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Fabián Flores-Bazán, Nicolás Hadjisavvas, Felipe Lara

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Fabián Flores-Bazán[†] Nicolas Hadjisavvas[‡] Felipe Lara[§]

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Abstract

Recently, the concepts of second order asymptotic directions and functions have been introduced and applied to global and vector optimization problems. In this work, we establish some new properties for these two concepts. In particular, in case of a convex set, a complete characterization of the second order asymptotic cone is given. Also, formulas that permit the easy computation of the second order asymptotic function of a convex function are established. It is shown that the second order asymptotic function provides a finer description of the behavior of functions at infinity, than the first order asymptotic function.

Key words. Asymptotic cone; recession cone; asymptotic function; second order asymptotic cone; second order asymptotic function.

1 Introduction

The concept of asymptotic (or recession) directions of a set has been introduced almost 100 years ago [14], and then it was rediscovered by Debreu [2], where its use concerns the closedness of the sum of any two closed sets. Such a notion may be conceived as a main tool to describe the asymptotic behaviour of the set at infinity along these particular directions, so it is of

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†Departamento de Ingeniería Matemática, CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile (fflores@ing-mat.udec.cl)

‡Department of Product and Systems Design Engineering, University of the Aegean, Syros, Greece (nhad@aegean.gr)

§Departamento de Ingeniería Matemática, CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile (felipelara@udec.cl)

primary importance for dealing with unbounded sets, as it gives rise to the "asymptotic analysis" approach. A vector $u \neq 0$, say with $\|u\| = 1$, is by definition an asymptotic direction of a set K if its direction is the limit of directions of a sequence of vectors $\{x_k\}_{k\in\mathbb{N}}$ in K such that $\|x_k\| \to +\infty$; equivalently, if there exists a sequence $\{x_k\}$ in K such that $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to u$. In many interesting cases, the vectors x_k do not approach the line $\mathbb{R}u$ containing u; actually, their distance from it converges to $+\infty$. How exactly do the directions of x_k approach the direction of u? One way to answer this question, is to consider the projections $\langle x_k, u\rangle\, u$ of x_k onto u, and the differences $x_k - \langle x_k, u\rangle\, u$. Then, if $\|x_k - \langle x_k, u\rangle\, u\| \to +\infty$, one can find the limit, if any, of the directions of the latter vectors. These are exactly the canonic (second order) directions, which are orthogonal to u. From this line of reasoning, and in order to have a finer tool in the study of the behavior of sets and functions at infinity, very recently the authors in [8] introduced, more generally, the notion of second order direction for sets, as the limit of directions of a sequence of vectors $x_k - \alpha_k u$ with $\alpha_k \to +\infty$, $\frac{x_k}{\|x_k\|} \to \frac{u}{\|u\|}$ and $\|x_k - \alpha_k u\| \to +\infty$, which show how the vector x_k is "seen" from a vector $\alpha_k u$. The precise definition for sets (and functions) will be given in the two following sections.

We point out that (first order) asymptotic cones in infinite dimensional spaces were considered in [4, 5], see also [10]; whereas in spaces of finite dimension, we refer to [1, 13]. More recent applications may be found in [3, 6, 7, 11] and references therein.

A related concept which is motivated mainly by minimization problems, is the concept of asymptotic function. A careful analysis of the behaviour of the asymptotic function (associated to objective) along the asymptotic directions of the feasible set is crucial for a study of the existence of minima. Similarly to the first order case, a second order asymptotic function can be defined.

We believe that in the same way as the first-order asymptotic analysis proved to be a powerful in the study of sets and functions at infinity, the second-order approach will yield finer results in optimization, economics, engineering, etc. Indeed, it was shown in [8] that these second order notions may be used to establish necessary or sufficient conditions for optimality, characterize the efficient points in vector optimization, or provide criteria for the closedness of the sum of closed sets, in cases where the results using the first order asymptotic notions are not adequate. It is worthwhile mentioning that in [8], mainly the general non convex case was treated, and no formula was provided for the convex case. This latter situation will be discussed in detail in the present paper, so we will see that there are very simple and attractive formulas that provide the corresponding asymptotic notions for sets and functions.

We also must mention that the meaning of our concept of asymptotic cone is different from that considered, for instance, in [9]. For us the term asymptotic means far away, in contrast to the mentioned in that paper, compare (3) and (4) in [9, Definition 2.1] and our Proposition 3.4(c).

One of the main results of the present paper is a characterization of the second order asymptotic cone in the case of a convex set. Based on this characterization, we will show several properties of such cones, and give formulas for the second order asymptotic function, that permit an easy computation.

The structure of the paper is as follows. The basic definitions and notations are given in the next section. Section 3 contains some preliminary results on the properties of second order asymptotic cones, and also their relation to the so-called canonic directions. In addition, the convex case is discussed in details. For example, it is known that for a closed convex set $K \subseteq \mathbb{R}^n$, its asymptotic cone K^∞ is the set of vectors $u \in \mathbb{R}^n$ such that $x_0 + tu \in K$ for every $x_0 \in K$ and $t > 0$. As we shall see, given a convex (not necessarily closed) set K and an element x_0 of its relative interior $\operatorname{ri} K$, the second order asymptotic cone of K with respect to u, which will be denoted by $K^{\infty 2}[u]$, is the set of all $v \in \mathbb{R}^n$ such that for every $s > 0$, $x_0 + tu + sv \in K$ for all t sufficiently large (Proposition 3.4); actually, this property implies both $u \in K^\infty$ and $v \in K^{\infty 2}[u]$. In the special case of polyhedral sets, we may have $x_0 \in K$ instead of $x_0 \in \operatorname{ri} K$, and $K^{\infty 2}[u]$ has a simple expression (Remark 3.8 and Proposition 3.10).

The above characterization, together with some other similar ones, will be used to obtain information about the structure of the second order asymptotic cone, as well as several of its properties. Section 4 is devoted to the second order asymptotic function. Using the characterizations found in Section 3, we obtain formulas which permit an easy calculation of this function in case the original function is (not necessarily lower semicontinuus) convex, as we show with some examples.

It should be noted that the notation differs with respect to the ones introduced in [8]. Also, the definition of second order asymptotic function (which in [8] was called lower second order asymptotic function) is different, even if it is proven to be equivalent to the one in [8]. The changes were deemed useful in order to approach the more usual notation and definitions of the corresponding first order notions. Similar changes affect the canonic directions.

2 Basic definitions

We denote the duality pairing between two elements of $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. The subspace generated by a vector $u \in \mathbb{R}^n$ is denoted by $\mathbb{R}u$; the subspace orthogonal to it, $u^\perp$. Given $K \subseteq \mathbb{R}^n$, its closure is denoted by $\overline{K}$, its boundary by bd K, its topological interior by int K, its relative interior by $\operatorname{ri} K$, and its convex hull by $\operatorname{co}(K)$. We set $\operatorname{cone}(K) = \bigcup_{t\geq 0} tK$ and $\overline{\operatorname{cone}}(K) = \overline{\bigcup_{t\geq 0} tK}$.

For $K \subseteq \mathbb{R}^n$, its first order asymptotic cone (or just asymptotic cone) is defined by

$$K^\infty = \{u \in \mathbb{R}^n : \ \ \exists\, t_k \to +\infty, \ \ \exists\, x_k \in K, \ \frac{x_k}{t_k} \to u\}.$$

In case K is a closed convex set it is known that

$$K^\infty = \{u \in \mathbb{R}^n : \ \ x_0 + \lambda u \in K, \ \ \forall\, \lambda \geq 0\} \text{ for any } x_0 \in K. \tag{1}$$

If K is convex without being necessarily closed, then there is a similar characterization of K^∞ by using elements of $\operatorname{ri} K$:

$$K^\infty = \{u \in \mathbb{R}^n : \ \ x_0 + \lambda u \in \operatorname{ri} K, \ \ \forall\, \lambda \geq 0\} \text{ for any } x_0 \in \operatorname{ri} K; \tag{2}$$

see for instance Proposition 2.1.8 in [1].

A function f is called proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\operatorname{dom} f \neq \emptyset$, where $\operatorname{dom} f \doteq \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$. Given any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of f is defined as the function $f^\infty : \ \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfying

$$\operatorname{epi} f^\infty = (\operatorname{epi} f)^\infty, \tag{3}$$

where $\operatorname{epi} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leq t\}$ is the epigraph of f. Consequently, when f is a convex and lower semicontinuous function, we have $\forall x_0 \in f^{-1}(\mathbb{R})$,

$$f^\infty(u) = \lim_{t\to+\infty} \frac{f(x_0 + tu) - f(x_0)}{t} = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t} \tag{4}$$

and

$$f^\infty(u) = \sup_{x\in\operatorname{dom} f} \left(f(x+u) - f(x)\right). \tag{5}$$

From (3), we obtain the following formula for f^∞ in the general case, given $u \in \mathbb{R}^n$,

$$f^\infty(u) = \inf\Big\{ \liminf_{k\to\infty} \frac{f(x_k)}{t_k} : x_k \in \operatorname{dom} f, t_k \to +\infty, \frac{x_k}{t_k} \to u \Big\}.$$

If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$, f^∞ denotes the asymptotic function of f, where we extend f to the whole $\mathbb{R}^n$ by setting $f(x) = +\infty$ if $x \in \mathbb{R}^n \setminus K$. More detailed information on asymptotic sets and functions may be found in [13].

