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Analysis of a vorticity-based fully-mixed formulation for the 3D Brinkman-Darcy problem*

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Abstract

We propose and analyze a fully-mixed finite element method to numerically approximate the flow patterns of a viscous fluid within a highly permeable medium, described by Brinkman equations, and its interaction with pure porous media flow under Darcy's law. The system is formulated in terms of velocity and pressure of the porous medium, together with vorticity, velocity and pressure of the fluid. In addition, the tangential component of the vorticity is supposed to vanish on the whole boundary of the fluid, whereas null normal components of both velocities are assumed on the respective boundaries, except on the interface where suitable transmission conditions are considered. In this way, the derivation of the corresponding mixed variational formulation leads to a Lagrange multiplier enforcing the pressure continuity across the interface, whereas mass balance results from essential boundary conditions on each domain. As a consequence, a typical saddle-point operator equation is obtained, and hence the classical Babuška-Brezzi theory is applied to establish the well-posedness of the continuous and discrete schemes. In particular, we remark that the continuous and discrete inf-sup conditions of the main bilinear form are proved by using suitably chosen injective operators to get lower bounds of the corresponding suprema, which constitutes a previously known technique, recently denominated T-coercivity. In turn, and consistent with the above, the stability of the Galerkin scheme requires that the **curl** of the finite element subspace approximating the vorticity be contained in the space where the discrete velocity of the fluid lives, which yields Raviart-Thomas and Nédélec finite element subspaces as feasible choices. Then we show that the aforementioned constraint can be avoided by augmenting the mixed formulation with a residual arising from the Brinkman momentum equation. Finally, several numerical examples illustrating the good performance of the methods and confirming the theoretical rates of convergence are reported.

Key words: Brinkman equation, Darcy equation, vorticity-based formulation, mixed finite elements, error analysis.

Mathematics Subject Classifications (1991): 65N30, 65N12, 76D07, 65N15

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1 Introduction

This paper is motivated by the numerical approximation of flow patterns in an heterogeneous media composed by a porous medium, where Darcy equations govern the flow behavior of an incompressible fluid, and a much more permeable domain, where the viscous laminar flow regime can be described by the linear Brinkman model. The two domains are separated by an essentially fixed interface. Such a scenario is often encountered in e.g. the modelling of surface and subsurface flow in porous media, petroleum reservoirs, or perfusion of physiological fluids into soft tissues, focusing typically on filtration or other similar processes of interest. We are also interested in accurately recovering the additional vorticity field (vectorial for three-dimensional flows and perpendicular to the plane of the flow and therefore considered scalar in 2D), which yields better information on circulation effects of the free fluid, sometimes observed near interfaces.

At the interface of the two domains, and depending on the specific form of the problem at hand, one typically requires preservation of physical quantities such as normal velocities, normal stresses, and so on. An abundant body of literature is devoted to different ways of treating the interface conditions, from both mathematical and numerical perspectives. These basically include sequential substructuring methods, where decoupled subproblems are solved on each subdomain, followed by an updating of the interface field values, then using these values as boundary data to solve a local problem on the other subdomain, and iterating in some adequate manner (see e.g. [7, 15, 28, 31]); and monolithic, fully coupled approaches where all sought fields are computed at once, for instance by a single operator acting on the two media or with the aid of Lagrange multipliers specifically designed to impose continuity of fields to be conserved across the interface (see for instance [6, 13, 17, 18]). Our method follows the latter strategy. Up to our knowledge, the coupling of Brinkman and Darcy flows has been only addressed in terms of the primal unknowns of velocity and pressure [11, 17, 28]. Vorticity-based formulations for the Stokes-Darcy coupling were introduced in [9] and later studied in [8, 16]. An advantage of this kind of formulations is that the vorticity can be accessed directly, without postprocessing and it is straightforward to include non-inertial effects by simply modifying initial and boundary data [4, 5]. For instance, for external flows it is known that boundary conditions are better suited for vorticity than for e.g. pressure. Moreover, in many flow regimes the vorticity is concentrated in a specific region of the domain, which suggests the use of vorticity as guide to mesh refinement.

The contents of the paper are organized as follows. In the remainder of the present section we recall basic terminology and some properties of functional spaces, and introduce further standard notations. In Section 2 we describe the coupled problem of interest and derive a first version of its mixed variational formulation. The solvability analysis of the later is carried out in Section 3. We first identify the non-trivial solutions of the associated homogeneous problem, and then reformulate the original continuous formulation in order to be able to guarantee unique solvability of it. The classical Babuška-Brezzi is then applied in such a way that the continuous inf-sup conditions of the main bilinear form are established by employing a known approach that has been recently referred as T-coercivity. Then, in Section 4 we introduce the associated Galerkin scheme and adapt the arguments from the continuous case to prove that, under suitable assumptions on the finite element subspaces involved, it is well-posed. In particular, the **curl** of the subspace approximating the vorticity must be contained in the space where the discrete velocity of the fluid lives, and hence Raviart-Thomas and Nédélec finite elements for velocities and vorticity, respectively, become feasible choices. In turn, the pressures and the Lagrange multiplier are approximated, respectively, by discontinuous and continuous piecewise polynomials. Next, in Section 5 we modify the mixed formulation by incorporating a residual arising from the Brinkman momentum equation, and show that the resulting augmented scheme, yielding a strongly elliptic main bilinear form, does not require the aforementioned constraint. Finally, several numerical examples illustrating the good performance of the mixed finite element methods and confirming the theoretical rates of convergence are provided in Section 6.

We end this section by specifying some notations to be employed throughout the paper. In particular, we utilize standard simplified terminology for Sobolev spaces and norms. For instance, if $\mathcal{O} \subseteq \mathbb{R}^3$ is a domain, $\mathcal{S} \subseteq \mathbb{R}^3$ is a Lipschitz surface, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) \,:=\, [\mathrm{H}^r(\mathcal{O})]^3 \quad \text{and} \quad \mathbf{H}^r(\mathcal{S}) \,:=\, [\mathrm{H}^r(\mathcal{S})]^3 \,.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$ and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$ and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $\mathrm{H}^r(\mathcal{O})$ and $\mathbf{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ (for $\mathrm{H}^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ to denote H^3. In turn, in the realm of mixed methods (see [12]) one usually needs the Hilbert spaces

$$\mathbf{H}(\mathrm{div};\mathcal{O}) \,:=\, \Big\{\boldsymbol{v} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\boldsymbol{v} \in \mathrm{L}^2(\mathcal{O})\Big\}\,, \quad \mathbf{H}(\mathbf{curl};\mathcal{O}) \,:=\, \Big\{\boldsymbol{v} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathbf{curl}\,\boldsymbol{v} \in \mathbf{L}^2(\mathcal{O})\Big\}\,,$$

normed, respectively, with

$$\|\boldsymbol{v}\|_{\mathrm{div};\mathcal{O}} := \Big\{\|\boldsymbol{v}\|_{0,\mathcal{O}}^2 \,+\, \|\,\mathrm{div}\,\boldsymbol{v}\|_{0,\mathcal{O}}^2\Big\}^{1/2}\,, \quad \|\boldsymbol{v}\|_{\mathbf{curl};\mathcal{O}} := \Big\{\|\boldsymbol{v}\|_{0,\mathcal{O}}^2 \,+\, \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\mathcal{O}}^2\Big\}^{1/2}\,,$$

where, for any vector field $\boldsymbol{v} \,:=\, (v_1, v_2, v_3)^{\mathtt{t}} \in \mathbf{L}^2(\mathcal{O})$,

$$\mathrm{div}\,\boldsymbol{v} \,:=\, \sum_{i=1}^{3} \partial_i v_i \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{v} \,:=\, \nabla \times \boldsymbol{v} \,=\, \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}\,.$$

In addition, we also recall the orthogonal decomposition

$$\mathrm{L}^2(\mathcal{O}) \,=\, \mathrm{L}_0^2(\mathcal{O}) \,\oplus\, P_0(\mathcal{O})\,, \tag{1.1}$$

where $P_0(\mathcal{O})$ is the space of constant functions on $\mathcal{O}$, and

$$\mathrm{L}_0^2(\mathcal{O}) \,=\, P_0(\mathcal{O})^{\perp} \,:=\, \Big\{q \in \mathrm{L}^2(\mathcal{O}) : \quad \int_{\mathcal{O}} q = 0\Big\}\,. \tag{1.2}$$

Equivalently, each $q \in \mathrm{L}^2(\mathcal{O})$ can be uniquely decomposed as $q \,=\, q_0 \,+\, c$, with

$$q_0 \,:=\, q \,-\, \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} q \,\in\, \mathrm{L}_0^2(\mathcal{O}) \quad \text{and} \quad c \,:=\, \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} q \,\in\, \mathbb{R}\,. \tag{1.3}$$

Certainly, $\mathrm{L}_0^2(\mathcal{O})$ is endowed with the usual norm of $\mathrm{L}^2(\mathcal{O})$, and it is easy to see that there holds

$$\|q\|_{0,\mathcal{O}}^2 \,=\, \|q_0\|_{0,\mathcal{O}}^2 \,+\, |\mathcal{O}|\,c^2\,. \tag{1.4}$$

Finally, in what follows $\mathbf{0}$ stands for a generic null vector (including the null functional and operator), and we use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values in different occurrences.

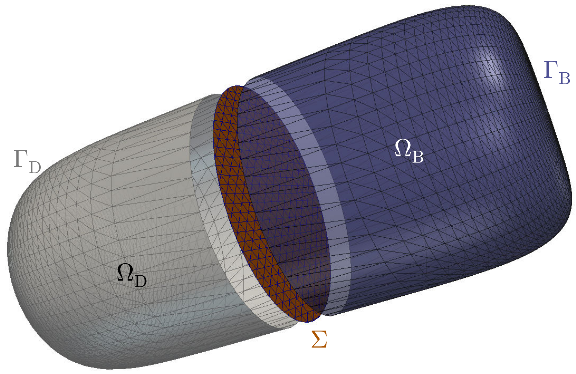

Figure 2.1: Sketch of the domains occupied by the incompressible fluid and by the porous medium (Ω_{B} and Ω_{D}, respectively), interface Σ, and corresponding boundaries.

2 The coupled problem and its mixed formulation

We first let Ω_{B} and Ω_{D} be bounded and simply connected polyhedral Lipschitz domains in $\mathbb{R}^3$ such that $\partial\Omega_{\mathrm{B}} \cap \partial\Omega_{\mathrm{D}} =: \Sigma \neq \emptyset$ and $\Omega_{\mathrm{B}} \cap \Omega_{\mathrm{D}} = \emptyset$, and set $\Omega := \overline{\Omega}_B \cup \overline{\Omega}_D$ with boundary $\Gamma = \partial\Omega$ split into Γ_{B} and Γ_{D} (see the sketch in Figure 2.1). Note that the interface Σ between Ω_{B} and Ω_{D} does not necessarily coincide with $\partial\Omega_{\mathrm{B}}$ (as it was assumed in e.g. [8, 16]). Then, given source terms $\boldsymbol{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ and $\boldsymbol{f}_{\mathrm{B}} \in \mathbf{L}^2(\Omega_{\mathrm{B}})$, we are interested in the Brinkman-Darcy coupled problem, which is formulated in what follows in terms of the fluid velocity $\boldsymbol{u}_{\mathrm{B}}$, the fluid pressure p_{B}, the fluid vorticity $\boldsymbol{\omega}_{\mathrm{B}}$, the Darcy velocity $\boldsymbol{u}_{\mathrm{D}}$, and the Darcy pressure p_{D}. More precisely, the sets of equations in the Brinkman and Darcy domains Ω_{B} and Ω_{D}, are given, respectively, by

$$\left.\begin{aligned} \alpha\,\boldsymbol{u}_{\mathrm{B}} + \nu\,\mathbf{curl}\,\boldsymbol{\omega}_{\mathrm{B}} + \nabla p_{\mathrm{B}} &= \boldsymbol{f}_{\mathrm{B}} \\ \boldsymbol{\omega}_{\mathrm{B}} - \mathbf{curl}\,\boldsymbol{u}_{\mathrm{B}} &= \mathbf{0} \\ \operatorname{div}\boldsymbol{u}_{\mathrm{B}} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\mathrm{B}}\,, \tag{2.1}$$

and

$$\left.\begin{aligned} \mu\boldsymbol{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} &= \boldsymbol{f}_{\mathrm{D}} \\ \operatorname{div}\boldsymbol{u}_{\mathrm{D}} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \tag{2.2}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, $\mu > 0$ depends on this viscosity and on the permeability of the porous medium, which is assumed to be homogeneous, and $\alpha > 0$ is a parameter related to the relaxation time (typically proportional to the inverse of the timestep after a Rothe type time discretization). In turn, the corresponding transmission conditions become

$$\boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} \quad \text{and} \quad p_{\mathrm{D}} = p_{\mathrm{B}} \quad \text{on} \quad \Sigma\,, \tag{2.3}$$

where $\boldsymbol{n}$ stands for the outward normal at Ω_{B} and Ω_{D}, whereas the boundary conditions reduce to

$$\boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_{\mathrm{B}} = \Sigma \cup \Gamma_{\mathrm{B}}\,, \quad \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{B}}\,, \quad \text{and} \quad \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,. \tag{2.4}$$

We now aim to derive the mixed variational formulation of (2.1) - (2.4). We begin by testing the first equation in (2.1) with functions in the space

$$\mathbf{H}_{\mathrm{B}}(\operatorname{div};\Omega_{\mathrm{B}}) := \Big\{ \boldsymbol{v}_{\mathrm{B}} \in \mathbf{H}(\operatorname{div};\Omega_{\mathrm{B}}) : \quad \boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{B}} \Big\}\,.$$

To this end, we need to recall that the fact that $\boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n} = 0$ on Γ_{B} guarantees that $\boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n}|_{\Sigma}$ belongs to $\mathrm{H}^{-1/2}(\Sigma)$ for each $\boldsymbol{v}_{\mathrm{B}} \in \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}})$ (see the beginning of Section 3 below for further details on this issue). In this way, integrating by parts and using the respective boundary conditions, we find that

$$\alpha \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{v}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} - \int_{\Omega_{\mathrm{B}}} p_{\mathrm{B}} \,\mathrm{div}\, \boldsymbol{v}_{\mathrm{B}} + \langle \boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n}, \lambda \rangle_{\Sigma} = \int_{\Omega_{\mathrm{B}}} \boldsymbol{f}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} \quad \forall\, \boldsymbol{v}_{\mathrm{B}} \in \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}}), \tag{2.5}$$

where, thanks to the second transmission condition in (2.3), we have introduced the auxiliary unknown

$$\lambda := p_{\mathrm{D}}|_{\Sigma} = p_{\mathrm{B}}|_{\Sigma} \in \mathrm{H}^{1/2}(\Sigma),$$

and $\langle \cdot, \cdot \rangle_{\Sigma}$ denotes the duality pairing of $\mathrm{H}^{-1/2}(\Sigma)$ and $\mathrm{H}^{1/2}(\Sigma)$ with respect to the $\mathrm{L}^2(\Sigma)$-inner product. Furthermore, it will become clear below that λ can also be seen as the Lagrange multiplier enforcing the continuity of pressure across the interface Σ. Next, we define

$$\mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}) := \Big\{ \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{B}}) : \quad \boldsymbol{z}_{\mathrm{B}} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega_{\mathrm{B}} \Big\},$$

so that testing the second equation in (2.1) with functions in this space, and integrating by parts, we obtain

$$\int_{\Omega_{\mathrm{B}}} \boldsymbol{\omega}_{\mathrm{B}} \cdot \boldsymbol{z}_{\mathrm{B}} - \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} = 0 \qquad \forall\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}). \tag{2.6}$$

In turn, the third equation in (2.1) is initially tested as

$$\int_{\Omega_{\mathrm{B}}} q_{\mathrm{B}} \,\mathrm{div}\, \boldsymbol{u}_{\mathrm{B}} = 0 \qquad \forall\, q_{\mathrm{B}} \in \mathrm{L}^2(\Omega_{\mathrm{B}}). \tag{2.7}$$

On the other hand, in order to deal with the equations in the Darcy domain, we now set

$$\mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}) := \Big\{ \boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \Big\},$$

and test the first equation of (2.2) with functions in this space. Thus, integrating by parts, using the corresponding boundary conditions, noting that the normal $\boldsymbol{n}$ on Σ points inward Ω_{D}, and recalling that $\lambda := p_{\mathrm{D}}|_{\Sigma}$, we get

$$\mu \int_{\Omega_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} - \int_{\Omega_{\mathrm{D}}} p_{\mathrm{D}} \,\mathrm{div}\, \boldsymbol{v}_{\mathrm{D}} - \langle \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}, \lambda \rangle_{\Sigma} = \int_{\Omega_{\mathrm{D}}} \boldsymbol{f}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \qquad \forall\, \boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}). \tag{2.8}$$

In addition, similarly as for the incompressibility condition in Ω_{B}, the second equation in (2.2) is initially tested as

$$\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \,\mathrm{div}\, \boldsymbol{v}_{\mathrm{D}} = 0 \qquad \forall\, q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}). \tag{2.9}$$

We end the present derivation with the weak imposition of the essential transmission condition given by the first equation in (2.3), that is

$$\langle \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} - \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n}, \xi \rangle_{\Sigma} = 0 \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma). \tag{2.10}$$

Consequently, reordering (2.5) - (2.10) in a suitable way, namely placing each set of equations $\{(2.5), (2.6), (2.8)\}$ and $\{(2.7), (2.9), (2.10)\}$ into a single equation each, we arrive at the mixed formulation of (2.1) - (2.4): Find $\vec{\boldsymbol{u}} := (\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}) \in \mathbf{H}$ and $\vec{p} := (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda) \in \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathbf{b}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathbf{F}(\vec{\boldsymbol{v}}) \qquad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{H}, \\ \mathbf{b}(\vec{\boldsymbol{u}}, \vec{q}) &= \mathbf{G}(\vec{q}) \qquad \forall\, \vec{q} := (q_{\mathrm{B}}, q_{\mathrm{D}}, \xi) \in \mathbf{Q}, \end{aligned} \tag{2.11}$$

where

$$\mathbf{H} := \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}}) \times \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}) \times \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \mathbf{Q} := \mathrm{L}^2(\Omega_{\mathrm{B}}) \times \mathrm{L}^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)\,,$$

$\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ are the bilinear forms defined by

$$\begin{aligned} \mathbf{a}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) := &\ \alpha \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{\omega}_{\mathrm{B}} \cdot \boldsymbol{z}_{\mathrm{B}} + \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{v}_{\mathrm{B}} \cdot \mathbf{curl}\,\boldsymbol{\omega}_{\mathrm{B}} \\ & - \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\,\boldsymbol{z}_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \qquad \forall\,(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \in \mathbf{H} \times \mathbf{H}\,, \end{aligned} \tag{2.12}$$

$$\mathbf{b}(\vec{\boldsymbol{v}},\vec{q}) := -\int_{\Omega_{\mathrm{B}}} q_{\mathrm{B}}\,\mathrm{div}\,\boldsymbol{v}_{\mathrm{B}} - \int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}}\,\mathrm{div}\,\boldsymbol{v}_{\mathrm{D}} + \langle \boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n} - \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}, \xi \rangle_{\Sigma} \qquad \forall\,(\vec{\boldsymbol{v}},\vec{q}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{2.13}$$

and $\mathbf{F} \in \mathbf{H}'$ and $\mathbf{G} \in \mathbf{Q}'$ are the functionals defined by

$$\mathbf{F}(\vec{\boldsymbol{v}}) := \int_{\Omega_{\mathrm{B}}} \boldsymbol{f}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \int_{\Omega_{\mathrm{D}}} \boldsymbol{f}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \quad \forall\,\vec{\boldsymbol{v}} \in \mathbf{H}\,, \quad \text{and} \quad \mathbf{G} = \mathbf{0}\,. \tag{2.14}$$

