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Abstract
In this paper, we are concerned with a finite volume method for a model with cross-diffusion of the indirect transmission of an epidemic disease between two spatially distributed host populations having non-coincident spatial domains, transmission occurring through a contaminated environment. The mobility of each class is assumed to be influenced by the gradient of the other classes. We propose a Finite Volume scheme and proved the well-posedness, nonnegativity and convergence of the discrete solution. The convergence proof is based on deriving a series of a priori estimates and by using a general L^p compactness criterion. Additionally, we address the questions of existence of weak solutions and existence and uniqueness of classical solution by using, respectively, a regularization method and an interpolation results between Banach spaces. The proofs of these results are given in the Appendix. Finally, the numerical scheme is illustrated by some examples.

Keywords : reaction-diffusion system, nonlocal cross-diffusion, weak solution, classical solution, finite volume scheme, discrete compactness.
AMS Subject Classifications: 35K57, 35M10, 35A05

1. Introduction

In recent times there has been much activity concerning the development and analysis of reaction-diffusion systems with cross-diffusion, see for more details Levin and Segel, [24], Okubo [30], Mimura and Murray [28], Mimura and Kawasaki [27], Mimura and Yamaguti [29], Galiano et al [19, 20], Bendahmane et al [5, 10] (see also [6, 8, 9, 11]), and many other authors.
This paper is devoted to the mathematical analysis of indirectly transmitted diseases with cross-diffusion. We assume that a disease is transmitted between two host populations distributed over non-coincident spatial domains, transmission occurring through a contaminated environment.
We assume that there are two independent host populations, H_1 and H_2, spatially distributed over two spatial domains, Ω_1 and Ω_2 in $\mathbb{R}^3$, $\Omega_1 \cap \Omega_2 \neq \emptyset$. Each population is subdivided into three subclasses, susceptibles, infectives and recovered. The susceptible class consists of individuals who are capable of becoming infected and the infective class consists of individuals who have contracted

the disease and are capable of transmitting it. Susceptible individuals in the host population H_1 can contract the disease from cross contacts with infected hosts from H_1 or with the environment. Individuals in the host population H_2 are infected by contact with the environment but there is neither cross infection from infected hosts from H_2 nor crisscross infection with H_1.

Our state variables $(\varphi(t,x), \psi(t,x), \chi(t,x))$ represent population densities of the subclasses of susceptible, infective and recovered individuals from the total population $H_1 = \varphi + \psi + \chi$ while $(u(t,x), v(t,x), w(t,x))$ represent population densities of the susceptible, infective and recovered subclasses of the total population $H_2 = u + v + w$ at time t and position x. The variable $c(x,t)$, represents the proportion of contaminated habitat or environment. The first population will follow a logistic dynamic with a spatially dependent birth-rate, $b(x)$, identical in each subclass, offspring being susceptible at birth because the disease is assumed to be benign in H_1. A spatially and density dependent mortality rate, $m(x) + k(x)H_1$, is considered allowing a spatially variable carrying capacity. Let $1/\lambda_i$ be the duration of the infective stage in population H_i, $i = 1, 2$. A fixed proportion $0 \le w_1 \le 1$ of infective individuals from H_1 become permanently immune, a proportion $0 \le 1 - w_1 \le 1$ reentering the susceptible class.

We assume also that the disease can be lethal in the second host population with a fixed survival rate, $0 \le \varepsilon \le 1$. In the system (1.2)-(1.3), $d_{ij} : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying: there exist constants $M_{ij}, C > 0$ such that for $i = 1, 2$ and $j = 1, 2, 3$,

$$M_{ij} \le d_{ij} \quad \text{and} \quad |d_{ij}(I_1) - d_{ij}(I_2)| \le C\,|I_1 - I_2| \quad \text{for all} \quad I_1, I_2 \in \mathbb{R}. \tag{1.1}$$

A prototype of a nonlinear system that governs our model is,

$$\begin{cases} \partial_t\varphi - d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\,dx\Big)\Delta\varphi - \operatorname{div}\Big((k_{11}\varphi + \psi + \chi)\nabla\varphi + \varphi\nabla\psi + \varphi\nabla\chi\Big) \\ \qquad\qquad = F_1(x, \varphi, \psi, \chi, c), \\ \partial_t\psi - d_{12}\Big(\displaystyle\int_{\Omega_1} \psi\,dx\Big)\Delta\psi - \operatorname{div}\Big(\psi\nabla\varphi + (\varphi + k_{12}\psi + \chi)\nabla\psi + \psi\nabla\chi\Big) \\ \qquad\qquad = F_2(x, \varphi, \psi, \chi, c), \\ \partial_t\chi - d_{13}\Big(\displaystyle\int_{\Omega_1} \chi\,dx\Big)\Delta\chi - \operatorname{div}\Big(\chi\nabla\varphi + \chi\nabla\psi + (\varphi + \psi + k_{13}\chi)\nabla\chi\Big) \\ \qquad\qquad = F_3(x, \varphi, \psi, \chi, c), \end{cases} \tag{1.2}$$

in $Q_{1,T} = (0,T) \times \Omega_1$ the time-space cylinder,

$$\begin{cases} \partial_t u - d_{21}\Big(\displaystyle\int_{\Omega_2} u\,dx\Big)\Delta u - \operatorname{div}\Big((k_{21}u + v + w)\nabla u + u\nabla v + u\nabla w\Big) \\ \qquad\qquad = G_1(x, u, v, w, c), \\ \partial_t v - d_{22}\Big(\displaystyle\int_{\Omega_2} v\,dx\Big)\Delta v - \operatorname{div}\Big(v\nabla u + (u + k_{22}v + w)\nabla v + v\nabla w\Big) \\ \qquad\qquad = G_2(x, u, v, w, c), \\ \partial_t w - d_{23}\Big(\displaystyle\int_{\Omega_2} w\,dx\Big)\Delta w - \operatorname{div}\Big(w\nabla u + w\nabla v + (u + v + k_{23}w)\nabla w\Big) \\ \qquad\qquad = G_3(x, u, v, w, c), \end{cases} \tag{1.3}$$

in $Q_{2,T} = (0,T) \times \Omega_2$, and

$$\partial_t c = K(x, \psi, v, c), \tag{1.4}$$

in $Q_T = (\Omega_1 \cup \Omega_2) \times (0,T)$. Systems (1.2) and (1.3) are supplemented with no-flux boundary conditions:

$$\begin{cases} \Big(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\,dx\Big)\nabla\varphi + (k_{11}\varphi + \psi + \chi)\nabla\varphi + \varphi\nabla\psi + \varphi\nabla\chi\Big)\cdot\eta_1 = 0, \\ \Big(d_{12}\Big(\displaystyle\int_{\Omega_1} \psi\,dx\Big)\nabla\psi + \psi\nabla\varphi + (\varphi + k_{12}\psi + \chi)\nabla\psi + \psi\nabla\chi\Big)\cdot\eta_1 = 0, \\ \Big(d_{13}\Big(\displaystyle\int_{\Omega_1} \chi\,dx\Big)\nabla\chi + \chi\nabla\varphi + \chi\nabla\psi + (\varphi + \psi + k_{13}\chi\Big)\cdot\eta_1 = 0, \end{cases} \tag{1.5}$$

on $(0,T)\times\partial\Omega_1$,

$$
\begin{cases}
\Big(d_{21}\Big(\int_{\Omega_2} u\,dx\Big)\nabla u+(k_{21}u+v+w)\nabla u+u\nabla v+u\nabla w\Big)\cdot\eta_2=0,\\
\Big(d_{22}\Big(\int_{\Omega_2} v\,dx\Big)\nabla v+v\nabla u+(u+k_{22}v+w)\nabla v+v\nabla w\Big)\cdot\eta_2=0,\\
\Big(d_{23}\Big(\int_{\Omega_2} w\,dx\Big)\nabla w+w\nabla u+w\nabla v+(u+v+k_{23}w\Big)\cdot\eta_2=0,
\end{cases}
\tag{1.6}
$$

on $(0,T)\times\partial\Omega_2$, η_i being the outer unit normal to Ω_i along its boundary $\partial\Omega_i$ for $i=1,2$, and with nonnegative initial data:

$$
\xi(0,x)=\xi_0(x)\geq 0,\qquad \xi=\varphi,\psi,\chi,u,v,w,c,
\tag{1.7}
$$

on their respective spatial domains of definition: Ω_1 for (φ,ψ,χ), Ω_2 for (u,v,w) and $\Omega_1\cup\Omega_2$ for c, $0\leq c_0(x)\leq 1$. Herein,
(1.8)

$$
\begin{cases}
F_1(x,\varphi,\psi,\chi,c)=-\sigma_{11}(x)\dfrac{\varphi\psi}{H_1}-\sigma_{31}(x)c\varphi+(1-w_1)\lambda_1\psi+b(x)H_1-(m(x)+k(x)H_1)\varphi,\\
F_2(x,\varphi,\psi,\chi,c)=\sigma_{11}(x)\dfrac{\varphi\psi}{H_1}+\sigma_{31}(x)c\varphi-\lambda_1\psi-(m(x)+k(x)H_1)\psi,\\
F_3(x,\varphi,\psi,\chi,c)=w_1\lambda_1\psi-(m(x)+k(x)H_1)\chi,\\
G_1(x,u,v,w,c)=-\sigma_{32}(x)cu,\\
G_2(x,u,v,w,c)=\sigma_{32}(x)cu-\lambda_2 v,\\
G_3(x,u,v,w,c)=\epsilon\lambda_2 v,\\
K(x,\psi,v,c)=\sigma_{13}(x)(1-c)\tilde{\psi}+\sigma_{23}(x)(1-c)\tilde{v}-\delta(x)c,\\
H_1=\varphi+\psi+\chi;
\end{cases}
$$

where $\tilde{\psi}$ and $\tilde{v}$, are the prolongation by zero of ψ and v, on Ω_1 and Ω_2, respectively.

In the above model we assume that individuals move from a higher to lower concentration region. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. Positive cross-diffusion term denotes one species tends to move in the direction of lower concentration of another species. The dynamics of interacting population with self and cross-diffusion are investigated by several researchers (see [34, 25, 19, 22] and the references therein).

The most interesting and real cases for this model are in dimension 3, but in 2-dimension it also has a realistic interpretation.
We want to mention that in (1.2)-(1.3), the diffusion rates $d_{ij}>0$ are supposed to depend on the whole of each population in the domain rather than on the local density for $i=1,2$ and $j=1,2,3$. This means that the diffusion of individuals is guided by the global state of the population in the medium. For instance, if we want to model species having the tendency to leave crowded zones, a natural assumption would be to assume that d_{ij} is an increasing function of its argument. On the other hand, if we are dealing with species attracted by the growing population in, one will suppose that the nonlocal diffusion d_{ij} decreases.
Note that, the parabolic (and elliptic) equations with nonlocal diffusion terms has already been studied from a theoretical point of view by several authors. First, in 1997, M. Chipot and B. Lovat [13] studied the existence and uniqueness of the solutions for a scalar parabolic equation with a nonlocal diffusion term. Existence-uniqueness and long time behavior for other class of nonlocal nonlinear parabolic equations and systems are studied in [1, 12, 31]. In passing we want to mention that when cross-diffusion is ignored, this model is similar to this in [18].

For technical reasons we assume that the coefficients k_{ij} satisfy

$$
k_{ij}>1 \text{ for } i=1,2,\ j=1,2,3,
\tag{1.9}
$$

this assumption will be used to prove the convergence of the numerical solution of the scheme to a weak solution.

A major difficulty for the analysis of the system (1.2)-(1.4) is the strong coupling in the highest derivatives. Note that standard parabolic theory is not directly aplicable to the reaction-diffusion system (1.2)-(1.4) due to the cross-diffusion terms. We point out that our problem (1.2)-(1.4) is strongly nonlinear and so no maximum principle applies.

2. Weak and classical solutions

Before stating our results concerning the weak and the classical solutions, we collect some preliminary material, including relevant notations and conditions imposed on the data of our problem. Let Ω_i be a bounded, open subsets of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega_i$; η_i is the unit outward normal to Ω_i on $\partial\Omega_i$, for $i = 1, 2$. Next, $|\Omega_i|$ is the Lebesgue measure of Ω_i for $i = 1, 2$. We denote by $H^1(\Omega_i)$ the Sobolev space of functions $u : \Omega_i \to \mathbb{R}$ for which $u \in L^2(\Omega_i)$ and $\nabla u \in L^2(\Omega_i; \mathbb{R}^3)$ for $i = 1, 2$. For $1 \le p \le +\infty$, $\| \cdot \|_{L^p(\Omega_i)}$ is the usual norm in $L^p(\Omega_i)$ for $i = 1, 2$; then

$$L^p_+(\Omega_i) = \{u : \Omega_i \longrightarrow \mathbb{R}_+ \text{ measurable and } \int_{\Omega_i} |u(x)|^p dx < +\infty\},$$

$$L^\infty_+(\Omega_i) = \{u : \Omega_i \longrightarrow \mathbb{R}_+ \text{ measurable and } \sup_{x\in\Omega_i} |u(x)| < +\infty\}.$$

If X is a Banach space, $a < b$ and $1 \le p \le +\infty$, $L^p(a, b; X)$ denotes the space of all measurable functions $u : (a, b) \longrightarrow X$ such that $\| u(\cdot) \|_X$ belongs to $L^p(a, b)$.

Next T is a positive number and

$$Q_{i,t} = \Omega_i \times (0, t),\ \Sigma_{i,t} = \partial\Omega_i \times (0, t) \text{ and } Q_t = (0, T) \times (\Omega_1 \cup \Omega_2),$$

for $i = 1, 2$ and for $0 < t \le T$.

Our basic requirements are the following

$$\sigma_{13}, \sigma_{23}, \delta \in L^\infty_+(Q_T), \tag{2.1}$$

$$m < b \text{ and } b, m, k, \sigma_{i1} \in L^\infty_+(Q_{1,T}), \text{ for } i = 1, 3, \tag{2.2}$$

$$\sigma_{32} \in L^\infty_+(Q_{2,T}). \tag{2.3}$$

Now we define what we mean by weak solutions of the system (1.2)-(1.4). We also supply our main existence results.

Definition 2.1. *A weak solution of (1.2)-(1.4) is a set of nonnegative functions (φ, ψ, χ), (u, v, w) and c such that,*

$c \in C([0, T], L^2(\Omega_1 \cup \Omega_2))$, $0 \le c(t, x) \le 1$ *a.e. in* Q_T,
$(\varphi, \psi, \chi) \in L^2(0, T; H^1(\Omega_1, \mathbb{R}^3))$, $(u, v, w) \in L^2(0, T; H^1(\Omega_2, \mathbb{R}^3))$,
$(\partial_t\varphi, \partial_t\psi, \partial_t\chi) \in L^2(0, T; (W^{1,\infty}(\Omega_1, \mathbb{R}^3))^*)$ *and* $(\partial_t u, \partial_t v, \partial_t w) \in L^2(0, T; (W^{1,\infty}(\Omega_2, \mathbb{R}^3))^*)$,
$\xi(0) = \xi_0$ *a.e. in* Ω_1, $\Theta(0) = \Theta_0$ *a.e. in* Ω_2 *and* $c(0) = c_0$ *a.e. in* $\Omega_1 \cup \Omega_2$, *for* $\xi = (\varphi, \psi, \chi)$ *and* $\Theta = (u, v, w)$, *and satisfying*

(2.4)

$$\int_0^T \langle \partial_t \varphi, \phi_1 \rangle_1 \, dt + \int\int_{Q_{1,T}} \left(d_{11}\Big(\int_{\Omega_1} \varphi \, dx\Big) \nabla \varphi + (k_{11}\varphi + \psi + \chi)\nabla\varphi + \varphi\nabla\psi + \varphi\nabla\chi \right) \cdot \nabla\phi_1 \, dx \, dt$$
$$= \int\int_{Q_{1,T}} F_1(x,\varphi,\psi,\chi,c)\phi_1 \, dx \, dt,$$
$$\int_0^T \langle \partial_t \psi, \phi_2 \rangle_1 \, dt + \int\int_{Q_{1,T}} \left(d_{12}\Big(\int_{\Omega_1} \psi \, dx\Big) \nabla \psi + \psi\nabla\varphi + (\varphi + k_{12}\psi + \chi)\nabla\psi + \psi\nabla\chi \right) \cdot \nabla\phi_2 \, dx \, dt$$
$$= \int\int_{Q_{1,T}} F_2(x,\varphi,\psi,\chi,c)\phi_2 \, dx \, dt,$$
$$\int_0^T \langle \partial_t \chi, \phi_3 \rangle_1 \, dt + \int\int_{Q_{1,T}} \left(d_{13}\Big(\int_{\Omega_1} \chi \, dx\Big) \nabla \chi + \chi\nabla\varphi + \chi\nabla\psi + (\varphi + \psi + k_{13}\chi)\nabla\chi \right) \cdot \nabla\phi_2 \, dx \, dt$$
$$= \int\int_{Q_{1,T}} F_3(x,\varphi,\psi,\chi,c)\phi_3 \, dx \, dt,$$
$$\int_0^T \langle \partial_t u, \Theta_1 \rangle_2 \, dt + \int\int_{Q_{2,T}} \left(d_{21}\Big(\int_{\Omega_2} u \, dx\Big) \nabla u + (k_{21}u + v + w)\nabla u + u\nabla v + u\nabla w \right) \cdot \nabla\Theta_1 \, dx \, dt$$
$$= \int\int_{Q_{2,T}} G_1(x,u,v,w,c)\Theta_1 \, dx \, dt,$$
$$\int_0^T \langle \partial_t v, \Theta_2 \rangle_2 \, dt + \int\int_{Q_{2,T}} \left(d_{22}\Big(\int_{\Omega_2} v \, dx\Big) \nabla v + v\nabla u + (u + k_{22}v + w)\nabla v + v\nabla w \right) \cdot \nabla\Theta_2 \, dx \, dt$$
$$= \int\int_{Q_{2,T}} G_2(x,u,v,w,c)\Theta_2 \, dx \, dt,$$
$$\int_0^T \langle \partial_t w, \Theta_3 \rangle_2 \, dt + \int\int_{Q_{2,T}} \left(d_{23}\left(\int_{\Omega_2} w \, dx\right) \nabla w + w\nabla u + w\nabla v + (u + v + k_{23}w)\nabla w \right) \cdot \nabla\Theta_3 \, dx \, dt$$
$$= \int\int_{Q_{2,T}} G_3(x,u,v,w,c)\Theta_3 \, dx \, dt,$$
$$-\int\int_{Q_T} c \, \partial_t \Gamma \, dx \, dt - \int_{\Omega_1 \cup \Omega_2} c_0(x)\Gamma(0,x) \, dx = \int\int_{Q_T} K(x,\psi,v,c)\Gamma \, dx \, dt,$$

for all $\phi_i \in L^2(0,T; W^{1,\infty}(\Omega_1))$, $\Theta_i \in L^2(0,T; W^{1,\infty}(\Omega_2))$ *for* $i = 1,2,3$, *and* $\Gamma \in \mathcal{D}([0,T) \times (\Omega_1 \cup \Omega_2))$*. Here,* $\langle \cdot, \cdot \rangle_i$ *denotes the duality pairing between* $W^{1,\infty}(\Omega_i)$ *and* $(W^{1,\infty}(\Omega_i))^*$ *for* $i = 1,2$.

Theorem 2.1. *Assume conditions (1.9) and (2.1)-(2.3) hold. If* $(\varphi_0, \psi_0, \chi_0) \in L^2(\Omega_1, \mathbb{R}^3)$, $(u_0, v_0, w_0) \in L^2(\Omega_2, \mathbb{R}^3)$ *and* $c_0 \in L^2(\Omega_1 \cup \Omega_2)$, $0 \le c_0 \le 1$, *then the problem (1.2)-(1.4) possesses a weak solution.*

To prove Theorem 2.1, we first prove existence of solutions to the approximate problem (A.1)-(A.3) below by applying the Schauder fixed-point theorem (in an appropriate functional setting). Then, having proved existence for the aproximate system, the final goal is to send the regularization parameter ε to zero to fabricate weak solutions of the original systems (1.2)-(1.4). Convergence is achieved by means of a priori estimates and compactness arguments. The proof of Theorem 2.1 is given in Appendix A.

Note that we have not been able to prove uniqueness of weak solutions because of the presence of nonlinear lower-order terms (cross-diffusion terms) in our model (1.2)-(1.4).

Let us define
$\tilde{d}_{11}(t) = d_{11}\left(\int_{\Omega_i} \varphi(t,x)\,dx\right)$, $\tilde{d}_{12}(t) = d_{12}\left(\int_{\Omega_i} \psi(t,x)\,dx\right)$, $\tilde{d}_{13}(t) = d_{13}\left(\int_{\Omega_i} \chi(t,x)\,dx\right)$, and, $\tilde{d}_{21}(t) = d_{21}\left(\int_{\Omega_i} u(t,x)\,dx\right)$, $\tilde{d}_{22}(t) = d_{22}\left(\int_{\Omega_i} v(t,x)\,dx\right)$, $\tilde{d}_{23}(t) = d_{23}\left(\int_{\Omega_i} w(t,x)\,dx\right)$. Concerning global existence of classical solutions, the second main result is summarized in the following theorem.

