UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Linear stochastic Schrödinger equations with unbounded coefficients

Franco Fagnola, Carlos M. Mora

PREPRINT 2010-08

SERIE DE PRE-PUBLICACIONES

Linear stochastic Schrödinger equations with unbounded coefficients

Franco Fagnola and Carlos M. Mora

Abstract. We deal with stochastic evolution equations that describe the dynamics of open quantum systems. In particular, we focus on physical systems with infinite dimensional space states such as bosons and atoms. Using resolvent approximations, we obtain a general sufficient condition for the existence and uniqueness of regular solutions to the linear stochastic Schrödinger equations driven by cylindrical Brownian motions. From this we get a new criterion for the existence and uniqueness of weak (probabilistic) regular solutions to the non-linear stochastic Schrödinger equations. These stochastic evolution equations on complex Hilbert spaces govern quantum measurement processes. We apply our results to physical systems involving, essentially, measurements of the position of particles.

Mathematics Subject Classification (2000). Primary 60H15; Secondary 60H30, 81C20, 46L55.

Keywords. Open quantum systems, Stochastic Schrödinger equations, existence and uniqueness of solutions, regularity of solutions, non-commutative dynamical systems, measurement processes.

1. Introduction

This paper develops stochastic Schrödinger equations with infinite dimensional space states. In other words, we are interested in stochastic evolution equations arising from open quantum systems formed, for example, by bosons, atoms and/or

---

This research was partially supported by FONDECYT Grants 7090088 and 1070686.
C.M.M. has been partially supported also by PBCT-ACT 13, FONDAP - BASAL projects CMM, Universidad de Chile, and by Centro de Investigación en Ingeniería Matemática (CI^2MA) .

infinitely many particles. First, using resolvent approximations we establish a general sufficient condition for the existence and uniqueness of regular solutions to

$$X_t(\xi) = \xi + \int_0^t G(s)\, X_s(\xi)\, ds + \sum_{k=1}^{\infty} \int_0^t L_k(s)\, X_s(\xi)\, dW_s^k. \tag{1.1}$$

Here, $\left(W^k\right)_{k\in\mathbb{N}}$ is a sequence of real valued independent Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\right)$, X is a pathwise continuous adapted stochastic processes taking values on a complex separable Hilbert space $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$, and $(G(t))_{t\geq 0}, (L_k(t))_{t\geq 0}$ are given families of linear operators in $\mathfrak{h}$ satisfying formally

$$G(t) = -iH(t) - \frac{1}{2}\sum_{k=1}^{\infty} L_k(t)^* L_k(t), \tag{1.2}$$

with $H(t)$ self-adjoint operator. Second, we obtain the well-posedness of (1.1) for a concrete class of physical systems. We deal, for example, with regularized versions of the Hydrogen atom in interaction with heat baths.

A motivation for this article come from the relevant role playing by the stochastic evolution equation with respect to cylindrical Brownian motion (1.1) in the study of open quantum systems. In fact, $\mathbb{E}\langle X_t(\xi), \cdot\rangle X_t(\xi)$ represents the density operator at time t. From the mathematical point of view, with the help of (1.1) we can deduce properties of the deterministic operator equations describing the evolution of open quantum systems (see, e.g., [25, 26]). Moreover, (1.1) constitute an important tool for proving the existence and uniqueness of solutions to the so-called non-linear stochastic Schrödinger equations (i.e., the stochastic partial differential equation (4.1) given below). These non-linear stochastic evolution equations govern, for instance, quantum measurement processes (see, e.g., [18, 32]) and allow the numerical simulation of the evolution of the mean values of quantum observables (see, e.g. [8, 22, 28]).

In [20], Holevo obtained the existence of weak (topological) solutions to (1.1). Previously, approximating dissipative stochastic evolution equations by coercive ones, Rozovskii [30] proved the existence and uniqueness of weak (topological) regular solutions for a general class of dissipative linear stochastic evolution equations on Hilbert spaces. Applying the Galerkin method directly to (1.1), [21, 23] showed that (1.1) has a unique strong regular solution. It is worth mentioning that non-commutative versions of (1.1) driven by a finite number of quantum noises have been treated, for example, in [7, 17].

In Subsection 3.1, we establish the existence and uniqueness of strong (topological) regular solutions to (1.1) under general hypotheses. Indeed, by means of resolvent approximations, Subsection 6.2 obtains the existence of strong regular solutions to (1.1) under general hypotheses. Furthermore, Subsection 3.2 examines the martingale and Markov properties of $X(\xi)$.

Section 3 and the corresponding Subsection 6.2 refine the techniques to show the existence of strong regular solutions to (1.1). From this we obtain a general sufficient condition for the existence and uniqueness of strong regular solutions to (1.1), which strengthens the applications of the stochastic Schrödinger equations to real physical systems. In fact, Sections 3 and 4 lead to a new criterion for the existence and uniqueness of weak (probabilistic) regular solutions to the non-linear stochastic Schrödinger equations. Using Section 3, [26] proves the well-posedness of the mean value of unbounded observables (like number, position and momentum operators) with respect to the solutions of the quantum master equations. Under the light of Section 3, Section 5 develops a class of open quantum systems formulated in coordinate representation. Moreover, the resolvent approximations used in Subsection 6.2 can yield new results on the existence of invariant density operators for the quantum master equations (this work is in progress).

In Section 5 we restrict our attention to models that describe, essentially, quantum non-demolition measurements of position. These open quantum systems have been studied in detail in the physical literature (see, e.g., [8, 14, 18, 32]), and can be reproduced in the laboratory using mechanical detectors. In particular, we verify that their equations of motion in coordinate representation satisfy the assumptions of Section 3. We also illustrate that our results applied to atoms whose Coulomb potentials have been regularized. In addition to that these physical systems have interest by themselves, their development is a step towards the understanding of open quantum systems whose Hamiltonian operators have singularities (see, e.g., [1, 10] for early works in this direction).

For the reader convenience, Section 2 recalls notation and Section 6 is devoted to proofs.

2. Notation

In this article, $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ is a separable complex Hilbert space whose scalar product $\langle \cdot, \cdot \rangle$ is linear in the second variable and anti-linear in the first one. We write $\mathcal{D}(A)$ for the domain of A, whenever A is a linear operator in $\mathfrak{h}$. If $\mathfrak{X}$, $\mathfrak{Z}$ are normed spaces, then we denote by $\mathfrak{L}(\mathfrak{X}, \mathfrak{Z})$ the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ and we define $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$. We set $[A, B] = AB - BA$ whenever A, B are operators in $\mathfrak{h}$. By $\mathcal{B}(\mathfrak{Y})$ we mean the set of all Borel set of the topological space $\mathfrak{Y}$.

Suppose that C be a self-adjoint positive operator in $\mathfrak{h}$. For any $x, y \in \mathcal{D}(C)$ we define $\langle x, y \rangle_C = \langle x, y \rangle + \langle Cx, Cy \rangle$ and $\|x\|_C = \sqrt{\langle x, x \rangle_C}$ is the graph norm of C. We use the symbol $L^2(\mathbb{P}, \mathfrak{h})$ to denote the set of all square integrable random variables from $(\Omega, \mathfrak{F}, \mathbb{P})$ to $(\mathfrak{h}, \mathfrak{B}(\mathfrak{h}))$. Moreover, $L^2_C(\mathbb{P}, \mathfrak{h})$ stands for the set of all $\xi \in L^2(\mathbb{P}, \mathfrak{h})$ such that $\xi \in \mathcal{D}(C)$ a.s. and $\mathbb{E}\|\xi\|_C^2 < \infty$. We define $\pi_C : \mathfrak{h} \to \mathfrak{h}$ to be

$$\pi_C(x) = \begin{cases} x, & \text{if } x \in \mathcal{D}(C) \\ 0, & \text{if } x \notin \mathcal{D}(C) \end{cases} .$$

We denote by $L^2(\mathbb{R}^n, \mathbb{C})$, with $n \in \mathbb{N}$, the set of all square integrable functions from $\mathbb{R}^n$ (equipped with the Lebesgue measure ν) to $\mathbb{C}$. In case $g : \mathbb{R}^n \mapsto \mathbb{C}$ is

Borel measurable, to simplify notation we continue to write g for the operator in $L^2(\mathbb{R}^n, \mathbb{C})$ given by $f \mapsto gf$. Let $C^k(\mathbb{R}^n, \mathbb{C})$ (respectively $C^k(\mathbb{R}^n, \mathbb{R})$) be the set of all functions from $\mathbb{R}^n$ to $\mathbb{C}$ (respectively $\mathbb{R}$) with continuous partial derivatives up to order k. In addition, $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ stands for the set of all functions from $\mathbb{R}^n$ to $\mathbb{C}$ having compact support and continuous partial derivatives of any order. If $f : \mathbb{R}^n \mapsto \mathbb{C}$, then $\partial f / \partial x_k$ denotes the partial derivative of f with respect to the kth coordinate of it's input variable, ∇f stands for the gradient of f and $\triangle f$ is the Laplacian of f. The symbol $\bullet$ denotes the pointwise product between two vectors in $\mathbb{C}^n$.

In what follows, the letter K denotes generic constants. We will write $K(\cdot)$ for different non-decreasing non-negative functions on the interval $[0, \infty[$ when no confusion can arise.

3. Strong regular solutions to the stochastic Schrödinger equations

3.1. Linear stochastic Schrödinger equations

As in [23], we consider the following notion of strong regular solution to (1.1).

Hypothesis 1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$ such that:*

(H1.1) *For each $k \in \mathbb{N}$: (i) $\mathcal{D}(C) \subset \mathcal{D}(L_k(t))$ whenever $t \geq 0$; and (ii) $L_k(\cdot) \circ \pi_C$ is measurable as a function from $([0, \infty[\times \mathfrak{h}, \mathcal{B}([0, \infty[\times \mathfrak{h}))$ to $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$.*

(H1.2) *For all $t \geq 0$, $\mathcal{D}(C) \subset \mathcal{D}(G(t))$. Moreover, $G(\cdot) \circ \pi_C$ is measurable as map from $([0, \infty[\times \mathfrak{h}, \mathcal{B}([0, \infty[\times \mathfrak{h}))$ to $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$.*

Definition 3.1. *Let Hypothesis 1 hold. Assume that $\mathbb{T}$ is either $[0, \infty[$ or the interval $[0, T]$, with $T \in \mathbb{R}_+$. An adapted process $(X_t(\xi))_{t \in \mathbb{T}}$ taking values in $\mathfrak{h}$ with continuous sample paths is called strong C-solution of (1.1) on $\mathbb{T}$ with initial datum ξ if and only if:*

- *For any $t \in \mathbb{T}$, $\mathbb{E}\left\|X_t(\xi)\right\|^2 \leq \mathbb{E}\left\|\xi\right\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ a.s. and*
$$\sup_{s \in [0,t]} \mathbb{E}\left\|C X_s(\xi)\right\|^2 < \infty.$$
- *$\mathbb{P}$-a.s. for all $t \in \mathbb{T}$,*
$$X_t(\xi) = \xi + \int_0^t G(s) \pi_C(X_s(\xi))\, ds + \sum_{k=1}^{\infty} \int_0^t L_k(s) \pi_C(X_s(\xi))\, dW_s^k.$$

We next provide a sufficient condition for the existence and uniqueness of strong C-solutions to (1.1).