3 Solvability analysis of the mixed formulation

In this section we analyze the solvability of (2.11). For this purpose, we first recall some definitions and technical results concerning Sobolev spaces on Γ_{D}, Γ_{B}, and Σ. We begin by mentioning that, given $\eta \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$, its restriction to Γ_{D}, say $\eta|_{\Gamma_{\mathrm{D}}}$, is defined as

$$\langle \eta|_{\Gamma_{\mathrm{D}}}, \rho \rangle_{\Gamma_{\mathrm{D}}} := \langle \eta, E_{\mathrm{D},0}(\rho) \rangle_{\partial\Omega_{\mathrm{D}}} \qquad \forall\,\rho \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})\,,$$

where $E_{\mathrm{D},0} : \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}}) \to \mathrm{L}^2(\partial\Omega_{\mathrm{D}})$ is the extension by zero in $\Sigma := \partial\Omega_{\mathrm{D}} \backslash \Gamma_D$, and

$$\mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}}) := \Big\{ \rho \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}}) : \quad E_{\mathrm{D},0}(\rho) \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}}) \Big\}\,,$$

which is endowed with the natural norm $\|\rho\|_{1/2,00,\Gamma_{\mathrm{D}}} := \|E_{\mathrm{D},0}(\rho)\|_{1/2,\partial\Omega_{\mathrm{D}}}$. It is quite clear, then, that $\eta|_{\Gamma_{\mathrm{D}}}$ belongs to $\mathrm{H}^{-1/2}_{00}(\Gamma_{\mathrm{D}})$, the dual of $\mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})$, and that $\eta = 0$ on Γ_{D} (equivalently $\eta|_{\Gamma_{\mathrm{D}}} = 0$) if and only if

$$\langle \eta, E_{\mathrm{D},0}(\rho) \rangle_{\partial\Omega_{\mathrm{D}}} = 0 \qquad \forall\,\rho \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})\,.$$

Hereafter, $\langle \cdot,\cdot \rangle_{\Gamma_{\mathrm{D}}}$ (resp. $\langle \cdot,\cdot \rangle_{\partial\Omega_{\mathrm{D}}}$) stands for the duality pairing of $\mathrm{H}^{-1/2}_{00}(\Gamma_{\mathrm{D}})$ and $\mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})$ (resp. $\mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$ and $\mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$) with respect to the $\mathrm{L}^2(\Gamma_{\mathrm{D}})$ (resp. $\mathrm{L}^2(\partial\Omega_{\mathrm{D}})$) inner product. Furthermore, it is not difficult to show (see, e.g. [18, Section 2]) that there holds the decomposition

$$H^{1/2}(\partial\Omega_{\mathrm{D}}) = E_{\mathrm{D}}\big(\mathrm{H}^{1/2}(\Sigma)\big) \oplus E_{\mathrm{D},0}\big(\mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})\big)\,,$$

where $E_{\mathrm{D}} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$ is the bounded linear extension defined by $E_{\mathrm{D}}(\xi) := z_\xi|_{\partial\Omega_{\mathrm{D}}} \quad \forall\,\xi \in \mathrm{H}^{1/2}(\Sigma)$, with $z_\xi \in \mathrm{H}^1(\Omega_{\mathrm{D}})$ being the unique weak solution of the boundary value problem with mixed boundary conditions:

$$\Delta z_\xi = 0 \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \quad z_\xi = \xi \quad \text{on} \quad \Sigma\,, \quad \nabla z_\xi \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,.$$

In this way, given $\varphi \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$, there exist unique $\xi_\varphi \in \mathrm{H}^{1/2}(\Sigma)$ and $\rho_\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})$ such that

$$\varphi = E_{\mathrm{D}}(\xi_\varphi) + E_{\mathrm{D},0}(\rho_\varphi)\,, \tag{3.1}$$

and hence

$$\langle \eta, \varphi \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \eta, E_{\mathrm{D}}(\xi_\varphi) \rangle_{\partial\Omega_{\mathrm{D}}} \,+\, \langle \eta, E_{\mathrm{D},0}(\rho_\varphi) \rangle_{\partial\Omega_{\mathrm{D}}} \,, \tag{3.2}$$

which can be rewritten as

$$\langle \eta, \varphi \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \eta_{\Sigma}, \xi_\varphi \rangle_{\Sigma} \,+\, \langle \eta_{\mathrm{D}}, \rho_\varphi \rangle_{\Gamma_{\mathrm{D}}} \,,$$

where $\eta_{\Sigma} \in \mathrm{H}^{-1/2}(\Sigma)$ and $\eta_{\mathrm{D}} \in \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{D}})$ are defined accordingly. In addition, it is clear from (3.1) and the definitions of E_{D} and $E_{\mathrm{D},0}$ that actually $\xi_\varphi = \varphi|_{\Sigma}$ for each $\varphi \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})$. In particular, when $\eta = 0$ on Γ_{D}, the foregoing equations yield $\langle \eta, \varphi \rangle_{\partial\Omega_{\mathrm{D}}} \,=\, \langle \eta, E_{\mathrm{D}}(\xi_\varphi) \rangle_{\partial\Omega_{\mathrm{D}}} =: \langle \eta_{\Sigma}, \xi_\varphi \rangle_{\Sigma} \,=\, \langle \eta_{\Sigma}, \varphi|_{\Sigma} \rangle_{\Sigma}$, and hence η can be identified with a functional $\eta_{\Sigma} \in \mathrm{H}^{-1/2}(\Sigma)$. In other words, one simply says that $\eta|_{\Sigma} = \eta_{\Sigma} \,\in\, \mathrm{H}^{-1/2}(\Sigma)$. Note that an interesting application of this result arises when we consider $\boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})$ and define $\eta := \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$. In fact, since $\boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n} = 0$ on Γ_{D}, we readily deduce that $\boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}|_{\Sigma} \in \mathrm{H}^{-1/2}(\Sigma)$. Moreover, the analogue conclusion obtained by exchanging Ω_{D}, Γ_{D}, and $\mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})$ by Ω_{B}, Γ_{B}, and $\mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}})$, respectively, is precisely what we used in Section 2 for the derivation of (2.5).

We are now in position to provide the following preliminary result, which establishes a continuous inf-sup condition on $\mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}) \times \big(\mathrm{L}_0^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)\big)$.

Lemma 3.1 *There exits $\beta_{\mathrm{D}} > 0$ such that*

$$S_{\mathrm{D}}(q_{\mathrm{D}}, \xi) \,:=\, \sup_{\substack{\boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}) \\ \boldsymbol{v}_{\mathrm{D}} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \,\mathrm{div}\, \boldsymbol{v}_{\mathrm{D}} \,+\, \langle \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}, \xi \rangle_{\Sigma}}{\|\boldsymbol{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,\geq\, \beta_{\mathrm{D}} \Big\{ \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \,+\, \|\xi\|_{1/2,\Sigma} \Big\} \tag{3.3}$$

for all $(q_{\mathrm{D}}, \xi) \,\in\, \mathrm{L}_0^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)$.

Proof. It proceeds almost verbatim as the 2D version provided in [24, Lemma 3.3]. However, for sake of completeness, most details are given in what follows. Indeed, the first part of the proof reduces to show that the operator $\mathrm{div} : \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}) \to \mathrm{L}_0^2(\Omega_{\mathrm{D}})$ is surjective, for which, given $q_{\mathrm{D}} \in \mathrm{L}_0^2(\Omega_{\mathrm{D}})$, it suffices to define the pre-image $\boldsymbol{v}_{\mathrm{D}} := \nabla z \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})$, where $z \in \mathrm{H}^1(\Omega_{\mathrm{D}})$ is the unique weak solution of the Neumann boundary value problem

$$\Delta z \,=\, q_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \quad \nabla z \cdot \boldsymbol{n} \,=\, 0 \quad \text{on} \quad \partial\Omega_{\mathrm{D}}\,, \quad \int_{\Omega_{\mathrm{D}}} z = 0\,.$$

In this way, since

$$S_{\mathrm{D}}(q_{\mathrm{D}}, \xi) \,\geq\, \sup_{\substack{\boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}) \\ \boldsymbol{v}_{\mathrm{D}} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \,\mathrm{div}\, \boldsymbol{v}_{\mathrm{D}}}{\|\boldsymbol{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \,-\, \|\xi\|_{1/2,\Sigma}\,,$$

the aforementioned surjectivity implies the existence of $C > 0$ such that

$$S_{\mathrm{D}}(q_{\mathrm{D}}, \xi) \,\geq\, C\, \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \,-\, \|\xi\|_{1/2,\Sigma}\,. \tag{3.4}$$

In turn, the main ingredient of the second part has to do with the construction of a proper extension of an arbitrary $\phi \in \mathrm{H}^{-1/2}(\Sigma)$ to a functional $\eta \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$. More precisely, given $\xi \in \mathrm{H}^{1/2}(\Sigma)$, we consider $\phi \in \mathrm{H}^{-1/2}(\Sigma)$ and, following the previous analysis and notations, we simply define $\eta \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{D}})$ as

$$\langle \eta, \varphi \rangle_{\partial\Omega_{\mathrm{D}}} \,:=\, \langle \phi, \xi_\varphi \rangle_{\Sigma} \,=\, \langle \phi, \varphi|_{\Sigma} \rangle_{\Sigma} \qquad \forall\, \varphi \in \mathrm{H}^{1/2}(\partial\Omega_{\mathrm{D}})\,,$$

which yields $\|\eta\|_{-1/2,\partial\Omega_{\rm D}} \le \|\phi\|_{-1/2,\Sigma}$. It follows straightforwardly from (3.1) and (3.2) that

$$\langle \eta, E_{\rm D,0}(\rho)\rangle_{\partial\Omega_{\rm D}} = 0 \quad \forall\, \rho \in {\rm H}^{1/2}_{00}(\Gamma_{\rm D}) \quad \text{and} \quad \langle \eta, E_{\rm D}(\xi)\rangle_{\partial\Omega_{\rm D}} = \langle \phi, \xi\rangle_\Sigma \quad \forall\, \xi \in {\rm H}^{1/2}(\Sigma)\,,$$

which says, equivalently, that $\eta = 0$ on $\Gamma_{\rm D}$ and $\eta = \phi$ on Σ. Next, we let $z_\eta \in {\rm H}^1(\Omega_{\rm D})$ be the unique weak solution of the boundary value problem

$$\Delta z_\eta = \frac{1}{|\Omega_{\rm D}|}\langle \eta, 1\rangle_{\partial\Omega_{\rm D}} \quad \text{in} \quad \Omega_{\rm D}\,, \quad \nabla z_\eta \cdot \boldsymbol{n} = \eta \quad \text{on} \quad \partial\Omega_{\rm D}\,, \quad \int_{\Omega_{\rm D}} z_\eta = 0\,,$$

define $\boldsymbol{w}_{\rm D} := \nabla z_\eta$ in $\Omega_{\rm D}$, and observe that ${\rm div}\,\boldsymbol{w}_{\rm D} = \frac{1}{|\Omega_{\rm D}|}\langle \eta, 1\rangle_{\partial\Omega_{\rm D}} \in P_0(\Omega_{\rm D})$ (which yields $\boldsymbol{w}_{\rm D} \in \mathbf{H}({\rm div};\Omega_{\rm D})$), $\boldsymbol{w}_{\rm D}\cdot n = \eta$ on $\partial\Omega_{\rm D}$, and $\|\boldsymbol{w}_{\rm D}\|_{{\rm div};\Omega_{\rm D}} \le C\,\|\eta\|_{-1/2,\partial\Omega_{\rm D}} \le C\,\|\phi\|_{-1/2,\Sigma}$. It follows that $\boldsymbol{w}_{\rm D} \in \mathbf{H}_{\rm D}({\rm div};\Omega_{\rm D})$ and hence

$$S_{\rm D}(q_{\rm D},\xi) \ge \frac{\left|\displaystyle\int_{\Omega_{\rm D}} q_{\rm D}\,{\rm div}\,\boldsymbol{w}_{\rm D} + \langle \boldsymbol{w}_{\rm D}\cdot\boldsymbol{n}, \xi\rangle_\Sigma\right|}{\|\boldsymbol{w}_{\rm D}\|_{{\rm div};\Omega_{\rm D}}} = \frac{|\langle \phi,\xi\rangle_\Sigma|}{\|\boldsymbol{w}_{\rm D}\|_{{\rm div};\Omega_{\rm D}}} \ge c\,\frac{|\langle \phi,\xi\rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}}\,,$$

which, being valid for any $\phi \in {\rm H}^{-1/2}(\Sigma)$, implies that $S_{\rm D}(q_{\rm D},\xi) \ge c\,\|\xi\|_{1/2,\Sigma}$. This inequality and (3.4) yield (3.3), thus completing the proof.

□

The following result is basically a "*mirror reflection*" through Σ of the previous lemma.

Lemma 3.2 *There exits $\beta_{\rm B} > 0$ such that*

$$S_{\rm B}(q_{\rm B},\xi) := \sup_{\substack{\boldsymbol{v}_{\rm B}\in\mathbf{H}_{\rm B}({\rm div};\Omega_{\rm B})\\ \boldsymbol{v}_{\rm B}\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\rm B}} q_{\rm B}\,{\rm div}\,\boldsymbol{v}_{\rm B} - \langle \boldsymbol{v}_{\rm B}\cdot\boldsymbol{n},\xi\rangle_\Sigma}{\|\boldsymbol{v}_{\rm B}\|_{{\rm div};\Omega_{\rm B}}} \ge \beta_{\rm B}\Big\{\|q_{\rm B}\|_{0,\Omega_{\rm B}} + \|\xi\|_{1/2,\Sigma}\Big\} \qquad (3.5)$$

for all $(q_{\rm B},\xi) \in {\rm L}^2_0(\Omega_{\rm B})\times{\rm H}^{1/2}(\Sigma)$.

Proof. It proceeds exactly as the proof of Lemma 3.1 by replacing $\Omega_{\rm D}$, $\Gamma_{\rm D}$, and $\mathbf{H}_{\rm D}({\rm div};\Omega_{\rm D})$ by $\Omega_{\rm B}$, $\Gamma_{\rm B}$, and $\mathbf{H}_{\rm B}({\rm div};\Omega_{\rm B})$, respectively.

□

Lemma 3.1 and 3.2 imply the following continuous inf-sup condition for $\mathbf{b}$.

Lemma 3.3 *There exits $\beta > 0$ such that*

$$S(\vec{q}) := \sup_{\substack{\vec{v}\in\mathbf{H}\\ \vec{v}\neq\mathbf{0}}} \frac{\mathbf{b}(\vec{v},\vec{q})}{\|\vec{v}\|_{\mathbf{H}}} \ge \beta\Big\{\|q_{\rm B,0}\|_{0,\Omega_{\rm B}} + \|q_{\rm D,0}\|_{0,\Omega_{\rm D}} + \|\xi - c_{\rm B}\|_{1/2,\Sigma} + \|\xi - c_{\rm D}\|_{1/2,\Sigma}\Big\} \qquad (3.6)$$

for all $\vec{q} := (q_{\rm B}, q_{\rm D}, \xi) \in \mathbf{Q}$, where, according to (1.1), $q_{\rm B} = q_{\rm B,0} + c_{\rm B}$ and $q_{\rm D} = q_{\rm D,0} + c_{\rm D}$, with $q_{\rm B,0} \in {\rm L}^2_0(\Omega_{\rm B})$, $q_{\rm D,0} \in {\rm L}^2_0(\Omega_{\rm D})$, and $c_{\rm B} := \frac{1}{|\Omega_{\rm B}|}\int_{\Omega_{\rm B}} q_{\rm B}$, $c_{\rm D} := \frac{1}{|\Omega_{\rm D}|}\int_{\Omega_{\rm D}} q_{\rm D} \in \mathbb{R}$.

Proof. Given $\vec{q} := (q_{\rm B}, q_{\rm D}, \xi) \in \mathbf{Q}$, with $q_{\rm B}$ and $q_{\rm D}$ decomposed as indicated above, we integrate by parts in $\Omega_{\rm B}$ and $\Omega_{\rm D}$, respectively, to deduce that

$$\int_{\Omega_{\rm B}} q_{\rm B}\,{\rm div}\,\boldsymbol{v}_{\rm B} - \langle \boldsymbol{v}_{\rm B}\cdot\boldsymbol{n},\xi\rangle_\Sigma = \int_{\Omega_{\rm B}} q_{\rm B,0}\,{\rm div}\,\boldsymbol{v}_{\rm B} - \langle \boldsymbol{v}_{\rm B}\cdot\boldsymbol{n},\xi - c_{\rm B}\rangle_\Sigma \qquad \forall\,\boldsymbol{v}_{\rm B} \in \mathbf{H}_{\rm B}({\rm div};\Omega_{\rm B})\,,$$

and

$$\int_{\Omega_D} q_D \operatorname{div} \boldsymbol{v}_D + \langle \boldsymbol{v}_D \cdot \boldsymbol{n}, \xi \rangle_\Sigma = \int_{\Omega_D} q_{D,0} \operatorname{div} \boldsymbol{v}_D + \langle \boldsymbol{v}_D \cdot \boldsymbol{n}, \xi - c_D \rangle_\Sigma \qquad \forall\, \boldsymbol{v}_D \in \mathbf{H}_D(\operatorname{div};\Omega_D).$$

Hence, bearing in mind the definitions of the bilinear form $\mathbf{b}$ (cf. (2.13)) and the operators S_D and S_B (cf. (3.3), (3.5)), and employing the foregoing equations, we easily find that

$$S(\vec{q}) \geq S_D(q_{D,0}, \xi - c_D) \qquad \text{and} \qquad S(\vec{q}) \geq S_B(q_{B,0}, \xi - c_B)$$

Consequently, these inequalities and straightforward applications of Lemmata 3.1 and 3.2 imply (3.6) and complete the proof.

□

Having proved a first property for $\mathbf{b}$, we now observe that the bilinear form $\mathbf{a}$ satisfies a positiveness condition. More precisely, it follows directly from its definition (cf. (2.12)) that

$$\mathbf{a}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}) = \alpha \|\boldsymbol{v}_B\|^2_{0,\Omega_B} + \nu \|\boldsymbol{z}_B\|^2_{0,\Omega_B} + \mu \|\boldsymbol{v}_D\|^2_{0,\Omega_D} \qquad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}_B, \boldsymbol{z}_B, \boldsymbol{v}_D) \in \mathbf{H}. \tag{3.7}$$

A first result concerning the solvability of our mixed formulation (2.11) is established next.

Theorem 3.1 *Let* $(\vec{\boldsymbol{u}}, \vec{p}) := \big((\boldsymbol{u}_B, \boldsymbol{\omega}_B, \boldsymbol{u}_D), (p_B, p_D, \lambda)\big) \in \mathbf{H} \times \mathbf{Q}$ *be a solution of the homogeneous problem associated to* (2.11), *that is with* $\mathbf{F} = \mathbf{G} = \mathbf{0}$. *Then* $\vec{\boldsymbol{u}} = \mathbf{0}$ *and there exists* $c \in \mathbb{R}$ *such that* $\vec{p} = (c, c, c)$.

Proof. We first notice from the second equation of (2.11) with $\vec{q} = \vec{p}$ that $\mathbf{b}(\vec{\boldsymbol{u}}, \vec{p}) = 0$, and hence, taking $\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}}$ in the first equation of (2.11) and using the identity (3.7), we arrive at

$$0 = \mathbf{a}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{u}}) = \alpha \|\boldsymbol{u}_B\|^2_{0,\Omega_B} + \nu \|\boldsymbol{\omega}_B\|^2_{0,\Omega_B} + \mu \|\boldsymbol{u}_D\|^2_{0,\Omega_D},$$

from which it follows that $\vec{\boldsymbol{u}} = \mathbf{0}$. In this way, the first equation of (2.11) becomes now $\mathbf{b}(\vec{\boldsymbol{v}}, \vec{p}) = 0$ for all $\vec{\boldsymbol{v}} \in \mathbf{Q}$, which, according to the continuous inf-sup condition for $\mathbf{b}$ given by Lemma 3.3, yields $p_{B,0} = 0$, $p_{D,0} = 0$, and $\lambda = \frac{1}{|\Omega_B|} \int_{\Omega_B} p_B = \frac{1}{|\Omega_D|} \int_{\Omega_D} p_D =: c \in \mathbb{R}$, so that $p_B = p_{B,0} + c = c$ and $p_D = p_{D,0} + c = c$.