Theorem 2.2. *(Strong solutions) Assume that* $\tilde{d}_{11}(t) = \tilde{d}_{12}(t) = \tilde{d}_{13}(t)$, $\tilde{d}_{21}(t) = \tilde{d}_{22}(t) = \tilde{d}_{23}(t)$ *for a.e.* $t \in (0,T)$, $k_{ij} = 2$ *for* $i = 1,2$, $j = 1,2,3$, *and (2.1)-(2.3) hold. Let* $(\varphi_0, \psi_0, \chi_0) \in C^{2+\theta}(\overline{\Omega}_1, \mathbb{R}^3)$ *and* $(u_0, v_0, w_0) \in C^{2+\theta}(\overline{\Omega}_2, \mathbb{R}^3)$ *for some* $\theta \in (0,1)$, *satisfying,* $\nabla\varphi_0 \cdot \eta_1 = \nabla\psi_0 \cdot \eta_1 = \nabla\chi_0 \cdot \eta_1 = 0$ *on* $\partial\Omega_1$ *and* $\nabla u_0 \cdot \eta_2 = \nabla v_0 \cdot \eta_2 = \nabla w_0 \cdot \eta_2 = 0$ *on* $\partial\Omega_2$. *Then the system (1.2)-(1.4) has a unique, classical, global nonnegative solution* $(\varphi, \psi, \chi) \in C^{\frac{2+\theta}{2}, 2+\theta}([0,+\infty) \times \overline{\Omega}_1, \mathbb{R}^3)$ *and* $(u, v, w) \in C^{\frac{2+\theta}{2}, 2+\theta}([0,+\infty) \times \overline{\Omega}_2, \mathbb{R}^3)$. *Furthermore, there are constants* $C_1, C_2 > 0$ *(dependent upon the initial data and the coefficients) such that,*

$$
\begin{aligned}
&0 \le \varphi(t,x), \psi(t,x), \chi(t,x) \le C_1 \textit{ for all } x \in \overline{\Omega}_1 \textit{ and } t > 0,\\
&0 \le u(t,x), v(t,x), w(t,x) \le C_2 \textit{ for all } x \in \overline{\Omega}_2 \textit{ and } t > 0.
\end{aligned}
\tag{2.5}
$$

The proof of Theorem 2.2 is based in a series of a priori estimates of the solutions in Banach spaces, especially the boundness of the solutions in L^∞, and then we apply the Sobolev embedding and standard regularity results of parabolic equations (see e.g. [14, 21]). Appendix B contains the proof of Theorem 2.2.

The plan of the paper is as follows: In Section 3, we propose a Finite Volume Scheme. We prove the existence and convergence of the discrete solution in Section 4. Finally, in Section 5 we give some numerical examples.

3. Finite Volume Approximation

3.1. **Admissible mesh.** In this work we assume that $\Omega_i \subset \mathbb{R}^d$, for $i = 1,2$, $d = 2$ (respectively, $d = 3$) is an open bounded polygonal (resp., polyhedral) connected domain with boundary $\partial\Omega_i$ for $i = 1,2$. We consider a family $\mathcal{T}_{i,h}$ of admissible meshes of the domain Ω_i, $i = 1,2$ consisting of disjoint open and convex polygons (resp., polyhedra) called control volumes and $\mathcal{T}_h$ is the union of $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,h}$ ($\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$). The parameter h has the sense of an upper bound for the maximum diameter of the control volumes in $\mathcal{T}_{i,h}$, $i = 1,2$. Whenever $\mathcal{T}_{i,h}$, $i = 1,2$ is fixed, we will drop the subscript h in the notation. Of course, the mesh should be admissible in the sense of [15].

A generic volume in $\mathcal{T}_{i,h}$ is denoted by K_i, $i = 1,2$. For all $K_i \in \mathcal{T}_{i,h}$, we denote by $|K_i|$ the d-dimensional Lebesgue measure of K_i. For a given finite volume K_i, we denote by $N(K_i)$ the set of neighbors of K_i which have a common interface with K_i; a generic neighbor of K_i is often denoted by L_i. For all $L_i \in N(K_i)$, we denote by σ_{K_i,L_i} the interface between K_i and L_i; we denote by η_{K_i,L_i} the unit normal vector to σ_{K_i,L_i} outward to K_i. We have $\eta_{L_i,K_i} = -\eta_{K_i,L_i}$. For an interface σ_{K_i,L_i}, $|\sigma_{K_i,L_i}|$ will denote its $(d-1)$-dimensional measure, $i = 1,2$.

By saying that $\mathcal{T}_{i,h}$ is admissible, we mean that there exists a family $(x_{K_i})_{K_i \in \mathcal{T}_{i,h}}$ such that the straight line $\overline{x_{K_i} x_{L_i}}$ is orthogonal to the interface σ_{K_i,L_i}. The point x_{K_i} is referred to as the center of K_i. In the case where $\mathcal{T}_{i,h}$ is a simplicial mesh of Ω_i (a triangulation, in dimension $d = 2$), one takes for x_{K_i} the center of the circumscribed ball of K_i. We also require that $\eta_{K_i,L_i} \cdot (x_{L_i} - x_{K_i}) > 0$ (in the case of simplicial meshes, this restriction amounts to the Delaunay condition, see e.g. Ref. [15]). The "diamond" constructed from the neighbor centers x_{K_i}, x_{L_i} and the interface σ_{K_i,L_i} is denoted by T_{K_i,L_i}; e.g. in the case $x_{K_i} \in K_i$, $x_{L_i} \in L_i$, T_{K_i,L_i} is the convex hull of x_{K_i},x_{L_i} and σ_{K_i,L_i}). We have $\Omega_i = \cup_{K_i \in \mathcal{T}_{i,h}} \Big(\cup_{L_i \in N(K_i)} \overline{T}_{K_i,L_i} \Big)$, $i = 1,2$.

We require local regularity restrictions on the family of meshes $\mathcal{T}_{i,h}$; namely, for $i = 1,2$,

$$
\exists \gamma > 0 \qquad \forall h \;\; \forall K_i \in \mathcal{T}_{i,h} \; \forall L_i \in N(K_i) \quad \operatorname{diam}(K_i) + \operatorname{diam}(L_i) \le \gamma d_{K_i,L_i}, \tag{3.1}
$$

$$
\exists \gamma > 0 \qquad \forall h \;\; \forall K_i \in \mathcal{T}_{i,h} \; \forall L_i \in N(K_i) \quad |\sigma_{K,L}| d_{K,L} \;\le\; \gamma\, |K|. \tag{3.2}
$$

Herein, d_{K_i,L_i} is the distance between x_{K_i} and x_{L_i} for $i = 1,2$.

A discrete function on the mesh $\mathcal{T}_{i,h}$ is a set $(w_{K_i})_{K_i \in \mathcal{T}_{i,h}}$ for $i = 1,2$. Whenever convenient, we identify it with the piecewise constant function $w_{i,h}$ on Ω_i such that $w_{i,h}|_{K_i} = w_{K_i}$. Finally, the discrete gradient $\nabla_h w_{i,h}$ of a constant per control volume function $w_{i,h}$ is defined as the constant per diamond T_{K_i,L_i} function, $\mathbb{R}^d$-valued, with the values

$$
\Big(\nabla_h w_{i,h}\Big)\Big|_{T_{K_i,L_i}} = \nabla_{K_i,L_i} w_{i,h} := d\, \frac{w_{L_i} - w_{K_i}}{d_{K_i,L_i}} \eta_{K_i,L_i} \text{ for } i = 1,2. \tag{3.3}
$$

Remark 3.1. *Because we consider no flux boundary condition, we do not need to distinguish between interior and exterior control volumes; only inner interfaces between volumes are needed in order to formulate the scheme.*

3.2. **Approximation of the nonlocal cross-diffusion model.** To discretize (1.2-1.8), we choose an admissible discretization of $Q_{i,T}$ consisting of an admissible mesh $\mathcal{T}_{i,h}$ of Ω_i and of a time step size $\Delta t_h > 0$; both Δt_h and the size $\max_{K_i \in \mathcal{T}_{i,h}} \operatorname{diam}(K_i)$ tend to zero as $h \to 0$. We define $N_h > 0$ as the smallest integer such that $(N_h + 1)\Delta t_h \geq T$, and set $t^n := n\Delta t_h$ for $n \in \{0, \dots, N_h\}$. Whenever Δt_h is fixed, we will drop the subscript h in the notation.

Furthermore, we denote for $i = 1, 2, 3$,

$$
\begin{aligned}
F^{n+1}_{i,K_1} &= F_i(x^{n+1^+}_{K_1}, \varphi^{n+1^+}_{K_1}, \psi^{n+1^+}_{K_1}, \chi^{n+1^+}_{K_1}, c^{n+1^+}_{K}),\\
G^{n+1}_{i,K_2} &= G_i(x^{n+1^+}_{K_2}, u^{n+1^+}_{K_2}, v^{n+1^+}_{K_2}, w^{n+1^+}_{K_2}, c^{n+1^+}_{K}),\\
K^{n+1}_{K} &= K(x^{n+1^+}_{K}, \psi^{n+1^+}_{K}, v^{n+1^+}_{K}, c^{n+1^+}_{K}),
\end{aligned} \tag{3.4}
$$

where $K_j \in \Omega_j$ for $j = 1, 2$, and $K \in \Omega_1 \cup \Omega_2$.
To approximate the cross-diffusive terms, we introduce the terms $\mathcal{M}^{n+1}_{1_{ij,K_1,L_1}}$ and $\mathcal{M}^{n+1}_{2_{ij,K_2,L_2}}$. Herein, we make the choice

$$
\mathcal{M}^{n+1}_{1_{ij,K_1,L_1}} := \mathcal{M}_{1_{ij}}\Big(\min\{\varphi^{n+1^+}_{K_1}, \varphi^{n+1^+}_{L_1}\}, \min\{\psi^{n+1^+}_{K_1}, \psi^{n+1^+}_{L_1}\}, \min\{\chi^{n+1^+}_{K_1}, \chi^{n+1^+}_{L_1}\}\Big) \tag{3.5}
$$

$$
\mathcal{M}^{n+1}_{2_{ij,K_2,L_2}} := \mathcal{M}_{2_{ij}}\Big(\min\{u^{n+1^+}_{K_2}, u^{n+1^+}_{L_2}\}, \min\{v^{n+1^+}_{K_2}, v^{n+1^+}_{L_2}\}, \min\{w^{n+1^+}_{K_2}, w^{n+1^+}_{L_2}\}\Big) \tag{3.6}
$$

where $\phi^{n+1^+}_j = \max(0, \phi^{n+1}_j)$ for $\phi = \varphi, \psi, \chi, u, v, w$.

Remark 3.2. *Note that the choice of the minimum in the discretization of $\mathcal{M}^{n+1}_{1_{ij,K_1,L_1}}$ and $\mathcal{M}^{n+1}_{2_{ij,K_2,L_2}}$ for $i \neq j$ and $i, j = 1, 2, 3$, is imposed to justify the non-negativity of our discrete solution. Moreover, the choice of the diagonal terms $\mathcal{M}^{n+1}_{1_{ii,K_1,L_1}}$ and $\mathcal{M}^{n+1}_{2_{ii,K_2,L_2}}$ for $i = 1, 2, 3$, is made in order to preserve, at the discrete level, the structure of the cross-diffusion matrix $\mathcal{M}_1$ and $\mathcal{M}_2$.*

The discrete initial conditions are given by:

$$
\varphi^0_{K_1} = \frac{1}{|K_1|}\int_{K_1} \varphi_0(x)\,dx, \quad \psi^0_{K_1} = \frac{1}{|K_1|}\int_{K_1} \psi_0(x)\,dx, \quad \chi^0_{K_1} = \frac{1}{|K_1|}\int_{K_1} \chi_0(x)\,dx, \tag{3.7}
$$

$$
u^0_{K_2} = \frac{1}{|K_2|}\int_{K_2} u_0(x)\,dx, \quad v^0_{K_2} = \frac{1}{|K_2|}\int_{K_2} v_0(x)\,dx, \quad w^0_{K_2} = \frac{1}{|K_2|}\int_{K_2} w_0(x)\,dx, \tag{3.8}
$$

$$
c^0_{K} = \frac{1}{|K|}\int_{K} c_0(x)\,dx, \tag{3.9}
$$

We use the following implicit finite volume scheme to advance the numerical solution from t^n to $t^{n+1} = t^n + \Delta t,$:

Determine $(\varphi^{n+1}_{K_1}, \psi^{n+1}_{K_1}, \chi^{n+1}_{K_1})_{K_1\in\mathcal{T}_{1,h}}$, $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})_{K_2\in\mathcal{T}_{2,h}}$ and $(c^{n+1}_K)_{K\in\mathcal{T}_h}$such that

$$
\begin{aligned}
&|K_1|\frac{\varphi^{n+1}_{K_1}-\varphi^n_{K_1}}{\Delta t} - d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\varphi^n_{K_{1_0}}\right)\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}(\varphi^{n+1}_{L_1}-\varphi^{n+1}_{K_1})\\
&\qquad -\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\Big[\mathcal{M}^{n+1}_{1_{11,K_1,L_1}}(\varphi^{n+1}_{L_1}-\varphi^{n+1}_{K_1}) + \mathcal{M}^{n+1}_{1_{12,K_1,L_1}}(\psi^{n+1}_{L_1}-\psi^{n+1}_{K_1})\\
&\qquad + \mathcal{M}^{n+1}_{1_{13,K_1,L_1}}(\chi^{n+1}_{L_1}-\chi^{n+1}_{K_1})\Big] = |K_1|F^{n+1}_{1,K_1},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&|K_1|\frac{\psi^{n+1}_{K_1}-\psi^n_{K_1}}{\Delta t} - d_{12}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\psi^n_{K_{1_0}}\right)\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}(\psi^{n+1}_{L_1}-\psi^{n+1}_{K_1})\\
&\qquad -\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\Big[\mathcal{M}^{n+1}_{1_{21,K_1,L_1}}(\varphi^{n+1}_{L_1}-\varphi^{n+1}_{K_1}) + \mathcal{M}^{n+1}_{1_{22,K_1,L_1}}(\psi^{n+1}_{L_1}-\psi^{n+1}_{K_1})\\
&\qquad + \mathcal{M}^{n+1}_{1_{23,K_1,L_1}}(\chi^{n+1}_{L_1}-\chi^{n+1}_{K_1})\Big] = |K_1|F^{n+1}_{2,K_1},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&|K_1|\frac{\chi^{n+1}_{K_1}-\chi^n_{K_1}}{\Delta t} - d_{13}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\chi^n_{K_{1_0}}\right)\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}(\chi^{n+1}_{L_1}-\chi^{n+1}_{K_1})\\
&\qquad -\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\Big[\mathcal{M}^{n+1}_{1_{31,K_1,L_1}}(\varphi^{n+1}_{L_1}-\varphi^{n+1}_{K_1}) + \mathcal{M}^{n+1}_{1_{32,K_1,L_1}}(\psi^{n+1}_{L_1}-\psi^{n+1}_{K_1})\\
&\qquad + \mathcal{M}^{n+1}_{1_{33,K_1,L_1}}(\chi^{n+1}_{L_1}-\chi^{n+1}_{K_1})\Big] = |K_1|F^{n+1}_{3,K_1},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&|K_2|\frac{u^{n+1}_{K_2}-u^n_{K_2}}{\Delta t} - d_{21}\left(\sum_{K_{2_0}\in\mathcal{T}_{2,h}}|K_{2_0}|u^n_{K_{2_0}}\right)\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}(u^{n+1}_{L_2}-u^{n+1}_{K_2})\\
&\qquad -\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}\Big[\mathcal{M}^{n+1}_{2_{11,K_2,L_2}}(u^{n+1}_{L_2}-u^{n+1}_{K_2}) + \mathcal{M}^{n+1}_{2_{12,K_2,L_2}}(v^{n+1}_{L_2}-v^{n+1}_{K_2})\\
&\qquad + \mathcal{M}^{n+1}_{2_{13,K_2,L_2}}(w^{n+1}_{L_2}-w^{n+1}_{K_2})\Big] = |K_2|G^{n+1}_{1,K_2},
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&|K_2|\frac{v^{n+1}_{K_2}-v^n_{K_2}}{\Delta t} - d_{22}\left(\sum_{K_{2_0}\in\mathcal{T}_{2,h}}|K_{2_0}|v^n_{K_{2_0}}\right)\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}(v^{n+1}_{L_2}-v^{n+1}_{K_2})\\
&\qquad -\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}\Big[\mathcal{M}^{n+1}_{2_{21,K_2,L_2}}(u^{n+1}_{L_2}-u^{n+1}_{K_2}) + \mathcal{M}^{n+1}_{2_{22,K_2,L_2}}(v^{n+1}_{L_2}-v^{n+1}_{K_2})\\
&\qquad + \mathcal{M}^{n+1}_{2_{23,K_2,L_2}}(w^{n+1}_{L_2}-w^{n+1}_{K_2})\Big] = |K_2|G^{n+1}_{2,K_2},
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&|K_2|\frac{w^{n+1}_{K_2}-w^n_{K_2}}{\Delta t} - d_{23}\left(\sum_{K_{2_0}\in\mathcal{T}_{2,h}}|K_{2_0}|w^n_{K_2}\right)\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}(w^{n+1}_{L_2}-w^{n+1}_{K_2})\\
&\qquad -\sum_{L_2\in N(K_2)}\frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}}\Big[\mathcal{M}^{n+1}_{2_{31,K_2,L_2}}(u^{n+1}_{L_2}-u^{n+1}_{K_2}) + \mathcal{M}^{n+1}_{2_{32,K_2,L_2}}(v^{n+1}_{L_2}-v^{n+1}_{K_2})\\
&\qquad + \mathcal{M}^{n+1}_{2_{33,K_2,L_2}}(w^{n+1}_{L_2}-w^{n+1}_{K_2})\Big] = |K_2|G^{n+1}_{3,K_2},
\end{aligned}
\tag{3.15}
$$

$$
|K|\frac{c^{n+1}_K-c^n_K}{\Delta t} = |K|\mathbf{K}^{n+1}_K,
\tag{3.16}
$$

for all $K_i \in \mathcal{T}_{i,h},\ i = 1, 2$ and $n \in [0, N_h]$. Herein

$$
\begin{aligned}
\mathcal{M}^{n+1}_{1_{11,K_1,L_1}} &:= k_{11} \min\{\varphi^{n+1^+}_{K_1}, \varphi^{n+1^+}_{L_1}\} + \min\{\psi^{n+1^+}_{K_1}, \psi^{n+1^+}_{L_1}\} + \min\{\chi^{n+1^+}_{K_1}, \chi^{n+1^+}_{L_1}\},\\
\mathcal{M}^{n+1}_{1_{22,K_1,L_1}} &:= \min\{\varphi^{n+1^+}_{K_1}, \varphi^{n+1^+}_{L}\} + k_{12} \min\{\psi^{n+1^+}_{K_1}, \psi^{n+1^+}_{L_1}\} + \min\{\chi^{n+1^+}_{K_1}, \chi^{n+1^+}_{L_1}\},\\
\mathcal{M}^{n+1}_{1_{33,K_1,L_1}} &:= \min\{\varphi^{n+1^+}_{K_1}, \varphi^{n+1^+}_{L_1}\} + \min\{\psi^{n+1^+}_{K_1}, \psi^{n+1^+}_{L_1}\} + k_{13} \min\{\chi^{n+1^+}_{K_1}, \chi^{n+1^+}_{L_1}\},\\
\mathcal{M}^{n+1}_{1_{12,K_1,L_1}} &= \mathcal{M}^{n+1}_{1_{13,K_1,L_1}} := \min\{\varphi^{n+1^+}_{K_1}, \varphi^{n+1^+}_{L_1}\},\\
\mathcal{M}^{n+1}_{1_{21,K_1,L_1}} &= \mathcal{M}^{n+1}_{1_{23,K_1,L_1}} := \min\{\psi^{n+1^+}_{K_1}, \psi^{n+1^+}_{L_1}\},\\
\mathcal{M}^{n+1}_{1_{31,K_1,L_1}} &= \mathcal{M}^{n+1}_{1_{32,K_1,L_1}} := \min\{\chi^{n+1^+}_{K_1}, \chi^{n+1^+}_{L_1}\},
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
\mathcal{M}^{n+1}_{2_{11,K_2,L_2}} &:= k_{21} \min\{u^{n+1^+}_{K_2} u^{n+1^+}_{L_2}\} + \min\{v^{n+1^+}_{K_2}, v^{n+1^+}_{L_2}\} + \min\{w^{n+1^+}_{K_2}, w^{n+1^+}_{L_2}\},\\
\mathcal{M}^{n+1}_{2_{22,K_2,L_2}} &:= \min\{u^{n+1^+}_{K_2}, u^{n+1^+}_{L_2}\} + k_{22} \min\{v^{n+1^+}_{K_2}, v^{n+1^+}_{L_2}\} + \min\{w^{n+1^+}_{K_2}, w^{n+1^+}_{L_2}\},\\
\mathcal{M}^{n+1}_{2_{33,K_2,L_2}} &:= \min\{u^{n+1^+}_{K_2}, u^{n+1^+}_{L_2}\} + \min\{v^{n+1^+}_{K_2}, v^{n+1^+}_{L_2}\} + k_{23} \min\{w^{n+1^+}_{K_2}, w^{n+1^+}_{L_2}\},\\
\mathcal{M}^{n+1}_{2_{12,K_2,L_2}} &= \mathcal{M}^{n+1}_{2_{13,K_2,L_2}} := \min\{u^{n+1^+}_{K_2}, u^{n+1^+}_{L_2}\},\\
\mathcal{M}^{n+1}_{2_{21,K_2,L_2}} &= \mathcal{M}^{n+1}_{2_{23,K_2,L_2}} := \min\{v^{n+1^+}_{K_2}, v^{n+1^+}_{L_2}\},\\
\mathcal{M}^{n+1}_{2_{31,K_2,L_2}} &= \mathcal{M}^{n+1}_{2_{32,K_2,L_2}} := \min\{w^{n+1^+}_{K_2}, w^{n+1^+}_{L_2}\},
\end{aligned}
\tag{3.18}
$$

Note that the boundary condition is taken into account implicitly. Indeed, the parts of ∂K_i that lie in $\partial\Omega_i$ do not contribute to the $\sum_{L_i \in N(K_i)}$ terms, which means that the flux zero is imposed on the external edges of the mesh.

The set of values $(\varphi^{n+1}_{K_1}, \psi^{n+1}_{K_1}, \chi^{n+1}_{K_1})_{K_1 \in \mathcal{T}_{1,h}, n \in [0,N_h]}$, $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})_{K_2 \in \mathcal{T}_{2,h}, n \in [0,N_h]}$ and $(c^{n+1}_K)_{K \in \mathcal{T}_h, n \in [0,N_h]}$ satisfying (3.7) - (3.16) will be called a discrete solution.

The existence of solutions to the our discrete scheme is given in the following theorem.

