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NUMERICAL SOLUTION OF A TRANSIENT NON-LINEAR AXISYMMETRIC EDDY CURRENT MODEL WITH NON-LOCAL BOUNDARY CONDITIONS

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This paper deals with an axisymmetric transient eddy current problem in conductive nonlinear magnetic media. This means that the relation between the magnetic field and the magnetic induction, the so-called H-B curve, is nonlinear. The source of the problem is the magnetic flux across a meridian section of the device, which leads to a parabolic nonlinear problem with nonlocal boundary conditions. First, by applying some abstract results, we prove the existence and uniqueness of the solution to a weak formulation written in terms of the magnetic field. Then, we compute the numerical solution of the problem by using a finite element method combined with a backward Euler time discretization. We derive error estimates in appropriate norms for both the semidiscrete (in space) and the fully discrete problems. Finally, we show numerical results which allow us to confirm the theoretical estimates and to assess the performance of the proposed scheme.

Keywords: transient eddy current problem; electromagnetic losses; nonlinear magnetic materials; non local boundary conditions; finite element method; error estimates.

AMS Subject Classification: 65N30 (78A55 78M10)

1. Introduction

An important challenge to bear in mind in the analysis and design of electrical machines is the accurate computation of the power losses in the ferromagnetic components of the core. These losses determine the efficiency of the device and have a significant influence on its operating cost.

At the macroscopic level, two main type of losses can be distinguished: hysteresis losses, which are related to the intrinsic nature of magnetic materials, and eddy current losses, due to the Joule effect.[6]

There are numerous publications devoted to obtain analytical simplified expressions to approximate the different losses, which are only valid under assumptions that often do not hold in practice (see, for instance, Refs. 6 and 7). Numerical modeling is an interesting alternative to overcome these limitations and, thus, we can find several works focused on the computation of hysteresis and eddy current losses (see Refs. 10, 15, 23 and references therein).

A first step in the computation of this kind of loses is the numerical solution of the underlying electromagnetic problem. This requires solving the quasi-static Maxwell's partial differential equations, a well established subject, even in the three-dimensional (3D) case where edge finite elements are very useful.[1] This issue was studied in Refs. 18 and 19 in terms of the magnetic field and in absence of hysteresis effects. In these references the 3D problem is posed on a bounded conducting domain and homogeneous Dirichlet boundary conditions are assumed. Moreover current sources are not taken into account, the only source term being the initial condition. A time semi-discretization scheme is proposed and analyzed to approximate this problem.

However, major difficulties arise from the fact that cores are laminated to reduce the eddy current losses. Thus, to account for the detailed geometry, extremely fine meshes should be needed, which becomes unaffordable. To overcome this difficulty, one can find different strategies based on the use of the so-called equivalent conductivity[4,11,13,15] or on homogenization techniques.[9] In this paper, we are interested in an alternative approach proposed by Van Keer et al.,[22,23] which consists in computing the electromagnetic field in a cross-section of the laminated device, orthogonal to the direction of the enforced flux. The models introduced in Cartesian and cylindrical coordinates in Refs. 22 and 23 include also hysteresis effects and lead to parabolic nonlinear problems which, to the authors' knowledge, have not yet been analyzed from the mathematical or numerical points of view.

We will address these issues in the axisymmetric case without including hysteresis effects. The behavior of the material is defined by an anhysteretic H-B curve. We prove that a weak formulation of this problem in terms of the magnetic field has a unique solution. Then, we propose a numerical scheme to approximate the solution of this problem for which we obtain error estimates.

The paper is organized as follows. In Section 2, we describe the transient axisymmetric eddy current model and introduce the nonlinear parabolic partial differential equation to be solved. Next, in Section 3, we obtain a weak formulation of the problem. The existence of solution is proved by applying results for abstract nonlinear parabolic equations. Section 4 is devoted to numerical methods. A space semi-discretization by finite elements is introduced and, then, a backward Euler scheme is applied for time discretization. Error estimates for both schemes are obtained. Finally, numerical results that confirm the theoretical estimates are shown in Section 5.

2. The transient nonlinear eddy current model

The eddy current model is an approximation of the full Maxwell system of equations obtained by neglecting the displacement currents in Ampère's law. This simplified model is suitable for most electrical engineering applications (the so-called low-frequency regime), for instance, in the numerical simulation of electrical machines working at power frequencies. The eddy current model reads

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{2.2}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{2.3}$$

where we have used standard notation in electromagnetism: $\mathbf{H}$ is the magnetic field, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic induction and $\mathbf{E}$ the electric field. In order to obtain a closed system we need to add constitutive laws. Assuming that the materials are electrically linear but magnetically nonlinear, we have

$$\mathbf{J} = \sigma\mathbf{E}, \tag{2.4}$$

$$\mathbf{B} = \boldsymbol{\mathcal{B}}(\mathbf{H}). \tag{2.5}$$

Equation (2.4) is Ohm's law, where σ denotes the electrical conductivity of the medium. In the magnetic constitutive relation (2.5), $\boldsymbol{\mathcal{B}}$ is in general a nonlinear mapping. Two extreme cases are the following: linear isotropic materials, for which this mapping reduces to $\boldsymbol{\mathcal{B}}(\mathbf{H}) = \mu\mathbf{H}$ with μ being the constant magnetic permeability, and ferromagnetic materials where hysteresis phenomena may occur, in which case the H-B relation exhibits a history-dependent behavior. Our analysis allows for a nonlinear magnetic material, that will be represented through an anhysteretic H-B curve, which could have a very steep slope. This choice is a simplification frequently used for soft magnetic materials by electrical engineers (see, for instance, Ref. 21).

Equations (2.1), (2.2) and (2.4), lead to the following vector partial differential equation in conductors:

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\left(\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}\right) = \mathbf{0}. \tag{2.6}$$

Our aim is to solve this together with the nonlinear constitutive equation (2.5).

2.1. *Axisymmetric eddy current model with enforced magnetic flux*

Let us consider a cylindrical coordinate system (r, θ, z) and denote by $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ the corresponding unit vectors of the local orthonormal basis as sketched in Fig. 1 (left). We suppose that the computational domain $\widetilde{\Omega}$ has cylindrical symmetry and that the current sources are independent of the azimuth θ and do not have azimuthal component, so that on each meridian section these currents lie on this section. In

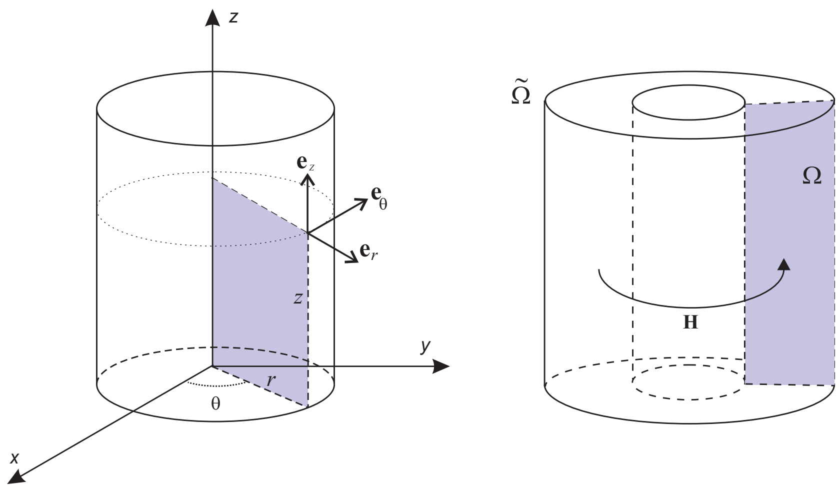

Fig. 1. Cylindrical coordinate system (left) and sketch of the domain (right).

such a case, none of the electromagnetic fields depend on θ and, furthermore, from Faraday's law (2.2), $\mathbf{B}$ has to be of the form,

$$\mathbf{B}(r,z,t) = B(r,z,t)\mathbf{e}_\theta. \tag{2.7}$$

Since we are assuming that the material is isotropic, the magnetic field $\mathbf{H}$ must be of the same form as $\mathbf{B}$, namely,

$$\mathbf{H}(r,z,t) = H(r,z,t)\mathbf{e}_\theta, \tag{2.8}$$

and the H-B relation reads

$$B(r,z,t) = \mathcal{B}(r,z,H(r,z,t)), \tag{2.9}$$

with $\mathcal{B}(r,z,\cdot)$ being a nonlinear mapping in $\mathbb{R}$ for each (r,z). Dependence of $\mathcal{B}$ in coordinates (r,z) is permitted to allow for computational domains including different materials. We notice that any field of the form (2.7) is divergence-free, so that (2.3) is automatically satisfied. Moreover, since

$$\mathbf{curl}\,\mathbf{H}(r,z,t) = -\frac{\partial H}{\partial z}(r,z,t)\,\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH)(r,z,t)\,\mathbf{e}_z, \tag{2.10}$$

equation (2.6) leads to

$$\frac{\partial B}{\partial t} - \frac{\partial}{\partial r}\left(\frac{1}{\sigma r}\frac{\partial(rH)}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H}{\partial z}\right) = 0. \tag{2.11}$$

This equation holds in any meridian section Ω of the domain $\widetilde{\Omega}$ for all time $t \in [0,T]$ ($T > 0$ fixed). To have a well-posed nonlinear parabolic problem we must add to equations (2.11) and (2.9) an initial condition

$$B(r,z,0) = B_0(r,z) \quad \text{in } \Omega, \tag{2.12}$$

and suitable boundary conditions on the boundary $\Gamma := \partial\Omega$.