□

As a straightforward consequence of Theorem 3.1 we conclude that whenever (2.11) has solution, it is not unique. Therefore, in order to overcome this drawback, we need to remove the constant $c \in \mathbb{R}$ from the solutions of the associated homogeneous system, for which from now on we propose to look for the unknown $\vec{p}$ in the space

$$\mathbf{Q}_0 := \mathrm{L}^2_0(\Omega_B) \times \mathrm{L}^2(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma). \tag{3.8}$$

Alternatively, one could also consider $\mathbf{Q}_0 := \mathrm{L}^2(\Omega_B) \times \mathrm{L}^2_0(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma)$ or $\mathbf{Q}_0 := \mathrm{L}^2(\Omega_B) \times \mathrm{L}^2(\Omega_D) \times \mathrm{H}^{1/2}_0(\Sigma)$, where $\mathrm{H}^{1/2}_0(\Sigma) := \big\{\xi \in \mathrm{H}^{1/2}(\Sigma) : \quad \langle 1, \xi \rangle_\Sigma = 0\big\}$.

Throughout the rest of the paper we stay with (3.8) and consider, instead of (2.11), the following mixed formulation: Find $\vec{\boldsymbol{u}} := (\boldsymbol{u}_B, \boldsymbol{\omega}_B, \boldsymbol{u}_D) \in \mathbf{H}$ and $\vec{p} := (p_B, p_D, \lambda) \in \mathbf{Q}_0$ such that

$$\begin{aligned} \mathbf{a}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathbf{b}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathbf{F}(\vec{\boldsymbol{v}}) \qquad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}_B, \boldsymbol{z}_B, \boldsymbol{v}_D) \in \mathbf{H}, \\ \mathbf{b}(\vec{\boldsymbol{u}}, \vec{q}) &= \mathbf{G}(\vec{q}) \qquad \forall\, \vec{q} := (q_B, q_D, \xi) \in \mathbf{Q}_0. \end{aligned} \tag{3.9}$$

Note that the second equation of (2.11), which is tested against $\vec{q} \in \mathbf{Q}$, is equivalent to the present second equation of (3.9), which is tested against $\vec{q} \in \mathbf{Q}_0$. In fact, one implication is obvious because of the inclusion $\mathbf{Q}_0 \subseteq \mathbf{Q}$. Conversely, assume that the second equation of (3.9) holds. Then, given $c \in \mathbb{R}$, we integrate by parts and, noting that $(\mathbf{0}, q_{\mathrm{D}} - c, \xi - c) \in \mathbf{Q}_0 := \mathrm{L}^2_0(\Omega_{\mathrm{B}}) \times \mathrm{L}^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)$, we find that

$$\mathbf{b}(\vec{\boldsymbol{u}}, (c, q_{\mathrm{D}}, \xi)) \,=\, \mathbf{b}(\vec{\boldsymbol{u}}, (\mathbf{0}, q_{\mathrm{D}} - c, \xi - c)) \,=\, 0\,,$$

which yields $\mathbf{b}(\vec{\boldsymbol{u}}, (q_{\mathrm{B}}, q_{\mathrm{D}}, \xi)) = 0 = \mathbf{G}(\vec{q})$ for all $\vec{q} := (q_{\mathrm{B}}, q_{\mathrm{D}}, \xi) \in \mathbf{Q}$, thus confirming that the second equation of (2.11) holds.

We now aim to establish the well-posedness of (3.9) by applying the classical Babuška-Brezzi theory. We begin with the inf-sup condition for $\mathbf{b}$ on $\mathbf{H} \times \mathbf{Q}_0$.

Lemma 3.4 *There exists $\widetilde{\beta} > 0$ such that*

$$S(\vec{q}) \,:=\, \sup_{\substack{\vec{\boldsymbol{v}} \in \mathbf{H} \\ \vec{\boldsymbol{v}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\boldsymbol{v}}, \vec{q})}{\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}} \,\geq\, \widetilde{\beta}\, \|\vec{q}\|_{\mathbf{Q}} \qquad \forall\, \vec{q} \in \mathbf{Q}_0\,. \tag{3.10}$$

Proof. Given $\vec{q} = (q_{\mathrm{B}}, q_{\mathrm{D}}, \xi) \in \mathbf{Q}_0 := \mathrm{L}^2_0(\Omega_{\mathrm{B}}) \times \mathrm{L}^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)$, we obtain from Lemma 3.3 that

$$S(\vec{q}) \,:=\, \sup_{\substack{\vec{\boldsymbol{v}} \in \mathbf{H} \\ \vec{\boldsymbol{v}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\boldsymbol{v}}, \vec{q})}{\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}} \,\geq\, \beta \Big\{ \|q_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \,+\, \|q_{\mathrm{D},0}\|_{0,\Omega_{\mathrm{D}}} \,+\, \|\xi\|_{1/2,\Sigma} \,+\, \|\xi - c_{\mathrm{D}}\|_{1/2,\Sigma} \Big\}\,, \tag{3.11}$$

where, according to (1.1), $q_{\mathrm{D}} = q_{\mathrm{D},0} + c_{\mathrm{D}}$, with $q_{\mathrm{D},0} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})$ and $c_{\mathrm{D}} := \frac{1}{|\Omega_{\mathrm{D}}|} \int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \in \mathbb{R}$. In turn, a simple application of the triangle inequality shows that

$$|\Sigma|\, |c_{\mathrm{D}}| \,=\, \|c_{\mathrm{D}}\|_{1/2,\Sigma} \,\leq\, \|\xi\|_{1/2,\Sigma} \,+\, \|\xi - c_{\mathrm{D}}\|_{1/2,\Sigma}\,,$$

which, combined with (3.11) and the fact that $\|q_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}} = \|q_{\mathrm{D},0}\|^2_{0,\Omega_{\mathrm{D}}} + |\Omega_{\mathrm{D}}|\, c_{\mathrm{D}}^2$ (cf. (1.4)), imply (3.10) and finish the proof.

$\square$

Next, we address the coerciveness of $\mathbf{a}$ on the kernel $\mathbf{V}$ of $\mathbf{b}$. Indeed, we first deduce from the definitions of $\mathbf{b}$ (cf. (2.13)) and $\mathbf{Q}_0$ (cf. (3.8)) that

$$\mathbf{V} \,=\, \mathbf{V}_{\mathrm{B,D}} \cap \mathbf{V}_{\Sigma}\,, \tag{3.12}$$

with

$$\mathbf{V}_{\mathrm{B,D}} \,:=\, \Big\{ \vec{\boldsymbol{v}} \,:=\, (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{H} : \quad \operatorname{div} \boldsymbol{v}_{\mathrm{B}} \in P_0(\Omega_{\mathrm{B}}) \quad \text{and} \quad \operatorname{div} \boldsymbol{v}_{\mathrm{D}} = 0 \text{ in } \Omega_{\mathrm{D}} \Big\}\,, \tag{3.13}$$

and

$$\mathbf{V}_{\Sigma} \,:=\, \Big\{ \vec{\boldsymbol{v}} \,:=\, (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{H} : \quad \boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n} = \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n} \quad \text{on} \quad \Sigma \Big\}\,. \tag{3.14}$$

Lemma 3.5 *There exists $\widetilde{\varrho} > 0$ such that*

$$\sup_{\substack{\vec{\boldsymbol{w}} \in \mathbf{V} \\ \vec{\boldsymbol{w}} \neq \mathbf{0}}} \frac{\mathbf{a}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})}{\|\vec{\boldsymbol{w}}\|_{\mathbf{H}}} \,\geq\, \widetilde{\varrho}\, \|\vec{\boldsymbol{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{v}} \in \mathbf{V}\,, \tag{3.15}$$

and

$$\sup_{\substack{\vec{\boldsymbol{v}} \in \mathbf{V} \\ \vec{\boldsymbol{v}} \neq \mathbf{0}}} \frac{\mathbf{a}(\vec{\boldsymbol{v}}, \vec{\boldsymbol{w}})}{\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}} \,\geq\, \widetilde{\varrho}\, \|\vec{\boldsymbol{w}}\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{w}} \in \mathbf{V}\,, \tag{3.16}$$

Proof. We begin by recalling from [23, Lemma 3.2] that there exists $\varrho_0 > 0$ such that

$$\|\boldsymbol{v}_{\rm B}\|_{0,\Omega_{\rm B}} \,\ge\, \varrho_0\,\|\boldsymbol{v}_{\rm B}\|_{{\rm div},\Omega_{\rm B}} \quad \forall\,\boldsymbol{v}_{\rm B}\in \mathbf{H}({\rm div};\Omega_{\rm B}) \quad \text{such that} \quad {\rm div}\,\boldsymbol{v}_{\rm B}\in P_0(\Omega_{\rm B})\,. \tag{3.17}$$

Hence, thanks to the foregoing inequality and (3.7), we find that

$$\mathbf{a}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \,\ge\, \widetilde{\varrho}_1\left\{\|\boldsymbol{v}_{\rm B}\|^2_{{\rm div};\Omega_{\rm B}} \,+\, \|\boldsymbol{z}_{\rm B}\|^2_{0,\Omega_{\rm B}} \,+\, \|\boldsymbol{v}_{\rm D}\|^2_{{\rm div};\Omega_{\rm D}}\right\} \quad \forall\,\vec{\boldsymbol{v}} := (\boldsymbol{v}_{\rm B},\boldsymbol{z}_{\rm B},\boldsymbol{v}_{\rm D})\in \mathbf{V}_{\rm B,D}\,, \tag{3.18}$$

with $\widetilde{\varrho}_1 := \min\{\alpha\varrho_0^2,\nu,\mu\} > 0$. Next, given a particular $\vec{\boldsymbol{v}} := (\boldsymbol{v}_{\rm B},\boldsymbol{z}_{\rm B},\boldsymbol{v}_{\rm D})\in\mathbf{V}$, we certainly have $\boldsymbol{z}_{\rm B}\in\mathbf{H}_0(\mathbf{curl};\Omega_{\rm B})$, and thus, due to a well-known result (see, e.g. [25, Chapter I, Section 2.3, Remark 2.5]), there holds $\mathbf{curl}\;\boldsymbol{z}_{\rm B}\in\mathbf{H}_0({\rm div};\Omega_{\rm B})$, where

$$\mathbf{H}_0({\rm div};\Omega_{\rm B}) \,:=\, \Big\{\boldsymbol{v}_{\rm B}\in\mathbf{H}({\rm div};\Omega_{\rm B}):\quad \boldsymbol{v}_{\rm B}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_{\rm B}\Big\}\,.$$

In this way, denoting

$$T_0(\vec{\boldsymbol{v}}) \,:=\, (\mathbf{curl}\;\boldsymbol{z}_{\rm B},\boldsymbol{z}_{\rm B},\mathbf{0})\,,$$

which clearly belongs to $\mathbf{V}$, we find, according to the definition of $\mathbf{a}$ (cf. (2.12)), that

$$\mathbf{a}(\vec{\boldsymbol{v}},T_0(\vec{\boldsymbol{v}})) \,=\, (\alpha-\nu)\int_{\Omega_{\rm B}}\boldsymbol{v}_{\rm B}\cdot\mathbf{curl}\,\boldsymbol{z}_{\rm B} \,+\, \nu\,\|\,\mathbf{curl}\,\boldsymbol{z}_{\rm B}\|^2_{0,\Omega_{\rm B}} \,+\, \nu\,\|\boldsymbol{z}_{\rm B}\|^2_{0,\Omega_{\rm B}}\,,$$

which, applying Cauchy-Schwarz's inequality and simple algebraic manipulations, yields

$$\mathbf{a}(\vec{\boldsymbol{v}},T_0(\vec{\boldsymbol{v}})) \,\ge\, -\,\frac{|\alpha-\nu|^2}{2\nu}\,\|\boldsymbol{v}_{\rm B}\|^2_{0,\Omega_{\rm B}} \,+\, \frac{\nu}{2}\,\|\,\mathbf{curl}\,\boldsymbol{z}_{\rm B}\|^2_{0,\Omega_{\rm B}} \,+\, \nu\,\|\boldsymbol{z}_{\rm B}\|^2_{0,\Omega_{\rm B}}\,. \tag{3.19}$$

Therefore, introducing now $T(\vec{\boldsymbol{v}}) := c\,\vec{\boldsymbol{v}} + c_0\,T_0(\vec{\boldsymbol{v}})$, with suitable chosen positive constants c and c_0 (depending on $\widetilde{\varrho}_1$, α, and ν), and utilizing (3.18) and (3.19), we obtain that

$$T(\vec{\boldsymbol{v}})\in\mathbf{V}\,,\quad \|T(\vec{\boldsymbol{v}})\|_{\mathbf{H}} \,\le\, C\,\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}\,,\quad \text{and} \quad \mathbf{a}(\vec{\boldsymbol{v}},T(\vec{\boldsymbol{v}})) \,\ge\, \widetilde{\varrho}_2\,\|\vec{\boldsymbol{v}}\|^2_{\mathbf{H}}\,,$$

with C and $\widetilde{\varrho}_2$ positive constants depending on $\widetilde{\varrho}_1$, α, and ν as well. Then, we can write

$$\sup_{\substack{\vec{\boldsymbol{w}}\in\mathbf{V}\\ \vec{\boldsymbol{w}}\neq\mathbf{0}}}\frac{\mathbf{a}(\vec{\boldsymbol{v}},\vec{\boldsymbol{w}})}{\|\vec{\boldsymbol{w}}\|_{\mathbf{H}}} \,\ge\, \frac{\mathbf{a}(\vec{\boldsymbol{v}},T(\vec{\boldsymbol{v}}))}{\|T(\vec{\boldsymbol{v}})\|_{\mathbf{H}}}\,,$$

which, due to the foregoing estimates, gives (3.15). On the other hand, introducing the operator $\widetilde{T}:\mathbf{H}\to\mathbf{H}$ as $\widetilde{T}(\vec{\boldsymbol{v}}) := (-\boldsymbol{v}_{\rm B},\boldsymbol{z}_{\rm B},-\boldsymbol{v}_{\rm D}) \quad \forall\,\vec{\boldsymbol{v}} := (\boldsymbol{v}_{\rm B},\boldsymbol{z}_{\rm B},\boldsymbol{v}_{\rm D})\in\mathbf{H}$, we realize that $\|\widetilde{T}(\vec{\boldsymbol{v}})\|_{\mathbf{H}} = \|\vec{\boldsymbol{v}}\|_{\mathbf{H}}$, $\mathbf{a}(\vec{\boldsymbol{v}},\vec{\boldsymbol{w}}) = \mathbf{a}(\widetilde{T}(\vec{\boldsymbol{w}}),\widetilde{T}(\vec{\boldsymbol{v}})) \quad \forall\,\vec{\boldsymbol{v}},\,\vec{\boldsymbol{w}}\in\mathbf{H}$, $\widetilde{T}(\vec{\boldsymbol{v}})\in\mathbf{V} \quad \forall\,\vec{\boldsymbol{v}}\in\mathbf{V}$, and $\widetilde{T}:\mathbf{V}\to\mathbf{V}$ is an isomorphism. It follows easily that

$$\sup_{\substack{\vec{\boldsymbol{v}}\in\mathbf{V}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}}\frac{\mathbf{a}(\vec{\boldsymbol{v}},\vec{\boldsymbol{w}})}{\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}} \,=\, \sup_{\substack{\vec{\boldsymbol{v}}\in\mathbf{V}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}}\frac{\mathbf{a}(\widetilde{T}(\vec{\boldsymbol{w}}),\widetilde{T}(\vec{\boldsymbol{v}}))}{\|\widetilde{T}(\vec{\boldsymbol{v}})\|_{\mathbf{H}}} \,=\, \sup_{\substack{\vec{\boldsymbol{v}}\in\mathbf{V}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}}\frac{\mathbf{a}(\widetilde{T}(\vec{\boldsymbol{w}}),\vec{\boldsymbol{v}})}{\|\vec{\boldsymbol{v}}\|_{\mathbf{H}}}\,,$$

which, thanks to (3.15), yields (3.16) and completes the proof.

$\square$

As a consequence of the previous analysis we can state the following main result.

Theorem 3.2 *Assume that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ *and* $\mathbf{f}_{\mathrm{B}} \in \mathbf{L}^2(\Omega_{\mathrm{B}})$. *Then there exists a unique* $(\vec{\boldsymbol{u}}, \vec{p}) := \big((\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}), (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda)\big) \in \mathbf{H} \times \mathbf{Q}_0$ *solution of the modified mixed formulation* (3.9). *Moreover, there exists* $C > 0$ *such that*

$$\|\vec{\boldsymbol{u}}\|_{\mathbf{H}} + \|\vec{p}\|_{\mathbf{Q}} \leq C \Big\{ \|\boldsymbol{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\boldsymbol{f}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \Big\} . \tag{3.20}$$

Proof. Thanks to Lemma 3.4 and 3.5, the proof is a straightforward application of the continuous Babuška-Brezzi theory. In particular, it is clear from the definition of $\mathbf{F}$ (cf. (2.14)) that $\|\mathbf{F}\|_{\mathbf{H}'}$ is bounded by the right hand side of (3.20).

□

We end this section by remarking that the way of proving the inf-sup conditions for the bilinear form $\mathbf{a}$ (cf. Lemma 3.5), namely using suitable operators T and $\widetilde{T}$ to get a lower bound of the suprema involved, corresponds basically to what has been recently denominated in the literature as T-coercivity (see, e.g. [10], [14]). Nevertheless, the same idea, without any particular name of it, had been employed previously at least in the context of fluid-solid interaction problems (see, e.g. [20], [21], and [22]).

4 The mixed finite element method

In this section we introduce and analyze a mixed finite element scheme for (3.9). More precisely, we first define the associated Galerkin scheme and establish suitable assumptions on the finite element subspaces ensuring that it becomes well posed. Then, we provide specific examples satisfying the required hypotheses. In what follows, given an integer $k \geq 0$ and a subset S of $\mathbb{R}^3$, we denote by $P_k(S)$ the space of polynomials in S of total degree $\leq k$. In addition, according to the notation introduced in Section 1, we let $\mathbf{P}_k(S) = [P_k(S)]^3$.