Theorem 3.1. *Assume that* $(\varphi_0, \psi_0, \chi_0) \in (L^2(\Omega_1, \mathbb{R}^3))^+$, $(u_0, v_0, w_0) \in (L^2(\Omega_2, \mathbb{R}^3))^+$ *and* $c_0 \in (L^2(\Omega_1 \cup \Omega_2))^+$ *satisfying* $0 \le c_0 \le 1$. *Let* $(\varphi^{n+1}_{K_1}, \psi^{n+1}_{K_1}, \chi^{n+1}_{K_1})_{K_1 \in \mathcal{T}_{1,h}, n \in [0,N_h]}$, $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})_{K_2 \in \mathcal{T}_{2,h}, n \in [0,N_h]}$ *and* $(c^{n+1}_K)_{K \in \mathcal{T}_h, n \in [0,N_h]}$ *be the discrete solution generated by the finite volume scheme* (3.7)-(3.16) *on a family of meshes satisfying* (3.1), (3.2). *Then, as* $h \to 0$, *the discrete solution converges (along a subsequence) a.e. on* $\Omega_{i,T}$ *to a limit* (φ, ψ, χ), (u, v, w) *and* c *which is a weak solution of* (1.2)-(1.8).

4. A priori estimates and existence

4.1. **Nonnegativity.** We have the following lemma.

Lemma 4.1. *Let* $(\varphi^{n+1}_{K_1}, \psi^{n+1}_{K_1}, \chi^{n+1}_{K_1})_{K_1 \in \mathcal{T}_{1,h}, n \in [0,N_h]}$, $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})_{K_2 \in \mathcal{T}_{2,h}, n \in [0,N_h]}$ *and* $(c^{n+1}_K)_{K \in \mathcal{T}_h, n \in [0,N_h]}$ *be a solution of the finite volume scheme* (3.7)-(3.16). *Then,* $(\varphi^{n+1}_{K_1}, \psi^{n+1}_{K_1}, \chi^{n+1}_{K_1})_{K_1 \in \mathcal{T}_{1,h}, n \in [0,N_h]}$, $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})_{K_2 \in \mathcal{T}_{2,h}, n \in [0,N_h]}$ *and* $(c^{n+1}_K)_{K \in \mathcal{T}_h, n \in [0,N_h]}$ *are nonnegative. Moreover* $c^{n+1}_K \le 1$ *for all* $K \in \mathcal{T}_h$ *and* $n \in [0, N_h]$.

Proof. We prove the nonnegativity by induction, that for all $n \in [0, N_h]$, $\min\{\varphi^{n+1}_{K_1}\}_{K_1 \in \mathcal{T}_{1,h}} \ge 0$. The proof for the other components is analogous.

For $n \geq 0$, we fix K_1 such that $\varphi_{K_1}^{n+1} = \min\{\varphi_{L_1}^{n+1}\}_{L_1 \in \mathcal{T}_{1,h}}$. We multiply equation (3.10) by $-\Delta t \varphi_{K_1}^{n+1^-}$ to deduce

(4.1)

$$-|K_1|\,\varphi_{K_1}^{n+1^-}(\varphi_{K_1}^{n+1} - \varphi_{K_1}^{n}) = -d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}} |K_{1_0}|\varphi_{K_{1_0}}^{n}\right)\Delta t \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}(\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1})\,\varphi_{K_1}^{n+1^-}$$
$$- \Delta t \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\Bigg[\mathcal{M}_{1_{11,K_1,L_1}}^{n+1}(\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) + \mathcal{M}_{1_{12,K_1,L_1}}^{n+1}(\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1})$$
$$+ \mathcal{M}_{1_{13,K_1,L_1}}^{n+1}(\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\Bigg]\varphi_{K_1}^{n+1^-} - \Delta t|K_1|\,F_{1,K_1}^{n+1}\,\varphi_{K_1}^{n+1^-}.$$

Taking into account the non-negativity of $\mathcal{M}_{1_{11,K_1,L_1}}^{n+1}$ and by the choice of K_1 , we get

$$\Delta t \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\left(d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}} |K_{1_0}|\varphi_{K_{1_0}}^{n}\right) + \mathcal{M}_{1_{11,K_1,L_1}}^{n+1}\right)(\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1})\,\varphi_{K_1}^{n+1^-} \geq 0.$$

Moreover, by the choice (3.5) of $\mathcal{M}_{1_{12,K_1,L_1}}^{n+1}$ and $\mathcal{M}_{1_{13,K_1,L_1}}^{n+1}$, we obtain

$$\Delta t \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\left[\mathcal{M}_{1_{12,K_1,L_1}}^{n+1}(\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) + \mathcal{M}_{1_{13,K_1,L_1}}^{n+1}(\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\right]\varphi_{K_1}^{n+1^-} = 0,$$

Similarly, by the definition of F_{1,K_1}^{n+1} we have

(4.2)

$$F_{1,K_1}^{n+1}\,\varphi_{K_1}^{n+1^-} = \Bigg(-\sigma_{11}(x)\varphi_{K_1}^{n+1^+}\psi_{K_1}^{n+1^+}/H_1{}^+ - \sigma_{31}(x)c_{K_1}^{n+1^+}\varphi_{K_1}^{n+1^+} + (1-w_1)\lambda_1\psi_{K_1}^{n+1^+}$$
$$+ b(x)H_1^+ - (m(x) + k(x)H_1^+)\varphi_{K_1}^{n+1^+}\Bigg)\varphi_{K_1}^{n+1^-} \geq 0.$$

Finally, we use the identity $\varphi_{K_1}^{n+1} = (\varphi_{K_1}^{n+1})^+ - (\varphi_{K_1}^{n+1})^-$ and the nonnegativity of $\varphi_{K_1}^0$ to deduce from (4.1) and (4.2) that $(\varphi_{K_1}^{n+1})^- = 0$. By induction in n, we infer that

$$\varphi_{L_1}^{n+1} \geq 0 \text{ for all } n \in [0, N_h] \text{ and } L_1 \in \mathcal{T}_{1,h}.$$

Along the same lines as $(\varphi_{K_1}^{n+1})_{K_1\in\mathcal{T}_{1,h},n\in[0,N_h]}$, we obtain the nonnegativity of the discrete solutions $(\psi_{K_1}^{n+1}, \chi_{K_1}^{n+1})_{K_1\in\mathcal{T}_{1,h},n\in[0,N_h]}$, $(u_{K_2}^{n+1}, v_{K_2}^{n+1}, w_{K_2}^{n+1})_{K_2\in\mathcal{T}_{2,h},n\in[0,N_h]}$ and $(c_K^{n+1})_{K\in\mathcal{T}_h,n\in[0,N_h]}$.

Finally, in order to prove (by induction) that $c_K^{n+1} \leq 1$, we take K such that c_K realizes $\max(c_L^{n+1})_{L\in\mathcal{T}}$. Multiplying the equation (3.16) by $(c_K^{n+1} - 1)^+$, with the same arguments as in the above proof we find that $(c_K^{n+1} - 1)^+ \leq 0$. □

4.2. **A priori estimates.** Now, our goal is to establish some a priori (discrete energy) estimates for the finite volume scheme.

Proposition 4.1. *Let* $(\varphi_{K_1}^{n+1}, \psi_{K_1}^{n+1}, \chi_{K_1}^{n+1})_{K_1\in\mathcal{T}_{1,h},n\in[0,N_h]}$, $(u_{K_2}^{n+1}, v_{K_2}^{n+1}, w_{K_2}^{n+1})_{K_2\in\mathcal{T}_{2,h},n\in[0,N_h]}$ *and* $(c_K^{n+1})_{K\in\mathcal{T}_h,n\in[0,N_h]}$ *be a solution of the finite volume scheme* (3.7) - (3.16)*. Then there exist a constant* $C_i > 0$, $(i = 1, \ldots, 6)$*, depending on the initial conditions, parameters of the nonlinearities,*

Ω_i *and* T, $\quad (i=1,2)$.

(4.3)

$$\max_{[0,N_h]} \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\varphi_{K_1}^{n+1}\right|^2 + \max_{[0,N_h]} \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\psi_{K_1}^{n+1}\right|^2 + \max_{[0,N_h]} \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\chi_{K_1}^{n+1}\right|^2 \le C_1,$$

$$\sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\varphi_{K_1}^{n+1}\right|^3 + \sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\psi_{K_1}^{n+1}\right|^3 + \sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} |K_1| \left|\chi_{K_1}^{n+1}\right|^3 \le C_1,$$

$$\begin{aligned} \sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \Big(\left|\varphi_{K_1}^{n+1}-\varphi_{L_1}^{n+1}\right|^2 + \left|\psi_{K_1}^{n+1}-\psi_{L_1}^{n+1}\right|^2 \\ + \left|\chi_{K_1}^{n+1}-\chi_{L_1}^{n+1}\right|^2 \Big) \le C_2, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \overline{\mathcal{M}}_{1_{K_1,L_1}}^{n+1} \Big(\left|\varphi_{K_1}^{n+1}-\varphi_{L_1}^{n+1}\right|^2 \\ + \left|\psi_{K_1}^{n+1}-\psi_{L_1}^{n+1}\right|^2 + \left|\chi_{K_1}^{n+1}-\chi_{L_1}^{n+1}\right|^2 \Big) \le C_3, \end{aligned} \tag{4.5}$$

and

$$\max_{[0,N_h]} \sum_{K_2\in\mathcal{T}_{2,h}} |K_2| \left|u_{K_2}^{n+1}\right|^2 + \max_{[0,N_h]} \sum_{K_2\in\mathcal{T}_{2,h}} |K_2| \left|v_{K_2}^{n+1}\right|^2 + \max_{[0,N_h]} \sum_{K_2\in\mathcal{T}_{2,h}} |K_2| \left|w_{K_2}^{n+1}\right|^2 \le C_4, \tag{4.6}$$

$$\begin{aligned} \sum_{n=0}^{N_h} \Delta t \sum_{K_2\in\mathcal{T}_{2,h}} \sum_{L_2\in N(K_2)} \frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}} \Big(\left|u_{K_2}^{n+1}-u_{L_2}^{n+1}\right|^2 + \left|v_{K_2}^{n+1}-v_{L_2}^{n+1}\right|^2 \\ + \left|w_{K_2}^{n+1}-w_{L_2}^{n+1}\right|^2 \Big) \le C_5, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \sum_{n=0}^{N_h} \Delta t \sum_{K_2\in\mathcal{T}_{2,h}} \sum_{L_2\in N(K_2)} \frac{|\sigma_{K_2,L_2}|}{d_{K_2,L_2}} \overline{\mathcal{M}}_{2_{K_2,L_2}}^{n+1} \Big(\left|u_{K_2}^{n+1}-u_{L_2}^{n+1}\right|^2 \\ + \left|v_{K_2}^{n+1}-v_{L_2}^{n+1}\right|^2 + \left|w_{K_2}^{n+1}-w_{L_2}^{n+1}\right|^2 \Big) \le C_6, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \overline{\mathcal{M}}_{1_{K_1,L_1}}^{n+1} &= \min\{\varphi_{K_1}^{n+1^+}, \varphi_{L_1}^{n+1^+}\} + \min\{\psi_{K_1}^{n+1^+}, \psi_{L_1}^{n+1^+}\} + \min\{\chi_{K_1}^{n+1^+}, \chi_{L_1}^{n+1^+}\}, \\ \overline{\mathcal{M}}_{2_{K_2,L_2}}^{n+1} &= \min\{u_{K_2}^{n+1^+}, u_{L_2}^{n+1^+}\} + \min\{v_{K_2}^{n+1^+}, v_{L_2}^{n+1^+}\} + \min\{w_{K_2}^{n+1^+}, w_{L_2}^{n+1^+}\}. \end{aligned}$$

Proof. We multiply (3.10), (3.11) and (3.12) by $\Delta t\varphi_{K_1}^{n+1}$, $\Delta t\psi_{K_1}^{n+1}$ and $\Delta t\chi_{K_1}^{n+1}$, respectively, and add together the outcomes. Summing the resulting equation over K_1 and n yields

$$S_1+S_2+S_3+S_4=0,$$

where

$$S_1 = \sum_{n=0}^{N_h} \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \Big((\varphi_{K_1}^{n+1} - \varphi_{K_1}^{n})\varphi_{K_1}^{n+1} + (\psi_{K_1}^{n+1} - \psi_{K_1}^{n})\psi_{K_1}^{n+1} + (\chi_{K_1}^{n+1} - \chi_{K_1}^{n})\chi_{K_1}^{n+1} \Big),$$

$$\begin{aligned} S_2 = -\sum_{n=0}^{N_h} \Delta t \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \Bigg[& d_{11} \Bigg(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \varphi_{K_{1_0}}^{n} \Bigg) (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1})\varphi_{K_1}^{n+1} \\ & + d_{12} \Bigg(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \psi_{K_{1_0}}^{n} \Bigg) (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1})\psi_{K_1}^{n+1} \\ & + d_{13} \Bigg(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \chi_{K_{1_0}}^{n} \Bigg) (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\chi_{K_1}^{n+1} \Bigg], \end{aligned}$$

$$\begin{aligned} S_3 = -\sum_{n=0}^{N_h} \Delta t \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \Bigg(& \Big[\mathcal{M}_{1_{11,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) \\ & + \mathcal{M}_{1_{12,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) + \mathcal{M}_{1_{13,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1}) \Big] \varphi_{K_1}^{n+1} \\ & + \Big[\mathcal{M}_{1_{21,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) + \mathcal{M}_{1_{22,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) \\ & + \mathcal{M}_{1_{23,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1}) \Big] \psi_{K_1}^{n+1} \\ & + \Big[\mathcal{M}_{1_{31,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) + \mathcal{M}_{1_{32,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) \\ & + \mathcal{M}_{1_{33,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1}) \Big] \chi_{K_1}^{n+1} \Bigg), \end{aligned}$$

$$S_4 = -\sum_{n=0}^{N_h} \Delta t \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \Big(F_{1,K_1}^{n+1} \varphi_{K_1}^{n+1} + F_{2,K_1}^{n+1} \psi_{K_1}^{n+1} + F_{3,K_1}^{n+1} \chi_{K_1}^{n+1} \Big).$$

Observe that, using the inequality "$a(a-b) \geq \frac{1}{2}(a^2 - b^2)$", we obtain

$$\begin{aligned} S_1 & \geq \frac{1}{2} \sum_{n=0}^{N_h} \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \Big(\big|\varphi_{K_1}^{n+1}\big|^2 - \big|\varphi_{K_1}^{n}\big|^2 + \big|\psi_{K_1}^{n+1}\big|^2 - \big|\psi_{K_1}^{n}\big|^2 + \big|\chi_{K_1}^{n+1}\big|^2 - \big|\chi_{K_1}^{n}\big|^2 \Big) \\ & = \frac{1}{2} \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \left(\left|\varphi_{K_1}^{N_h+1}\right|^2 - \big|\varphi_{K_1}^{0}\big|^2 + \left|\psi_{K_1}^{N_h+1}\right|^2 - \big|\psi_{K_1}^{0}\big|^2 + \left|\chi_{K_1}^{N_h+1}\right|^2 - \big|\chi_{K_1}^{0}\big|^2 \right), \end{aligned}$$

Gathering by edges, we obtain

$$\begin{aligned} S_2 = \sum_{n=0}^{N_h} \Delta t \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \Bigg(& \frac{d_{11} \Big(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \varphi_{K_{1_0}}^{n} \Big)}{2} \big|\varphi_{K_1}^{n+1} - \varphi_{L_1}^{n+1}\big|^2 \\ & + \frac{d_{12} \Big(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \psi_{K_{1_0}}^{n} \Big)}{2} \big|\psi_{K_1}^{n+1} - \psi_{L_1}^{n+1}\big|^2 \\ & + \frac{d_{13} \Big(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \chi_{K_{1_0}}^{n} \Big)}{2} \big|\chi_{K_1}^{n+1} - \chi_{L_1}^{n+1}\big|^2 \Bigg). \end{aligned}$$

Next, using (A.7) where $f_\varepsilon^+(\varphi)$, $f_\varepsilon^+(\psi)$, and $f_\varepsilon^+(\chi)$ are replaced by $\min\{\varphi_{K_1}^{n+1^+}, \varphi_{L_1}^{n+1^+}\}$, $\min\{\psi_{K_1}^{n+1^+}, \psi_{L_1}^{n+1^+}\}$ and $\min\{\psi_{K_1}^{n+1^+}, \psi_{L_1}^{n+1^+}\}$ respectively, we deduce

$$S_3 \geq c\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\overline{\mathcal{M}}_{1_{K_1,L_1}}^{n+1}\Big(\big|\varphi_{K_1}^{n+1}-\varphi_{L_1}^{n+1}\big|^2 + \big|\psi_{K_1}^{n+1}-\psi_{L_1}^{n+1}\big|^2 + \big|\chi_{K_1}^{n+1}-\chi_{L_1}^{n+1}\big|^2\Big).$$

Now we use the nonnegativity of $\varphi_{K_1}^{n+1}, \psi_{K_1}^{n+1}$ and $\chi_{K_1}^{n+1}$, and the discrete expressions of F_1, F_2, F_3 to deduce

$$\begin{aligned} S_4 \geq &- C\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}|K_1|\Big(\big|\varphi_{K_1}^{n+1}\big|^2+\big|\psi_{K_1}^{n+1}\big|^2+\big|\chi_{K_1}^{n+1}\big|^2\Big)\\ &+ C'\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}|K_1|\Big(\big|\varphi_{K_1}^{n+1}\big|^3+\big|\psi_{K_1}^{n+1}\big|^3+\big|\chi_{K_1}^{n+1}\big|^3\Big),\end{aligned}$$

for some constants $C, C' > 0$. Collecting the previous inequalities we obtain

(4.9)

$$\begin{aligned} &\frac{1}{2}\sum_{K_1\in\mathcal{T}_{1,h}}|K_1|\left(\left|\varphi_{K_1}^{N_h+1}\right|^2-\left|\varphi_{K_1}^0\right|^2+\left|\psi_{K_1}^{N_h+1}\right|^2-\left|\psi_{K_1}^0\right|^2+\left|\chi_{K_1}^{N_h+1}\right|^2-\left|\chi_{K_1}^0\right|^2\right)\\ &\quad + C'\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}|K_1|\Big(\big|\varphi_{K_1}^{n+1}\big|^3+\big|\psi_{K_1}^{n+1}\big|^3+\big|\chi_{K_1}^{n+1}\big|^3\Big)\\ &\quad + \sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\left(\frac{d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\varphi_{K_{1_0}}^n\right)}{2}\big|\varphi_{K_1}^{n+1}-\varphi_{L_1}^{n+1}\big|^2\right.\\ &\qquad\left.+\frac{d_{12}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\psi_{K_{1_0}}^n\right)}{2}\big|\psi_{K_1}^{n+1}-\psi_{L_1}^{n+1}\big|^2+\frac{d_{13}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\chi_{K_{1_0}}^n\right)}{2}\big|\chi_{K_1}^{n+1}-\chi_{L_1}^{n+1}\big|^2\right)\\ &\quad + c\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}\sum_{L_1\in N(K_1)}\frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\overline{\mathcal{M}}_{1_{ij,K,L}}^{n+1}\Big(\big|\varphi_{K_1}^{n+1}-\varphi_{L_1}^{n+1}\big|^2\\ &\qquad\qquad + \big|\psi_{K_1}^{n+1}-\psi_{L_1}^{n+1}\big|^2+\big|\chi_{K_1}^{n+1}-\chi_{L_1}^{n+1}\big|^2\Big).\\ &\leq C\sum_{n=0}^{N_h}\Delta t\sum_{K_1\in\mathcal{T}_{1,h}}|K_1|\Big(\big|\varphi_{K_1}^{n+1}\big|^2+\big|\psi_{K_1}^{n+1}\big|^2+\big|\chi_{K_1}^{n+1}\big|^2\Big).\end{aligned}$$

By an application of the discrete Gronwall inequality, (4.3) follows from (4.9). Consequently, (4.9) implies the estimates (4.4)–(4.5). Along the same lines as $(\varphi_{K_1}^{n+1}, \psi_{K_1}^{n+1}, \chi_{K_1}^{n+1})_{K_1\in\mathcal{T}_{1,h}, n\in[0,N_h]}$, we obtain the estimates (4.6), (4.7) and (4.8) for the discrete solutions $(u_{K_2}^{n+1}, v_{K_2}^{n+1}, w_{K_2}^{n+1})$. This concludes the proof of Proposition 4.1. □

4.3. **Existence of a solution for the finite volume scheme.** The existence of a solution for the finite volume scheme is given in the following proposition.