The application that has motivated this paper is the computation of eddy current losses in laminated media. Thus, following the work of Van Keer et al.,[22] we will impose the magnetic flux $b(t)$ flowing through the meridian section Ω of the domain (see Fig. 1, (right)). This leads to the nonlocal source condition

$$\int_\Omega B(r,z,t)\; dr\; dz = b(t). \tag{2.13}$$

Moreover, we have also to impose that there is no current flux through the boundary of Ω; namely, $\mathbf{curl}\,\mathbf{H}\cdot\mathbf{n} = \mathbf{J}\cdot\mathbf{n} = 0$ on Γ, where $\mathbf{n}$ is the unit normal to Γ. Hence, from (2.10), it is straightforward to obtain that the tangential derivative of (rH) has to vanish on Γ. Therefore, provided Γ is connected, for each $t \in [0,T]$ $(rH(t))$ has to be a constant (which depends on t) on the whole Γ. Consequently, there exists an (unknown) function $\psi(t)$ which varies in time but is space independent on Γ such that

$$rH(r,z,t) = \psi(t) \quad \text{on } \Gamma. \tag{2.14}$$

All together, the resulting axisymmetric problem reads:

Problem 2.1. *Find $H(r,z,t)$ and $B(r,z,t)$ such that*

$$\frac{\partial B}{\partial t} - \frac{\partial}{\partial r}\left(\frac{1}{\sigma r}\frac{\partial(rH)}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H}{\partial z}\right) = f \qquad \text{in } \Omega\times(0,T), \tag{2.15}$$

$$B(r,z,t) = \mathcal{B}(r,z,H(r,z,t)) \quad \text{in } \Omega\times(0,T), \tag{2.16}$$

$$rH(r,z,t) = \psi(t) \quad \text{on } \Gamma\times(0,T), \tag{2.17}$$

$$\int_\Omega B(r,z,t)\; dr\; dz = b(t) \quad \text{in } (0,T), \tag{2.18}$$

$$B(r,z,0) = B_0(r,z) \quad \text{in } \Omega, \tag{2.19}$$

where $\sigma(r,z,t)$, $f(r,z,t)$, $b(t)$ and $B_0(r,z)$ are given data and $\psi(t)$ is unknown.

Remark 2.1. We include in (2.15) a right-hand side f to consider a more general parabolic problem, although in the case of the eddy current model f is zero. Moreover, we consider a space and time dependent electrical conductivity σ because in practical applications it is a function of temperature which, in its turn, is a time dependent field.

Problem 2.1 has been proposed and numerically solved in Ref. 22 in a more general setting including hysteresis. The goal of the present paper is to study the well-posedness and the numerical approximation of this problem.

3. Mathematical analysis

In this section, we derive a weak formulation for Problem 2.1 and prove that it is well posed.

3.1. *Functional spaces and preliminary results*

We recall some weighted Sobolev spaces typical in axisymmetric problems. We refer to Refs. 14 and 5 for more details. For the sake of simplicity, partial derivatives will be also denoted by ∂_r, ∂_z and so on.

Let $\Omega \subset \{(r,z) \in \mathbb{R}^2 : \ r > 0\}$ be a Lipschitz bounded simply connected open set. Let $\mathrm{L}^p_r(\Omega)$ denote the weighted Lebesgue space of all measurable functions u defined in Ω for which

$$\|u\|^p_{\mathrm{L}^p_r(\Omega)} := \int_\Omega |u|^p \, r \, dr \, dz < \infty \qquad 1 \leq p < \infty.$$

The weighted Sobolev space $\mathrm{H}^k_r(\Omega)$ consists of all functions in $\mathrm{L}^2_r(\Omega)$ whose derivatives up to order k are also in $\mathrm{L}^2_r(\Omega)$. We define the norms and semi-norms in the standard way; for instance,

$$|u|^2_{\mathrm{H}^1_r(\Omega)} := \int_\Omega \left(|\partial_r u|^2 + |\partial_z u|^2 \right) r \, dr \, dz.$$

Let $\mathrm{L}^2_{1/r}(\Omega)$ denote the set of all measurable functions u defined in Ω for which

$$\|u\|^2_{\mathrm{L}^2_{1/r}(\Omega)} := \int_\Omega \frac{|u|^2}{r} \, dr \, dz < \infty.$$

We also define $\mathrm{H}^k_{1/r}(\Omega)$ as before.

Finally, we introduce the function space $\widehat{\mathrm{H}}^1_r(\Omega)$ defined by

$$\widehat{\mathrm{H}}^1_r(\Omega) = \{u \in \mathrm{L}^2_r(\Omega) : \ \partial_r(ru) \in \mathrm{L}^2_{1/r}(\Omega), \ \partial_z u \in \mathrm{L}^2_r(\Omega)\},$$

which is a Hilbert space with the norm

$$\|u\|_{\widehat{\mathrm{H}}^1_r(\Omega)} := \left(\|u\|^2_{\mathrm{L}^2_r(\Omega)} + \|\partial_r(ru)\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \|\partial_z u\|^2_{\mathrm{L}^2_r(\Omega)} \right)^{1/2}.$$

Remark 3.1. For Ω being the meridian section of a 3D axisymmetric domain $\widetilde{\Omega}$, the space $\widehat{\mathrm{H}}^1_r(\Omega)$ can be considered as an axisymmetric version of the 3D space $\mathbf{H}(\mathbf{curl}, \widetilde{\Omega})$. More precisely, from the expression of the **curl** operator in cylindrical coordinates it is immediate to see that $G(r,z) \in \widehat{\mathrm{H}}^1_r(\Omega)$ if and only if $\mathbf{G}(r,z,\theta) = G(r,z)\mathbf{e}_\theta(\theta) \in \mathbf{H}(\mathbf{curl}, \widetilde{\Omega})$.

3.2. *Weak formulation*

Before stating a weak formulation of Problem 2.1, we notice that if the boundary of Ω intersect the symmetry axis ($r = 0$), then $\psi(t)$ should be identically zero because r vanishes there. In such a case, (2.17) would become a homogeneous Dirichlet boundary condition. However, this does not happen in the application that motivates this problem in which the domain is well separated from the symmetry axis (see Ref. 22). This is the reason why, from now on, we will assume that $\inf\{r > 0 : (r,z) \in \Omega\} > 0$ and, hence, $\mathrm{L}^2_r(\Omega)$ and $\mathrm{L}^2_{1/r}(\Omega)$ are both identical to $\mathrm{L}^2(\Omega)$. Similarly, $\widehat{\mathrm{H}}^1_r(\Omega)$ is identical to $\mathrm{H}^1(\Omega)$.

Let us introduce the following closed subspace of $\widehat{\mathrm{H}}^1_r(\Omega)$:

$$\mathcal{W} := \{G \in \widehat{\mathrm{H}}^1_r(\Omega) : (rG)|_\Gamma \text{ is constant}\}. \tag{3.1}$$

Since $\widehat{\mathrm{H}}^1_r(\Omega)$ is densely and compactly included in $\mathrm{L}^2_r(\Omega)$, the same is true for $\mathcal{W}$ (the density because $\mathcal{W} \supset \mathcal{D}(\Omega)$). Thus, if we identify $\mathrm{L}^2_r(\Omega)$ with its topological dual, we have that $\mathcal{W} \subset \mathrm{L}^2_r(\Omega) \subset \mathcal{W}'$. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{W},\mathcal{W}'}$ the corresponding duality paring.

In order to obtain a weak formulation, first we integrate (2.15) in Ω and use Gauss theorem to write

$$\frac{d}{dt}\int_\Omega B(r,z,t)\, dr\, dz - \int_\Gamma \frac{1}{\sigma r}\left(\frac{\partial(rH)}{\partial r} n_r + \frac{\partial(rH)}{\partial z} n_z\right) d\Gamma = \int_\Omega f\, dr\, dz,$$

where $\mathbf{n} = n_r\mathbf{e}_r + n_z\mathbf{e}_z$ is the outward unit normal vector to Γ. Hence, by using (2.18) we deduce that

$$\int_\Gamma \frac{1}{\sigma r}\left(\frac{\partial(rH)}{\partial r} n_r + \frac{\partial(rH)}{\partial z} n_z\right) d\Gamma = b'(t) - \int_\Omega f\, dr\, dz. \tag{3.2}$$

Next, we multiply (2.15) by (rG), G being any test function in $\mathcal{W}$, integrate in Ω and use a Green's formula. From the resulting expression and (3.2), we easily obtain the following weak formulation for Problem 2.1:

Problem 3.1. *Given* $b \in \mathrm{H}^1(0,T)$*,* $f \in \mathrm{L}^2(0,T;\mathcal{W}')$ *and* $B_0 \in \mathrm{L}^2_r(\Omega)$*, find* $H \in \mathrm{L}^2(0,T;\mathcal{W})$ *and* $B \in \mathrm{H}^1(0,T;\mathcal{W}')$ *such that*

$$\left\langle \frac{\partial B}{\partial t}, G\right\rangle_{\mathcal{W},\mathcal{W}'} + \int_\Omega \frac{1}{\sigma r}\left(\frac{\partial(rH)}{\partial r}\frac{\partial(rG)}{\partial r} + \frac{\partial(rH)}{\partial z}\frac{\partial(rG)}{\partial z}\right) dr\, dz$$

$$= \langle f, G\rangle_{\mathcal{W},\mathcal{W}'} + \left(b'(t) - \langle f, r^{-1}\rangle_{\mathcal{W},\mathcal{W}'}\right)(rG)|_\Gamma \quad \forall G \in \mathcal{W},\ a.e.\ t \in (0,T),$$

$$B(r,z,t) = \mathcal{B}(r,z,H(r,z,t)) \quad a.e.\ in\ \Omega \times (0,T),$$

$$B(r,z,0) = B_0(r,z) \quad a.e.\ in\ \Omega.$$

Notice that $\langle f, r^{-1}\rangle_{\mathcal{W},\mathcal{W}'}$ is well defined because $r^{-1} \in \mathcal{W}$.

3.3. *Existence of solution*

We introduce the following hypotheses and notations that will be used to prove the existence of a solution to the above problem.

