\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\,.
\end{aligned}$$

Next, employing the estimates for $\boldsymbol{u}$, φ, and $\boldsymbol{u}_h$ given by (3.27), (3.36), and (4.7), respectively, and then performing some algebraic manipulations, we find that

$$\begin{aligned}
\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,+\, \|(\varphi,\lambda) - (\varphi_h,\lambda_h)\| \,&\le\, \widehat{C}_{\rm ST}\,\mathrm{dist}\Big((\varphi,\lambda),\mathbb{H}_h^\varphi\times\mathbf{H}_h^\lambda\Big)\\
+\quad & C_{\rm ST}\,\Big(1 + c_1(\Omega)\,(\kappa_1^2+1)^{1/2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}\Big)\,\mathrm{dist}\Big((\boldsymbol{\sigma},\boldsymbol{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\Big)\\
+\quad & \mathbf{C}(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D)\,\Big\{\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\| \,+\, \|(\varphi,\lambda) - (\varphi_h,\lambda_h)\|\Big\}\,,
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
\mathbf{C}(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D) \quad &:= \quad \max\Big\{\mathbf{C}_1(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D),\mathbf{C}_2(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D)\Big\}\,,\\
\mathbf{C}_1(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D) \quad &:= \quad \Big\{r\,C_1 + C_2\Big\}\,\|\boldsymbol{g}\|_{\infty,\Omega} + C_1\,\Big\{\|\boldsymbol{u}_D\|_{0,\Gamma} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\Big\}\,,\\
\mathbf{C}_2(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D) \quad &:= \quad C_3\,\Big\{r\,\|\,\boldsymbol{g}\,\|_{\infty,\Omega} \,+\, \|\,\boldsymbol{u}_D\,\|_{0,\Gamma} \,+\, \|\,\boldsymbol{u}_D\,\|_{1/2,\Gamma}\Big\} \,+\, C_4\,\|\,\varphi_D\,\|_{1/2,\Gamma}\,,
\end{aligned}$$

and the constants C_1, C_2, C_3, and C_4, are given by

$$C_1 \,:=\, \widehat{C}_{\rm ST}\,c_2(\Omega)\,c_{\mathbf{S}}\,,\quad C_2 \,:=\, C_{\rm ST}\,(\mu^2+\kappa_2^2)^{1/2}\,,$$

$$C_3 \,:=\, c_{\mathbf{S}}\,\Big\{\widehat{C}_{\rm ST}\,c_2(\Omega) \,+\, r\,c_{\widetilde{\mathbf{S}}} \,+\, C_{\rm ST}\,c_1(\Omega)\,(\kappa_1^2\,+\,1)^{1/2}\Big\}\,,\quad \text{and}\quad C_4 \,:=\, \widehat{C}_{\rm ST}\,c_2(\Omega)\,c_{\widetilde{\mathbf{S}}}$$

In this way, since the expression multiplying $\mathrm{dist}\Big((\boldsymbol{\sigma},\boldsymbol{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\Big)$ in (5.12) is already controlled by constants, parameters, and data only, and since the constants $\mathbf{C}_i(\boldsymbol{g},\boldsymbol{u}_D,\varphi_D)$, $i\in\{1,2\}$, depend linearly on the data $\boldsymbol{g}$, $\boldsymbol{u}_D$, and φ_D, we conclude from the foregoing analysis the following main result.

Theorem 5.5 *Asume that the data $\boldsymbol{g}$, $\boldsymbol{u}_D$ and φ_D are such that*

$$\mathbf{C}_i(\boldsymbol{g}, \boldsymbol{u}_D, \varphi_D) \,\le\, \frac{1}{2} \qquad \forall\, i \,\in\, \{1, 2\}\,. \tag{5.13}$$

Then, there exits a positive constant C_5, depending only on parameters, data and other constants, all of them independent of h, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\| \,+\, \|(\varphi, \lambda) - (\varphi_h, \lambda_h)\| \\ &\le\, C_5 \left\{ \operatorname{dist}\Big((\boldsymbol{\sigma}, \boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\Big) \,+\, \operatorname{dist}\Big((\varphi, \lambda), \mathbb{H}_h^{\varphi} \times \mathbf{H}_h^{\lambda}\Big) \right\}. \end{aligned} \tag{5.14}$$