4.1 Preliminaries and main results

We begin by selecting a set of arbitrary discrete spaces, namely

$$\begin{gathered} \mathbf{H}_h^{\mathrm{B}} \subseteq \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}}), \qquad \mathbf{H}_{0,h}^{\mathrm{B}} \subseteq \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}), \qquad \mathbf{H}_h^{\mathrm{D}} \subseteq \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}), \\ \mathrm{Q}_h^{\mathrm{B}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{B}}), \qquad \mathrm{Q}_h^{\mathrm{D}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}), \quad \text{and} \quad \mathrm{Q}_h^{\Sigma} \subseteq \mathrm{H}^{1/2}(\Sigma). \end{gathered} \tag{4.1}$$

In addition, in order to deal with the mean value condition for the Brinkman pressure p_{B}, and also to handle the assumptions guaranteeing the discrete inf-sup condition for $\mathbf{b}$, we need to define

$$\mathrm{Q}_{h,0}^{\mathrm{B}} := \mathrm{Q}_h^{\mathrm{B}} \cap \mathrm{L}_0^2(\Omega_{\mathrm{B}}) \quad \text{and} \quad \mathrm{Q}_{h,0}^{\mathrm{D}} := \mathrm{Q}_h^{\mathrm{D}} \cap \mathrm{L}_0^2(\Omega_{\mathrm{D}}). \tag{4.2}$$

Hence, setting the global spaces

$$\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}} \times \mathbf{H}_{0,h}^{\mathrm{B}} \times \mathbf{H}_h^{\mathrm{D}} \quad \text{and} \quad \mathbf{Q}_{0,h} := \mathrm{Q}_{h,0}^{\mathrm{B}} \times \mathrm{Q}_h^{\mathrm{D}} \times \mathrm{Q}_h^{\Sigma}, \tag{4.3}$$

the Galerkin scheme for (3.9) becomes: Find $\vec{\boldsymbol{u}}_h := (\boldsymbol{u}_h^{\mathrm{B}}, \boldsymbol{\omega}_h^{\mathrm{B}}, \boldsymbol{u}_h^{\mathrm{D}}) \in \mathbf{H}_h$ and $\vec{p}_h := (p_h^{\mathrm{B}}, p_h^{\mathrm{D}}, \lambda_h) \in \mathbf{Q}_{0,h}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h) + \mathbf{b}(\vec{\boldsymbol{v}}_h, \vec{p}_h) &= \mathbf{F}(\vec{\boldsymbol{v}}_h) \qquad \forall\, \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{v}_h^{\mathrm{D}}) \in \mathbf{H}_h, \\ \mathbf{b}(\vec{\boldsymbol{u}}_h, \vec{q}_h) &= \mathbf{G}(\vec{q}_h) \qquad \forall\, \vec{q}_h := (q_h^{\mathrm{B}}, q_h^{\mathrm{D}}, \xi_h) \in \mathbf{Q}_{0,h}. \end{aligned} \tag{4.4}$$

We now aim to derive general hypotheses on the finite element subspaces introduced in (4.1) ensuring, by means of the discrete Babuška-Brezzi theory, that the Galerkin scheme (4.4) becomes well-posed. Our approach consists of adapting to the present discrete case the arguments employed in Section 3 for the analysis of the continuous problem, mainly those from the proofs of Lemmas 3.4 and 3.5. We begin by observing that in order to have meaningful spaces $\mathrm{Q}_{h,0}^{\mathrm{B}}$ and $\mathrm{Q}_{h,0}^{\mathrm{D}}$ (cf. (4.2)), we need to be able to eliminate constants polynomials from $\mathrm{Q}_h^{\mathrm{B}}$ and $\mathrm{Q}_h^{\mathrm{D}}$. This request is certainly satisfied if we assume that:

(H.0) $P_0(\Omega_{\mathrm{B}}) \subseteq \mathrm{Q}_h^{\mathrm{B}}$ *and* $P_0(\Omega_{\mathrm{D}}) \subseteq \mathrm{Q}_h^{\mathrm{D}}$,

which, in turn, yields the analogue orthogonal decompositions suggested by (1.1), that is

$$\mathrm{Q}_h^{\mathrm{B}} = \mathrm{Q}_{h,0}^{\mathrm{B}} \oplus P_0(\Omega_{\mathrm{B}}) \quad \text{and} \quad \mathrm{Q}_h^{\mathrm{D}} = \mathrm{Q}_{h,0}^{\mathrm{D}} \oplus P_0(\Omega_{\mathrm{D}}) . \tag{4.5}$$

Next, according to the same arguments utilized in the proof of Lemma 3.4, which actually are determined by those employed in the proofs of Lemmata 3.1, 3.2, and 3.3, we realize that in order to show the discrete inf-sup condition for $\mathbf{b}$ on $\mathbf{H}_h \times \mathbf{Q}_{0,h}$, we need to assume the following hypothesis:

(H.1) *there holds* $P_0(\Sigma) \subseteq \mathrm{Q}_h^{\Sigma}$ *and there exist* $\widetilde{\beta}_{\mathrm{B}}$, $\widetilde{\beta}_{\mathrm{D}} > 0$, *independent of* h, *such that*

$$S_h^{\mathrm{B}}(q_h^{\mathrm{B}}, \xi_h) := \sup_{\substack{\boldsymbol{v}_h^{\mathrm{B}} \in \mathbf{H}_h^{\mathrm{B}} \\ \boldsymbol{v}_h^{\mathrm{B}} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{B}}} q_h^{\mathrm{B}} \operatorname{div} \boldsymbol{v}_h^{\mathrm{B}} - \langle \boldsymbol{v}_h^{\mathrm{B}} \cdot \boldsymbol{n}, \xi_h \rangle_\Sigma}{\|\boldsymbol{v}_h^{\mathrm{B}}\|_{\operatorname{div};\Omega_{\mathrm{B}}}} \geq \widetilde{\beta}_{\mathrm{B}} \left\{ \|q_h^{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} + \|\xi_h\|_{1/2,\Sigma} \right\} \tag{4.6}$$

for all $(q_h^{\mathrm{B}}, \xi_h) \in \mathrm{Q}_{h,0}^{\mathrm{B}} \times \mathrm{Q}_h^{\Sigma}$, *and*

$$S_h^{\mathrm{D}}(q_h^{\mathrm{D}}, \xi_h) := \sup_{\substack{\boldsymbol{v}_h^{\mathrm{D}} \in \mathbf{H}_h^{\mathrm{D}} \\ \boldsymbol{v}_h^{\mathrm{D}} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q_h^{\mathrm{D}} \operatorname{div} \boldsymbol{v}_h^{\mathrm{D}} + \langle \boldsymbol{v}_h^{\mathrm{D}} \cdot \boldsymbol{n}, \xi_h \rangle_\Sigma}{\|\boldsymbol{v}_h^{\mathrm{D}}\|_{\operatorname{div};\Omega_{\mathrm{D}}}} \geq \widetilde{\beta}_{\mathrm{D}} \left\{ \|q_h^{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\xi_h\|_{1/2,\Sigma} \right\} \tag{4.7}$$

for all $(q_h^{\mathrm{D}}, \xi_h) \in \mathrm{Q}_{h,0}^{\mathrm{D}} \times \mathrm{Q}_h^{\Sigma}$.

On the other hand, we now look at the discrete kernel of $\mathbf{b}$, which is defined by

$$\mathbf{V}_h := \left\{ \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{v}_h^{\mathrm{D}}) \in \mathbf{H}_h : \quad \mathbf{b}(\vec{\boldsymbol{v}}_h, \vec{q}_h) = 0 \quad \forall\, \vec{q}_h := (q_h^{\mathrm{B}}, q_h^{\mathrm{D}}, \xi_h) \in \mathbf{Q}_{0,h} \right\} . \tag{4.8}$$

Actually, in order to have a more explicit definition of $\mathbf{V}_h$, similarly as obtained for the continuous kernel $\mathbf{V}$ (cf. (3.12)), we now introduce the following assumption

(H.2) $\operatorname{div} \mathbf{H}_h^{\mathrm{B}} \subseteq \mathrm{Q}_h^{\mathrm{B}}$ *and* $\operatorname{div} \mathbf{H}_h^{\mathrm{D}} \subseteq \mathrm{Q}_h^{\mathrm{D}}$,

which, together with (4.3) and (4.5), implies

$$\mathbf{V}_h = \mathbf{V}_{\mathrm{B,D}}^h \cap \mathbf{V}_\Sigma^h , \tag{4.9}$$

with

$$\mathbf{V}_{\mathrm{B,D}}^h := \left\{ \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{v}_h^{\mathrm{D}}) \in \mathbf{H}_h : \quad \operatorname{div} \boldsymbol{v}_h^{\mathrm{B}} \in P_0(\Omega_{\mathrm{B}}) \quad \text{and} \quad \operatorname{div} \boldsymbol{v}_h^{\mathrm{D}} = 0 \text{ in } \Omega_{\mathrm{D}} \right\} , \tag{4.10}$$

and

$$\mathbf{V}_\Sigma^h := \left\{ \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\rm B}, \boldsymbol{z}_h^{\rm B}, \boldsymbol{v}_h^{\rm D}) \in \mathbf{H}_h : \quad \langle \boldsymbol{v}_h^{\rm B} \cdot \boldsymbol{n} - \boldsymbol{v}_h^{\rm D} \cdot \boldsymbol{n}, \xi_h \rangle_\Sigma = 0 \quad \forall\, \xi_h \in {\rm Q}_h^\Sigma \right\}. \tag{4.11}$$

Since $\mathbf{V}_{\rm B,D}^h \subseteq \mathbf{V}_{\rm B,D}$ (cf. (3.13)), it is clear that inequality (3.18) is also valid in $\mathbf{V}_{\rm B,D}^h$ and hence in the discrete kernel $\mathbf{V}_h$. Consequently, in order to show the discrete coerciveness of $\mathbf{a}$ on $\mathbf{V}_h$ by adapting the procedure utilized in the proof of Lemma 3.5, it just remains to assume the following hypothesis

(H.3) $\mathbf{curl}\ \mathbf{H}_{0,h}^{\rm B} \subseteq \mathbf{H}_h^{\rm B}$.

Having established hypotheses **(H.0)**, **(H.1)**, **(H.2)**, and **(H.3)**, we now reconfirm that they suffice to show that our Galerkin scheme (4.4) is well-posed and convergent. We begin with the discrete inf-sup condition for $\mathbf{b}$.

Lemma 4.1 *There exists $\widehat{\beta} > 0$, independent of h, such that*

$$S_h(\vec{q}_h) := \sup_{\substack{\vec{\boldsymbol{v}}_h \in \mathbf{H}_h \\ \vec{\boldsymbol{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\boldsymbol{v}}_h, \vec{q}_h)}{\|\vec{\boldsymbol{v}}_h\|_{\mathbf{H}}} \geq \widehat{\beta}\, \|\vec{q}_h\|_{\mathbf{Q}} \qquad \forall\, \vec{q}_h \in \mathbf{Q}_{0,h}\,. \tag{4.12}$$

Proof. Given $\vec{q}_h := (q_h^{\rm B}, q_h^{\rm D}, \xi_h) \in \mathbf{Q}_{0,h}$, we let $q_{h,0}^{\rm D} \in {\rm Q}_{h,0}^{\rm D}$ and $c_{\rm D} \in \mathbb{R}$ such that $q_h^{\rm D} = q_{h,0}^{\rm D} + c_{\rm D}$. Then, reasoning as in the proof of Lemma 3.3, which in this case reduces to integrate by parts in $\Omega_{\rm D}$ only (since $q_h^{\rm B}$ is already in ${\rm Q}_{h,0}^{\rm B}$), we find, using the notations from **(H.1)**, that

$$S_h(\vec{q}_h) \geq S_h^{\rm B}(q_h^{\rm B}, \xi_h) \quad \text{and} \quad S_h(\vec{q}_h) \geq S_h^{\rm D}(q_{h,0}^{\rm D}, \xi_h - c_{\rm D})\,.$$

In this way, since thanks to the first assumption in **(H.1)** we have that $\xi_h - c_{\rm D}$ belongs to ${\rm Q}_h^\Sigma$, the foregoing inequalities and a straightforward application of the discrete inf-sup conditions (4.6) and (4.7), imply

$$S_h(\vec{q}_h) \geq \frac{1}{2}\left(\widetilde{\beta}_{\rm B} + \widetilde{\beta}_{\rm D}\right) \left\{ \|q_h^{\rm B}\|_{0,\Omega_{\rm B}} + \|\xi_h\|_{1/2,\Sigma} + \|q_{h,0}^{\rm D}\|_{0,\Omega_{\rm D}} + \|\xi_h - c_{\rm D}\|_{1/2,\Sigma} \right\}.$$

The proof is concluded by employing the triangle inequality, exactly as we did for Lemma 3.4. □

The discrete inf-sup condition for $\mathbf{a}$ on $\mathbf{V}_h$ is proved next. Since $\mathbf{V}_h$ is finite dimensional, it suffices to show one of the discrete analogues of the inequalities provided in Lemma 3.5.

Lemma 4.2 *There exists $\widehat{\varrho} > 0$, independent of h, such that*

$$\sup_{\substack{\vec{\boldsymbol{w}}_h \in \mathbf{V}_h \\ \vec{\boldsymbol{w}}_h \neq \mathbf{0}}} \frac{\mathbf{a}(\vec{\boldsymbol{v}}_h, \vec{\boldsymbol{w}}_h)}{\|\vec{\boldsymbol{w}}_h\|_{\mathbf{H}}} \geq \widehat{\varrho}\, \|\vec{\boldsymbol{v}}_h\|_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{v}}_h \in \mathbf{V}_h\,. \tag{4.13}$$

Proof. Given $\vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\rm B}, \boldsymbol{z}_h^{\rm B}, \boldsymbol{v}_h^{\rm D}) \in \mathbf{V}_h$, we know from (3.18) that

$$\mathbf{a}(\vec{\boldsymbol{v}}_h, \vec{\boldsymbol{v}}_h) \geq \widetilde{\varrho}_1 \left\{ \|\boldsymbol{v}_h^{\rm B}\|_{{\rm div};\Omega_{\rm B}}^2 + \|\boldsymbol{z}_h^{\rm B}\|_{0,\Omega_{\rm B}}^2 + \|\boldsymbol{v}_h^{\rm D}\|_{{\rm div};\Omega_{\rm D}}^2 \right\}.$$

In addition, thanks to the result in [25, Chapter I, Section 2.3, Remark 2.5] and our assumption **(H.3)**, we find that $\mathbf{curl}\,\boldsymbol{z}_h^{\mathrm{B}} \in \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{B}}) \cap \mathbf{H}_h^{\mathrm{B}}$, and hence $T_0(\vec{\boldsymbol{v}}_h) := (\mathbf{curl}\,\boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \mathbf{0})$ clearly belongs to $\mathbf{V}_h$ (cf. (4.9)). The rest of the proof proceeds as in Lemma 3.5. Moreover, it is easy to realize that the constants c and c_0 defining now $T(\vec{\boldsymbol{v}}_h) := c\,\vec{\boldsymbol{v}}_h + c_0\,T_0(\vec{\boldsymbol{v}}_h)$ can be taken exactly as those chosen in the proof of that lemma, so that the resulting constant $\widehat{\varrho}$ of the present result coincides with $\widetilde{\varrho}$ in (3.15) and (3.16). □

The following main result is a direct consequence of the previous analysis.

Theorem 4.1 *Assume that $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ and $\mathbf{f}_{\mathrm{B}} \in \mathbf{L}^2(\Omega_{\mathrm{B}})$. In addition, suppose that* **(H.0)**, **(H.1)**, **(H.2)**, *and* **(H.3)** *hold. Then there exists a unique $(\vec{\boldsymbol{u}}_h, \vec{p}_h) := \big((\boldsymbol{u}_h^{\mathrm{B}}, \boldsymbol{\omega}_h^{\mathrm{B}}, \boldsymbol{u}_h^{\mathrm{D}}), (p_h^{\mathrm{B}}, p_h^{\mathrm{D}}, \lambda_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$ solution of the Galerkin scheme (4.4). Moreover, there exist C_1, $C_2 > 0$, independent of h, such that*

$$\|\vec{\boldsymbol{u}}_h\|_{\mathbf{H}} + \|\vec{p}_h\|_{\mathbf{Q}} \,\le\, C_1 \Big\{ \|\boldsymbol{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\boldsymbol{f}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \Big\}, \tag{4.14}$$

and

$$\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\le\, C_2 \Big\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) \Big\}. \tag{4.15}$$

Proof. Thanks to Lemma 4.1 and 4.2, the proof results as a straightforward application of the discrete Babuška-Brezzi theory. □

4.2 Specific finite element subspaces

We now specify concrete examples of finite element subspaces satisfying the hypotheses introduced in the previous section. For this purpose, we now let $\mathcal{T}_h$ be a regular family of triangulations of $\bar{\Omega}_{\mathrm{B}} \cup \bar{\Omega}_{\mathrm{D}}$ by tetrahedra K of diameter h_K with mesh size $h := \max\{h_K : \quad K \in \mathcal{T}_h\}$, such that $\mathcal{T}_h(\Omega_\star) := \Big\{ K \in \mathcal{T}_h : \quad K \subseteq \bar{\Omega}_\star \Big\}$ is a triangulation of $\Omega_\star$ for each $\star \in \big\{\mathrm{B}, \mathrm{D}\big\}$. Then, we denote by $\mathcal{T}_h(\Sigma)$ the triangulation on Σ induced by $\mathcal{T}_h$ (either from Ω_{B} or Ω_{D}). Also, for reasons that will become clear below, we introduce an independent triangulation $\mathcal{T}_{\tilde{h}}(\Sigma)$ of Σ by triangles $\widetilde{T}$ of diameter $h_{\widetilde{T}}$, and define $\tilde{h} := \max\big\{h_{\widetilde{T}} : \quad \widetilde{T} \in \mathcal{T}_{\tilde{h}}(\Sigma)\big\}$.

4.2.1 Definition of subspaces

We first introduce the finite element subspaces

$$\begin{aligned}
\mathbf{H}_h^\star &:= \Big\{ \boldsymbol{v}_h^\star \in \mathbf{H}_\star(\mathrm{div};\Omega_\star) : \quad \boldsymbol{v}_h^\star|_K \in \mathbb{RT}_0(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_\star) \Big\}, \\
\mathrm{Q}_h^\star &:= \Big\{ q_h \in \mathrm{L}^2(\Omega_\star) : \quad q_h|_K \in P_0(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_\star) \Big\}, \\
\mathrm{Q}_{h,0}^\star &:= \mathrm{Q}_h^\star \cap \mathrm{L}_0^2(\Omega_\star),
\end{aligned}$$

where $\star \in \{\mathrm{B}, \mathrm{D}\}$, and for any $K \in \mathcal{T}_h(\Omega_\star)$

$$\mathbb{RT}_0(K) := \mathbf{P}_0(K) \oplus P_0(K)\,\boldsymbol{x}$$

is the local Raviart-Thomas space of lowest order. In addition, we set

$$\mathbf{H}_{0,h}^{\mathrm{B}} := \Big\{ \boldsymbol{z}_h^{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}) : \quad \boldsymbol{z}_h^{\mathrm{B}}|_K \in \mathbb{ND}_1(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_{\mathrm{B}}) \Big\},$$

where for any $K \in \mathcal{T}_h(\Omega_{\rm B})$

$$\mathbb{ND}_1(K) := \mathbf{P}_0(K) \oplus \mathbf{P}_0(K) \times \boldsymbol{x}$$

is the local edge space of Nédélec, that is

$$\mathbb{ND}_1(K) := \Big\{ w : K \to \mathbb{C}^3 : \quad w(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} \quad \forall\, \boldsymbol{x} \in K,\, \boldsymbol{a},\, \boldsymbol{b} \in \mathbb{C}^3 \Big\}.$$

Finally for the interface Σ we consider the finite element subspace

$$\mathrm{Q}_{\tilde h}^{\Sigma} := \Big\{ \lambda_{\tilde h} \in \mathcal{C}^0(\Sigma) : \quad \lambda_{\tilde h}|_{\tilde T} \in P_1(\tilde T) \quad \forall\, \tilde T \in \mathcal{T}_{\tilde h}(\Sigma) \Big\}.$$

It is easy to check that these subspaces satisfy the hypotheses **(H.0)**, **(H.2)** and **(H.3)**.