Hypothesis 2. *Let C satisfy Hypothesis 1. In addition assume that:*

(H2.1) *For all $t \geq 0$ and $x \in \mathcal{D}(C)$ $\left\|G(t) x\right\|^2 \leq K(t) \left\|x\right\|_C^2$.*

(H2.2) *For every natural number k there exists a non-decreasing positive function K_k on $[0, \infty[$ satisfying $\left\|L_k(t) x\right\|^2 \leq K_k(t) \left\|x\right\|_C^2$ for all $x \in \mathcal{D}(C)$ and $t \geq 0$.*

(H2.3) *There exists a non-decreasing non-negative function α and a core $\mathfrak{D}_1$ of C^2 for which*

$$2\Re \left\langle C^2 x, G(t)\, x \right\rangle + \sum_{k=1}^{\infty} \left\| C L_k(t)\, x \right\|^2 \leq \alpha(t) \left\| x \right\|_C^2$$

whenever $x \in \mathfrak{D}_1$ and $t \geq 0$.

(H2.4) *There exists a core $\mathfrak{D}_2$ of C such that for any x in $\mathfrak{D}_2$ and $t \geq 0$,*

$$2\Re \left\langle x, G(t)\, x \right\rangle + \sum_{k=1}^{\infty} \left\| L_k(t)\, x \right\|^2 \leq 0.$$

Theorem 3.1. *Assume that Hypothesis* 2 *holds. Let ξ be a $\mathfrak{F}_0$-measurable random variable of $L^2_C(\mathbb{P}, \mathfrak{h})$. Then (*1.1*) has a unique strong C-solution $(X_t(\xi))_{t\geq 0}$ with initial datum ξ. Moreover,*

$$\mathbb{E} \left\| C X_t(\xi) \right\|^2 \leq \exp\left(t\alpha(t)\right) \left(\mathbb{E} \left\| C\xi \right\|^2 + t\alpha(t)\, \mathbb{E} \left\| \xi \right\|^2 \right).$$

Proof. Deferred to Subsection 6.2. □

In many practical situations like the autonomous case, the following lemma guarantees the measurability of $G(\cdot) \circ \pi_C$ and $L_k(\cdot) \circ \pi_C$ required in Hypothesis 1.

Lemma 3.1. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $L \in \mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$. Then $L \circ \pi_C : (\mathfrak{h}, \mathcal{B}(\mathfrak{h})) \to (\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable.*

Proof. Deferred to Subsection 6.1. □

Remark 3.1. *Suppose that L is a closable operator in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(L)$, where C is aself-adjoint positive operator in $\mathfrak{h}$. Applying the closed graph theorem gives $L \in \mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$.*

Remark 3.2. *Hypothesis* 2 *is a refined version of non-explosion criteria that guarantee the Markov property of the quantum dynamical semigroups (see, e.g,* [10, 11, 16]*). Hypothesis* 2 *can be verified, for instance, in quantum oscillators* [26]*, quantum measurement processes* [23] *and infinite many particle systems* [23]*.*

3.2. Martingale and Markov properties

Subsection 3.1 makes it legitimate to assume Condition H3.2 given below. Indeed, Theorem 3.1 shows that it applies to many physical open quantum systems.

Hypothesis 3. *Let Hypothesis* 1 *and Condition H2.1 hold. Suppose that:*

(H3.1) *For all $x \in \mathcal{D}(C)$ and $t \geq 0$ we have $2\Re \left\langle x, G(t)\, x \right\rangle + \sum_{k=1}^{\infty} \left\| L_k(t)\, x \right\|^2 = 0$.*

(H3.2) *Let $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$ be $\mathfrak{F}_0$-measurable. Then for all $T > 0$, (*1.1*) has a unique strong C-solution on $[0, T]$ with initial datum ξ.*

Remark 3.3. *Condition H3.1 of Hypothesis* 3 *is a weak version of (*1.2*). Relation (*1.2*) arises formally from physical situations where we can expect that the solutions of the quantum master equations have trace* 1 *at any instant. Nevertheless, (*1.2*) is not a sufficient condition for the minimal quantum dynamical semigroup to be identity preserving (see, e.g., Section 3.5 of* [16] *for a counterexample).*

We next prove the martingale property $\|X(\xi)\|^2$ under Hypothesis 3, which establishes essentially the conservative property of the open quantum systems.

Theorem 3.2. *Suppose that Hypothesis* 3 *holds and that* $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$*. Then* $\left(\|X_t(\xi)\|^2\right)_{t\geq 0}$ *is a martingale. Moreover, for any measurable bounded function* $f : (\mathfrak{h}, \mathfrak{B}(\mathfrak{h})) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ *and* $t \geq s \geq 0$ *we have*

$$\begin{aligned} \mathbb{E}\left(f\left(X_t(\xi)\right) \diagup \mathfrak{F}_s\right) &= \mathbb{E}\left(f\left(X_t(\xi)\right) \diagup X_s(\xi)\right) \\ &= \int_{\mathfrak{h}} f(z)\, P_{s,t}\left(X_s(\xi), dz\right), \end{aligned} \tag{3.1}$$

where $P_{s,t}(x,\cdot)$ *is the Dirac measure* δ_x *provided that* $x \notin \mathcal{D}(C)$*, and* $P_{s,t}(x,\cdot)$ *is the distribution at time* t *of the strong* C*-solution of (*1.1*) with initial datum at time* s *equal to* $x \in \mathcal{D}(C)$*.*

Proof. Deferred to Subsection 6.3. □

4. Non-linear stochastic Schrödinger equations

For any $y \in \mathfrak{h}$ and $t \geq 0$, we choose (by abuse of notation)

$$L_k(s,y) = L_k(s)\,\pi_C(y) - \Re\left\langle y, L_k(s)\,\pi_C(y)\right\rangle y,$$

and

$$\begin{aligned} G(s,y) &= G(s)\,\pi_C(y) \\ &\quad + \sum_{k=1}^{\infty}\left(\Re\left\langle y, L_k(s)\,\pi_C(y)\right\rangle L_k(s)\,\pi_C(y) - \frac{1}{2}\Re^2\left\langle y, L_k(s)\,\pi_C(y)\right\rangle y\right). \end{aligned}$$

Then, the non-linear stochastic Schrödinger equation

$$Y_t = Y_0 + \int_0^t G(s,Y_s)\,ds + \sum_{k=1}^{\infty}\int_0^t L_k(s,Y_s)\,dW_s^k \tag{4.1}$$

describes quantum measurement processes (see, e.g., [2, 3, 5, 6, 14, 18, 32]) and may represent objective (independent of any observer) trajectories of quantum systems.

We now report our careful verification that we can use the same arguments as in the proof of Theorem 1 of [24] for establishing the existence and uniqueness of solutions to the stochastic evolution equation driven by a standard cylindrical Brownian motion (4.1) under Hypothesis 3.

Definition 4.1. *Let C satisfy Hypothesis 1. Suppose that $\mathbb{T}$ is either $[0,+\infty[$ or $[0,T]$ provided $T \in [0,+\infty[$. We say that $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{T}}, \mathbb{Q}, (Y_t)_{t\in\mathbb{T}}, \left(W_t^k\right)_{t\in\mathbb{T}}^{k\in\mathbb{N}}\right)$ is a solution of class C of (4.1) with initial distribution θ on the interval $\mathbb{T}$ if and only if:*

- $\left(W^k\right)_{k\in\mathbb{N}}$ *is a sequence of real valued independent Brownian motions on the filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{T}}, \mathbb{Q}\right)$.*
- $(Y_t)_{t\in\mathbb{T}}$ *is an $\mathfrak{h}$-valued process with continuous sample paths such that the law of Y_0 coincides with θ and $\mathbb{Q}\left(\|Y_t\| = 1 \text{ for all } t \in \mathbb{T}\right) = 1$.*
- *For every $t \in \mathbb{T}$, $Y_t \in \mathcal{D}(C)$ $\mathbb{Q}$-a.s. and $\sup_{s\in[0,t]} \mathbb{E}_{\mathbb{Q}} \|CY_s\|^2 < \infty$.*
- *$\mathbb{Q}$-a.s., $\left(Y, \left(W^k\right)_{k\in\mathbb{N}}\right)$ satisfies (4.1) for all $t \in \mathbb{T}$.*

For abbreviation, we simply say $\left(\mathbb{Q}, (Y_t)_{t\in\mathbb{T}}, (W_t)_{t\in\mathbb{T}}\right)$ is a C-solution of (4.1) when no confusion can arise.

Theorem 4.1. *Let C satisfy Hypothesis 3. Suppose that θ is a probability measure on $\mathfrak{B}(\mathfrak{h})$ such that $\theta\left(\mathcal{D}(C) \cap \{y \in \mathfrak{h} : \|y\| = 1\}\right) = 1$ and $\int_{\mathfrak{h}} \|Cy\|^2 \theta(dx) < \infty$. Then (4.1) has a unique C-solution $\left(\mathbb{Q}, (Y_t)_{t\geq 0}, (B_t)_{t\geq 0}\right)$ with initial law θ.*

Proof. Theorem 3.2 allows to use the same analysis as in the proof of Theorem 1 of [24] to show our statement. □

It is worth pointing out that in the proof of Theorem 4.1 we use the following construction of C-solutions of (4.1) on finite intervals.

Theorem 4.2. *Adopt the assumptions of Theorem 4.1. In addition, let $(X_t(\xi))_{t\geq 0}$ be the C-strong solution of (1.1), where ξ is distributed according to θ. Define $\mathbb{Q} = \|X_T(\xi)\|^2 \cdot \mathbb{P}$, where $T \in]0,+\infty[$. For any $t \in [0,T]$, we set*

$$Y_t = \begin{cases} X_t(\xi) / \|X_t(\xi)\|, & \text{if } X_t(\xi) \neq 0 \\ 0, & \text{if } X_t(\xi) = 0 \end{cases}$$

and

$$B_t^k = W_t^k - \int_0^t \frac{1}{\|X_s(\xi)\|^2} d\left[W^k, X(\xi)\right]_s,$$

with $k \in \mathbb{N}$. Then $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{Q}, (Y_t)_{t\in[0,T]}, \left(B_t^k\right)_{t\in[0,T]}^{k\in\mathbb{N}}\right)$ is a C-solution of (4.1) with initial distribution θ.

Proof. By Theorem 3.2, our assertion can be obtained proceeding along the same lines as in the proof of Proposition 1 of [24]. □

5. Concrete physical systems

This section focusses on the following general model, which describes spinless particles in coordinate representation.