Proposition 4.2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be admissible discretizations of $Q_{1,T}$ and $Q_{2,T}$ respectively. Then, the discrete problem* (3.7) - (3.16) *admits at least one solution* $(\varphi_{K_1}^{n+1}, \psi_{K_1}^{n+1}, \chi_{K_1}^{n+1})_{K_1\in\mathcal{T}_{1,h}, n\in[0,N_h]}$, $(u_{K_2}^{n+1}, v_{K_2}^{n+1}, w_{K_2}^{n+1})_{K_2\in\mathcal{T}_{2,h}, n\in[0,N_h]}$ *and* $(c_K^{n+1})_{K\in\mathcal{T}_h, n\in[0,N_h]}$

Proof. First we introduce the Hilbert space

$$E_h = H_h(\Omega_1)\times H_h(\Omega_1)\times H_h(\Omega_1),$$

of triples $\mathbf{u}_h^{n+1} = (\varphi_h^{n+1}, \psi_h^{n+1}, \chi_h^{n+1})$ of discrete functions on Ω_1. We denote by $H_h(\Omega_1) \subset L^2(\Omega_1)$ the space of functions which are piecewise constant on each control volume K_1. We defined the norm

$$\begin{aligned} \|\mathbf{u}_h^{n+1}\|_{E_h}^2 \quad := \quad & \Big(|\varphi_h^{n+1}|_{H_h(\Omega_1)}^2 + |\psi_h^{n+1}|_{H_h(\Omega_1)}^2 + |\chi_h^{n+1}|_{H_h(\Omega_1)}^2\Big) \\ & + \Big(\|\varphi_h^{n+1}\|_{L^2(\Omega_1)}^2 + \|\psi_h^{n+1}\|_{L^2(\Omega_1)}^2 + \|\chi_h^{n+1}\|_{L^2(\Omega_1)}^2\Big), \end{aligned}$$

where the discrete seminorm $|\cdot|_{H_h(\Omega_1)}^2$ of $w_h \in H_h(\Omega_1)$ is given by

$$|w_h|_{H_h(\Omega_1)}^2 := \frac{1}{2} \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} |T_{K_1,L_1}| \left|\frac{w_{L_1} - w_{K_1}}{d_{K_1,L_1}}\right|^2,$$

and the $L^2(\Omega_1)$ norm of $w_h \in H_h(\Omega_1)$ is given by

$$\|w_h\|_{L^2(\Omega_1)}^2 := \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \; |w_{K_1}|^2.$$

Let $\Phi_h = (\varphi_{1,h}, \varphi_{2,h}, \varphi_{3,h}) \in E_h$ and define the discrete bilinear forms

$$B_h(\mathbf{u}_h^{n+1}, \Phi_h) = \sum_{K_1 \in \mathcal{T}_{1,h}} |K_1| \, (\varphi_{K_1}^{n+1} \varphi_{1,K_1} + \psi_{K_1}^{n+1} \varphi_{2,K_1} + \chi_{K_1}^{n+1} \varphi_{3,K_1}),$$

$$\begin{aligned} & a_{1,h}(\mathbf{u}_h^{n+1}, \Phi_h) = \\ & \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \left(\frac{d_{11}\left(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \varphi_{K_{1_0}}^n\right)}{2} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1})(\varphi_{1,L_1} - \varphi_{1,K_1}) \right. \\ & + \frac{d_{12}\left(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \psi_{K_{1_0}}^n\right)}{2} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1})(\varphi_{2,L_1} - \varphi_{2,K_1}) \\ & \left. + \frac{d_{13}\left(\sum_{K_{1_0} \in \mathcal{T}_{1,h}} |K_{1_0}| \chi_{K_{1_0}}^n\right)}{2} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})(\varphi_{3,L_1} - \varphi_{3,K_1}) \right) \end{aligned}$$

Similarly, for given matrices $\mathcal{M}_h^{n+1} := \left(\left(\mathcal{M}_{1_{ij,K_1,L_1}}^{n+1}\right)_{1 \le i,j \le 3}\right)_{K \in \mathcal{T}_h, L \in N(K)}$, define the bilinear form

$$\begin{aligned} a_{2,h}(\mathcal{M}_h^{n+1}(\mathbf{u}_h^{n+1})\mathbf{u}_h^{n+1}, \Phi_h) = \frac{1}{2} \sum_{K_1 \in \mathcal{T}_{1,h}} \sum_{L_1 \in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}} \Bigg(& \Big[\mathcal{M}_{1_{11,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) \\ + \mathcal{M}_{1_{12,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) + \mathcal{M}_{1_{13,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\Big] & (\varphi_{1,L_1}^{n+1} - \varphi_{1,K_1}^{n+1}) \\ + \Big[\mathcal{M}_{1_{21,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) + \mathcal{M}_{1_{22,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) & \\ + \mathcal{M}_{1_{23,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\Big] (\varphi_{2,L_1}^{n+1} - \varphi_{2,K_1}^{n+1}) & \\ + \Big[\mathcal{M}_{1_{31,K_1,L_1}}^{n+1} (\varphi_{L_1}^{n+1} - \varphi_{K_1}^{n+1}) + \mathcal{M}_{1_{32,K_1,L_1}}^{n+1} (\psi_{L_1}^{n+1} - \psi_{K_1}^{n+1}) & \\ + \mathcal{M}_{1_{33,K_1,L_1}}^{n+1} (\chi_{L_1}^{n+1} - \chi_{K_1}^{n+1})\Big] (\varphi_{3,L_1}^{n+1} - \varphi_{3,K_1}^{n+1}) \Bigg). & \end{aligned}$$

Multiplying (3.10), (3.11) and (3.12) by $\varphi_{1,K}$, $\varphi_{2,K}$ and $\varphi_{3,K}$, respectively, summing in $K \in \mathcal{T}_h$, we get the equation

$$\begin{aligned} \frac{1}{\Delta t}\Big(B_h(\mathbf{u}_h^{n+1}, \Phi_h) - B_h(\mathbf{u}_h^n, \Phi_h)\Big) + a_{1,h}(\mathbf{u}_h^{n+1}, \Phi_h) + a_{2,h}(\mathcal{M}_h^{n+1}(\mathbf{u}_h^{n+1}); \mathbf{u}_h^{n+1}, \Phi_h) & \\ + B_h(R_h(\mathbf{u}_h^{n+1}), \Phi_h) = 0, & \end{aligned}$$

here the entries $\mathcal{M}^{n+1}_{1_{ij,K_1,L_1}}$ of $\mathcal{M}^{n+1}_h$ are defined from $\mathbf{u}^{n+1}_h$ with the help of formulas (3.5). Furthermore, $R_h(\mathbf{u}^{n+1}_h) := (F^{n+1}_{1,h}, F^{n+1}_{2,h}, F^{n+1}_{3,h})$ with the discrete functions $F^{n+1}_{1,h}, F^{n+1}_{2,h}, F^{n+1}_{3,h}$ defined from $\mathbf{u}^{n+1}_h$ by formulas (3.4). It is clear that, $\mathbf{u}^n_h$ being given, there exists a solution $\mathbf{u}^{n+1}_h$ of the above equation if and only if there exists a discrete solution of (3.7)-(3.12) at the time step $(n+1)$. Now we define, by duality, the mapping $\mathcal{P}$ from E_h into itself:

$$\forall \Phi_h \in E_h \quad [\mathcal{P}(\mathbf{u}^{n+1}_h), \Phi_h] = \frac{1}{\Delta t}(B_h(\mathbf{u}^{n+1}_h, \Phi_h) - B_h(\mathbf{u}^n_h, \Phi_h)) + a_{1,h}(\mathbf{u}^{n+1}_h, \Phi_h) + a_{2,h}(\mathcal{M}^{n+1}_h(\mathbf{u}^{n+1}_h); \mathbf{u}^{n+1}_h, \Phi_h) + B_h(R_h(\mathbf{u}^{n+1}_h), \Phi_h).$$

Now, using Lemma 4.1, Proposition 4.1, and an application of Young's inequality to deduce

$$[\mathcal{P}(\mathbf{u}^{n+1}_h), \mathbf{u}^{n+1}_h] \geq C \left\|\mathbf{u}^{n+1}_h\right\|^2_{E_h} - C' \left\|\mathbf{u}^{n+1}_h\right\|_{E_h} - C'' \geq 0 \text{ for } \left\|\mathbf{u}^{n+1}_h\right\|_{E_h} \text{ large enough,}$$

for some constants $C, C', C'' > 0$. We deduce that

$$[\mathcal{P}(\mathbf{u}^{n+1}_h), \mathbf{u}^{n+1}_h] > 0 \quad \text{for } \left\|\mathbf{u}^{n+1}_h\right\|_{E_h} \text{ large enough.}$$

This implies (see for e.g. [26] and [33]): there exists $\mathbf{u}^{n+1}_h$ such that

$$\mathcal{P}(\mathbf{u}^{n+1}_h) = 0.$$

Thus $\mathbf{u}^{n+1}_h$ does exist. Then, we obtain the existence of at least one solution to the scheme $\mathbf{u}^{n+1}_h$.

Along the same lines as $\mathbf{u}^{n+1}_h$, we obtain the existence of the discrete solutions $(u^{n+1}_{K_2}, v^{n+1}_{K_2}, w^{n+1}_{K_2})$ and c^{n+1}_K.

□

4.4. **Convergence of the scheme.** In this section, we prove that the family of discrete solutions $\mathbf{u_{1,h}} = (\varphi_h, \psi_h, \chi_h)$ and $\mathbf{u_{2,h}} = (u_h, v_h, w_h)$ are relatively compact in L^1. We first apply the following lemma (see the proof of this lemma in Appendix A in [7]):

Lemma 4.2. *Let Ω be an open domain in $\mathbb{R}^d$, $T > 0$, $\Omega_T = (0,T)\times\Omega$. Let $(\mathcal{T}^h)_h$ be an admissible family of meshes of Ω satisfying the restriction (3.1); let $(\Delta t^h)_h$ be the associated time steps.*

For all $h > 0$, assume that discrete functions $\left(u^{n+1}_h\right)_{n\in[0,N_h]}$, $\left(f^{n+1}_h\right)_{n\in[0,N_h]}$ and discrete fields $\left(\vec{\mathcal{F}}^{n+1}_h\right)_{n\in[0,N_h]}$ satisfy the discrete evolution equations

$$\text{for } n \in [0, N_h], \quad \frac{u^{n+1}_h - u^n_h}{\Delta t} = \mathrm{div}_h\,[\vec{\mathcal{F}}^{n+1}_h] + f^{n+1}_h \tag{4.10}$$

with a family $(u^0_h)_h$ of initial data. Assume that for all $\Omega' \Subset \Omega$, there exists a constant $M(\Omega')$ such that

$$\sum\nolimits_{n=0}^{N_h} \Delta t \left\| u^{n+1}_h \right\|_{L^1(\Omega')} + \sum\nolimits_{n=0}^{N_h} \Delta t \left\| f^{n+1}_h \right\|_{L^1(\Omega')} + \sum\nolimits_{n=0}^{N_h} \Delta t \left\| \vec{\mathcal{F}}^{n+1}_h \right\|_{L^1(\Omega')} \leq M(\Omega'). \tag{4.11}$$

and, moreover,

$$\sum\nolimits_{n=0}^{N_h} \Delta t \left\| \nabla_h u^{n+1}_h \right\|_{L^1(\Omega')} \leq M(\Omega'). \tag{4.12}$$

Assume that the family $(u^h_0)_h$ is bounded in $L^1_{loc}(\Omega)$. Then there exists a measurable function u on Ω_T such that, along a subsequence,

$$\sum\nolimits_{n=0}^{N_h} \sum\nolimits_{K\in\mathcal{T}_h} u^{n+1}_K \mathbb{1}_{(t^n,t^{n+1}]\times K} \longrightarrow u \quad \text{in } L^1_{loc}([0,T]\times\Omega) \ \text{ as } h \to 0.$$

Denoted by $\mathcal{M}^h_\kappa$ the 3×3 matrix on $Q_{\kappa,T}$ with the entries $\mathcal{M}^h_{\kappa_{ij}}$ given by

$$\mathcal{M}^h_{\kappa_{ij}} := \frac{1}{2}\sum_{n=0}^{N_h} \sum_{K_\kappa\in\mathcal{T}_{\kappa,h}} \sum_{L_\kappa\in N(K_\kappa)} \mathcal{M}^{n+1}_{\kappa_{ij,K_\kappa,L_\kappa}} : \mathbb{1}_{(t^n,t^{n+1}]\times T_{K_\kappa,L_\kappa}} \qquad \text{for } \kappa = 1, 2.$$

Observe that we may consider that the evolution of the first component $(\varphi_h^{n+1})_{n\in[0,N_h]}$ the solution (3.10) is governed by the system of discrete equations

$$\frac{\varphi_{K_1}^{n+1}-\varphi_{K_1}^{n}}{\Delta t}=\frac{1}{|K_1|}\sum_{L_1\in N(K_1)}|\sigma_{K_1,L_1}|\,\vec{\mathcal{F}}_{K_1,L_1}^{n+1}\cdot\nu_{K_1,L_1}+f_{K_1}^{n+1}. \tag{4.13}$$

Herein,

$$\begin{aligned}
f_K^{n+1} :=&F_{1,K_1}^{n+1},\\
\vec{\mathcal{F}}_{K,L}^{n+1} :=&d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\varphi_{K_{1_0}}^n\right)\frac{\varphi_{L_1}^{n+1}-\varphi_{K_1}^{n+1}}{d_{K_1,L_1}}\nu_{K_1,L_1}+\mathcal{M}_{1_{11},K_1,L_1}^{n+1}\frac{\varphi_{L_1}^{n+1}-\varphi_{K_1}^{n+1}}{d_{K_1,L_1}}\nu_{K_1,L_1}\\
&+\mathcal{M}_{1_{12},K_1,L_1}^{n+1}\frac{\psi_{L_1}^{n+1}-\psi_{K_1}^{n+1}}{d_{K_1,L_1}}\nu_{K_1,L_1}+\mathcal{M}_{1_{13},K_1,L_1}^{n+1}\frac{\chi_{L_1}^{n+1}-\chi_{K_1}^{n+1}}{d_{K_1,L_1}}\nu_{K_1,L_1}\\
\equiv&\frac{1}{d}\Bigg[d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}}|K_{1_0}|\varphi_{K_{1_0}}^n\right)\nabla_{K_1,L_1}\varphi_h^{n+1}+\mathcal{M}_{1_{11},K_1,L_1}^{n+1}\nabla_{K_1,L_1}\varphi_h^{n+1}\\
&+\mathcal{M}_{1_{12},K_1,L_1}^{n+1}\nabla_{K_1,L_1}\psi_h^{n+1}+\mathcal{M}_{1_{13},K_1,L_1}^{n+1}\nabla_{K_1,L_1}\chi_h^{n+1}\Bigg].
\end{aligned}$$

It is easy to see that equations (4.13) have the form (4.10) required in Lemma 4.2.

Observe that from Lemma 4.1 and Proposition 4.1, the local L^1 bounds (4.11) and (4.12) are verified. Consequently from Lemma 4.2, there exit $\mathbf{u}_1\in(L^2(0,T;H^1(\Omega_1))^3$, $\mathbf{u}_2\in(L^2(0,T;H^1(\Omega_2))^3$ $c\in L^\infty(Q_T)$ and subsequences of $\mathbf{u}_{1,h}=(\varphi_h,\psi_h,\chi_h)$, $\mathbf{u}_{2,h}=(u_h,v_h,w_h)$, c_h, not labelled, such that, as $h\to 0$,

(4.14)

(i) $\mathbf{u}_{\kappa,h}\to\mathbf{u}_\kappa$ strongly in $(L^1(Q_{\kappa,T}))^3$ and a.e. in $Q_{\kappa,T}$,

(ii) $\nabla_h\mathbf{u}_{\kappa,h}\longrightarrow\nabla\mathbf{u}_\kappa$ weakly in $(L^2(Q_{\kappa,,T}))^{3\times 3}$,

(iii) $\mathcal{M}_\kappa^h\nabla_h\mathbf{u}_{\kappa,h}\longrightarrow\mathcal{M}_\kappa(\mathbf{u}_\kappa)\nabla\mathbf{u}_\kappa$ weakly in $(L^1(Q_{\kappa,T}))^{3\times 3}$,

(iv) $(F_i(\mathbf{u}_{1,h},c_h),G_i(\mathbf{u}_{2,h},c_h),K(\psi_h,v_h,c_h))\longrightarrow(F(\mathbf{u}_1,c),G(\mathbf{u}_2,c),K(\psi,v,c))$ strongly in $(L^1(Q_{1,T}))^3$, $(L^1(Q_{2,T}))^3$ and $(L^1(Q_T))^3$ repectively,

(v) $c_h\to c$ strongly in $L^2(Q_T)$,

for $\kappa=1,2$ and $i=1,2,3$.

4.5. **Convergence Analysis.** Our final goal is to show that the limit functions $\mathbf{u}_1=(\varphi,\psi,\chi)$, $\mathbf{u}_2=(u,v,w)$ and c constructed in (4.14) constitute a weak solution of our cross-diffusion system. We start by passing to the limit in (3.10) to get the first equality in Definition 2.1, the arguments for the passage to the limit in the rest of the equalities (2.4) are entirely similar. Let $\phi\in\mathcal{D}([0,T)\times\overline{\Omega_1})$.

Set $\phi_{K_1}^n:=\phi(t^n,x_{K_1})$ for all $K_1\in\mathcal{T}_{1,h}$ and $n\in[0,N_h+1]$. We multiply the discrete equation (3.10) by $\Delta t\phi_K^{n+1}$. Summing the result over $K_1\in\mathcal{T}_{1,h}$ and $n\in[0,N_h]$, yields

$$S_1^h+S_2^h+S_3^h=S_4^h,$$

where

$$
\begin{aligned}
S_1^h =& \sum_{n=0}^{N_h} \sum_{K_1\in\mathcal{T}_{1,h}} |K_1|\,(\varphi_{K_1}^{n+1}-\varphi_{K_1}^{n})\,\phi_{K_1}^{n+1},\\
S_2^h =& -\sum_{n=0}^{N_h} \Delta t\, d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}} |K_{1_0}|\varphi_{K_{1_0}}^{n}\right) \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}(\varphi_{L_1}^{n+1}-\varphi_{K_1}^{n+1})\,\phi_{K_1}^{n+1},\\
S_3^h =& -\sum_{n=0}^{N_h} \Delta t \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} \frac{|\sigma_{K_1,L_1}|}{d_{K_1,L_1}}\Bigg[\mathcal{M}_{1_{11},K_1,L_1}^{n+1}(\varphi_{L_1}^{n+1}-\varphi_{K_1}^{n+1})+\mathcal{M}_{1_{12},K_1,L_1}^{n+1}(\psi_{L_1}^{n+1}-\psi_{K_1}^{n+1})\\
&+\mathcal{M}_{1_{13},K_1,L_1}^{n+1}(\chi_{L_1}^{n+1}-\chi_{K_1}^{n+1})\Bigg]\,\phi_{K_1}^{n+1},\\
S_4^h =& \sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathcal{T}_{1,h}} |K_1| F_{1,K_1}^{n+1}\,\phi_{K_1}^{n+1}.
\end{aligned}
$$

An integration-by-parts and keeping in mind that $\phi_K^{N_h+1}=0$ for all $K_1\in\mathcal{T}_{1,h}$, we get from (4.14) (i) the convergence (along a subsequence)

$$
\lim_{h\to 0} S_1^h = -\int_0^T\int_{\Omega_1} \varphi\partial_t\phi - \int_{\Omega_1}\varphi_0\phi(0,\cdot).
$$

Gathering by edges and using the definition (3.3) of ∇_h, we have

$$
\begin{aligned}
S_2^h =& \frac{1}{2}\sum_{n=0}^{N_h}\Delta t d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}} |K_{1_0}|\varphi_{K_{1_0}}^{n}\right)\times\\
& \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} \frac{1}{d}|\sigma_{K_1,L_1}|\,d_{K_1,L_1}\;\; d\,\frac{\varphi_{L_1}^{n+1}-\varphi_{K_1}^{n+1}}{d_{K_1,L_1}}\;\frac{\phi_{L_1}^{n+1}-\phi_{K_1}^{n+1}}{d_{K_1,L_1}}\\
=& \frac{1}{2}\sum_{n=0}^{N_h}\Delta t d_{11}\left(\sum_{K_{1_0}\in\mathcal{T}_{1,h}} |K_{1_0}|\varphi_{K_{1_0}}^{n}\right)\times\\
& \sum_{K_1\in\mathcal{T}_{1,h}} \sum_{L_1\in N(K_1)} |T_{K_1,L_1}|\,\Big(\nabla_{K_1,L_1}\varphi_h^{n+1}\cdot\nu_{K_1,L_1}\Big)\,\Big(\nabla\phi(t^{n+1},\overline{x_{K_1,L_1}})\cdot\nu_{K_1,L_1}\Big),
\end{aligned}
$$

where $\overline{x_{K_1,L_1}}$ is some point on the segment with the endpoints x_{K_1}, x_{L_1}. Since the values of ∇_{K_1,L_1} are directed by ν_{K_1,L_1}, we have

$$
\Big(\nabla_{K_1,L_1}\varphi_h^{n+1}\cdot\eta_{K_1,L_1}\Big)\,\Big(\nabla\phi(t^{n+1},\overline{x_{K_1,L_1}})\cdot\nu_{K_1,L_1}\Big)\equiv\nabla_{K_1,L_1}\varphi_h^{n+1}\cdot\nabla\phi(t^{n+1},\overline{x_{K_1,L_1}}).
$$

Moreover, each term corresponding to T_{K_1,L_1} appears twice in the above formula,

$$
S_2^h=\int_0^T d_{11}\Big(\int_{\Omega_1}\varphi_h(t,x)\,dx\Big)\int_{\Omega_1}\nabla_h\varphi_h\cdot(\nabla\phi)_h,
$$

where

$$
(\nabla\phi)_h|_{(t^n,t^{n+1}]\times T_{K_1,L_1}} := \nabla\phi(t^{n+1},\overline{x_{K_1,L_1}}).
$$

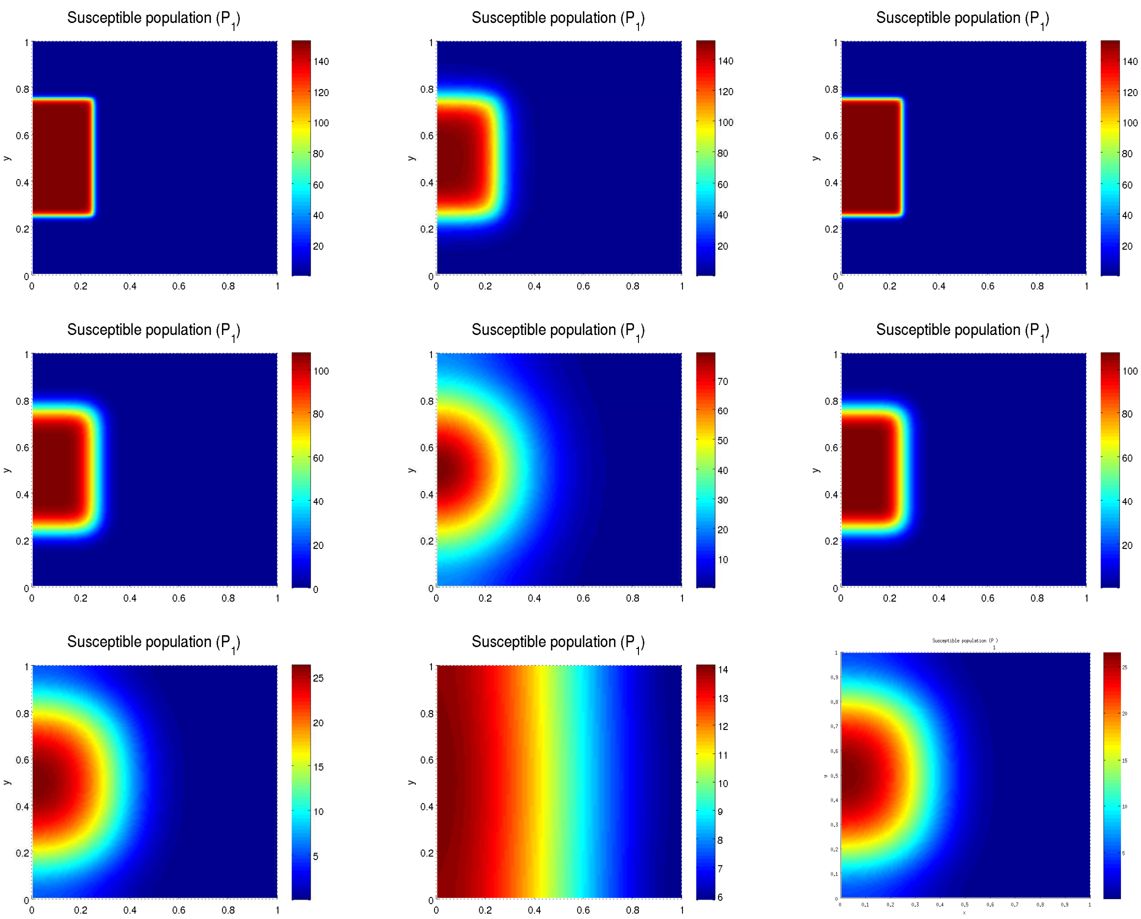
Observe that from the continuity of $\nabla\phi$ we get $(\nabla\phi)_h\to\nabla\phi$ in $L^\infty(Q_{1,T})$. Hence using the sharp Sobolev embedding and the interpolation between $L^2(0,T;L^{2^*}(\Omega_1))$ and $L^\infty(0,T;L^2(\Omega_1))$, and the weak L^2 convergence of $\nabla_h\varphi_h$ to $\nabla\varphi$, we pass to the limit in S_2^h and S_3^h, as $h\to 0$. Then, again along a subsequence, we have

$$\lim_{h\to 0} S_2^h = \int_0^T d_{11}\Big(\int_{\Omega_1} \varphi(t,x)\,dx\Big)\int_{\Omega_1} \nabla\varphi\cdot\nabla\phi,$$

$$\lim_{h\to 0} S_3^h = \iint_{\Omega_T} \Big(\mathcal{M}_{1_{12}}(\varphi,\psi,\chi)\nabla\varphi + \mathcal{M}_{1_{12}}(\varphi,\psi,\chi)\nabla\psi + \mathcal{M}_{1_{13}}(\varphi,\psi,\chi)\nabla\chi\Big)\cdot\nabla\phi.$$

Finally, using (4.14) (iv), we deduce that S_4^h converges to $\displaystyle\iint_{Q_{1,T}} F_1(\varphi,\psi,\chi)\,\phi$ as $h\to 0$. Gathering the obtained results, we justify the first equality in Definition 2.1. Reasoning along the same lines as above, we conclude that also the rest of the equalities in Definition 2.1 hold. This concludes the proof of Theorem 3.1

5. Numerical Results



a) constant diffusion b) non local diffusion c) cross diffusion

Figure 1. Example 1. Disease-free populations (P_1) for different times, $t = 0.01, 0.1, 1$.