Definition 2.1 *Given a nonempty set $K \subseteq \mathbb{R}^n$ and $u \in \mathbb{R}^n$, we say that $v \in \mathbb{R}^n$ is a second order asymptotic direction of K at u if there are sequences $x_k \in K$, s_k and $t_k \in \mathbb{R}$, with $s_k, t_k \to +\infty$ such that,*

$$v = \lim_{k\to+\infty} \left(\frac{x_k}{s_k} - t_k u\right). \tag{6}$$

The set of all such elements v is denoted by $K^{\infty 2}[u]$.

Equivalently, $v \in K^{\infty 2}[u]$ if for each $k \in \mathbb{N}$ there exist $s_k > k$ and $t_k > k$ such that

$$\left| \frac{x_k}{s_k} - t_k u - v \right| < \frac{1}{k}. \tag{7}$$

Note that if (6) holds for s_k, t_k, x_k as in the definition, then

$$\lim_{k \to +\infty} \frac{x_k}{s_k t_k} = u.$$

Consequently, one has necessarily $u \in K^{\infty}$.

The set $K^{\infty 2}[u]$ is a cone, termed the second order asymptotic cone of K at u. It is nonempty exactly when $u \in K^{\infty}$ [8]. $K^{\infty 2}[u]$ is always closed, and if $u = 0$ then $K^{\infty 2}[0] = K^{\infty}$.

Let us fix a direction $u \in \mathbb{R}^n$ for which $f^{\infty}(u)$ is finite; then $u \in (\operatorname{dom} f)^{\infty}$ and $(u, f^{\infty}(u)) \in (\operatorname{epi} f)^{\infty}$. Set $A = (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right]$. Then $(v, \alpha) \in A$ iff there exist sequences $(x_k, \alpha_k) \in \operatorname{epi} f$ and $s_k, t_k \to +\infty$ such that

$$\frac{(x_k, \alpha_k)}{s_k} - t_k \left(u, f^{\infty}(u)\right) \to (v, \alpha). \tag{8}$$

In this case, for every $h > 0$, $(x_k, \alpha_k + s_k h) \in \operatorname{epi} f$ and

$$\frac{(x_k, \alpha_k + s_k h)}{s_k} - t_k \left(u, f^{\infty}(u)\right) \to (v, \alpha + h).$$

Thus, $(v, \alpha + h) \in A$ for every $h > 0$. Since A is a closed cone, this means that A is the epigraph of some lsc, positively homogeneous function. We call this function second order asymptotic function of f at u and we denote its value at v by $f^{\infty 2}(u; v)$, that is,

$$\text{epi } f^{\infty 2}(u; \cdot) = (\text{epi } f)^{\infty 2}[(u, f^{\infty}(u))]. \tag{9}$$

This yields the following straightforward result.

Proposition 2.2 *For $u \in \mathbb{R}^n$ satisfying $f^{\infty}(u) \in \mathbb{R}$, the function $f^{\infty 2}(u; \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, defined as in (9), is lsc and positively homogeneous; it satisfies $f^{\infty 2}(u; 0) = 0$ or $-\infty$, while $f^{\infty 2}(u; 0) = 0$ if and only if $f^{\infty 2}(u; \cdot)$ is proper.*

3 Second order asymptotic cones

We start by establishing various properties for the second order asymptotic cone of any set, including its link to canonic directions. Then we provide some characterizations of the second order asymptotic cone in case the set is convex.

3.1 Preliminary results

Next proposition collects some basic properties of the second order asymptotic cone for any set, although they partly already appear in [8].

Proposition 3.1 *Let $\emptyset \neq K \subseteq \mathbb{R}^n$. The following assertions hold.*

(a) *If $K_0 \subseteq K$, then $K_0^{\infty 2}[u] \subseteq K^{\infty 2}[u]$ for all $u \in \mathbb{R}^n$.*

(b) *$K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ for all $u \in K^{\infty}$.*

(c) *$K^{\infty 2}[u] = \overline{K}^{\infty 2}[u]$ for all $u \in K^{\infty}$.*

(d) *For all $u \in K^{\infty}$, $(K^{\infty 2}[u])^{\infty} = K^{\infty 2}[u]$ and $K^{\infty 2}[u] = (K^{\infty 2}[u])^{\infty 2}[u]$.*

(e) *If $u \in K^{\infty}$, then $K^{\infty 2}[u] \subseteq \overline{\mathbb{R}_+ K - \mathbb{R}_+ u}$.*

(f) *If $u \in \mathrm{ri}\ K^{\infty}$, then $\mathrm{aff}\ K^{\infty} = K^{\infty} - K^{\infty} \subseteq K^{\infty 2}[u]$.*

(g) *Let $A, B \subseteq \mathbb{R}^n$, with $u_1 \in A^{\infty}, u_2 \in B^{\infty}$, then*

$$(A \times B)^{\infty 2}[(u_1, u_2)] = A^{\infty 2}[u_1] \times B^{\infty 2}[u_2].$$

(h) *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a family of sets and $u \in \mathbb{R}^n$. Then*

$$\bigcup_{i \in I} K_i^{\infty 2}[u] \subseteq \Big(\bigcup_{i \in I} K_i\Big)^{\infty 2}[u].$$

Equality holds when $|I| < +\infty$.

(i) *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a family of sets satisfying $\bigcap_{i \in I} K_i \neq \emptyset$ and $u \in \mathbb{R}^n$. Then*

$$\Big(\bigcap_{i \in I} K_i\Big)^{\infty 2}[u] \subseteq \bigcap_{i \in I} K_i^{\infty 2}[u].$$

Proof.

(a) and (b) were proved in Proposition 2.2 [8]. (c), (e) and (g) are straightfoward.

(d): The first equality is obvious since $K^{\infty 2}[u]$ is a closed cone. To show inclusion $(\supseteq)$ in the second equality, let $w \in (K^{\infty 2}[u])^{\infty 2}[u]$. In view of (7), for each $k \in \mathbb{N}$ there exist $w_k \in K^{\infty 2}[u]$ and $s_k > k$, $t_k > k$ such that $\left|\frac{w_k}{s_k} - t_k u - w\right| < \frac{1}{k}$; moreover, there exist $w'_k \in K$ and $s'_k > k$, $t'_k > k$ such that $\left|\frac{w'_k}{s'_k} - t'_k u - w_k\right| < \frac{1}{k}$. Dividing the second inequality by s_k and adding the to first we deduce that

$$\left|\frac{w'_k}{s'_k s_k} - \left(t_k + \frac{t'_k}{s_k}\right) - w\right| < \frac{1}{k} + \frac{1}{k s_k} \leq \frac{1}{2k}.$$

Since $s'_k s_k > k$ and $t_k + \frac{t'_k}{s_k} > k$, we infer that $w \in K^{\infty 2}[u]$.

($\subseteq$) Let $w \in K^{\infty 2}[u]$. For every sequence $t_k \to +\infty$, we have $w + t_k u \in K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ by property (b), so $t_k w + t_k^2 u \in K^{\infty 2}[u]$ since $K^{\infty 2}[u]$ is a cone. From

$$w = \lim_{k\to+\infty} (\frac{t_k w + t_k^2 u}{t_k} - t_k u)$$

it then follows that $w \in (K^{\infty 2}[u])^{\infty 2}[u]$.

(f) Given $v \in \text{aff } K^{\infty}$, we have that for all $k \in \mathbb{N}$ large enough, $u + \frac{1}{k} v \in K^{\infty}$. Since K^{∞} is a cone, $ku + v \in K^{\infty}$. Consequently one can find $x_k \in K$ and $t_k > k$ such that $\left| \frac{x_k}{t_k} - ku - v \right| < \frac{1}{k}$. In view of (7), this implies that $v \in K^{\infty 2}[u]$.

(h): ($\subseteq$) is an obvious consequence of property (a).

($\supseteq$) ($|I| < \infty$). Let $w \in (\bigcup_{i=1}^m K_i)^{\infty 2}[u]$. Then there exist sequences $\{x_k\}_{k\in\mathbb{N}} \subseteq \bigcup_{i=1}^m K_i$ and $s_k, t_k \to +\infty$ with $w = \lim_{k\to\infty} \left(\frac{x_k}{s_k} - t_k u \right)$. Since I is finite, there exists i_0 and a subsequence $\{x_{k_l}\}_{l\in\mathbb{N}}$ such that $x_{k_l} \in K_{i_0}$ for all $l \in \mathbb{N}$. Hence, $w \in K_{i_0}^{\infty 2}[u] \subseteq \bigcup_{i=1}^m K_i^{\infty 2}[u]$.

(i) is again a trivial consequence of property (b). ■

As we will see in the next section, the reverse inclusion does not hold in general, but it does hold in some cases.

We now study canonic directions, which are of special importance. Given $u \neq 0$, a second order asymptotic direction $v \in K^{\infty 2}[u]$ is called canonic if $\langle u, v \rangle = 0$. The set of canonic directions $K^{\nu}[u]$ is thus $K^{\infty 2}[u] \cap u^{\perp}$. Note that for every $u \neq 0$ and $\alpha > 0$, $K^{\infty 2}[\alpha u] = K^{\infty 2}[u]$ (see for instance Proposition 2.2(vi) in [8]). Thus, $K^{\infty 2}[u] = K^{\infty 2}[\frac{u}{\|u\|}]$ and $K^{\nu}[u] = K^{\nu}[\frac{u}{\|u\|}]$, so we can always restrict ourselves to the case $\|u\| = 1$.