Proof. It suffices to realize from (5.13) that $\mathbf{C}(\boldsymbol{g}, \boldsymbol{u}_D, \varphi_D) \,\le\, \frac{1}{2}$, which, combined with (5.12), yields

$$\begin{aligned} \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\| \,+\, \|(\varphi, \lambda) - (\varphi_h, \lambda_h)\| \,&\le\, 2\, \widehat{C}_{\mathrm{ST}} \operatorname{dist}\Big((\varphi, \lambda), \mathbb{H}_h^{\varphi} \times \mathbf{H}_h^{\lambda}\Big) \\ +\quad 2\, C_{\mathrm{ST}} \Big(1 + c_1(\Omega)\, (\kappa_1^2 + 1)^{1/2}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\Big) &\operatorname{dist}\Big((\boldsymbol{\sigma}, \boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\Big)\,. \end{aligned}$$

The rest of the proof reduces to employ the upper bounds for $\|\boldsymbol{u}\|_{1,\Omega}$ and $\|\boldsymbol{u}_h\|_{1,\Omega}$. □

Finally, we complete our a priori error analysis with the rates of convergence of the Galerkin scheme when the specific finite element subspaces introduced in Section 4.3 are employed.

Theorem 5.6 *In addition to the hypotheses of Theorems* 3.9, 4.9*, and* 5.5*, assume that there exists $s > 0$ such that $\boldsymbol{\sigma} \,\in\, \mathbb{H}^s(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \,\in\, \mathbf{H}^s(\Omega)$, $\boldsymbol{u} \,\in\, \mathbf{H}^{s+1}(\Omega)$, $\varphi \,\in\, \mathrm{H}^{s+1}(\Omega)$, and $\lambda \,\in\, \mathrm{H}^{-1/2+s}(\Gamma)$, and that the finite element subspaces are defined by* (4.15)*,* (4.16)*,* (4.17)*, and* (4.18)*. Then, there exist $C \,>\, 0$, independent of h and $\widetilde{h}$, such that for all $h \,\le\, C_0\, \widetilde{h}$ there holds*

$$\begin{aligned} \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\| \,+\, \|(\varphi, \lambda) - (\varphi_h, \lambda_h)\| \,&\le\, C\, \widetilde{h}^{\min\{s,k+1\}}\, \|\lambda\|_{-1/2+s,\Gamma} \\ +\quad C\, h^{\min\{s,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{s,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \,&+\, \|\boldsymbol{u}\|_{s+1,\Omega} \,+\, \|\varphi\|_{s+1,\Omega} \Big\}\,. \end{aligned} \tag{5.15}$$

Proof. It follows from the Céa estimate (5.14) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\boldsymbol{u}})$, $(\mathbf{AP}_h^{\varphi})$ and $(\mathbf{AP}_{\widetilde{h}}^{\boldsymbol{\lambda}})$ specified in Section 4.3. □

We end this section by remarking that, for practical purposes, particularly for the implementation of the examples reported below in Section 6, the restriction on the meshsizes is verified in an heuristic sense only. More precisely, since the constant C_0 involved there is actually unknown, we simply assume $C_0 \,=\, 1/2$ and consider a partition of Γ with a meshsize $\widetilde{h}$ given approximately by the double of h. The numerical results to be provided in that section will confirm the suitability of this choice.

6 Numerical results

In this section we present two examples illustrating the performance of our augmented mixed-primal finite element scheme (4.2) on a set of quasi-uniform triangulations of the corresponding domains and considering the finite element spaces introduced in Section 4.3. Our implementation is based on a *FreeFem++* code (see [23]), in conjunction with the direct linear solver UMFPACK (see [14]).