Model 1. *Let $\mathfrak{h} = L^2(\mathbb{R}^n, \mathbb{C})$, with $n \in \mathbb{N}$. Suppose that t is an arbitrary non-negative real number. We consider the Hamiltonian*

$$H(t) = -\alpha \triangle + g(t, \cdot),$$

where $\alpha > 0$ and $g : ([0, \infty[\times \mathbb{R}^n, \mathcal{B}([0, \infty[\times \mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable function. For each $k = 1, \ldots, m$ we set

$$L_k(t) = \phi_k(t, \cdot),$$

with $\phi_k : ([0, \infty[\times \mathbb{R}^n, \mathcal{B}([0, \infty[\times \mathbb{R}^n)) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ measurable. Moreover, we choose $L_k = 0$ whenever $k > m$. Define

$$G(t) = -iH(t) - \frac{1}{2}\sum_{k=1}^{m} |\phi_k(t, \cdot)|^2.$$

Hypothesis 4. *In the context of Model 1 we suppose that for all $t \geq 0$ and $x \in \mathbb{R}^n$ we have $|g(t, x)| \leq K(t)\left(1 + |x|^2\right)$ and $|\nabla g(t, x)| \leq K(t)(1 + |x|)$. Moreover, we assume that for any $k = 1, \ldots, m$ we have: $\phi_k(t, \cdot) \in C^2(\mathbb{R}, \mathbb{C})$,*

$$\max\left\{|\triangle \phi_k(t, x)|, \left|\overline{\phi_k(t, x)} \triangle \phi_k(t, x)\right|\right\} \leq K(t)\left(|x|^2 + 1\right) \tag{5.1}$$

and

$$\max\left\{|\phi_k(t, x)|, |\nabla \phi_k(t, x)|, \left|\overline{\phi_k(t, x)} \nabla \phi_k(t, x)\right|\right\} \leq K(t)(|x| + 1). \tag{5.2}$$

whenever $t \geq 0$ and $x \in \mathbb{R}^n$.

By means of Theorems 3.1 and 4.1, we are able to obtain the well-posedness of Model 1.

Theorem 5.1. *Adopt the framework of Model 1 and Hypothesis 4. Let $C = -\triangle + |x|^2$. Suppose that ξ is a $\mathfrak{F}_0$-measurable random variable taking values in $L^2(\mathbb{R}^n, \mathbb{C})$ such that $\mathbb{E}\|\xi\|^2 = 1$ and $\mathbb{E}\|C\xi\|^2 < \infty$. Then (1.1) has a unique strong C-solution with initial datum ξ. Moreover, (4.1) has a unique C-solution whose initial law coincides with the distribution of ξ.*

Proof. Deferred to Subsection 6.4. □

In order to illustrate the physical nature of Model 1, we next present a simple process of continuous measurement of position.

Example 1. *In the context of Model 1 we select $n = m = 1$ and $\alpha = 1/(2M)$, with $M > 0$. Moreover, we take $g(t, x) = 0$ and $\phi_1(t, x) = \sigma x$, where σ is a positive real number.*

Remark 5.1. *Example 1 describes the dynamics of the continuous measurement of position of a free quantum particle (see, e.g., [4, 9]). This process can be observed with mechanical detectors. For a construction of a general class of models of spontaneous localizations in space we refer the reader to, for instance, [18]. Moreover, Example 1 with $g(t, x) = 0$ replaced by $g(t, x) = \kappa x^2$, with $\kappa > 0$, models the position measurement of a harmonically bound particle.*

The next particular model is due to Singh and Rost [33]. It arises from the application of intense linearly polarized laser to the hydrogen atom.

Example 2. *In the framework of Model* 1*, we consider* $n = 1$*,* $\alpha = 1/2$ *and* $g(t, x) = V(x) + xF(t)$*, where* $V(x) = -1/\sqrt{x^2 + a^2}$ *and*

$$F(t) = F_0 \sin(\beta t + \delta) \begin{cases} \sin(\pi t/(2\tau)), & \text{if } t < \tau \\ 1, & \text{if } \tau \leq t \leq T - \tau \\ \cos^2(\pi(t + \tau - T)/(2\tau)), & \text{if } T - \tau \leq t \leq T \end{cases} .$$

Here $\beta, \delta \in \mathbb{R}$ *and* a, F_0, τ, T *are positive constants. Select* $m = 1$ *and* $\phi_1(t, x) = \sigma x$ *for all* $x \in \mathbb{R}$*, with* $\sigma > 0$*.*

Remark 5.2. *Example* 2 *simulates the evolution of the electron of the hydrogen atom under the influence of a laser field* $F(t)$*. In this context,* V *is a soft core potential that approximates the Coulomb potential of the atom.*

From [19] we have that our final example provides the evolution of a quantum system in a fluctuating trap.

Example 3. *Adopt Model* 1 *with* $n = m = 1$*,* $\alpha = 1/(2M)$ *and* $g(t, x) = \kappa x^2$*, where* $M > 0$ *and* $\kappa > 0$*. Set* $\phi_1(t, x) = -i\sigma x$*, whenever* $\sigma > 0$*.*

6. Proofs

6.1. Proof of Lemma 3.1

We first characterize the domain of C by means of the Yosida approximation of $-C$.

Lemma 6.1. *Let* C *be a self-adjoint positive operator in* $\mathfrak{h}$*. For any* $n \in \mathbb{N}$ *we set* $R_n = n(n + C)^{-1}$*. Then*

$$\mathcal{D}(C) = \{x \in \mathfrak{h} : (CR_n x)_n \text{ converges}\} = \left\{ x \in \mathfrak{h} : \sup_{n \in \mathbb{N}} \|CR_n x\| < \infty \right\}.$$

Proof. Since $-C$ is dissipative and self-adjoint, for all $x \in \mathcal{D}(C)$ we have

$$CR_n x \longrightarrow_{n \to \infty} Cx \tag{6.1}$$

(see, e.g., [15, 27]). Thus $\mathcal{D}(C) \subset \{x \in \mathfrak{h} : (CR_n x)_n \text{ converges}\}$.

Now, assume that $(\|CR_n x\|)_{n \in \mathbb{N}}$ is bounded. Using the Banach-Alaoglu theorem we deduce that there exists a subsequence $(CR_{n_k} x)_{k \in \mathbb{N}}$ which converges weakly to a vector $z \in \mathfrak{h}$. Since $R_n x \longrightarrow_{n \to \infty} x$, for any $y \in \mathcal{D}(C)$ we have

$$\langle x, Cy \rangle = \lim_{k \to \infty} \langle R_{n_k} x, Cy \rangle = \lim_{k \to \infty} \langle CR_{n_k} x, y \rangle = \langle z, y \rangle .$$

Hence $x \in \mathcal{D}(C^*)$ $(= \mathcal{D}(C))$, and so $\{x \in \mathfrak{h} : \sup_{n \in \mathbb{N}} \|CR_n x\| < \infty\} \subset \mathcal{D}(C)$. $\square$

Proof of Theorem 3.2. Let R_n be as in Lemma 6.1. Then $CR_n \in \mathfrak{L}(\mathfrak{h})$. Using Lemma 6.1 we obtain that $\mathcal{D}(C)$ is a Borel set of $\mathfrak{h}$, and so $\pi_C : (\mathfrak{h}, \mathcal{B}(\mathfrak{h})) \to (\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable. On the other hand, the range of R_n is a subset of $\mathcal{D}(C)$. Therefore $LR_n \in \mathfrak{L}(\mathfrak{h})$ by $L \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Thus $LR_n \circ \pi_C$ is measurable. Combining $R_n \longrightarrow_{n\to\infty} I$ with (6.1) yields $LR_n \circ \pi_C \longrightarrow_{n\to\infty} L \circ \pi_C$, which implies the measurability of $L \circ \pi_C$. □

6.2. Proof of Theorem 3.1

Although it is not the purpose of this paper to discuss mild regular solutions of (1.1), through this subsection we develop preliminary results for studying this kind of solutions to (1.1). To this end, we introduce the following conditions.

Hypothesis 5. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Assume that:*

(H5.1) *For all $t \geq 0$, $\mathcal{D}(C^2) \subset \mathcal{D}(G(t))$ and $\|G(t)x\|^2 \leq K(t)\|x\|_{C^2}^2$ whenever $x \in \mathcal{D}(C^2)$.*

(H5.2) *The function $G(\cdot) \circ \pi_{C^2}$ from $[0, \infty[\times \mathfrak{h}$ equipped with its Borel σ-algebra to $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable.*

(H5.3) *There exists a core $\mathfrak{D}_2$ of C^2 such that for all x in $\mathfrak{D}_2$ and $t \geq 0$,*

$$2\Re\langle x, G(t)x\rangle + \sum_{k=1}^{\infty} \|L_k(t)x\|^2 \leq 0.$$

Remark 6.1. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Since $\mathcal{D}(C^2) \subset \mathcal{D}(C)$, the closed graph theorem gives $C \in \mathfrak{L}\left(\left(\mathcal{D}(C^2), \|\cdot\|_{C^2}\right), \mathfrak{h}\right)$. Thus, Condition H2.1 of Hypothesis* 2 *implies Condition H5.1 of Hypothesis* 5*. Moreover, using Lemma* 6.1 *we obtain that $\pi_{C^2} : (\mathfrak{h}, \mathcal{B}(\mathfrak{h})) \to (\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable, and so Condition H1.2 leads to Condition H5.2.*

Since $\mathcal{D}(C^2)$ is core for C, Condition H5.3 implies Condition H2.4. Nevertheless, we can extend the inequality of Condition H2.4 to $\mathcal{D}(C)$ provided that Hypothesis 2 holds.

Lemma 6.2. *Under Conditions H2.1, H2.2 and H2.4 of Hypothesis* 2*, for all x in $\mathcal{D}(C)$ we have $2\Re\langle x, G(t)x\rangle + \sum_{k=1}^{\infty} \|L_k(t)x\|^2 \leq 0$.*

Proof. The assertion follows from the definition of core and Fatou's lemma. □

We next extend the inequalities given in Conditions H2.3 and H5.2 to $\mathcal{D}(C^2)$.

Lemma 6.3. *Suppose that C satisfies Conditions H2.2, H2.3 of Hypothesis* 2 *and Condition H5.1 of Hypothesis* 5*. Let x belong to $\mathcal{D}(C^2)$ and $t \geq 0$. Then $L_k(t)x \in \mathcal{D}(C)$ for any $k \in \mathbb{N}$, and*

$$2\Re\left\langle C^2x, G(t)x\right\rangle + \sum_{k=1}^{\infty} \|CL_k(t)x\|^2 \leq \alpha(t)\|x\|_C^2. \tag{6.2}$$

Proof. Since $\mathfrak{D}_1$ is a core of C^2, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathfrak{D}_1$ converging to x such that $C^2 x_n \longrightarrow_{n\to\infty} C^2 x$. Combining $G(t) \in \mathfrak{L}\left(\left(\mathcal{D}\left(C^2\right), \|\cdot\|_{C^2}\right), \mathfrak{h}\right)$ with Condition H2.3 yields

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} \|CL_k(t)\, x_n\|^2 \leq \alpha(t)\, \|x\|_C^2 - 2\Re \left\langle C^2 x, G(t)\, x \right\rangle. \tag{6.3}$$

Using (6.3), the Banach-Alaoglu theorem and diagonalization arguments we deduce that $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $\left(x_{n(l)}\right)_{l\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$,

$$CL_k(t)\, x_{n(l)} \longrightarrow_{l\to\infty} z_k \qquad \textit{weakly in } \mathfrak{h}, \tag{6.4}$$

with $z_k \in \mathfrak{h}$. According to Remark 6.1 and Condition H2.2, we have

$$L_k(t)\, x_{n(l)} \longrightarrow_{l\to\infty} L_k(t)\, x.$$

Since C is closed, (6.4) implies $L_k(t)\, x \in \mathcal{D}(C)$ and $z_k = CL_k(t)\, x$. Hence

$$\|CL_k(t)\, x\| \leq \liminf_{l\to\infty} \left\|CL_k(t)\, x_{n(l)}\right\|. \tag{6.5}$$

From (6.3), (6.5) and Fatou's lemma we obtain (6.2). $\square$

Lemma 6.4. *Let Conditions H5.1 and H5.3 of Hypothesis* 5 *hold. Under Condition H2.2 of Hypothesis* 2*, for all x in $\mathcal{D}\left(C^2\right)$ we have*

$$2\Re \langle x, G(t)\, x\rangle + \sum_{k=1}^{\infty} \|L_k(t)\, x\|^2 \leq 0.$$

Proof. By Remark 6.1, combining the definition of core with Fatou's lemma we deduce our claim. $\square$

We now approximate (1.1) by stochastic evolutions equations with bounded coefficients. To this end, we use the Yosida approximation of $-C^2$.