In this section we give numerical results from our finite volume scheme. We take the domains as follow $\Omega_1 = (0,1)\times(0,1)$ and $\Omega_2 = (0.5,1.5)\times(0,1)$, such that $\Omega_1\cap\Omega_2 = (0.5,1.0)\times(0,1) \neq \emptyset$. We consider here a uniform mesh in both domains, given by a Cartesian grid with $N_{x_i}\times N_{y_i}$, $i=1,2$ control volumes. Obviously, it is possible to consider unstructured meshes, but we will take here to an uniform mesh $\Omega_R = \{K_{ij}\in\Omega | K_{ij} = (i-1)N_x, iN_x)(j-1)N_y, jN_y), i = 1,\dots,N_x, j =$

$1, \dots, N_y\}$, for simplicity of the simulated models. The discretization in time is given by Nt $=$ 500 time steps for $T = 0.5$. That is, $\delta t = T/Nt$ and $m(K) = 1/(NxNy)$. The parameters of the model are given by

$$\sigma_{11} = 0.8, \quad \sigma_{31} = 0.7, \quad \omega_1 = 0.1, \quad \lambda_1 = 12, \quad b = 0.03, \quad m = 0.01, \quad k = 0.03,$$
$$\sigma_{32} = 0.9, \quad \lambda_2 = 36, \quad \epsilon = 1.0, \quad \sigma_{13} = 0.3, \quad \sigma_{23} = 0.3, \quad \delta = 52.$$

Additionally, if we compute with constant diffusion, we will take $d_{ij} = 0.01$ for $i = 1, 2$ and $j = 1, 2, 3$. The nonlocal diffusion depends on the total population as we mentioned before. In the case of simulation with nonlocal diffusion terms, they are given by a simple choice of the functions d_{ij} for $i = 1, 2$ and $j = 1, 2, 3$ equal to

$$d_{ij}\left(\int_{\Omega_i} \zeta \, dx\right) = 0.01 \int_{\Omega_i} \zeta \, dx,$$

where $\zeta = \varphi, \psi, \chi, u, v, w$. For the cross diffusion parameters we take $k_{11} = k_{21} = 1$ and $k_{12} = k_{13} = k_{22} = k_{23} = 0.1$ (up to a rescaling with respect to (1.9)). It is important to mention that differences between constant and cross diffusion with this parameters of k_{ij} are slight.

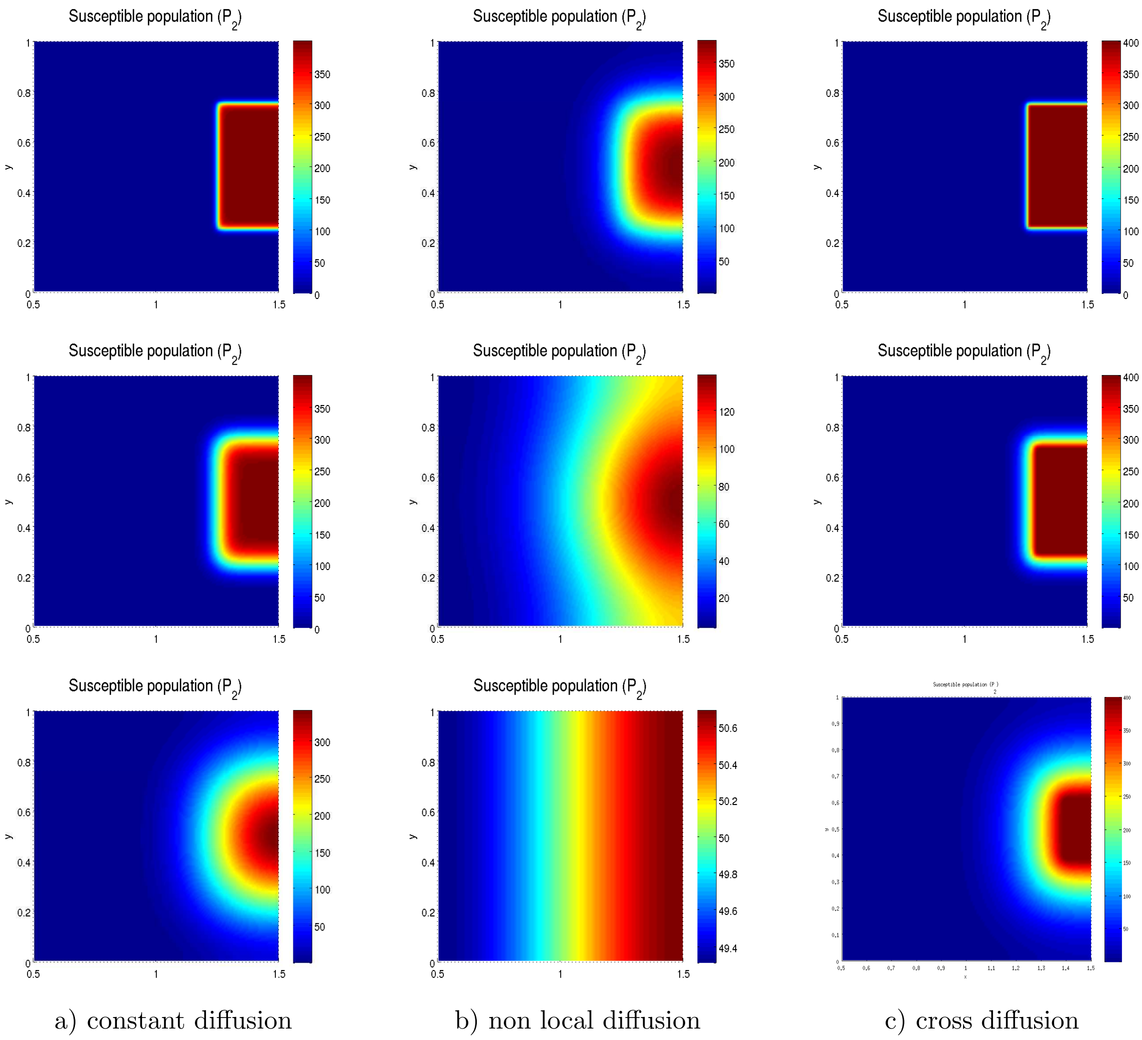
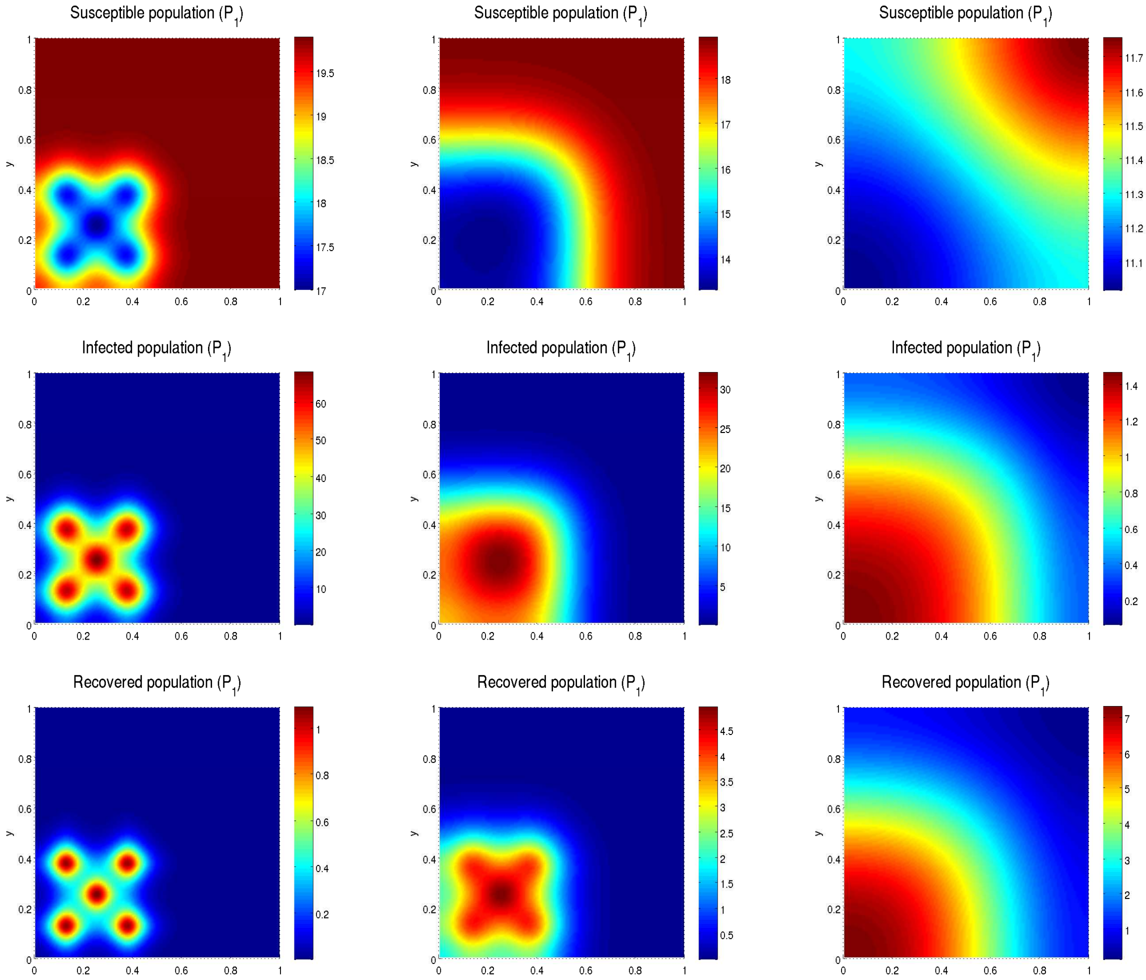


FIGURE 2. Example 1. Disease-free populations (P_2) for different times, $t = 0.01, 0.1, 1$.

5.1. Example 1. Behavior of disease-free populations.



a) $t = 0.01$ b) $t = 0.1$ c) $t = 0.01$

FIGURE 3. Example 2. Indirectly transmitted disease (P_1) with non-local diffusion, for different times. $t = 0.01, 0.1, 1$.

In this first example, we want to study the behavior of disease-free populations. To do this, we remove the contaminant and the presence of infected and recovered populations, simply as imposing initial data $\psi(x,y,o) = v(x,y,0) = \chi(x,y,0) = w(x,y,0) = c(x,y,0) = 0$. This makes that our system of 7 equations is reduced to just two decoupled equations:

$$\partial_t\varphi - d_{11}\Big(\int_{\Omega_1}\varphi\,dx\Big)\Delta\varphi - \operatorname{div}\Big(k_{11}\varphi\nabla\varphi\Big) = (b-m)\varphi - k\varphi^2, \qquad \text{in } Q_{1,T}$$

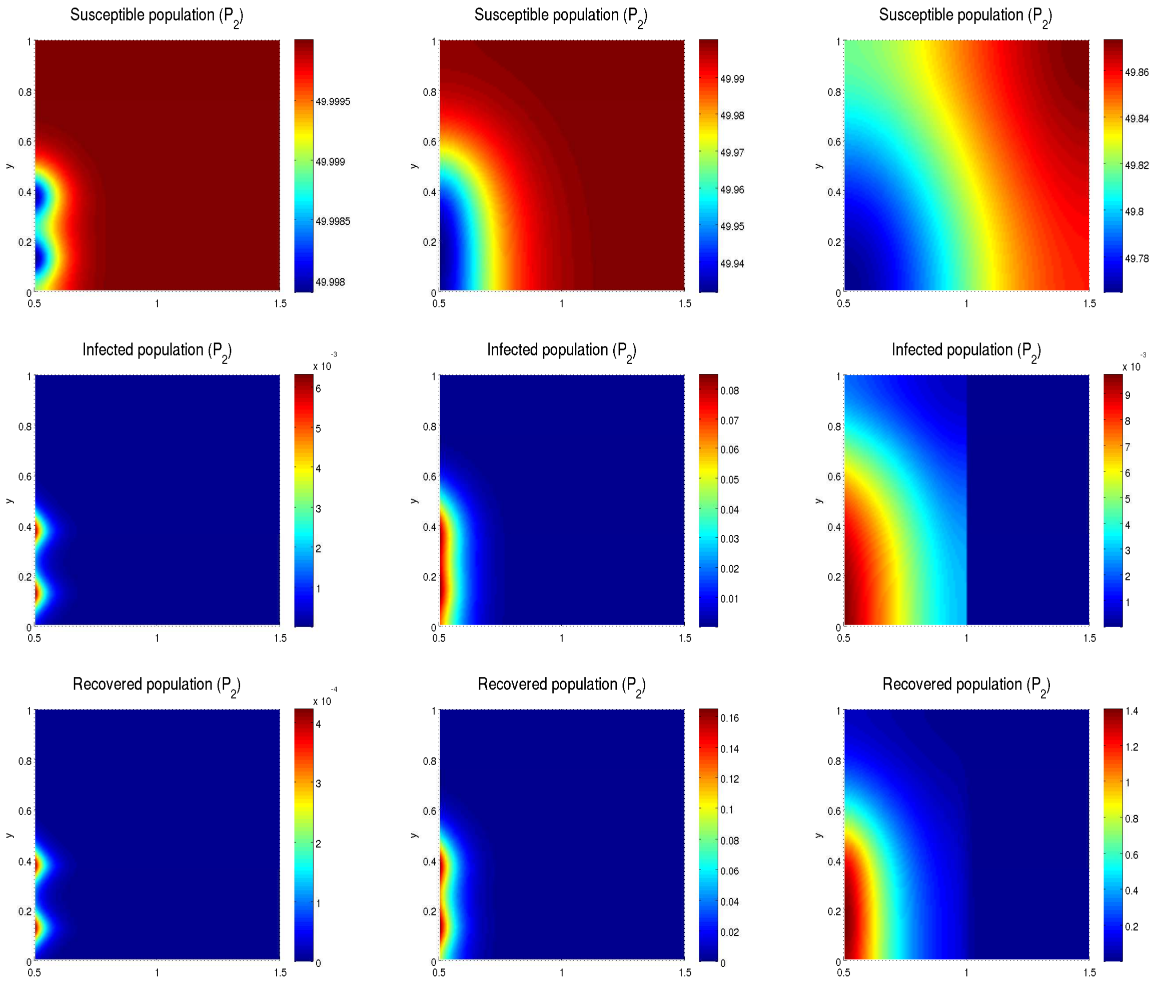
$$\partial_t u - d_{21}\Big(\int_{\Omega_2} u\,dx\Big)\Delta u - \operatorname{div}\Big(k_{21}u\nabla u\Big) = 0, \qquad \text{in } Q_{2,T}.$$

May be, these two equations have not the complexity of the original system of seven nonlinear equations that we are interested in studying, but for simplicity, we want to highlight the difference in behavior of the solutions for different types of diffusion, constant, non-local and cross-diffusion. In this sense our scenario is as follows: we simulate the meeting of the two susceptible populations

φ and u, although at the beginning both populations are separated. The population φ initially located at the left side of its domain, and the population u on the right side of its respective domain. For this, we assume that initially only susceptible populations are different from zero. More precisely, the initial data are given by

$$\varphi(x,y,0) = \begin{cases} 160 & x \in (0,0.25)\times(0.25,0.75) \\ 0 & \text{in other case} \end{cases}$$
$$u(x,y,0) = \begin{cases} 400 & x \in (1.25,1.5)\times(0.25,0.75) \\ 0 & \text{in other case} \end{cases}$$

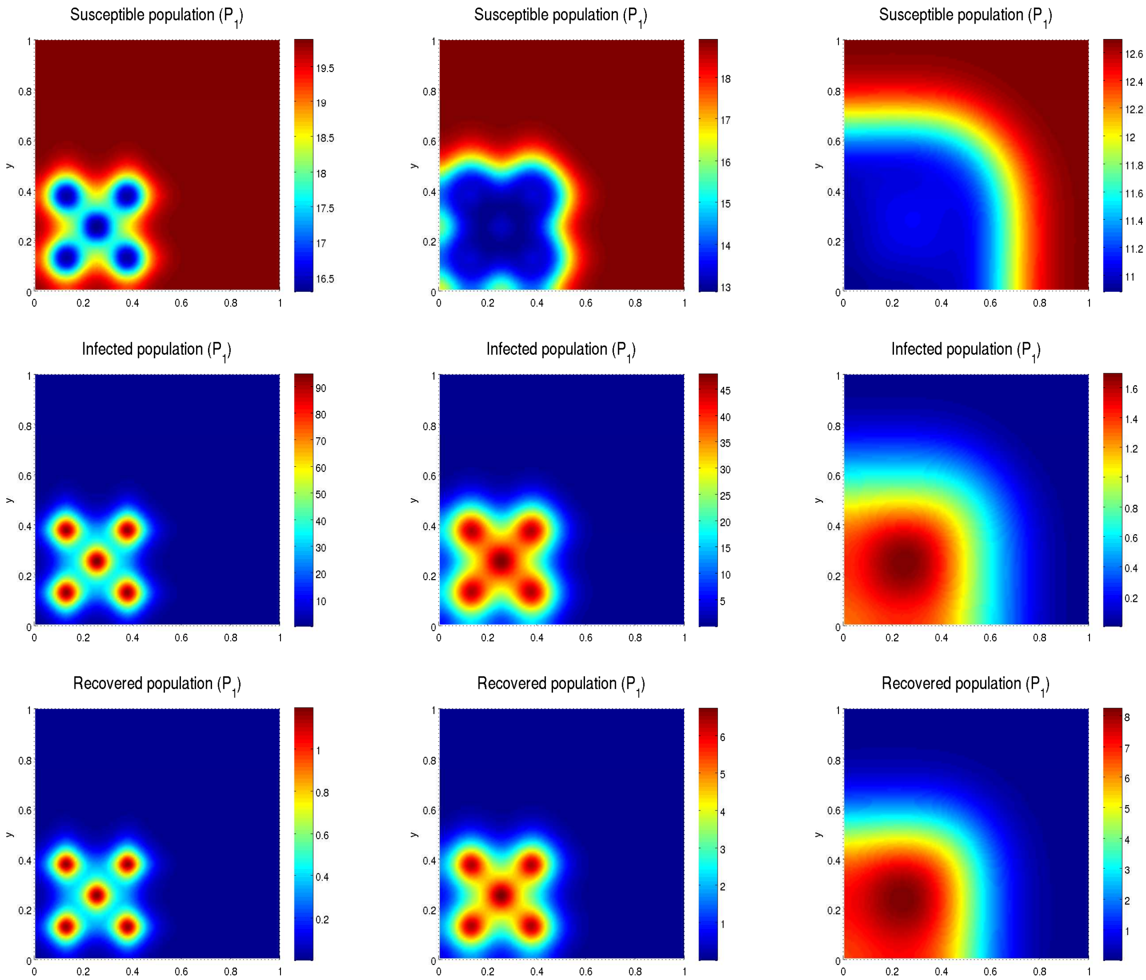
In figure 1 and 2, we can observe the dynamics of both populations for different times, $t = 0.01, 0.1, 1$. As time goes on, both populations begin to move toward the center of the domain $\Omega_1 \cap \Omega_2 = (0.5, 1.0) \times (0, 1)$ due to diffusion terms. The meeting is effective in all three cases, but is most evident in the case of nonlocal diffusion (column 2 of Figures 1 and 2). In the case of nonlocal diffusion, it increases in proportion to the total population of each susceptible population , generating a first contact between $P1$ and $P2$ about the time, $t = 0.1$ (see central pictures, in column 2 and row 2, for both Figures 1 and 2). Then, at time $t = 1.0$ both populations are thoroughly mixed.



a) $t = 0.01$ b) $t = 0.1$ c) $t = 0.01$

FIGURE 4. Example 2. Indirectly transmitted disease (P_2) with non-local diffusion, for different times. $t = 0.01, 0.1, 1$.