In what follows $P_{u^\perp}$ denotes the projection on $u^{\perp}$.

Proposition 3.2 *For every $u \in K^{\infty}\backslash\{0\}$, $K^{\infty 2}[u] = \mathbb{R}u + K^{\nu}[u]$. Thus, $K^{\nu}[u] = P_{u^\perp}(K^{\infty 2}[u])$*

Proof. We may assume that $\|u\| = 1$. To show inclusion $\subseteq$, write any $v \in K^{\infty 2}[u]$ as $v = \langle u, v \rangle u + (v - \langle v, u \rangle u)$. Then $\langle u, v \rangle u \in \mathbb{R}u$ and $v - \langle v, u \rangle u \in u^{\perp}$. In addition, $v - \langle v, u \rangle u \in K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ by Proposition 3.1(b). Thus inclusion $\subseteq$ follows. The opposite inclusion follows from $\mathbb{R}u + K^{\nu}[u] \subseteq \mathbb{R}u + K^{\infty 2}[u] = K^{\infty 2}[u]$.

The second assertion is a consequence of the first one. ■

We note that in [8] canonic directions were defined differently; we presented here another definition which is simpler. The equivalence of the two definitions is a consequence of Proposition 2.8(ii)(iii) in [8].

It is interesting that the canonic directions in $K^{\nu}[u]$ are first order directions of the projection of K onto $u^{\perp}$:

Proposition 3.3 *Let $u \in K^{\infty}\backslash\{0\}$. Then $K^{\nu}[u] \subseteq (P_{u^\perp} K)^{\infty}$.*

Proof. Take any $v \in K^\nu[u]$. Then there exist sequences $\{x_n\} \subseteq K$ and $t_n, s_n \to +\infty$ such that (6) holds. It follows that

$$v = P_{u^\perp} v = \lim \left(\frac{P_{u^\perp} x_n}{t_n} - s_n P_{u^\perp} u \right) = \lim \frac{P_{u^\perp} x_n}{t_n} \in (P_{u^\perp} K)^\infty.$$

■

Note that in general $K^\nu[u] \neq (P_{u^\perp} K)^\infty$. For instance, if $K \subseteq \mathbb{R}^2$ is the set $\{(x_1, 0) : x_1 \in \mathbb{R}\} \cup \{(0, x_2) : x_2 \in \mathbb{R}\}$ and $u = (0, 1)$, then $u^\perp = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, $(P_{u^\perp} K)^\infty = u^\perp$ but $K^\nu[u] = \{0\}$.

3.2 The case of convex sets

In case K is a convex (not necessarily closed) subset of $\mathbb{R}^n$, we have a characterization of $K^{\infty 2}[u]$ which reminds the one for K^∞ given by (1). This will permit us to show some properties of the second order asymptotic cone of convex sets.

Proposition 3.4 *Let $K \subseteq \mathbb{R}^n$ be convex and $x \in \operatorname{ri} K$. Then the following assertions are equivalent.*

(*a*) $u \in K^\infty$ *and* $v \in K^{\infty 2}[u]$.

(*b*) *for all* $s > 0$ *there exists* $\bar{t} > 0$ *such that for every* $t > \bar{t}$, $x + tu + sv \in K$.

(*c*) *there exist sequences* $s_n \to +\infty$, $t_n \to +\infty$ *such that* $x + s_n t_n u + s_n v \in K$.

Proof. $(a) \Rightarrow (b)$. Let $x \in \operatorname{ri} K$ be arbitrary. Let P be the projection on $u^\perp$. Write $x = t_1 u + Px$ and $v = t_2 u + Pv$. By Theorem 6.6 in [12], $\operatorname{ri} PK = P(\operatorname{ri} K)$, hence $Px \in \operatorname{ri} PK$. By Propositions 3.2 and 3.3, $Pv \in K^\nu[u] \subseteq (PK)^\infty$.

Since PK is convex, from (2) we deduce that for every $s > 0$, $Px + sPv \in \operatorname{ri} PK$. Hence there exists $y \in \operatorname{ri} K$ such that $Py = Px + sPv$. Write $y = t_3 u + Py$. Substituting Py, Px and Pv we deduce $y = x + \bar{t} u + sv$ where $\bar{t} = t_3 - t_1 - st_2$. From $y \in \operatorname{ri} K$ and $u \in K^\infty$ we infer that for every $t > \bar{t}$, $x + tu + sv = y + (t - \bar{t})u \in \operatorname{ri} K$. Since this is true for all $t > \bar{t}$, it is clear that we can choose some $\bar{t}_1 > 0$ such that for all $t > \bar{t}_1$, $x + tu + sv \in \operatorname{ri} K$.

$(b) \Rightarrow (c)$. If (b) holds for some $x \in \operatorname{ri} K$, then set $s_k = k$ and choose λ_k large enough, say $\lambda_k > k^2$, such that $x + \lambda_k u + kv \in K$. Define $t_k = \lambda_k / k$. Then (c) holds for the same x.

$(c) \Rightarrow (a)$. If (c) holds for some x, then for the sequence $a_k \doteq x + s_k t_k u + s_k v$ we will have

$$\frac{a_k}{s_k t_k} = \frac{x}{s_k t_k} + \frac{v}{t_k} + u \to u,$$

so $u \in K^\infty$. In addition,

$$\frac{a_k}{s_k} - t_k u = \frac{x}{s_k} + v \to v,$$

hence $v \in K^{\infty 2}[u]$. ■

Note that if (b) is true for some $x \in \operatorname{ri} K$, then this implies (a), which in turn implies (b) and (c) for every $x \in \operatorname{ri} K$. Hence if (b) is true for some $x \in \operatorname{ri} K$, then it is true for all.

We also note, as is clear from the proof, that (b) in Proposition 3.4 may be replaced by

(b') for all $s > 0$ there exists $\bar{t} > 0$ such that for every $t > \bar{t}$, $x + tu + sv \in \operatorname{ri} K$.

We also have some equivalent characterizations, when we know that $u \in K^\infty$.

Proposition 3.5 *Let $K \subseteq \mathbb{R}^n$ be convex and $u \in K^\infty$. Given $v \in \mathbb{R}^n$, the following are equivalent.*

(a) $v \in K^{\infty 2}[u]$.

(b) *For every (equivalently, for some) $x \in \operatorname{ri} K$ and every $s > 0$, there exists $t \in \mathbb{R}$ such that $x + tu + sv \in K$.*

(c) *For every $x \in \operatorname{ri} K$, there exists $t \in \mathbb{R}$ such that $x + tu + v \in K$.*

Proof. If (a) holds, then (b) holds in view of Proposition 3.4. Conversely, if (b) holds for some $x \in \operatorname{ri} K$, given $s > 0$ choose $t \in \mathbb{R}$ so that $x + 2tu + 2sv \in K$. Then $x + tu + sv = \frac{1}{2}x + \frac{1}{2}(x + 2tu + 2sv) \in \operatorname{ri} K$. Thus for every $t' > t$, $x + t'u + sv = x + tu + sv + (t' - t)u \in K$. Hence (a) holds by Proposition 3.4.

So we have only to prove that (c) implies (b). Assume that (c) holds and let $s > 0$. Given $x \in \operatorname{ri} K$, there exists $t_1 \in \mathbb{R}$ such that $x + t_1 u + v \in K$. Then

$$x + \frac{t_1}{2}u + \frac{1}{2}v \in \left]x, x + t_1 u + v\right[,$$

so $x + \frac{t_1}{2}u + \frac{1}{2}v \in \operatorname{ri} K$. By using again (c) on $x + \frac{t_1}{2}u + \frac{1}{2}v$ we can find $t_2 \in \mathbb{R}$ such that $x + (\frac{t_1 + t_2}{2})u + 2\frac{v}{2} \in \operatorname{ri} K$. Using induction, we conclude that for every $k \in \mathbb{N}$ we can find $t'_k \in \mathbb{R}$ such that $x + t'_k u + \frac{k}{2}v \in \operatorname{ri} K$. Take k large enough so that $\frac{k}{2} > s$, and set $\lambda = \frac{2s}{k} \in]0, 1[$. Then $x + \lambda t'_k u + sv \in \left]x, x + t'_k u + \frac{k}{2}v\right[$. This implies $x + \lambda t'_k u + sv \in K$ and (b) holds. ■

As a first application of the characterization, we show:

Proposition 3.6 *Let $K \subseteq \mathbb{R}^n$ be convex.*

(*a*) $K^{\infty 2}[u]$ *is convex for all* $u \in K^\infty$.

(*b*) $K^\infty \subseteq K^{\infty 2}[u]$ *for all* $u \in K^\infty$.

(*c*) *If* $u \in K^\infty \cap (-K^\infty)$, *then* $K^{\infty 2}[u] = K^\infty$.

(*d*) *If* $u \in K^\infty$ *then* $K^\infty - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$, *and so if additionaly* K *is a cone, we get* $K^{\infty 2}[u] = \overline{K - \mathbb{R}_+ u}$.

Proof. (*a*) Fix $x \in \mathrm{ri}\ K$. If $v_1, v_2 \in K^{\infty 2}[u]$, then for every $s > 0$ we can find t_1, t_2 such that for all $t \geq t_i$, $x + tu + sv_i \in K$. Since K is convex, for every $t > \max\{t_1, t_2\}$ and $\lambda \in]0,1[$ we have $x + tu + s((1-\lambda)v_1 + \lambda v_2) \in K$. Thus $(1-\lambda)v_1 + \lambda v_2 \in K^{\infty 2}[u]$.