Definition 6.1. *Let Conditions H1.1, H2.2, H5.1, H5.2 hold. Assume that: (i) $W^1, W^2, \ldots,$ are real valued independent Wiener processes on $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\right)$; (ii) ξ is a $\mathfrak{F}_0$-measurable random variable belonging to $L^2(\mathbb{P}, \mathfrak{h})$; and (iii) n is a natural number. Choose $\widetilde{R}_n = n\left(n + C^2\right)^{-1}$. Then, we define X^n to be the unique continuous solution of*

$$X_t^n = \xi + \int_0^t G^n(s)\, X_s^n ds + \sum_{k=1}^{n} \int_0^t L_k^n(s)\, X_s^n dW_s^k, \tag{6.6}$$

where $G^n(s) = \widetilde{R}_n G(s)\, \widetilde{R}_n$ and $L_k^n(s) = L_k(s)\, \widetilde{R}_n$, with $s \geq 0$ and $k \in \mathbb{N}$.

Remark 6.2. *The range of $\widetilde{R}_n$ is a subset of $\mathcal{D}\left(C^2\right)$ and $C^2\widetilde{R}_n \in \mathfrak{L}(\mathfrak{h})$. By $\left\|\widetilde{R}_n\right\| \leq 1$, Conditions H2.2 and H5.1 implies that $G^n(t)$ and $L_k^n(t)$, with $k = 1, \ldots, n$, are bounded operators in $\mathfrak{h}$ whose norms are uniformly bounded on compact time intervals. It follows that X^n is well-defined.*

The following three lemmata provide a priori estimates for X^n.

Lemma 6.5. *Suppose that C satisfies Conditions H1.1, H2.2, H5.1-H5.3. Let $t \geq 0$. Then $\mathbb{E} \left\| X_t^n \right\|^2 \leq \mathbb{E} \left\| \xi \right\|^2$. Moreover, for all $x \in \mathfrak{h}$ we have*

$$2\Re \left\langle x, G^n\left(t\right) x \right\rangle + \sum_{k=1}^{\infty} \left\| L_k^n\left(t\right) x \right\|^2 \leq 0. \tag{6.7}$$

Proof. Since the range of $\widetilde{R}_n$ is a subset of $\mathcal{D}\left(C^2\right)$, Lemma 6.4 leads to (6.7). Using complex Itô's formula yields

$$\left\| X_t^n \right\|^2 \leq \left\| \xi \right\|^2 + \sum_{k=1}^{n} \int_0^t 2\Re \left\langle X_s^n, L_k^n\left(s\right) X_s^n \right\rangle dW_s^k. \tag{6.8}$$

Set $\tau_j = \inf\left\{t \geq 0 : \left\| X_t^n \right\| > j\right\}$. By (6.8), $\mathbb{E} \left\| X_{t \wedge \tau_j}^n \right\|^2 \leq \mathbb{E} \left\| \xi \right\|^2$. Because X^n is pathwise continuous, $\tau_j \nearrow \infty$ as $j \to \infty$. Applying Fatou's lemma we get $\mathbb{E} \left\| X_t^n \right\|^2 \leq \liminf_{j \to \infty} \mathbb{E} \left\| X_{t \wedge \tau_j}^n \right\|^2 \leq \mathbb{E} \left\| \xi \right\|^2$. □

Lemma 6.6. *Adopt Conditions H1.1, H2.2, H2.3, H5.1-H5.3. Suppose that ξ is a $\mathfrak{F}_0$-measurable random variable of $L_C^2\left(\mathbb{P}, \mathfrak{h}\right)$. Then*

$$\mathbb{E} \left\| C X_t^n \right\|^2 \leq \exp\left(t\alpha\left(t\right)\right) \left(\mathbb{E} \left\| C\xi \right\|^2 + t\alpha\left(t\right) \mathbb{E} \left\| \xi \right\|^2 \right). \tag{6.9}$$

Proof. Due to $\left\| C G^n\left(t\right) \right\| \leq \left\| C \widetilde{R}_n \right\| \left\| G\left(t\right) \widetilde{R}_n \right\|$, combining Condition H2.1 with Remark 6.1 gives

$$\left\| C G^n\left(t\right) \right\| \leq K\left(t\right). \tag{6.10}$$

Applying Lemma 6.3 we obtain

$$2\Re \left\langle C^2 \widetilde{R}_n x, G\left(t\right) \widetilde{R}_n x \right\rangle + \sum_{k=1}^{\infty} \left\| C L_k^n\left(t\right) x \right\|^2 \leq \alpha\left(t\right) \left\| \widetilde{R}_n x \right\|_C^2$$

for all $x \in \mathfrak{h}$. It follows that for any $k \in \mathbb{N}$ we have

$$\left\| C L_k^n\left(t\right) \right\| \leq K_k\left(t\right), \tag{6.11}$$

where $K_k\left(t\right)$ is a non-decreasing non-negative function.

Let y belong to $\mathcal{D}\left(C\right)$ and $t \geq 0$. We proceed to show that

$$2\Re \left\langle Cy, C G^n\left(t\right) y \right\rangle + \sum_{k=1}^{n} \left\| C L_k^n\left(t\right) y \right\|^2 \leq \alpha\left(t\right) \left\| y \right\|_C^2. \tag{6.12}$$

To this end, fix $x \in \mathcal{D}\left(C^2\right)$. Then $\widetilde{R}_n C^2 x = C^2 \widetilde{R}_n x$. Using $\widetilde{R}_n$ is self-adjoint we deduce that

$$\left\langle Cx, C G^n\left(t\right) x \right\rangle = \left\langle C^2 \widetilde{R}_n x, G\left(t\right) \widetilde{R}_n x \right\rangle,$$

and so Lemma 6.3 leads to

$$2\Re \left\langle Cx, C G^n\left(t\right) x \right\rangle + \sum_{k=1}^{n} \left\| C L_k^n\left(t\right) x \right\|^2 \leq \alpha\left(t\right) \left\| \widetilde{R}_n x \right\|_C^2.$$

As $\left\|\widetilde{R}_n\right\| \leq 1$ and C commutates with $\widetilde{R}_n$, $\left\|\widetilde{R}_n x\right\|_C^2 \leq \|x\|_C^2$. Since $\mathcal{D}\left(C^2\right)$ is a core of C, a passage to the limit now gives (6.12).

From (6.10), (6.11) and Lemma 6.5 we obtain

$$\mathbb{E}\left\|CG^n(t) X_t^n\right\|^2 \leq \left\|CG^n(t)\right\|^2 \mathbb{E}\|\xi\|^2 \leq K(t)\,\mathbb{E}\|\xi\|^2$$

and

$$\mathbb{E}\left\|CL_k^n(t) X_t^n\right\|^2 \leq \left\|CL_k^n(t)\right\|^2 \mathbb{E}\|\xi\|^2 \leq K_k(t)\,\mathbb{E}\|\xi\|^2 .$$

Then according to, for instance, Propositions 1.6 and 4.15 of [12], we have $CX_t^n = Y_t^n$ *a.s.* for any $t \geq 0$, where

$$Y^n = C\xi + \int_0^{\cdot} CG^n(s) X_s^n ds + \sum_{k=1}^n \int_0^{\cdot} CL_k^n(s) X_s^n dW_s^k .$$

Choose $\tau_j = \inf\left\{t \geq 0 : \left\|Y_t^n\right\| > j\right\}$. Lema 6.5 implies

$$\mathbb{E}\left|\Re\left\langle Y_{s\wedge\tau_j}^n, CL_k^n(s) X_s^n\right\rangle\right|^2 \leq j^2 \left\|CL_k^n(s)\right\|^2 \mathbb{E}\|\xi\|^2 .$$

Then, applying Itô's formula yields

$$\mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 = \mathbb{E}\|C\xi\|^2 + \mathbb{E}\int_0^{t\wedge\tau_j}\left(2\Re\left\langle Y_s^n, CG^n(s) X_s^n\right\rangle + \sum_{k=1}^n \left\|CL_k^n(s) X_s^n\right\|^2\right) ds.$$

Since $Y_s^n = CX_s^n$ a.s., combining (6.12) with Lema 6.5 we have

$$\mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 \leq \mathbb{E}\|C\xi\|^2 + \alpha(t)\int_0^t \mathbb{E}\left\|CX_s^n\right\|^2 ds + t\alpha(t)\,\mathbb{E}\|\xi\|^2 .$$

This gives

$$\mathbb{E}\left\|Y_t^n\right\|^2 \leq \liminf_{j\to\infty} \mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 \leq \mathbb{E}\|C\xi\|^2 + t\alpha(t)\,\mathbb{E}\|\xi\|^2 + \alpha(t)\int_0^t \mathbb{E}\left\|Y_s^n\right\|^2 ds.$$

Hence, the Gronwall-Bellman lemma leads to (6.9). □

Lemma 6.7. *Fix $T > 0$. Under the assumptions of Theorem 3.1,*

$$\mathbb{E}\left\|X_t^n - X_s^n\right\|^2 \leq K_{T,\xi}(t-s), \tag{6.13}$$

where $0 \leq s \leq t < T$ and $K_{T,\xi}$ is a constant depending of T and ξ.

Proof. Consider $\tau_j = \inf\left\{t \geq 0 : \left\|X_t^n\right\| > j\right\}$. Using Itô's formula we have

$$\begin{aligned}&\mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \\ &= \mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j}\left(2\Re\left\langle X_r^n - X_{s\wedge\tau_j}^n, G^n(r) X_r^n\right\rangle + \sum_{k=1}^n \left\|L_k^n(r) X_r^n\right\|^2\right) ds.\end{aligned}$$

By Condition H2.1, combining $\left\|\widetilde{R}_n\right\| \leq 1$ with $\widetilde{R}_n C \subset C\widetilde{R}_n$ yields

$$\left\|G^n(t) x\right\|^2 \leq K(t)\,\|x\|_C^2$$

for all $x \in \mathcal{D}(C)$. Since X_s^n belongs to $\mathcal{D}(C)$ a.s., from (6.7) we deduce

$$\mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \leq -\mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j} 2\Re\left\langle X_s^n, G^n(r)\, X_r^n\right\rangle dr$$
$$\leq K(t)\,\mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j} \|X_s^n\|\,\|X_r^n\|_C\, dr.$$

Fatou's lemma now implies

$$\mathbb{E}\|X_t^n - X_s^n\|^2 \leq \liminf_{j\to\infty}\mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \leq K(t)\int_s^t \sqrt{\mathbb{E}\|X_r^n\|_C^2}\sqrt{\mathbb{E}\|X_s^n\|^2}dr.$$

Applying Lemmata 6.5 and 6.6 we obtain

$$\mathbb{E}\|X_t^n - X_s^n\|^2 \leq K_T(t-s)\sqrt{\mathbb{E}\|\xi\|^2}\sqrt{\mathbb{E}\|C\xi\|^2 + \mathbb{E}\|\xi\|^2 + 1}.$$

□

Finally, we obtain a strong C-solution of (1.1) by means of a limit procedure.