H.1: $\mathcal{B}(r,z,u)$ is the derivative with respect to u of a (differentiable) normal convex integrand α defined in $\Omega \times \mathbb{R}$ (see, for instance, Ref. 3); i.e.,

$$\mathcal{B}(r,z,u) := \partial_u \alpha(r,z,u) \quad \forall u \in \mathbb{R}, \forall (r,z) \in \Omega. \tag{3.3}$$

Moreover, we assume that α satisfies the following conditions:

- there exist $\beta_1 \in \mathrm{L}^2_r(\Omega)$ and $\beta_2 \in \mathrm{L}^1_r(\Omega)$ such that
$$\alpha(r,z,u) \geq \beta_1(r,z)u + \beta_2(r,z) \quad \forall u \in \mathbb{R}, \forall (r,z) \in \Omega;$$
- for each $w \in \mathrm{L}^2_r(\Omega)$, $\alpha(\cdot,\cdot,w(\cdot,\cdot)) \in \mathrm{L}^1_r(\Omega)$.

H.2: There exist two positive constants N_1 and N_2 such that
$$|\mathcal{B}(r,z,u)| \leq N_1|u| + N_2 \quad \forall u \in \mathbb{R}, \forall (r,z) \in \Omega.$$

H.3: $\mathcal{B}(r,z,u)$ is strongly monotone with respect to u uniformly in Ω; i.e., there exists a strictly positive constant ω such that
$$(\mathcal{B}(r,z,u) - \mathcal{B}(r,z,v))(u-v) \geq \omega|u-v|^2 \quad \forall u,v \in \mathbb{R},\ \forall (r,z) \in \Omega.$$

H.4: $\sigma : (0,T) \longrightarrow \mathrm{L}^\infty(\Omega)$ is measurable and there exist constants σ_* and σ^* such that
$$0 < \sigma_* \leq \sigma(r,z,t) \leq \sigma^* \quad \forall (r,z) \in \Omega, \text{ a.e. } t \in (0,T).$$

H.5: There exists $H_0 \in \mathcal{W}$ such that $B_0(r,z) = \mathcal{B}(r,z,H_0(r,z))$ a.e. in Ω.

Note that, as a consequence of H.5 and H.2, $B_0 \in \mathrm{L}^2_r(\Omega)$.

Let us introduce the function $\varphi : \mathrm{L}^2_r(\Omega) \to \mathbb{R}$ defined by
$$\varphi(H) := \int_\Omega \alpha(r,z,H(r,z))\, r\, dr\, dz, \quad H \in \mathrm{L}^2_r(\Omega), \tag{3.4}$$
which is well defined because of the last property in H.1. Then, from the assumptions on α, φ is a differentiable convex function in $\mathrm{L}^2_r(\Omega)$ (see Ref. 2) and its differential, which we denote $\partial\varphi$, satisfies
$$\partial\varphi(H)(r,z) = \partial_u\alpha(r,z,H(r,z)) = \mathcal{B}(r,z,H(r,z)) \quad (r,z) \in \Omega,\ \forall H \in \mathrm{L}^2_r(\Omega), \tag{3.5}$$
the last equality because of (3.3).

On the other hand, for each $t \in [0,T]$, let us denote by $a_t(\cdot,\cdot)$ the bilinear form defined by
$$a_t(H,G) := \int_\Omega \frac{1}{\sigma(\cdot,t)} \left(\frac{1}{r}\frac{\partial(rH)}{\partial r}\, \frac{1}{r}\frac{\partial(rG)}{\partial r} + \frac{\partial H}{\partial z}\frac{\partial G}{\partial z} \right) r\, dr\, dz \quad H,G \in \widehat{\mathrm{H}}^1_r(\Omega). \tag{3.6}$$

From H.4, we have the following result whose proof is straightforward.

Lemma 3.1. *The bilinear forms $a_t : \widehat{\mathrm{H}}^1_r(\Omega) \times \widehat{\mathrm{H}}^1_r(\Omega) \to \mathbb{R}$, $t \in [0,T]$, are continuous uniformly in t. Moreover, they satisfy the Gårding's inequality*
$$a_t(G,G) + \lambda\|G\|^2_{\mathrm{L}^2_r(\Omega)} \geq \gamma\|G\|^2_{\widehat{\mathrm{H}}^1_r(\Omega)} \quad \forall G \in \widehat{\mathrm{H}}^1_r(\Omega), \tag{3.7}$$
with $\lambda = \gamma = 1/\sigma^*$.

Let us introduce $R \in \mathrm{L}^2(0,T;\mathcal{W}')$ defined by
$$\langle R(t), G\rangle_{\mathcal{W},\mathcal{W}'} := \langle f(t), G\rangle_{\mathcal{W},\mathcal{W}'} + \left(b'(t) - \langle f(t), r^{-1}\rangle_{\mathcal{W},\mathcal{W}'}\right)(rG)|_\Gamma,$$
for all $G \in \mathcal{W}$, a.e. $t \in (0,T)$.

Now we are in position to prove that Problem 3.1 has a solution.

Theorem 3.1. *Let us assume hypotheses H.1 to H.5. Then, Problem 3.1 has a solution.*

Proof. We will derive this result as a consequence of Theorem 2 from Ref. 12. With this aim, first we rewrite Problem 3.1 as follows:

Find $H \in \mathrm{L}^2(0,T;\mathcal{W})$ and $B \in \mathrm{H}^1(0,T;\mathcal{W}')$ such that

$$\frac{\partial B}{\partial t}(t) + A(t)H(t) = R(t), \quad \text{a.e. } t \in (0,T), \tag{3.8}$$

$$B(t) = \partial\varphi(H(t)), \quad \text{a.e. } t \in (0,T), \tag{3.9}$$

$$B(0) = B_0, \tag{3.10}$$

where, for a.e. $t \in (0,T)$, $A(t) : \mathcal{W} \to \mathcal{W}'$ is the linear operator induced by $a_t(\cdot,\cdot)$; namely,

$$\langle A(t)H, G\rangle_{\mathcal{W}',\mathcal{W}} := a_t(H,G) \quad \forall H, G \in \mathcal{W}.$$

Notice that from H.5 and (3.5) we have $B_0 = \partial\varphi(H_0)$. In order to apply Theorem 2 from Ref. 12 to Problem (3.8)–(3.10), we must check all the hypotheses of this theorem. Some of them are void (C.1 to C.4), or automatically satisfied (A.2, A.6) or consequence of the other hypotheses in our case (A.3, A.4), mainly because φ is time independent (cf. Remark 1 from Ref. 12). In what follows we check the remaining ones:

A.1: As stated above, in our case φ is differentiable and convex.

A.5: From H.3 and (3.5), $\partial\varphi$ is strongly monotone; namely,

$$(\partial\varphi(H_1) - \partial\varphi(H_2), H_1 - H_2)_{\mathrm{L}^2_r(\Omega)} \geq \omega \|H_1 - H_2\|^2_{\mathrm{L}^2_r(\Omega)} \quad \forall H_1, H_2 \in \mathrm{L}^2_r(\Omega).$$

A.7: From H.2 and (3.5),

$$\|\partial\varphi(H)\|_{\mathrm{L}^2_r(\Omega)} \leq N_1 \|H\|_{\mathrm{L}^2_r(\Omega)} + N_2 \quad \forall H \in \mathrm{L}^2_r(\Omega).$$

B.1: $A(t)$ is maximal monotone in $\mathcal{W}$, because it is a linear bounded operator and $a_t(G,G) \geq 0$ for all $G \in \mathcal{W}$ (see, for instance, Ref. 2). Moreover, we also have from the definition of $A(t)$ that

$$\|A(t)H\|_{\mathcal{W}'} \leq \frac{1}{\sigma_*} \|H\|_{\widehat{\mathrm{H}}^1_r(\Omega)} \quad \forall H \in \mathcal{W}, \text{ a.e. } t \in (0,T).$$

B.2: It follows from the assumption that $\sigma : (0,T) \to \mathrm{L}^\infty(\Omega)$ is measurable (cf. H.4) and the fact that $A(t)$ is a linear bounded operator.

B.3: It is a consequence of Gårding's inequality from Lemma 3.1.

Thus, all the hypothesis of Theorem 2 from Ref. 12 are fulfilled and we are allowed to apply it to Problem (3.8)–(3.10) to conclude the proof. □

Remark 3.2. As a consequence of H.2, the solution of Problem 3.1 also satisfies $B \in \mathrm{L}^2(0,T;\mathrm{L}^2_r(\Omega))$.

Remark 3.3. The above existence result is independent of the slope of the H-B curve; even an infinite slope is allowed.

Remark 3.4. The previous theorem yields the existence of solution to Problem 3.1. If the electrical conductivity σ does not depend on time, we can also conclude the uniqueness. Indeed, let H_1 and H_2 be two solutions to Problem 3.1; then, for a.e. $t \in (0,T)$,

$$\left\langle \frac{\partial \mathcal{B}(H_1(t))}{\partial t} - \frac{\partial \mathcal{B}(H_2(t))}{\partial t}, G \right\rangle_{\mathcal{W},\mathcal{W}'} + a(H_1(t) - H_2(t), G) = 0 \quad \forall G \in \mathcal{W},$$

where $a(\cdot,\cdot)$ denotes the bilinear form defined in (3.6) for σ independent of t. By integrating this equation with respect to time, choosing $G = H_1(t) - H_2(t)$ as test function and using the monotonicity of $\mathcal{B}$, we obtain

$$\omega \|H_1(t) - H_2(t)\|^2_{\mathrm{L}^2_r(\Omega)} + a\left(\int_0^t (H_1 - H_2)(s)\, ds, H_1(t) - H_2(t)\right) \leq 0.$$

Thus, by integrating in $(0,T)$, using the equality

$$\begin{aligned} 2\int_0^T a\left(\int_0^t (H_1 - H_2)(s)\, ds, H_1(t) - H_2(t)\right)\, dt \\ = a\left(\int_0^T (H_1 - H_2)(t)\, dt, \int_0^T (H_1 - H_2)(t)\, dt\right) \end{aligned} \tag{3.11}$$

and taking into account that $a(\cdot,\cdot)$ is positive semi-definite, we conclude that $H_1 = H_2$.