5.2. **Example 2. Indirectly transmitted disease.**

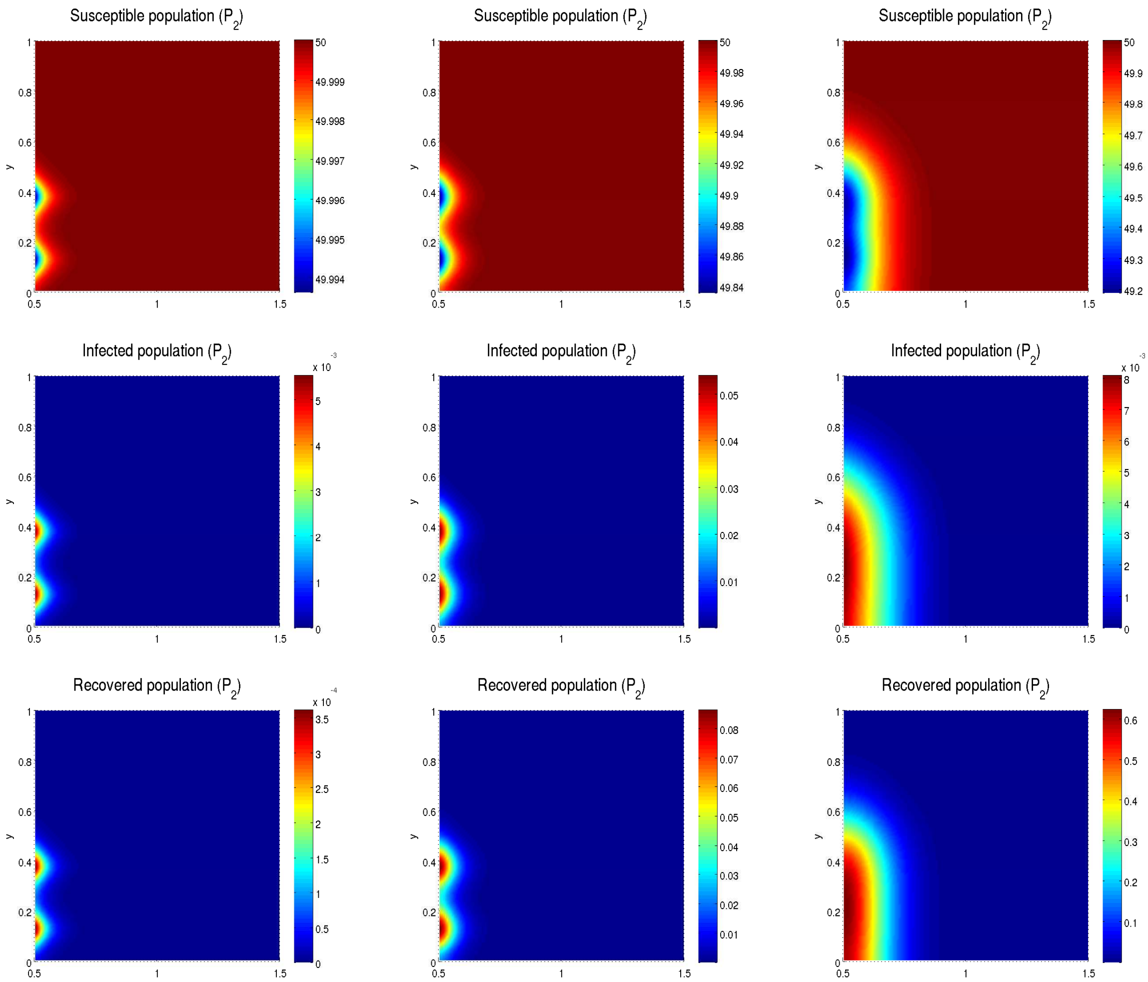


a) $t = 0.01$ b) $t = 0.1$ c) $t = 0.01$

FIGURE 5. Example 2. Indirectly transmitted disease (P_1) with cross-diffusion, for different times. $t = 0.01, 0.1, 1$.

In this Example 2, we want to study how localized sources of infection in the population $P1$, could be affect the population $P2$, which is not initially involved in the transmission of the disease, but who comes to be, due to the diffusive effects, and the pollutant which acts on the intersection of the domains.

For them, we consider initially constant population of susceptible individuals in each domain Ω_1 and Ω_2, and a focus of infected individuals at the left end of the domain Ω_1, modeled by sums



a) $t = 0.01$ b) $t = 0.1$ c) $t = 0.01$

FIGURE 6. Example 2. Indirectly transmitted disease (P_2) with cross-diffusion, for different times. $t = 0.01, 0.1, 1$.

of hyperbolic secant. More precisely, our initial condition in this case is given by

$$\varphi(x,y,0) = 20 \qquad (x,y) \in \Omega_1,$$
$$\psi(x,y,0) = 100 \sum_{1}^{5} sech(25(x-x_j))sech(25(y-y_j)); \qquad (x,y) \in \Omega_1,$$
$$\chi(x,y,0) = 0 \qquad (x,y) \in \Omega_1,$$

where $(x_1, y_1) = (0.252, 0.252)$, $(x_2, y_2) = (0.126, 0.126)$, $(x_3, y_3) = (0.126, 0.378)$, $(x_4, y_4) = (0.378, 0.126)$, and $(x_5, y_5) = (0.378, 0.378)$.
On the other hand, the initial conditions for u, v, w, are

$$u(x,y,0) = 50 \qquad (x,y) \in \Omega_2,$$
$$v(x,y,0) = 0 \qquad (x,y) \in \Omega_2,$$
$$w(x,y,0) = 0 \qquad (x,y) \in \Omega_2,$$

The initial condition for the contaminant is given by $c(x,y,0) = 0$ for all $(x,y) \in \Omega_1 \cup \Omega_2$.

These initial conditions for the both populations given before mean that at $t = 0$, for the first population, we assume that the susceptible population is constant in the domain and there is no presence of recovered population. Moreover, it is possible to observe, 5 pockets of high density infected population which are located in the quadrant $[0, 0.5] \times [0, 0.5]$ which will diffuse the epidemic disease on the rest of the domain contaminating the environment, for this reason the second population will be infected. Since at $t = 0$, for the second population, we assume that the susceptible population is constant in the entire domain and there is no presence of infected individuals nor presence of recovered individuals. We compare simulations between non-local diffusion and cross-diffusion.

First, in Figures 3, we can observe the effect of non local diffusion on Population 1 for different times. In Figure 4, we can observed the effect of non local and cross diffusion on Population 2 for different times. The difference is more perceptible for susceptible individuals, for infective and recovered individuals there is no much qualitative difference but it is possible to observed quantitative variations. It is important to note here that the population $P2$ is affected because infected individuals of the population $P1$ arrived to the area of interaction due to the diffusion, and the contaminant begins to act.

From figures 5 and 6, we observe the same for the cross-diffusion. On the other hand, 6 shows that in the case of cross diffusion $P1$ population, although equally altered by the disease is not so much as in the case of non-local diffusion (Figure 4). Observe also that there is discontinuity in the evolution of the population infected P2 (shown in column 3, row 2 and 3, Figure 4). This discontinuity is due to the contaminant acts strongly in the population infected, and only acts to the boundary $x = 1$. The non-local diffusion infected individual is not enough to observe migration of the population towards $x > 1$. However, the susceptible population becomes extinct smoothly (without apparent discontinuity) in the area of contamination, and this because of the migration of this same population due to the diffusion.

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Appendix A. Existence of weak solutions

A.1. **Existence of solutions for the approximate problems.** This subsection is devoted to proving existence of solutions to the approximate problem of systems (1.2)-(1.4). The existence proof is based on the Shauder fixed-point theorem, a priori estimates, and the compactness method. The approximation systems read:
(A.1)

$$\begin{cases} \partial_t\varphi - d_{11}\Big(\displaystyle\int_{\Omega_1}\varphi\,dx\Big)\Delta\varphi - \operatorname{div}\Big((k_{11}f_\varepsilon^+(\varphi)+f_\varepsilon^+(\psi)+f_\varepsilon^+(\chi))\nabla\varphi + f_\varepsilon^+(\varphi)\nabla\psi + f_\varepsilon^+(\varphi)\nabla\chi\Big) \\ \qquad = F_{1,\varepsilon}(x,\varphi^+,\psi^+,\chi^+,c^+), \\ \partial_t\psi - d_{12}\Big(\displaystyle\int_{\Omega_1}\psi\,dx\Big)\Delta\psi - \operatorname{div}\Big(f_\varepsilon^+(\psi)\nabla\varphi + (f_\varepsilon^+(\varphi)+k_{12}f_\varepsilon^+(\psi)+f_\varepsilon^+(\chi))\nabla\psi + f_\varepsilon^+(\psi)\nabla\chi\Big) \\ \qquad = F_{2,\varepsilon}(x,\varphi^+,\psi^+,\chi^+,c^+), \\ \partial_t\chi - d_{13}\Big(\displaystyle\int_{\Omega_1}\chi\,dx\Big)\Delta\chi - \operatorname{div}\Big(f_\varepsilon^+(\chi)\nabla\varphi + f_\varepsilon^+(\chi)\nabla\psi + (f_\varepsilon^+(\varphi)+f_\varepsilon^+(\psi)+k_{13}f_\varepsilon^+(\chi))\nabla\chi\Big) \\ \qquad = F_{3,\varepsilon}(x,\varphi^+,\psi^+,\chi^+,c^+), \end{cases}$$

in $Q_{1,T} = (0,T) \times \Omega_1$,
(A.2)

$$\begin{cases} \partial_t u - d_{21}\Big(\displaystyle\int_{\Omega_2} u\,dx\Big)\Delta u - \operatorname{div}\Big((k_{21} f_\varepsilon^+(u) + f_\varepsilon^+(v) + f_\varepsilon^+(w))\nabla u + f_\varepsilon^+(u)\nabla v + f_\varepsilon^+(u)\nabla w\Big) \\ \qquad\qquad = G_{1,\varepsilon}(x, u^+, v^+, w^+, c^+), \\ \partial_t v - d_{22}\Big(\displaystyle\int_{\Omega_2} v\,dx\Big)\Delta v - \operatorname{div}\Big(f_\varepsilon^+(v)\nabla u + (f_\varepsilon^+(u) + k_{22} f_\varepsilon^+(v) + f_\varepsilon^+(w))\nabla v + f_\varepsilon^+(v)\nabla w\Big) \\ \qquad\qquad = G_{2,\varepsilon}(x, u^+, v^+, w^+, c^+), \\ \partial_t w - d_{23}\Big(\displaystyle\int_{\Omega_2} w\,dx\Big)\Delta w - \operatorname{div}\Big(f_\varepsilon^+(w)\nabla u + f_\varepsilon^+(w)\nabla v + (f_\varepsilon^+(u) + f_\varepsilon^+(v) + k_{23} f_\varepsilon^+(w))\nabla w\Big) \\ \qquad\qquad = G_{3,\varepsilon}(x, u^+, v^+, w^+, c^+), \end{cases}$$

in $Q_{2,T} = (0,T) \times \Omega_2$, and

$$\partial_t c = K(x, \psi^+, v^+, c^+), \tag{A.3}$$

in $Q_T = (0,T) \times (\Omega_1 \cup \Omega_2)$. Herein, $\varepsilon > 0$ is a small number,

$$\begin{cases} F_{i,\varepsilon} = \dfrac{F_i}{1 + \varepsilon\,|F_i|} \text{ and } G_{i,\varepsilon} = \dfrac{G_i}{1 + \varepsilon\,|G_i|} \text{ for } i = 1,2,3, \\ \\ f_\varepsilon(a) = \dfrac{a}{1 + \varepsilon\,|a|} \text{ and } b^+ = \max(0,b) \text{ for any } a, b \in \mathbb{R}. \end{cases} \tag{A.4}$$

We supplement (A.1), (A.2) and (A.3) with no-flux boundary conditions (1.5)-(1.6) and initial data (1.7).

Observe that one can replace (A.1) and (A.2) by
(A.5)

$$\begin{cases} \partial_t \varphi - d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\,dx\Big)\Delta\varphi - \operatorname{div}(\alpha_{1,1}\nabla\varphi + \alpha_{1,2}\nabla\psi + \alpha_{1,3}\nabla\chi) = F_{1,\varepsilon}(x, \varphi^+, \psi^+, \chi^+, c^+), \\ \partial_t \psi - d_{12}\Big(\displaystyle\int_{\Omega_1} \psi\,dx\Big)\Delta\psi - \operatorname{div}(\alpha_{2,1}\nabla\varphi + \alpha_{2,2}\nabla\psi + \alpha_{2,3}\nabla\chi) = F_{2,\varepsilon}(x, \varphi^+, \psi^+, \chi^+, c^+), \\ \partial_t \chi - d_{13}\Big(\displaystyle\int_{\Omega_1} \chi\,dx\Big)\Delta\chi - \operatorname{div}(\alpha_{3,1}\nabla\varphi + \alpha_{3,2}\nabla\psi + \alpha_{3,3}\nabla\chi) = F_{3,\varepsilon}(x, \varphi^+, \psi^+, \chi^+, c^+), \end{cases}$$

and
(A.6)

$$\begin{cases} \partial_t u - d_{21}\Big(\displaystyle\int_{\Omega_2} u\,dx\Big)\Delta u - \operatorname{div}(\beta_{1,1}\nabla u + \beta_{1,2}\nabla v + \beta_{1,3}\nabla w) = G_{1,\varepsilon}(x, u^+, v^+, w^+, c^+), \\ \partial_t v - d_{22}\Big(\displaystyle\int_{\Omega_2} v\,dx\Big)\Delta v - \operatorname{div}(\beta_{2,1}\nabla u + \beta_{2,2}\nabla v + \beta_{2,3}\nabla w) = G_{2,\varepsilon}(x, u^+, v^+, w^+, c^+), \\ \partial_t w - d_{23}\Big(\displaystyle\int_{\Omega_2} w\,dx\Big)\Delta w - \operatorname{div}(\beta_{3,1}\nabla u + \beta_{3,2}\nabla v + \beta_{3,3}\nabla w) = G_{3,\varepsilon}(x, u^+, v^+, w^+, c^+), \end{cases}$$

respectively. Herein, the diffusion matrix $\mathcal{M}_i$ for $i = 1,2$,

$$\begin{aligned} \mathcal{M}_1 &= \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} (k_{11} f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi) + f_\varepsilon^+(\chi)) & f_\varepsilon^+(\varphi) & f_\varepsilon^+(\varphi) \\ f_\varepsilon^+(\psi) & (f_\varepsilon^+(\varphi) + k_{12} f_\varepsilon^+(\psi) + f_\varepsilon^+(\chi)) & f_\varepsilon^+(\psi) \\ f_\varepsilon^+(\chi) & f_\varepsilon^+(\chi) & (f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi) + k_{13} f_\varepsilon^+(\chi)) \end{pmatrix}, \end{aligned}$$

and

$$\mathcal{M}_2 = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{pmatrix}$$
$$= \begin{pmatrix} (k_{21} f_\varepsilon^+(u) + f_\varepsilon^+(v) + f_\varepsilon^+(w)) & f_\varepsilon^+(u) & f_\varepsilon^+(u) \\ f_\varepsilon^+(v) & (f_\varepsilon^+(u) + k_{22} f_\varepsilon^+(v) + f_\varepsilon^+(w)) & f_\varepsilon^+(v) \\ f_\varepsilon^+(w) & f_\varepsilon^+(w) & (f_\varepsilon^+(u) + f_\varepsilon^+(v) + k_{23} f_\varepsilon^+(w)) \end{pmatrix},$$

is uniformly nonnegative: using condition (1.9) and the inequality $ab \geq -\frac{a^2}{2} - \frac{b^2}{2}$ for all $a, b \in \mathbb{R}$ one gets:

(A.7)

$$\begin{aligned} \xi^T \mathcal{M}_1 \xi =& \Big(k_{11} f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi) + f_\varepsilon^+(\chi)\Big)\xi_1^2 + \Big(k_{12} f_\varepsilon^+(\psi) + f_\varepsilon^+(\varphi) + f_\varepsilon^+(\chi)\Big)\xi_2^2 \\ &+ \Big(k_{13} f_\varepsilon^+(\chi) + f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi)\Big)\xi_3^2 + f_\varepsilon^+(\varphi)(\xi_2 + \xi_3)\xi_1 \\ &+ f_\varepsilon^+(\psi)(\xi_1 + \xi_3)\xi_2 + f_\varepsilon^+(\chi)(\xi_1 + \xi_2)\xi_3 \\ \geq& \Big((k_{11} - 1) f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi)/2 + f_\varepsilon^+(\chi)/2\Big)\xi_1^2 + \Big(f_\varepsilon^+(\varphi)/2 + (k_{12} - 1) f_\varepsilon^+(\psi) f_\varepsilon^+(\chi)/2\Big)\xi_2^2 \\ &+ \Big(f_\varepsilon^+(\varphi)/2 + f_\varepsilon^+(\psi)/2 + (k_{13} - 1) f_\varepsilon^+(\chi)\Big)\xi_3^2 \\ \geq& c\Big(f_\varepsilon^+(\varphi) + f_\varepsilon^+(\psi) + f_\varepsilon^+(\chi)\Big)\Big(\xi_1^2 + \xi_2^2 + \xi_3^2\Big) \geq 0, \end{aligned}$$

and

(A.8)

$$\begin{aligned} \xi^T \mathcal{M}_2 \xi \geq& \Big((k_{21} - 1) f_\varepsilon^+(u) + f_\varepsilon^+(v)/2 + f_\varepsilon^+(w)/2\Big)\xi_1^2 + \Big(f_\varepsilon^+(u)/2 + (k_{22} - 1) f_\varepsilon^+(v) + f_\varepsilon^+(w)/2\Big)\xi_2^2 \\ &+ \Big(f_\varepsilon^+(u)/2 + f_\varepsilon^+(v)/2 + (k_{23} - 1) f_\varepsilon^+(w)\Big)\xi_3^2 \\ \geq& c\Big(f_\varepsilon^+(u) + f_\varepsilon^+(v) + f_\varepsilon^+(w)\Big)\Big(\xi_1^2 + \xi_2^2 + \xi_3^2\Big) \geq 0, \end{aligned}$$

for some constant $c > 0$ and for any $\xi = (\xi_1, \xi_2, \xi_3) \in R^3$. We shall frequently use (A.7) and (A.8) to prove the existence (and nonnegativity) of weak solutions.

A.2. **Existence result to the fixed problem.** In this subsection, we omit the dependence of the solutions on the parameter ε. We prove, for each fixed $\varepsilon > 0$, the existence of solutions to the fixed problem (A.1)-(A.2), by applying the Schauder fixed-point theorem. Since we use Schauder fixed-point theorem, we need to introduce the following closed subsets of the Banach space $L^2(Q_T, \mathbb{R}^n)$:

$$\mathcal{A}_i = \{U_i = (u_{i,1}, u_{i,2}, u_{i,3}) \in L^2(Q_{i,T}, \mathbb{R}^3) : \|U_i\|_{L^\infty(0,T;L^2(\Omega_i)) \cap L^2(0,T;H^1(\Omega_i))} \leq C_{\mathcal{A}_i}\}, \tag{A.9}$$

for $i = 1, 2$, where $C_{\mathcal{A}_1} > 0$ and $C_{\mathcal{A}_2} > 0$ are two constants that will be defined below. With $(\overline{\varphi}, \overline{\psi}, \overline{\chi}) \in \mathcal{A}_1$ and $(\overline{u}, \overline{v}, \overline{w}) \in \mathcal{A}_2$ fixed, let (φ, ψ, χ), (u, v, w) and c be the unique weak solution of the systems

(A.10)

$$\begin{cases} \partial_t \varphi - d_{11}\Big(\int_{\Omega_1} \overline{\varphi}\, dx\Big)\Delta\varphi - div\Big((k_{11} f_\varepsilon^+(\overline{\varphi}) + f_\varepsilon^+(\overline{\psi}) + f_\varepsilon^+(\overline{\chi}))\nabla\varphi + f_\varepsilon^+(\overline{\varphi})\nabla\psi + f_\varepsilon^+(\overline{\varphi})\nabla\chi\Big) \\ \qquad = F_{1,\varepsilon}(x, \overline{\varphi}^+, \overline{\psi}^+, \overline{\chi}^+, c^+), \\ \partial_t \psi - d_{12}\Big(\int_{\Omega_1} \overline{\psi}\, dx\Big)\Delta\psi - div\Big(f_\varepsilon^+(\overline{\psi})\nabla\varphi + (f_\varepsilon^+(\overline{\varphi}) + k_{12} f_\varepsilon^+(\overline{\psi}) + f_\varepsilon^+(\overline{\chi}))\nabla\psi + f_\varepsilon^+(\overline{\psi})\nabla\chi\Big) \\ \qquad = F_{2,\varepsilon}(x, \overline{\varphi}^+, \overline{\psi}^+, \overline{\chi}^+, c^+), \\ \partial_t \chi - d_{13}\Big(\int_{\Omega_1} \overline{\chi}\, dx\Big)\Delta\chi - div\Big(f_\varepsilon^+(\overline{\chi})\nabla\varphi + f_\varepsilon^+(\overline{\chi})\nabla\psi + (f_\varepsilon^+(\overline{\varphi}) + f_\varepsilon^+(\overline{\psi}) + k_{13} f_\varepsilon^+(\overline{\chi}))\nabla\chi\Big) \\ \qquad = F_{3,\varepsilon}(x, \overline{\varphi}^+, \overline{\psi}^+, \overline{\chi}^+, c^+), \end{cases}$$

in $Q_{1,T} = (0,T)\times\Omega_1$,

(A.11)
$$\begin{cases}\partial_t u - d_{21}\Big(\displaystyle\int_{\Omega_2} \overline{u}\,dx\Big)\Delta u - div\Big((k_{21} f_\varepsilon^+(\overline{u}) + f_\varepsilon^+(\overline{v}) + f_\varepsilon^+(\overline{w}))\nabla u + f_\varepsilon^+(\overline{u})\nabla v + f_\varepsilon^+(\overline{u})\nabla w\Big)\\ \qquad\qquad = G_{1,\varepsilon}(x,\overline{u}^+,\overline{v}^+,\overline{w}^+,c^+),\\ \partial_t v - d_{22}\Big(\displaystyle\int_{\Omega_2} \overline{v}\,dx\Big)\Delta v - div\Big(f_\varepsilon^+(\overline{v})\nabla u + (f_\varepsilon^+(\overline{u}) + k_{22} f_\varepsilon^+(\overline{v}) + f_\varepsilon^+(\overline{w}))\nabla v + f_\varepsilon^+(\overline{v})\nabla w\Big)\\ \qquad\qquad = G_{2,\varepsilon}(x,\overline{u}^+,\overline{v}^+,\overline{w}^+,c^+),\\ \partial_t w - d_{23}\Big(\displaystyle\int_{\Omega_2} \overline{w}\,dx\Big)\Delta w - div\Big(f_\varepsilon^+(\overline{w})\nabla u + f_\varepsilon^+(\overline{w})\nabla v + (f_\varepsilon^+(\overline{u}) + f_\varepsilon^+(\overline{v}) + k_{23} f_\varepsilon^+(\overline{w}))\nabla w\Big)\\ \qquad\qquad = G_{3,\varepsilon}(x,\overline{u}^+,\overline{v}^+,\overline{w}^+,c^+),\end{cases}$$

in $Q_{2,T} = (0,T)\times\Omega_2$, and

$$\partial_t c = K(x,\overline{\psi}^+,\overline{v}^+,c^+), \tag{A.12}$$

in $Q_T = (0,T)\times(\Omega_1\cup\Omega_2)$.