Regarding the implementation of the iterative methods, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\boldsymbol{u}}$, H_h^{φ} and H_h^{λ} and tol is a fixed tolerance to be specified on each example. For each example shown below we simply take $(\boldsymbol{u}_h^0, \varphi_h^0) = (\mathbf{0}, 0)$ as initial guess, and we choose the parameters $\kappa_1 = \mu$, $\kappa_2 = \mu^2$ and $\kappa_2 = \mu^2$, which clearly satisfy the hypotheses of Lemma 4.8, with $\delta = \mu$.

We now introduce some additional notation. The individual errors are denoted by:

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}, \quad \mathsf{e}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}, \quad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega},$$

$$\mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega}, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{0,\Gamma},$$

where p is the exact pressure of the fluid and p_h is the postprocessed discrete pressure suggested by the formulae given in (2.5) and (3.3), namely,

$$p_h = -\frac{1}{n}\,\mathrm{tr}\Big\{\boldsymbol{\sigma}_h + c_h\mathbb{I} + (\boldsymbol{u}_h \otimes \boldsymbol{u}_h)\Big\}, \quad \text{with} \quad c_h := -\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{u}_h \otimes \boldsymbol{u}_h).$$

Moreover, it is not difficult to show that there exists $C > 0$, independent of h, such that

$$\|p - p_h\|_{0,\Omega} \leq C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\right\},$$

which says that the rate of convergence of p_h is the same provided by (5.15) (cf. Theorem 5.6).

Next, we let $r(\boldsymbol{\sigma})$, $r(\boldsymbol{u})$, $r(p)$, $r(\varphi)$, and $r(\lambda)$ be the experimental rates of convergence given by

$$r(\boldsymbol{\sigma}) := \frac{\log(\mathsf{e}(\boldsymbol{\sigma})/\mathsf{e}'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad r(\boldsymbol{u}) := \frac{\log(\mathsf{e}(\boldsymbol{u})/\mathsf{e}'(\boldsymbol{u}))}{\log(h/h')}, \quad r(p) := \frac{\log(\mathsf{e}(p)/\mathsf{e}'(p))}{\log(h/h')},$$
$$r(\varphi) := \frac{\log(\mathsf{e}(\varphi)/\mathsf{e}'(\varphi))}{\log(h/h')}, \quad r(\lambda) := \frac{\log(\mathsf{e}(\lambda)/\mathsf{e}'(\lambda))}{\log(\tilde{h}/\tilde{h}')}.$$

where h and h', ($\tilde{h}$ and $\tilde{h}'$ for λ) denote two consecutive meshsizes with errors e and e'.

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega := (-1/2, 3/2)\times(0, 2)$. We consider the viscosity $\mu = 1$, the thermal conductivity $\mathbb{K} = e^{x_1+x_2}\mathbb{I}\ \forall\,(x_1, x_2) \in \Omega$, and the external force $\mathbf{g} = (0, -1)^t$. Then, the terms on the right-hand sides are adjusted so that the exact solution is given by the functions

$$\begin{aligned}
\varphi(x_1, x_2) &= x_1^2(x_2^2 + 1),\\
\boldsymbol{u}(x_1, x_2) &= \begin{pmatrix} 1 - e^{\lambda x_1}\cos(2\pi x_2) \\ \frac{\lambda}{2\pi}e^{\lambda x_1}\sin(2\pi x_2) \end{pmatrix},\\
p(x_1, x_2) &= -\frac{1}{2}e^{2\lambda x_1} + \bar{p},
\end{aligned}$$

where

$$\lambda := \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-2} + 16\pi^2}}.$$

and the constant $\bar{p}$ is such that $\int_\Omega p = 0$. Notice that $(\boldsymbol{u}, p)$ is the well known analytical solution for the Navier-Stokes problem obtained by Kovasznay in [27], which presents a boundary layer at $\{-1/2\} \times (0,2)$.