4. Numerical analysis

In this section we propose a numerical method to approximate the solution to Problem 3.1. In order to obtain error estimates for this numerical method, from now on we consider the following additional assumptions:

H.6 σ is time independent.
H.7 $\mathcal{B}(r,z,u)$ is uniformly Lipschitz continuous with respect to u, namely: there exists a positive constant L such that

$$|\mathcal{B}(r,z,u) - \mathcal{B}(r,z,v)| \leq L|u - v| \quad \forall\, u, v \in \mathbb{R}, \forall (r,z) \in \Omega.$$

Notice that H.2 immediately follows from H.7.

To impose the constraint of (rH) being constant on Γ (cf. (2.14)) we proceed as in Ref. 22: we make a change of unknown and write the equations in terms of $\widetilde{H} := rH$ and $\widetilde{B} := rB$.

With this end, we introduce some additional notation. First notice that $G \in \widehat{\mathrm{H}}^1_r(\Omega)$ if and only if $\widetilde{G} := rG \in \mathrm{H}^1_{1/r}(\Omega)$. Hence, $G \in \mathcal{W}$ is and only if $\widetilde{G}$ belongs to the following space:

$$\mathcal{Y} := \left\{ Y \in \mathrm{H}^1_{1/r}(\Omega) : \ Y|_\Gamma \text{ is constant} \right\},$$

which we endow with the $\mathrm{H}^1_{1/r}(\Omega)$-norm. Since, $\mathrm{H}^1_{1/r}(\Omega)$ is densely included in $\mathrm{L}^2_{1/r}(\Omega)$, if we identify $\mathrm{L}^2_{1/r}(\Omega)$ with its dual space, we have

$$\mathcal{Y} \subset \mathrm{L}^2_{1/r}(\Omega) \subset \mathcal{Y}'.$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathcal{Y}'$ and $\mathcal{Y}$.

From now on, we fix the data of Problem 3.1: $b \in \mathrm{H}^1(0,T)$, $f \in \mathrm{L}^2(0,T;\mathcal{W}')$ and $B_0 \in \mathrm{L}^2_r(\Omega)$, and define $\widetilde{R} \in \mathrm{L}^2(0,T;\mathcal{Y}')$ and $\widetilde{B}_0 \in \mathrm{L}^2_{1/r}(\Omega)$ by

$$\langle \widetilde{R}(t), \widetilde{G} \rangle := \langle f(t), r^{-1}\widetilde{G} \rangle_{\mathcal{W},\mathcal{W}'} + \left(b'(t) - \langle f(t), r^{-1} \rangle_{\mathcal{W},\mathcal{W}'} \right) (\widetilde{G}|_\Gamma) \quad \widetilde{G} \in \mathcal{Y}\,, \text{a.e. } t \in [0,T],$$

and

$$\widetilde{B}_0(r,z) := rB_0(r,z) \quad (r,z) \in \Omega.$$

Moreover, let

$$\widetilde{\mathcal{B}}(r,z,u) := r\mathcal{B}(r,z,r^{-1}u) \quad (r,z) \in \Omega\,, u \in \mathbb{R}.$$

It is easy to check that $\widetilde{\mathcal{B}}$ is also strongly monotone and Lipschitz continuous, namely: there exists positive constants ω and L (the same as in H.3 and H.7) such that

$$(\widetilde{\mathcal{B}}(r,z,u) - \widetilde{\mathcal{B}}(r,z,v))(u-v) \geq \omega |u-v|^2 \quad \forall u,v \in \mathbb{R},\ \forall (r,z) \in \Omega \tag{4.1}$$

and

$$|\widetilde{\mathcal{B}}(r,z,u) - \widetilde{\mathcal{B}}(r,z,v)| \leq L|u-v| \quad \forall\, u,v \in \mathbb{R}\,, \forall (r,z) \in \Omega. \tag{4.2}$$

Finally, let us introduce the bilinear form $\widetilde{a}(\cdot,\cdot) : \mathrm{H}^1_{1/r}(\Omega) \times \mathrm{H}^1_{1/r}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{a}(\widetilde{G}_1, \widetilde{G}_2) := a_t(r^{-1}\widetilde{G}_1, r^{-1}\widetilde{G}_2) = \int_\Omega \frac{1}{\sigma r} \left(\frac{\partial \widetilde{G}_1}{\partial r} \frac{\partial \widetilde{G}_2}{\partial r} + \frac{\partial \widetilde{G}_1}{\partial z} \frac{\partial \widetilde{G}_2}{\partial r} \right) dr\, dz$$

for $\widetilde{G}_1, \widetilde{G}_2 \in \mathrm{H}^1_{1/r}(\Omega)$. Notice that now, because of H.6, a_t actually does not depend on t. As a consequence of Lemma 3.1 we have the following result.

Lemma 4.1. *The bilinear form $\widetilde{a}$ is continuous and satisfies the Gårding's inequality*

$$\widetilde{a}(\widetilde{G}, \widetilde{G}) + \lambda \|\widetilde{G}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \geq \gamma \|\widetilde{G}\|^2_{\mathrm{H}^1_{1/r}(\Omega)} \quad \forall\, \widetilde{G} \in \mathrm{H}^1_{1/r}(\Omega),$$

with $\lambda = \gamma = 1/\sigma^*$.

Under assumptions H.1 to H.5 we have shown that Problem 3.1 has a solution. Moreover, under the same assumptions and H.6, it is easy to prove that (H, B) is the unique solution to Problem 3.1 if and only if $(\widetilde{H}, \widetilde{B})$ is a solution to the following:

Problem 4.1. *Find $\widetilde{H} \in \mathrm{L}^2(0,T;\mathcal{Y})$ and $\widetilde{B} \in \mathrm{H}^1(0,T;\mathcal{Y}')$ such that*

$$\left\langle \frac{\partial \widetilde{B}}{\partial t}, \widetilde{G} \right\rangle + \widetilde{a}(\widetilde{H}, \widetilde{G}) = \langle \widetilde{R}, \widetilde{G} \rangle \quad \forall \widetilde{G} \in \mathcal{Y},\ a.e.\ t \in (0,T),$$

$$\widetilde{B}(r,z,t) = \widetilde{\mathcal{B}}(r,z,\widetilde{H}(r,z,t)) \quad a.e.\ in\ \Omega \times (0,T),$$

$$\widetilde{B}(r,z,0) = \widetilde{B}_0(r,z) \quad a.e.\ in\ \Omega.$$

4.1. *Space discretization*

We introduce in this section a space semi-discretization of Problem 4.1 and obtain an optimal order error estimate in the $\mathrm{L}^2(0,T,\mathrm{L}^2_{1/r}(\Omega))$-norm. The following analysis is inspired in Ref. 19 and on the classical numerical analysis of linear parabolic equations (see, for instance, Ref. 20).

To begin with, from now on we assume Ω is a convex polygon. We associate a family of partitions $\{\mathcal{T}_h\}_{h>0}$ of Ω into triangles, where h denotes the mesh size (i.e., the maximal length of the sides of the triangulation). Let $\mathcal{Y}_h := \mathcal{V}_h \cap \mathcal{Y}$, where $\mathcal{V}_h$ denotes the space of continuous piecewise linear finite elements. By using this finite element space, we are led to the following discretization of Problem 4.1.

Problem 4.2. *Find $\widetilde{H}_h \in \mathrm{L}^2(0,T;\mathcal{Y}_h)$ and $\widetilde{B}_h \in \mathrm{H}^1(0,T;\mathcal{Y}')$, satisfying*

$$\left\langle \frac{\partial \widetilde{B}_h}{\partial t}, \widetilde{G}_h \right\rangle + \widetilde{a}(\widetilde{H}_h, \widetilde{G}_h) = \langle \widetilde{R}, \widetilde{G}_h \rangle \quad \forall \widetilde{G}_h \in \mathcal{Y}_h\,,\ a.e.\ t \in (0,T),$$

$$\widetilde{B}_h(r,z,t) = \widetilde{\mathcal{B}}(r,z,\widetilde{H}_h(r,z,t)) \quad a.e.\ in\ \ \Omega \times (0,T),$$

$$\widetilde{B}_h(r,z,0) = \widetilde{B}_{0h}(r,z) \quad a.e.\ in\ \Omega,$$

where we assume that there exists $\widetilde{H}_{0h} \in \mathcal{Y}_h$ such that

$$\widetilde{B}_{0h}(r,z) = \widetilde{\mathcal{B}}(r,z,\widetilde{H}_{0h}(r,z)) \quad a.e.\ in\ \Omega. \tag{4.3}$$

A convenient $\widetilde{H}_{0h}$ has to be used for the solution of Problem 4.2 to approximate that of Problem 4.1. A possible (theoretical) choice is the Scott-Zhang interpolant of $\widetilde{H}_0 := rH_0$ (see Ref. 17) which preserves its constant values on Γ.