(*b*) Given $x \in \mathrm{ri}\ K$, we note that for every $u, v \in K^\infty$, and for every $s, t > 0$, $x + su \in \mathrm{ri}\, K$ so $x + tu + sv \in K$. Thus by the characterization of Proposition 3.4, $v \in K^{\infty 2}[u]$.

(*c*) Let $x \in \mathrm{ri}\ K$ and $v \in K^{\infty 2}[u]$. For every $s > 0$, we can find t such that $x + tu + sv \in K$. Since $-u \in K^\infty$, $(x + tu + sv) + t(-u) \in K$. Thus $x + sv \in K$ for all $s > 0$, so $v \in K^\infty$. This shows that $K^{\infty 2}[u] \subseteq K^\infty$. Using (*b*) we obtain the equality.

(*d*) Since K is convex, $K^\infty \subseteq K^{\infty 2}[u]$ by part (*b*). Proposition 3.1(*b*) implies that $\mathbb{R}u \subseteq K^{\infty 2}[u]$, and therefore $K^\infty - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$ due to the convexity of the cone $K^{\infty 2}[u]$. In case K is a cone, the equality follows from Proposition 3.1(*e*). ■

Another application is that the inclusion in Proposition 3.3 is an equality for convex sets:

Proposition 3.7 *Let* $K \in \mathbb{R}^n$ *be nonempty and convex,* $u \in K^\infty \backslash \{0\}$ *and* P *be the projection on* $u^\perp$. *Then* $K^\nu[u] = (PK)^\infty$ *and, consequently,* $K^{\infty 2}[u] = \mathbb{R}u + (PK)^\infty$.

Proof. We already know by Proposition 3.3 that for any nonempty set K, $K^\nu[u] \subseteq (PK)^\infty$. Take any $v \in (PK)^\infty$. If $x \in \mathrm{ri}\, K$ then $Px \in \mathrm{ri}\,(PK)$. As in the proof of implication (*a*) $\Rightarrow$ (*b*) in Proposition 3.4, we infer that for every $s > 0$ there exists $\bar{t} > 0$ such that for $t > \bar{t}$, $x + tu + sv \in K$. Hence $v \in K^{\infty 2}[u]$. Since $\langle v, u \rangle = 0$, v is canonic, so $K^\nu[u] = (PK)^\infty$. ■

In other words, the canonic directions of K at u are exactly the first order directions of the projection of K on $u^\perp$. The second order directions are the vectors whose projection on $u^\perp$, are first order directions of PK.

Remark 3.8 The assumption $x \in \mathrm{ri}\ K$ is used in the proof of (*a*) $\Rightarrow$ (*b*) of Proposition 3.4, in order to ensure that $Px \in \mathrm{ri}\, PK$, so that $Px + sPv \in \mathrm{ri}\, PK$ since $Pv \in (PK)^\infty$. Whenever K is a closed set such that PK is also closed, this assumption is not needed and the same proof shows that one can take simply $x \in K$. Such a case occurs for example when K is polyhedral.

The following counterexample shows that $x \in \text{ri } K$ cannot be replaced by $x \in K$ in the general case.

Example 3.9 Consider the cone

$$K = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 2x_2x_3, \quad x_3 \ge 0\}.$$

This is the cone generated by the circle $x_1^2 + (x_2 - 1)^2 \le 1$, $x_3 = 1$. Then $K^\infty = K$, and $u \doteq (0,0,1) \in K$. If we set $P = P_{u^\perp}$, then

$$\begin{aligned} PK &= \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_2 > 0\} \cup \{(0,0,0)\} \\ (PK)^\infty &= \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_2 \ge 0\} \\ K^{\infty 2}[u] &= \mathbb{R}u + (PK)^\infty = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \ge 0\}. \end{aligned}$$

By taking $v = (1,0,0)$ we see that $u \in K^\infty$, $v \in K^{\infty 2}[u]$, $0 \in K$ but $0 + tu + sv = (s, 0, t) \notin K$ for any $s > 0$, t arbitrary. Hence condition (b) in Proposition 3.4 does not hold.

Further, we show that the inclusion in Proposition 3.1(i) is an equality in case of finitely many convex sets satisfying a regularity condition.

Proposition 3.10 *Let $\{K_i\}_{i\in I} \subseteq \mathbb{R}^n$ be a finite family of convex sets such that $\bigcap_{i\in I} \text{ri } K_i \neq \emptyset$. If $u \in (\bigcap_{i\in I} K_i)^\infty$, then*

$$(\bigcap_{i\in I} K_i)^{\infty 2}[u] = \bigcap_{i\in I} K_i^{\infty 2}[u]. \tag{10}$$

Proof. We only need to prove $(\supseteq)$.
Let $v \in \bigcap_{i\in I} K_i^{\infty 2}[u]$. Choose $x_0 \in \bigcap_{i\in I} \text{ri } K_i$. Then, given $i \in I$ and $s > 0$, there exist $\overline{t_i} \ge 0$ such that for all $t_i > \overline{t_i}$, $x_0 + t_i u + sv \in K_i^{\infty 2}[u]$. Let $\bar{t} = \max_{i\in I}\{\overline{t_i}\}$, then $x_0 + tu + sv \in \bigcap_{i\in I} K_i$, for all $t > \bar{t}$. By Proposition 3.4, $v \in (\bigcap_{i\in I} K_i)^{\infty 2}[u]$. ■

Remark 3.11 a) The previous proposition is not true for an infinite family, even of polyhedral sets. For example, take for every $m \in \mathbb{N}$, $K_m \doteq \{(x_1, x_2) : x_2 \ge m\,|x_1|\}$. Clearly $\bigcap\limits_{m\in\mathbb{N}} K_m = \mathbb{R}_+(0,1)$, and $(\bigcap\limits_{m\in\mathbb{N}} K_m)^{\infty 2}[u] = \mathbb{R}u$, while $\bigcap\limits_{m\in\mathbb{N}} K_m^{\infty 2}[u] = \mathbb{R}^2$.

b) The proposition is also not true in general if $\bigcap\limits_{i=1}^{k} \text{ri}\, K_i = \emptyset$, even for a finite family of closed convex sets. Consider for example the cones K_1 and K_2 generated, respectively, by the circles $x_1^2 + (x_2 - 1)^2 \le 1$, $x_3 = 1$ and $x_1^2 + (x_2 + 1)^2 \le 1$, $x_3 = 1$ (see example 3.9). The cones have the half axis $K = \mathbb{R}_+ u$, $u = (0,0,1)$ as common generatrix. It is easy to see that $K_1 \cap K_2 = K$. From Example 3.9 we see that $K_1^{\infty 2}[u] = \{(x_1, x_2, x_3) : x_2 \ge 0\}$ and, likewise, $K_2^{\infty 2}[u] = \{(x_1, x_2, x_3) : x_2 \le 0\}$. Hence, $\mathbb{R}[u] = K^{\infty 2}[u] \subsetneq K_1^{\infty 2}[u] \cap K_2^{\infty 2}[u]$.

When the set is polyhedral, the second order asymptotic cone has a simple expression. In what follows, set for $i = 1, 2, \ldots, m$,

$$H_i = \{x : \ \langle a_i, x\rangle \le \alpha_i\},$$

for some $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{R}$. Then $H_i^\infty = \{x : \ \langle a_i, x\rangle \le 0\}$. Given $u \in (\bigcap_{i=1}^m H_i)^\infty$, set $I_1 = \{i \in \{1, 2, \ldots, m\} : \ u \in \text{bd } H_i^\infty\}$ (set of active inequalities for u) and $I_2 = \{i \in \{1, 2, \ldots, m\} : \ u \in \text{int } H_i^\infty\}$. The next proposition gives an expression for $K^{\infty 2}[u]$ when K is polyhedral. As usual, the intersection of an empty family of subsets of $\mathbb{R}^n$ is considered to be the whole space $\mathbb{R}^n$.

Proposition 3.12 *Assume for $i = 1, 2, \ldots, m$, that H_i is a halfspace as above, and that $\bigcap_{i=1}^m H_i \ne \emptyset$. If $u \in (\bigcap_{i=1}^m H_i)^\infty$, then*

$$(\bigcap_{i=1}^m H_i)^{\infty 2}[u] = \bigcap_{i \in I_1} H_i^\infty = \bigcap_{i=1}^m H_i^{\infty 2}[u],$$

where $I_1 = \{i \in \{1, 2, \ldots, m\} : \ u \in \text{bd } H_i^\infty\}$.

Proof. By Proposition 3.1(f), $H_i^{\infty 2}[u] = \mathbb{R}^n$ for all $i \in I_2$. For $i \in I_1$, $u \in \text{bd } H_i^\infty$ means that $u \in H_i^\infty \cap (-H_i^\infty)$. By Proposition 3.6($c$), $H_i^\infty = H_i^{\infty 2}[u]$ from which the second equality follows.