In Table 1 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the finite element spaces introduced in Section 4.3 with $k = 0$ and $k = 1$, and solving the nonlinear problem with the fixed-point iteration provided in Section 4.2 with a tolerance $tol = 1E{-}8$. We observe there that the rate of convergence $O(h^{k+1})$ predicted by Theorem 5.6 (when $s = k+1$) is attained in all the cases. Next, in Figures 1, 2 and 3 we display (to the left) the approximate temperature, the approximate velocity magnitude and vector field, and the approximate pressure, respectively, and we compare them with their corresponding exact counterparts (to the right). All the figures were built using the $\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximation with $N = 177320$ degrees of freedom. In all the cases we observe that the finite element subspaces employed provide very accurate approximations to the unknowns, showing a good behaviour on the boundary layer.

In our second example we illustrate a more realistic situation in which the exact solution is unknown. Here, we consider the geometry $\Omega = (-1,1) \times (-1,2)$, the viscosity fluid $\mu = 1$, the thermal conductivity $\mathrm{K} = \mathbb{I}$, the external force $\mathbf{g} = (0,-1)^t$, and the boundary data

$$\boldsymbol{u}_D(x_1,x_2) = 0 \quad \text{and} \quad \varphi_D(x_1,x_2) := (x_1+1)e^{x_1 x_2} \quad \text{on} \quad \Gamma.$$

Notice, that φ_D attains its maximum value at $(x_1,x_2) = (1,1)$, whereas $\varphi_D = 0$ on $\{-1\} \times (-1,1)$. In Table 2 we summarize the convergence history for a sequence of uniform triangulations, considering a $\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximation and a tolerance $tol = 1E{-}8$. There, the errors and experimental rates of convergence are computed by considering the discrete solution obtained with a finer mesh ($N = 2822774$) as the exact solution. We observe that the rate of convergence $O(h)$ is attained by all the unknowns. Next, in Figure 4 we display the approximates temperature (left) and pressure (right) whereas in Figure 5 we show the first and second components of the velocity (bottom) together with the velocity magnitude and the velocity vector field (top). All the figures were obtained with N=177644 degrees of freedom. We can observe that the discrete temperature and velocity preserve the prescribed boundary conditions.

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ERRORS AND RATES OF CONVERGENCE FOR THE MIXED-PRIMAL
$RT_0 - P_1 - P_1 - P_0$ APPROXIMATION

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathsf{e}(p)$	$r(p)$
806	0.3802	73.0680	–	39.1463	–	4.8682	–
2934	0.1901	44.1852	0.7257	21.5882	0.8586	2.7057	0.5248
11321	0.0968	24.3903	0.8578	11.3580	0.9271	1.4606	1.1140
44313	0.0530	11.6299	1.2664	5.2548	1.3180	0.7152	1.4531
177320	0.0266	5.7070	1.0322	2.5486	1.0492	0.3541	1.1283
700032	0.0142	2.8348	1.1174	1.2442	1.1452	0.0798	1.1611

N	h	$\mathsf{e}(\varphi)$	$r(\varphi)$	$\tilde{h}$	$\mathsf{e}(\lambda)$	$r(\lambda)$	Iterations
806	0.3802	1.3109	–	0.5000	88.1781	–	13
2934	0.1901	0.5472	1.2606	0.2500	45.3437	0.9595	17
11321	0.0968	0.2581	1.0845	0.1250	22.1691	1.0323	18
44313	0.0530	0.1305	1.1660	0.0625	10.8920	1.0253	19
177320	0.0266	0.0639	1.0348	0.0312	5.3797	1.0177	19
700032	0.0142	0.0318	1.1131	0.0156	2.6694	1.0110	20

ERRORS AND RATES OF CONVERGENCE FOR THE MIXED-PRIMAL
$RT_1 - P_2 - P_2 - P_1$ APPROXIMATION

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathsf{e}(p)$	$r(p)$
2686	0.3802	28.7866	–	9.9080	–	12.8970	–
10078	0.1901	9.0869	1.6635	3.2510	1.6077	3.4669	1.8953
39550	0.0968	2.5644	1.9156	0.8685	1.9985	0.9029	2.0370
156158	0.0530	0.5872	2.3887	0.1913	2.4518	0.2070	2.3867
627678	0.0266	0.1429	2.0490	0.0442	2.1239	0.0475	2.1352