On the other hand, for purposes of the analysis, we also need to define

$$\Phi_h(\Sigma) := \Big\{ \psi_h \in \mathrm{L}^2(\Sigma) : \quad \psi_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h(\Sigma) \Big\}.$$

4.2.2 Approximation properties

In what follows $\star$ is a mute symbol taken in $\{\mathrm{B}, \mathrm{D}\}$. We let $\Pi_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h^\star$ be the usual Raviart-Thomas interpolation operator, that is, given a sufficiently smooth vector field $\boldsymbol{v} : \Omega_\star \to \mathbb{R}^3$, we define $\Pi_h^\star(\boldsymbol{v})$ as the only element of $\mathbf{H}_h^\star$ such that

$$\int_F \Pi_h^\star(\boldsymbol{v}) \cdot \boldsymbol{n} = \int_F \boldsymbol{v} \cdot \boldsymbol{n} \quad \forall\, F \in \mathcal{E}_h^\star, \tag{4.16}$$

where $\mathcal{E}_h^\star$ is the set of faces of the triangulation $\mathcal{T}_h(\Omega_\star)$. We now recall some properties of $\Pi_h^\star$ and its local counterparts $\Pi_K^\star$ for each $K \in \mathcal{T}_h(\Omega_\star)$ (see, e.g [19]):

- **(a)** $\Pi_h^\star$ is well defined in $\mathbf{H}^\delta(\Omega_\star) \cap \mathbf{H}(\mathrm{div}; \Omega_\star)$ for any $\delta \in (0,1)$.
- **(b)** There holds $\mathrm{div}\, \Pi_h^\star(\boldsymbol{v}) = \mathcal{P}_h^\star(\mathrm{div}\, \boldsymbol{v})$, where $\mathcal{P}_h^\star : \mathrm{L}^2(\Omega_\star) \to \mathrm{Q}_h^\star$ is the orthogonal projector. Equivalently
$$\int_{\Omega_\star} q_h\, \mathrm{div}\, \Pi_h^\star(\boldsymbol{v}) = \int_{\Omega_\star} q_h\, \mathrm{div}\,(\boldsymbol{v}) \qquad \forall\, q_h \in \mathrm{Q}_h^\star.$$
- **(c)** For each face F of K there holds $\Pi_K^\star(\boldsymbol{v}) \cdot \boldsymbol{n}_F = \mathcal{P}_F(\boldsymbol{v} \cdot \boldsymbol{n}_F)$, where $\boldsymbol{n}_F$ is the unit outward normal on F and $\mathcal{P}_F : \mathrm{L}^2(F) \to P_0(F)$ is the orthogonal projector.
- **(d)** Given $\delta \in (0,1)$ and $\boldsymbol{v} \in \mathbf{H}^\delta(\Omega_\star) \cap \mathbf{H}(\mathrm{div}; \Omega_\star)$, there holds
$$\|\boldsymbol{v} - \Pi_K^\star(\boldsymbol{v})\|_{0,K} \le C h_K^\delta \Big\{ |\boldsymbol{v}|_{\delta,K} + \|\mathrm{div}\,(\boldsymbol{v})\|_{0,K} \Big\} \qquad \forall\, K \in \mathcal{T}_h(\Omega_\star). \tag{4.17}$$

Next, for any $\epsilon > 0$ we introduce the Sobolev space

$$\mathbf{H}^\epsilon(\mathbf{curl}; \Omega_{\rm B}) := \left\{ \boldsymbol{v} \in \mathbf{H}^\epsilon(\Omega_{\rm B}) : \quad \mathbf{curl}\ \boldsymbol{v} \in \mathbf{H}^\epsilon(\Omega_{\rm B}) \right\},$$

and endow it with its Hilbertian norm

$$\|\boldsymbol{v}\|_{\mathbf{H}^\epsilon(\mathbf{curl}; \Omega_{\rm B})} := \Big\{ \|\boldsymbol{v}\|_{\epsilon,\Omega_{\rm B}}^2 + \|\,\mathbf{curl}(\boldsymbol{v})\|_{\epsilon,\Omega_{\rm B}}^2 \Big\}^{1/2}.$$

Then for each face F of $\mathcal{T}_h(\Omega_{\rm B})$ we let $\boldsymbol{t}_F$ be a unit tangential vector on F. It follows from [3, Lemma 4.7] that if $\epsilon > 1/2$ the interpolation operator $\Pi_h : \mathbf{H}^\epsilon(\mathbf{curl};\Omega_{\rm B}) \to \mathbf{H}^{\rm B}_{0,h}$ associated with the face finite element, which is characterized by

$$\int_F \Pi_h(\boldsymbol{v})\cdot\boldsymbol{t}_F \;=\; \int_F \boldsymbol{v}\cdot\boldsymbol{t}_F \quad \forall \text{ faces } F \text{ of } \mathcal{T}_h(\Omega_{\rm B}),$$

is well defined and uniformly bounded. In addition, the following property of Π_h holds.

Lemma 4.3 *There exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{v} - \Pi_h(\boldsymbol{v})\|_{\mathbf{curl};\Omega_{\rm B}} \le C h^\epsilon \|\boldsymbol{v}\|_{\mathbf{H}^\epsilon(\mathbf{curl};\Omega_{\rm B})} \tag{4.18}$$

for all $\boldsymbol{v} \in \mathbf{H}^\epsilon(\mathbf{curl};\Omega_{\rm B})$ and for all $\epsilon \in (1/2, 1]$.

Proof. See, [2, Proposition 5.6]. $\square$

The approximation properties of the finite element subspaces involved are then established as follows (see, e.g [12], [27], [30]):

($\mathrm{AP}_h^{\boldsymbol{u}_\star}$) there exists $C > 0$, independent of h, such that for each $\delta \in (0,1]$ and for each $\boldsymbol{v} \in \mathbf{H}^\delta(\Omega_\star)$, with $\mathrm{div}\,(\boldsymbol{v}) \in \mathbf{H}^\delta(\Omega_\star)$, there holds

$$\|\boldsymbol{v} - \Pi_h^\star(\boldsymbol{v})\|_{\mathrm{div};\Omega_\star} \le C h^\delta \Big\{ \|\boldsymbol{v}\|_{\delta,\Omega_\star} + \|\mathrm{div}\,(\boldsymbol{v})\|_{\delta,\Omega_\star} \Big\} \qquad (\star \in \{\mathrm{B},\mathrm{D}\}).$$

($\mathrm{AP}_h^{p_\star}$) there exists $C > 0$, independent of h, such that for each $\delta \in (0,1]$ and for each $q \in \mathrm{H}^\delta(\Omega_\star)$, there holds

$$\|q - \mathcal{P}_h^\star(q)\|_{0,\Omega_\star} \le C h^\delta \|q\|_{\delta,\Omega_\star} \qquad (\star \in \{\mathrm{B},\mathrm{D}\}).$$

($\mathrm{AP}_h^{\boldsymbol{\omega}_{\rm B}}$) there exists $C > 0$, independent of h, such that for each $\delta \in (1/2, 1]$ and for each $\boldsymbol{z}_{\rm B} \in \mathbf{H}^\delta(\mathbf{curl};\Omega_{\rm B})$, there holds

$$\|\boldsymbol{z}_{\rm B} - \Pi_h(\boldsymbol{z}_{\rm B})\|_{\mathbf{curl};\Omega_{\rm B}} \le C\, h^\delta\, \|\boldsymbol{z}_{\rm B}\|_{\mathbf{H}^\delta(\mathbf{curl};\Omega_{\rm B})}.$$

($\mathrm{AP}_{\tilde h}^{\lambda}$) there exists $C > 0$, independent of $\tilde h$, such that for each $\delta \in (0,1]$ and for each $\xi \in \mathrm{H}^{1/2+\delta}(\Sigma)$, there holds

$$\|\xi - \mathcal{P}_{\tilde h}(\xi)\|_{1/2,\Sigma} \le C\, \tilde h^\delta\, \|\xi\|_{1/2+\delta,\Sigma},$$

where $\mathcal{P}_{\tilde h} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{Q}_{\tilde h}^\Sigma$ is the orthogonal projector.

($\mathrm{AP}_h{}^{\psi}$) there exists $C > 0$, independent of h, such that for each $\delta \in (0,1]$ and for each $\varphi \in \mathrm{H}^{-1/2+\delta}(\Sigma)$, there holds

$$\|\varphi - \mathcal{P}_h^{-1/2}(\varphi)\|_{-1/2,\Sigma} \le C\, h_\Sigma^\delta\, \|\varphi\|_{-1/2+\delta,\Sigma},$$

where $\mathcal{P}_h^{-1/2} : \mathrm{H}^{-1/2}(\Sigma) \to \Phi_h(\Sigma)$ is the orthogonal projector.

4.2.3 Stable discrete liftings

In this section, as usual we let $\star$ be a mute symbol taken in $\{\mathrm{B},\mathrm{D}\}$, and provide sufficient conditions for the existence of a stable discrete lifting $\mathcal{L}_h : \Phi_h(\Sigma) \to \mathbf{H}_h^\star$. To this end, we proceed as in [19, Theorem 4.1], and assume first that $\mathcal{T}_h(\Omega_\star)$ is quasi-uniform in a neighborhood of Σ. This means that there exists a neighborhood of Σ, say Ω_Σ, and a constant $c > 0$, independent of h, such that, denoting

$$\mathcal{T}_{h,\Sigma}^\star := \{K \in \mathcal{T}_h(\Omega_\star) : \quad K \cap \Omega_\Sigma \neq \emptyset\}, \tag{4.19}$$

there holds

$$\max_{K\in\mathcal{T}_{h,\Sigma}^\star} h_K \leq c \min_{K\in\mathcal{T}_{h,\Sigma}^\star} h_K .$$

Now, because of the regularity of $\mathcal{T}_h(\Omega_\star)$, the quasi-uniformity assumption around Σ implies that the partition $\mathcal{T}_h(\Sigma)$ inherited from $\mathcal{T}_h(\Omega_\star)$ is quasi-uniform as well, which implies that $\Phi_h(\Sigma)$ satisfies the inverse inequality (see, [19, Lemma 4.6])

$$\|\psi_h\|_{-1/2+\delta,\Sigma} \leq C\, h_\Sigma^{-\delta} \|\psi_h\|_{-1/2,\Sigma} \qquad \forall\, \psi_h \in \Phi_h(\Sigma), \qquad \forall\, \delta \in [0,1/2], \tag{4.20}$$

where $h_\Sigma := \max\big\{h_T : \quad T \in \mathcal{T}_h(\Sigma)\big\}$.

Lemma 4.4 *There exist a linear operator $\mathcal{L}_h : \Phi_h(\Sigma) \to \mathbf{H}_h^\star$ and a constant $C_{\mathcal{L}} > 0$, independent of h, such that for each $\psi_h \in \Phi_h(\Sigma)$ there hold*

$$\mathcal{L}_h(\psi_h)\cdot\boldsymbol{n} = \psi_h \quad \text{on} \quad \Sigma, \quad \|\mathcal{L}_h(\psi_h)\|_{\mathrm{div},\Omega_\star} \leq C_{\mathcal{L}}\, \|\psi_h\|_{-1/2,\Sigma}, \quad \text{and} \quad \operatorname{div}\mathcal{L}_h(\psi_h) \in P_0(\Omega_\star).$$

Proof. Let $\psi_h \in \Phi_h(\Sigma)$, and let $z \in \mathrm{H}^1(\Omega_\star)$ be the unique solution of the boundary value problem

$$\Delta z \,=\, \frac{1}{|\Omega_\star|}\langle\psi_h, 1\rangle_\Sigma \quad \text{in} \quad \Omega_\star, \qquad \nabla z\cdot\boldsymbol{n} \,=\, \left\{\begin{array}{ll} \psi_h & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_\star \end{array}\right., \qquad \int_{\Omega_\star} z \,=\, 0.$$

The corresponding continuous dependence result says that $\|z\|_{1,\Omega_\star} \leq c_1\, \|\psi_h\|_{-1/2,\Sigma}$. In turn, the elliptic regularity result (cf. [26]) establishes that there exists $\delta > 0$ such that

$$\|z\|_{1+\delta,\Omega_\star} \leq c_2\, \|\psi_h\|_{-1/2+\delta,\Sigma}.$$

In addition, notice that

$$\operatorname{div}(\nabla z) \,=\, \Delta z \,=\, \frac{1}{|\Omega_\star|}\langle\psi_h, 1\rangle_\Sigma \in \mathbb{R} \quad \text{in} \quad \Omega_\star.$$

It follows that $\nabla z \in \mathbf{H}^\delta(\Omega_\star) \cap \mathbf{H}(\mathrm{div};\Omega_\star)$, and hence we can define

$$\mathcal{L}_h(\psi_h) \,:=\, \Pi_h^\star(\nabla z).$$

Next, from properties **(b)** and **(c)** of the Raviart-Thomas interpolation operator, we find that

$$\operatorname{div}\mathcal{L}_h(\psi_h) \,=\, \operatorname{div}\Pi_h^\star(\nabla z) \,=\, \mathcal{P}_h^\star(\operatorname{div}\nabla z) \,=\, \mathcal{P}_h^\star(\Delta z) \,=\, \frac{1}{|\Omega_\star|}\langle\psi_h, 1\rangle_\Sigma \quad \text{in} \quad \Omega_\star, \tag{4.21}$$

and

$$\mathcal{L}_h(\psi_h)\cdot\boldsymbol{n} \,=\, \Pi_h^\star(\nabla z)\cdot\boldsymbol{n} \,=\, \mathcal{P}_{h,\Sigma}(\nabla z\cdot\boldsymbol{n}) \,=\, \mathcal{P}_{h,\Sigma}(\psi_h) \,=\, \psi_h \quad \text{on} \quad \Sigma\,, \tag{4.22}$$

where $\mathcal{P}_{h,\Sigma} : \mathrm{L}^2(\Sigma) \to \Phi_h(\Sigma)$ is the orthogonal projector. It remains to show that $\mathcal{L}_h$ is uniformly bounded. To this end, we first observe that

$$\|\mathcal{L}_h(\psi_h)\|^2_{\mathrm{div},\Omega_\star} = \|\mathcal{L}_h(\psi_h)\|^2_{0,\Omega_\star} + \left\|\frac{1}{|\Omega_\star|}\langle\psi_h,1\rangle_\Sigma\right\|^2_{0,\Omega_\star} \leq \|\mathcal{L}_h(\psi_h)\|^2_{0,\Omega_\star} + c_3\,\|\psi_h\|^2_{-1/2,\Sigma}. \tag{4.23}$$

Now, we divide $\Omega_\star$ into two regions

$$\Omega^1_{\star,h} := \cup\Big\{K \in \mathcal{T}_h(\Omega_\star) : \quad K \notin \mathcal{T}^\star_{h,\Sigma}\Big\} \subseteq \Omega_\star\backslash\Omega_\Sigma, \qquad \Omega^2_{\star,h} := \Omega_\star\backslash\Omega^1_{\star,h} = \cup\Big\{K \in \mathcal{T}^\star_{h,\Sigma}\Big\},$$

where we recall that $\mathcal{T}^\star_{h,\Sigma} := \Big\{K \in \mathcal{T}_h(\Omega_\star) : \quad K \cap \Omega_\Sigma \neq \emptyset\Big\}$. Then, since $\Omega_\star\backslash\Omega_\Sigma$ is strictly contained in $\Omega_\star$, the interior elliptic regularity result [29, Theorem 4.16] implies that $z|_{\Omega_\star\backslash\Omega_\Sigma} \in \mathrm{H}^2(\Omega_\star\backslash\Omega_\Sigma)$ and

$$\|z\|_{2,\Omega_\star\backslash\Omega_\Sigma} \leq c_4\,\|\psi_h\|_{-1/2,\Sigma}.$$

It follows that

$$\begin{aligned}
\|\mathcal{L}_h(\psi_h)\|_{0,\Omega_\star} &\leq \|\mathcal{L}_h(\psi_h)\|_{0,\Omega^1_{\star,h}} + \|\mathcal{L}_h(\psi_h)\|_{0,\Omega^2_{\star,h}} \\
&= \|\Pi^\star_h(\nabla z)\|_{0,\Omega^1_{\star,h}} + \|\Pi^\star_h(\nabla z)\|_{0,\Omega^2_{\star,h}} \\
&\leq c_5\,\|\nabla z\|_{1,\Omega^1_{\star,h}} + \|\nabla z\|_{0,\Omega^2_{\star,h}} + \|\nabla z - \Pi^\star_h(\nabla z)\|_{0,\Omega^2_{\star,h}} \\
&\leq c_5\,\|z\|_{2,\Omega^1_{\star,h}} + \|z\|_{1,\Omega^2_{\star,h}} + \|\nabla z - \Pi^\star_h(\nabla z)\|_{0,\Omega^2_{\star,h}} \\
&\leq c_5\,c_4\,\|\psi_h\|_{-1/2,\Sigma} + c_1\,\|\psi_h\|_{-1/2,\Sigma} + \|\nabla z - \Pi^\star_h(\nabla z)\|_{0,\Omega^2_{\star,h}}.
\end{aligned} \tag{4.24}$$

On the other hand, applying estimate (4.17) and inverse inequality (4.20), we obtain that

$$\begin{aligned}
\|\nabla z - \Pi^\star_h(\nabla z)\|^2_{0,\Omega^2_{\star,h}} &= \sum_{K\in\mathcal{T}^\star_{h,\Sigma}} \|\nabla z - \Pi^\star_K(\nabla z)\|^2_{0,K} \\
&\leq c_6 \sum_{K\in\mathcal{T}^\star_{h,\Sigma}} h_K^{2\delta}\Big\{|\nabla z|^2_{\delta,K} + \left\|\frac{1}{|\Omega_\star|}\langle\psi_h,1\rangle_\Sigma\right\|^2_{0,K}\Big\} \\
&\leq c_7 \max_{K\in\mathcal{T}^\star_{h,\Sigma}} h_K^{2\delta}\Big\{\|z\|^2_{1+\delta,\Omega^2_{\star,h}} + \|\psi_h\|^2_{-1/2,\Sigma}\Big\} \\
&\leq c_7 \max_{K\in\mathcal{T}^\star_{h,\Sigma}} h_K^{2\delta}\Big\{\|z\|^2_{1+\delta,\Omega_\star} + \|\psi_h\|^2_{-1/2,\Sigma}\Big\} \\
&\leq c_8 \max_{K\in\mathcal{T}^\star_{h,\Sigma}} h_K^{2\delta}\Big\{\|\psi_h\|^2_{-1/2+\delta,\Sigma} + \|\psi_h\|^2_{-1/2,\Sigma}\Big\} \\
&\leq c_8 \max_{K\in\mathcal{T}^\star_{h,\Sigma}} h_K^{2\delta}\Big\{h_\Sigma^{-2\delta}\,\|\psi_h\|^2_{-1/2,\Sigma} + \|\psi_h\|^2_{-1/2,\Sigma}\Big\} \\
&\leq c_9\,\|\psi_h\|^2_{-1/2,\Sigma}\,,
\end{aligned} \tag{4.25}$$

where the fact that $h_K \leq c\,h_\Sigma \quad \forall\,K \in \mathcal{T}^\star_{h,\Sigma}$ has been used in the last inequality. In this way, from (4.23), (4.24) and (4.25) we conclude that

$$\|\mathcal{L}_h(\psi_h)\|_{\mathrm{div},\Omega_\star} \leq C_{\mathcal{L}}\,\|\psi_h\|_{-1/2,\Sigma} \qquad \forall\,\psi_h \in \Phi_h(\Sigma),$$

which, together with the identities (4.21) and (4.22), complete the proof. □

We remark at this point that the quasi-uniformity assumption of $\mathcal{T}_h(\Omega_*)$ around Σ, which is needed here for the stable discrete lifting provided by Lemma 4.4, has been removed recently in [1, Theorem 2.1] for the case of locally refined meshes, when the lifting is from the whole boundary of the given domain. However, it is not clear from the analysis in [1] whether that result is also valid for a discrete lifting from part of the boundary (as it is required in the present case).

We now assume that the family of independent triangulations $\mathcal{T}_{\tilde{h}}(\Sigma)$ is also quasi-uniform, which implies that $\mathrm{Q}_{\tilde{h}}^{\Sigma}$ satisfies the inverse inequality, that is there exists a constant $C > 0$, independent of $\tilde{h}$, such that for each $\delta \in [0, 1)$ there holds (cf. [20, Lemma 7.4])

$$\|\xi\|_{1/2+\delta,\Sigma} \le C\,\tilde{h}^{-\delta}\|\xi\|_{1/2,\Sigma} \qquad \forall\,\xi \in \mathrm{Q}_{\tilde{h}}^{\Sigma}. \tag{4.26}$$

Then, we have the following result.