The existence of solution to the above problem is given by the following lemma:

Lemma 4.2. *There exists a unique solution to Problem 4.2*

Proof. Let $\{\widetilde{\varphi}_i\}_{i=1}^K$ be a basis of $\mathcal{Y}_h$, then for all $t \in [0,T]$, a solution $\widetilde{H}_h$ to Problem 4.2, can be written as follows:

$$\widetilde{H}_h(r,z,t) = \sum_{i=1}^{K} \alpha_i(t)\widetilde{\varphi}_i(r,z) \qquad (r,z) \in \Omega. \tag{4.4}$$

Similarly, we write

$$\widetilde{H}_{0h}(r,z) = \sum_{i=1}^{K} \alpha_i^0 \widetilde{\varphi}_i(r,z) \qquad (r,z) \in \Omega.$$

We set $\boldsymbol{\alpha}(t) := (\alpha_i(t))_{1\le i\le K}$, $t \in [0,T]$, and $\boldsymbol{\alpha}_0 := (\alpha_i^0)_{1\le i\le K}$. By choosing $\widetilde{G}_h = \widetilde{\varphi}_j, j = 1,\ldots,K$, in Problem 4.2, we obtain the following nonlinear system of differential equations:

$$\frac{d}{dt}\mathbf{C}\left(\boldsymbol{\alpha}(t)\right) + \mathbf{D}\boldsymbol{\alpha}(t) = \mathbf{R}(t) \quad \text{a.e. } t \in [0,T], \tag{4.5}$$

$$\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \tag{4.6}$$

where the nonlinear function $\mathbf{C} : \mathbb{R}^K \to \mathbb{R}^K$, the matrix $\mathbf{D} := (D_{i,j})_{1\le i,j\le K}$ and the vector $\mathbf{R}(t) := (R_i(t))_{1\le i\le K}$ are defined by

$$\mathbf{C}\left(\boldsymbol{\alpha}\right)_i := \int_\Omega \frac{1}{r}\widetilde{\mathcal{B}}\left(r, z, \sum_{j=1}^{K} \widetilde{\varphi}_j(r,z)\alpha_j\right)\widetilde{\varphi}_i(r,z)\, dr\, dz,$$

$$D_{i,j} := \widetilde{a}\left(\widetilde{\varphi}_i, \widetilde{\varphi}_j\right) \quad \text{and} \quad R_i(t) := \left\langle \widetilde{R}(t), \widetilde{\varphi}_i \right\rangle.$$

In order to prove the existence of solution to (4.5)-(4.6), we make a change of variable: we define $\psi_i(t) := \int_0^t \alpha_i(s)\, ds$, so that $\alpha_i = d\psi_i/dt$. Then, integrating in time (4.5), we obtain

$$\mathbf{C}\left(\frac{d\boldsymbol{\psi}}{dt}(t)\right) + \mathbf{D}\boldsymbol{\psi}(t) = \int_0^t \mathbf{R}(s)\, ds - \mathbf{C}\left(\boldsymbol{\alpha}_0\right) \quad \text{a.e. } t \in [0,T],$$

$$\boldsymbol{\psi}(0) = \mathbf{0},$$

were $\boldsymbol{\psi} := (\psi_i)_{1\le i\le K}$.

Since $\widetilde{\mathcal{B}}$ is strongly monotone and Lipschitz continuous (cf. (4.1) and (4.2)), it is straightforward to show that $\mathbf{C}$ is strongly monotone and Lipschitz continuous, too. Therefore, $\mathbf{C}$ is invertible and $\mathbf{C}^{-1}$ is also Lipschitz continuous. Hence, the system above has a unique solution $\boldsymbol{\psi} \in C^1(0,T;\mathbb{R}^K)$ (see, for instance, Ref. 8), $\boldsymbol{\alpha} = d\boldsymbol{\psi}/dt$ is the unique solution to (4.5)-(4.6) and $\widetilde{H}_h$ given by (4.4) that to Problem 4.2. □

In what follows we will prove error estimates for this semi-discrete problem. With this aim, let us introduce the so-called elliptic projector $P_h : \mathcal{Y} \cap \mathrm{H}_0^1(\Omega) \to \mathcal{Y}_h \cap \mathrm{H}_0^1(\Omega)$, defined for $u \in \mathrm{H}_0^1(\Omega)$ by

$$\widetilde{a}(P_h u, w_h) = \widetilde{a}(u, w_h) \quad \forall w_h \in \mathcal{Y}_h \cap \mathrm{H}_0^1(\Omega).$$

The following lemma yields an error estimate for $P_h u$. Its proof, based on Galerkin orthogonality and a duality argument, is standard. From now on, we suppose that C is a strictly positive constant independent of h and Δt (the time step that will be introduced below).

Lemma 4.3. *There exists $C > 0$ such that, for all $u \in \mathrm{H}^2_{1/r}(\Omega) \cap \mathrm{H}^1_0(\Omega)$,*

$$\|P_h u - u\|_{\mathrm{L}^2_{1/r}(\Omega)} + h\|P_h u - u\|_{\mathrm{H}^1_{1/r}(\Omega)} \leq C h^2 \|u\|_{\mathrm{H}^2_{1/r}(\Omega)}.$$

Next, we define the operator $\widetilde{P}_h : \mathcal{Y} \to \mathcal{Y}_h$ by

$$\widetilde{P}_h v := P_h(v - (v|_\Gamma)) + (v|_\Gamma) \quad \forall v \in \mathcal{Y}.$$

It is easy to show that

$$\widetilde{a}(\widetilde{P}_h v, v_h) = \widetilde{a}(v, v_h) \quad \forall v_h \in \mathcal{Y}_h. \tag{4.7}$$

Moreover, from Lemma 4.3 we have the following result.

Lemma 4.4. *There exists $C > 0$ such that, for all $u \in \mathrm{H}^2_{1/r}(\Omega) \cap \mathcal{Y}$,*

$$\|\widetilde{P}_h u - u\|_{\mathrm{L}^2_{1/r}(\Omega)} + h\|\widetilde{P}_h u - u\|_{\mathrm{H}^1_{1/r}(\Omega)} \leq C\, h^2\, \|u\|_{\mathrm{H}^2_{1/r}(\Omega)}.$$

Now we are in position to obtain an error estimate for the above semi-discrete problem.

Theorem 4.1. *Let $\widetilde{H}$ and $\widetilde{H}_h$ be the solutions to Problems 4.1 and 4.2, respectively. If $\widetilde{H} \in \mathrm{L}^2(0,T;\mathrm{H}^2_{1/r}(\Omega))$, then there exists $C > 0$ such that*

$$\|\widetilde{H}_h - \widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))} \leq C\Big\{h^2\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{H}^2_{1/r}(\Omega))} + \|\widetilde{H}_0 - \widetilde{H}_{0h}\|_{\mathrm{L}^2_{1/r}(\Omega)}\Big\}. \tag{4.8}$$

Proof. We proceed by means of a classical technique for parabolic equations. Let us write

$$\widetilde{H}(t) - \widetilde{H}_h(t) = \Big(\widetilde{H}(t) - \widetilde{P}_h\widetilde{H}(t)\Big) + \Big(\widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\Big). \tag{4.9}$$

Notice that the term $\widetilde{H}(t) - \widetilde{P}_h\widetilde{H}(t)$ can be bounded as in Lemma 4.4. To estimate the other one, we test Problem 4.1 with $\widetilde{G}_h \in \mathcal{Y}_h$, subtract from Problem 4.2 and integrate in time. Thus we obtain, for $t \in (0,T]$

$$\int_\Omega \frac{1}{r}(\widetilde{B} - \widetilde{B}_h)(t)\widetilde{G}_h\, dr\, dz + \widetilde{a}\left(\int_0^t (\widetilde{H} - \widetilde{H}_h)(s)\, ds, \widetilde{G}_h\right) = \int_\Omega \frac{1}{r}(\widetilde{B}_0 - \widetilde{B}_{0h})\widetilde{G}_h\, dr\, dz.$$

Hence, from (4.7) we arrive at

$$\begin{aligned}\int_\Omega \frac{1}{r}(\widetilde{\mathcal{B}}(\widetilde{P}_h\widetilde{H}(t)) - \widetilde{\mathcal{B}}(\widetilde{H}_h(t)))\widetilde{G}_h\, dr\, dz + \widetilde{a}\left(\int_0^t (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(s)\, ds,\, \widetilde{G}_h\right)\\ = \int_\Omega \frac{1}{r}(\widetilde{B}_0 - \widetilde{B}_{0h})\widetilde{G}_h\, dr\, dz + \int_\Omega \frac{1}{r}(\widetilde{\mathcal{B}}(\widetilde{P}_h\widetilde{H}(t)) - \widetilde{\mathcal{B}}(\widetilde{H}(t)))\widetilde{G}_h\, dr\, dz.\end{aligned}$$

Now we take $\widetilde{G}_h := \widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)$. Integrating in time, using the strong monotonicity and Lipschitz continuity of $\widetilde{\mathcal{B}}$ (cf. (4.1) and (4.2)) and Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned}
&\frac{\omega}{2}\int_0^T \|\widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\,dt \\
&\qquad + \int_0^T \widetilde{a}\left(\int_0^t (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(s)\,ds, \widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\right)\,dt \\
&\quad \le \frac{TL^2}{\omega}\|\widetilde{H}_0 - \widetilde{H}_{0h}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \frac{L^2}{\omega}\int_0^T \|\widetilde{P}_h\widetilde{H}(t) - \widetilde{H}(t)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\,dt.
\end{aligned} \tag{4.10}$$

To estimate the right-hand side above we use Lemma 4.4, whereas, for the left-hand side we use the following equality (analogous to (3.11))

$$\begin{aligned}
&\int_0^T \widetilde{a}\left(\int_0^t (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(s)\,ds, \widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\right)\,dt \\
&\qquad = \frac{1}{2}\,\widetilde{a}\left(\int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt, \int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt\right)
\end{aligned} \tag{4.11}$$

and the fact that

$$\begin{aligned}
&\frac{\omega}{4}\int_0^T \|\widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\,dt + \beta\left\|\int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)} \\
&\quad \le \frac{\omega}{2}\int_0^T \|\widetilde{P}_h\widetilde{H}(t) - \widetilde{H}_h(t)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\,dt \\
&\qquad + \frac{1}{2}\,\widetilde{a}\left(\int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt, \int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt\right),
\end{aligned} \tag{4.12}$$

for some positive constant β, which follows from (4.11) and Lemma 4.1. Thus, from (4.10) and (4.12) we obtain

$$\begin{aligned}
&\|\widetilde{P}_h\widetilde{H} - \widetilde{H}_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))} + \left\|\int_0^T (\widetilde{P}_h\widetilde{H} - \widetilde{H}_h)(t)\,dt\right\|_{\mathrm{H}^1_{1/r}(\Omega)} \\
&\quad \le C\Big\{h^2\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{H}^2_{1/r}(\Omega))} - \|\widetilde{H}_0 - \widetilde{H}_{0h}\|_{\mathrm{L}^2_{1/r}(\Omega)}\Big\}.
\end{aligned} \tag{4.13}$$