N	h	$\mathsf{e}(\varphi)$	$r(\varphi)$	$\tilde{h}$	$\mathsf{e}(\lambda)$	$r(\lambda)$	Iterations
2686	0.3802	0.1358	–	0.5000	10.0095	–	25
10078	0.1901	0.0240	2.5018	0.2500	2.5666	1.9634	19
39550	0.0968	0.0045	2.5203	0.1250	0.6438	1.9953	19
156158	0.0530	0.0009	2.5911	0.0625	0.1609	2.0006	20
627678	0.0266	0.0002	2.2535	0.0312	0.0402	2.0010	20

Table 1: EXAMPLE 1: Degrees of freedom, meshsizes, errors, rates of convergence and number of iterations for the mixed $RT_0 - P_1 - P_1 - P_0$ and $RT_1 - P_2 - P_2 - P_1$ approximations of the boussinesq equations.

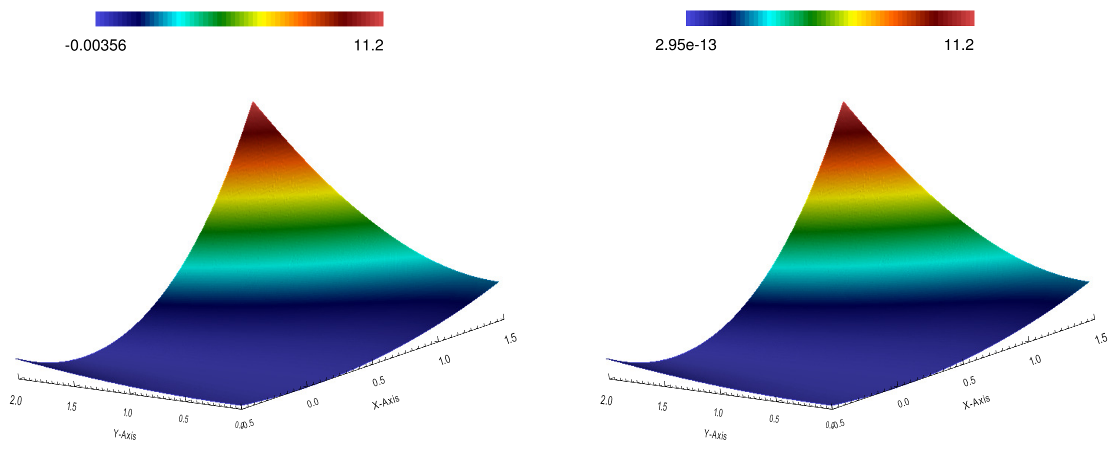

Figure 1: Example 1: φ_h (left) and φ (right) with $N = 177320$ ($\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$).

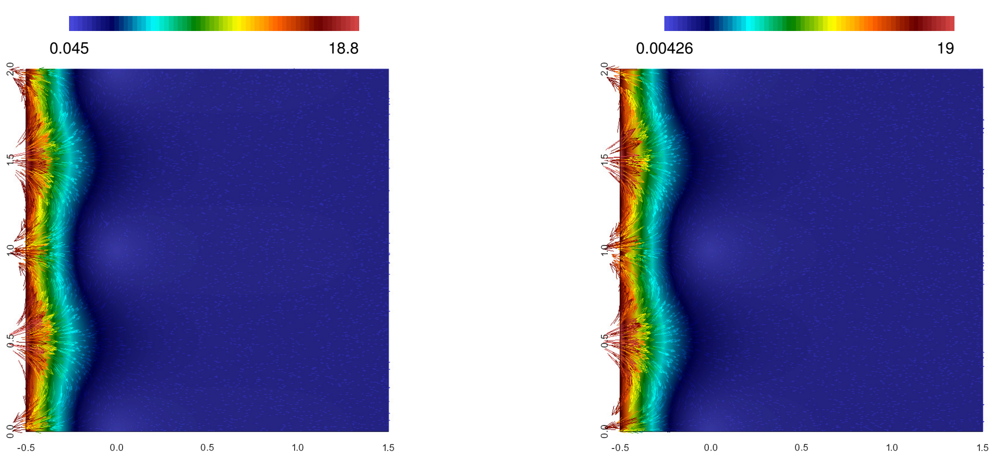

Figure 2: Example 1: velocitiy magnitudes $|\boldsymbol{u}_h|$ (left) and $|\boldsymbol{u}|$ (right) and velocity vector fields with $N = 177320$ ($\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$).