To show the first equality, we use Proposition 3.5(b): Fix any $x \in \text{ri}(\bigcap_{i=1}^m H_i)$. Note that $\langle a_i, x\rangle \le \alpha_i$ for all $i \in I_1 \cup I_2$, $\langle a_i, u\rangle = 0$ for all $i \in I_1$ and $\langle a_i, u\rangle < 0$ for all $i \in I_2$. Given $v \in \mathbb{R}^n$, we have that $v \in (\bigcap_{i=1}^m H_i)^{\infty 2}[u]$ if and only if for all $s > 0$ there exists $t > 0$ such that $x + tu + sv \in \bigcap_{i=1}^m H_i$, i.e., $\langle a_i, x + tu + sv\rangle \le \alpha_i$ for all i. For $i \in I_2$ it always holds that $\langle a_i, x + tu + sv\rangle \le \alpha_i$ if t is sufficiently large. For $i \in I_1$, $\langle a_i, x + tu + sv\rangle = \langle a_i, x + sv\rangle \le \alpha_i$ holds for large $s > 0$ if and only if $\langle a_i, v\rangle \le 0$, i.e., $v \in \bigcap_{i \in I_1} H_i^\infty$. This shows the first equality. ■

It follows immediately that in case of polyhedral sets, equality (10) holds without any assumption on the relative interiors:

Corollary 3.13 *Let $\{K_i\}_{i\in I}$ be a finite family of polyhedral sets. If $\bigcap_{i\in I} K_i \ne \emptyset$, then $(\bigcap_{i\in I} K_i)^{\infty 2}[u] = \bigcap_{i\in I} K_i^{\infty 2}[u]$.*

4 Second order asymptotic functions

We first give a formula for $f^{\infty 2}(u; \cdot)$ for any proper function f, followed by some basic properties linking second and first order asymptotic notions. Afterwards, the case of a convex function is considered, and we provide various formulas for $f^{\infty 2}(u; \cdot)$.

4.1 Some preliminaries

From (9) we derive the next formula.

Proposition 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and $u \in \mathbb{R}^n$ be such that $f^\infty(u)$ is finite. Then for every $v \in \mathbb{R}^n$,*

$$f^{\infty 2}(u; v) = \inf \Big\{ \liminf_{k\to\infty} \left(\frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) : \\ x_k \in \operatorname{dom} f, s_k, t_k \to +\infty, \ \frac{x_k}{s_k} - t_k u \to v \Big\} \quad (11)$$

Proof. Let $g(v)$ be the expression at the right-hand side of (11). We will show that $f^{\infty 2}(u; \cdot)$ and g have the same epigraph.

If $(v, \alpha) \in \operatorname{epi} f^{\infty 2}(u; \cdot)$, i.e., $(v, \alpha) \in (\operatorname{epi} f)^{\infty 2} [(u, f^\infty(u))]$, then by Definition 2.1 there exist sequences $(x_k, \alpha_k) \in \operatorname{epi} f$, $s_k, t_k \to +\infty$ such that $\frac{x_k}{s_k} - t_k u \to v$ and $\frac{\alpha_k}{s_k} - t_k f^\infty(u) \to \alpha$.

Since $f(x_k) \le \alpha_k$, it follows that

$$\liminf \left(\frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) \le \lim \left(\frac{\alpha_k}{s_k} - t_k f^\infty(u) \right) = \alpha.$$

Hence $g(v) \le \alpha$, i.e., $(v, \alpha) \in \operatorname{epi} g$.

Conversely, assume that $(v, \alpha) \in \operatorname{epi} g$. Then for every $\varepsilon > 0$, $g(v) < \alpha + \varepsilon$. It follows that there exist sequences $x_k \in \operatorname{dom} f$, $s_k, t_k \to +\infty$ such that

$$\frac{x_k}{s_k} - t_k u \to v, \quad (12)$$

$$\liminf \left(\frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) < \alpha + \varepsilon. \quad (13)$$

By taking a subsequence if necessary, we may assume that the lim inf is actually lim. Let $\gamma_k \in \mathbb{R}$ be such that

$$\frac{f(x_k) + \gamma_k}{s_k} - t_k f^\infty(u) = \alpha + \varepsilon, \quad (14)$$

that is,

$$\gamma_k = s_k \left(\alpha + \varepsilon - \left(\frac{f(x_k)}{s_k} - t_k f^\infty(u) \right) \right).$$

Relation (13) implies that $\gamma_k > 0$ for large k. Hence $(x_k, f(x_k) + \gamma_k) \in \operatorname{epi} f$. From (12) and (14) we deduce that

$$\frac{(x_k, f(x_k) + \gamma_k)}{s_k} - t_k(u, f^\infty(u)) \to (v, \alpha + \varepsilon)$$

hence $(v, \alpha + \varepsilon) \in (\operatorname{epi} f)^{\infty 2}[(u, f^{\infty}(u))] = \operatorname{epi} f^{\infty 2}(u; \cdot)$ for every $\varepsilon > 0$. Since the second order cone is closed, we deduce that $(v, \alpha) \in \operatorname{epi} f^{\infty 2}(u; \cdot)$. ■

Note that in [8] the second order asymptotic function was defined directly through formula (11), was called "lower second order asymptotic function", and was denoted by $R''_{-} f(u; \cdot)$.

In the special case $u = 0$ formula (11) implies that $f^{\infty 2}(0; v) = f^{\infty}(v)$ for all $v \in \mathbb{R}$. This is also a consequence of the fact that the functions $f^{\infty 2}(0; \cdot)$ and f^{∞} have the same epigraph, in view of the equality $K^{\infty 2}[0] = K^{\infty}$ for $K = \operatorname{epi} f$.

We have a simple inequality between $f^{\infty}(u)$ and $f^{\infty 2}(u; u)$:

Proposition 4.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then $f^{\infty 2}(u; u) \leq f^{\infty}(u)$, for every $u \in \mathbb{R}^n$ such that $f^{\infty}(u) \in \mathbb{R}$.*

Proof. By Proposition 3.1, $u \in K^{\infty 2}[u]$ whenever $u \in K^{\infty}$. Consequently, $(u, f^{\infty}(u)) \in (\operatorname{epi} f)^{\infty 2}[(u, f^{\infty}(u))] = \operatorname{epi} f^{\infty 2}(u; \cdot)$. This implies immediately that $f^{\infty 2}(u; u) \leq f^{\infty}(u)$. ■

In the study of minimization problems, one must analyze the behaviour of the objective function along unbounded minimizing sequences $\{x_k\}$. Given an objective function f, a control of the growth rate of the quotient $\dfrac{f(x_k)}{\|x_k\|}$ is provided by $f^{\infty}(u)$ whenever $\dfrac{x_k}{\|x_k\|} \to u$. We will show that still the second order asymptotic function $f^{\infty 2}$ provides a finer description of the growth of f at infinity. To see this, let $\{x_k\}$ be a sequence in $\operatorname{dom} f$ with $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u$, then

$$\liminf \frac{f(x_k)}{\|x_k\|} \geq f^{\infty}(u). \tag{15}$$

In other words, if $f^{\infty}(u) \in \mathbb{R}$, then the rate of growth of $f(x_k)$ is at least the rate of growth of $\|x_k\| f^{\infty}(u)$. However, this does not mean that $f(x_k) - \|x_k\| f^{\infty}(u)$ is bounded from below, i.e., that

$$\liminf \left(f(x_k) - \|x_k\| f^{\infty}(u)\right) > -\infty. \tag{16}$$

It can be easily seen that (16) is a stronger statement than (15). In fact, the second order asymptotic function gives a necessary condition for (16) to hold.

Proposition 4.3 *Let f be arbitrary and $u \in \mathbb{R}^n$ be such that $\|u\| = 1$ and $f^{\infty}(u) \in \mathbb{R}$. If (16) holds for every sequence $x_k \in \operatorname{dom} f$ such that $\|x_k\| \to +\infty$, $\frac{x_k}{\|x_k\|} \to u$, then $f^{\infty 2}(u; u) = f^{\infty}(u)$.*

Proof. By Proposition 4.2, $f^{\infty 2}(u; u) \leq f^{\infty}(u)$. Assume that $f^{\infty 2}(u; u) < f^{\infty}(u)$. Then there exist sequences $x_k \in \operatorname{dom} f$, $s_k, t_k \to +\infty$, such that

$\frac{x_k}{s_k} - t_k u \to u$ and $\frac{f(x_k)}{s_k} - t_k f^\infty(u) - f^\infty(u) \to -\alpha$ with $\alpha \in]0, +\infty]$. From $\frac{x_k}{s_k} - (t_k+1)u \to 0$ follows that $\|x_k\| \to +\infty$, $\frac{\|x_k\|}{s_k} - (t_k+1)\|u\| \to 0$ and

$$\frac{x_k}{\|x_k\|} = \frac{x_k}{s_k(t_k+1)} \frac{s_k(t_k+1)}{\|x_k\|} \to u.$$

In addition,

$$\begin{aligned} -\alpha &= \lim \left(\frac{f(x_k)}{s_k} - (t_k+1) f^\infty(u) \right) \\ &= \lim \left(\frac{f(x_k)}{s_k} - \frac{\|x_k\|}{s_k \|u\|} f^\infty(u) \right). \end{aligned}$$

Using $s_k \to +\infty$ and $\|u\| = 1$ we obtain $\lim (f(x_k) - \|x_k\| f^\infty(u)) = -\infty$. This contradicts our assumption, so $f^{\infty 2}(u; u) = f^\infty(u)$. ∎

The converse is not true, even for a convex function.