Observe that for any fixed $\overline{\psi}\in L^2(Q_{1,T})$ and $\overline{v}\in L^2(Q_{2,T})$, problem (A.12) is uniformly ODE, so we have immediately:

Lemma A.1. *If $c_0\in L^\infty_+(\Omega)$ and $0\le c_0\le 1$, then (A.12) has a unique solution $c\in L^\infty_+(Q_T)\cap C(0,T;L^2(\Omega))$, satisfying:*

$$\begin{aligned}&0\le c(t,x)\le 1 \text{ for a.e. } (t,x)\in Q_T,\\ &\|c\|_{C(0,T;L^2(\Omega))}\le C,\end{aligned} \tag{A.13}$$

where $C>0$ is a constant which depends only on $\|c_0\|_{L^\infty(\Omega)}$, $\|\sigma_{13}\|_{L^\infty(Q_{1,T})}$, $\|\sigma_{23}\|_{L^\infty(Q_{2,T})}$, $\|\overline{\psi}\|_{L^2(Q_{1,T})}$, $\|c\|_{L^2(Q_{1,T})}$ and $|Q_T|$.

Remark A.1. *Note that the first estimate in* (A.13) *follows from the maximum principle.*

A.2.1. *The fixed-point method.* Now, we introduce a map $L_i:\mathcal{A}_i\to\mathcal{A}_i$ for $i=1,2$ such that $L_1(\overline{\varphi},\overline{\psi},\overline{\chi}) = (\varphi,\psi,\chi)$ and $L_2(\overline{u},\overline{v},\overline{w}) = (u,v,w)$, where (φ,ψ,χ) and (u,v,w) solve (A.10) and (A.11) respectively. By using the Schauder fixed-point theorem, we prove that the maps L_1 and L_2 have a fixed point for (A.10) and (A.11).

First, let us show that L_i is a continuous mapping for $i=1,2$. For this, letting $(\overline{\varphi}_\ell,\overline{\psi}_\ell,\overline{\chi}_\ell)_\ell$ and $(\overline{u}_\ell,\overline{v}_\ell,\overline{w}_\ell)_\ell$ be sequences in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Next, we let $(\overline{\varphi},\overline{\psi},\overline{\chi})\in\mathcal{A}_1$ and $(\overline{u},\overline{v},\overline{w})\in\mathcal{A}_2$ be such that $(\overline{\varphi}_\ell,\overline{\psi}_\ell,\overline{\chi}_\ell)_\ell\to(\overline{\varphi},\overline{\psi},\overline{\chi})$ in $L^2(Q_{1,T},\mathbb{R}^3)$ and $(\overline{u}_\ell,\overline{v}_\ell,\overline{w}_\ell)_\ell\to(u,v,w)$ in $L^2(Q_{2,T},\mathbb{R}^3)$ as $\ell\to\infty$. Define $(\varphi_\ell,\psi_\ell,\chi_\ell) = L_1(\overline{\varphi}_\ell,\overline{\psi}_\ell,\overline{\chi}_\ell)$ and $(u_\ell,v_\ell,w_\ell) = L_2(\overline{u}_\ell,\overline{v}_\ell,\overline{w}_\ell)$. The goal is to show that $(\varphi_\ell,\psi_\ell,\chi_\ell)$ converges to $L_1(\overline{\varphi},\overline{\psi},\overline{\chi})$ in $L^2(Q_{1,T},\mathbb{R}^3)$ and (u_ℓ,v_ℓ,w_ℓ) converges to $L_2(\overline{u},\overline{v},\overline{w})$ in $L^2(Q_{2,T},\mathbb{R}^3)$. Next, we need the following lemma:

Lemma A.2. *The solutions $(\varphi_\ell,\psi_\ell,\chi_\ell)$ and (u_ℓ,v_ℓ,w_ℓ) to systems (A.10) and (A.11) respectively satisfy:*

(i) *The sequences $(\varphi_\ell,\psi_\ell,\chi_\ell)_\ell$ and $(u_\ell,v_\ell,w_\ell)_\ell$ are bounded in $L^2(0,T;H^1(\Omega_1,\mathbb{R}^3))\cap L^\infty(0,T;L^2(\Omega_1,\mathbb{R}^3))$ and in $L^2(0,T;H^1(\Omega_2,\mathbb{R}^3))\cap L^\infty(0,T;L^2(\Omega_2,\mathbb{R}^3))$, respectively.*
(ii) *The sequences $(\varphi_\ell,\psi_\ell,\chi_\ell)_\ell$ and $(u_\ell,v_\ell,w_\ell)_\ell$ are relatively compact in $L^2(Q_{1,T},\mathbb{R}^3)$ and in $L^2(Q_{2,T},\mathbb{R}^3)$, respectively.*

Proof. (*i*) We multiply the first, the second and the third equation in (A.10) by φ_ℓ, ψ_ℓ and χ_ℓ respectively, integrate over Ω_1, using (A.7), and definition of $F_{i,\varepsilon}$ in A.4, yields

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_1}\Big(|\varphi_\ell|^2+|\psi_\ell|^2+|\chi_\ell|^2\Big)dx + d\int_{\Omega_1}\Big(|\nabla\varphi_\ell|^2+|\nabla\psi_\ell|^2+|\nabla\chi_\ell|^2\Big)dx\\ &\qquad\le C\int_{\Omega_1}\Big(|\varphi_\ell|^2+|\psi_\ell|^2+|\chi_\ell|^2\Big)dx,\end{aligned} \tag{A.14}$$

for some constant $C > 0$. Herein, $d = \min(M_{11}, M_{12}, M_{13})$ (recall that M_{ij} is defined in (1.1) for $i = 1, 2$ and $j = 1, 2, 3$). In view of Gronwall's inequality it follows from (A.14) that,

$$\sup_{t\in(0,T)} \int_{\Omega_1} \left(|\varphi_\ell|^2 + |\psi_\ell|^2 + |\chi_\ell|^2\right) dx \le \exp(CT) \|\varphi_0 + \psi_0 + \chi_0\|_{L^2(\Omega_1)}, \tag{A.15}$$

which proves the first part of (i).
From (A.14) and (A.15) one may also conclude that,

$$\int\int_{Q_{1,T}} (|\nabla\varphi_\ell|^2 + |\nabla\psi_\ell|^2 + |\nabla\chi_\ell|^2) dx\, dt \le \frac{T\exp(CT)}{d} \|\varphi_0 + \psi_0 + \chi_0\|_{L^2(\Omega_1)}, \tag{A.16}$$

yielding (i).

(*ii*) Finally multiplying the first, the second and the third equation (A.10) by $\varphi_1, \varphi_2, \varphi_3 \in L^2(0, T; H^1(\Omega))$, respectively and using the boundedness of f_ε^+ and $F_{i,\varepsilon}$ for $i = 1, 2, 3$, and (A.16) there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} &\left|\int_0^T \langle \partial_t\varphi_\ell, \varphi_1 \rangle\, dt\right| + \left|\int_0^T \langle \partial_t\psi_\ell, \varphi_2 \rangle\, dt\right| + \left|\int_0^T \langle \partial_t\chi_\ell, \varphi_3 \rangle\, dt\right| \\ &\quad \le C(\varepsilon) \sum_{i=1}^{3} \|\varphi_i\|_{L^2(0,T;H^1(\Omega))}. \end{aligned} \tag{A.17}$$

so we get (ii) for $(\varphi_\ell, \psi_\ell, \chi_\ell)_\ell$.
Then, (ii) is a consequence of (i) and the uniform boundedness (A.17) of $(\varphi_\ell, \psi_\ell, \chi_\ell)_\ell$ in $L^2(0, T; (H^1(\Omega_1, \mathbb{R}^3))')$
Reasoning along the same lines for $(\varphi_\ell, \psi_\ell, \chi_\ell)_\ell$ yield (i) and (ii) for $(u_\ell, v_\ell, w_\ell)_\ell$.

□

Remark A.2. *Note that it is easy to deduce from Lemma A.2 that the constants $C_{\mathcal{A}_1} > 0$ and $C_{\mathcal{A}_2} > 0$ (consult (A.9)) are defined as follows:*

$$C_{\mathcal{A}_1} = \frac{(d + T)\exp(CT)}{d} \|\varphi_0 + \psi_0 + \chi_0\|_{L^2(\Omega_1)}$$

and

$$C_{\mathcal{A}_2} = \frac{(d' + T)\exp(CT)}{d'} \|u_0 + v_0 + w_0\|_{L^2(\Omega_2)}$$

for some constant $C > 0$. Herein, $d' = \min(M_{21}, M_{22}, M_{23})$.

From Lemma A.2, there exist functions $(\varphi_\ell, \psi_\ell, \chi_\ell) \in L^2(0, T; H^1(\Omega_1, \mathbb{R}^3))$ and $(u_\ell, v_\ell, w_\ell) \in L^2(0, T; H^1(\Omega_2, \mathbb{R}^3))$such that, up to extracting subsequences if necessary,

$$(\varphi_\ell, \psi_\ell, \chi_\ell) \to (\varphi, \psi, \chi) \text{ in } (L^2(Q_{1,T}))^3 \text{ strongly}, \qquad (u_\ell, v_\ell, w_\ell) \to (u, v, w) \text{ in } (L^2(Q_{2,T}))^3 \text{ strongly},$$

and from this the continuity of L_i on $\mathcal{A}_i$ follows for $i = 1, 2$.

We observe that, from Lemma A.2, $L_i(\mathcal{A}_i)$ is bounded in the set

$$\mathcal{E}_i = \left\{u \in L^2(0, T; H^1(\Omega_i, \mathbb{R}^3)) \;:\; \partial_t u \in L^2(0, T; (H^1(\Omega_i, \mathbb{R}^3))^*)\right\}, \tag{A.18}$$

for $i = 1, 2$. By the results of [32], $\mathcal{E}_i \hookrightarrow L^2(Q_{i,T}, \mathbb{R}^3)$ is compact for $i = 1, 2$, thus L_i is compact for $i = 1, 2$. Now, by the Schauder fixed point theorem, the operators L_1 and L_2 have a fixed points $(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$, respectively, such that $L_1(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon) = (\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$ and $L_2(u_\varepsilon, v_\varepsilon, w_\varepsilon) = (u_\varepsilon, v_\varepsilon, w_\varepsilon)$. Then there exists a solution $(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$, $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ and c_ε of

(A.19)

$$\begin{aligned} &\int_0^T \langle \partial_t\varphi_\varepsilon, \phi_1\rangle_1\, dt + \int\int_{Q_{1,T}} \Big(d_{11}\Big(\int_{\Omega_1} \varphi_\varepsilon\, dx\Big)\nabla\varphi_\varepsilon + (k_{11} f_\varepsilon^+(\varphi_\varepsilon) + f_\varepsilon^+(\psi_\varepsilon) + f_\varepsilon^+(\chi_\varepsilon))\nabla\varphi_\varepsilon + f_\varepsilon^+(\varphi_\varepsilon)\nabla\psi_\varepsilon \\ &\qquad + f_\varepsilon^+(\varphi_\varepsilon)\nabla\chi_\varepsilon\Big)\cdot\nabla\phi_1\, dx\, dt = \int\int_{Q_{1,T}} F_{1,\varepsilon}(x, \varphi_\varepsilon^+, \psi_\varepsilon^+, \chi_\varepsilon^+, c)\phi_1\, dx\, dt, \end{aligned}$$

(A.20)
$$\int_0^T \langle \partial_t \psi_\varepsilon, \phi_2\rangle_1 \, dt + \int\int_{Q_{1,T}} \Big(d_{12}\Big(\int_{\Omega_1} \psi_\varepsilon \, dx\Big)\nabla\psi_\varepsilon + f_\varepsilon^+(\psi_\varepsilon)\nabla\varphi_\varepsilon + (f_\varepsilon^+(\varphi_\varepsilon) + k_{12} f_\varepsilon^+(\psi_\varepsilon) + f_\varepsilon^+(\chi_\varepsilon))\nabla\psi_\varepsilon + f_\varepsilon^+(\psi_\varepsilon)\nabla\chi_\varepsilon\Big)\cdot\nabla\phi_2 \, dx\, dt = \int\int_{Q_{1,T}} F_{2,\varepsilon}(x,\varphi_\varepsilon^+,\psi_\varepsilon^+,\chi_\varepsilon^+,c)\phi_2 \, dx\, dt,$$

(A.21)
$$\int_0^T \langle \partial_t \chi_\varepsilon, \phi_3\rangle_1 \, dt + \int\int_{Q_{1,T}} \Big(d_{13}\Big(\int_{\Omega_1} \chi_\varepsilon \, dx\Big)\nabla\chi_\varepsilon + f_\varepsilon^+(\chi_\varepsilon)\nabla\varphi_\varepsilon + f_\varepsilon^+(\chi_\varepsilon)\nabla\psi_\varepsilon + (f_\varepsilon^+(\varphi_\varepsilon) + f_\varepsilon^+(\psi_\varepsilon) + k_{13} f_\varepsilon^+(\chi_\varepsilon))\nabla\chi_\varepsilon\Big)\cdot\nabla\phi_2 \, dx\, dt = \int\int_{Q_{1,T}} F_{3,\varepsilon}(x,\varphi_\varepsilon^+,\psi_\varepsilon^+,\chi_\varepsilon^+,c)\phi_3 \, dx\, dt,$$

(A.22)
$$\int_0^T \langle \partial_t u_\varepsilon, \Theta_1\rangle_2 \, dt + \int\int_{Q_{2,T}} \Big(d_{21}\Big(\int_{\Omega_2} u_\varepsilon \, dx\Big)\nabla u_\varepsilon + (k_{21} f_\varepsilon^+(u_\varepsilon) + f_\varepsilon^+(v_\varepsilon) + f_\varepsilon^+(w_\varepsilon))\nabla u_\varepsilon + f_\varepsilon^+(u_\varepsilon)\nabla v_\varepsilon + f_\varepsilon^+(u_\varepsilon)\nabla w_\varepsilon\Big)\cdot\nabla\Theta_1 \, dx\, dt = \int\int_{Q_{2,T}} G_{1,\varepsilon}(x,u_\varepsilon^+,v_\varepsilon^+,w_\varepsilon^+,c)\Theta_1 \, dx\, dt,$$

(A.23)
$$\int_0^T \langle \partial_t v_\varepsilon, \Theta_2\rangle_2 \, dt + \int\int_{Q_{2,T}} \Big(d_{22}\Big(\int_{\Omega_2} v_\varepsilon \, dx\Big)\nabla v_\varepsilon + f_\varepsilon^+(v_\varepsilon)\nabla u_\varepsilon + (f_\varepsilon^+(u_\varepsilon) + k_{22} f_\varepsilon^+(v_\varepsilon) + f_\varepsilon^+(w_\varepsilon))\nabla v_\varepsilon + f_\varepsilon^+(v_\varepsilon)\nabla w_\varepsilon\Big)\cdot\nabla\Theta_2 \, dx\, dt = \int\int_{Q_{2,T}} G_{2,\varepsilon}(x,u_\varepsilon^+,v_\varepsilon^+,w_\varepsilon^+,c)\Theta_2 \, dx\, dt,$$

(A.24)
$$\int_0^T \langle \partial_t w_\varepsilon, \Theta_3\rangle_2 \, dt + \int\int_{Q_{2,T}} \Big(d_{23}\Big(\int_{\Omega_2} w_\varepsilon \, dx\Big)\nabla w_\varepsilon + f_\varepsilon^+(w_\varepsilon)\nabla u_\varepsilon + f_\varepsilon^+(w_\varepsilon)\nabla v_\varepsilon + (f_\varepsilon^+(u_\varepsilon) + f_\varepsilon^+(v_\varepsilon) + k_{23} f_\varepsilon^+(w_\varepsilon))\nabla w_\varepsilon\Big)\cdot\nabla\Theta_3 \, dx\, dt = \int\int_{Q_{2,T}} G_{3,\varepsilon}(x,u_\varepsilon^+,v_\varepsilon^+,w_\varepsilon^+,c)\Theta_3 \, dx\, dt,$$

(A.25)
$$\int\int_{Q_T} \partial_t c_\varepsilon \,\Gamma \, dx\, dt = \int\int_{Q_T} K(x,\psi_\varepsilon,v_\varepsilon,c_\varepsilon)\Gamma \, dx\, dt,$$

for all $\phi_i \in L^2(0,T;H^1(\Omega_1))$, $\Theta_i \in L^2(0,T;H^1(\Omega_2))$ for $i = 1,2,3$, and $\Gamma \in L^2(0,T;H^1(\Omega_1\cup\Omega_2))$.

A.3. **Existence of weak solutions.** Note that since problem (A.12) is uniformly ODE, the estimates (A.13) holds with c replaced by c_ε.

From Section A.1, we know there exist sequences $(\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon)_{\varepsilon>0}$, $(u_\varepsilon,v_\varepsilon,w_\varepsilon)_{\varepsilon>0}$ and c_ε of solutions to (A.1), (A.2), (A.3). We have now the following series of a priori estimates.

Lemma A.3. *Assume conditions (1.9) and (2.1)-(2.3) hold. If $\varphi_0,\psi_0,\chi_0 \in L^2_+(\Omega_1)$ and $u_0,v_0,w_0 \in L^2_+(\Omega_2)$, then the solutions $(\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon)$ and $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ are nonnegative. Moreover, there exist constants $c_1,c_2,c_3,c_4 > 0$ not depending on ε such that*

(A.26)
$$\|(\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon)\|_{L^\infty(0,T;L^2(\Omega_1,\mathbb{R}^3))} + \|(u_\varepsilon,v_\varepsilon,w_\varepsilon)\|_{L^\infty(0,T;L^2(\Omega_2,\mathbb{R}^3))} \le c_1,$$

(A.27)
$$\|F_{i,\varepsilon}(\cdot,\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon,c_\varepsilon)\|_{L^1(Q_{1,T})} + \|G_{i,\varepsilon}(\cdot,u_\varepsilon,v_\varepsilon,w_\varepsilon,c_\varepsilon)\|_{L^1(Q_{2,T})} \le c_2,$$

for $i = 1,2,3$,

(A.28)
$$\|\nabla\varphi_\varepsilon\|_{L^2(Q_{1,T})} + \|\nabla\psi_\varepsilon\|_{L^2(Q_{1,T})} + \|\nabla\chi_\varepsilon\|_{L^2(Q_{1,T})} + \|\nabla u_\varepsilon\|_{L^2(Q_{2,T})} + \|\nabla v_\varepsilon\|_{L^2(Q_{2,T})} + \|\nabla w_\varepsilon\|_{L^2(Q_{2,T})} \le c_3,$$

(A.29)
$$\|\partial_t\varphi_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_1))^*)} + \|\partial_t\psi_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_1))^*)} + \|\partial_t\chi_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_1))^*)} + \|\partial_t u_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_2))^*)} + \|\partial_t v_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_2))^*)} + \|\partial_t w_\varepsilon\|_{L^2(0,T,(W^{1,\infty}(\Omega_2))^*)} \le c_4.$$

Proof. In the weak formulation (A.19)-(A.21) we take $\phi_1 = -\varphi_\varepsilon^-$, $\phi_2 = -\psi_\varepsilon^-$ and $\phi_3 = -\chi_\varepsilon^-$, and we integrate over Ω_1 instead $Q_{1,T}$, we get from (A.7)

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_1}\Big(\left|\varphi_\varepsilon^-\right|^2 + \left|\psi_\varepsilon^-\right|^2 + \left|\chi_\varepsilon^-\right|^2\Big)\,dx \le 0. \tag{A.30}$$

This yields the nonnegativity of $(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$. Reasoning along the same lines for $(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$ yields the nonnegativity of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$.