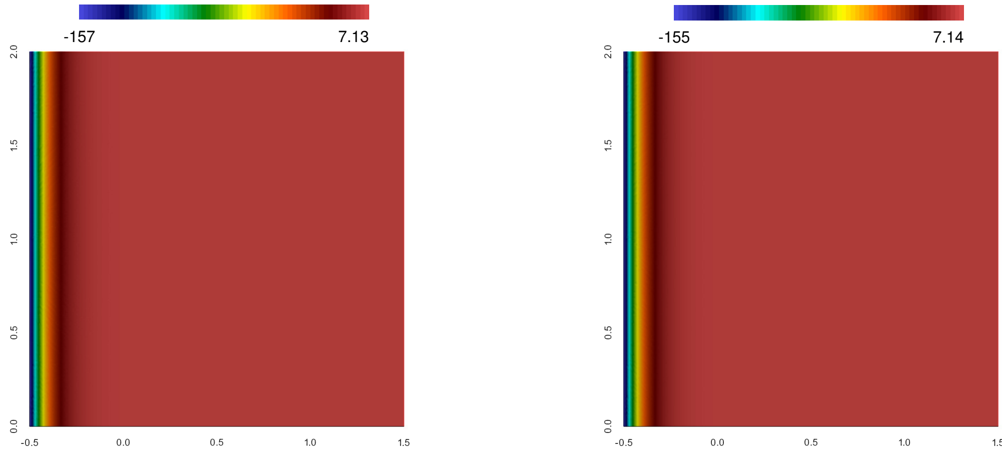

Figure 3: Example 1: postprocessed discrete pressure p_h (left) and exact pressure (right) with $N = 177320$ ($\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$).

ERRORS AND RATES OF CONVERGENCE FOR THE MIXED-PRIMAL
$\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ APPROXIMATION

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathsf{e}(p)$	$r(p)$
815	0.4129	0.3208	–	0.7711	–	0.1830	–
2997	0.1901	0.1593	0.9539	0.3621	0.9744	0.0918	0.8898
11357	0.0968	0.0759	1.1342	0.1663	1.1525	0.0406	1.2062
44412	0.0527	0.0394	1.1540	0.0853	1.0984	0.0199	1.1783
177644	0.0307	0.0196	1.3091	0.0419	1.3123	0.0099	1.2795
701022	0.0150	0.0105	0.9685	0.0211	0.9599	0.0054	0.8523

N	h	$\mathsf{e}(\varphi)$	$r(\varphi)$	$\tilde{h}$	$\mathsf{e}(\lambda)$	$r(\lambda)$	Iterations
815	0.4129	0.7301	–	0.2500	1.8899	–	9
2997	0.1901	0.3461	0.9620	0.1250	1.1122	0.7649	8
11357	0.0968	0.1589	1.1526	0.0625	0.5825	0.9331	9
44412	0.0527	0.0806	1.1180	0.0312	0.2992	0.9609	9
177644	0.0307	0.0401	1.2887	0.0156	0.1487	1.0091	9
701022	0.0150	0.0205	0.9433	0.0078	0.0695	1.0977	9

Table 2: EXAMPLE 2: Degrees of freedom, meshsizes, errors, rates of convergence and number of iterations for the mixed $\mathrm{RT}_0 - \mathrm{P}_1 - \mathrm{P}_1 - \mathrm{P}_0$ approximations of the boussinesq equations with unknown solution.