Therefore, (4.8) follows from (4.9), (4.13) and Lemma 4.4, and we conclude the proof. □

Remark 4.1. If $\widetilde{H}_0 \in \mathrm{H}^2_{1/r}(\Omega)$, then we can use the Lagrange interpolant of $\widetilde{H}_0$ as $\widetilde{H}_{0h}$, and in such a case, we have

$$\|\widetilde{H}_h - \widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))} \le Ch^2\Big\{\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{H}^2_{1/r}(\Omega))} + \|\widetilde{H}_0\|_{\mathrm{H}^2_{1/r}(\Omega)}\Big\}.$$

Remark 4.2. It is straightforward to obtain from (4.9), Lemma 4.4 and (4.13) the following $\mathrm{L}^2(0,T;\mathrm{H}^1_{1/r}(\Omega))$-type error estimate:

$$\sup_{t\in[0,T]}\left\|\int_0^t(\widetilde{H}_h-\widetilde{H})(s)\,ds\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_{1/r}(\Omega))}\le C\left\{\|\widetilde{H}_0-\widetilde{H}_{0h}\|_{\mathrm{L}^2_{1/r}(\Omega)}+h\|\widetilde{H}\|_{\mathrm{L}^2(0,T;\mathrm{H}^2_{1/r}(\Omega))}\right\}.$$

4.2. *Full discretization*

In this section we introduce a time discretization of Problem 4.2 by means of a backward Euler scheme and prove its convergence. We consider a uniform partition $\{t^i:=i\Delta t,\, i=0,\dots,M\}$ of $[0,T]$, with time step $\Delta t:=T/M$, $M\in\mathbb{N}$. The notation $\bar\partial z^{i+1}$ refers to the difference quotient

$$\bar\partial z^{i+1}:=\frac{z^{i+1}-z^i}{\Delta t}.$$

We consider the following further assumption on the data of the problem:

H.8 $f\in\mathrm{H}^1(0,T;\mathcal{W}')$.

A full discretization of Problem 4.1 stands as follows:

Problem 4.3. *For $i=0,\dots,M-1$, find $\widetilde{H}_h^{i+1}\in\mathcal{Y}_h$ and $\widetilde{B}_h^{i+1}\in\mathrm{L}^2_{1/r}(\Omega)$ satisfying*

$$\int_\Omega\frac{1}{r}\bar\partial\widetilde{B}_h^{i+1}\widetilde{G}_h\,dr\,dz+\widetilde{a}(\widetilde{H}_h^{i+1},\widetilde{G}_h)=\langle\widetilde{R}^{i+1},\widetilde{G}_h\rangle\quad\forall\widetilde{G}_h\in\mathcal{Y}_h,\tag{4.14}$$

$$\widetilde{B}_h^{i+1}(r,z)=\widetilde{\mathcal{B}}(r,z,\widetilde{H}_h^{i+1}(r,z))\quad a.e.\ in\ \Omega,\tag{4.15}$$

$$\widetilde{B}_h^{0}(r,z)=\widetilde{B}_{0h}(r,z)\quad a.e.\ in\ \Omega,\tag{4.16}$$

where $\widetilde{B}_{0h}$ is as in (4.3). In the problem above, we have used $\widetilde{R}^{i+1}\in\mathcal{Y}'$, defined by

$$\langle\widetilde{R}^{i+1},\widetilde{G}\rangle:=\langle f(t^{i+1}),r^{-1}\widetilde{G}\rangle_{\mathcal{W},\mathcal{W}'}+\left(\bar\partial b(t^{i+1})-\langle f(t^{i+1}),r^{-1}\rangle_{\mathcal{W},\mathcal{W}'}\right)(\widetilde{G}|_\Gamma)\qquad G\in\mathcal{Y},$$

to approximate $\widetilde{R}(t^{i+1})$, $i=0,\dots,M-1$.

The existence of solution at each time step is guaranteed by the following lemma.

Lemma 4.5. *There exists a unique solution to Problem 4.3.*

Proof. For each $i=0,\dots,M-1$, we rewrite (4.14) as follows:

$$\mathcal{Z}(\widetilde{H}_h^{i+1})=\widetilde{R}^{i+1}|_{\mathcal{Y}_h}+\widetilde{F}^{i+1}\qquad\text{in }\mathcal{Y}_h',\tag{4.17}$$

with $\mathcal{Z}:\mathcal{Y}_h\to\mathcal{Y}_h'$ defined by

$$\langle\mathcal{Z}(\widetilde{H}_h^{i+1}),\widetilde{G}_h\rangle_{\mathcal{Y}_h,\mathcal{Y}_h'}:=\int_\Omega\frac{1}{r}\widetilde{\mathcal{B}}(r,z,\widetilde{H}_h^{i+1}(r,z))\widetilde{G}_h\,dr\,dz+\Delta t\,\widetilde{a}(\widetilde{H}_h^{i+1},\widetilde{G}_h)\quad\widetilde{G}_h\in\mathcal{Y}_h$$

and $\widetilde{F}^{i+1} \in \mathcal{Y}_h^{'}$ by

$$\langle \widetilde{F}^{i+1}, \widetilde{G}_h \rangle_{\mathcal{Y}_h, \mathcal{Y}_h^{'}} := \int_\Omega \frac{1}{r} \widetilde{\mathcal{B}}(r, z, \widetilde{H}_h^i(r,z)) \widetilde{G}_h \, dr \, dz \quad \forall \widetilde{G}_h \in \mathcal{Y}_h.$$

Since $\widetilde{\mathcal{B}}$ is strongly monotone and Lipschitz continuous (cf. (4.1) and (4.2)) and

$$\widetilde{a}(\widetilde{G}_h, \widetilde{G}_h) \geq \frac{1}{\sigma^*} | \widetilde{G}_h |^2_{\mathrm{H}^1_{1/r}(\Omega)} \quad \forall \widetilde{G}_h \in \mathcal{Y}_h,$$

we have that $\mathcal{Z} : \mathcal{Y}_h \to \mathcal{Y}_h^{'}$ is a strongly monotone, Lipschitz continuous operator. Thus, applying the Banach fixed-point technique, it can be shown that the equation (4.17) ($i = 0, \ldots, M-1$) has a unique solution. (see, for instance, Ref. 16). □

The following theorem provides an error estimate for the fully-discrete problem.

Theorem 4.2. *Let $\widetilde{H}$ and $\widetilde{H}_h^{i+1}$ be the solutions to Problems 4.1 and 4.3, respectively. If $\widetilde{H} \in \mathrm{H}^1(0,T; \mathrm{H}^2_{1/r}(\Omega))$, then there exists $C > 0$ such that*

$$\left(\sum_{i=0}^{M-1} \Delta t \| \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1} \|^2_{\mathrm{L}^2_{1/r}(\Omega)} \right)^{1/2}$$
$$\leq C \Big\{ (\Delta t + h^2) \| \widetilde{H} \|_{\mathrm{H}^1(0,T;\mathrm{H}^2_{1/r}(\Omega))} + \| \widetilde{H}_0 - \widetilde{H}_{0h} \|_{\mathrm{L}^2_{1/r}(\Omega)} + \Delta t \, \| f \|_{\mathrm{H}^1(0,T;\mathcal{W}')} \Big\}.$$

Proof. We write as in the proof of Theorem 4.1

$$\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1} = \left(\widetilde{H}(t^{i+1}) - \widetilde{P}_h \widetilde{H}(t^{i+1}) \right) + \left(\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1} \right) \tag{4.18}$$

and focus on estimating the second term. First, by taking $\widetilde{G} = \widetilde{G}_h$ in Problem 4.1, integrating from 0 to $t^{l+1} \in (0,T]$ and using (4.7), we obtain

$$\begin{aligned} \int_\Omega \frac{1}{r} \widetilde{\mathcal{B}}(\widetilde{H}(t^{l+1})) \, \widetilde{G}_h \, dr \, dz + \Delta t \, \widetilde{a} \left(\sum_{i=0}^{l} \widetilde{P}_h \widetilde{H}(t^{i+1}), \widetilde{G}_h \right) \\ = \widetilde{a} \left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t) \, dt, \widetilde{G}_h \right) + \left\langle \int_0^{t^{l+1}} \widetilde{R}(t) \, dt, \widetilde{G}_h \right\rangle \\ + \int_\Omega \frac{1}{r} \, \widetilde{B}_0 \, \widetilde{G}_h \, dr \, dz \qquad \forall \widetilde{G}_h \in \mathcal{Y}_h, \end{aligned} \tag{4.19}$$

with $\widetilde{H}_{\Delta t}$ being the piecewise constant interpolant of $\widetilde{H}$ (i.e., $\widetilde{H}_{\Delta t}(t^0) := \widetilde{H}(t^0)$ and $\widetilde{H}_{\Delta t}(t) := \widetilde{H}(t^i)$, $t \in (t^{i-1}, t^i]$). Then, by summing up (4.14) for $i = 0, \ldots, l$, with

$l \in \{0, \ldots, M-1\}$, and subtracting from (4.19), we have

$$\begin{aligned}
&\int_\Omega \frac{1}{r}(\widetilde{\mathcal{B}}(\widetilde{P}_h \widetilde{H}(t^{l+1})) - \widetilde{\mathcal{B}}(\widetilde{H}_h^{l+1}))\widetilde{G}_h \, dr \, dz + \Delta t \, \widetilde{a}\left(\sum_{i=0}^{l}(\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}), \widetilde{G}_h\right) \\
&\quad = \int_\Omega \frac{1}{r}(\widetilde{B}_0 - \widetilde{B}_{0h})\widetilde{G}_h \, dr \, dz + \int_\Omega \frac{1}{r}(\widetilde{\mathcal{B}}(\widetilde{P}_h \widetilde{H}(t^{l+1})) - \widetilde{\mathcal{B}}(\widetilde{H}(t^{l+1})))\widetilde{G}_h \, dr \, dz \\
&\qquad + \widetilde{a}\left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t) \, dt, \, \widetilde{G}_h\right) + \left\langle \int_0^{t^{l+1}} \widetilde{R}(t) \, dt - \Delta t \sum_{i=0}^{l} \widetilde{R}^{i+1}, \widetilde{G}_h \right\rangle .
\end{aligned} \tag{4.20}$$

The last term above can be written as follows: for all $\widetilde{G}_h \in \mathcal{Y}_h$

$$\left\langle \int_0^{t^{l+1}} \widetilde{R}(t) \, dt - \Delta t \sum_{i=0}^{l} \widetilde{R}^{i+1}, \widetilde{G}_h \right\rangle = \int_0^{t^{l+1}} \left\langle E_f(t), \widetilde{G} \right\rangle \, dt$$

where $E_f \in \mathrm{L}^2(0,T,\mathcal{Y}')$ is defined, a.e. $t \in (0,T)$ by

$$\begin{aligned}
\left\langle E_f(t), \widetilde{G} \right\rangle &:= \left\langle (f - f_{\Delta t})(t), r^{-1}\widetilde{G} \right\rangle_{\mathcal{W},\mathcal{W}'} \\
&\quad - \left\langle (f - f_{\Delta t})(t), r^{-1} \right\rangle_{\mathcal{W},\mathcal{W}'} (\widetilde{G}|_\Gamma)
\end{aligned}$$

for $\widetilde{G} \in \mathcal{Y}$, with $f_{\Delta t}$ being the piecewise constant interpolant of f defined as above. Notice that, clearly,

$$\|E_f\|_{\mathrm{L}^2(0,T;\mathcal{Y}')} \leq C \, \|f - f_{\Delta t}\|_{\mathrm{L}^2(0,T;\mathcal{W}')} . \tag{4.21}$$

Now, by choosing $\widetilde{G}_h = \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}$ in (4.20) and using the monotonicity and Lipschitz continuity of $\widetilde{\mathcal{B}}$ (cf. (4.1) and (4.2)) and Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned}
&\frac{\omega}{2}\|\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \Delta t \, \widetilde{a}\left(\sum_{i=0}^{l}(\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}), \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\right) \\
&\quad \leq \frac{L^2}{\omega}\|\widetilde{H}_0 - \widetilde{H}_{0h}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \frac{L^2}{\omega}\|\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}(t^{l+1})\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \\
&\qquad + \widetilde{a}\left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t) \, dt, \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\right) \\
&\qquad + \left\langle \int_0^{t^{l+1}} E_f(t) \, dt, \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} \right\rangle .
\end{aligned}$$

Summing up the above equation for $l = 0, \dots, M-1$, we obtain

$$\begin{aligned}
&\frac{\omega}{2} \sum_{l=0}^{M-1} \Delta t \, \|\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \\
&\qquad + \Delta t^2 \sum_{l=0}^{M-1} \widetilde{a} \left(\sum_{i=0}^{l} (\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}), \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} \right) \\
&\quad \le \frac{L^2 T}{\omega} \|\widetilde{H}_0 - \widetilde{H}_{0h}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \frac{L^2 \Delta t}{\omega} \sum_{l=0}^{M-1} \|\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}(t^{l+1})\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \\
&\qquad + \Delta t \sum_{l=0}^{M-1} \widetilde{a} \left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t) \, dt, \, \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} \right) \\
&\qquad + \Delta t \sum_{l=0}^{M-1} \left\langle \int_0^{t^{l+1}} E_f(t) \, dt, \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} \right\rangle . \qquad (4.22)
\end{aligned}$$

First, we will deal with the left-hand side above. We rewrite its second term by using the following identity, for $l \ge 1$:

$$\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} = \sum_{i=0}^{l} (\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}) - \sum_{i=0}^{l-1} (\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}). \qquad (4.23)$$

Thus we obtain a discrete version of (4.11), namely,

$$\begin{aligned}
&\Delta t^2 \sum_{l=0}^{M-1} \widetilde{a} \left(\sum_{i=0}^{l} (\widetilde{P}_h \widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}), \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1} \right) \\
&\quad = \frac{1}{2} \widetilde{a} \left(\Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}), \Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}) \right).
\end{aligned}$$

Using this and Lemma 4.1, we obtain the following estimates for the left-hand side of (4.22): there exists $\beta > 0$ (which depends on ω, T and σ^*) such that

$$\begin{aligned}
&\frac{\omega}{2} \sum_{l=0}^{M-1} \Delta t \|\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \\
&\qquad + \frac{1}{2} \widetilde{a} \left(\Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}), \Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}) \right) \\
&\quad \ge \frac{\omega}{4} \sum_{l=0}^{M-1} \Delta t \| \widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \\
&\qquad + \beta \left\| \Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h \widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}) \right\|^2_{\mathrm{H}^1_{1/r}(\Omega)} . \qquad (4.24)
\end{aligned}$$

Next, we estimate the right-hand side of (4.22). The second term will be easily bounded by means of Lemma 4.4. For the third term we use (4.23) and summation by parts to obtain

$$\begin{aligned}
&\Delta t \sum_{l=0}^{M-1} \widetilde{a}\left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t)\,dt,\, \widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\right) \\
&\quad = \widetilde{a}\left(\int_0^{T} (\widetilde{H}_{\Delta t} - \widetilde{H})(t)\,dt,\, \Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1})\right) \\
&\qquad - \sum_{l=0}^{M-2} \widetilde{a}\left(\int_{t^{l+1}}^{t^{l+2}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t)\,dt,\, \Delta t \sum_{i=0}^{l} (\widetilde{P}_h\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1})\right).
\end{aligned}$$

Hence, using the continuity of $\widetilde{a}$ and Young's inequality, we obtain that for all $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$\begin{aligned}
&\Delta t \sum_{l=0}^{M-1} \widetilde{a}\left(\int_0^{t^{l+1}} (\widetilde{H}_{\Delta t} - \widetilde{H})(t)\,dt,\, \widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\right) \\
&\quad \leq C_\alpha \|\widetilde{H}_{\Delta t} - \widetilde{H}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1_{1/r}(\Omega))} + \frac{\alpha}{2}\left\|\Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1})\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)} \\
&\qquad + \frac{1}{2}\sum_{l=0}^{M-2}\left\|\Delta t \sum_{i=0}^{l} (\widetilde{P}_h\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1})\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)}.
\end{aligned} \tag{4.25}$$

For the last term of (4.22), we proceed analogously to obtain

$$\begin{aligned}
&\Delta t \sum_{l=0}^{M-1} \left\langle \int_0^{t^{l+1}} E_f(t)\,dt, \widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\right\rangle \\
&\quad \leq C_\alpha \|E_f\|^2_{\mathrm{L}^2(0,T;\mathcal{Y}')} + \frac{\alpha}{2}\left\|\Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1})\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)} \\
&\qquad + \frac{1}{2}\sum_{l=0}^{M-2}\left\|\Delta t \sum_{i=0}^{l} (\widetilde{P}_h\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1})\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)}.
\end{aligned} \tag{4.26}$$

By taking $\alpha := \beta/2$, replacing (4.24)-(4.26) in (4.22) and using (4.21) and the discrete Gronwall's inequality, we arrive at

$$\begin{aligned}
&\sum_{l=0}^{M-1} \Delta t \| \widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \left\|\Delta t \sum_{l=0}^{M-1} (\widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}_h^{l+1})\right\|^2_{\mathrm{H}^1_{1/r}(\Omega)} \\
&\quad \leq C\left\{ \|\widetilde{H}_{\Delta t} - \widetilde{H}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1_{1/r}(\Omega))} + \|f - f_{\Delta t}\|^2_{\mathrm{L}^2(0,T;\mathcal{W}')} \right. \\
&\qquad \left. + \|\widetilde{H}_0 - \widetilde{H}_{0h}\|^2_{\mathrm{L}^2_{1/r}(\Omega)} + \Delta t \sum_{l=0}^{M-1} \|\widetilde{P}_h\widetilde{H}(t^{l+1}) - \widetilde{H}(t^{l+1})\|^2_{\mathrm{L}^2_{1/r}(\Omega)} \right\}.
\end{aligned} \tag{4.27}$$

Thus, the result follows from (4.18), Lemma 4.4 and classical approximation results for the piecewise constant interpolant. □

Remark 4.3. As noted in Remark 4.1, if $\widetilde{H}_0 \in \mathrm{H}^2_{1/r}(\Omega)$, then the Lagrange interpolant of $\widetilde{H}_0$ can be used as $\widetilde{H}_{0h}$ and, in such a case, we conclude that

$$\left(\sum_{i=0}^{M-1} \Delta t \|\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1}\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\right)^{1/2} \leq C\Big\{(\Delta t + h^2)\|\widetilde{H}\|_{\mathrm{H}^1(0,T;\mathrm{H}^2_{1/r}(\Omega))} + h^2\|\widetilde{H}_0\|_{\mathrm{H}^2_{1/r}(\Omega)} + \Delta t\,\|f\|_{\mathrm{H}^1(0,T;\mathcal{W}')}\Big\}.$$

Remark 4.4. A result analogous to that of Remark 4.2 also holds true. In fact, from (4.18), Lemma 4.4 and (4.27) it is straightforward to prove that

$$\max_{l\in\{1,\ldots,M\}} \left\|\sum_{i=0}^{l-1} \Delta t(\widetilde{H}(t^{i+1}) - \widetilde{H}_h^{i+1})\right\|_{\mathrm{H}^1_{1/r}(\Omega)} \leq C\Big\{(\Delta t + h)\|\widetilde{H}\|_{\mathrm{H}^1(0,T;\mathrm{H}^2_{1/r}(\Omega))} + \|\widetilde{H}_0 - \widetilde{H}_{0h}\|_{\mathrm{L}^2_{1/r}(\Omega)} + \Delta t\|f\|_{\mathrm{H}^1(0,T;\mathcal{W}')}\Big\}.$$

5. Numerical results

In this section we report some numerical results obtained with a Fortran code, which implements the numerical method described above. In order to analyze the convergence properties of the numerical scheme, we apply it to a test problem with a known analytical solution.