Observe that

$$|F_{1,\varepsilon}(\cdot,\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon,c)| + |F_{2,\varepsilon}(\cdot,\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon,c)| + |F_{3,\varepsilon}(\cdot,\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon,c)| \le C(|\varphi_\varepsilon|^2 + |\psi_\varepsilon|^2 + |\chi_\varepsilon|^2),$$

and

$$|G_{1,\varepsilon}(\cdot,u_\varepsilon,v_\varepsilon,w_\varepsilon,c)| + |G_{2,\varepsilon}(\cdot,u_\varepsilon,v_\varepsilon,w_\varepsilon,c)| + |G_{3,\varepsilon}(\cdot,u_\varepsilon,v_\varepsilon,w_\varepsilon,c)| \le C(|u_\varepsilon|^2 + |v_\varepsilon|^2 + |w_\varepsilon|^2),$$

for some constant $C > 0$. Now we exploit this and (A.26) to deduce (A.27)

By the (weak) lower semicontinuity properties of norms, the estimates (A.15) and (A.16) hold with $(\varphi_\ell, \psi_\ell, \chi_\ell)$ and (u_ℓ, v_ℓ, w_ℓ) replaced by $(\varphi_\varepsilon, \psi_\varepsilon, \chi_\varepsilon)$ and $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$. Moreover, the constants c_1, c_3 are independent of ε (consult the proof of Lemma A.2).

Finally using the weak formulation (A.19), we deduce from (A.26) and (A.28): for all $\phi_1 \in L^2(0,T;W^{1,\infty}(\Omega_1))$

(A.31)
$$\begin{aligned}
&\left|\int_0^T \langle \partial_t\varphi_\varepsilon, \phi_1\rangle_1\,dt\right| \\
&\quad\le \sup_{t\in[0,T]}\left|d_{11}\Big(\int_{\Omega_1}\varphi_\varepsilon\,dx\Big)\right| \|\nabla\varphi_\varepsilon\|_{L^2(Q_{1,T})}\|\nabla\phi_1\|_{L^2(Q_{1,T})} \\
&\qquad + C\Big(\|\varphi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\psi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))}\Big) \\
&\qquad\quad \times\Big(\|\nabla\varphi_\varepsilon\|_{L^2(Q_{1,T})} + \|\nabla\psi_\varepsilon\|_{L^2(Q_{1,T})} + \|\nabla\chi_\varepsilon\|_{L^2(Q_{1,T})}\Big)\|\nabla\phi_1\|_{L^2(0,T;L^\infty(\Omega_1))} \\
&\qquad\quad + C'\Big(1 + \|\varphi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\psi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_1))}\Big) \\
&\qquad\quad \times\Big(1 + \|\varphi_\varepsilon\|_{L^2(Q_{1,T})} + \|\psi_\varepsilon\|_{L^2(Q_{1,T})} + \|\chi_\varepsilon\|_{L^2(Q_{1,T})}\Big)\|\phi_1\|_{L^2(0,T;L^\infty(\Omega_1))} \\
&\quad\le C''\,\|\phi_1\|_{L^2(0,T;W^{1,\infty}(\Omega_1))},
\end{aligned}$$

for some constant $C, C', C'' > 0$ independent of ε. From this we deduce the bound

$$\|\partial_t\varphi_\varepsilon\|_{L^2(0,T;(W^{1,\infty}(\Omega))^*)} \le C''. \tag{A.32}$$

Reasoning along the same lines for φ_ε yields (A.32) for ψ_ε, χ_ε, u_ε, v_ε and w_ε . □

In view of Lemma A.3 and Aubin's lemma, we can assume there exist limit functions $(\varphi, \psi, \chi, u, v, w)$ such that as $\varepsilon \to 0$ the following convergences hold (modulo extraction of subsequences, which we do not bother to relabel):

(A.33)
$$\begin{cases}
(\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon) \to (\varphi,\psi,\chi) \text{ a.e. in } Q_{1,T}, \text{ and strongly in } L^2(Q_{1,T},\mathbb{R}^3), \text{ weakly in } L^2(0,T;H^1(\Omega_1,\mathbb{R}^3)), \\
(u_\varepsilon,v_\varepsilon,w_\varepsilon) \to (u,v,w) \text{ a.e. in } Q_{2,T}, \text{ and strongly in } L^2(Q_{2,T},\mathbb{R}^3), \text{ weakly in } L^2(0,T;H^1(\Omega_2,\mathbb{R}^3)), \\
F_{i,\varepsilon}(\cdot,\varphi_\varepsilon,\psi_\varepsilon,\chi_\varepsilon,c_\varepsilon) \to F_i(\cdot,\varphi,\psi,\chi,c) \text{ a.e. in } Q_{1,T} \text{ and strongly in } L^1(Q_{1,T}) \\
\text{ and } G_{i,\varepsilon}(\cdot,u_\varepsilon,v_\varepsilon,w_\varepsilon,c_\varepsilon) \to G_i(\cdot,u,v,w,c) \text{ a.e. in } Q_{2,T} \text{ and strongly in } L^2(Q_{2,T}),
\end{cases}$$

for $i = 1, 2, 3$. Additionally, $(\partial_t\varphi_\varepsilon, \partial_t\psi_\varepsilon, \partial_t\chi_\varepsilon) \to (\partial_t\varphi, \partial_t\psi, \partial_t\chi)$ and $(\partial_t u_\varepsilon, \partial_t v_\varepsilon, \partial_t w_\varepsilon) \to (\partial_t u, \partial_t v, \partial_t w)$ weakly in $L^2(0,T;(W^{1,\infty}(\Omega_1,\mathbb{R}^3))^*)$ and $L^2(0,T;(W^{1,\infty}(\Omega_2,\mathbb{R}^3))^*)$, respectively.

An application of Young and Hölder inequalities we get

$$
\begin{aligned}
\|f_\varepsilon(\varphi_\varepsilon)-\varphi\|_{L^2(Q_{1,T})} &\le \sqrt{2}\,\|\varphi_\varepsilon-\varphi\|_{L^2(Q_{1,T})}+\sqrt{2}\left\|\frac{\varepsilon\varphi_\varepsilon\varphi}{1+\varepsilon\varphi_\varepsilon}\right\|_{L^2(Q_{1,T})}\\
&\le \sqrt{2}\,\|\varphi_\varepsilon-\varphi\|_{L^2(Q_{1,T})}+\sqrt{2}\left\|\frac{\varepsilon\varphi_\varepsilon\varphi}{(1+\varepsilon\varphi_\varepsilon)^{2/3}(\varepsilon\varphi_\varepsilon)^{1-2/3}}\right\|_{L^2(Q_{1,T})}\\
&\le \sqrt{2}\,\|\varphi_\varepsilon-\varphi\|_{L^2(Q_{1,T})}+\sqrt{2}\varepsilon^{2/3}\left\|\varphi_\varepsilon^{2/3}\varphi\right\|_{L^2(Q_{1,T})}\\
&\le \sqrt{2}\,\|\varphi_\varepsilon-\varphi\|_{L^2(Q_{1,T})}+\sqrt{2}\varepsilon^{2/3}\,\|\varphi_\varepsilon\|^{2/3}_{L^\infty(0,T;L^2(\Omega_1))}\\
&\qquad\times\|\varphi_\varepsilon\|_{L^2(0,T;L^6(\Omega_1))}.
\end{aligned}
\tag{A.34}
$$

Thanks to the Sobolev embedding ($H^1(\Omega_1)\subset L^6(\Omega_1)$) we deduce from (A.34)

$$
f_\varepsilon(\varphi_\varepsilon)\to\varphi \text{ a.e. in } Q_{1,T} \text{ and strongly in } L^r(Q_{1,T}) \text{ for all } 1\le r\le 2.
\tag{A.35}
$$

In the same way we get

$$
\begin{aligned}
&(f_\varepsilon(\psi_\varepsilon),f_\varepsilon(\chi_\varepsilon))\to(\psi,\chi) \text{ a.e. in } Q_{1,T} \text{ and strongly in } L^r(Q_{1,T}),\\
&(f_\varepsilon(u_\varepsilon),f_\varepsilon(v_\varepsilon),f_\varepsilon(w_\varepsilon))\to(u,v,w) \text{ a.e. in } Q_{2,T} \text{ and strongly in } L^r(Q_{2,T}),
\end{aligned}
\tag{A.36}
$$

for all $1\le r\le 2$. Finally, by passing to the limit $\varepsilon\to 0$ in the weak formulation (A.19)-(A.25), with $\phi_i\in L^2(0,T;W^{1,\infty}(\Omega_1))$, $\Theta_i\in L^2(0,T;W^{1,\infty}(\Omega_2))$ for $i=1,2,3$, and $\Gamma\in\mathcal{D}([0,T)\times(\Omega_1\cup\Omega_2))$, we obtain in this way that the limit $(\varphi,\psi,\chi,u,v,w,c)$ is a solution of system (1.2)-(1.4) in the sense of Definition 2.1.

Appendix B. Existence and uniqueness of the classical solution (proof of Theorem 2.2)

In this proof we adapt the result obtained in [9] (and the references therein) to the system (1.2), (1.3) and (1.4) with (2.1)-(2.3) and

$$
\begin{aligned}
&k_{ij}=2 \text{ for } i=1,2,\ j=1,2,3,\ d_{11}\Big(\int_{\Omega_1}\varphi\,dx\Big)=d_{12}\Big(\int_{\Omega_1}\psi\,dx\Big)=d_{13}\Big(\int_{\Omega_1}\chi\,dx\Big),\\
&d_{21}\Big(\int_{\Omega_2}u\,dx\Big)=d_{22}\Big(\int_{\Omega_2}v\,dx\Big)=d_{23}\Big(\int_{\Omega_2}w\,dx\Big).
\end{aligned}
\tag{B.1}
$$

Now, we describe the steps to show existence of strong solutions and boundness of solutions.

Observe that for any fixed $(\varphi_0,\psi_0,\chi_0)\in(W^{1,p}(\Omega_1))^3$ ($p>3$), there exists a maximal existence time $T\in(0,+\infty]$ such that the system (1.2) has a unique solution $(\varphi,\psi,\chi)\in(C(0,T;W^{1,p}(\Omega_1)\cap C^\infty(\overline{\Omega}))^3$ (see the result of Amann [2] (see also [3, 4]) for more detrails).

Now, letting $w=\varphi+\psi+\chi$ and summing the equations in (1.2), the result is (recall that (B.1) holds)

$$
\begin{cases}
\partial_t w-\operatorname{div}\left(\left(d_{11}\Big(\int_{\Omega_1}\varphi\,dx\Big)+2\,w\right)\nabla w\right)=b(x)H_1-(m(x)+k(x)H_1)H_1 & \text{in } Q_{1,T},\\
\nabla w\cdot\eta=0 \quad \text{on } (0,T)\times\partial\Omega_1,\quad w(0,x)=w_0(x)=\varphi_0+\psi_0+\chi_0, & x\in\Omega.
\end{cases}
\tag{B.2}
$$

An application of maximum principle to (B.2) (see for e.g. [21]), we deduce that there exists a constant C_0 depending upon the initial data so that

$$
0\le\varphi(t,x),\psi(t,x),\chi(t,x)\le C_0 \text{ for all } x\in\overline{\Omega}_1 \text{ and } t>0.
\tag{B.3}
$$

Since $d_{11}\Big(\int_{\Omega_1}\varphi\,dx\Big)+2w\in L^\infty([0,T)\times\Omega)$ and

$$
b(x)H_1-(m(x)+k(x)H_1)H_1\in L^\infty(\overline{[0,T)\times\Omega_1}),
$$

we can apply the Hölder continuity result to (B.2) (see Theorem 1.3, p. 43 in [14] and we get

$$
w\in C^{\frac{\beta}{2},\beta}(\overline{[0,T)\times\Omega_1}) \text{ for some } \beta\in(0,1).
\tag{B.4}
$$

Now we let $\tilde{w} = d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) w + w^2$. Clearly $\tilde{w}$ satisfies

(B.5)
$$\begin{cases} \partial_t \tilde{w} - \Big(d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + 2\,w\Big)\Delta \tilde{w} = -\Big(d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + 2\,w\Big)(b(x) - (m(x) + k(x)H_1))H_1 \\ \qquad\qquad -w\partial_t d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) \qquad \text{in } Q_{1,T}, \\ \nabla \tilde{w}\cdot\eta = 0 \quad \text{on } (0,T)\times\partial\Omega_1, \quad \tilde{w}(0,x) = \tilde{w}_0(x) = d_{11}\Big(\int_{\Omega_1} \varphi_0\, dx\Big) w_0 + w_0^2, \qquad x \in \Omega_1. \end{cases}$$

To continue we will need the following lemma proved in [21] (see Theorem 9.1 in [21] for more details).

Lemma B.1. *Let $3 < q < +\infty$. Suppose u is a solution to the following problem*
$$\begin{cases} \partial_t u - a(t,x)\Delta u = f(t,x) \text{ in } Q_T, \\ \nabla u\cdot\eta = 0 \quad \text{on } (0,T)\times\partial\Omega, \quad u(0,x) = u_0(x), \qquad x\in\Omega. \end{cases}$$
where $T < +\infty$ and a is a positive bounded continuous function on $\overline{Q_T}$. Suppose that $f \in L^q(Q_T)$. Then there exists a constant C_q depending on the bounds of a, Ω T and q such that
$$\|u\|_{W_q^{2,1}(Q_T)} \le C_q\Big(\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\Omega)}\Big)$$
where u_0 satisfies the compatibility condition $\nabla u_0\cdot\eta = 0$ on $\partial\Omega$.

Herein $W_q^{2,1}(Q_T) = \{u : \|u\|_{W_q^{2,1}(Q_T)} \le C\}$ and
$$\|u\|_{W_q^{2,1}(Q_T)} = \left(\int\int_{Q_T} \Big(|u|^q + |\nabla u|^q + |\Delta u|^q + |\partial_t u|^q\Big)\, dx\, dt\right)^{\frac{1}{q}}.$$

Observe that $0 < d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + 2w \in L^\infty(\overline{[0,T)\times\Omega_1})$ and
$$(d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + 2\,w)(b(x) - (m(x) + k(x)H_1))H_1 - w\partial_t d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) \in L^\infty(\overline{[0,T)\times\Omega_1}).$$

Thus we can use Lemma B.1 together with $w \in C^{\frac{\beta}{2},\beta}(\overline{[0,T)\times\Omega_1})$ to obtain

(B.6)
$$\begin{aligned} \|\tilde{w}\|_{W_q^{2,1}(Q_{1,T})} &\le C\left(\left\|(d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + 2\,w)(b(x) - (m(x) + k(x)H_1))H_1 - w\partial_t d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big)\right\|_{L^q(Q_{1,T})}\right. \\ &\qquad \left. + \left\|d_{11}\Big(\int_{\Omega_1} \varphi_0\, dx\Big) w_0 + w_0^2\right\|_{W^{2-\frac{2}{q},q}(\Omega_1)}\right) \\ &\le C', \text{ for all } 3 < q < +\infty, \end{aligned}$$

for some constants $C, C' > 0$. An application of Sobolev embedding theorem for parabolic equation (see Lemma 3.3, p. 80 in [21]), we get $\tilde{w} \in C^{\frac{1+\rho}{2},1+\rho}(\overline{Q_{1,T}})$ for any $0 < \rho < 1$. Using this and $\tilde{w} = dw + w^2$, we get

$$w \in C^{\frac{1+\rho}{2},1+\rho}(\overline{Q_{1,T}}) \text{ for any } 0 < \rho < 1. \tag{B.7}$$

Observe that we can write (1.2) in the following form:

$$\begin{cases} \partial_t\varphi - \operatorname{div}\Big((d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + w)\nabla\varphi + \varphi\nabla w\Big) = F_1(x,\varphi,\psi,\chi,c) \text{ in } Q_{1,T}, \\ \partial_t\psi - \operatorname{div}\Big((\int_{\Omega_1} d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + w)\nabla\psi + \psi\nabla w\Big) = F_2(x,\varphi,\psi,\chi,c) \text{ in } Q_{1,T}, \\ \partial_t\chi - \operatorname{div}\Big((d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big) + w)\nabla\chi + \chi\nabla w\Big) = F_3(x,\varphi,\psi,\chi,c) \text{ in } Q_{1,T}, \\ \nabla\varphi\cdot\eta = \nabla\psi\cdot\eta = \nabla\chi\cdot\eta = 0 \quad \text{on } \partial\Omega_1, \\ \varphi(0,x) = \varphi_0(x),\ \psi(0,x) = \psi_0(x) \text{ and } \chi(0,x) = \chi_0(x). \qquad x\in\Omega_1, \end{cases} \tag{B.8}$$

Now we apply the Hölder continuity result to (B.8) (see e.g. Theorem 1.3, p. 43 in [14]), we deduce

$$\varphi, \psi, \chi \in C^{\frac{\alpha}{2},\alpha}(\overline{Q_{1,T}}) \text{ for some } 0 < \alpha < 1. \tag{B.9}$$

Herein we have used that φ, ψ, χ, and $\nabla\varphi, \nabla\psi, \nabla\chi$ are bounded because of (B.3) and (B.7).

Observe that $(\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}) = \Bigg(\Big(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)\varphi, (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)\psi, (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)\chi\Bigg)$ satisfies the following system

(B.10)

$$\begin{cases} \partial_t\tilde{\varphi} - (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\Delta\tilde{\varphi} = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)F_1(x,\varphi,\psi,\chi,c) \\ \qquad\qquad\qquad\qquad +\varphi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big) \quad \text{in } Q_{1,T}, \\ \partial_t\tilde{\psi} - (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\Delta\tilde{\psi} = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)F_2(x,\varphi,\psi,\chi,c) \\ \qquad\qquad\qquad\qquad +\psi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big) \quad \text{in } Q_{1,T}, \\ \partial_t\tilde{\chi} - (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\Delta\tilde{\chi} = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)F_3(x,\varphi,\psi,\chi,c) \\ \qquad\qquad\qquad\qquad +\chi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big) \quad \text{in } Q_{1,T}, \\ \nabla\tilde{\varphi}\cdot\eta = \nabla\tilde{\psi}\cdot\eta = \nabla\tilde{\chi}\cdot\eta = 0 \quad \text{on } \partial\Omega_1, \\ \tilde{\varphi}(0,x) = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi_0\, dx\Big)+w_0(x))\varphi_0(x),\ \tilde{\psi}(0,x) = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi_0\, dx\Big)+w_0(x))\psi_0(x), \\ \text{and } \tilde{\chi}(0,x) = (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi_0\, dx\Big)+w_0(x))\chi_0(x), \qquad x \in \Omega_1. \end{cases}$$

Therefore from the definition of $\tilde{w}$ $\Big($recall that $\tilde{w} = d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)w + w^2\Big)$, (B.3) and (B.6), we deduce

$$\|w\|_{W_q^{2,1}(Q_{1,T})} \le C, \text{ for all } 3 < q < +\infty, \tag{B.11}$$

for some constant $C > 0$. In particular, we have $\partial_t w \in L^q(Q_{1,T})$ for all $1 < q < +\infty$. Using this, $w \in C^{\frac{\beta}{2},\beta}(\overline{Q_{1,T}})$, $\Bigg(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)F_1(x,\varphi,\psi,\chi,c) + \varphi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big)\Bigg) \in L^q(\overline{Q_{1,T}})$, $\Bigg(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)F_2(x,\varphi,\psi,\chi,c) + \psi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big)\Bigg) \in L^q(\overline{Q_{1,T}})$, $\Bigg(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w\Big)$ $F_3(x,\varphi,\psi,\chi,c) + \chi\partial_t\Big(w + d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)\Big)\Bigg) \in L^q(\overline{Q_{1,T}})$ and Lemma B.1, we get

$$\|\tilde{\varphi}\|_{W_q^{2,1}(Q_{1,T})} + \left\|\tilde{\psi}\right\|_{W_q^{2,1}(Q_{1,T})} + \|\tilde{\chi}\|_{W_q^{2,1}(Q_{1,T})} \le C, \text{ for all } 3 < q < +\infty, \tag{B.12}$$

for some constant $C > 0$.

Next we use the definition of $\tilde{\varphi}$, $\tilde{\psi}$, and $\tilde{\chi}$ (recall that $(\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}) = \Bigg((d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\varphi, (d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\psi,$ $(d_{11}\Big(\displaystyle\int_{\Omega_1} \varphi\, dx\Big)+w)\chi\Bigg)$ to deduce from (B.3), (B.11), (B.12) and the Sobolev embedding theorem for parabolic equation (see Lemma 3.3, p. 80 in [21])

$$\varphi, \psi, \chi \in C^{\frac{1+\rho}{2},1+\rho}(\overline{Q_{1,T}}) \text{ for any } 0 < \rho < 1. \tag{B.13}$$

Exploiting (B.7) and (B.13), and using the Schauder estimate (see Theorem 5.3, p. 320-321 in [21]) applied to (B.5), we deduce

$$\tilde{w} \in C^{\frac{2+\theta}{2},2+\theta}(\overline{Q_{1,T}}) \text{ for some } \theta \in (0,1).$$

This implies that (recall that $\tilde{w} = d_{11}\Big(\int_{\Omega_1} \varphi\, dx\Big)w + w^2$)

$$w \in C^{\frac{2+\theta}{2},2+\theta}(\overline{Q_{1,T}}) \text{ and } \partial_t w \in C^{\frac{\theta}{2},\theta}(\overline{Q_{1,T}}).$$

Then an another application of Schauder estimates to (B.10), we get

$$\tilde{\varphi}, \tilde{\psi}, \tilde{\chi} \in C^{\frac{2+\theta}{2},2+\theta}(\overline{Q_{1,T}}). \tag{B.14}$$

Finally, we use the definition of $(\tilde{\varphi}, \tilde{\psi}, \tilde{\chi})$ to deduce from (B.14)

$$\varphi, \psi, \chi \in C^{\frac{2+\theta}{2},2+\theta}(\overline{Q_{1,T}}). \tag{B.15}$$

Finally the solution (φ, ψ, χ) exists globally in time. Reasoning along the same lines for (φ, ψ, χ) yields:

$$(u, v, w) \in C^{\frac{2+\theta}{2},2+\theta}([0,+\infty) \times \overline{\Omega}_2, \mathbb{R}^3).$$

This concludes the proof of Theorem 2.2.

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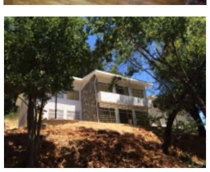

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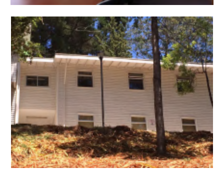

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