Definition 6.2. *For any natural number n, we define $\left(\mathfrak{G}_s^{\xi,n}\right)_{s\geq 0}$ to be the filtration that satisfies the usual hypotheses generated by ξ and $W^1,\ldots,W^n$. Let t be a non-negative real number. By $\mathfrak{G}_t^{\xi,W}$ we mean the σ-algebra generated by $\cup_{n\in\mathbb{N}}\mathfrak{G}_t^{\xi,n}$. As usual, $\mathfrak{G}_{t+}^{\xi,W} = \cap_{\epsilon>0}\mathfrak{G}_{t+\epsilon}^{\xi,W}$.*

Lemma 6.8. *Let C satisfy Conditions H1.1, H2.2, H2.3, H5.1-H5.3. Suppose that $\xi \in L_C^2(\mathbb{P},\mathfrak{h})$ and that inequality (6.13) holds. Fix $T>0$. Then, we can extract from any subsequence of $(X^n(\xi))_{n\in\mathbb{N}}$ a subsequence $(X^{n_k}(\xi))_{k\in\mathbb{N}}$ for which there exists a $\left(\mathfrak{G}_{t+}^{\xi,W}\right)_{t\in[0,T]}$-predictable process $(Z_t(\xi))_{t\in[0,T]}$ such that for any $t\in[0,T]$,*

$$X_t^{n_k}(\xi) \longrightarrow_{k\to\infty} Z_t(\xi) \qquad \textit{weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right). \tag{6.14}$$

Proof. Let $(\chi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2\left(\left(\Omega,\mathfrak{G}_T^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$. Combining the Cauchy-Schwarz inequality with (6.13) we obtain the equicontinuity of the family of complex functions $\left(\mathbb{E}\langle\chi_j, X^n(\xi)\rangle\right)_{n\in\mathbb{N}}$, with $j\in\mathbb{N}$. Using Lemma 6.5, the Arzelà-Ascoli theorem and diagonalization arguments we deduce that can extract from any subsequence of $(X^n(\xi))_{n\in\mathbb{N}}$ a subsequence $(X^{n_k}(\xi))_{k\in\mathbb{N}}$ such that $\mathbb{E}\langle\chi_j, X^{n_k}(\xi)\rangle$ is uniformly convergent in $[0,T]$ for any $j\in\mathbb{N}$. Lemma 6.5 now shows that $X_t^{n_k}(\xi)$ is weakly convergent in $L^2\left(\left(\Omega,\mathfrak{G}_T^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$ for any $t\in[0,T]$. Since $X_t^{n_k}(\xi)$ is $\mathfrak{G}_t^{\xi,W}$-measurable, for any $t\in[0,T]$ there exists a $\mathfrak{G}_t^{\xi,W}$-measurable random variable ψ_t satisfying

$$X_t^{n_k}(\xi) \longrightarrow_{k\to\infty} \psi_t \qquad \textit{weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right). \tag{6.15}$$

Assume that $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathfrak{h}$. According to (6.15) we have

$$\langle e_j, X_t^{n_k}(\xi)\rangle \longrightarrow_{k\to\infty} \langle e_j,\psi_t\rangle \quad \textit{weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathbb{C}\right),$$

Applying (6.13) and Fatou's lemma yields

$$\mathbb{E}\left|\langle e_j, \psi_t - \psi_s\rangle\right|^2 \leq \liminf_{k\to\infty} \mathbb{E}\left|\langle e_j, X_t^{n_k}(\xi) - X_t^{n_k}(\xi)\rangle\right|^2 \leq K_{T,\xi}(t-s).$$

It follows that $\langle e_j, \psi\rangle$ has a $\left(\mathfrak{G}_{t+}^{\xi,W}\right)_{t\in[0,T]}$-predictable version $\widetilde{\langle e_j, \psi\rangle}$ (see, e.g., Proposition 3.6 of [12]). We define $\mathfrak{a}$ to be the set of all (t,ω) belonging to $[0,T]\times\Omega$ such that $\sum_{j=1}^{n} \widetilde{\langle e_j, \psi\rangle}_t(\omega)\, e_j$ converge as n goes to ∞. The proof is completed by choosing

$$Z_t(\xi)(\omega) = \begin{cases} \sum_{j=1}^{\infty} \widetilde{\langle e_j, \psi\rangle}_t(\omega)\, e_j, & \text{if } (t,\omega)\in\mathfrak{a} \\ 0, & \text{if } (t,\omega)\notin\mathfrak{a} \end{cases}.$$

Thus $Z(\xi)$ becomes a version of ψ. □

Lemma 6.9. *Adopt the assumptions and notation of Lemma* 6.8. *Let t belong to $[0,T]$. Then $\mathbb{E}\left\|Z_t(\xi)\right\|^2 \leq \mathbb{E}\left\|\xi\right\|^2$, $Z_t(\xi)\in Dom(C)$ a.s., and*

$$\mathbb{E}\left\|CZ_t(\xi)\right\|^2 \leq \exp\left(\alpha(t)\,t\right)\left(\mathbb{E}\left\|C\xi\right\|^2 + \alpha(t)\,t\mathbb{E}\left\|\xi\right\|^2\right). \tag{6.16}$$

Moreover, for all $j\in\mathbb{N}$ we have

$$L_j^{n_k}(t)\, X_t^{n_k}(\xi) \longrightarrow_{k\to\infty} L_j(t)\, Z_t(\xi) \quad \textit{weakly in } L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right). \tag{6.17}$$

If in addition $G(t)\in\mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$, then

$$G^{n_k}(t)\, X_t^{n_k}(\xi) \longrightarrow_{k\to\infty} G(t)\, Z_t(\xi) \quad \textit{weakly in } L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right). \tag{6.18}$$

Proof. Combining Lemma 6.5 with (6.14) we obtain $\mathbb{E}\left\|Z_t(\xi)\right\|^2 \leq \mathbb{E}\left\|\xi\right\|^2$. Consider U in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)$. Since $\left\|\widetilde{R}_n\right\| \leq 1$ and $\widetilde{R}_n \longrightarrow_{n\to\infty} I$, the dominated convergence theorem shows that $\widetilde{R}_n U \longrightarrow_{n\to\infty} U$ in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)$. Using (6.14) yields

$$\mathbb{E}\left\langle U, \widetilde{R}_{n_k} X_t^{n_k}(\xi)\right\rangle = \mathbb{E}\left\langle \widetilde{R}_{n_k} U, X_t^{n_k}(\xi)\right\rangle \longrightarrow_{k\to\infty} \mathbb{E}\left\langle U, Z_t(\xi)\right\rangle. \tag{6.19}$$

Let $L\in\mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$. Set $L^n = L\widetilde{R}_n$. Due to $\left\|\widetilde{R}_n\right\| \leq 1$ and $\widetilde{R}_n C \subset C\widetilde{R}_n$, applying Lemma 6.6 and the Banach-Alaoglu theorem we deduce that any subsequence of $\left(X_t^{n_k}(\xi)\right)_{k\in\mathbb{N}}$ contains a subsequence denoted (to shorten notation) by $\left(X_t^{n_l}(\xi)\right)_{l\in\mathbb{N}}$ such that $\left(L^{n_l} X_t^{n_l}(\xi)\right)_{l\in\mathbb{N}}$ and $\left(C\widetilde{R}_{n_l} X_t^{n_l}(\xi)\right)_{l\in\mathbb{N}}$ are weakly convergent in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)$. Hence $\left(\widetilde{R}_{n_l} X_t^{n_l}(\xi), L^{n_l} X_t^{n_l}(\xi), C\widetilde{R}_{n_l} X_t^{n_l}(\xi)\right)$ converges weakly in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}^3\right)$ by (6.19).

Since $L\in\mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$, $\mathcal{D}(C)\times L(\mathcal{D}(C))\times C(\mathcal{D}(C))$ is a closed set in $\mathfrak{h}^3 = \mathfrak{h}\times\mathfrak{h}\times\mathfrak{h}$. Therefore $\left\{(\eta, A\eta, L\eta) : \eta\in L_C^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)\right\}$ is a closed linear linear manifold of $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}^3\right)$. Thus, this set is closed with respect to the weak topology of $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}^3\right)$ (see, e.g., Theorem 3.12 of [31]). Using (6.19) we now get that $\left(\widetilde{R}_{n_l} X_t^{n_l}(\xi), L^{n_l} X_t^{n_l}(\xi), C\widetilde{R}_{n_l} X_t^{n_l}(\xi)\right)$ converges weakly to $(Z_t(\xi), LZ_t(\xi), CZ_t(\xi))$ as $l \to \infty$ in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}^3\right)$. This leads to

$$L^{n_k} X_t^{n_k}(\xi) \longrightarrow_{k\to\infty} L_j Z_t(\xi) \quad \textit{weakly in } L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right), \tag{6.20}$$

and so (6.17) holds by Condition H2.2. Taking $L = C$ in (6.20) and using Lemma 6.6 we get (6.16).

Suppose that $G(t) \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. By (6.20), $G(t)\widetilde{R}_{n_k} X_t^{n_k}(\xi)$ converges weakly to $G(t) Z_t(\xi)$ as $n \to \infty$ in $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)$. Hence

$$\mathbb{E}\left\langle \widetilde{R}_{n_k} U, G(t)\widetilde{R}_{n_k} X_t^{n_k}(\xi)\right\rangle \longrightarrow_{k\to\infty} \mathbb{E}\left\langle U, G(t) Z_t(\xi)\right\rangle,$$

where $U \in L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathfrak{h}\right)$. This gives (6.18), because $\widetilde{R}_n$ is self-adjoint. □

Remark 6.3. *Let χ be an element of $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,m}, \mathbb{P}\right), \mathbb{C}\right)$, with $t \in [0,T]$. Then, there exist $\left(\mathfrak{G}_s^{\xi,m}\right)_{s\geq 0}$-predictable processes $H^1, \cdots, H^m$ such that:*

(i) *$H^1, \cdots, H^m \in L^2(([0,T]\times\Omega, \nu\otimes\mathbb{P}), \mathbb{C})$, where ν denotes the Lesbesgue mesure on $\mathfrak{B}([0,T])$.*
(ii) *$\chi = \mathbb{E}\left(\chi|\mathfrak{G}_0^{\xi,m}\right) + \sum_{j=1}^m \int_0^t H_s^j dW_s^j$.*

Lemma 6.10. *Assume the setting of Theorem 3.1. Suppose that $(X^{n_k}(\xi))_{k\in\mathbb{N}}$ and χ are as in Lemma 6.8 and Remark 6.3 respectively. Let $x \in \mathfrak{h}$. Then*

$$\begin{aligned}&\lim_{k\to\infty} \mathbb{E}\left\langle \chi x, \sum_{j=1}^{n_k} \int_0^t L_j^{n_k}(s) X_s^{n_k}(\xi)\, dW_s^j\right\rangle \\ &= \mathbb{E}\left\langle \chi x, \sum_{j=1}^{\infty} \int_0^t L_j(s)\, \pi(Z_s(\xi))\, dW_s^j\right\rangle.\end{aligned}$$

Proof. Throughout this proof, $H^1, \cdots, H^m$ are as in Remark 6.3. From Condition H2.1 and Lemma 6.2 we get

$$\sum_{k=1}^{\infty} \|L_k(t) y\|^2 \leq K(t)\|y\|_C^2. \tag{6.21}$$

for all y in $\mathcal{D}(C)$ and $t \geq 0$.