We consider the eddy current Problem 2.1 defined in the meridian section Ω :=[0.06,0.18] × [0,0.06], where the dimensions are given in meters. The right-hand side f is chosen so that

$$H = \frac{150\exp(t)}{r}\sin\left(\frac{\pi r}{0.06}\right)\sin\left(\frac{\pi z}{0.06}\right)$$

is the solution to the problem. Notice that $\widetilde{H} = rH$ is constant (actually it vanishes) on the boundary of the domain.

We consider a nonlinear material whose magnetization is given by its anhysteretic H-B curve defined by

$$\mathcal{B}(H) := \mu_0 H + \frac{2J_s}{\pi}\arctan\left(\frac{\pi(\mu_r - 1)\mu_0 H}{2\ J_s}\right), \tag{5.1}$$

where $\mu_0 = 4\pi\times10^{-7}\,\mathrm{Hm}^{-1}$, $\mu_r = 3000$ and $J_s = 1.89$ T. This curve, whose positive part is shown in Fig. 2, is very similar to the first magnetization curve of laminated steels (cf. Ref. 21). The value of the electrical conductivity is $\sigma = 4 \times 10^6$ (Ohm m)$^{-1}$.

The problem has been solved in the time interval [0,2] so that the values of the solution H vary approximately between -12000 and 12000 A/m. Hence, the nonlinear part of the curve is clearly attained (see Fig. 2).

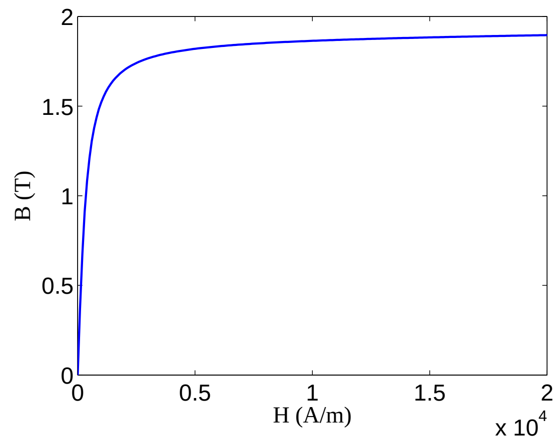

Fig. 2. Positive part of the H-B curve.

The numerical method has been applied with several successively refined meshes and time-steps. The nonlinear system arising at each time step have been solved with a Newton's iteration. A sufficiently small tolerance has been chosen (10^{-4}), so that the error of this iteration be negligible. The computed approximate solutions have been compared with the analytical one by calculating the percentual relative error for $\widetilde{H}$ and $\mathbf{grad}\,\widetilde{H}$ in the $\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))$-norm by means of

$$\mathcal{E}_h^{\Delta t}(\tilde{H}) := 100 \frac{\left(\sum_{k=1}^M \Delta t \|\widetilde{H}(t^k) - \widetilde{H}_h^k\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\right)^{1/2}}{\left(\sum_{k=1}^M \Delta t \|\widetilde{H}(t^k)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\right)^{1/2}},$$

$$\mathcal{E}_h^{\Delta t}\,(\mathbf{grad}\,\widetilde{H}) := 100 \frac{\left(\sum_{k=1}^M \Delta t \|\,\mathbf{grad}\,\widetilde{H}(t^k) - \,\mathbf{grad}\,\widetilde{H}_h^k\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\right)^{1/2}}{\left(\sum_{k=1}^M \Delta t\,\|\,\mathbf{grad}\,\widetilde{H}(t^k)\|^2_{\mathrm{L}^2_{1/r}(\Omega)}\right)^{1/2}}.$$

Table 1 shows the relative errors for $\widetilde{H}$ at different levels of discretization. We notice that by taking a small enough time-step Δt one can observe the behavior of the error with respect to the space discretization (see the row corresponding to $\Delta t/128$). On the other hand, by considering a small enough mesh-size h, one can inspect the order of convergence with respect to Δt (see the column corresponding to $h/16$). In this example, we observe an order of convergence $\mathcal{O}(h^2 + \Delta t)$ for $\widetilde{H}$, which is the one expected from the theoretical analysis (cf. Remark 4.3).

In Table 2 we show the percentual relative errors for $\mathbf{grad}\,\widetilde{H}$ in the $\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))$-norm. In this case, the space discretization error dominates the time discretization one, even for the finest mesh. In fact, an order $\mathcal{O}(h)$ can be observed for both time steps. Let us remark that we have not proved theoretically this experimental result (note that the estimates in Remark 4.4 are in a different norm).

Table 1. Relative error (%) for $\widetilde{H}$: $\mathcal{E}_h^{\Delta t}(\widetilde{H})$.

	h	$h/2$	$h/4$	$h/8$	$h/16$
Δt	$\boxed{13.85}$	2.91	0.63	0.65	0.75
$\Delta t/2$	14.04	3.14	0.61	0.30	0.38
$\Delta t/4$	14.14	$\boxed{3.25}$	0.70	0.15	0.18
$\Delta t/8$	14.19	3.32	0.77	0.15	0.08
$\Delta t/16$	14.21	3.36	$\boxed{0.80}$	0.17	0.04
$\Delta t/32$	14.22	3.37	0.82	0.19	0.04
$\Delta t/64$	14.21	3.38	0.83	$\boxed{0.20}$	0.04
$\Delta t/128$	14.20	3.38	0.83	0.20	$\boxed{0.04}$

Table 2. Relative error (%) for $\mathbf{grad}\,\widetilde{H}$: $\mathcal{E}_h^{\Delta t}(\mathbf{grad}\,\widetilde{H})$.

	h	$h/2$	$h/4$	$h/8$	$h/16$
Δt	94.04	49.88	25.33	12.73	6.41
$\Delta t/2$	94.13	49.95	25.36	12.73	6.38

Once the order of convergence is checked, we report in one single figure the simultaneous dependence on h and Δt for $\widetilde{H}$ in $\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))$-norm by proceeding in the following way: we choose initial coarse values of h and Δt and, for each successively refined mesh, we take a value of Δt proportional to h^2 (see the values within boxes in Table 1). Fig. 3 shows a log-log plot of the corresponding relative errors for $\widetilde{H}$ in the $\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))$-norm versus the number of degrees of freedom (d.o.f.). The slope of the curve shows an order of convergence $\mathcal{O}(h^2) = \mathcal{O}(h^2 + \Delta t)$. In a similar way, Fig. 4 shows an order $\mathcal{O}(h + \Delta t)$ for $\mathbf{grad}\,\widetilde{H}$ in the $\mathrm{L}^2(0,T;\mathrm{L}^2_{1/r}(\Omega))$-norm.

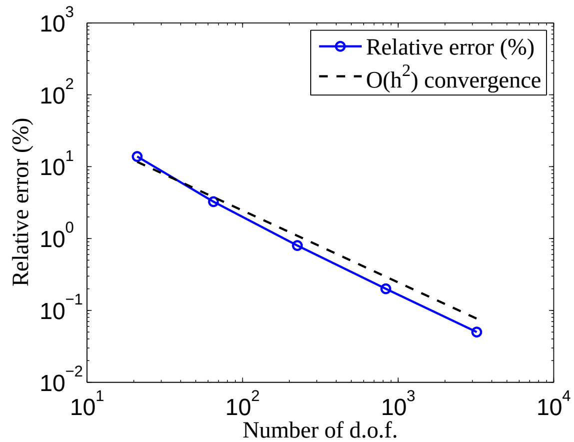

Fig. 3. $\mathcal{E}_h^{\Delta t}(\widetilde{H})$ versus number of d.o.f. (log-log scale), $\Delta t = Ch^2$.

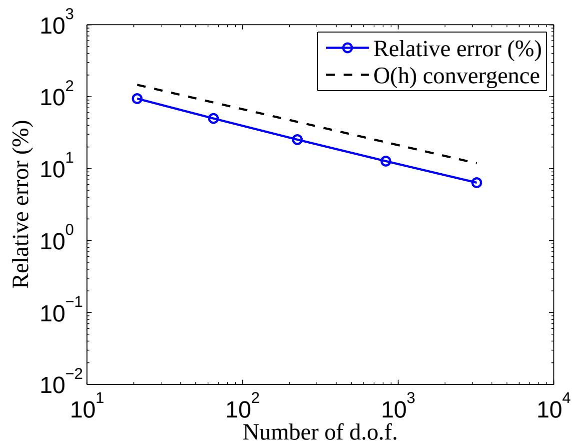

Fig. 4. $\mathcal{E}_h^{\Delta t}(\mathbf{grad}\,\widetilde{H})$ versus number of d.o.f. (log-log scale), $\Delta t = Ch$.

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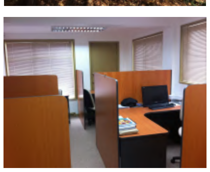

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