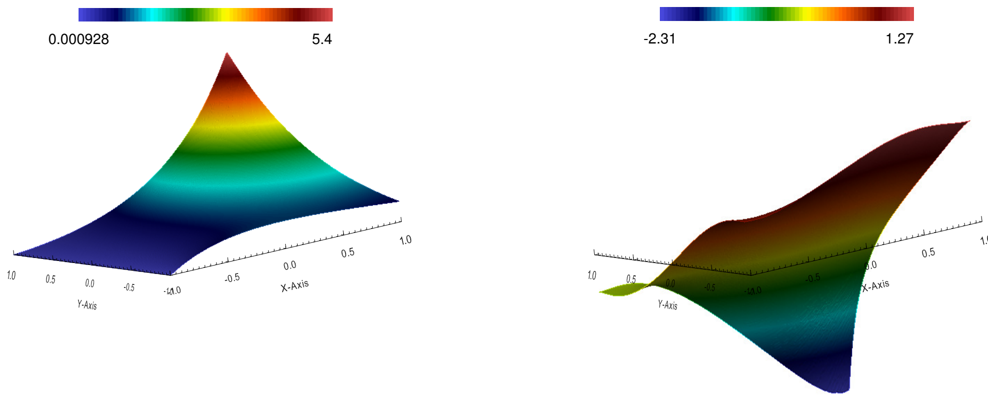

Figure 4: Example 1: p_h (left) and φ_h (right) with $N = 177320$

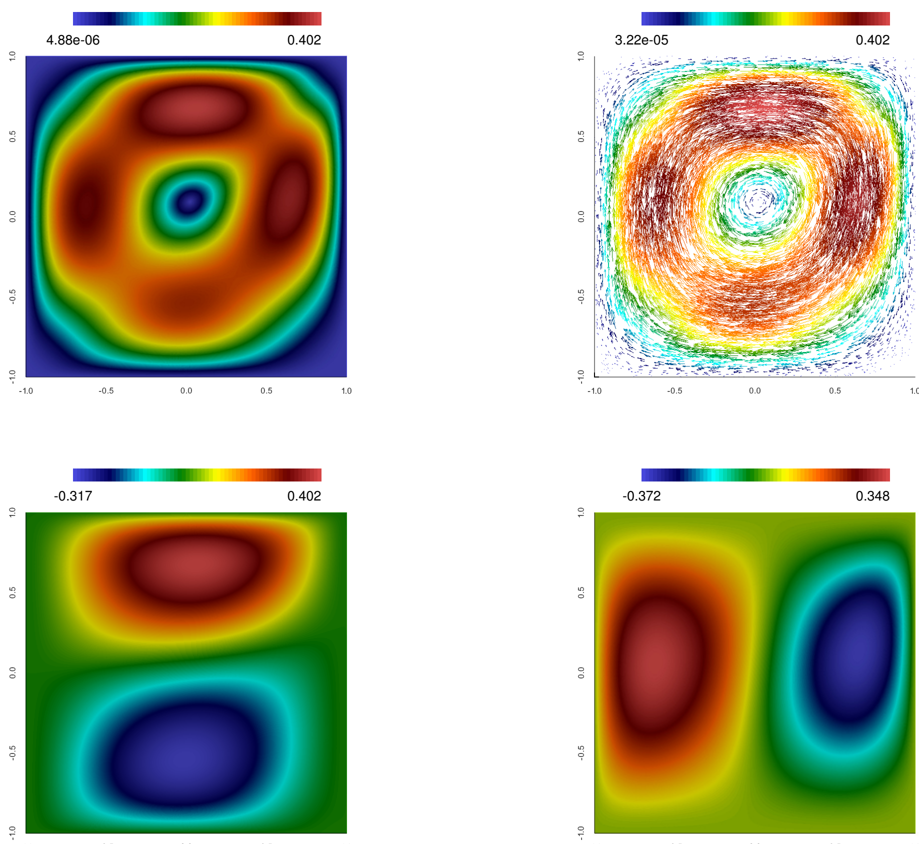

Figure 5: Example 2: velocity magnitude (top left), velocity vector field (top right), first component of $\boldsymbol{u}_h$ (bottom left) and second component fo $\boldsymbol{u}_h$ (bottom right) with $N = 701022$

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