Using Lemma 6.5, basic properties of stochastic integrals and Fubini's theorem we deduce that for all $n \geq m$,

$$\mathbb{E}\chi\left\langle x, \sum_{j=1}^{n} \int_0^t L_j^n(s) X_s^n(\xi)\, dW_s^j\right\rangle = \sum_{j=1}^{m} \int_0^t \mathbb{E} H_s^j \left\langle x, L_j^n(s) X_s^n(\xi)\right\rangle ds.$$

Due to (6.17), $\left\|\widetilde{R}_n\right\| \leq 1$ and $\widetilde{R}_n C \subset C\widetilde{R}_n$, Lemmata 6.5 and 6.6 allow to use the dominated convergence theorem to obtain

$$\int_0^t \mathbb{E}H_s^j \left\langle x, L_j^{n_k}(s) X_s^{n_k}(\xi)\right\rangle ds \longrightarrow_{k\to\infty} \int_0^t \mathbb{E}H_s^j \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle ds,$$

for any $j = 1, \ldots, m$. Hence

$$\lim_{k\to\infty} \mathbb{E}\left\langle \chi x, \sum_{j=1}^{n_k} \int_0^t L_j^{n_k}(s) X_s^{n_k}(\xi)\, dW_s^j \right\rangle = \sum_{j=1}^{m} \int_0^t \mathbb{E}H_s^j \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle ds.$$

Combining (6.21) with Lemmata 6.5 and 6.6 we deduce that

$$\sum_{j=1}^{m} \int_0^t \mathbb{E}H_s^j \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle ds = \sum_{j=1}^{n} \mathbb{E}\chi \int_0^t \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle dW_s^j$$

whenever $n \geq m$, and that

$$\sum_{j=1}^{n} \int_0^t L_j(s)\pi(Z_s(\xi))\, dW_s^j \longrightarrow_{n\to\infty} \sum_{j=1}^{\infty} \int_0^t L_j(s)\pi(Z_s(\xi))\, dW_s^j$$

in $L^2(\mathbb{P}, \mathfrak{h})$. This gives

$$\sum_{j=1}^{\infty} \mathbb{E}\chi \int_0^t \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle dW_s^j = \sum_{j=1}^{m} \int_0^t \mathbb{E}H_s^j \left\langle x, L_j(s)\pi(Z_s(\xi))\right\rangle ds.$$

□

Lemma 6.11. *Adopt the assumptions of Theorem* 3.1. *Assume that T and Z are defined as in Lemma* 6.8. *Then for all $t \in [0, T]$ we have*

$$Z_t(\xi) = \xi + \int_0^t G(s)\pi_C(Z_s(\xi))\, ds + \sum_{k=1}^{\infty} \int_0^t L_k(s)\pi_C(Z_s(\xi))\, dW_s^k \quad a.s.$$

Proof. Consider $x \in \mathfrak{h}$ and let $(X^{n_k})_{k\in\mathbb{N}}$ be as in Lemma 6.8. According to Lemma 6.10 we have

$$\lim_{k\to\infty} \mathbb{E}\left\langle \chi x, \sum_{j=1}^{n_k} \int_0^t L_j^{n_k}(s) X_s^{n_k}(\xi)\, dW_s^j \right\rangle$$
$$= \mathbb{E}\left\langle \chi x, \sum_{j=1}^{\infty} \int_0^t L_j(s)\pi(Z_s(\xi))\, dW_s^j \right\rangle.$$

Since $\mathbb{E}\left(\chi \diagup \mathfrak{G}_s^{\xi,W}\right) \in L^2(\mathbb{P}, \mathbb{C})$, using (6.18) and Lemmata 6.5, 6.6 we obtain

$$\int_0^t \mathbb{E}\left(\left\langle x, G^{n_k}(s) X_s^{n_k}(\xi)\right\rangle \mathbb{E}\left(\chi \diagup \mathfrak{G}_s^{\xi,W}\right)\right) ds$$
$$\longrightarrow_{k\to\infty} \int_0^t \mathbb{E}\left(\left\langle x, G(s)\pi(Z_s(\xi))\right\rangle \mathbb{E}\left(\chi \diagup \mathfrak{G}_s^{\xi,W}\right)\right) ds.$$

To this end we apply the dominated convergence theorem. Hence (6.14) implies

$$\begin{aligned}&\mathbb{E}\chi \left\langle x, Z_t(\xi)\right\rangle \qquad (6.22)\\&= \mathbb{E}\chi \left\langle x, \xi + \int_0^t G(s)\,\pi(Z_s(\xi))\,ds + \sum_{j=1}^{\infty}\int_0^t \left\langle x, L_j(s)\,\pi(Z_s(\xi))\right\rangle dW_s^j \right\rangle.\end{aligned}$$

Using a monotone class theorem (e.g., Theorem I.21 of [13]) we extend the range of validity of (6.22) from $\chi \in L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,m}, \mathbb{P}\right), \mathbb{C}\right)$ to any bounded χ belonging to $L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P}\right), \mathbb{C}\right)$, which completes the proof. □

Proof of Theorem 3.1. Consider $T > 0$. First, we combine Lemma 6.2 with Itô's formula to deduce that there exists at most one strong C-solution of (1.1) on $[0,T]$ (see proof of Lemma 2.2 of [23] for details). Second, for all $t \in [0,T]$, we set

$$Z_t^T = \xi + \int_0^t G(s)\,\pi_C(Z_s(\xi))\,ds + \sum_{k=1}^{\infty}\int_0^t L_k(s)\,\pi_C(Z_s(\xi))\,dW_s^k,$$

where $Z(\xi)$ is described as in Lemma 6.8. According to Lemma 6.11 we have that Z^T is a continuous version of $Z(\xi)$. Hence Z^T is a strong C-solution of (1.1) on $[0,T]$, and so Z^T is the unique one.

Finally, we define $\widetilde{\Omega}$ to be the set of all ω satisfying $Z_t^n(\omega) = Z_t^{n+1}(\omega)$ for all $n \in \mathbb{N}$ and any $t \in [0,n]$. For any $t \geq 0$, we choose

$$X_t(\xi) = \begin{cases} Z_t^n(\omega), & if\ \omega \in \widetilde{\Omega}\ and\ t \in [0,n] \\ 0, & if\ \omega \notin \widetilde{\Omega} \end{cases}.$$

Thus $X(\xi)$ is the unique strong C-solution of (1.1) on $[0,\infty[$. □

6.3. Proof of Theorem 3.2

Proof of Theorem 3.2. From Condition H3.2 of Hypothesis 3 we deduce that (1.1) has a unique strong C-solution $X(\xi)$ in $[0,\infty[$. Using the same arguments as in the proof of Theorem 3.1 of [23] we obtain

$$\mathbb{E}\left\|X_t(\xi)\right\|^2 = \mathbb{E}\left\|\xi\right\|^2 \qquad (6.23)$$

for all $t \geq 0$. For the reader's convenience, we next outline the proof of (6.23). First, we set

$$\widehat{\tau}_n = \inf\left\{ t \leq 0 : \int_0^t \mathbf{1}_{]0,\infty[}\left(\left\|X_s(\xi)\right\|\right) \frac{\left\|CX_s(\xi)\right\|^2}{\left\|X_s(\xi)\right\|^2} ds > n \right\} \wedge T,$$

where $T > 0$ and $\mathbf{1}$ stands for the indicator function. Using Condition H3.1, Itô's formula and Novikov's criterion we can assert that $\left(\left\|X_{t\wedge\widehat{\tau}_n}(\xi)\right\|^2\right)_{t\in[0,T]}$ is a martingale. Second, we consider the stopping time

$$\widetilde{\tau}_n = T \wedge \inf\left\{ t \in [\widehat{\tau}_n, T] : \left\|X_t(\xi)\right\| < \left\|X_{\widehat{\tau}_n}(\xi)\right\|/2 \right\}.$$

The martingale property of $\left\|X^{\widehat{\tau}_n}(\xi)\right\|^2$ leads to

$$\begin{aligned}
&\mathbb{E}\left(\|X_{t\wedge\widehat{\tau}_n}(\xi)\|^2 \mathbf{1}_{\{\widehat{\tau}_n<t\}}\right)\\
&\leq \frac{1}{n}\mathbb{E}\left(\|X_{t\wedge\widehat{\tau}_n}(\xi)\|^2 \mathbf{1}_{\{\widehat{\tau}_n<t\}} \int_0^{t\wedge\widetilde{\tau}_n} \mathbf{1}_{]0,\infty[}(\|X_s(\xi)\|) \frac{\|CX_s(\xi)\|^2}{\|X_s(\xi)\|^2} ds\right)\\
&\leq \frac{3}{n}\int_0^t \mathbb{E}\left(\|CX_s(\xi)\|^2\right) ds.
\end{aligned}$$

We thus get $\mathbb{E}\left(\|X_{t\wedge\widehat{\tau}_n}(\xi)\|^2 \mathbf{1}_{\{\widehat{\tau}_n<t\}}\right) \to_{n\to\infty} 0$, and so $\mathbb{E}\left(\|X_t(\xi)\|^2\right) \geq \mathbb{E}\left(\|\xi\|^2\right)$, which implies (6.23) due to $\mathbb{E}\|X_t(\xi)\|^2 \leq \mathbb{E}\|\xi\|^2$.

Assume that $T>0$ and $n\in\mathbb{N}$. We now choose

$$\tau_n = \inf\{t\geq 0 : \|X_t(\xi)\| > n\} \wedge T.$$

Combining Condition H3.1 with Itô's formula we obtain

$$\|X_{t\wedge\tau_n}(\xi)\|^2 = \|\xi\|^2 + \sum_{k=1}^{\infty}\int_0^{t\wedge\tau_n} 2\Re\langle X_s(\xi), L_k(s) X_s(\xi)\rangle dW_s^k. \tag{6.24}$$

From Conditions H2.1 and H3.1 we have

$$\sum_{k=1}^{\infty}\mathbb{E}\int_0^{\tau_n} \left(\Re\langle X_s(\xi), L_k(s) X_s(\xi)\rangle\right)^2 ds \leq K_{n,T}\left(1+\mathbb{E}\|\xi\|_C^2\right),$$

where $K_{n,T}$ is a constant depending of n and T. Hence (6.24) shows that $\|X^{\tau_n}(\xi)\|^2$ is a martingale. Applying Fatou's lemma yields the supermartingale property of $\left(\|X_t(\xi)\|^2\right)_{t\in[0,T]}$. Therefore $\left(\|X_t(\xi)\|^2\right)_{t\in[0,T]}$ is a martingale by (6.23).

Finally, we can prove (3.1), that is the Markov property of $X_t(\xi)$, using techniques of well-posed martingale problems (see, e.g., proof of Theorem 3 of [24]). □

6.4. Proof of Theorem 5.1

We start by proving a basic inequality (i.e., relation (6.25) given below).