Lemma 4.5 *There exist C_0, $\beta > 0$, independent of h_Σ and $\tilde{h}$, such that for all $h_\Sigma \le C_0\,\tilde{h}$ there holds*

$$\sup_{\substack{\psi_h \in \Phi_h(\Sigma)\\ \psi_h \neq 0}} \frac{\langle \psi_h, \xi_{\tilde{h}}\rangle_\Sigma}{\|\psi_h\|_{-1/2,\Sigma}} \ge \beta\,\|\xi_{\tilde{h}}\|_{1/2,\Sigma} \qquad \forall\,\xi_{\tilde{h}} \in \mathrm{Q}_{\tilde{h}}^{\Sigma}. \tag{4.27}$$

Proof. We proceed similarly as in [19, Lemma 4.11]. In fact, given $\xi_{\tilde{h}} \in \mathrm{Q}_{\tilde{h}}^{\Sigma}\backslash\{0\}$, we let $z \in \mathrm{H}^1(\Omega_\star)$ be the unique solution of the boundary value problem with mixed boundary conditions:

$$-\Delta z + z = 0 \quad \text{in} \quad \Omega_\star, \quad z = \xi_{\tilde{h}} \quad \text{on} \quad \Sigma, \quad \nabla z \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_\star.$$

Notice that the corresponding continuous dependence result gives

$$\|z\|_{1,\Omega_\star} \le C_1\,\|\xi_{\tilde{h}}\|_{1/2,\Sigma}, \tag{4.28}$$

and thanks to the trace theorem and a simple integration by parts procedure, we also have that

$$C_2\,\|\xi_{\tilde{h}}\|_{1/2,\Sigma}^2 \le \|z\|_{1,\Omega_\star}^2 = \langle \nabla z \cdot \boldsymbol{n}, \xi_{\tilde{h}}\rangle_\Sigma. \tag{4.29}$$

On the other hand, since $\mathrm{Q}_{\tilde{h}}^{\Sigma} \subset \mathrm{H}^1(\Sigma)$, we obtain that $z \in \mathrm{H}^{1+\delta}(\Omega_\star)$ for some $\delta > 0$ (see [26]), and there holds

$$\|\nabla z \cdot \boldsymbol{n}\|_{-1/2+\delta,\Sigma} \le C_3\|z\|_{1+\delta,\Omega_\star} \le C_4\,\|\xi_{\tilde{h}}\|_{1/2+\delta,\Sigma}. \tag{4.30}$$

We now let $\psi_h^* := \mathcal{P}_h^{-1/2}(\nabla z \cdot \boldsymbol{n}) \in \Phi_h(\Sigma)$. Then, applying the approximation property (AP_h^{ψ}), the regularity estimate (4.30), and the inverse inequality (4.26), we deduce that

$$\|\nabla z \cdot \boldsymbol{n} - \psi_h^*\|_{-1/2,\Sigma} \le C_5\,h_\Sigma^\delta\|\nabla z \cdot \boldsymbol{n}\|_{-1/2+\delta,\Sigma} \le C_6\,h_\Sigma^\delta\|\xi_{\tilde{h}}\|_{1/2+\delta,\Sigma} \le C_7\left(\frac{h_\Sigma}{\tilde{h}}\right)^\delta\|\xi_{\tilde{h}}\|_{1/2,\Sigma}.$$

Next, using that $\|\nabla z\|_{\mathrm{div},\Omega_\star} = \|z\|_{1,\Omega_\star}$, it follows that

$$\|\psi_h^*\|_{-1/2,\Sigma} = \|\mathcal{P}_h^{-1/2}(\nabla z \cdot \boldsymbol{n})\|_{-1/2,\Sigma} \le \|\nabla z \cdot \boldsymbol{n}\|_{-1/2,\Sigma} \le \|\nabla z\|_{\mathrm{div},\Omega_\star} = \|z\|_{1,\Omega_\star}\,,$$

which together with the estimate (4.28), imply

$$\|\psi_h^*\|_{-1/2,\Sigma} \le C_8\,\|\xi_{\tilde{h}}\|_{1/2,\Sigma}.$$

Now, using (4.29) and the foregoing estimates, we find that

$$\begin{aligned}
\langle \psi_h^*, \xi_{\tilde{h}}\rangle_\Sigma &= \langle \nabla z \cdot \boldsymbol{n}, \xi_{\tilde{h}}\rangle_\Sigma - \langle \nabla z \cdot \boldsymbol{n} - \psi_h^*, \xi_{\tilde{h}}\rangle_\Sigma \\
&\ge \left\{C_2 - C_7\Big(\frac{h_\Sigma}{\tilde{h}}\Big)^\delta\right\}\|\xi_{\tilde{h}}\|_{1/2,\Sigma}^2
\end{aligned}$$

$$\geq \quad \left\{ \frac{C_2}{C_8} - \frac{C_7}{C_8} \Big(\frac{h_\Sigma}{\tilde{h}} \Big)^\delta \right\} \|\xi_{\tilde{h}}\|_{1/2,\Sigma} \, \|\psi^*_{\tilde{h}}\|_{-1/2,\Sigma}.$$

Consequently, we can write

$$\sup_{\substack{\psi_h \in \Phi_h(\Sigma) \\ \psi_h \neq 0}} \frac{\langle \psi_h, \xi_{\tilde{h}} \rangle_\Sigma}{\|\psi_h\|_{-1/2,\Sigma}} \geq \frac{\langle \psi^*_h, \xi_{\tilde{h}} \rangle_\Sigma}{\|\psi^*_h\|_{-1/2,\Sigma}} \geq \left\{ \frac{C_2}{C_8} - \frac{C_7}{C_8} \Big(\frac{h_\Sigma}{\tilde{h}} \Big)^\delta \right\} \|\xi_{\tilde{h}}\|_{1/2,\Sigma},$$

from which, taking $h_\Sigma \leq C_0 \, \tilde{h}$ with $C_0 := \left(\frac{C_2}{2C_7} \right)^{1/\delta}$, we conclude the proof. □

4.2.4 Verification of the discrete inf-sup conditions

We are now in a position to prove the discrete inf-sup conditions required by hypotheses **(H.1)**. To this end, we assume from now on that $\mathcal{T}_h(\Omega_{\mathrm{D}})$ and $\mathcal{T}_h(\Omega_{\mathrm{B}})$ are quasi-uniform in a neighborhood Ω_Σ of Σ, and that $\mathcal{T}_{\tilde{h}}(\Sigma)$ is quasi-uniform.

Lemma 4.6 *There exist C_0, $\widetilde{\beta}_{\mathrm{D}} > 0$, independent of h, h_Σ and $\tilde{h}$, such that for all $h_\Sigma \leq C_0 \, \tilde{h}$, there holds*

$$S^{\mathrm{D}}_h(q^{\mathrm{D}}_h, \xi_{\tilde{h}}) := \sup_{\substack{\boldsymbol{v}^{\mathrm{D}}_h \in \mathbf{H}^{\mathrm{D}}_h \\ \boldsymbol{v}^{\mathrm{D}}_h \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q^{\mathrm{D}}_h \operatorname{div} \boldsymbol{v}^{\mathrm{D}}_h \, + \, \langle \boldsymbol{v}^{\mathrm{D}}_h \cdot \boldsymbol{n}, \xi_{\tilde{h}} \rangle_\Sigma}{\|\boldsymbol{v}^{\mathrm{D}}_h\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq \widetilde{\beta}_{\mathrm{D}} \left\{ \|q^{\mathrm{D}}_h\|_{0,\Omega_{\mathrm{D}}} \, + \, \|\xi_{\tilde{h}}\|_{1/2,\Sigma} \right\} \tag{4.31}$$

for all $(q^{\mathrm{D}}_h, \xi_{\tilde{h}}) \in \mathrm{Q}^{\mathrm{D}}_{h,0} \times \mathrm{Q}^{\Sigma}_{\tilde{h}}$.

Proof. We begin by observing that

$$S^{\mathrm{D}}_h(q^{\mathrm{D}}_h, \xi_{\tilde{h}}) \geq \sup_{\substack{\boldsymbol{v}^{\mathrm{D}}_h \in \mathbf{H}^{\mathrm{D}}_h \\ \boldsymbol{v}^{\mathrm{D}}_h \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q^{\mathrm{D}}_h \operatorname{div} \boldsymbol{v}^{\mathrm{D}}_h}{\|\boldsymbol{v}^{\mathrm{D}}_h\|_{\mathrm{div};\Omega_{\mathrm{D}}}} - \|\xi_{\tilde{h}}\|_{1/2,\Sigma}.$$

Then according to the results in [12, Chapter IV] (see also [19, Section 4.2]), we know that there exists $C_{\mathrm{D}} > 0$, independent of h, h_Σ and $\tilde{h}$, such that

$$\sup_{\substack{\boldsymbol{v}^{\mathrm{D}}_h \in \mathbf{H}^{\mathrm{D}}_h \\ \boldsymbol{v}^{\mathrm{D}}_h \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q^{\mathrm{D}}_h \operatorname{div} \boldsymbol{v}^{\mathrm{D}}_h}{\|\boldsymbol{v}^{\mathrm{D}}_h\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \geq C_{\mathrm{D}} \, \|q^{\mathrm{D}}_h\|_{0,\Omega_{\mathrm{D}}} \qquad \forall \, q^{\mathrm{D}}_h \in \mathrm{Q}^{\mathrm{D}}_{0,h},$$

and hence

$$S^{\mathrm{D}}_h(q^{\mathrm{D}}_h, \xi_{\tilde{h}}) \geq C_{\mathrm{D}} \, \|q^{\mathrm{D}}_h\|_{0,\Omega_{\mathrm{D}}} - \|\xi_{\tilde{h}}\|_{1/2,\Sigma} \qquad \forall \, (q^{\mathrm{D}}_h, \xi_{\tilde{h}}) \in \mathrm{Q}^{\mathrm{D}}_{h,0} \times \mathrm{Q}^{\Sigma}_{\tilde{h}}. \tag{4.32}$$

On the other hand, we know from Lemma 4.4, that there exist a linear operator $\mathcal{L}_h : \Phi_h(\Sigma) \to \mathbf{H}^{\mathrm{D}}_h$ and a constant $C_{\mathcal{L}} > 0$, independent of h, such that for each $\psi_h \in \Phi_h(\Sigma)$ there hold

$$\mathcal{L}_h(\psi_h) \cdot \boldsymbol{n} = \psi_h \quad \text{on} \quad \Sigma, \quad \|\mathcal{L}_h(\psi_h)\|_{\mathrm{div},\Omega_{\mathrm{D}}} \leq C_{\mathcal{L}} \, \|\psi_h\|_{-1/2,\Sigma}, \quad \text{and} \quad \operatorname{div} \mathcal{L}_h(\psi_h) \in P_0(\Omega_{\mathrm{D}}).$$

In this way, we deduce that

$$S_h^{\mathrm{D}}(q_h^{\mathrm{D}},\xi_{\tilde h}) \,\geq\, \frac{\displaystyle\int_{\Omega_{\mathrm{D}}} q_h^{\mathrm{D}}\,\mathrm{div}\,\mathcal{L}_h(\psi_h) \,+\, \langle \mathcal{L}_h(\psi_h)\cdot\boldsymbol{n},\xi_{\tilde h}\rangle_\Sigma}{\|\mathcal{L}_h(\psi_h)\|_{\mathrm{div};\Omega_{\mathrm{D}}}} \qquad \forall\,\psi_h \,\in\, \Phi_h(\Sigma),$$

from which, using that $\mathrm{div}\,\mathcal{L}_h(\psi_h) \,\in\, P_0(\Omega_{\mathrm{D}})$ and that $q_h^{\mathrm{D}} \,\in\, \mathrm{Q}_{h,0}^{\mathrm{D}}$, it follows that

$$S_h^{\mathrm{D}}(q_h^{\mathrm{D}},\xi_{\tilde h}) \,\geq\, \frac{|\langle \mathcal{L}_h(\psi_h)\cdot\boldsymbol{n},\xi_{\tilde h}\rangle_\Sigma|}{\|\mathcal{L}_h(\psi_h)\|_{\mathrm{div},\Omega_{\mathrm{D}}}} \,\geq\, \frac{1}{C_{\mathcal{L}}}\frac{|\langle \psi_h,\xi_{\tilde h}\rangle_\Sigma|}{\|\psi_h\|_{-1/2,\Sigma}} \qquad \forall\,\psi_h \,\in\, \Phi_h(\Sigma),$$

and hence

$$S_h^{\mathrm{D}}(q_h^{\mathrm{D}},\xi_{\tilde h}) \,\geq\, \frac{1}{C_{\mathcal{L}}} \sup_{\substack{\psi_h\in\Phi_h(\Sigma)\\ \psi_h\neq 0}} \frac{\langle \psi_h,\xi_{\tilde h}\rangle_\Sigma}{\|\psi_h\|_{-1/2,\Sigma}} \qquad \forall\,\xi_{\tilde h}\in \mathrm{Q}_{\tilde h}^{\Sigma}. \tag{4.33}$$

Therefore, (4.33) and a straightforward application of Lemma 4.5 imply the existence of $\widetilde{C}_{\mathrm{D}} > 0$, independent of h, h_Σ and $\tilde h$, such that for all $h_\Sigma \leq C_0\tilde h$ there holds

$$S_h^{\mathrm{D}}(q_h^{\mathrm{D}},\xi_{\tilde h}) \,\geq\, \widetilde{C}_{\mathrm{D}}\,\|\xi_{\tilde h}\|_{1/2,\Sigma} \qquad \forall\,\xi_{\tilde h}\in \mathrm{Q}_{\tilde h}^{\Sigma}. \tag{4.34}$$

Finally, it easy to see that estimates (4.32) and (4.34) imply the discrete inf-sup condition (4.31), thus finishing the proof. □

Lemma 4.7 *There exist C_0, $\tilde\beta_{\mathrm{B}} > 0$, independent of h, h_Σ and $\tilde h$, such that for all $h_\Sigma \leq C_0\,\tilde h$ there holds*

$$S_h^{\mathrm{B}}(q_h^{\mathrm{B}},\xi_{\tilde h}) \,:=\, \sup_{\substack{\boldsymbol{v}_h^{\mathrm{B}}\in\mathbf{H}_h^{\mathrm{B}}\\ \boldsymbol{v}_h^{\mathrm{B}}\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_{\mathrm{B}}} q_h^{\mathrm{B}}\,\mathrm{div}\,\boldsymbol{v}_h^{\mathrm{B}} \,-\, \langle \boldsymbol{v}_h^{\mathrm{B}}\cdot\boldsymbol{n},\xi_{\tilde h}\rangle_\Sigma}{\|\boldsymbol{v}_h^{\mathrm{B}}\|_{\mathrm{div};\Omega_{\mathrm{B}}}} \,\geq\, \tilde\beta_{\mathrm{B}}\left\{\,\|q_h^{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \,+\, \|\xi_{\tilde h}\|_{1/2,\Sigma}\right\} \tag{4.35}$$

for all $(q_h^{\mathrm{B}},\xi_{\tilde h}) \,\in\, \mathrm{Q}_{h,0}^{\mathrm{B}}\times\mathrm{Q}_{\tilde h}^{\Sigma}$.

Proof. It proceeds exactly as the proof of Lemma 4.6 by replacing Ω_{D}, Γ_{D}, $\mathrm{Q}_{h,0}^{\mathrm{D}}$ and $\mathbf{H}_h^{\mathrm{D}}$ by Ω_{B}, Γ_{B}, $\mathrm{Q}_{h,0}^{\mathrm{B}}$ and $\mathbf{H}_h^{\mathrm{B}}$, respectively. □

The following theorem provides the rate of convergence of our Galerkin scheme (4.4).

Theorem 4.2 *Let $\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}}\times\mathbf{H}_{0,h}^{\mathrm{B}}\times\mathbf{H}_h^{\mathrm{D}}$ and $\mathrm{Q}_{h,0} := \mathrm{Q}_{h,0}^{\mathrm{B}}\times\mathrm{Q}_h^{\mathrm{D}}\times\mathrm{Q}_{\tilde h}^{\Sigma}$ be the subspaces specified above, and let $(\vec{\boldsymbol{u}},\vec{p}) \,:=\, ((\boldsymbol{u}_{\mathrm{B}},\boldsymbol{\omega}_{\mathrm{B}},\boldsymbol{u}_{\mathrm{D}}),(p_{\mathrm{B}},p_{\mathrm{D}},\lambda)) \in \mathbf{H}\times\mathbf{Q}_0$ and $(\vec{\boldsymbol{u}}_h,\vec{p}_h) \,:=\, ((\boldsymbol{u}_h^{\mathrm{B}},\boldsymbol{\omega}_h^{\mathrm{B}},\boldsymbol{u}_h^{\mathrm{D}}),(p_h^{\mathrm{B}},p_h^{\mathrm{D}},\lambda_{\tilde h})) \in \mathbf{H}_h\times\mathbf{Q}_{0,h}$ be the unique solutions of the continuous and discrete problems (3.9) and (4.4), respectively. Assume that $\boldsymbol{u}_\star \in \mathbf{H}^\delta(\Omega_\star), \mathrm{div}\,\boldsymbol{u}_\star \in \mathrm{H}^\delta(\Omega_\star), p_\star \in \mathrm{H}^\delta(\Omega_\star)$ where $\star \in \{\mathrm{B},\mathrm{D}\}$, $\boldsymbol{\omega}_{\mathrm{B}} \in \mathbf{H}^\delta(\mathbf{curl};\Omega_{\mathrm{B}})$ and $\lambda \in \mathrm{H}^{1/2+\delta}(\Sigma)$, for some $\delta \in (1/2,1]$. Then, there exists $C > 0$ and $\widetilde{C} > 0$ independent of h and $\tilde h$ such that*

$$\begin{aligned}\|(\vec{\boldsymbol{u}},\vec{p}) - (\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}} \leq\; & Ch^\delta\Big\{\|\boldsymbol{u}_{\mathrm{B}}\|_{\delta,\Omega_{\mathrm{B}}} + \|\mathrm{div}\,(\boldsymbol{u}_{\mathrm{B}})\|_{\delta,\Omega_{\mathrm{B}}} \,+\, \|\boldsymbol{\omega}_{\mathrm{B}}\|_{\mathbf{H}^\delta(\mathbf{curl};\Omega_{\mathrm{B}})} \,+\, \|\boldsymbol{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}}\\ & + \|\mathrm{div}\,(\boldsymbol{u}_{\mathrm{D}})\|_{\delta,\Omega_{\mathrm{D}}} \,+\, \|p_{\mathrm{B}}\|_{\delta,\Omega_{\mathrm{B}}} + \|p_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}}\Big\} \,+\, \widetilde{C}\tilde h^\delta\|\lambda\|_{\delta+1/2,\Sigma}.\end{aligned}$$

Proof. It follows from the Céa estimate (4.15) and the approximation properties $(\mathrm{AP}_h^{\boldsymbol{u}_\star})$, $(\mathrm{AP}_h^{p_\star})$, $(\mathrm{AP}_h^{\boldsymbol{\omega}_{\mathrm{B}}})$ and $(\mathrm{AP}_h^{\lambda})$. □

We end this section by remarking that the analysis from Section 4.2 can be extended without difficulties, to Raviart-Thomas and Nédélec spaces of higher order.

5 An augmented mixed formulation

In this section we propose an augmented variational formulation of problem (3.9). Indeed, though many finite element subspaces $\mathbf{H}_{0,h}^{\mathrm{B}} \subseteq \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}})$ and $\mathbf{H}_h^{\mathrm{B}} \subseteq \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}})$ do satisfy **(H.3)**, we would like to explore the possibility of getting rid of that assumption. To this end, we suggest to enrich the mixed variational formulation (3.9) with a residual arising from the Brinkman momentum equation in (2.1). More precisely, we include into the variational problem (3.9) the following Galerkin least-squares equation in Ω_{B}:

$$\kappa \int_{\Omega_{\mathrm{B}}} (\alpha\, \boldsymbol{u}_{\mathrm{B}} + \nu\, \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} + \nabla p_{\mathrm{B}} - \boldsymbol{f}_{\mathrm{B}}) \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} = 0 \quad \forall\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}})\,, \tag{5.1}$$

where κ is a positive parameter to be specified later. Actually, integrating by parts, and using again that $\mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathrm{div};\Omega_{\mathrm{B}})$ for each $\boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}})$ (cf. [25, Chapter I, Section 2.3, Remark 2.5]), we easily find that

$$\int_{\Omega_{\mathrm{B}}} \nabla p_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} \;=\; 0 \qquad \forall\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}})\,,$$

whence (5.1) can be recast in the form

$$\kappa\, \alpha \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} + \kappa\, \nu \int_{\Omega_{\mathrm{B}}} \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} = \kappa \int_{\Omega_{\mathrm{B}}} \boldsymbol{f}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} \quad \forall\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}). \tag{5.2}$$

In this way, adding (5.2) to the first equation of (3.9), we obtain the following augmented variational formulation: Find $\vec{\boldsymbol{u}} := (\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}) \in \mathbf{H}$ and $\vec{p} := (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda) \in \mathbf{Q}_0$ such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathcal{B}(\vec{\boldsymbol{v}},\vec{p}) &= \mathcal{F}(\vec{\boldsymbol{v}}) \qquad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{H}\,, \\ \mathcal{B}(\vec{\boldsymbol{u}},\vec{q}) &= \mathcal{G}(\vec{q}) \qquad \forall\, \vec{q} := (q_{\mathrm{B}}, q_{\mathrm{D}}, \xi) \in \mathbf{Q}_0\,, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) &:= \alpha \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{\omega}_{\mathrm{B}} \cdot \boldsymbol{z}_{\mathrm{B}} + \kappa\, \nu \int_{\Omega_{\mathrm{B}}} \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} \\ &+ \nu \int_{\Omega_{\mathrm{B}}} \boldsymbol{v}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} + \big(\kappa\, \alpha - \nu\big) \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \qquad \forall\, (\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \in \mathbf{H} \times \mathbf{H}\,, \end{aligned} \tag{5.4}$$

$$\mathcal{F}(\vec{\boldsymbol{v}}) := \int_{\Omega_{\mathrm{B}}} \boldsymbol{f}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \int_{\Omega_{\mathrm{D}}} \boldsymbol{f}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} + \kappa \int_{\Omega_{\mathrm{B}}} \boldsymbol{f}_{\mathrm{B}} \cdot \mathbf{curl}\, \boldsymbol{z}_{\mathrm{B}} \qquad \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}\,, \tag{5.5}$$

$\mathcal{B} = \mathbf{b}$, and $\mathcal{G} = \mathbf{G} = \mathbf{0}$.

In what follows we address the solvability of (5.3). We first observe that the continuous inf-sup condition for $\mathcal{B}$ on $\mathbf{H} \times \mathbf{Q}_0$ is already proved by Lemma 3.4. In turn, the continuous kernel of $\mathcal{B}$ is certainly given by $\mathbf{V}$ (cf. (3.12) - (3.14)). Then, we have the following result establishing the ellipticity of $\mathcal{A}$ on $\mathbf{V}_{\mathrm{B,D}}$ and hence on $\mathbf{V}$.

Lemma 5.1 *Assume that the stabilization parameter $\kappa \in \big(0, 2\delta\big)$ with $\delta \in \left(0, \dfrac{2\nu}{\alpha}\right)$. Then, there exists $\varrho > 0$, depending on κ and δ, such that*

$$\mathcal{A}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \;\geq\; \varrho\, \|\vec{\boldsymbol{v}}\|_{\mathbf{H}}^2 \qquad \forall\, \vec{\boldsymbol{v}} \in \mathbf{V}_{\mathrm{B,D}}\,. \tag{5.6}$$

Proof. Given $\vec{\boldsymbol{v}} := (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{V}_{\mathrm{B,D}}$, we obtain from the definition of $\mathcal{A}$ (cf. (5.4)) and the Cauchy-Schwarz inequality, that

$$\begin{aligned}\mathcal{A}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \,&=\, \alpha\,\|\boldsymbol{v}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \nu\,\|\boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \kappa\,\nu\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \kappa\,\alpha\int_{\Omega_{\mathrm{B}}} \boldsymbol{v}_{\mathrm{B}}\cdot\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}} \,+\, \mu\,\|\boldsymbol{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}\\ &\geq\, \alpha\,\|\boldsymbol{v}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \nu\,\|\boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \kappa\,\nu\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} - \kappa\alpha\|\boldsymbol{v}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}}\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} + \mu\,\|\boldsymbol{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}.\end{aligned}$$

Next, for each $\delta > 0$ we find that

$$-\kappa\,\alpha\,\|\boldsymbol{v}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}}\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \,\geq\, -\,\frac{\kappa\,\alpha}{2\delta}\,\|\boldsymbol{v}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,-\, \frac{\delta\,\kappa\,\alpha}{2}\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}},$$

which, replaced back into the foregoing estimate, yields

$$\mathcal{A}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \,\geq\, \alpha\,\left(1-\frac{\kappa}{2\delta}\right)\,\|\boldsymbol{v}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \nu\,\|\boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \kappa\,\left(\nu\,-\,\frac{\delta\,\alpha}{2}\right)\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \mu\,\|\boldsymbol{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}.$$

Next, using (3.17) and noting that $\|\boldsymbol{v}_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{B}}} \,=\, \|\boldsymbol{v}_{\mathrm{D}}\|^2_{\mathrm{div};\Omega_{\mathrm{D}}}$, we obtain

$$\mathcal{A}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \,\geq\, \alpha\,\left(1\,-\,\frac{\kappa}{2\delta}\right)\,\varrho_0^2\,\|\boldsymbol{v}_{\mathrm{B}}\|^2_{\mathrm{div},\Omega_{\mathrm{B}}} \,+\, \nu\,\|\boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \kappa\,\left(\nu\,-\,\frac{\delta\,\alpha}{2}\right)\,\|\,\mathbf{curl}\ \boldsymbol{z}_{\mathrm{B}}\|^2_{0,\Omega_{\mathrm{B}}} \,+\, \mu\,\|\boldsymbol{v}_{\mathrm{D}}\|^2_{\mathrm{div},\Omega_{\mathrm{D}}}. \tag{5.7}$$

Hence, since $1\,-\,\frac{\kappa}{2\,\delta} > 0$ and $\nu\,-\,\frac{\delta\,\alpha}{2} > 0$, we conclude that

$$\mathcal{A}(\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}) \,\geq\, \varrho\|\vec{\boldsymbol{v}}\|^2_{\mathbf{H}} \quad \vec{\boldsymbol{v}} \,\in\, \mathbf{V}_{\mathrm{B,D}},$$

where $\varrho \,:=\, \min\Big\{\alpha\left(1-\frac{\kappa}{2\delta}\right)\varrho_0^2, \nu, \kappa\left(\nu-\frac{\delta\,\alpha}{2}\right), \mu\Big\}$. □

Note that, taking in particular $\kappa = \delta = \dfrac{\nu}{\alpha}$, we obtain the optimal ellipticity constant

$$\varrho \,:=\, \frac{1}{2}\,\min\Big\{\alpha\varrho_0^2, 2\nu, \kappa\nu, 2\mu\Big\}.$$

The foregoing analysis yields the following main result.

Theorem 5.1 *Assume that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, $\mathbf{f}_{\mathrm{B}} \in \mathbf{L}^2(\Omega_{\mathrm{B}})$, *and that* κ *satisfies the assumption from Lemma* 5.1. *Then there exists a unique* $(\vec{\boldsymbol{u}}, \vec{p}) \,:=\, \big((\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}), (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda)\big) \in \mathbf{H}\times\mathbf{Q}_0$ *solution of the augmented mixed formulation* (5.3). *Moreover, there exists* $C > 0$ *such that*

$$\|\vec{\boldsymbol{u}}\|_{\mathbf{H}} \,+\, \|\vec{p}\|_{\mathbf{Q}} \,\leq\, C\,\Big\{\|\boldsymbol{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \,+\, \|\boldsymbol{f}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}}\Big\}\,. \tag{5.8}$$

Proof. Thanks to Lemmata 3.4 and 5.1, the proof is a straightforward application of the continuous Babuška-Brezzi theory. □

We now look at the Galerkin scheme of (5.3). More precisely, employing the same generic finite elements subspaces and related notations introduced in Section 4.1, we now consider the augmented mixed finite element scheme: Find $\vec{\boldsymbol{u}}_h \,:=\, (\boldsymbol{u}_h^{\mathrm{B}}, \boldsymbol{\omega}_h^{\mathrm{B}}, \boldsymbol{u}_h^{\mathrm{D}}) \in \mathbf{H}_h$ and $\vec{p}_h \,:=\, (p_h^{\mathrm{B}}, p_h^{\mathrm{D}}, \lambda_h) \in \mathbf{Q}_{0,h}$ such that

$$\begin{aligned}\mathcal{A}(\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h) \,+\, \mathcal{B}(\vec{\boldsymbol{v}}_h, \vec{p}_h) \quad &= \quad \mathcal{F}(\vec{\boldsymbol{v}}_h) \qquad \forall\,\vec{\boldsymbol{v}}_h \,:=\, (\boldsymbol{v}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{v}_h^{\mathrm{D}}) \in \mathbf{H}_h\,,\\ \mathcal{B}(\vec{\boldsymbol{u}}_h, \vec{q}_h) \quad &= \quad \mathcal{G}(\vec{q}_h) \qquad \forall\,\vec{q}_h \,:=\, (q_h^{\mathrm{B}}, q_h^{\mathrm{D}}, \xi_h) \in \mathbf{Q}_{0,h}\,.\end{aligned} \tag{5.9}$$

Then, assuming that hypotheses **(H.0)**, **(H.1)**, and **(H.2)** from Section 4 are satisfied, we certainly deduce that $\mathcal{B}$ verifies the discrete inf-sup condition on $\mathbf{H}_h \times \mathbf{Q}_{0,h}$ (cf. Lemma 4.1), the discrete kernel of $\mathcal{B}$ is given again by $\mathbf{V}_h = \mathbf{V}^h_{\mathrm{B,D}} \cap \mathbf{V}^h_\Sigma$ (cf. (4.9) - (4.11)), and hence, since $\mathbf{V}^h_{\mathrm{B,D}}$ is contained in $\mathbf{V}_{\mathrm{B,D}}$, the bilinear form $\mathcal{A}$ is elliptic in $\mathbf{V}^h_{\mathrm{B,D}}$ (cf. Lemma 5.1) and therefore in $\mathbf{V}_h$. Consequently, a straightforward application of the discrete Babuška-Brezzi theory allows to conclude the following result.

Theorem 5.2 *Assume that* $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$ *and* $\mathbf{f}_{\mathrm{B}} \in \mathbf{L}^2(\Omega_{\mathrm{B}})$. *In addition, suppose that* **(H.0)**, **(H.1)**, *and* **(H.2)** *hold. Then there exists a unique* $(\vec{\boldsymbol{u}}_h, \vec{p}_h) := \big((\boldsymbol{u}^{\mathrm{B}}_h, \boldsymbol{\omega}^{\mathrm{B}}_h, \boldsymbol{u}^{\mathrm{D}}_h), (p^{\mathrm{B}}_h, p^{\mathrm{D}}_h, \lambda_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$ *solution of the augmented Galerkin scheme* (5.9). *Moreover, there exist* C_1, $C_2 > 0$, *independent of* h, *such that*

$$\|\vec{\boldsymbol{u}}_h\|_{\mathbf{H}} + \|\vec{p}_h\|_{\mathbf{Q}} \le C_1 \Big\{ \|\boldsymbol{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\boldsymbol{f}_{\mathrm{B}}\|_{0,\Omega_{\mathrm{B}}} \Big\}, \tag{5.10}$$

and

$$\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}} \le C_2 \Big\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) + \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) \Big\}. \tag{5.11}$$

6 Numerical results

In this section we provide three computer experiments confirming the convergence rates anticipated by Theorem 4.2 and illustrating the applicability of the method in surface-subsurface flow problems.

6.1 Accuracy of the mixed and augmented formulations on two embedded cubes

We start by evaluating the convergence of the fully-mixed and the augmented finite element methods applied to (2.1)-(2.2) and defined on the two cubes $\Omega_{\mathrm{B}} = [-r_B, r_B]^3$ and $\Omega_{\mathrm{D}} = [-r_D, r_D]^3$, with $r_D = \frac{1}{2}, r_B = \frac{3}{20}$. Notice that this particular domain configuration does not fall exactly in the theoretical framework analyzed in this paper. However, both the continuous and discrete study could be carried out using the analogous tools as those used here. We employ the model parameters $\alpha = \mu = 1, \nu = 0.01$, yielding the stabilization constant $\kappa = 2\nu/\alpha = 0.02$ suggested by Lemma 5.1. The convergence of the method is assessed by computing errors between the following manufactured smooth exact solutions

$$\boldsymbol{\omega}_{\mathrm{B}}(x_1,x_2,x_3) = \begin{pmatrix} -3\pi\sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ 3\pi\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ 0 \end{pmatrix}, \boldsymbol{u}(x_1,x_2,x_3) = \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3) \\ \sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \\ -2\sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \end{pmatrix},$$

$$p(x_1,x_2,x_3) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3),\ \boldsymbol{u}_{\mathrm{B}} = \boldsymbol{u}|_{\Omega_{\mathrm{B}}},\ \boldsymbol{u}_{\mathrm{D}} = \boldsymbol{u}|_{\Omega_{\mathrm{D}}},\ p_{\mathrm{B}} = p|_{\Omega_{\mathrm{B}}},\ p_{\mathrm{D}} = p|_{\Omega_{\mathrm{D}}},\ \lambda = p|_{\Sigma},$$

and their finite element approximations using a $\mathbb{RT}_0 - \mathbb{ND}_1 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1$ family on a sequence of successively refined tetrahedral meshes $\mathcal{T}_{h_{Bi}}$ and $\mathcal{T}_{h_{Di}}$ of sizes $h_{Bi} = r_B 2^{1-i}$ and $h_{Di} = r_D 2^{-i}$, respectively, $i = 0, 1, \ldots$. We adequately choose forcing terms $\boldsymbol{f}_{\mathrm{B}} = \alpha\boldsymbol{u}_{\mathrm{B}} + \mathbf{curl}\,\boldsymbol{\omega}_{\mathrm{B}} + \nabla p_{\mathrm{B}}$, $\boldsymbol{f}_{\mathrm{D}} = \mu\boldsymbol{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}}$, and suitable *nonhomogeneous* slip velocity on $\partial\Omega$ and nonhomogeneous Dirichlet data for the tangential vorticity on $\partial\Omega_{\mathrm{B}}$, such that (2.1)-(2.2) holds. For sake of convenience we define a conforming partition for Σ, that is $\mathcal{T}_{\tilde{h}} = \mathcal{T}_h$. The approximate solutions are depicted in Figure 6.1 and the error history, written in terms of the quantities

$$e(\boldsymbol{u}_{\mathrm{B}}) := \|\boldsymbol{u}_{\mathrm{B}} - \boldsymbol{u}_{\mathrm{B}h}\|_{\mathrm{div},\Omega_{\mathrm{B}}},\ e(\boldsymbol{\omega}_{\mathrm{B}}) := \|\boldsymbol{\omega}_{\mathrm{B}} - \boldsymbol{\omega}_{\mathrm{B}h}\|_{\mathbf{curl},\Omega_{\mathrm{B}}},\ e(\boldsymbol{u}_{\mathrm{D}}) := \|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_{\mathrm{D}h}\|_{\mathrm{div},\Omega_{\mathrm{D}}},$$

$$e(p_{\mathrm{B}}) := \|p_{\mathrm{B}} - p_{\mathrm{B}h}\|_{0,\Omega_{\mathrm{B}}},\ e(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D}h}\|_{0,\Omega_{\mathrm{D}}},\ e(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma},\ r(\cdot) := \frac{\log(e(\cdot)/\hat{e}(\cdot))}{\log(h/\hat{h})},$$

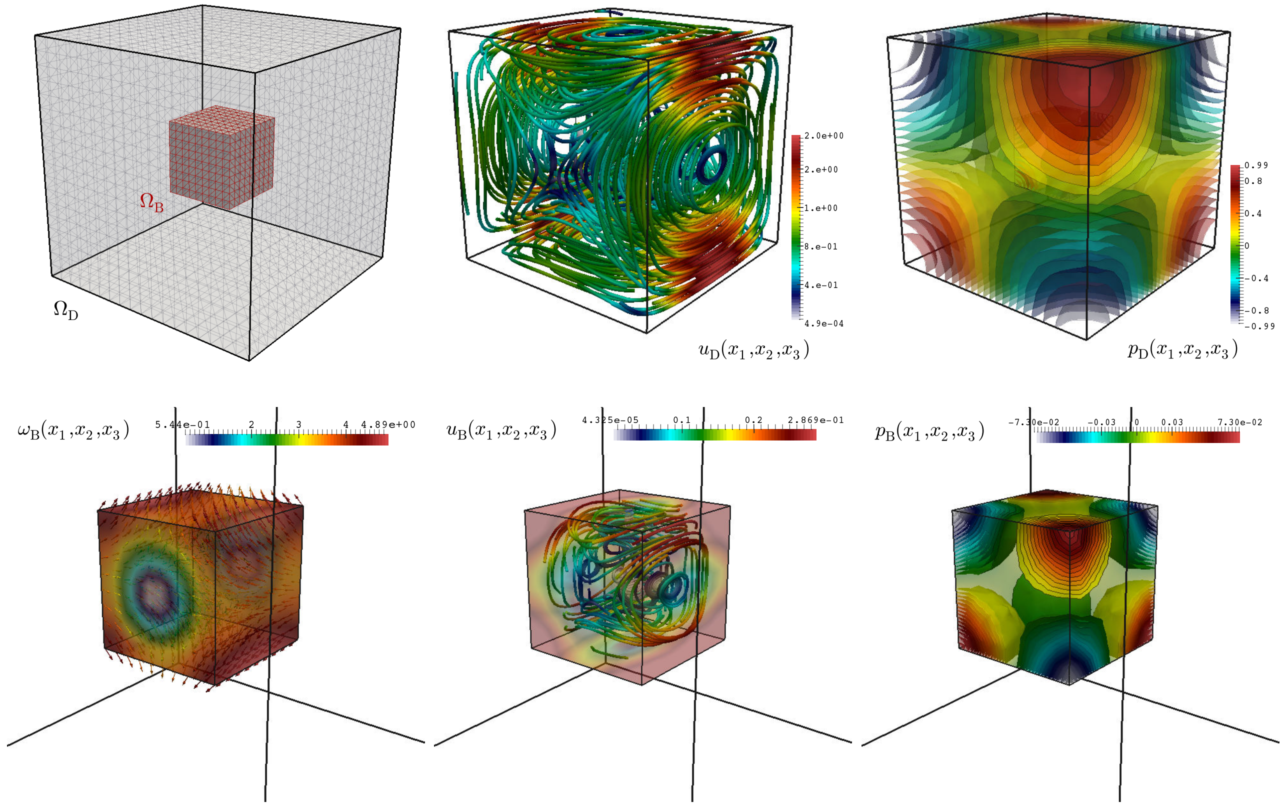

Figure 6.1: Example 1: Two-domain geometry and mesh (top left), approximated Darcy velocity streamlines (top middle), approximated Darcy pressure isosurfaces (top right), zoom of approximated Brinkman vorticity vectors (bottom left), zoom of approximated Brinkman velocity streamlines (bottom middle), and isosurfaces of the computed Brinkman pressure (bottom right).

are reported in Table 6.1, where $e, \hat{e}$ denote errors computed on two consecutive meshes of sizes $h = \max\{h_B, h_D\}$ and $\hat{h}$. We observe that both methods deliver optimal convergence rates for vorticity, velocity and pressure in the corresponding norms.

6.2 Flow into a cracked porous medium

Our second example focuses on the simulation of flow in a porous medium with a smoothed V-shaped crack, similar to the 2D simulations presented in the Stokes-Darcy examples of [8, Section 7.1] and [13, Section 6.3]. The full domain is the box $\Omega = [0,2] \times [0,0.2] \times [0,1]$, the Brinkman domain on the top is $0.75 \leq x_1 \leq 1.25$ and goes down to $x_3 = 0.5$. Viscosity and porosity correspond to the case of water flowing in a mixture of calcarenite and sand: $\nu = 0.01, \mu = 10000$, and we set $\alpha = 0.001$. The external forces on both domains correspond to gravity $\boldsymbol{f}_\mathrm{D} = \boldsymbol{f}_\mathrm{B} = (0,0,-0.98)^\mathtt{t}$, and a constant flowrate $\boldsymbol{u}_\mathrm{D} \cdot \boldsymbol{n} = (10,0,0)^\mathtt{t} \cdot \boldsymbol{n}$, is imposed on the right wall $\Gamma_\mathrm{D}^\mathrm{in}$, at $x_1 = 0$ (see sketch in figure 6.2), representing a subsurface flow in the x_1-direction. Normal Darcy velocities are set to zero everywhere else on Γ_D. As in [8] we impose a smooth vorticity profile on the top of Γ_B $\boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{n} = (0, 1/16 - (x_1-1)^2, 0)^\mathtt{t} \times \boldsymbol{n}$, which takes into account the wind on the surface, and we also

h	$e(\boldsymbol{u}_\mathrm{B})$	$r(\boldsymbol{u}_\mathrm{B})$	$e(\boldsymbol{\omega}_\mathrm{B})$	$r(\boldsymbol{\omega}_\mathrm{B})$	$e(\boldsymbol{u}_\mathrm{D})$	$r(\boldsymbol{u}_\mathrm{D})$	$e(p_\mathrm{B})$	$r(p_\mathrm{B})$	$e(p_\mathrm{D})$	$r(p_\mathrm{D})$	$e(\lambda)$	$r(\lambda)$
Fully mixed scheme (4.4)												
0.70711	1.02802	–	0.08636	–	0.65565	–	0.00404	–	0.64650	–	0.51608	–
0.38079	0.66329	0.63216	0.04547	0.86511	0.30143	0.96758	0.00167	0.86588	0.24919	1.54026	0.37415	0.94712
0.30610	0.45239	1.30206	0.03253	1.13929	0.21869	1.46952	0.00130	1.44871	0.12438	1.18266	0.22579	0.93668
0.18503	0.29153	0.93254	0.02240	0.79149	0.16048	0.61483	0.00073	1.21498	0.05130	1.15932	0.17396	0.95387
0.14412	0.18023	1.02275	0.01527	0.85417	0.10498	0.89264	0.00042	1.15692	0.02337	1.14603	0.10985	0.96014
0.05487	0.11716	0.95707	0.00833	0.95944	0.05076	0.99002	0.00027	0.96289	0.01207	0.98594	0.05139	0.96765
0.03564	0.07258	0.97681	0.00561	0.98670	0.03123	0.99197	0.00019	0.98441	0.00896	0.99455	0.02987	0.97732
Augmented mixed scheme (5.9)												
0.70711	1.02681	–	0.08574	–	0.65549	–	0.00416	–	0.64537	–	0.51899	–
0.38079	0.62020	0.98418	0.04306	0.87429	0.26781	0.94902	0.00158	0.95434	0.24503	0.94234	0.38461	0.97729
0.30610	0.42963	0.99011	0.02763	0.91368	0.17061	0.96547	0.00109	0.93939	0.10942	0.96471	0.21733	1.07908
0.18503	0.27689	0.94556	0.01916	0.94842	0.12903	0.95084	0.00066	0.98741	0.04873	0.96933	0.16430	0.98544
0.14412	0.16540	0.96134	0.01344	0.96083	0.08211	0.95171	0.00039	0.98177	0.01998	0.96297	0.08127	0.97476
0.05487	0.10214	0.98608	0.00703	0.95798	0.04714	0.90989	0.00024	0.97506	0.00987	0.97250	0.04550	0.97732
0.03564	0.06071	0.96110	0.00416	0.98465	0.02595	1.01103	0.00016	0.98411	0.00593	1.00141	0.02831	0.97460

Table 6.1: Example 1: Error history associated to fully mixed (top rows) and augmented (bottom rows) $\mathbb{RT}_0 - \mathbb{ND}_1 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1$ discretizations of (2.1)-(2.2) on a 3D domain.

assume a compatible normal velocity on that same surface $\boldsymbol{u}_\mathrm{B} \cdot \boldsymbol{n} = (0, 0, -x_1/16 + [(x_1-1)^3]/3)^\mathtt{t} \cdot \boldsymbol{n}$. Everywhere else we set zero normal fluid velocity and zero tangential vorticity. A tetrahedral mesh with conforming interface is generated having 57426 vertices and 307544 elements, which in total correspond to 962639 degrees of freedom for $\mathbb{RT}_0 - \mathbb{ND}_1 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_1$ finite elements. Figure 6.2 depicts the domain configuration along with the approximate solutions, matching qualitatively the results from [8, 13].

6.3 Perpendicular infiltration trhough a porous medium

In the last test we present a model of coupled surface and subsurface flow where the top domain is the flow region and the bottom half of the domain represents e.g. an aquifer. On the top left octant of Ω_B, denoted by $\Gamma_\mathrm{B}^\mathrm{in}$, we consider an inflow rate of $\boldsymbol{u}_\mathrm{B} \cdot \boldsymbol{n} = -0.01$ and on $\Gamma_\mathrm{D}^\mathrm{out}$ (see the domain sketch in Figure 6.3) we set an outflow of fluid at rate $\boldsymbol{u}_\mathrm{B} \cdot \boldsymbol{n} = 0.01$. Also on $\Gamma_\mathrm{B}^\mathrm{in}$, we impose a smooth vorticity $\boldsymbol{\omega}_\mathrm{B} \times \boldsymbol{n} = (0, -0.01x_1x_2x_3, 0)^\mathtt{t} \times \boldsymbol{n}$. On the remainder of $\partial\Omega$ we set zero normal velocities and tangential vorticity. As in the previous example, we take into account the gravity force acting on both domains $\boldsymbol{f}_\mathrm{D} = \boldsymbol{f}_\mathrm{B} = (0, 0, -0.98)^\mathtt{t}$, and employ the model parameters $\alpha = 10, \nu = 0.001, \mu = 10000$. The mesh for Ω consists of 32768 vertices and 191452 tetrahedral elements representing 700835 degrees of freedom. As expected, from Figure 6.3 we observe flow patterns entering the domain through $\Gamma_\mathrm{B}^\mathrm{in}$, percolating through Σ, and leaving the domain through $\Gamma_\mathrm{D}^\mathrm{out}$.

References

[1] M. Ainsworth, J. Guzmán and F.-J. Sayas, *Discrete extension operators for mixed finite element spaces on locally refined meshes.* arXiv:1406.5534v1 [math.NA]

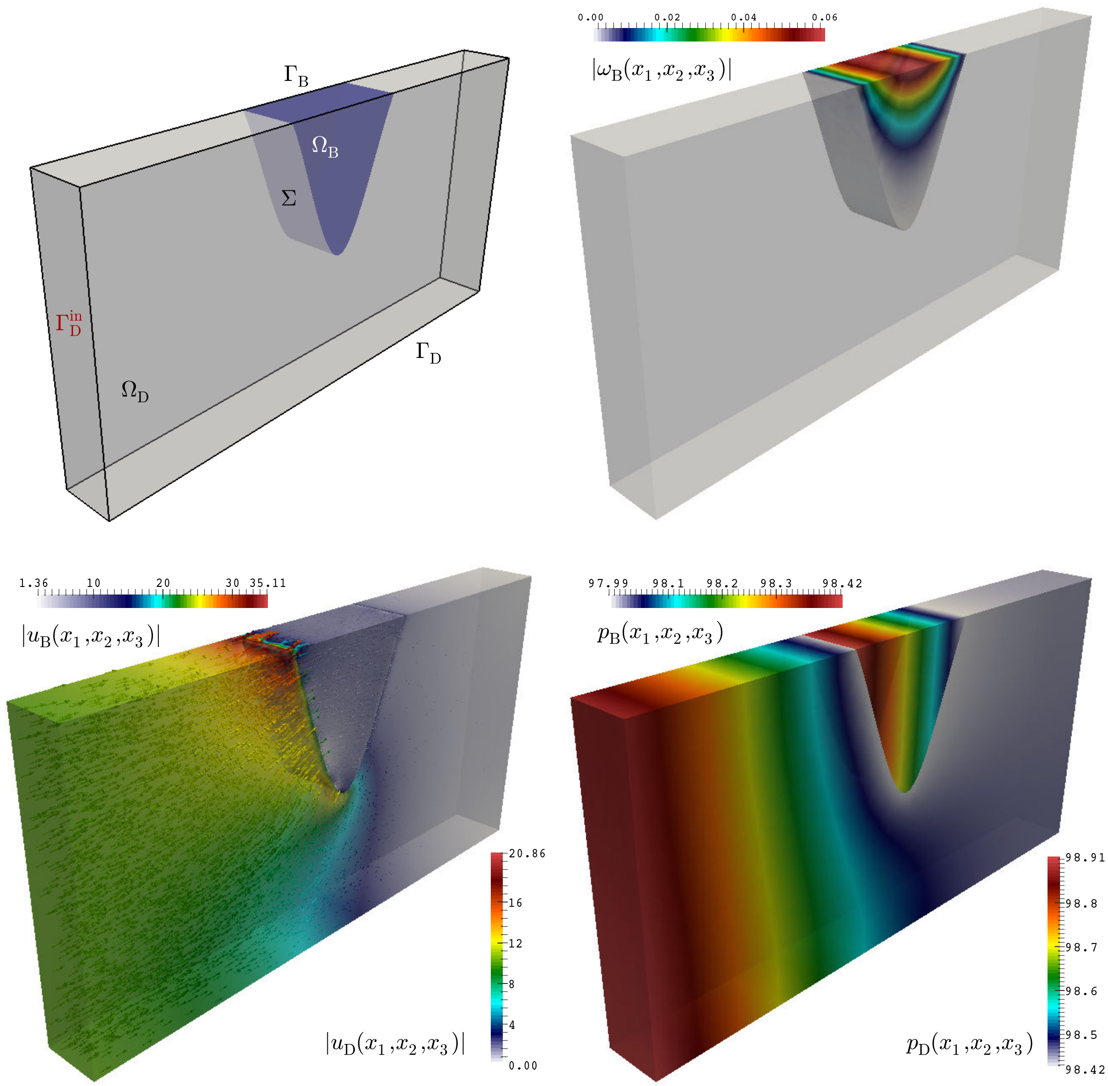

Figure 6.2: Example 2: Two-domain geometry and boundaries (top left), approximated Brinkman vorticity magnitude (top right), approximated velocity magnitude and vectors (bottom left), and computed pressure profiles (bottom right) for the Brinkman-Darcy coupling.

[2] A. Alonso and A. Valli, *An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations*. Math. Comput., 68 (1999) 607–631.

[3] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in tree-dimensional nonsmooth domains*. Math. Methods. Appl. Sci., 21 (1998) 823–864.

[4] V. Anaya, G.N. Gatica, D. Mora, and R. Ruiz-Baier, *Augmented velocity - vorticity - pressure formulations for the Brinkman equations*. Preprint 2014–11, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, (2014) [available from `http://www.ci2ma.udec.cl`]

[5] V. Anaya, D. Mora, R. Oyarzúa, and R. Ruiz-Baier, *A priori and a posteriori error analysis for a vorticity-based mixed formulation of the generalized Stokes equations*. Preprint 2014–20, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, (2014) [available from `http://www.ci2ma.udec.cl`]

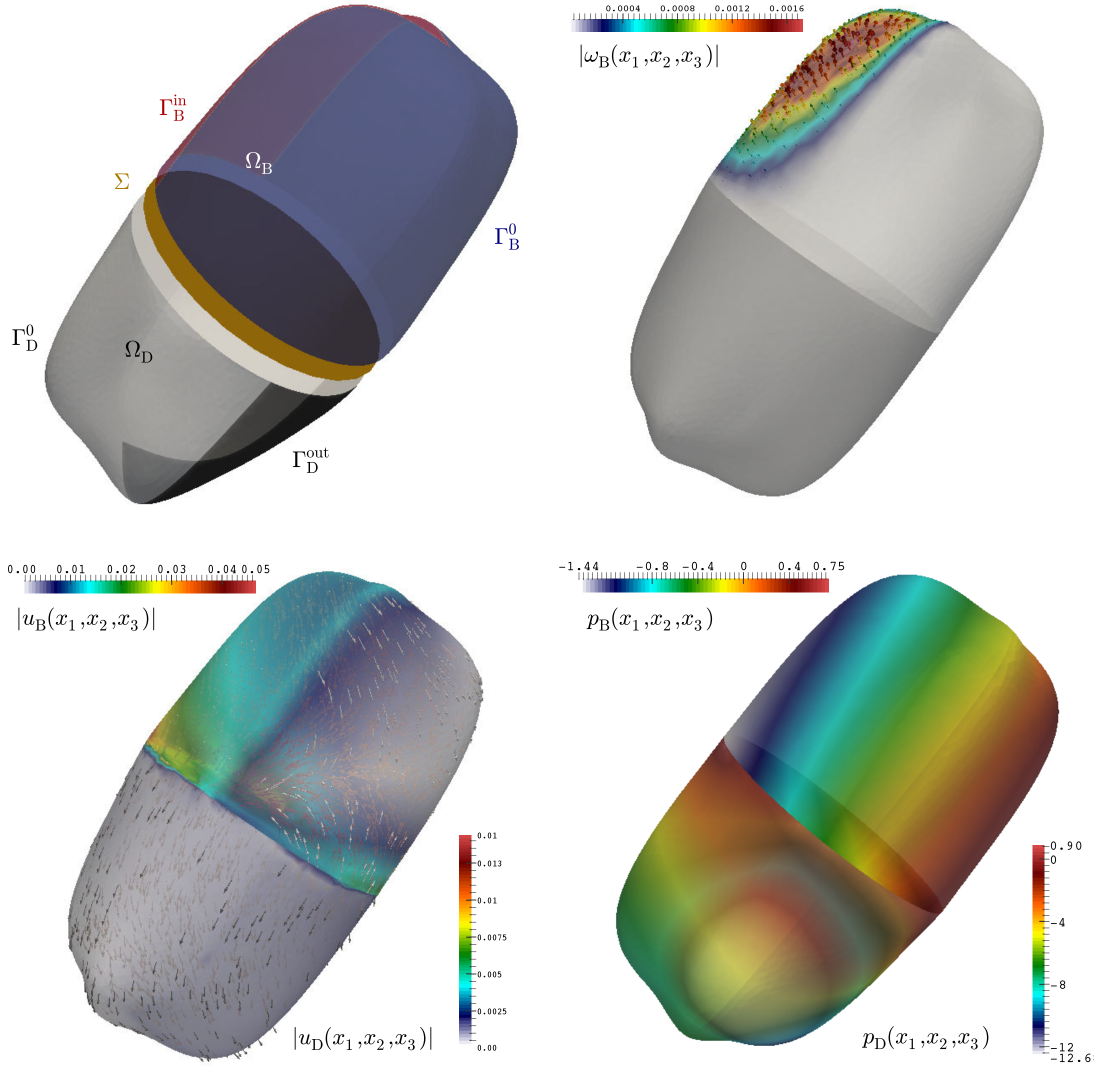

Figure 6.3: Example 3: Two-domain geometry and boundaries (top left), approximated Brinkman vorticity magnitude and vectors (top right), approximated velocity magnitude and vectors (bottom left), and computed pressure profiles (bottom right) for the Brinkman-Darcy coupling.

[6] T. Arbogast and D.S. Brunson, *A computatonal method for approximating a Darcy-Stokes system governing a vuggy porous medium*, Comput. Geosci., 11(3) (2007) 207–218.

[7] L. Badea, M. Discacciati, and A. Quarteroni, *Numerical analysis of the Navier-Stokes/Darcy coupling*, Numer. Math., 115 (2010) 195–227.

[8] C. Bernardi, F. Hecht, and F.Z. Nouri, *A new finite-element discretization of the Stokes problem coupled with the Darcy equations*. IMA J. Numer. Anal., 30 (2010) 61–93.

[9] C. Bernardi, F. Hecht, and O. Pironneau, *Coupling Darcy and Stokes equations for porous media with cracks*. ESAIM: Math. Model. Numer. Anal., 39(1) (2005) 7–35.

[10] A.-S. Bonnet-Ben Dhia, L. Chesnel, and P. Ciarlet, *T-coercivity for scalar interface problems between dielectrics and metamaterials*. ESAIM Math. Model. Numer. Anal. 46(6) (2012) 1363–1387.

[11] M. Braack, and F. Schieweck, *Equal-order finite elements with local projection stabilization for the Darcy-Brinkman equations*. Comput. Methods Appl. Mech. Engrg., 200(9-12) (2011) 1126–1136.

[12] F. Brezzi, and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer Verlag, New York, 1991.

[13] J. Camaño, G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier, and P. Venegas-Tapia, *New fully-mixed finite element methods for the Stokes-Darcy coupling*. Preprint 2014–18, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, (2014) [available from `http://www.ci2ma.udec.cl`]

[14] L. Chesnel and P. Ciarlet, *T-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients*. Numer. Math., 124(1) (2013) 1–129.

[15] M. Discacciati, and A. Quarteroni, *Navier-Stokes/Darcy coupling: modeling, analysis, and numerical approximation*. Rev. Mat. Complut. 22 (2009) 315–426.

[16] F. El Chami, G. Mansour, and T. Sayah, *Error studies of the coupling Darcy-Stokes system with velocity-pressure formulation*. Calcolo, 49 (2012) 73–93.

[17] V.J. Ervin, E.W. Jenkins, and S. Sun, *Coupled generalized nonlinear Stokes flow with flow through a porous medium*, SIAM J. Numer. Anal., 47(2) (2009) 929–952.

[18] J. Galvis and M. Sarkis, *Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations*. Electr. Trans. Numer. Anal., 26 (2007) 350–384.

[19] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*. Springer-Verlag, Berlin, 2014.

[20] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A coupled mixed finite element method for the interaction problem between and electromagnetic field and elastic body*. SIAM J. Numer. Anal., 48(4) (2010) 1338–1368.

[21] G.N. Gatica, A. Márquez, and S. Meddahi, *Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid-solid interaction problem*. SIAM J. Numer. Anal., 45(5) (2007) 2072–2097.

[22] G.N. Gatica, A. Márquez, and S. Meddahi, *Analysis of the coupling of Lagrange and Arnold-Falk-Winther finite elements for a fluid-solid interaction problem in 3D*. SIAM J. Numer. Anal., 50(3) (2012) 1648–1674.

[23] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas, *Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem*. Math. Comput., 80(276) (2011) 1911–1948.

[24] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas, *A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow*. IMA J. Numer. Anal., 32(3) (2012) 845–887.

[25] V. Girault, and P. A. Raviart, *Finite element methods for Navier-Stokes equations. Theory and algorithms*. Springer-Verlag, Berlin, 1986.

[26] P. Grisvart, *Singularities in Boundary Value Problems. Recherches en Mathmatiques Appliques*. Springer, Berlin, vol. 22, 1992.

[27] R. Hiptmair, *Finite elements in computational electromagnetism*. Acta Numer., 11 (2002) 237–339.

[28] M. Lesinigo, C. D'Angelo, and A. Quarteroni, *A multiscale Darcy-Brinkman model for fluid flow in fractured porous media*. Numer. Math., 117 (2011) 717–752.

[29] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge, 2000.

[30] P. Monk, *Finite Element Methods for Maxwell's Equations*. Oxford University Press, New York, 2003.

[31] B. Rivière, *Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems*. J. Sci. Comput. 22(1) (2005) 479–500.

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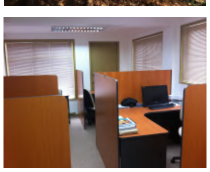

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