Example 4.4 Define f on $\mathbb{R}^2$ by

$$f(\alpha, \beta) = \begin{cases} -\sqrt{\beta}, & \beta \geq 0 \\ +\infty, & \beta < 0 \end{cases}.$$

Then f is convex and lsc. For every $x \in \operatorname{ri}\operatorname{dom} f$ the function $f(x + t(1,0))$ does not depend on t, so one can easily see that $f^\infty(1,0) = 0 = f^{\infty 2}((1,0);(1,0))$. However, if we take $x_k = (k, \sqrt{k})$ then we can check that $\|x_k\| \to +\infty$, $\frac{x_k}{\|x_k\|} \to (1,0)$ but

$$\lim (f(x_k) - \|x_k\| f^\infty(1,0)) = -\infty.$$

As we will see in the next section (cf. Remark 4.14), whenever f is a convex function and $f^\infty(u) = f^{\infty 2}(u;u)$, then given a sequence $\{x_k\}$ such that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u$, we are sure that (16) holds for sequences that belong to a line of the form $x + tu$, $t > 0$ for some $x \in \operatorname{ri}\operatorname{domf}$. For more general sequences, (16) might not hold.

Given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ let $S_\lambda \doteq \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ be its sublevel set. Next proposition shows the relation between the zero-level set of $f^{\infty 2}(u; \cdot)$ and the second order cone of the level set of f.

Proposition 4.5 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function and $\lambda \geq \inf f$. If $u \in \mathbb{R}^n$ with $f^\infty(u) = 0$, then*

$$(S_\lambda)^{\infty 2}[u] \subseteq \{w : f^{\infty 2}(u; w) \leq 0\}.$$

Proof. Let $v \in (S_\lambda)^{\infty 2}[u]$. Then there exist $x_k \in S_\lambda, t_k, s_k \to +\infty$ such that $\frac{x_k}{s_k} - t_k u \to v$. Thus

$$\frac{f(x_k)}{s_k} - t_k f^\infty(u) = \frac{f(x_k)}{s_k} \leq \frac{\lambda}{s_k} \to 0,$$

and $f^{\infty 2}(u; v) \leq 0$. ∎

4.2 The case of convex functions

Whenever f is convex, $f^{\infty 2}(u;\cdot)$ is convex too since $(\mathrm{epi}\, f)^{\infty 2}\,[(u, f^{\infty}(u))]$ is convex by Proposition 3.6. In this case, $f^{\infty 2}(u;\cdot)$ has a simpler form, as we will see. In preparation of what follows, we first show a formula for f^{∞} which is analogous to (4), but does not assume that f is lower semicontinuous. We will make use of the following result, see [12, Lemma 7.3].

Proposition 4.6 *For any proper and convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$,*

$$\mathrm{ri}\,\mathrm{epi}\, f = \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathrm{ri}\,\mathrm{dom}\, f, \mu > f(x)\}. \tag{17}$$

Proposition 4.7 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then*

(*a*) *Given $x_0 \in \mathrm{ri}\,\mathrm{dom}\, f$ and $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}$, one has*

$$(u, \mu) \in \mathrm{epi}\, f^{\infty} \iff (x_0, f(x_0)) + t(u, \mu) \in \mathrm{epi}\, f,\ \forall t > 0.$$

(*b*) *For every $x_0 \in \mathrm{ri}\,\mathrm{dom}\, f$, $u \in \mathbb{R}^n$,*

$$f^{\infty}(u) = \lim_{t\to+\infty} \frac{f(x_0 + tu) - f(x_0)}{t} = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

Proof. (*a*) For every $\beta > f(x_0)$, one has $(x_0, \beta) \in \mathrm{ri}\,\mathrm{epi}\, f$. Hence, if $(u, \mu) \in \mathrm{epi}\, f^{\infty} = (\mathrm{epi}\, f)^{\infty}$ then $(x_0, \beta) + t(u, \mu) \in \mathrm{epi}\, f$, $\forall t > 0$. The last inclusion means that $f(x_0+tu) \le \beta+t\mu$. Since this is true for all $\beta > f(x_0)$, we deduce that $f(x_0 + tu) \le f(x_0) + t\mu$, i.e., $(x_0, f(x_0)) + t(u, \mu) \in \mathrm{epi}\, f$. The converse is similar.
(*b*) is proved by using (*a*), exactly as the analogous equalities when f is lsc.
■

It follows from the above proposition that whenever f is a proper convex function, $f^{\infty}(0) = 0$ so f^{∞} is also proper.

Corollary 4.8 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, k$, be convex functions such that $\bigcap_{i=1}^{k} \mathrm{ri}\,\mathrm{dom}\, f_i \neq \emptyset$. Then*

$$\left(\sum_{i=1}^{k} f_i\right)^{\infty} = \sum_{i=1}^{k} f_i^{\infty}$$

$$\left(\max_{1\le i\le k} f_i\right)^{\infty} = \max_{1\le i\le k} f_i^{\infty}.$$

Proof. Since $\mathrm{dom} \sum_{i=1}^{k} f_i = \bigcap_{i=1}^{k} \mathrm{dom}\, f_i = \mathrm{dom}\max_i f_i$, from the assumption we get that $\bigcap_{i=1}^{k} \mathrm{ri}\,\mathrm{dom}\, f_i = \mathrm{ri}\,\mathrm{dom} \sum_{i=1}^{k} f_i = \mathrm{ri}\,\mathrm{dom}\max_i f_i$. We then apply Proposition 4.7(b). ■

We now establish some useful monotonicity properties.

Lemma 4.9 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex.*

(*a*) *For every $x \in \operatorname{ri}\operatorname{dom} f$ and u such that $f^\infty(u)$ is finite, the function $g(t) := f(x+tu) - tf^\infty(u)$ is decreasing on $]0,+\infty[$.*

(*b*) *If $(y,\delta) \in \operatorname{epi} f$, then for every $v \in \mathbb{R}^n$ the function $s \to \frac{f(y+sv)-\delta}{s}$ is increasing on $]0,+\infty[$.*

(*c*) *Let $x \in \operatorname{ri}\operatorname{dom} f$, $\beta \geq f(x)$, $f^\infty(u)$ be finite and $v \in (\operatorname{dom} f)^{\infty 2}[u]$. If we set*

$$k_\beta(s,t) = \frac{f(x+tu+sv) - tf^\infty(u) - \beta}{s}, \qquad s > 0, t > 0 \tag{18}$$

then the function $s \to \lim_{t\to+\infty} k_\beta(s,t)$ is increasing. Consequently for all $\beta \geq f(x)$,

$$\begin{aligned}\lim_{s\to+\infty}\lim_{t\to+\infty} k_{f(x)}(s,t) &= \lim_{s\to+\infty}\lim_{t\to+\infty} k_\beta(s,t) = \sup_{s>0}\inf_{t>0} k_\beta(s,t) \\ &= \sup_{s>0}\inf_{t>0} k_{f(x)}(s,t).\end{aligned} \tag{19}$$

Proof. (*a*) Let $t' > t > 0$. Since $x \in \operatorname{ri}\operatorname{dom} f$, we have $x + tu \in \operatorname{ri}\operatorname{dom} f$ and $x + t'u \in \operatorname{ri}\operatorname{dom} f$. Setting $x_1 = x + tv$, we know by Proposition 4.7 that

$$\frac{f(x+t'v) - f(x+tv)}{t'-t} = \frac{f(x_1 + (t'-t)v) - f(x_1)}{t'-t} \leq f^\infty(u).$$

From this we obtain $f(x+t'v) - t'f^\infty(u) \leq f(x+tv) - tf^\infty(u)$.
(*b*) The function is the sum of two increasing functions:

$$\frac{f(y+sv)-\delta}{s} = \frac{f(y+sv)-f(y)}{s} + \frac{f(y)-\delta}{s}.$$

(*c*) Using (*a*) we deduce that for every $s > 0$, $\lim_{t\to+\infty} k_\beta(s,t)$ exists and is equal to $\inf_{t>0} k_\beta(s,t)$. Let $s' > s > 0$. By using Proposition 3.4 on $(\operatorname{dom} f)^{\infty 2}[u]$ we deduce that there exists $\bar{t} > 0$ such that for all $t \geq \bar{t}$, $x + tu + sv \in \operatorname{dom} f$ and $x + tu + s'v \in \operatorname{dom} f$. Then by Proposition 4.7, $(x+tu, \beta + tf^\infty(u)) = (x,\beta) + t(u, f^\infty(u)) \in \operatorname{epi} f$ since $(u, f^\infty(u)) \in (\operatorname{epi} f)^\infty$. Using (*b*) for $y = x + tu$, $\delta = \beta + tf^\infty(u)$ we obtain

$$\frac{f(x+tu+sv) - tf^\infty(u) - \beta}{s} \leq \frac{f(x+tu+s'v) - tf^\infty(u) - \beta}{s'}, \qquad \forall t \geq \bar{t}.$$

Taking the limit as $t \to +\infty$ we find that $\lim_{t\to+\infty} k_\beta(s,t) \leq \lim_{t\to+\infty} k_\beta(s',t)$, i.e., the function $s \to \lim_{t\to+\infty} k_\beta(s,t)$ is increasing. Thus,

$$\lim_{s\to+\infty}\lim_{t\to+\infty} k_\beta(s,t) = \sup_{s>0}\inf_{t>0} k_\beta(s,t), \qquad \forall \beta \geq f(x). \tag{20}$$

On the other hand, it is clear that

$$\lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x+tu+sv)-tf^{\infty}(u)-\beta}{s}=\lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x+tu+sv)-tf^{\infty}(u)}{s}$$

hence $\lim_{s\to+\infty}\lim_{t\to+\infty}k_{\beta}(s,t)=\lim_{s\to+\infty}\lim_{t\to+\infty}k_{f(x)}(s,t)$. From this and (20) we deduce the equalities (19). ■

Proposition 4.10 *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be proper and convex and $x\in\operatorname{ri}\operatorname{dom}f$. For every u such that $f^{\infty}(u)$ is finite and $v\in(\operatorname{dom}f)^{\infty 2}[u]$,*

$$f^{\infty 2}(u;v)=\sup_{s>0}\inf_{t>0}\frac{f(x+tu+sv)-tf^{\infty}(u)-f(x)}{s} \tag{21}$$

$$=\lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x+tu+sv)-tf^{\infty}(u)-f(x)}{s} \tag{22}$$

Proof. Take any $\beta>f(x)$. Then $(x,\beta)\in\operatorname{ri}\operatorname{epi}f$. Define $k_{\beta}(s,t)$ as in (18). We show that $\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)\le f^{\infty 2}(u;v)$ by showing that for every $\alpha\in\mathbb{R}$, $f^{\infty 2}(u;v)\le\alpha$ implies $\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)\le\alpha$.

Since $\operatorname{epi}f$ is convex, using Proposition 3.5 we have the following implications:

$$\begin{aligned}
f^{\infty 2}(u;v)\le\alpha &\Rightarrow (v,\alpha)\in(\operatorname{epi}f)^{\infty 2}[(u,f^{\infty}(u))]\\
&\Rightarrow \forall s>0,\exists t>0,(x,\beta)+t(u,f^{\infty}(u))+s(v,\alpha)\in\operatorname{epi}f\\
&\Rightarrow \forall s>0,\exists t>0,(x+tu+sv,\beta+tf^{\infty}(u)+s\alpha)\in\operatorname{epi}f\\
&\Rightarrow \forall s>0,\exists t>0,f(x+tu+sv)\le\beta+tf^{\infty}(u)+s\alpha\\
&\Rightarrow \forall s>0,\exists t>0,k_{\beta}(s,t)\le\alpha\\
&\Rightarrow \sup_{s>0}\inf_{t>0}k_{\beta}(s,t)\le\alpha.
\end{aligned}$$

We now show that $f^{\infty 2}(u;v)\le\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)$ by showing that for every $\alpha\in\mathbb{R}$, $\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)<\alpha$ implies $f^{\infty 2}(u;v)\le\alpha$. Following the previous implications in reverse order, we obtain

$$\begin{aligned}
\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)<\alpha &\Rightarrow \forall s>0,\exists t>0,k_{\beta}(s,t)<\alpha\\
&\Rightarrow \forall s>0,\exists t>0,(x+tu+sv,\beta+tf^{\infty}(u)+s\alpha)\in\operatorname{epi}f\\
&\Rightarrow \forall s>0,\exists t>0,(x,\beta)+t(u,f^{\infty}(u))+s(v,\alpha)\in\operatorname{epi}f\\
&\Rightarrow (v,\alpha)\in(\operatorname{epi}f)^{\infty 2}[(u,f^{\infty}(u))]\Rightarrow f^{\infty 2}(u;v)\le\alpha.
\end{aligned}$$

It follows that $f^{\infty 2}(u;v)=\sup_{s>0}\inf_{t>0}k_{\beta}(s,t)$ for every $\beta>f(x)$. Using (19) we deduce equalities (22) and (21). ■

A formula analogous to (5) holding for f^{∞} also holds.

Proposition 4.11 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex. For every u such that $f^\infty(u)$ is finite and $v \in (\operatorname{dom} f)^{\infty 2}[u]$,*

$$f^{\infty 2}(u;v) = \sup_{x \in \operatorname{ri}\operatorname{dom} f} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - f(x)\right).$$

Proof. As in Proposition 4.10, we use a representation of $(\operatorname{epi} f)^{\infty 2}[(u, f^\infty(u))]$. Let us show first that

$$f^{\infty 2}(u;v) = \sup_{(x,\beta) \in \operatorname{ri}\operatorname{epi} f} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - \beta\right). \tag{23}$$

Assume that $\alpha \geq f^{\infty 2}(u;v)$, i.e., $(v,\alpha) \in \operatorname{epi} f^{\infty 2}(u;\cdot) = (\operatorname{epi} f)^{\infty 2}[(u, f^\infty(u))]$. By the equivalence $(a) \Longleftrightarrow (c)$ of Proposition 3.5, for every $(x,\beta) \in \operatorname{ri}\operatorname{epi} f$ there exists $t > 0$ such that $(x,\beta) + t(u, f^\infty(u)) + (v,\alpha) \in \operatorname{epi} f$. This amounts to

$$\forall\, (x,\beta) \in \operatorname{ri}\operatorname{epi} f, \exists t > 0 : f(x+tu+v) - tf^\infty(u) - \beta \leq \alpha$$

or

$$\sup_{(x,\beta) \in \operatorname{ri}\operatorname{epi} f} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - \beta\right) \leq \alpha.$$

Since this is true for every $\alpha \geq f^{\infty 2}(u;v)$ we deduce inequality $\geq$ in (23). The reverse inequality is deduced similarly, by taking any α such that

$$\sup_{(x,\beta) \in \operatorname{ri}\operatorname{epi} f} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - \beta\right) < \alpha$$

and deducing, using again Proposition 3.5, that $f^{\infty 2}(u;v) \leq \alpha$. Thus, equation (23) holds. Note that $(x,\beta) \in \operatorname{ri}\operatorname{epi} f$ if and only if $x \in \operatorname{ri}\operatorname{dom} f$ and $\beta > f(x)$. Hence,

$$\begin{aligned} f^{\infty 2}(u;v) &= \sup_{x \in \operatorname{ri}\operatorname{dom} f} \sup_{\beta > f(x)} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - \beta\right) \\ &= \sup_{x \in \operatorname{ri}\operatorname{dom} f} \sup_{\beta > f(x)} \left(\inf_{t>0} \left(f(x+tu+v) - tf^\infty(u)\right) - \beta \right) \\ &= \sup_{x \in \operatorname{ri}\operatorname{dom} f} \inf_{t>0} \left(f(x+tu+v) - tf^\infty(u) - f(x)\right) \end{aligned}$$

which proves the proposition. ∎

Formula (22) comes in handy for calculating $f^{\infty 2}$ for convex functions.

Example 4.12 (a) Take $f(x) = \|x\|$. Then $f^\infty = f$. To calculate $f^{\infty 2}(u;v)$ we may consider that $u \neq 0$ since as we remarked, $f^{\infty 2}(0;v) = f^\infty(v)$. We use (22) with $x = 0$. We first calculate:

$$\begin{aligned}\lim_{t\to+\infty}\left(f(tu+sv)-tf^{\infty}(u)\right) &= \lim_{t\to+\infty}\left(\|tu+sv\|-t\|u\|\right)\\ &= \lim_{t\to+\infty}\frac{2ts\langle u,v\rangle+s^2\|v\|^2}{\|tu+sv\|+t\|u\|}\\ &= \lim_{t\to+\infty}\frac{2s\langle u,v\rangle+\frac{1}{t}s^2\|v\|^2}{\left\|u+\frac{s}{t}v\right\|+\|u\|}=\frac{s\langle u,v\rangle}{\|u\|}.\end{aligned}$$

Hence, $f^{\infty 2}(u;v) = \dfrac{\langle u,v\rangle}{\|u\|}$.

(b) Let f be the quadratic convex function $f(x) = \frac{1}{2}\langle Ax,x\rangle + \langle c,x\rangle + k$ where A is a symmetric positive semidefinite matrix, $c \in \mathbb{R}^n$ and $k \in \mathbb{R}$. It is known that $f^{\infty}(u) = \langle c,u\rangle$ if $u \in \ker A$, while $f^{\infty}(u) = +\infty$ if $u \notin \ker A$. An application of (22) yields immediately that $f^{\infty 2}(u;v) = f^{\infty}(v)$ for every $u \in \ker A$ and $v \in \mathbb{R}^n$.

(c) Consider $f(x) = (1+\langle Ax,x\rangle)^{\frac{1}{2}}$ where A is a symmetric positive semidefinite matrix. Then $f^{\infty}(u) = \langle Au,u\rangle^{\frac{1}{2}}$. Since $\langle u,Au\rangle = 0$ if and only if $u \in \ker A$, one can easily compute from (22) and obtain

$$f^{\infty 2}(u;v) = \langle Av,v\rangle^{\frac{1}{2}},\ \text{if } u\in\ker A;\ f^{\infty 2}(u;v) = \frac{\langle Au,v\rangle}{\langle Au,u\rangle^{\frac{1}{2}}},\ \text{if } u\notin\ker A.$$

(d) Let $g : \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ be proper and, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear and such that $A(\mathbb{R}^n)\cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be defined by $f(x) = g(Ax)$.

It is known that $f^{\infty}(u) \geq g^{\infty}(Au)$, $\forall u \in \mathbb{R}^n$ [1, Prop. 2.6.3]. For every u such that $f^{\infty}(u) = g^{\infty}(Au)$ and are finite, and every v,

$$\begin{aligned}f^{\infty 2}(u;v) &= \inf\{\liminf\left(\frac{g(Ax_k)}{s_k}-t_kf^{\infty}(u)\right) : \frac{x_k}{s_k}-t_ku\to v, t_k,s_k\to+\infty\}\\ &\geq \inf\{\liminf\left(\frac{g(y_k)}{s_k}-t_kg^{\infty}(Au)\right) : \frac{y_k}{s_k}-t_kAu\to Av, t_k,s_k\to+\infty\}\\ &= g^{\infty 2}(Au;Av).\end{aligned}$$

If in addition g is convex, in which case also f is convex, then $f^{\infty}(u) = g^{\infty}(Au)$, $\forall u \in \mathbb{R}^n$. Under our assumption, there exists $x_0 \in \mathbb{R}^n$ such that $Ax_0 \in \operatorname{ri}\operatorname{dom} g$. In this case, $x_0 \in \operatorname{ri}\operatorname{dom} f$. Then for every u such that $f^{\infty}(u)$ is finite and every v,

$$\begin{aligned}f^{\infty 2}(u;v) &= \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x_0+tu+sv)-tf^{\infty}(u)-f(x_0)}{s}\\ &= \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{g(A(x_0+tu+sv))-tg^{\infty}(Au)-g(Ax_0)}{s}\\ &= g^{\infty 2}(Au;Av).\end{aligned}$$

Note that in the examples (a), (b) and (c), $f^{\infty 2}(u,u) = f^{\infty}(u)$. This is not a coincidence, as shown by the next proposition.

Proposition 4.13 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $u \in \mathbb{R}^n$ be such that $f^{\infty}(u) \in \mathbb{R}$. The following hold:*

- (a) $f^{\infty 2}(u; v - ru) \leq f^{\infty}(v) - r f^{\infty}(u)$ *for every* $r \geq 0$ *and* $v \in \operatorname{dom} f^{\infty}$. *In particular,* $f^{\infty 2}(u; v) \leq f^{\infty}(v)$.
- (b) *If* $f^{\infty 2}(u; 0) = 0$, *then* $f^{\infty 2}(u; u) = f^{\infty}(u) = -f^{\infty 2}(u; -u)$.
- (c) *If* $f^{\infty 2}(u; 0) = -\infty$, *then* $f^{\infty 2}(u; u) = -\infty$.

Proof. (a) We apply the inclusion $K^{\infty} - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$ (cf Proposition 3.6(d)) to the set $K = \operatorname{epi} f$. It follows that

$$\operatorname{epi} f^{\infty} - \mathbb{R}_+(u, f^{\infty}(u)) \subseteq (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right] = \operatorname{epi} f^{\infty 2}(u; \cdot).$$

Thus,

$$(v - ru, t - r f^{\infty}(u)) \in \operatorname{epi} f^{\infty 2}(u; \cdot), \quad \forall r \geq 0, \forall (v, t) \in \operatorname{epi} f^{\infty}$$

which means

$$f^{\infty 2}(u; v - ru) \leq t - r f^{\infty}(u), \quad \forall r \geq 0, \forall (v, t) \in \operatorname{epi} f^{\infty}$$

proving (a).

(b) From (a) we obtain $f^{\infty 2}(u; u) \leq f^{\infty}(u)$ and $f^{\infty 2}(u; -u) \leq f^{\infty}(0) - f^{\infty}(u) = -f^{\infty}(u)$ (note that f^{∞}, being convex, lsc, and such that $f^{\infty}(u)$ is finite, never takes the value $-\infty$). The convexity of $f^{\infty 2}(u; \cdot)$ yields

$$0 = f^{\infty 2}(u; 0) \leq \frac{1}{2} f^{\infty 2}(u; u) + \frac{1}{2} f^{\infty 2}(u; -u) \leq 0.$$

Hence

$$f^{\infty 2}(u; u) = -f^{\infty 2}(u; -u) = f^{\infty}(u),$$

the desired result.

(c) If $f^{\infty 2}(u; 0) = -\infty$, then $f^{\infty 2}(u; u)$ cannot be finite. As it is bounded above by $f^{\infty}(u)$, necessarily $f^{\infty 2}(u; u) = -\infty$. ■

Remark 4.14 By Lemma 4.9, for every $x \in \operatorname{ri} \operatorname{dom} f$ and $u \in (f^{\infty})^{-1}(\mathbb{R})$, the function $t \to f(x + tu) - t f^{\infty}(u)$ is decreasing, hence $\lim_{t \to +\infty} (f(x + tu) - t f^{\infty}(u))$ exists. By inspecting formula (22) we see that $f^{\infty 2}(u; 0) = 0$ holds if and only if $\lim_{t \to +\infty} (f(x + tu) - t f^{\infty}(u)) \in \mathbb{R}$, while $f^{\infty 2}(u; 0) = -\infty$ holds if and only if $\lim_{t \to +\infty} (f(x + tu) - t f^{\infty}(u)) = -\infty$.

Consequently, if $f^{\infty 2}(u; u) \in \mathbb{R}$, then $\lim_{t \to +\infty} (f(x + tu) - t f^{\infty}(u)) \in \mathbb{R}$.

We also provide some more calculus rules.

Proposition 4.15 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, k$, be convex functions such that $\bigcap_{i=1}^{k} \operatorname{ri} \operatorname{dom} f_i \neq \emptyset$. For every $u \in \mathbb{R}^n$ such that $f_i^\infty(u) \in \mathbb{R}$ for all i, and every $v \in \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$ the following equality holds:*

$$\left(\max_{1 \leq i \leq k} f_i\right)^{\infty 2}(u; v) = \max_{1 \leq i \leq k} f_i^{\infty 2}(u; v)$$

Also, the equality

$$(f_1 + f_2 + \cdots + f_k)^{\infty 2}(u; v) = \sum_{i=1}^{k} f_i^{\infty 2}(u; v)$$

holds, provided that the right-hand side is defined, i.e., if $f_i^{\infty 2}(u; v) = +\infty$ for some i, then $f_j^{\infty 2}(u; v) > -\infty$ for all $j \neq i$.

Proof. Set $f = f_1 + f_2 + \cdots + f_k$. Then $\operatorname{dom} f = \bigcap_{i=1}^{k} \operatorname{dom} f_i$ and $\operatorname{ri} \operatorname{dom} f = \bigcap_{i=1}^{k} \operatorname{ri} \operatorname{dom} f_i$. By Proposition 3.10, $(\operatorname{dom} f)^{\infty 2}[u] = \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$. Take any $x \in \bigcap_{i=1}^{k} \operatorname{ri} \operatorname{dom} f_i$. For every $v \in \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$, using Corollary 4.8 we find

$$\begin{aligned}
\sum_{i=1}^{k} f_i^{\infty 2}(u; v) &= \sum_{i=1}^{k} \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{f_i(x + tu + sv) - t f_i^\infty(u) - f_i(x)}{s} \\
&= \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{f(x + tu + sv) - t f^\infty(u) - f(x)}{s} \\
&= f^{\infty 2}(u; v).
\end{aligned}$$

The proof of the other equality is similar. ∎

In case of convex functions and for $\lambda > \inf f$, the inclusion of Proposition 4.5 becomes an equality, as we now show.

Proposition 4.16 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $\lambda > \inf f$. If $u \in \mathbb{R}^n$ with $f^\infty(u) = 0$, then*

$$(S_\lambda)^{\infty 2}[u] = \{w : f^{\infty 2}(u; w) \leq 0\}.$$

Proof. Inclusion ($\subseteq$) follows from Proposition 4.5, so we have only to show ($\supseteq$).

Let $v \in \mathbb{R}^n$ with $f^{\infty 2}(u;v) \leq 0$. Since $\lambda > \inf f$ and f is convex, by Corollary 7.3.2 in [12] there exists $y \in \operatorname{ri}\operatorname{dom} f$ such that $f(y) < \lambda$. Then

$$\sup_{s>0} \inf_{t>0} \frac{f(y+tu+sv) - tf^{\infty}(u) - f(y)}{s} = f^{\infty 2}(u;v) \leq 0.$$

Thus for every $s > 0$,

$$\inf_{t>0} f(y+tu+sv) \leq f(y) < \lambda.$$

Since by Lemma 4.9 the function $t \to f(y+tu+sv)$ is nonincreasing, we deduce that for every $s > 0$ there exists $\bar{t} \geq 0$ such that $f(y+tu+sv) < \lambda$ for all $t \geq \bar{t}$, that is $y + tu + sv \in S_\lambda$ for all $t \geq \bar{t}$. One can easily see that we have also $y \in \operatorname{ri} S_\lambda$. Hence, $v \in (S_\lambda)^{\infty 2}[u]$. ∎

Conclusions. Having revisited the second order asymptotic cone and function, new formulae for those, in the convex case, are established. In a subsequent work we shall present applications to the minimization problem and, in particular, characterizations of the nonmeptiness and boundedness of the solution set will be established, in the quasiconvex case.

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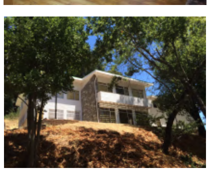

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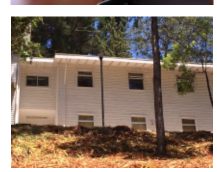
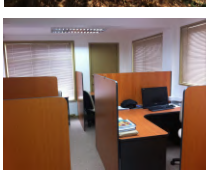

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
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