Lemma 6.12. *Let $f\in C_0^{\infty}(\mathbb{R},\mathbb{C})$. Then*

$$\left\|-\frac{d^2}{dx^2}f(x)\right\|^2 + \left\|x^2 f(x)\right\|^2 \leq \left\|\left(-\frac{d^2}{dx^2}+x^2\right)f(x)\right\|^2 + 2\|f\|^2,$$

where $\|\cdot\|$ stands for the norm in $L^2(\mathbb{R},\mathbb{C})$.

Proof. Throughout the proof, we restrict the domain of all operators to $C_0^{\infty}(\mathbb{R},\mathbb{C})$. Moreover, as usual in physics, we set $p=-i\frac{d}{dx}$ and $q=x$. Then

$$\left(p^2+q^2\right)^2 = p^4 + 2pq^2p + q^4 + p\left[p,q^2\right] + \left[q^2,p\right]p.$$

Since pq^2p is a non-negative operator,

$$\left(p^2+q^2\right)^2 \geq p^4 + q^4 + p\left[p,q^2\right] + \left[q^2,p\right]p.$$

Using $[p,q]=-i$ we get

$$\left[p,q^2\right]=[p,q]\,q+q\,[p,q]=-2iq.$$

Hence $p\left[p,q^2\right]+\left[q^2,p\right]p=-2$, and so

$$\left\langle f,\left(p^2+q^2\right)^2 f\right\rangle+2\left\|f\right\|^2\geq\left\|p^2 f\right\|^2+\left\|q^2 f\right\|^2,$$

which establishes the desire inequality. □

Lemma 6.13. *If $f\in C_0^{\infty}\left(\mathbb{R}^n,\mathbb{C}\right)$, then*

$$\left\|-\triangle f\right\|^2+\left\|\left|x\right|^2 f\left(x\right)\right\|^2\leq\left\|\left(-\triangle+\left|x\right|^2\right)f\left(x\right)\right\|^2+2n\left\|f\right\|^2, \tag{6.25}$$

where $\left\|\cdot\right\|$ denotes the norm in $L^2\left(\mathbb{R}^n,\mathbb{C}\right)$.

Proof. Let j and k be different natural numbers lying between 1 and n. According to the fact that the operator x_j commutes with the positive operator $-\partial^2/\partial x_k^2$, we have $\left\langle f,-\partial^2\left(x_j^2 f\right)/\partial x_k^2\right\rangle\geq 0$ and $\left\langle f,-x_j^2\partial^2\left(f\right)/\partial x_k^2\right\rangle\geq 0$. Hence

$$\begin{aligned}\left\langle f,\left(-\triangle+\left|x\right|^2\right)^2 f\right\rangle\geq&\sum_{k=1}^{n}\left\langle f,\left(-\frac{\partial^2}{\partial x_k^2}+x_k^2\right)^2 f\right\rangle\\&+\sum_{j,k=1,\ldots n;j\neq k}\left(\left\langle f,\frac{\partial^2}{\partial x_k^2}\frac{\partial^2}{\partial x_j^2}f\right\rangle+\left\langle f,x_k^2x_j^2 f\right\rangle\right).\end{aligned}$$

Using Lemma 6.12 yields (6.25). □

Using Lemma 6.13 and the integration by parts formula we deduce the next auxiliary result.

Lemma 6.14. *Let $f\in C_0^{\infty}\left(\mathbb{R},\mathbb{C}\right)$. Assume that $\varphi:[0,\infty[\times\mathbb{R}^n\mapsto\mathbb{C}^n$ is such that for any $t\in[0,\infty[$, $\varphi\left(t,\cdot\right)$ is Borel measurable and*

$$\left|\varphi\left(t,x\right)\right|\leq K\left(t\right)\left(1+\left|x\right|\right) \tag{6.26}$$

whenever $x\in\mathbb{R}^n$. Then for all $t\geq 0$,

$$\left\|\varphi\left(t,\cdot\right)\bullet\nabla f\right\|^2\leq K\left(t\right)\left(\left\|\left(-\triangle+\left|x\right|^2\right)f\left(x\right)\right\|^2+\left\|f\right\|^2\right). \tag{6.27}$$

Proof. Applying first the Cauchy-Schwarz inequality and then using (6.26) we deduce that

$$\begin{aligned}\left\|\varphi\left(t,\cdot\right)\bullet\nabla f\right\|^2&\leq n\sum_{k=1}^{n}\left\|\varphi_k\left(t,\cdot\right)\frac{\partial}{\partial x_k}f\right\|^2 \qquad (6.28)\\&\leq K\left(t\right)\sum_{k=1}^{n}\int_{\mathbb{R}^n}\left(1+\left|x\right|^2\right)\frac{\partial}{\partial x_k}f\left(x\right)\frac{\partial}{\partial x_k}\overline{f}\left(x\right)d\nu\left(x\right),\end{aligned}$$

where φ_k stands for the kth coordinate of φ.

Combining the integration by parts formula with the Cauchy-Schwarz inequality we get

$$\left|\sum_{k=1}^{n}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_k}f\frac{\partial}{\partial x_k}\overline{f}d\nu\right|=\left|\int_{\mathbb{R}^n}-\triangle f\overline{f}d\nu\right|\leq 2\left(\left\|-\triangle f\right\|^2+\left\|f\right\|^2\right). \tag{6.29}$$

Using again the integration by parts formula and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\int_{\mathbb{R}^n}\left|x\right|^2\frac{\partial}{\partial x_k}f\left(x\right)\frac{\partial}{\partial x_k}\overline{f}\left(x\right)d\nu\left(x\right)\right|\\
=&\left|\int_{\mathbb{R}^n}\left|x\right|^2\overline{f}\left(x\right)\left(-\triangle f\right)\left(x\right)d\nu\left(x\right)+2\sum_{k=1}^{n}\int_{\mathbb{R}^n}x_k\overline{f}\left(x\right)\frac{\partial}{\partial x_k}f\left(x\right)d\nu\left(x\right)\right|\\
\leq&\left\|\left|x\right|^2f\right\|\left\|-\triangle f\right\|+4\int_{\mathbb{R}^n}\left|x\right|^2\left|f\right|^2d\nu\left(x\right)+4\sum_{k=1}^{n}\int_{\mathbb{R}^n}\left|\frac{\partial}{\partial x_k}f\right|^2d\nu.
\end{aligned}$$

From (6.29) it follows that

$$\left|\sum_{k=1}^{n}\int_{\mathbb{R}^n}\left|x\right|^2\frac{\partial}{\partial x_k}f\left(x\right)\frac{\partial}{\partial x_k}\overline{f}\left(x\right)d\nu\left(x\right)\right|\leq\left(\left\|-\triangle f\right\|^2+\left\|\left|x\right|^2f\right\|^2+\left\|f\right\|^2\right). \tag{6.30}$$

According to (6.28)-(6.30) and Lemma 6.13 we have (6.27). □

Combining the integration by parts formula with density arguments we get the following simple equality.

Lemma 6.15. *Let $f\in C_0^{\infty}\left(\mathbb{R},\mathbb{C}\right)$. Suppose that φ is locally in $L^1\left(\mathbb{R},\mathbb{R}\right)$. Then*

$$\Re\left(i\int_{\mathbb{R}}\varphi\left(x\right)\overline{f\left(x\right)}\frac{d^2}{dx^2}f\left(x\right)dx\right)=0.$$

Proof. We first assume that $\varphi\in C^2\left(\mathbb{R},\mathbb{R}\right)$. Combining Leibnitz's formula with the integration by parts formula we deduce that

$$\begin{aligned}
\Re\left(i\int_{\mathbb{R}}\varphi\left(x\right)\overline{f\left(x\right)}\frac{d^2}{dx^2}f\left(x\right)dx\right)&=\Re\left(i\int_{\mathbb{R}}\left|f\left(x\right)\right|^2\frac{d^2}{dx^2}\varphi\left(x\right)dx\right)\\
&\quad-\Re\left(i\int_{\mathbb{R}}\varphi\left(x\right)f\left(x\right)\overline{\frac{d^2}{dx^2}f\left(x\right)}dx\right).
\end{aligned}$$

This implies $\Re\left(i\int_{\mathbb{R}}\varphi\left(x\right)\overline{f\left(x\right)}f''\left(x\right)dx\right)=0$. The proof is completed by using that $C^2\left(\mathbb{R},\mathbb{R}\right)$ is dense in $L^1_{loc}\left(\mathbb{R},\mathbb{R}\right)$. □

We now provide a tool for treating the dissipative terms of the right hand side of the inequality described in Condition H2.3.

Lemma 6.16. *Consider* $C = -\triangle + |x|^2$. *Let* $\varphi \in C^2(\mathbb{R}^n, \mathbb{C})$. *Then, for all* $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ *we have*

$$-\Re \left\langle C^2 f, |\varphi|^2 f \right\rangle + \|C\varphi f\|^2 \leq \|Cf\| \left(\|\bar{\varphi}[C,\varphi] f\| + \|[C,\bar{\varphi}]\varphi f\| \right) + \|[C,\varphi] f\|^2 .$$

Proof. Rearranging terms yields

$$\begin{aligned} &\langle C\varphi f, C\varphi f\rangle - \left\langle C^2 f, |\varphi|^2 f \right\rangle \\ &= \langle (\varphi C + [C,\varphi]) f, (\varphi C + [C,\varphi]) f\rangle - \left\langle Cf, \left(|\varphi|^2 C + \left[C, |\varphi|^2\right] \right) f \right\rangle . \end{aligned} \tag{6.31}$$

Substituting $\left[C, |\varphi|^2\right] = [C,\bar{\varphi}]\varphi + \bar{\varphi}[C,\varphi]$ into (6.31) we obtain

$$\langle C\varphi f, C\varphi f\rangle - \left\langle C^2 f, |\varphi|^2 f \right\rangle = \langle \bar{\varphi}[C,\varphi] f, Cf\rangle - \langle Cf, [C,\bar{\varphi}]\varphi f\rangle + \|[C,\varphi] f\|^2 .$$

Thus, the lemma follows from the Cauchy-Schwarz inequality. □

Proof of Theorem 5.1. Due to $|x|^2$ is locally in $L^2(\mathbb{R}^n, \mathbb{C})$, C is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ (see, e.g., Theorem X.29 of [29])). From Lemma 6.12 we obtain that $\mathcal{D}(C) = \mathcal{D}(-\triangle) \cap \mathcal{D}\left(|x|^2\right)$. By Hypothesis 4, Lemma 6.13 lead to Conditions H2.1 and H2.2. Moreover, applying a functional version of the monotone class theorem and the dominated convergence theorem we obtain Conditions H1.1 and H1.2.

Using the definition of G we deduce that for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,

$$2\Re \langle f, G(t) f\rangle + \sum_{k=1}^{m} \|L_k(t) f\|^2 = 0.$$

According to $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is a core of C we have that Condition H3.1 holds, as well as Condition H2.4.

Let $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Since

$$\Re \left\langle C^2 f, -iH(t) f \right\rangle = \Re \left\langle \left(-\triangle + |x|^2 \right)^2 f, -i\left(-\alpha |x|^2 + g(t,\cdot) \right) f \right\rangle ,$$

combining Lemma 6.15 with Fubini's theorem yields

$$\Re \left\langle C^2 f, -iH(t) f \right\rangle = -\Re i \left\langle (-\triangle)^2 f - \triangle\left(|x|^2 f \right), \left(-\alpha |x|^2 + g(t,\cdot) \right) f \right\rangle .$$

By $-\triangle$ is a non-negative operator, $\Re i \left\langle -\triangle\left(|x|^2 f \right), |x|^2 f \right\rangle = 0$. Lemma 6.15, together with Leibnitz formula and Fubini's theorem, now show

$$\Re i \left\langle \triangle\left(|x|^2 f \right), \left(-\alpha |x|^2 + g(t,\cdot) \right) f \right\rangle = 2\Re i \left\langle \nabla\left(|x|^2 \right) \bullet \nabla f, g(t,\cdot) f \right\rangle .$$

From the Cauchy-Schwarz inequality follows that

$$\left| \Re i \left\langle \triangle\left(|x|^2 f \right), \left(-\alpha |x|^2 + g(t,\cdot) \right) f \right\rangle \right| \leq 4 \left(\left\| \nabla\left(|x|^2 \right) \bullet \nabla f \right\|^2 + \|g(t,\cdot) f\|^2 \right) .$$

Due to $-\triangle$ is self-adjoint, according to Leibnitz's formula, Lemma 6.15 and Fubini's theorem we have

$$\Re i \left\langle (-\triangle)^2 f, \left(-\alpha |x|^2 + g(t,\cdot)\right) f \right\rangle = 2\Re i \left\langle -\triangle f, \nabla \left(-\alpha |x|^2 + g(t,\cdot)\right) \bullet \nabla f \right\rangle,$$

and so

$$\begin{aligned} &\left| \Re i \left\langle (-\triangle)^2 f, \left(-\alpha |x|^2 + g(t,\cdot)\right) f \right\rangle \right| \\ &\leq 4 \left(\|-\triangle f|^2 + \left\| \nabla \left(-\alpha |x|^2 + g(t,\cdot)\right) \bullet \nabla f \right|^2 \right). \end{aligned}$$

Using Hypothesis 4 and Lemma 6.14 we deduce that

$$\Re \left\langle C^2 f, -iH(t) f \right\rangle \leq K(t) \|f\|_C^2 .$$

Since $\phi_k(t,\cdot) \in C^2(\mathbb{R}^n, \mathbb{C})$,

$$[C, \phi_k(t,\cdot)] f = -f \triangle \phi_k(t,\cdot) - 2\nabla \phi_k(t,\cdot) \bullet \nabla f.$$

Hence

$$\overline{\phi_k(t,\cdot)} [C, \phi_k(t,\cdot)] f = -f \overline{\phi_k(t,\cdot)} \triangle \phi_k(t,\cdot) - 2\overline{\phi_k(t,\cdot)} \nabla \phi_k(t,\cdot) \bullet \nabla f$$

and

$$\begin{aligned} &\left[C, \overline{\phi_k(t,\cdot)}\right] \phi_k(t,\cdot) f \\ &= -f \phi_k(t,\cdot) \triangle \overline{\phi_k(t,\cdot)} - 2f |\nabla \phi_k(t,\cdot)|^2 - 2\phi_k(t,\cdot) \nabla \overline{\phi_k(t,\cdot)} \bullet \nabla f. \end{aligned}$$

By (5.1) and (5.2), Lemmata 6.12, 6.14 and 6.16 lead to

$$-\Re \left\langle C^2 f, |\phi_k(t,\cdot)|^2 f \right\rangle + \|C \phi_k(t,\cdot) f\|^2 \leq K(t) \|f\|_C^2 .$$

Therefore Condition H2.3 holds with $\mathfrak{D}_1 = C_0^\infty(\mathbb{R}, \mathbb{C})$, and so Hypothesis 2 holds. Thus, Theorems 3.1 and 4.1 show our statement. □

References

[1] R. Alicki, *On the scattering theory for quantum dynamical semigroups,* Ann. Inst. Henri Poincaré (A) Physique théorique **35** (1981), 97-103.

[2] A. Barchielli and V.P. Belavkin, *Measurements continuous in time and a posteriori states in quantum mechanics*, J. Phys. A: Math. Gen. **24** (1991), 1495–1514.

[3] A. Barchielli and M. Gregoratti, *Quantum trajectories and measurements in continuous time: the diffusive case*, Lecture Notes in Physics, **782**, Springer, Berlin, 2009.

[4] A. Bassi and D. Dürr, *On the long-time behavior of Hilbert space diffusion*, Europhys. Lett. **84** (2008), 10005.

[5] V. P. Belavkin, *A New Wave Equation for a Continuous Nondemolition Measurement*, Phys. Lett. A. **140** (1989), 355–358.

[6] V. P. Belavkin, *Quantum causality, decoherence, trajectories and information*, Rep. on Prog. on Phys. **65** (2002), 353–420.

[7] L. Bouten, R. van Handel and A. Silberfarb, *Approximation and limit theorems for quantum stochastic models with unbounded coefficients*, J. Funct. Anal. **254** (2008), 3123–3147.

[8] H.P. Breuer and F. Petruccione, *The theory of open quantum systems*, Oxford University Press, New York, 2002.

[9] C. M. Caves and G. J. Milburn, *Quantum mechanical model for continuous position measurements*, Phys. Rev. A **36** (1987), 5543.

[10] A.M. Chebotarev and F. Fagnola, *Sufficient conditions for conservativity of minimal quantum dynamical semigroups*, J. Funct. Anal. **153** (1998), 382–404.

[11] A. M. Chebotarev, J. Garcia and R. Quezada, *A priori estimates and existence theorems for the Lindblad equation with unbounded time-dependent coefficients*, in Recent trends in infinite dimensional non-commutative analysis, Publ. Res. Inst. Math. Sci. Kokyuroku **1035**, (1998), 44-65.

[12] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, Cambridge, 1992.

[13] C. Dellacherie and P.-A. Meyer, *Probabilities and Potential*, North-Holland Mathematics Studies **29** North-Holland, Amsterdam, 1978.

[14] L. Diósi, *Continuous quantum measurement and Itô formalism*, Phys. Lett. A **129** (1988), 419–423.

[15] K-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer, New York, 2000.

[16] F. Fagnola, *Quantum markov semigroups and quantum flows*, Proyecciones **18** (1999), 1–144.

[17] F. Fagnola and S.J. Wills, *Solving quantum stochastic differential equations with unbounded coefficients*, J. Funct. Anal. **198** (2003), 279–310.

[18] J. Gough and A. Sobolev, *Continuous measurement of canonical observables and limit stochastic Schrodinger equations*, Phys. Rev. A **69** (2004), 032107.

[19] T. Grotz, L. Heaney and W.T. Strunz, *Quantum dynamics in fluctuating traps: Master equation, decoherence, and heating*, Phys. Rev. A **74** (2006), 022102.

[20] A.S. Holevo, *On dissipative stochastic equations in a Hilbert space*, Probab. Theory Related Fields **104** (1996), 483–500.

[21] C.M. Mora, *Numerical simulation of stochastic evolution equations associated to quantum Markov semigroups*, Math. Comp. **73** (2004), 1393–1415.

[22] C.M. Mora, *Numerical solution of conservative finite-dimensional stochastic Schrödinger equations*, Ann. Appl. Probab. **15** (2005), 2144–2171.

[23] C.M. Mora and R. Rebolledo, *Regularity of solutions to linear stochastic Schrödinger equations*, Infinite Dimens. Anal. Quantum Probab. Rel. Topics **10** (2007), 237-259.

[24] C.M. Mora and R. Rebolledo, *Basic properties of non-linear stochastic Schrödinger equations driven by Brownian motions*, Ann. Appl. Probab. **18** (2008), 591—619.

[25] C.M. Mora, *Heisenberg evolution of quantum observables represented by unbounded operators*, J. Funct. Anal. **255** (2008), 3249–3273.

[26] C.M. Mora, *Regularity of solutions to quantum master equations: A stochastic approach*, Preprint, 2010.

[27] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

[28] I.C. Percival, *Quantum state diffusion*, Cambridge University Press, Cambridge, 1998.

[29] M. Reed and B. Simon, *Methods of modern mathematical physics. Vol II: Fourier analysis and self-adjointness*, Academic Press, San Diego, 1975.

[30] B.L. Rozovskii, *Stochastic evolution systems. Linear theory and applications to nonlinear filtering*, Kluwer, Dordrecht, 1990.

[31] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[32] A.J. Scott and G.J. Milburn, *Quantum nonlinear dynamics of continuously measured systems*, Phys. Rev. A **63** (2001), 42101.

[33] K.P. Singh and J.M. Rost, *Femtosecond photoionization of atoms under noise*, Phys. Rev. A **76** (2007), 063403.

Acknowledgments

C.M.M wishes to thank the Statistical and Applied Mathematical Sciences Institute (SAMSI) for its warm hospitality and financial support during a part of the period in which this paper was written.

Franco Fagnola
Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo Da Vinci 32
I-20133 Milano
Italy
e-mail: `franco.fagnola@polimi.it`

Carlos M. Mora
CI^{2}MA, Departamento de Ingeniería Matemática
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Casilla 160 C
Concepción
Chile
e-mail: `cmora@ing-mat.udec.cl`

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2009 - 2010

2009-21 Carlos D. Acosta, Raimund Bürger, Carlos E. Mejia: *Monotone difference schemes stabilized by discrete mollification for strongly degenerate parabolic equations*

2009-22 Gabriel N. Gatica, Salim Meddahi: *Finite element analysis of a time harmonic Maxwell problem with an impedance boundary condition*

2009-23 Veronica Anaya, Mostafa Bendahmane, Mauricio Sepúlveda: *Mathematical and numerical analysis for predator-prey system in a polluted environment*

2009-24 Julio Aracena, Eric Fanchon, Marco Montalva, Mathilde Noual: *Combinatorics on update digraphs of discrete networks*

2010-01 Stefan Berres, Raimund Bürger, Rodrigo Garces: *Centrifugal settling of flocculated suspensions: A sensitivity analysis of parametric model functions*

2010-02 Mauricio Sepúlveda: *Stabilization of a second order scheme for a GKdV-4 equation modelling surface water waves*

2010-03 Lourenco Beirao-Da-Veiga, David Mora: *A mimetic discretization of the Reissner-Mindlin plate bending problem*

2010-04 Alfredo Bermúdez, Carlos Reales, Rodolfo Rodríguez, Pilar Salgado: *Mathematical and numerical analysis of a transient eddy current axisymmetric problem involving velocity terms*

2010-05 Maria G. Armentano, Claudio Padra, Rodolfo Rodríguez, Mario Scheble: *An hp finite element adaptive method to compute the vibration modes of a fluid-solid coupled system*

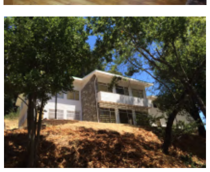
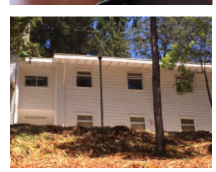
2010-06 Alfredo Bermúdez, Rodolfo Rodríguez, María L. Seoane: *A fictitious domain method for the numerical simulation of flows past sails*

2010-07 Carlo Lovadina, David Mora, Rodolfo Rodríguez: *A locking-free finite element method for the buckling problem of a non-homogeneous Timoshenko beam*

2010-08 Franco Fagnola, Carlos M. Mora: *Linear stochastic Schrödinger equations with unbounded coefficients*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl