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A mimetic discretization of the Reissner-Mindlin plate bending problem

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Abstract We present a mimetic approximation of the Reissner-Mindlin plate bending problem which uses deflections and rotations as discrete variables. The method applies to very general polygonal meshes, even with non matching or non convex elements. We prove linear convergence for the method uniformly in the plate thickness.

1 Introduction

The Mimetic Finite Difference (or MFD) method allows for the discretization of problems in partial differential equations using very general polygonal/polyhedral grids. The MFD scheme has been successfully employed for solving problems of continuum mechanics [40], electromagnetics [35], gas dynamics [24], and linear diffusion (see e.g. [36,41,37,14,15,34] and references therein).

Recently, a new approach to the MFD method has been proposed in [21]. Such approach, which interprets the MFD method as a generalization of the finite element method, seems to be more flexible both for developing the method and for the convergence analysis. This last generation of MFD should be more appropriately called Mimetic Discretization (MD) methods, since the original finite difference approach is abandoned. From the standpoint of finite elements, the fundamental idea of the mimetic discretization scheme

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becomes the following: the discrete variational problem is written directly in terms of the degrees of freedom and the underlying basis functions are not specified explicitly. Clearly, the differential operators and bilinear forms appearing in the problem must be suitably discretized in such a way that certain stability and consistency properties are satisfied. This approach allows for general polygonal/polyhedral meshes, even with non-matching and non-convex elements. Another remarkable fact is that the aforementioned forms can be practically constructed in a rather simple algebraic way.

The ideas and convergence analysis presented in [21] for the diffusion problem has been further developed in [17,10,38,7]. As previously mentioned, this analysis resulted also in new algebraic methods for building mass [23, 22] and stiffness [17] matrices on arbitrary-shaped elements for the linear diffusion problem. These algebraic methods have been developed also for higher order MFD methods [13,33]. A-posteriori error estimators have been analyzed in [6,12], while in [25,26] the authors introduced a post-processing technique and generalized some previous results. Moreover, a mimetic discretization of the Stokes problem following this new approach was presented in [8,11]. Finally, the mimetic discretization method has been shown to share strong similarities also with the finite volume method in [29], see also [28].

The aim of the present paper is to develop a Mimetic Discretization of the Reissner-Mindlin plate bending problem. This problem has attracted a large attention in the last decades both in the engineering and mathematical communities, mainly due to the large applicability of the model and the strong difficulties hidden in its numerical approximation. Nowadays there exists a large range of finite element schemes for the Reissner-Mindlin plate bending problem, the most famous and popular ones belonging without doubt to the Mixed Interpolation of Tensorial Components (MITC) class of methods [4,2]. The convergence analysis of the MITC elements has been covered in several papers from different points of view, see for instance [18,3,20,42,32, 43,31,5,39].

In the present paper, we propose a MD method which applies to general polygonal (even non-conforming or non-convex) meshes and which takes the steps from the MITC philosophy. The degrees of freedom for the (scalar) displacement variable are one for each mesh vertex, while for the (vectorial) rotation variable we adopt two degrees of freedom for each vertex plus an additional degree of freedom on each edge. Under certain assumptions on the mesh, such edge degrees of freedom can be dropped, leading to a method which uses only vertex d.o.f.s. both for the displacements and rotations. Taking inspiration from the MITC approach, the proposed scheme adopts a reduction of the shear energy in order to avoid locking. As it happens in mimetic discretizations, all the reduction and differential operators, bilinear forms and degrees of freedom must be defined carefully in order to correctly mimic the properties of the original problem.

The paper is organized as follows. In Section 2 we present the model problem. In Section 3, after introducing the discrete spaces, operators and bilinear forms, we describe the proposed method. In the rest of the paper we develop the error analysis. In order to do so, we take inspiration from the ideas of [42,1,20,39] which rewrite the discrete problem as a combination of different sub-problems via a discrete Helmholtz decomposition. We choose such approach because, although it is perhaps less direct than others, it has the advantage of unveiling the true structure of the problem. After introducing the equivalent discrete problem in Section 4, we derive the error analysis in Section 5. In the main Theorem 1, we finally prove the linear convergence of the method, uniformly in the thickness parameter t and under realistic regularity requirements for the solution.

2 The Reissner-Mindlin plate bending problem

Here and thereafter we use the following operator notation for any tensor field $\boldsymbol{\tau} = (\tau_{ij})\ i,j = 1,2$, any vector field $\boldsymbol{\eta} = (\eta_i)\ i = 1,2$ and any scalar field v:

$$\operatorname{div} \boldsymbol{\eta} := \partial_1 \eta_1 + \partial_2 \eta_2, \qquad \operatorname{rot} \boldsymbol{\eta} := \partial_1 \eta_2 - \partial_2 \eta_1, \qquad \nabla v := \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix},$$

$$\operatorname{curl} v := \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}, \qquad \mathbf{div}\, \boldsymbol{\tau} := \begin{pmatrix} \partial_1 \tau_{11} + \partial_2 \tau_{12} \\ \partial_1 \tau_{21} + \partial_2 \tau_{22} \end{pmatrix}, \qquad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}.$$

Throughout the paper we will use standard notations for Sobolev spaces, norms and semi-norms. Moreover, we will denote with c and C, with or without subscripts, tildes, or hats a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

Consider an elastic plate of thickness t such that $0 < t \leq \operatorname{diam}(\Omega)$, with reference configuration $\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, where Ω is a convex polygonal domain of $\mathbb{R}^2$ occupied by the midsection of the plate. The deformation of the plate is described by means of the Reissner-Mindlin model in terms of the rotations $\boldsymbol{\beta} = (\beta_1, \beta_2)$ of the fibers initially normal to the plate's midsurface, the scaled shear stresses $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, and the transverse displacement w. Assuming that the plate is clamped on its whole boundary $\partial\Omega$, the following strong equations describe the plate's response to conveniently scaled transversal load $g \in L^2(\Omega)$: find $(\boldsymbol{\beta}, w, \boldsymbol{\gamma})$ such that

$$\begin{cases} -\mathbf{div}\, \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}) - \boldsymbol{\gamma} = \mathbf{0} & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{\gamma} = g & \text{in } \Omega, \\ \boldsymbol{\gamma} = \kappa t^{-2}(\nabla w - \boldsymbol{\beta}) & \text{in } \Omega, \\ \boldsymbol{\beta} = \mathbf{0}, w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the tensor of bending moduli

$$\mathbb{C}\boldsymbol{\tau} := \frac{\mathbb{E}}{12(1-\nu^2)} \left((1-\nu)\boldsymbol{\tau} + \nu \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}\right),$$

with $\mathbb{E} > 0$ representing the Young modulus, $0 < \nu < 1/2$ being the Poisson ratio for the material and $\mathbf{I}$ indicating the second order identity tensor.

Let the $H_0^1(\Omega)^2$-elliptic bilinear form be given by

$$
\begin{aligned}
a(\boldsymbol{\beta},\boldsymbol{\eta}) &:= \int_\Omega \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) = \\
&= \frac{\mathbb{E}}{12(1-\nu^2)} \int_\Omega \left[(1-\nu)\boldsymbol{\varepsilon}(\boldsymbol{\beta}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \nu \operatorname{div}\boldsymbol{\beta} \operatorname{div}\boldsymbol{\eta}\right],
\end{aligned}
\tag{2}
$$

with $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1\le i,j\le 2}$ the standard strain tensor defined by $\varepsilon_{ij}(\boldsymbol{\beta}) := \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_i), 1 \le i,j \le 2$.

Then, the variational formulation of problem (1) reads: Find $(\boldsymbol{\beta}, w, \boldsymbol{\gamma}) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \times L^2(\Omega)^2$ such that

$$
\begin{cases}
a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\eta})_{0,\Omega} = (g,v)_{0,\Omega} & \forall(\boldsymbol{\eta}, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\
(\nabla w - \boldsymbol{\beta}, \boldsymbol{\delta})_{0,\Omega} - \kappa^{-1}t^2(\boldsymbol{\gamma},\boldsymbol{\delta})_{0,\Omega} = 0 & \forall \boldsymbol{\delta} \in L^2(\Omega)^2,
\end{cases}
\tag{3}
$$

where $\kappa := \mathbb{E}\mathbf{k}/2(1+\nu)$ is the shear modulus with $\mathbf{k}$ a correction factor usually taken as $5/6$ for clamped plates.

Using the Helmholtz decomposition for the shear term [18]

$$
\boldsymbol{\gamma} = \nabla\psi + \operatorname{curl} p, \tag{4}
$$

with $\psi \in H_0^1(\Omega)$ and $p \in H^1(\Omega) \cap L_0^2(\Omega)$, the same decomposition for the test function

$$
\boldsymbol{\delta} = \nabla\xi + \operatorname{curl} q,
$$

and integrating by parts, we easily infer that problem (3) is equivalent to the following: Find $(\psi, \boldsymbol{\beta}, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H^1(\Omega) \cap L_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$
\begin{cases}
(\nabla\psi, \nabla v)_{0,\Omega} = (g,v)_{0,\Omega} & \forall v \in H_0^1(\Omega), \\
a(\boldsymbol{\beta},\boldsymbol{\eta}) - (p, \operatorname{rot}\boldsymbol{\eta})_{0,\Omega} = (\nabla\psi, \boldsymbol{\eta})_{0,\Omega} & \forall \boldsymbol{\eta} \in H_0^1(\Omega)^2, \\
-(\operatorname{rot}\boldsymbol{\beta}, q)_{0,\Omega} - \kappa^{-1}t^2(\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in H^1(\Omega) \cap L_0^2(\Omega), \\
(\nabla w, \nabla\xi)_{0,\Omega} = (\boldsymbol{\beta}, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^2(\nabla\psi, \nabla\xi)_{0,\Omega} & \forall \xi \in H_0^1(\Omega).
\end{cases}
\tag{5}
$$

It can be easily checked that there is a unique solution for both variational problems considered above. In what follows we will make the following regularity assumption. The load term $g \in L^2(\Omega)$, all components of the solution $(\psi, \boldsymbol{\beta}, p, w)$ of (5) are in $H^2(\Omega)$ and it holds

$$
\|\psi\|_{2,\Omega} + \|\boldsymbol{\beta}\|_{2,\Omega} + \|p\|_{1,\Omega} + t\|p\|_{2,\Omega} + \|w\|_{2,\Omega} \le C\|g\|_{0,\Omega}, \tag{6}
$$

with C independent of t.

The above assumption is reasonable. We recall for instance the following regularity result (see [1]):

Proposition 1 *Let Ω be a convex polygon or a smoothly bounded domain in the plane. Then, for any $t \in (0, \operatorname{diam}(\Omega)]$ and $g \in L^2(\Omega)$, the condition* (6) *is satisfied.*

3 A mimetic discretization

In this section we present a mimetic discretization method for the Reissner-Mindlin plate bending problem.

3.1 Mesh notation and assumptions

Let Ω_h be a partition of the computational domain Ω into $\mathcal{N}(\Omega_h)$ polygons E. We assume that this partition is conformal, i.e. intersection of two different elements E_1 and E_2 is either a few mesh points, or a few mesh edges (two adjacent elements may share more than one edge) or empty. We allow Ω_h to contain non-convex and degenerate elements. For each polygon E, $|E|$ denotes its area, h_E denotes its diameter and

$$h := \max_{E \in \Omega_h} h_E.$$

We denote the set of mesh vertices and edges by $\mathcal{V}_h$ and $\mathcal{E}_h$, the set of internal vertices and edges by $\mathcal{V}_h^0$ and $\mathcal{E}_h^0$, the set of vertices and edges of a particular element E by $\mathcal{V}_h^E$ and $\mathcal{E}_h^E$, and the set of boundary vertices and edges by $\mathcal{V}_h^\partial$ and $\mathcal{E}_h^\partial$, respectively. Moreover, we denote a generic mesh vertex by v, a generic edge by e and its length both by h_e and $|\mathsf{e}|$.

A fixed orientation is also set for the mesh Ω_h, which is reflected by a unit normal vector $\mathbf{n}_\mathsf{e}$, $\mathsf{e} \in \mathcal{E}_h$, fixed once for all. Moreover, $\mathbf{t}_\mathsf{e}$ denotes the tangent vector defined as the anticlockwise rotation of $\mathbf{n}_\mathsf{e}$ by 90°.

For every polygon E and edge $\mathsf{e} \in \mathcal{E}_h^E$, we define a unit normal vector $\mathbf{n}_E^\mathsf{e}$ that points outside of E, and by $\mathbf{t}_E^\mathsf{e}$ the tangent vector as the anticlockwise rotation of $\mathbf{n}_E^\mathsf{e}$ by 90°.

The mesh is assumed to satisfy the following shape regularity properties, which have already been used in [17].

There exist

- an integer number N_s, which is independent of h;
- a real positive number ρ independent of h;
- a *compatible* sub-decomposition $\mathcal{T}_h$ of every Ω_h into shape-regular triangles,

such that

(H1) any polygon $E \in \Omega_h$ admits a decomposition $\mathcal{T}_h|_E$ formed by less than N_s triangles;

(H2) any triangle $T \in \mathcal{T}_h$ is shape-regular in the sense that the ratio between the radius r of the inscribed ball and the diameter h_T is bounded from below by ρ:

$$0 < \rho \leq \frac{r}{h_T}.$$

From (H1), (H2) there can be easily derived several useful properties that we list below:

(M1) the number of vertices and edges of every polygon E of Ω_h are *uniformly* bounded from above by two integer numbers N_{v} and N_{e}, which only depend on N_s;

(M2) there exists a real positive number σ_s, which only depends on N_s and ρ, such that

$$h_{\mathsf{e}} \geq \sigma_s h_E \qquad \text{and} \qquad |E| \geq \sigma_s h_E^2,$$

for every polygon E of every decomposition Ω_h, for every edge e of E.

(M3) there exists a constant C_a, only dependent on ρ and N_s, such that for every polygon E, for every edge e of E and for every function $\psi \in H^1(E)$ there holds the *trace inequality*:

$$||\psi||_{0,\mathsf{e}}^2 \leq C_a \Big(h_E^{-1} ||\psi||_{0,E}^2 + h_E |\psi|_{1,E}^2 \Big).$$

(M4) there exists a constant C_{app}^*, which is independent of h, such that for every E and for every function $\psi \in H^1(E)$ there exists a constant $\psi_0 \in \mathbb{R}$ such that

$$||\psi - \psi_0||_{0,E} \leq C_{app}^* h_E |\psi|_{1,E}.$$

(M5) there exists a constant C_{app}, which is independent of h, such that for every E and for every function $\psi \in H^2(E)$ there holds the *interpolation inequality*

$$||\psi - \psi_1||_{0,E} + h_E |\psi - \psi_1|_{1,E} \leq C_{app} h_E^2 |\psi|_{2,E},$$

where ψ_1 is the $L^2(E)$-orthogonal projection of ψ over the space of linear polynomials defined on E.

Note that (M4) and (M5) follow, for instance, from the extended Bramble-Hilbert lemma of [30,16]. We make also the following assumptions on the material data $\mathbb{E}$, ν.

(H3) The scalar functions $\mathbb{E}, \nu$ are piecewise constant with respect to the mesh Ω_h. Moreover, there exist two positive constants $C_\star$ and $C^\star$ such that $C_\star < \mathbb{E} < C^\star$ on the whole domain.

The above uniformity condition on $\mathbb{E}$ is standard, while the piecewise constant condition can be interpreted as an approximation of the data and is introduced only for simplicity. In the general case, it is sufficient to assume that $\mathbb{E}$ and ν are (piecewise) $W^{1,\infty}$ and to introduce an element-wise averaging in the data of the numerical scheme.

3.2 Degrees of freedom and interpolation operators

The discretization of problem (3) requires to discretize the scalar field of displacement and the vector fields of rotations and shears. In order to do so, we introduce the degrees of freedom for the numerical solution in accordance with the correspondance

$$w, v \in H_0^1(\Omega) \to w_h, v_h \in W_h,$$

$$\boldsymbol{\beta}, \boldsymbol{\eta} \in H_0^1(\Omega)^2 \to \boldsymbol{\beta}_h, \boldsymbol{\eta}_h \in H_h,$$

$$\boldsymbol{\gamma}, \boldsymbol{\delta} \in L^2(\Omega)^2 \to \boldsymbol{\gamma}_h, \boldsymbol{\delta}_h \in \Gamma_h,$$

where W_h represents the linear space of discrete displacement, H_h indicates the linear space of discrete rotations and Γ_h is the linear space of discrete shears.

The discrete space for transverse displacements W_h is defined as follows: a vector $v_h \in W_h$ consists of a collection of degrees of freedom

$$v_h := \{v^{\mathsf{v}}\}_{\mathsf{v} \in \mathcal{V}_h^0},$$

one per internal mesh vertex, e.g. to every vertex $\mathsf{v} \in \mathcal{V}_h^0$, we associate a real number v^{v}. The scalar v^{v} represents the nodal value of the underlying discrete scalar field of displacement. The number of unknowns is equal to the number of internal vertices.

The discrete space for rotations H_h is defined as follows: a vector $\boldsymbol{\eta}_h \in H_h$ is a collection of degrees of freedom

$$\boldsymbol{\eta}_h = \{\boldsymbol{\eta}^{\mathsf{v}}\}_{\mathsf{v} \in \mathcal{V}_h^0} \cup \{\eta_E^{\mathsf{e}}\}_{E \in \Omega_h, \mathsf{e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0},$$

i.e. we assign a vector $\boldsymbol{\eta}^{\mathsf{v}} \in \mathbb{R}^2$ per each vertex $\mathsf{v} \in \mathcal{V}_h^0$, and, for every element E in Ω_h, one real number $\eta_E^{\mathsf{e}} \in \mathbb{R}$ per each edge $\mathsf{e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0$. We make the following continuity assumption: for each edge e shared by two element E_1 and E_2, we have

$$\eta_{E_1}^{\mathsf{e}} = -\eta_{E_2}^{\mathsf{e}},$$

so that, in practice, we have only one degree of freedom per edge. The vector $\boldsymbol{\eta}^{\mathsf{v}}$ represents the nodal values of the underlying discrete vector field of rotations, while the scalar η_E^{e} represents a bubble-type correction to the tangent value of the discrete rotations on edges. The number of unknowns is equal to twice the number of internal vertices plus the number of internal edges.

Finally, the space for the discrete shear force Γ_h is defined as follows: to every element E in Ω_h and every edge $\mathsf{e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0$, we associate a number δ_E^{e}, i.e.

$$\boldsymbol{\delta}_h = \{\delta_E^{\mathsf{e}}\}_{E \in \Omega_h, \mathsf{e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0}.$$

We make the continuity assumption that for each edge e shared by two element E_1 and E_2, we have

$$\delta_{E_1}^{\mathsf{e}} = -\delta_{E_2}^{\mathsf{e}}.$$

The scalar δ_E^{e} represents the average on edges of the discrete shears in the tangential direction. The number, of unknowns is equal to the number of internal edges.

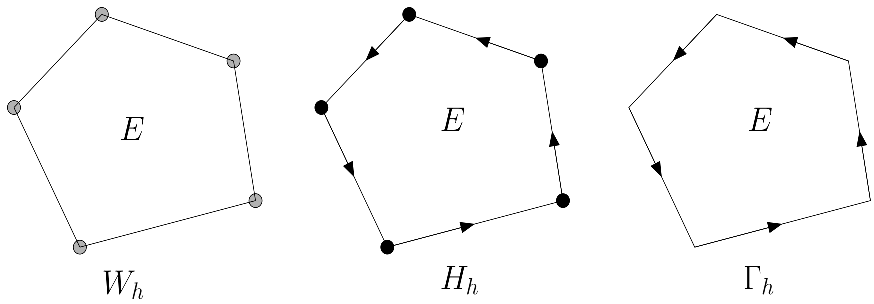

Fig. 1 Degrees of freedom for transverse displacements (left), rotations (center) and shear force (right).

We now define the following interpolation operators from the spaces of smooth enough functions to the discrete spaces W_h, H_h and Γ_h, respectively. For every function $v \in \mathcal{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)$, we define $v_{\rm I} \in W_h$ by

$$v_{\rm I}^{\sf v} := v({\sf v}) \quad \forall {\sf v} \in \mathcal{V}_h^0.$$

For every function $\boldsymbol{\eta} \in [\mathcal{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)]^2$, we define $\boldsymbol{\eta}_{\mathbf{I}} \in H_h$ by

$$\boldsymbol{\eta}_{\mathbf{I}}^{\sf v} := \boldsymbol{\eta}({\sf v}) \quad \forall {\sf v} \in \mathcal{V}_h^0,$$

$$(\eta_{\mathbf{I}})_E^{\sf e} := \frac{1}{|{\sf e}|} \int_{\sf e} \boldsymbol{\eta} \cdot \mathbf{t}_E^{\sf e} - \frac{1}{2} [\boldsymbol{\eta}_{\mathbf{I}}^{{\sf v}_1} + \boldsymbol{\eta}_{\mathbf{I}}^{{\sf v}_2}] \cdot \mathbf{t}_E^{\sf e} \quad \forall E \in \Omega_h \quad \forall {\sf e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0,$$

where ${\sf v}_1$ and ${\sf v}_2$ are the vertices of the edge ${\sf e}$.

For every function $\boldsymbol{\delta} \in H_0(\mathrm{rot}; \Omega) \cap [L^s(\Omega)]^2$, $s > 2$, we define $\boldsymbol{\delta}_{\rm II} \in \Gamma_h$ by

$$(\delta_{\rm II})_E^{\sf e} := \frac{1}{|{\sf e}|} \int_{\sf e} \boldsymbol{\delta} \cdot \mathbf{t}_E^{\sf e} \quad \forall E \in \Omega_h \quad \forall {\sf e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0.$$

For all $E \in \Omega_h$ in the sequel we will also make use of local interpolation operators $v_{{\rm I},E}, \boldsymbol{\eta}_{\mathbf{I},E}, \boldsymbol{\delta}_{{\rm II},E}$, with values in $W_h|_E, H_h|_E, \Gamma_h|_E$ respectively; such operators are simply the obvious restriction of the global ones to the element E for functions which are sufficiently regular on E.

Remark 1 Although all the discrete degrees of freedom live only on the internal vertices and edges, in the sequel we will often (implicitly) consider its extension to the boundary vertices and edges. In such case, the values associated to the degrees of freedom living on boundary vertices and edges must always be considered zero.

3.3 Discrete norms and operators

We endow the space W_h with the following norm

$$||v_h||_{W_h}^2 := \sum_{E \in \Omega_h} ||v_h||_{W_h,E}^2 = \sum_{E \in \Omega_h} |E| \sum_{{\sf e} \in \mathcal{E}_h^E} \left[\frac{1}{|{\sf e}|} (v^{{\sf v}_2} - v^{{\sf v}_1}) \right]^2, \qquad (7)$$

where v_1 and v_2 are the vertices of e, oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2.

In the space H_h, we consider the norm

$$|||\boldsymbol{\eta}_h|||_{H_h}^2 := \sum_{E\in\Omega_h} |||\boldsymbol{\eta}_h|||_{H_h,E}^2 = \sum_{E\in\Omega_h} |E| \sum_{\mathsf{e}\in\mathcal{E}_h^E} \left(\frac{1}{|\mathsf{e}|}\left(||\boldsymbol{\eta}^{\mathsf{v}_1} - \boldsymbol{\eta}^{\mathsf{v}_2}|| + |\eta_E^{\mathsf{e}}|\right)\right)^2, \tag{8}$$

where v_1 and v_2 are the vertices of the edge e, and $||\cdot||$ denotes the euclidean norm on vectors.

In the space Γ_h, we consider the following norm

$$||\boldsymbol{\delta}_h||_{\Gamma_h}^2 := \sum_{E\in\Omega_h} ||\boldsymbol{\delta}_h||_{\Gamma_h,E}^2 = \sum_{E\in\Omega_h} |E| \sum_{\mathsf{e}\in\mathcal{E}_h^E} |\delta_E^{\mathsf{e}}|^2. \tag{9}$$

The norms on W_h and H_h are $H^1(\Omega)$ type discrete semi-norms, which become norms due to the boundary conditions on the spaces. Indeed, the differences appearing in both norms represent gradients on edges and the scalings with respect to h_E are the correct ones to mimic an $H^1(E)$ local semi-norm. Note that for the edge degrees of freedom in H_h no difference is needed since such part represents a bubble correction. Finally, the norm for Γ_h is an $L^2(\Omega)$ type discrete norm.

In the sequel we will also use the following norm on H_h, which is a $||\boldsymbol{\varepsilon}(\cdot)||_{0,\Omega}$ type discrete norm:

$$||\boldsymbol{\eta}_h||_{H_h}^2 := \sum_{E\in\Omega_h} ||\boldsymbol{\eta}_h||_{H_h,E}^2 = \sum_{E\in\Omega_h} \min_{c\in\mathbb{R}} |||\boldsymbol{\eta}_h - c([-\bar{y},\bar{x}])_{\mathbf{I},E}|||_{H_h,E}^2, \tag{10}$$

where $(\bar{x},\bar{y})$ are local cartesian coordinates on E which are null on the barycenter of E, so that the function $[-\bar{y},\bar{x}]$ represents a (linearized) rotation around the barycenter. Moreover, we note that

$$||\boldsymbol{\eta}_h||_{H_h,E} \le |||\boldsymbol{\eta}_h|||_{H_h,E} \quad \forall\boldsymbol{\eta}_h \in H_h|_E. \tag{11}$$

We now introduce the operator ∇_h, defined from the set of nodal unknowns W_h to the set of edge unknowns Γ_h as follows:

$$\nabla_h : W_h \to \Gamma_h$$

$$(\nabla_h v_h)_E^{\mathsf{e}} := \frac{1}{|\mathsf{e}|}(v^{\mathsf{v}_2} - v^{\mathsf{v}_1}) \quad \forall E \in \Omega_h, \quad \forall \mathsf{e} \in \mathcal{E}_h^E \cap \mathcal{E}_h^0, \quad \forall v_h \in W_h,$$

where v_1 and v_2 are the vertices of e, oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2.

The operator ∇_h represents a discrete gradient on W_h. It is immediate to check that it holds

$$||v_h||_{W_h} = ||\nabla_h v_h||_{\Gamma_h}. \tag{12}$$

We consider also a reduction operator, defined from the discrete space of rotations H_h to the set of edge unknowns Γ_h as follows:

$$\Pi_h : H_h \to \Gamma_h$$

$$(\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} := \eta_E^{\mathsf{e}} + \frac{1}{2}[\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{t}_E^{\mathsf{e}} \quad \forall E \in \Omega_h, \quad \forall \mathsf{e} \in \mathcal{E}_h^E, \quad \forall \boldsymbol{\eta}_h \in H_h,$$

where v_1 and v_2 are the vertices of e, oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2.

3.4 Scalar products and bilinear forms

We equip the space Γ_h with a suitable scalar product, defined as follows:

$$[\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h]_{\Gamma_h} := \sum_{E \in \Omega_h} [\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h]_{\Gamma_h, E}, \tag{13}$$

where $[\cdot,\cdot]_{\Gamma_h,E}$ is a discrete scalar product on the element E.

Following [21], we introduce the following assumptions:

(S1) There exist two positive constants c_1 and c_2 independent of h such that, for every $\boldsymbol{\delta}_h \in \Gamma_h$ and each $E \in \Omega_h$, we have

$$c_1 ||\boldsymbol{\delta}_h||^2_{\Gamma_h,E} \le [\boldsymbol{\delta}_h, \boldsymbol{\delta}_h]_{\Gamma_h,E} \le c_2 ||\boldsymbol{\delta}_h||^2_{\Gamma_h,E}. \tag{14}$$

(S2) For every element E, every scalar linear function p_1 on E and every $\boldsymbol{\delta}_h \in \Gamma_h$, we have

$$[(\operatorname{curl} p_1)_{\mathrm{II}}, \boldsymbol{\delta}_h]_{\Gamma_h,E} = \int_E p_1 (\operatorname{rot}_{\Gamma_h} \boldsymbol{\delta}_h)_E - \sum_{\mathsf{e} \in \mathcal{E}_h^E} \delta_E^{\mathsf{e}} \int_{\mathsf{e}} p_1 \tag{15}$$

where the operator $(\operatorname{rot}_{\Gamma_h} \boldsymbol{\delta}_h)_E := \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} \delta_E^{\mathsf{e}} |\mathsf{e}|$ will be better detailed in Section 4.

The above scalar product mimics an L^2 type scalar product on the underlying space, i.e.

$$[\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h]_{\Gamma_h,E} \sim \int_E \widetilde{\boldsymbol{\gamma}}_h \cdot \widetilde{\boldsymbol{\delta}}_h,$$

where, roughly speaking, $\widetilde{\boldsymbol{\gamma}}_h, \widetilde{\boldsymbol{\delta}}_h$ denote regular functions living on E which "extend the data" $\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h$ inside the element. In this sense, property (S1) mimics the coercivity of the scalar product and the correct scaling with respect to the element size, while property (S2) is a consistency condition which asserts that the scalar product respects integration by parts when tested with the curl of linear functions.

We denote with $a_h(\cdot,\cdot) : H_h \times H_h \to \mathbb{R}$ the discretization of the bilinear form $a(\cdot,\cdot)$, defined as follows:

$$a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) = \sum_{E \in \Omega_h} a_h^E(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) \quad \forall \boldsymbol{\beta}_h, \boldsymbol{\eta}_h \in H_h, \tag{16}$$

where $a_h^E(\cdot,\cdot)$ is a symmetric bilinear form on each element E, mimicking

$$a_h^E(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) \sim \int_E \mathbb{C}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{\beta}}_h) : \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{\eta}}_h) \ .$$

Similarly to the previous case, we introduce two assumptions for the local bilinear form $a_h^E(\cdot,\cdot)$. The first one represents the coercivity (up to the kernel) and the correct scaling of the local forms.

($\mathsf{S1}_a$) there exist two positive constants $\tilde{c}_1$ and $\tilde{c}_2$ independent of h such that, for every $\boldsymbol{\eta}_h \in H_h$ and each $E \in \Omega_h$, we have

$$\tilde{c}_1 ||\boldsymbol{\eta}_h||^2_{H_h,E} \leq a_h^E(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) \leq \tilde{c}_2 ||\boldsymbol{\eta}_h||^2_{H_h,E}. \tag{17}$$

In order to introduce the second condition, we observe beforehand that, using an integration by parts,

$$\begin{aligned}\int_E \mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) &= \sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} (\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1})\mathbf{n}_E^{\mathsf{e}}) \cdot \boldsymbol{\eta} \\ &= \sum_{\mathsf{e}\in\mathcal{E}_h^E} \Big[\big(\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1})\mathbf{n}_E^{\mathsf{e}} \cdot \mathbf{n}_E^{\mathsf{e}}\big) \int_{\mathsf{e}} \boldsymbol{\eta} \cdot \mathbf{n}_E^{\mathsf{e}} + \big(\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1})\mathbf{n}_E^{\mathsf{e}} \cdot \mathbf{t}_E^{\mathsf{e}}\big) \int_{\mathsf{e}} \boldsymbol{\eta} \cdot \mathbf{t}_E^{\mathsf{e}} \Big]\end{aligned} \tag{18}$$

for all $E \in \Omega_h$, for all $\boldsymbol{\eta} \in [H^1(E)]^2$ and for all linear vector functions $\mathbf{p_1}$. Substituting the two integrals in the last line of (18) with an integration rule based on the available degrees of freedom gives our second condition

($\mathsf{S2}_a$) For every element E, every linear vector function $\mathbf{p_1}$ on E, and every $\boldsymbol{\eta}_h \in H_h$, it holds

$$\begin{aligned} a_h^E((\mathbf{p_1})_\mathbf{I}, \boldsymbol{\eta}_h) = \sum_{\mathsf{e}\in\mathcal{E}_h^E} \Big[& (\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1})\mathbf{n}_E^{\mathsf{e}} \cdot \mathbf{n}_E^{\mathsf{e}}) \left(\frac{|\mathsf{e}|}{2} [\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{n}_E^{\mathsf{e}} \right) \\ & + (\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{p_1})\mathbf{n}_E^{\mathsf{e}} \cdot \mathbf{t}_E^{\mathsf{e}}) \left(|\mathsf{e}|\eta_E^{\mathsf{e}} + \frac{|\mathsf{e}|}{2} [\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{t}_E^{\mathsf{e}} \right) \Big]. \end{aligned} \tag{19}$$

The meaning of the above consistency condition ($\mathsf{S2}_a$) is therefore that the discrete bilinear form respects integration by parts when tested with linear functions.

Remark 2 The scalar product and the bilinear form shown in this Section can be easily built element by element in a simple algebraic way. The details of such construction can be found in [23] for the scalar product (13) and in [8] for the bilinear form (16).

3.5 The discrete method

Finally, we are able to define the proposed mimetic discrete method for Reissner-Mindlin plates. Let the loading term

$$(g, v_h)_h := \sum_{E\in\Omega_h} \bar{g}|_E \sum_{i=1}^{k_E} v^{\mathsf{v}_i} \omega_E^i, \tag{20}$$

with $\mathsf{v}_1, \dots, \mathsf{v}_{k_E}$ are the vertices of E, $\bar{g}|_E := \frac{1}{|E|}\int_E g$, and $\omega_E^1, \dots, \omega_E^{k_E}$ are positive weights such that $\sum_{i=1}^{k_E} \omega_E^i = |E|$. The loading term above is an approximation of

$$(g, v_h)_h \sim \int_\Omega g\tilde{v},$$

which is exact for constant functions.

Then, the initial discretization of problem (3) reads:

Method 1 *Given* $g \in L^2(\Omega)$, *find* $(\boldsymbol{\beta}_h, w_h, \boldsymbol{\gamma}_h) \in H_h \times W_h \times \Gamma_h$ *such that*

$$\begin{cases} a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) + [\gamma_h, \nabla_h v_h - \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h} = (g, v_h)_h & \forall (\boldsymbol{\eta}_h, v_h) \in H_h \times W_h \\ [\nabla_h w_h - \Pi_h \boldsymbol{\beta}_h, \boldsymbol{\delta}_h]_{\Gamma_h} - \kappa^{-1} t^2 [\boldsymbol{\gamma}_h, \boldsymbol{\delta}_h]_{\Gamma_h} = 0 & \forall \boldsymbol{\delta}_h \in \Gamma_h, \end{cases}$$

It is immediate to check that the Method 1 is equivalent to the following one:

Method 2 *Given* $g \in L^2(\Omega)$, *find* $(\boldsymbol{\beta}_h, w_h) \in H_h \times W_h$ *such that*

$$a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) + \frac{\kappa}{t^2} [\nabla_h w_h - \Pi_h \boldsymbol{\beta}_h, \nabla_h v_h - \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h} = (g, v_h)_h$$

for all $(\boldsymbol{\eta}_h, v_h) \in H_h \times W_h$. The Method 2 is positive definite, see the observations below, and it involves less variables. Therefore, it is in general more suitable for practical implementation.

Due to assumptions $(\mathsf{S1})$ and $(\mathsf{S1}_a)$ the bilinear form appearing in Method 2 is clearly semi-positive definite on $W_h \times H_h$. Moreover, again due to $(\mathsf{S1})$, $(\mathsf{S1}_a)$ and the boundary conditions on W_h, H_h, it is easy to check that if

$$a_h(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) + \frac{\kappa}{t^2} [\nabla_h v_h - \Pi_h \boldsymbol{\eta}_h, \nabla_h v_h - \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h} = 0$$

then $\boldsymbol{\eta}_h$ and v_h are null. Therefore, Method 2 is positive definite and has a unique solution for all h and $t > 0$. For ease of exposition, the uniform stability of the Method with respect to h, t will be left as an implicit consequence of the error analysis that follows.

4 A Discrete Helmholtz decomposition

As in the continuous case, we will write an equivalent formulation of Method 1 based on a discrete Helmholtz decomposition. With this aim, we define an auxiliary discrete space Q_h defined as follows: every discrete scalar $q_h \in Q_h$ consists of one degree of freedom per each element E in Ω_h, e.g. to every element E, we associate a real number q_E,

$$q_h = \{q_E\}_{E \in \Omega_h},$$

satisfying the additional constraint that

$$\sum_{E \in \Omega_h} q_E |E| = 0. \tag{21}$$

The number of unknowns is equal to the number of elements minus one. For all $E \in \Omega_h$, q_E can be interpreted as the (constant) value on E of a global function $\widetilde{q}_h \in L^2_0(\Omega)$.

We define the following interpolation operator in Q_h: for every function $q \in L^2_0(\Omega)$, we define $q_\pi \in Q_h$ by

$$(q_\pi)_E := \frac{1}{|E|} \int_E q \quad \forall E \in \Omega_h.$$

It is immediate to check that q_π satisfies condition (21).

The space Q_h is endowed with the $L^2(\Omega)$ type scalar product

$$[p_h, q_h]_{Q_h} := \sum_{E \in \Omega_h} |E| \, p_E q_E \quad \forall p_h, q_h \in Q_h, \tag{22}$$

and with the norm

$$||q_h||^2_{Q_h} := [q_h, q_h]_{Q_h}.$$

We now observe that, for all $E \in \Omega_h$ and for all sufficiently regular functions $\boldsymbol{\delta}$, it holds

$$\frac{1}{|E|} \int_E \operatorname{rot} \boldsymbol{\delta} = \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} \int_{\mathsf{e}} \boldsymbol{\delta} \cdot \mathbf{t}_E^{\mathsf{e}}.$$

Consistently, we introduce the following operators which represent a discrete "rot" operator from Γ_h to Q_h and from H_h to Q_h, respectively

$$\operatorname{rot}_{\Gamma_h} : \Gamma_h \to Q_h$$

$$(\operatorname{rot}_{\Gamma_h} \boldsymbol{\delta}_h)_E := \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} \delta_E^{\mathsf{e}} |\mathsf{e}|, \tag{23}$$

and

$$\operatorname{rot}_{H_h} : H_h \to Q_h$$

$$(\operatorname{rot}_{H_h} \boldsymbol{\eta}_h)_E := \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} \left(\eta_E^{\mathsf{e}} + \frac{1}{2} [\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{t}_E^{\mathsf{e}} \right) |\mathsf{e}|, \tag{24}$$

where v_1 and v_2 are the vertices of e, oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2.

Using (23) and (24) it is easy to check the following commutative diagram properties hold

$$\operatorname{rot}_{\Gamma_h}(\boldsymbol{\delta}_{\mathrm{II}}) = (\operatorname{rot} \boldsymbol{\delta})_\pi, \tag{25}$$

$$\operatorname{rot}_{H_h}(\boldsymbol{\eta}_{\mathbf{I}}) = (\operatorname{rot} \boldsymbol{\eta})_\pi, \tag{26}$$

for all $\boldsymbol{\delta} \in H_0(\mathrm{rot};\Omega) \cap [L^s(\Omega)]^2$, $s > 2$ and $\boldsymbol{\eta} \in [\mathcal{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)]^2$. Moreover, we note that the operator $\operatorname{rot}_{\Gamma_h}$ satisfies $\operatorname{rot}_{\Gamma_h} \nabla_h v_h = 0$ for all $v_h \in W_h$. In fact,

$$(\operatorname{rot}_{\Gamma_h} \nabla_h v_h)_E = \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} (\nabla_h v_h)_E^{\mathsf{e}} |\mathsf{e}| = \frac{1}{|E|} \sum_{\mathsf{e} \in \mathcal{E}_h^E} (v^{\mathsf{v}_2} - v^{\mathsf{v}_1}) = 0, \tag{27}$$

since v_1 and v_2 are by definition the vertices of the edge e oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2. Furthermore, the following identity is easy to check

$$\operatorname{rot}_{H_h} \boldsymbol{\eta}_h = \operatorname{rot}_{\Gamma_h}(\Pi_h \boldsymbol{\eta}_h) \quad \forall \boldsymbol{\eta}_h \in H_h. \tag{28}$$

Using the definition above, we define a discretization of the "curl" operator as the adjoint to the discrete $\operatorname{rot}_{\Gamma_h}$ operator with respect to the scalar product (13) and (22), i.e.

$$\operatorname{curl}_h : Q_h \to \Gamma_h$$

$$[\boldsymbol{\delta}_h, \operatorname{curl}_h q_h]_{\Gamma_h} = [q_h, \operatorname{rot}_{\Gamma_h} \boldsymbol{\delta}_h]_{Q_h} \quad \forall q_h \in Q_h, \quad \forall \boldsymbol{\delta}_h \in \Gamma_h. \tag{29}$$

We have the following discrete Helmholtz decomposition.

Lemma 1 *For every $\boldsymbol{\delta}_h \in \Gamma_h$ there exists a unique $(\xi_h, q_h) \in W_h \times Q_h$ such that*

$$\boldsymbol{\delta}_h = \nabla_h \xi_h + \operatorname{curl}_h q_h. \tag{30}$$

Proof. Let $\boldsymbol{\delta}_h \in \Gamma_h$. In order to prove the lemma, we need to show the existence of $(\xi_h, q_h, \boldsymbol{\alpha}_h) \in W_h \times Q_h \times \Gamma_h$ such that

$$\begin{aligned} \boldsymbol{\delta}_h &= \nabla_h \xi_h + \boldsymbol{\alpha}_h, \\ [\boldsymbol{\alpha}_h, \mathbf{r}_h]_{\Gamma_h} &= [\operatorname{rot}_{\Gamma_h} \mathbf{r}_h, q_h]_{Q_h} \quad \forall \mathbf{r}_h \in \Gamma_h. \end{aligned} \tag{31}$$

Note that, applying the operator $\operatorname{rot}_{\Gamma_h}$ to both sides of $(31)_1$ and recalling (27), we get that the function $\boldsymbol{\alpha}_h$ must satisfy $\operatorname{rot}_{\Gamma_h}(\boldsymbol{\alpha}_h - \boldsymbol{\delta}_h) = 0$. Combined with $(31)_2$, this is equivalent to solve the following problem: Find $(\boldsymbol{\alpha}_h, q_h) \in \Gamma_h \times Q_h$ such that

$$\begin{aligned} [\boldsymbol{\alpha}_h, \mathbf{r}_h]_{\Gamma_h} - [q_h, \operatorname{rot}_{\Gamma_h} \mathbf{r}_h]_{Q_h} &= 0 & \forall \mathbf{r}_h \in \Gamma_h, \\ [\operatorname{rot}_{\Gamma_h} \boldsymbol{\alpha}_h, d_h]_{Q_h} &= [\operatorname{rot}_{\Gamma_h} \boldsymbol{\delta}_h, d_h]_{Q_h} & \forall d_h \in Q_h. \end{aligned} \tag{32}$$

This is a well posed problem as a consequence of the results in [21] for the diffusion problem in mixed form, simply changing $\mathcal{DIV}^d$ to $\operatorname{rot}_{\Gamma_h}$ and "rotating the fields 90°". Therefore, there exists a unique couple $(\boldsymbol{\alpha}_h, q_h) \in \Gamma_h \times Q_h$ which satisfies the two equations in (32).

As already mentioned, due to $(32)_1$ $\boldsymbol{\alpha}_h$ satisfies $(31)_2$, while, due to $(32)_2$, it holds $\operatorname{rot}_{\Gamma_h}(\boldsymbol{\alpha}_h - \boldsymbol{\delta}_h) = 0$. Therefore, what is left to prove is that for all $\mathbf{r}_h \in \Gamma_h$ with $\operatorname{rot}_{\Gamma_h} \mathbf{r}_h = 0$, it exists a unique $v_h \in W_h$ such that $\nabla_h v_h = \mathbf{r}_h$.

We will show this natural result rather briefly. Given any two nodes v_1 and v_2 of the mesh, we call $\gamma(\mathsf{v}_1, \mathsf{v}_2)$ a path from v_1 to v_2 made along (oriented) edges of the mesh, in such a way that each edge is never repeated. It is immediate to check that this can always be done, since all the vertices are connected along edges. Then, given $\mathbf{r}_h \in \Gamma_h$, we define $v_h \in W_h$ in the following way: We choose a node v_0 on the boundary and set $v^{\mathsf{v}_0} = 0$. For any other node v of the mesh, we define

$$v^{\mathsf{v}} = \sum_{\mathsf{e} \in \gamma(\mathsf{v}_0, \mathsf{v})} |\mathsf{e}| r_E^{\mathsf{e}} (\mathbf{t}_{\mathsf{e}}^{\gamma} \cdot \mathbf{t}_E^{\mathsf{e}}) \ , \tag{33}$$

where $\mathbf{t}_{\mathsf{e}}^{\gamma}$ is the tangent along each edge e oriented as the path. Note that, in (33), the element E that appears in r_E^{e} can be chosen as any one among the two elements that share the edge e (without changing the result).

In order to prove that the above construction is well defined, we must show that the value v^{v} does not depend on the particular path chosen. It is easy to check that this is equivalent to show that for any (oriented) circular path without repetition of edges $\gamma(\mathsf{v}, \mathsf{v})$, v vertex of Ω_h, it holds

$$\sum_{\mathsf{e} \in \gamma(\mathsf{v}, \mathsf{v})} |\mathsf{e}| r_E^{\mathsf{e}} (\mathbf{t}_{\mathsf{e}}^{\gamma} \cdot \mathbf{t}_E^{\mathsf{e}}) = 0 \ . \tag{34}$$

This can be proved by induction. Any circular path $\gamma(\mathsf{v}, \mathsf{v})$ as described above corresponds to the (oriented) boundary of a connected set of n elements, $n \in \mathbb{N}$. Accordingly, in the following we will write that a path is of class n

if it "surrounds" n elements. If the path is of class $n = 1$, then $\gamma(\mathsf{v},\mathsf{v})$ is the boundary of a single element E and, recalling that $\mathrm{rot}_{\Gamma_h}\,\mathbf{r}_h = 0$, we get

$$\sum_{\mathsf{e}\in\gamma(\mathsf{v},\mathsf{v})} |\mathsf{e}| r_E^{\mathsf{e}}(\mathbf{t}_{\mathsf{e}}^{\gamma}\cdot\mathbf{t}_E^{\mathsf{e}}) = \pm \sum_{\mathsf{e}\in\mathcal{E}_h^E} |\mathsf{e}| r_E^{\mathsf{e}} = 0\ . \tag{35}$$

Now, Assuming that equation (34) is true for any path of class n. Then, it is easy to show that any circular path $\gamma(\mathsf{v},\mathsf{v})$ of class $n+1$ without repetition of edges can be splitted as the sum of two circular paths without repetitions, respectively of class n and class 1. Therefore, the result follows by the induction hypothesis and (35). Therefore, v_h in (33) is well defined.

From definition (33) we get immediately

$$\frac{1}{|\mathsf{e}|}[v^{\mathsf{v}_2} - v^{\mathsf{v}_1}] = r_E^{\mathsf{e}} \quad \forall E\in\Omega_h,\ \forall\mathsf{e}\in\mathcal{E}_h^E, \tag{36}$$

where $\mathsf{v}_1 = \mathsf{v}_1(\mathsf{e})$ and $\mathsf{v}_2 = \mathsf{v}_2(\mathsf{e})$ have the usual meaning, simply by evaluating the left hand side as the difference along two ad-hoc chosen paths which differ only by the edge e. By definition, identity (36) implies $\nabla_h v_h = \mathbf{r}_h$. Moreover, by selecting a path along the boundary and recalling that the values of $\mathbf{r}_h$ on boundary edges are null, it correctly follows that v_h is null on all boundary nodes. Finally, the uniqueness of v_h follows immediately from the fact that the kernel of ∇_h on W_h reduces to the trivial one. □

By using the previous lemma, we can write

$$\boldsymbol{\gamma}_h = \nabla_h\psi_h + \mathrm{curl}_h\, p_h, \tag{37}$$

with $\psi_h\in W_h$ and $p_h\in Q_h$. By using the same decomposition for the test function

$$\boldsymbol{\delta}_h = \nabla_h\xi_h + \mathrm{curl}_h\, q_h,$$

we obtain that Method 1 is equivalent to the following problem:

Find $(\psi_h,\boldsymbol{\beta}_h,p_h,w_h)\in W_h\times H_h\times Q_h\times W_h$ such that

$$\begin{cases} [\nabla_h\psi_h,\nabla_h v_h]_{\Gamma_h} = (g,v_h)_h & \forall v_h\in W_h \\ a_h(\boldsymbol{\beta}_h,\boldsymbol{\eta}_h) - [\mathrm{curl}_h\,p_h,\Pi_h\boldsymbol{\eta}_h]_{\Gamma_h} = [\nabla_h\psi_h,\Pi_h\boldsymbol{\eta}_h]_{\Gamma_h} & \forall\boldsymbol{\eta}_h\in H_h, \\ -[\Pi_h\boldsymbol{\beta}_h,\mathrm{curl}_h\,q_h]_{\Gamma_h} - \kappa^{-1}t^2[\mathrm{curl}_h\,p_h,\mathrm{curl}_h\,q_h]_{\Gamma_h} = 0 & \forall q_h\in Q_h, \\ [\nabla_h w_h,\nabla_h\xi_h]_{\Gamma_h} = [\Pi_h\boldsymbol{\beta}_h,\nabla_h\xi_h]_{\Gamma_h} + \kappa^{-1}t^2[\nabla_h\psi_h,\nabla_h\xi_h]_{\Gamma_h} & \forall\xi_h\in W_h. \end{cases} \tag{38}$$

Using (29) and (28) we get that for all $q_h\in Q_h$ and $\boldsymbol{\eta}_h\in H_h$,

$$[\mathrm{curl}_h\,q_h,\Pi_h\boldsymbol{\eta}_h]_{\Gamma_h} = [q_h,\mathrm{rot}_{\Gamma_h}(\Pi_h\boldsymbol{\eta}_h)]_{Q_h} = [q_h,\mathrm{rot}_{H_h}\,\boldsymbol{\eta}_h]_{Q_h}. \tag{39}$$

Therefore, problem (38) finally becomes: find $(\psi_h,\boldsymbol{\beta}_h,p_h,w_h)\in W_h\times H_h\times Q_h\times W_h$ such that

$$\begin{cases} [\nabla_h\psi_h,\nabla_h v_h]_{\Gamma_h} = (g,v_h)_h & \forall v_h\in W_h, \\ a_h(\boldsymbol{\beta}_h,\boldsymbol{\eta}_h) - [p_h,\mathrm{rot}_{H_h}\,\boldsymbol{\eta}_h]_{Q_h} = [\nabla_h\psi_h,\Pi_h\boldsymbol{\eta}_h]_{\Gamma_h} & \forall\boldsymbol{\eta}_h\in H_h, \\ -[\mathrm{rot}_{H_h}\,\boldsymbol{\beta}_h,q_h]_{Q_h} - \kappa^{-1}t^2[\mathrm{curl}_h\,p_h,\mathrm{curl}_h\,q_h]_{\Gamma_h} = 0 & \forall q_h\in Q_h, \\ [\nabla_h w_h,\nabla_h\xi_h]_{\Gamma_h} = [\Pi_h\boldsymbol{\beta}_h,\nabla_h\xi_h]_{\Gamma_h} + \kappa^{-1}t^2[\nabla_h\psi_h,\nabla_h\xi_h]_{\Gamma_h} & \forall\xi_h\in W_h. \end{cases} \tag{40}$$

Problem (40), which is the combination of two Poisson-like problems (first and last lines) and a rotated Stokes-like problem (second plus third lines), is going to be used in the error analysis. Due to Lemma 1 the existence of a unique solution for problem (40) follows easily from that of Method 1.

5 Error estimates

In this section we estimate the error between the continuous problem (5) and the discrete problem (40). The main result of this section is the following bound.

Theorem 1 *Let $(\psi, \boldsymbol{\beta}, p, w)$ and $(\psi_h, \boldsymbol{\beta}_h, p_h, w_h)$ be the solutions of problems* (5) *and* (40)*, respectively. Let the regularity bound* (6) *holds. Then, there exists a constant C independent of h and t such that*

$$\|\psi_{\mathrm{I}} - \psi_h\|_{W_h} + \|\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h\|_{H_h} + \|p_\pi - p_h\|_{Q_h} \\ + t\|\operatorname{curl}_h p_\pi - \operatorname{curl}_h p_h\|_{\Gamma_h} + \|w_{\mathrm{I}} - w_h\|_{W_h} \le Ch\|g\|_{0,\Omega}.$$

The proof of the above result will follow by combining the three propositions 2, 3 and 4 shown in the following.

5.1 Error estimate for variable ψ.

From now on, given an element E we use the subscript $|_E$ to denote the restrictions of the involved unknowns to E. For instance $W_h|_E$ will denote the restriction of W_h to the nodes belonging to E.

Let ψ_h be the solution of the discrete problem $(40)_1$, ψ be the solution of the continuous problem $(5)_1$ and ψ_{I} its interpolant in W_h. Let ψ^ℓ be a piecewise linear discontinuous function on Ω which is an approximation of ψ. The restriction of ψ^ℓ to E, $\forall E \in \Omega_h$, is denoted by ψ^ℓ_E and is defined as the $L^2(E)$-projection of ψ onto the polynomials of degree ≤ 1. We will also consider the local interpolant $(\psi^\ell_E)_{\mathrm{I}} \in W_h|_E$ and a piecewise linear discontinuous function f^ℓ such that

$$\operatorname{curl} f^\ell_E = \nabla \psi^\ell_E \quad \forall E \in \Omega_h. \tag{41}$$

In the following we will need two lemmas which has been proved in [17]. The first one is a technical bound.

Lemma 2 *Let $\omega^1_E, \dots, \omega^{k_E}_E$ be positive weights such that $\sum_{i=1}^{k_E} \omega^i_E = |E|$, for all $E \in \Omega_h$ with k_E vertices. For every vertex $\mathsf{v}_1 \in \mathcal{V}^E_h$, and for every $v_h \in W_h|_E$ there exists a constant C independent of h, such that*

$$\sum_{i=1}^{k_E} [v^{\mathsf{v}_1} - v^{\mathsf{v}_i}]^2 \omega^i_E \le Ch_E^2 \|v_h\|^2_{W_h,E}.$$

The second lemma shows the existence of a stable lifting operator.

Lemma 3 *For all $E \in \Omega_h$, it exists a linear operator R_h^E, from the space of nodal unknown $W_h|_E$ into the Sobolev space $H^1(E) \cap \mathcal{C}^0(\bar{E})$, with the following properties:*

(P1) $(R_h^E v_h)(\mathsf{v}) = v^{\mathsf{v}} \quad \forall \mathsf{v} \in \mathcal{V}_h^E \quad \forall v_h \in W_h|_E$,

(P2) $R_h^E v_h|_{\mathsf{e}}$ *is a linear function* $\forall \mathsf{e} \in \mathcal{E}_h^E \quad \forall v_h \in W_h|_E$,

(P3) $|R_h^E v_h|_{1,E}^2 \leq C \|v_h\|_{W_h,E}^2 \quad \forall v_h \in W_h|_E$,

(P4) $\|R_h^E v_h - v^{\mathsf{v}}\|_{0,E}^2 \leq C h_E^2 \|v_h\|_{W_h,E}^2 \quad \forall \mathsf{v} \in \mathcal{V}_h^E \quad \forall v_h \in W_h|_E$.

We have the following result:

Proposition 2 *Let ψ and ψ_h be the solutions of problems* $(5)_1$ *and* $(40)_1$*, respectively. Let assumption* (6) *holds. Then, there exists a constant $C > 0$ independent of h and t such that*

$$\|\psi_{\mathrm{I}} - \psi_h\|_{W_h} \leq C h \|g\|_{0,\Omega}.$$

Proof. Using (12), property (S1), the first equation of problem (40) and adding and subtracting $(\psi_E^\ell)_{\mathrm{I}}$, we get

$$\begin{aligned}
c_1||\psi_{\mathrm{I}} - \psi_h||_{W_h}^2 &= c_1||\nabla_h(\psi_{\mathrm{I}} - \psi_h)||_{\Gamma_h}^2 = c_1 \sum_{E \in \Omega_h} ||\nabla_h(\psi_{\mathrm{I}} - \psi_h)||_{\Gamma_h,E}^2 \\
&\leq \sum_{E \in \Omega_h} [\nabla_h(\psi_{\mathrm{I}} - \psi_h), \nabla_h(\psi_{\mathrm{I}} - \psi_h)]_{\Gamma_h,E} \\
&= \sum_{E \in \Omega_h} [\nabla_h \psi_{\mathrm{I}} - \nabla_h(\psi_E^\ell)_{\mathrm{I}}, \nabla_h(\psi_{\mathrm{I}} - \psi_h)]_{\Gamma_h,E} - (g, \psi_{\mathrm{I}} - \psi_h)_h \\
&+ \sum_{E \in \Omega_h} [\nabla_h(\psi_E^\ell)_{\mathrm{I}}, \nabla_h(\psi_{\mathrm{I}} - \psi_h)]_{\Gamma_h,E}.
\end{aligned} \tag{42}$$

We continue with the last term in the above estimate. First, from the definitions of our interpolants, we have

$$\nabla_h(\psi_E^\ell)_{\mathrm{I}} = (\nabla \psi_E^\ell)_{\mathrm{II}} \quad \text{in} \quad \Gamma_h|_E, \tag{43}$$

thus, using (43), (41), property (S2) and the fact that $\mathrm{rot}_{\Gamma_h} \nabla_h v_h = 0$, we obtain

$$\begin{aligned}
\sum_{E \in \Omega_h} [\nabla_h(\psi_E^\ell)_{\mathrm{I}}, \nabla_h(\psi_{\mathrm{I}} - \psi_h)]_{\Gamma_h,E} &= \sum_{E \in \Omega_h} [(\mathrm{curl}\, f_E^\ell)_{\mathrm{II}}, \nabla_h(\psi_{\mathrm{I}} - \psi_h)]_{\Gamma_h,E} \\
&= - \sum_{E \in \Omega_h} \Big(\sum_{\mathsf{e} \in \mathcal{E}_h^E} (\nabla_h(\psi_{\mathrm{I}} - \psi_h))_E^{\mathsf{e}} \int_{\mathsf{e}} f_E^\ell \Big).
\end{aligned} \tag{44}$$

Let the global operator $R_h : W_h \to H_0^1(\Omega)$ be defined by $(R_h v_h)|_E = R_h^E(v_h|_E)$ for all $v_h \in W_h$ and for all $E \in \Omega_h$. Then, for each $v_h \in W_h|_E$ and each $E \in \Omega_h$, due to (P1) and (P2)

$$\begin{aligned}
(\nabla_h v_h)_E^{\mathsf{e}} &= \frac{1}{|\mathsf{e}|}(v^{\mathsf{v}_2} - v^{\mathsf{v}_1}) = \frac{1}{|\mathsf{e}|} \left(R_h^E v_h(\mathsf{v}_2) - R_h^E v_h(\mathsf{v}_1) \right) \\
&= \frac{1}{|\mathsf{e}|} \int_{\mathsf{e}} \nabla R_h^E v_h \cdot \mathbf{t}_E^{\mathsf{e}} = \nabla R_h^E v_h \cdot \mathbf{t}_E^{\mathsf{e}} \quad \forall \mathsf{e} \in \mathcal{E}_h^E,
\end{aligned}$$

where v_1 and v_2 are the vertices of e, oriented such that $\mathbf{t}_E^{\mathsf{e}}$ points from v_1 to v_2. Thus, it follows

$$\sum_{E\in\Omega_h}\Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E}(\nabla_h(\psi_{\mathrm{I}}-\psi_h))_E^{\mathsf{e}}\int_{\mathsf{e}}f_E^{\ell}\Big)=\sum_{E\in\Omega_h}\Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E}\int_{\mathsf{e}}f_E^{\ell}(\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\cdot\mathbf{t}_E^{\mathsf{e}})\Big). \tag{45}$$

Using an integration by parts on each element E for the last term of (45), applying again (41) and adding and subtracting the exact solution ψ, from (44) we get that

$$\begin{aligned}\sum_{E\in\Omega_h}[\nabla_h(\psi_E^{\ell})_{\mathrm{I}},\nabla_h(\psi_{\mathrm{I}}-\psi_h)]_{\Gamma_h,E}&=\sum_{E\in\Omega_h}\Big(\int_E\operatorname{curl}f_E^{\ell}\cdot\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\\&\quad-\int_E f_E^{\ell}\operatorname{rot}\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\Big)=\sum_{E\in\Omega_h}\int_E\nabla\psi_E^{\ell}\cdot\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\\&=\sum_{E\in\Omega_h}\int_E\nabla(\psi_E^{\ell}-\psi)\cdot\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)+\int_\Omega\nabla\psi\cdot\nabla R_h(\psi_{\mathrm{I}}-\psi_h).\end{aligned} \tag{46}$$

Therefore, using the first equation of problem (5), we obtain from (42) and (46)

$$\begin{aligned}c_1||\psi_{\mathrm{I}}-\psi_h||_{W_h}^2&\le\sum_{E\in\Omega_h}[\nabla_h\psi_{\mathrm{I}}-\nabla_h(\psi_E^{\ell})_{\mathrm{I}},\nabla_h(\psi_{\mathrm{I}}-\psi_h)]_{\Gamma_h,E}\\&\quad+\sum_{E\in\Omega_h}\int_E\nabla(\psi_E^{\ell}-\psi)\cdot\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\\&\quad+\big[(g,R_h(\psi_{\mathrm{I}}-\psi_h))_{0,\Omega}-(g,\psi_{\mathrm{I}}-\psi_h)_h\big]=T_1+T_2+T_3.\end{aligned} \tag{47}$$

For the first term in the above bound, by a Cauchy-Schwarz inequality and ($\mathsf{S1}$) give for all $E\in\Omega_h$

$$\begin{aligned}&[\nabla_h\psi_{\mathrm{I}}-\nabla_h(\psi_E^{\ell})_{\mathrm{I}},\nabla_h(\psi_{\mathrm{I}}-\psi_h)]_{\Gamma_h,E}\\&\qquad\le C||\nabla_h\psi_{\mathrm{I}}-\nabla_h(\psi_E^{\ell})_{\mathrm{I}}||_{\Gamma_h,E}||\nabla_h(\psi_{\mathrm{I}}-\psi_h)||_{\Gamma_h,E},\end{aligned}$$

which, using an approximation result (Lemma 6.3 from [17]), yields

$$[\nabla_h\psi_{\mathrm{I}}-\nabla_h(\psi_E^{\ell})_{\mathrm{I}},\nabla_h(\psi_{\mathrm{I}}-\psi_h)]_{\Gamma_h,E}\le\big(Ch_E^2|\psi|_{2,E}^2\big)^{1/2}||\nabla_h(\psi_{\mathrm{I}}-\psi_h)||_{\Gamma_h,E}. \tag{48}$$

Summing on the elements, from bound (48) it follows

$$T_1\le Ch|\psi|_{2,\Omega}||\nabla_h(\psi_{\mathrm{I}}-\psi_h)||_{\Gamma_h}\le Ch\|g\|_{0,\Omega}||\psi_{\mathrm{I}}-\psi_h||_{W_h}, \tag{49}$$

where in the last inequality, we have used (12) and (6).

For the second term in (47), by a Cauchy-Schwarz inequality and using ($\mathsf{M5}$) and ($\mathsf{P3}$), we get

$$\int_E\nabla(\psi_E^{\ell}-\psi)\cdot\nabla R_h^E(\psi_{\mathrm{I}}-\psi_h)\le Ch_E|\psi|_{2,E}||\psi_{\mathrm{I}}-\psi_h||_{W_h,E}.$$

Summing on the elements and using again (6), the above bound yields

$$T_2 \le Ch|\psi|_{2,\Omega}||\psi_{\mathrm{I}} - \psi_h||_{W_h} \le Ch\|g\|_{0,\Omega}||\psi_{\mathrm{I}} - \psi_h||_{W_h}. \tag{50}$$

Now, we bound T_3. It is easy to see that for each vertex $\mathsf{v} \in \mathcal{V}_h^E$, $E \in \Omega_h$, we have

$$\begin{aligned}\bar{g}|_E \sum_{i=1}^{k_E} (\psi_{\mathrm{I}} - \psi_h)^{\mathsf{v}} \omega_E^i &= \bar{g}|_E \sum_{i=1}^{k_E} (\psi_{\mathrm{I}}^{\mathsf{v}} - \psi^{\mathsf{v}}) \omega_E^i = \bar{g}|_E \int_E (\psi_{\mathrm{I}}^{\mathsf{v}} - \psi^{\mathsf{v}}) \\ &= \int_E g(\psi_{\mathrm{I}}^{\mathsf{v}} - \psi^{\mathsf{v}}).\end{aligned} \tag{51}$$

Thus, using the definition of the loading term $(\cdot,\cdot)_h$ in (20), adding and subtracting the term $\int_E g(\psi_{\mathrm{I}}^{\mathsf{v}_1} - \psi^{\mathsf{v}_1})$, where v_1 is any fixed vertex of E, for all $E \in \Omega_h$, from (51) we obtain

$$\begin{aligned} T_3 &= \int_\Omega g R_h(\psi_{\mathrm{I}} - \psi_h) - \sum_{E\in\Omega_h} \bar{g}|_E \sum_{i=1}^{k_E} (\psi_{\mathrm{I}}^{\mathsf{v}_i} - \psi^{\mathsf{v}_i}) \omega_E^i \\ &= \sum_{E\in\Omega_h} \int_E g \left(R_h^E(\psi_{\mathrm{I}} - \psi_h) - (\psi_{\mathrm{I}}^{\mathsf{v}_1} - \psi^{\mathsf{v}_1}) \right) \\ &\quad + \sum_{E\in\Omega_h} \bar{g}|_E \sum_{i=1}^{k_E} \left((\psi_{\mathrm{I}}^{\mathsf{v}_1} - \psi^{\mathsf{v}_1}) - (\psi_{\mathrm{I}}^{\mathsf{v}_i} - \psi^{\mathsf{v}_i}) \right) \omega_E^i. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} T_3 \le& \sum_{E\in\Omega_h} \|g\|_{0,E} \|R_h^E(\psi_{\mathrm{I}} - \psi_h) - (\psi_{\mathrm{I}}^{\mathsf{v}_1} - \psi_h^{\mathsf{v}_1})\|_{0,E} \\ &+ \sum_{E\in\Omega_h} \left(\sum_{i=1}^{k_E} \bar{g}|_E^2 \omega_E^i \right)^{1/2} \left(\sum_{i=1}^{k_E} [(\psi_{\mathrm{I}}^{\mathsf{v}_1} - \psi^{\mathsf{v}_1}) - (\psi_{\mathrm{I}}^{\mathsf{v}_i} - \psi^{\mathsf{v}_i})]^2 \omega_E^i \right)^{1/2}. \end{aligned}$$

Finally, from (P4), Lemma 2 and the fact that $|E|^2|\bar{g}|_E| \le \|g\|_{0,E}^2$, we obtain

$$T_3 \le Ch\|g\|_{0,\Omega}\|\psi_{\mathrm{I}} - \psi_h\|_{W_h}. \tag{52}$$

The result follow combining (47) with the above bounds for T_1, T_2, T_3.

□

5.2 Error estimate for variables $\boldsymbol{\beta}$ and p.

Now, let $(\boldsymbol{\beta}_h, p_h)$ be the solution of the discrete problem given by $(40)_{2-3}$, and $(\boldsymbol{\beta}, p)$ be the solution of the continuous problem given by $(5)_{2-3}$.

The following inf-sup condition holds

Lemma 4 *There exists $C > 0$ independent of h such that for every $q_h \in Q_h$ there exists $\boldsymbol{\eta}_h \in H_h$ satisfying:*

$$[\operatorname{rot}_{H_h} \boldsymbol{\eta}_h, q_h]_{Q_h} \geq C\|q_h\|_{Q_h},$$
$$||\boldsymbol{\eta}_h||_{H_h} \leq 1.$$

Proof. Changing $\mathcal{DIV}_h$ to $\operatorname{rot}_{\Gamma_h}$ and rotating the fields $90°$ in Lemma 4.2 of [11] prove the result. □

We introduce the following discrete bilinear form

$$A_h^t((\boldsymbol{\beta}_h, p_h), (\boldsymbol{\eta}_h, q_h)) := a_h(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) - [p_h, \operatorname{rot}_{H_h} \boldsymbol{\eta}_h]_{Q_h} - [\operatorname{rot}_{H_h} \boldsymbol{\beta}_h, q_h]_{Q_h} - \kappa^{-1}t^2[\operatorname{curl}_h p_h, \operatorname{curl}_h q_h]_{\Gamma_h}. \tag{53}$$

As a consequence of Lemma 4 and property $(\mathsf{S1}_a)$, following standard techniques of mixed finite element methods [19], it is easy to show the following stability estimate for the discrete Stokes-like problem given by $(40)_{2-3}$.

Lemma 5 *There exists $C > 0$ independent of h and t such that*

$$\sup_{\substack{\boldsymbol{\eta}_h \in H_h \\ q_h \in Q_h}} \frac{A_h^t((\boldsymbol{\beta}_h, p_h), (\boldsymbol{\eta}_h, q_h))}{\|\boldsymbol{\eta}_h\|_{H_h} + \|q_h\|_{Q_h} + t\|\operatorname{curl}_h q_h\|_{\Gamma_h}} \geq C(\|\boldsymbol{\beta}_h\|_{H_h} + \|p_h\|_{Q_h} + t\|\operatorname{curl}_h p_h\|_{\Gamma_h})$$

for all $(\boldsymbol{\beta}_h, p_h) \in H_h \times Q_h$, and where the sup *is taken on non-null couples of functions.*

The following lemma states the existence of a stable lifting operator also for the rotation variable.

Lemma 6 *For all $E \in \Omega_h$, it exists a linear operator $\mathbf{R}_h^E$ from the space $H_h|_E$ into the Sobolev space $[H^1(E) \cap \mathcal{C}^0(\bar{E})]^2$ with the following properties:*

(O1) $(\mathbf{R}_h^E \boldsymbol{\eta}_h)(\mathsf{v}) = \boldsymbol{\eta}^{\mathsf{v}} \quad \forall \mathsf{v} \in \mathcal{V}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E,$

(O2) $\|\varepsilon(\mathbf{R}_h^E \boldsymbol{\eta}_h)\|_{0,E}^2 \leq C\|\boldsymbol{\eta}_h\|_{H_h,E}^2 \quad \forall \boldsymbol{\eta}_h \in H_h|_E,$

(O3) $(\mathbf{R}_h^E \boldsymbol{\eta}_h|_{\mathsf{e}}) \cdot \mathbf{n}_E^{\mathsf{e}}$ *is a linear function* $\forall \mathsf{e} \in \mathcal{E}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E,$
$(\mathbf{R}_h^E \boldsymbol{\eta}_h|_{\mathsf{e}}) \cdot \mathbf{t}_E^{\mathsf{e}}$ *is a quadratic function* $\forall \mathsf{e} \in \mathcal{E}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E,$

(O4) $\displaystyle\int_{\mathsf{e}} (\mathbf{R}_h^E \boldsymbol{\eta}_h) \cdot \mathbf{t}_E^{\mathsf{e}} = |\mathsf{e}|\eta_E^{\mathsf{e}} + \frac{|\mathsf{e}|}{2}[\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{t}_E^{\mathsf{e}} \quad \forall \mathsf{e} \in \mathcal{E}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E,$

where as usual v_1 and v_2 are the vertices of the edge e.

The proof of the above lemma can be found in the Appendix. The lifting operator $\mathbf{R}_h^E$ is an extension of those in [11,17], with the additional important property of preserving linear functions.

Note that as a consequence of $(\mathsf{O3})$ it holds

$$\int_{\mathsf{e}} (\mathbf{R}_h^E \boldsymbol{\eta}_h) \cdot \mathbf{n}_E^{\mathsf{e}} = \frac{|\mathsf{e}|}{2}[\boldsymbol{\eta}^{\mathsf{v}_1} + \boldsymbol{\eta}^{\mathsf{v}_2}] \cdot \mathbf{n}_E^{\mathsf{e}} \quad \forall \mathsf{e} \in \mathcal{E}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h, \tag{54}$$

while, due to of (O4), we have

$$\int_E \operatorname{rot}(\mathbf{R}_h^E \boldsymbol{\eta}_h) = |E|(\operatorname{rot}_{H_h} \boldsymbol{\eta}_h)_E \quad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h. \tag{55}$$

Finally, we define the global operator $\mathbf{R}_h : H_h \to [H_0^1(\Omega)]^2$ by $(\mathbf{R}_h \boldsymbol{\eta}_h)|_E = \mathbf{R}_h^E(\boldsymbol{\eta}_h|_E)$ for all $\boldsymbol{\eta}_h \in H_h$ and for all $E \in \Omega_h$. The image of $\mathbf{R}_h$ is indeed in $[H_0^1(\Omega)]^2$ due to property (O3).

Now, we are able to state and prove our second convergence result.

Proposition 3 *Let $(\boldsymbol{\beta}, p)$ and $(\boldsymbol{\beta}_h, p_h)$ be the solutions of problems* $(5)_{2-3}$ *and* $(40)_{2-3}$*, respectively. Let the bound* (6) *holds. Then,*

$$\|\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h\|_{H_h} + \|p_\pi - p_h\|_{Q_h} + t\| \operatorname{curl}_h p_\pi - \operatorname{curl}_h p_h\|_{\Gamma_h} \le Ch\|g\|_{0,\Omega},$$

where C is independent of h and t.

Proof. We divide this rather long proof into two parts. In step 1 we bound the error as a sum of various terms, which will be bounded separately in step 2.

Step 1. From Lemma 5, we have that there exists $(\boldsymbol{\eta}_h, q_h) \in H_h \times Q_h$ such that

$$\|\boldsymbol{\eta}_h\|_{H_h} + \|q_h\|_{Q_h} + t\| \operatorname{curl}_h q_h\|_{\Gamma_h} \le 1 \tag{56}$$

and

$$\begin{aligned} C(\|\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h\|_{H_h} + \|p_\pi - p_h\|_{Q_h} &+ t\| \operatorname{curl}_h p_\pi - \operatorname{curl}_h p_h\|_{\Gamma_h}) \\ &\le A_h^t((\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h, p_\pi - p_h), (\boldsymbol{\eta}_h, q_h)). \end{aligned} \tag{57}$$

Now, we can rewrite the right hand side of (57), using $(40)_{2-3}$ as follows:

$$A_h^t((\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h, p_\pi - p_h), (\boldsymbol{\eta}_h, q_h)) = A_h^t((\boldsymbol{\beta}_{\mathbf{I}}, p_\pi), (\boldsymbol{\eta}_h, q_h)) - [\nabla_h \psi_h, \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h}.$$

Therefore, from (53), we have

$$\begin{aligned} A_h^t((\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h, p_\pi - p_h), (\boldsymbol{\eta}_h, q_h)) =& a_h(\boldsymbol{\beta}_{\mathbf{I}}, \boldsymbol{\eta}_h) - [p_\pi, \operatorname{rot}_{H_h} \boldsymbol{\eta}_h]_{Q_h} \\ & - [\operatorname{rot}_{H_h} \boldsymbol{\beta}_{\mathbf{I}}, q_h]_{Q_h} \\ & - \kappa^{-1} t^2 [\operatorname{curl}_h p_\pi, \operatorname{curl}_h q_h]_{\Gamma_h} \\ & - [\nabla_h \psi_h, \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h} \\ =& A_1 - A_2 - A_3 - A_4 - A_5. \end{aligned}$$

We also consider a piecewise linear discontinuous function $\boldsymbol{\beta}^\ell$ which is an approximation of $\boldsymbol{\beta}$ on each element E. The restriction of $\boldsymbol{\beta}^\ell$ to E, $E \in \Omega_h$, is denoted by $\boldsymbol{\beta}_E^\ell$ and is defined as the $L^2(E)$-projection of $\boldsymbol{\beta}$ onto the space of linear vector valued functions defined on E. We also consider the local interpolant $(\boldsymbol{\beta}_E^\ell)_{\mathbf{I}} \in H_h|_E$.

In that follows, we will manipulate the terms A_i, $i=1,\dots,5$. We begin with term A_1: adding and subtracting $(\boldsymbol{\beta}_E^\ell)_{\mathbf{I}}$, we obtain

$$\begin{aligned}A_1 &= \sum_{E\in\Omega_h}\Big(a_h^E(\boldsymbol{\beta}_{\mathbf{I}}-(\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h)+a_h^E((\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h)\Big)\\ &= B_1+\sum_{E\in\Omega_h}a_h^E((\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h).\end{aligned}$$

Using assumption $(\mathsf{S2}_a)$, we get

$$\begin{aligned}\sum_{E\in\Omega_h}a_h^E((\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h) = \sum_{E\in\Omega_h}\Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E}\Big[\Big(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell)\mathbf{n}_E^{\mathsf{e}}\cdot\mathbf{n}_E^{\mathsf{e}}\Big)\Big(\frac{|\mathsf{e}|}{2}(\boldsymbol{\eta}^{\mathsf{v}_1}+\boldsymbol{\eta}^{\mathsf{v}_2})\cdot\mathbf{n}_E^{\mathsf{e}}\Big)\\ +\Big(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell)\mathbf{n}_E^{\mathsf{e}}\cdot\mathbf{t}_E^{\mathsf{e}}\Big)\Big(|\mathsf{e}|\eta_E^{\mathsf{e}}+\frac{|\mathsf{e}|}{2}(\boldsymbol{\eta}^{\mathsf{v}_1}+\boldsymbol{\eta}^{\mathsf{v}_2})\cdot\mathbf{t}_E^{\mathsf{e}}\Big)\Big]\Big).\end{aligned}$$

First, from (54), $(\mathsf{O4})$ and then using an integration by parts, we obtain

$$\begin{aligned}\sum_{E\in\Omega_h}a_h^E((\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h) &= \sum_{E\in\Omega_h}\Bigg(\sum_{\mathsf{e}\in\mathcal{E}_h^E}\Big[\Big(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell)\mathbf{n}_E^{\mathsf{e}}\cdot\mathbf{n}_E^{\mathsf{e}}\Big)\int_{\mathsf{e}}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\cdot\mathbf{n}_E^{\mathsf{e}}\Big]\Bigg)\\ &\quad+\Big(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell)\mathbf{n}_E^{\mathsf{e}}\cdot\mathbf{t}_E^{\mathsf{e}}\Big)\int_{\mathsf{e}}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\cdot\mathbf{t}_E^{\mathsf{e}}\\ &= \sum_{E\in\Omega_h}\Bigg(\sum_{\mathsf{e}\in\mathcal{E}_h^E}\int_{\mathsf{e}}\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell)\mathbf{n}_E^{\mathsf{e}}\cdot\mathbf{R}_h^E\boldsymbol{\eta}_h\Bigg)=\sum_{E\in\Omega_h}\int_E\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell):\boldsymbol{\varepsilon}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\\ &= \sum_{E\in\Omega_h}\int_E\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}_E^\ell-\boldsymbol{\beta}):\boldsymbol{\varepsilon}(\mathbf{R}_h^E\boldsymbol{\eta}_h)+\int_\Omega\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}):\boldsymbol{\varepsilon}(\mathbf{R}_h\boldsymbol{\eta}_h)\\ &= B_2+\int_\Omega\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\beta}):\boldsymbol{\varepsilon}(\mathbf{R}_h\boldsymbol{\eta}_h).\end{aligned}\tag{58}$$

Using $(5)_2$, from (58) we get

$$\sum_{E\in\Omega_h}a_h^E((\boldsymbol{\beta}_E^\ell)_{\mathbf{I}},\boldsymbol{\eta}_h)=B_2+\int_\Omega p\,\mathrm{rot}(\mathbf{R}_h\boldsymbol{\eta}_h)+\int_\Omega\nabla\psi\cdot\mathbf{R}_h\boldsymbol{\eta}_h,$$

and thus

$$A_1=B_1+B_2+\int_\Omega p\,\mathrm{rot}(\mathbf{R}_h\boldsymbol{\eta}_h)+\int_\Omega\nabla\psi\cdot\mathbf{R}_h\boldsymbol{\eta}_h.$$

We continue with the term A_2. Using the definition of $[\cdot,\cdot]_{Q_h}$, (55) and adding and subtracting the exact solution p, we obtain

$$\begin{aligned}
A_2 &= \sum_{E\in\Omega_h} |E|(p_\pi)_E(\mathrm{rot}_{H_h}\,\boldsymbol{\eta}_h)_E = \sum_{E\in\Omega_h} \int_E \mathrm{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h)(p_\pi)_E \\
&= \sum_{E\in\Omega_h} \int_E \mathrm{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h)((p_\pi)_E - p) + \int_\Omega \mathrm{rot}(\mathbf{R}_h\boldsymbol{\eta}_h)p \\
&= B_3 + \int_\Omega \mathrm{rot}(\mathbf{R}_h\boldsymbol{\eta}_h)p.
\end{aligned}$$

Now, we rewrite A_3, using (26) as follows:

$$A_3 = [(\mathrm{rot}\,\boldsymbol{\beta})_\pi, q_h]_{Q_h} = -\kappa^{-1}t^2[(\mathrm{rot}(\mathrm{curl}\,p))_\pi, q_h]_{Q_h},$$

where in the last equality we have used that $\mathrm{rot}\,\boldsymbol{\beta} = -\kappa^{-1}t^2(\mathrm{rot}(\mathrm{curl}\,p))$ which is a consequence of (4) and $(1)_3$. Then, using (25) and (29), we get

$$A_3 = -\kappa^{-1}t^2[\mathrm{rot}_{\Gamma_h}((\mathrm{curl}\,p)_{\mathrm{II}}), q_h]_{Q_h} = -\kappa^{-1}t^2[(\mathrm{curl}\,p)_{\mathrm{II}}, \mathrm{curl}_h\, q_h]_{\Gamma_h}. \quad (59)$$

We now consider p^ℓ a piecewise linear discontinuous function which is an approximation of p on Ω. The restriction of p^ℓ to E, $E\in\Omega_h$, is denoted by p^ℓ_E and is defined as the $L^2(E)$-projection of p onto the polynomials of degree ≤ 1. Using (59), (29) and adding and subtracting the term $(\mathrm{curl}\,p^\ell_E)_{\mathrm{II}}$ on each element, we get

$$\begin{aligned}
A_4 + A_3 = \kappa^{-1}t^2\Big([p_\pi, \mathrm{rot}_{\Gamma_h}(\mathrm{curl}_h\, q_h)]_{Q_h} &- \sum_{E\in\Omega_h} [(\mathrm{curl}\,p^\ell_E)_{\mathrm{II}}, \mathrm{curl}_h\, q_h]_{\Gamma_h,E} \\
+ \sum_{E\in\Omega_h} [(\mathrm{curl}(p^\ell_E - p))_{\mathrm{II}}, \mathrm{curl}_h\, q_h]_{\Gamma_h,E}\Big).
\end{aligned}$$

From assumption $(\mathsf{S2})$ and the identity

$$[p_\pi, \mathrm{rot}_{\Gamma_h}(\mathrm{curl}_h\, q_h)]_{Q_h} = \sum_{E\in\Omega_h} \int_E p\ \ \mathrm{rot}_{\Gamma_h}(\mathrm{curl}_h\, q_h)|_E,$$

we obtain

$$\begin{aligned}
A_4 + A_3 &= \kappa^{-1}t^2\Big(\sum_{E\in\Omega_h} [(\mathrm{curl}(p^\ell_E - p))_{\mathrm{II}}, \mathrm{curl}_h\, q_h]_{\Gamma_h,E} \\
&+ \sum_{E\in\Omega_h} \Big[\int_E (p - p^\ell_E)(\mathrm{rot}_{\Gamma_h}(\mathrm{curl}_h\, q_h))_E + \sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} p^\ell_E (\mathrm{curl}_h\, q_h)^{\mathsf{e}}_E\Big]\Big) \\
&= B_4 + B_5 + B_6.
\end{aligned}$$

Thus, collecting all the previous bounds for terms A_i, $i = 1, \dots, 5$, we obtain the following inequality:

$$\begin{aligned} A_h^t((\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h, p_\pi - p_h), (\boldsymbol{\eta}_h, q_h)) =& A_1 - A_2 - A_3 - A_4 - A_5 \\ \leq& B_1 + B_2 - B_3 - B_4 - B_5 - B_6 \\ &+ \int_\Omega \nabla\psi \cdot \mathbf{R}_h \boldsymbol{\eta}_h - [\nabla_h \psi_h, \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h}. \end{aligned} \tag{60}$$

Defining

$$B_7 := \int_\Omega \nabla\psi \cdot \mathbf{R}_h \boldsymbol{\eta}_h - [\nabla_h \psi_h, \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h},$$

from (57) and (60) we get

$$C(\|\boldsymbol{\beta}_{\mathbf{I}} - \boldsymbol{\beta}_h\|_{H_h} + \|p_\pi - p_h\|_{Q_h} + t\|\operatorname{curl}_h p_\pi - \operatorname{curl}_h p_h\|_{\Gamma_h}) \leq \sum_{i=1}^{7} |B_i|. \tag{61}$$

Step 2. We bound each term B_i, $i = 1, \dots, 7$ with a constant C independent of h and t.

Estimate of $|B_1|$. Using assumption $(\mathsf{S1}_a)$, the Cauchy-Schwarz inequality, (11), (56), the estimates (4.31) and (4.36) from [11] and finally (6), we obtain

$$\begin{aligned} |B_1| &\leq C \sum_{E \in \Omega_h} \|\boldsymbol{\beta}_{\mathbf{I}} - (\boldsymbol{\beta}_E^\ell)_{\mathbf{I}}\|_{H_h,E} \|\boldsymbol{\eta}_h\|_{H_h,E} \\ &\leq C\Big(\sum_{E \in \Omega_h} \|\boldsymbol{\beta}_{\mathbf{I}} - (\boldsymbol{\beta}_E^\ell)_{\mathbf{I}}\|_{H_h,E}^2\Big)^{1/2} \Big(\sum_{E \in \Omega_h} \|\boldsymbol{\eta}_h\|_{H_h,E}^2\Big)^{1/2} \\ &\leq C\Big(\sum_{E \in \Omega_h} |||\boldsymbol{\beta}_{\mathbf{I}} - (\boldsymbol{\beta}_E^\ell)_{\mathbf{I}}|||_{H_h,E}^2\Big)^{1/2} \|\boldsymbol{\eta}_h\|_{H_h} \\ &\leq Ch\|\boldsymbol{\beta}\|_{2,\Omega} \leq Ch\|g\|_{0,\Omega}. \end{aligned}$$

Estimate of $|B_2|$. We apply the Cauchy-Schwarz inequality, the estimate of the interpolation error $(\mathsf{M5})$, property $(\mathsf{O2})$ of the lifting operator $\mathbf{R}_h^E(\cdot)$, (56) and (6); we obtain

$$\begin{aligned} |B_2| &\leq \sum_{E \in \Omega_h} |\boldsymbol{\beta} - \boldsymbol{\beta}_E^\ell|_{1,E} \|\boldsymbol{\varepsilon}(\mathbf{R}_h^E \boldsymbol{\eta}_h)\|_{0,E} \\ &\leq \Big(\sum_{E \in \Omega} |\boldsymbol{\beta} - \boldsymbol{\beta}_E^\ell|_{1,E}^2\Big)^{1/2} \Big(\sum_{E \in \Omega} \|\boldsymbol{\varepsilon}(\mathbf{R}_h^E \boldsymbol{\eta}_h)\|_{0,E}^2\Big)^{1/2} \\ &\leq \Big(\sum_{E \in \Omega} C_{app} h_E^2 |\boldsymbol{\beta}|_{2,E}^2\Big)^{1/2} \Big(\sum_{E \in \Omega} C\|\boldsymbol{\eta}_h\|_{H_h,E}^2\Big)^{1/2} \\ &\leq Ch\|\boldsymbol{\beta}\|_{2,\Omega} \|\boldsymbol{\eta}_h\|_{H_h} \leq Ch\|g\|_{0,\Omega}. \end{aligned}$$

Estimate of $|B_3|$. Using the Cauchy-Schwarz inequality, the estimate of the interpolation error (M4), the Korn inequality [27], property (O2) of the lifting operator $\mathbf{R}_h^E(\cdot)$, (56) and (6), we get

$$
\begin{aligned}
|B_3| &\leq \sum_{E\in\Omega_h} \|p-(p_\pi)_E\|_{0,E}\|\operatorname{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\|_{0,E} \\
&\leq \Big(\sum_{E\in\Omega}\|p-(p_\pi)_E\|_{0,E}^2\Big)^{1/2}\Big(\sum_{E\in\Omega}|\mathbf{R}_h^E\boldsymbol{\eta}_h|_{1,E}^2\Big)^{1/2} \\
&\leq \Big(\sum_{E\in\Omega} C_{app}^* h_E^2|p|_{1,E}^2\Big)^{1/2}|\mathbf{R}_h\boldsymbol{\eta}_h|_{1,\Omega} \\
&\leq Ch\|p\|_{1,\Omega}\|\boldsymbol{\varepsilon}(\mathbf{R}_h\boldsymbol{\eta}_h)\|_{0,\Omega} \leq Ch\|p\|_{1,\Omega}\Big(\sum_{E\in\Omega}\|\boldsymbol{\varepsilon}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\|_{0,E}^2\Big)^{1/2} \\
&\leq Ch\|p\|_{1,\Omega}\Big(\sum_{E\in\Omega}C\|\boldsymbol{\eta}_h\|_{0,E}^2\Big)^{1/2} \leq Ch\|p\|_{1,\Omega}\|\boldsymbol{\eta}_h\|_{H_h} \leq Ch\|g\|_{0,\Omega}.
\end{aligned}
$$

Estimate of $|B_4|$. Using assumption (S1), the Cauchy-Schwarz inequality and the definition of the norm $\|\cdot\|_{\Gamma_h,E}$, we get

$$
\begin{aligned}
|B_4| &\leq \kappa^{-1}t^2\sum_{E\in\Omega_h}\|(\operatorname{curl}(p_E^\ell-p))_{\mathrm{II}}\|_{\Gamma_h,E}\|\operatorname{curl}_h q_h\|_{\Gamma_h,E} \\
&\leq \kappa^{-1}t^2\Big(\sum_{E\in\Omega_h}\|(\operatorname{curl}(p_E^\ell-p))_{\mathrm{II}}\|_{\Gamma_h,E}^2\Big)^{1/2}\Big(\sum_{E\in\Omega_h}\|\operatorname{curl}_h q_h\|_{\Gamma_h,E}^2\Big)^{1/2} \\
&= \kappa^{-1}t^2\Big(\sum_{E\in\Omega_h}|E|\sum_{\mathsf{e}\in\mathcal{E}_h^E}\big|(\operatorname{curl}(p_E^\ell-p))_{\mathrm{II}})_E^{\mathsf{e}}\big|^2\Big)^{1/2}\|\operatorname{curl}_h q_h\|_{\Gamma_h}.
\end{aligned}
$$

Now, using the definition of the interpolant $(\cdot)_{\mathrm{II}}$, the Cauchy-Schwarz inequality, properties (M3), (M1) and the estimate of the interpolation error provided by (M5), yields

$$
\begin{aligned}
\sum_{\mathsf{e}\in\mathcal{E}_h^E}\big|(\operatorname{curl}(p_E^\ell-p))_{\mathrm{II}})_E^{\mathsf{e}}\big|^2 &= \sum_{\mathsf{e}\in\mathcal{E}_h^E}\Big|\frac{1}{|\mathsf{e}|}\int_{\mathsf{e}}\operatorname{curl}(p_E^\ell-p)\cdot\mathbf{t}_E^{\mathsf{e}}\Big|^2 \\
&\leq \sum_{\mathsf{e}\in\mathcal{E}_h^E}\frac{1}{|\mathsf{e}|}\int_{\mathsf{e}}\big|\operatorname{curl}(p_E^\ell-p)\cdot\mathbf{t}_E^{\mathsf{e}}\big|^2 \leq \sum_{\mathsf{e}\in\mathcal{E}_h^E}\frac{1}{|\mathsf{e}|}\Big(h_E^{-1}\|\operatorname{curl}(p_E^\ell-p)\|_{0,E}^2 \\
&+ h_E|\operatorname{curl}(p_E^\ell-p)|_{1,E}^2\Big) \leq C\frac{N_{\mathsf{e}}}{h_E}\left(h_E^{-1}|p_E^\ell-p|_{1,E}^2+h_E|p|_{2,E}^2\right) \\
&\leq C\frac{N_{\mathsf{e}}}{h_E}\left(h_E^{-1}C_{app}h_E^2|p|_{2,E}^2+h_E|p|_{2,E}^2\right) \leq C|p|_{2,E}^2.
\end{aligned}
$$

Therefore, using the above estimate, bound (56) and (6) we obtain

$$
|B_4| \leq \kappa^{-1}t^2\Big(\sum_{E\in\Omega_h}C|E||p|_{2,E}^2\Big)^{1/2}\|\operatorname{curl}_h q_h\|_{\Gamma_h} \leq Ch\|g\|_{0,\Omega}.
$$

Estimate of $|B_5|$. Due to (M2), the definitions of rot_{Γ_h} and $||\cdot||_{\Gamma_h}$ yield the following inverse estimate

$$|E|^{1/2}(\mathrm{rot}_{\Gamma_h}\,\boldsymbol{\delta}_h)_E \le Ch_E^{-1}||\boldsymbol{\delta}_h||_{\Gamma_h,E} \quad \forall \boldsymbol{\delta}_h \in \Gamma_h, \quad \forall E \in \Omega_h. \tag{62}$$

Therefore, using also the Cauchy-Schwarz inequality and the estimate of the interpolation error (M5), we obtain the following development:

$$\begin{aligned} |B_5| &\le \kappa^{-1}t^2 \sum_{E\in\Omega_h} \|p - p_E^\ell\|_{0,E}\, |E|^{1/2}(\mathrm{rot}_{\Gamma_h}(\mathrm{curl}_h\, q_h))_E \\ &\le \kappa^{-1}t^2 \sum_{E\in\Omega_h} h_E^2|p|_{2,E}\, h_E^{-1}||\,\mathrm{curl}_h\, q_h||_{\Gamma_h,E} \\ &\le C\kappa^{-1}t^2\Big(\sum_{E\in\Omega_h} h_E^2|p|_{2,E}^2\Big)^{1/2}\Big(\sum_{E\in\Omega_h} ||\,\mathrm{curl}_h\, q_h||_{\Gamma_h,E}^2\Big)^{1/2}. \end{aligned} \tag{63}$$

Finally, from (63), (56) and bound (6) it follows

$$|B_5| \le C\kappa^{-1}t^2h^2|p|_{2,\Omega}h^{-1}\|\,\mathrm{curl}_h\, q_h\|_{\Gamma_h} \le Ch\|g\|_{0,\Omega}.$$

Estimate of $|B_6|$. Using the same argument as in Lemma 5.3 of [21], bound (56) and (6), we can prove that

$$|B_6| \le Cht^2\|p\|_{2,\Omega}\|\,\mathrm{curl}_h\, q_h\|_{\Gamma_h} \le Ch\|g\|_{0,\Omega},$$

where C is independent of h and t.

Estimate of $|B_7|$. In order to estimate this term, we split it as follows. Adding and subtracting the terms $\nabla\psi_E^\ell$ and $\nabla_h(\psi_E^\ell)_{\mathrm{I}}$ on each element E, we get

$$B_7 = B_7^1 + B_7^2 + \sum_{E\in\Omega_h}\Big[\int_E \nabla\psi_E^\ell \cdot \mathbf{R}_h^E\boldsymbol{\eta}_h - [\nabla_h(\psi_E^\ell)_{\mathrm{I}}, \Pi_h\boldsymbol{\eta}_h]_{\Gamma_h,E}\Big], \tag{64}$$

where

$$B_7^1 = \sum_{E\in\Omega_h}\int_E \nabla(\psi - \psi_E^\ell)\cdot \mathbf{R}_h^E\boldsymbol{\eta}_h\,, \quad B_7^2 = \sum_{E\in\Omega_h}[\nabla_h((\psi_E^\ell)_{\mathrm{I}} - \psi_h), \Pi_h\boldsymbol{\eta}_h]_{\Gamma_h,E}. \tag{65}$$

Using (41) and (43), integrating by parts and finally using assumption (S2), from (64) we obtain

$$
\begin{aligned}
B_7 =& B_7^1 + B_7^2 + \sum_{E\in\Omega_h} \Big[\int_E \operatorname{curl} f_E^\ell \cdot \mathbf{R}_h^E \boldsymbol{\eta}_h - [(\operatorname{curl} f_E^\ell)_{\mathrm{II}}, \Pi_h \boldsymbol{\eta}_h]_{\Gamma_h,E} \Big] \\
=& B_7^1 + B_7^2 + \sum_{E\in\Omega_h} \Big[\int_E f_E^\ell \operatorname{rot}(\mathbf{R}_h^E \boldsymbol{\eta}_h) - \sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} f_E^\ell (\mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}}) \Big] \\
& - \sum_{E\in\Omega_h} \Big[\int_E f_E^\ell (\operatorname{rot}_{\Gamma_h}(\Pi_h \boldsymbol{\eta}_h))_E - \sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} f_E^\ell (\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} \Big] \\
=& B_7^1 + B_7^2 + \sum_{E\in\Omega_h} \Big[\int_E f_E^\ell (\operatorname{rot}(\mathbf{R}_h^E \boldsymbol{\eta}_h) - (\operatorname{rot}_{\Gamma_h}(\Pi_h \boldsymbol{\eta}_h))_E) \Big] \\
& + \sum_{E\in\Omega_h} \Big[\sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} f_E^\ell \left((\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} - (\mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}}) \right) \Big] \\
=& B_7^1 + B_7^2 + B_7^3 + B_7^4.
\end{aligned}
$$

Thus, in order to bound the term B_7, we have to bound each term B_7^i, $i = 1, 2, 3, 4$ separately.

Estimate of $|B_7^1|$. Using the Cauchy-Schwarz inequality, the estimate of the interpolation error (M5), the Korn inequality [27], property (O2), (56) and (6), we get

$$
\begin{aligned}
|B_7^1| \leq& \Big(\sum_{E\in\Omega_h} \|\nabla(\psi - \psi_E^\ell)\|_{0,E}^2 \Big)^{1/2} \Big(\sum_{E\in\Omega_h} \|\mathbf{R}_h^E \boldsymbol{\eta}_h\|_{0,E}^2 \Big)^{1/2} \\
\leq& Ch\|\psi\|_{2,\Omega} \|\boldsymbol{\varepsilon}(\mathbf{R}_h \boldsymbol{\eta}_h)\|_{0,\Omega} \leq Ch\|\psi\|_{2,\Omega} \|\boldsymbol{\eta}_h\|_{H_h} \leq Ch\|g\|_{0,\Omega}.
\end{aligned}
$$

Estimate of $|B_7^2|$. We begin this estimate by using assumption (S1), the Cauchy-Schwarz inequality, adding and subtracting $\nabla_h \psi_{\mathrm{I}}$, and applying the triangular inequality. We obtain

$$
\begin{aligned}
|B_7^2| \leq& \sum_{E\in\Omega_h} \|\nabla_h((\psi_E^\ell)_{\mathrm{I}} - \psi_h)\|_{\Gamma_h,E} \|\Pi_h \boldsymbol{\eta}_h\|_{\Gamma_h,E} \\
\leq& \Big(\sum_{E\in\Omega_h} \|\nabla_h((\psi_E^\ell)_{\mathrm{I}} - \psi_h)\|_{\Gamma_h,E}^2 \Big)^{1/2} \Big(\sum_{E\in\Omega_h} \|\Pi_h \boldsymbol{\eta}_h\|_{\Gamma_h,E}^2 \Big)^{1/2} \\
\leq& C\Big(\|\nabla_h(\psi_{\mathrm{I}} - \psi_h)\|_{\Gamma_h}^2 + \sum_{E\in\Omega_h} \|\nabla_h((\psi_E^\ell)_{\mathrm{I}} - \psi_{\mathrm{I}})\|_{\Gamma_h,E}^2 \Big)^{1/2} \|\Pi_h \boldsymbol{\eta}_h\|_{\Gamma_h}.
\end{aligned}
\tag{66}
$$

We now note that the following inequality holds, as shown in the Appendix:

$$
\|\Pi_h \boldsymbol{\theta}_h\|_{\Gamma_h} \leq C||\boldsymbol{\theta}_h||_{H_h} \quad \forall \boldsymbol{\theta}_h \in H_h. \tag{67}
$$

Therefore, first using identity (12), Proposition 2, Lemma 6.3 from [17] and (67) in (66), then applying the bounds (56) and (6) yields

$$\begin{aligned}|B_7^2| &\leq C\Big(Ch^2\|g\|_{0,\Omega}^2 + \sum_{E\in\Omega_h} Ch^2|\psi|_{2,E}^2\Big)^{1/2}||\boldsymbol{\eta}_h||_{H_h}\\ &\leq Ch\|g\|_{0,\Omega}\|\boldsymbol{\eta}_h\|_{H_h} \leq Ch\|g\|_{0,\Omega}.\end{aligned}$$

Estimate of $|B_7^3|$. From (28) and (55) it follows

$$\int_E \operatorname{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h) = \int_E (\operatorname{rot}_{\Gamma_h}(\Pi_h\boldsymbol{\eta}_h))_E = |E|(\operatorname{rot}_{\Gamma_h}(\Pi_h\boldsymbol{\eta}_h))_E. \tag{68}$$

Let $\bar{f}_E^\ell = \frac{1}{|E|}\int_E f_E^\ell$ for all $E\in\Omega_h$. Using identity (68) it follows

$$|B_7^3| = \Big|\sum_{E\in\Omega_h}\int_E (f_E^\ell - \bar{f}_E^\ell)\operatorname{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\Big|,$$

which, using a Cauchy-Schwarz inequality, gives

$$|B_7^3| \leq \sum_{E\in\Omega_h}\|f_E^\ell - \bar{f}_E^\ell\|_{0,E}\|\operatorname{rot}(\mathbf{R}_h^E\boldsymbol{\eta}_h)\|_{0,E}.$$

Now, using the estimate of the interpolation error (M4), the Korn inequality [27], the fact that $|f_E^\ell|_{1,E} = |\psi_E^\ell|_{1,E} \leq ||\psi||_{1,E}$, property (O2), (56) and finally (6), we obtain

$$\begin{aligned}|B_7^3| \leq&\Big(\sum_{E\in\Omega_h}\|f_E^\ell - \bar{f}_E^\ell\|_{0,E}^2\Big)^{1/2}\Big(\sum_{E\in\Omega_h}|\mathbf{R}_h^E\boldsymbol{\eta}_h|_{1,E}^2\Big)^{1/2}\\ \leq&\Big(\sum_{E\in\Omega_h}Ch_E^2|f_E^\ell|_{1,E}^2\Big)^{1/2}|\mathbf{R}_h\boldsymbol{\eta}_h|_{1,\Omega}\\ \leq&\Big(\sum_{E\in\Omega_h}Ch_E^2|\psi_E^\ell|_{1,E}^2\Big)^{1/2}\|\boldsymbol{\varepsilon}(\mathbf{R}_h\boldsymbol{\eta}_h)\|_{0,\Omega}\\ \leq&Ch\|\psi\|_{1,\Omega}\Big(\sum_{E\in\Omega_h}\|\boldsymbol{\eta}_h\|_{H_h,E}^2\Big)^{1/2} = Ch\|\psi\|_{1,\Omega}\|\boldsymbol{\eta}_h\|_{H_h} \leq Ch\|g\|_{0,\Omega}.\end{aligned}$$

Estimate of $|B_7^4|$. Similarly to the previous case, from (68) and the definition of $\operatorname{rot}_{\Gamma_h}$ in (23), we get

$$\int_{\mathsf{e}}\mathbf{R}_h^E\boldsymbol{\eta}_h\cdot\mathbf{t}_E^{\mathsf{e}} = \int_{\mathsf{e}}(\Pi_h\boldsymbol{\eta}_h)_E^{\mathsf{e}} = |\mathsf{e}|(\Pi_h\boldsymbol{\eta}_h)_E^{\mathsf{e}}. \tag{69}$$

Using this identity, the triangular inequality and the Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
|B_7^4| =& \Big| \sum_{E\in\Omega_h} \Big[\sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} f_E^\ell \left((\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} - (\mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}}) \right) \Big] \Big| \\
=& \Big| \sum_{E\in\Omega_h} \Big[\sum_{\mathsf{e}\in\mathcal{E}_h^E} \int_{\mathsf{e}} (f_E^\ell - \bar f_E^\ell) \left((\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} - (\mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}}) \right) \Big] \Big| \\
\le& \sum_{E\in\Omega_h} \Big[\sum_{\mathsf{e}\in\mathcal{E}_h^E} \| f_E^\ell - \bar f_E^\ell \|_{0,\mathsf{e}} \| \mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}} - (\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} \|_{0,\mathsf{e}} \Big] \\
\le& \sum_{E\in\Omega_h} \Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E} \| f_E^\ell - \bar f_E^\ell \|_{0,\mathsf{e}}^2 \Big)^{1/2} \Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E} \| \mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}} - (\Pi_h \boldsymbol{\eta}_h)_E^{\mathsf{e}} \|_{0,\mathsf{e}}^2 \Big)^{1/2}.
\end{aligned}
$$

Using (M3), (M4), one dimensional interpolation estimates, the fact that $|f_E^\ell|_{1,E} = |\psi_E^\ell|_{1,E} \le ||\psi||_{1,E}$ and a trace inequality, gives

$$
\begin{aligned}
|B_7^4| \le& \sum_{E\in\Omega_h} \Big(h_E^{-1} \| f_E^\ell - \bar f_E^\ell \|_{0,E}^2 + h_E |f_E^\ell|_{1,E}^2 \Big)^{1/2} \\
& \Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E} h_{\mathsf{e}} |\mathbf{R}_h^E \boldsymbol{\eta}_h \cdot \mathbf{t}_E^{\mathsf{e}}|_{1/2,\mathsf{e}}^2 \Big)^{1/2} \\
\le& C h^{1/2} \sum_{E\in\Omega_h} \Big(h_E |f_E^\ell|_{1,E}^2 \Big)^{1/2} \| \mathbf{R}_h^E \boldsymbol{\eta}_h \|_{1/2,\partial E} \\
\le& C h \sum_{E\in\Omega_h} |\psi_E^\ell|_{1,E} \| \mathbf{R}_h^E \boldsymbol{\eta}_h \|_{1,E}.
\end{aligned}
$$

The Cauchy-Schwarz inequality, the Korn inequality [27], property (O2), (56) and (6) now yield

$$
\begin{aligned}
|B_7^4| \le& C h |\psi|_{1,\Omega} \| \mathbf{R}_h \boldsymbol{\eta}_h \|_{1,\Omega} \le C h \|\psi\|_{1,\Omega} \| \boldsymbol{\varepsilon}(\mathbf{R}_h \boldsymbol{\eta}_h) \|_{0,\Omega} \\
\le& C h \|\psi\|_{1,\Omega} \Big(\sum_{E\in\Omega_h} \| \boldsymbol{\eta}_h \|_{H_h,E}^2 \Big)^{1/2} = C h \|\psi\|_{1,\Omega} \| \boldsymbol{\eta}_h \|_{H_h} \le C h \|g\|_{0,\Omega}.
\end{aligned}
$$

Combining (61) with all the above bounds for the B_i, $i = 1, .., 7$, gives the proof of the proposition. □

5.3 Error estimate for variable w.

Let w_h be the solution of the discrete problem $(40)_4$ and w be the solution of the continuous problem $(5)_4$. Using essentially the same arguments used to prove the error estimate for variable ψ, one can show the following bound:

$$
\begin{aligned}
c_1||w_{\mathrm{I}} - w_h||^2_{W_h} \leq \sum_{E\in\Omega_h} & [\nabla_h w_{\mathrm{I}} - \nabla_h (w_E^\ell)_{\mathrm{I}}, \nabla_h (w_{\mathrm{I}} - w_h)]_{\Gamma_h,E} \\
& + \sum_{E\in\Omega_h} \int_E \nabla(w_E^\ell - w)\cdot \nabla R_h^E (w_{\mathrm{I}} - w_h) \\
& +(\boldsymbol{\beta}, \nabla R_h(w_{\mathrm{I}} - w_h))_{0,\Omega} - [\Pi_h \boldsymbol{\beta}_h, \nabla_h(w_{\mathrm{I}} - w_h)]_{\Gamma_h} \\
& +\kappa^{-1}t^2(\nabla\psi, \nabla R_h(w_{\mathrm{I}} - w_h))_{0,\Omega} \\
& -\kappa^{-1}t^2[\nabla_h\psi_h, \nabla_h(w_{\mathrm{I}} - w_h)]_{\Gamma_h}.
\end{aligned}
\tag{70}
$$

From (70), repeating the same techniques used in Sections 5.1 and 5.2, the bounds for the deflection variable follow.

Proposition 4 *Let w and w_h be the solutions of problems* $(5)_4$ *and* $(40)_4$*, respectively. Let bound* (6) *holds. Then, there exists a constant $C > 0$ independent of h and t such that*

$$\|w_{\mathrm{I}} - w_h\|_{W_h} \leq Ch\|g\|_{0,\Omega}.$$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. The proof follows easily by combining Propositions 2, 3 and 4. □

Moreover, the following important remark holds.

Remark 3 The "bubble" edge degrees of freedom in the rotation space are added in order to guarantee the validity of Lemma 4, i.e. the stability of the discrete system, and do not enhance the approximation capabilities of H_h. In [9] the authors show that, under certain conditions on the adopted mesh, the nodal degrees of freedom alone are sufficient to derive Lemma 4. Such conditions on the mesh are not very strict, and include for example a large array of meshes made with polygons with more than 4 edges. Although the results of [9] are intended for the Stokes problem, a "rotation of 90^0" allows immediate application also to our case. Once Lemma 4 is proven, the rest of our proofs extend almost identically to the case with no edge degrees of freedom. Therefore, under the favorable mesh conditions of [9], it is easy to check that the same plate method presented here, but with the smaller rotation space

$$H_h = \{\boldsymbol{\eta}_h \mid \boldsymbol{\eta}_h = \{\boldsymbol{\eta}^{\mathsf{v}}\}_{\mathsf{v}\in\mathcal{V}_h^0}\},$$

is stable, and the same $O(h)$ error estimates hold. This is interesting since it allows to use the same degrees of freedom both for rotations and displacement.

6 Conclusions

We presented a mimetic discretization method for the Reissner-Mindlin plate bending problem. The fundamental idea of the mimetic discretization methodology lays in writing the variational problem directly in terms of the degrees of freedom, without specifying the underlying basis functions. The present scheme adopts one degree of freedom in each mesh vertex for the deflections, and two degrees of freedom in each mesh vertex for the rotations, plus an additional degree of freedom on each edge (that is not always needed). After building all the necessary tools, such as discrete bilinear forms and operators, we presented the method and proved linear convergence with respect to the mesh size, uniformly in the plate thickness. The latter result is achieved rewriting the discrete problem as a combination of different sub-problems via a discrete Helmholtz decomposition.

7 Appendix

In the first part of this section we briefly show, for all $E \in \Omega_h$, the existence of a lifting operator

$$\mathbf{R}_h^E \ : \ H_h|_E \longrightarrow [H^1(E) \cap \mathcal{C}^0(\bar{E})]^2$$

which satisfies the conditions in Lemma 6. In the second part we will prove bound (67).

Existence of a lifting operator. We will build the lifting operator in two steps taking full advantage of the results in [17,11]. Note that we can not use directly the (rotated) operator of [11] since it does not preserve linear functions, which is needed to prove (O2).

We start with a slightly modified construction of the lifting operator in [17], which we call $\widetilde{\mathbf{R}}_h^E$. Given $\boldsymbol{\eta}_h \in H_h|_E$, the vector function $\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h$ is globally continuous and piecewise linear on the sub-triangulation $\mathcal{T}_h$ and defined in the following way. On the vertices $\mathsf{v} \in \mathcal{V}_h^E$ we set $\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h(\mathsf{v}) = \boldsymbol{\eta}^{\mathsf{v}}$. On the remaining nodes of $\mathcal{T}_h$ that lay on the boundary, $\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h$ is defined by linear interpolation of the two vertex values of the edge. On the internal nodes of E, we do instead the following construction. Given any internal node v of $\mathcal{T}_h$, we call Ξ_{v} the set of nodes which share an edge with v and are different from v. Then, it is easy to check that v, which lays in the convex hull determined by the nodes $\{\bar{\mathsf{v}}\}_{\bar{\mathsf{v}} \in \Xi_{\mathsf{v}}}$, can be expressed (in a non unique way) as a weighted sum

$$\mathsf{v} = \sum_{\bar{\mathsf{v}} \in \Xi_{\mathsf{v}}} w_{\bar{\mathsf{v}}}^{\mathsf{v}} \, \bar{\mathsf{v}} \tag{71}$$

with $w_{\bar{\mathsf{v}}}^{\mathsf{v}}$ non-negative real numbers such that $\sum_{\bar{\mathsf{v}} \in \Xi_{\mathsf{v}}} w_{\bar{\mathsf{v}}}^{\mathsf{v}} = 1$. For each internal node v, we then enforce the condition

$$\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h(\mathsf{v}) - \sum_{\bar{\mathsf{v}} \in \Xi_{\mathsf{v}}} w_{\bar{\mathsf{v}}}^{\mathsf{v}} \, \widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h(\bar{\mathsf{v}}) = 0 \ .$$

This set of conditions provides a square linear system which determines the value of $\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h$ in the internal nodes. Indeed, it is immediate to verify that the associated matrix is an M-matrix, which in particular implies the existence of a unique solution and a discrete maximum principle. In addition, due to the identity (71), this operator preserves linear vector functions, in the sense that

$$\widetilde{\mathbf{R}}_h^E(\mathbf{p_1})_{\mathbf{I},E} = \mathbf{p_1} \qquad \text{for all linear vector functions } \mathbf{p_1} \text{ on } E\ .$$

Following the same argument as in [17], from the maximum principle it follows that the operator $\widetilde{\mathbf{R}}_h^E$ satisfies the following properties

(O'2) $|\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h|^2_{1,E} \leq C|||\boldsymbol{\eta}_h|||^2_{H_h,E} \qquad \forall \boldsymbol{\eta}_h \in H_h|_E,$

with C independent to the particular element E of the mesh family. Furthermore, by definition of $\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h$ it holds

(O'1) $(\widetilde{\mathbf{R}}_h^E \boldsymbol{\eta}_h)(\mathsf{v}) = \boldsymbol{\eta}^{\mathsf{v}} \quad \forall \mathsf{v} \in \mathcal{V}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h.$

(O'3) $\mathbf{R}_h^E \boldsymbol{\eta}_h|_{\mathsf{e}}$ is a linear (vector) polynomial for all $\mathsf{e} \in \mathcal{E}_h^E \quad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h$.

We then build our final lifting operator $\mathbf{R}_h^E$ as a correction of $\widetilde{\mathbf{R}}_h^E$ by the addition of tangential edge bubbles, as done in [11]. More precisely

$$\mathbf{R}_h^E = \widetilde{\mathbf{R}}_h^E + \mathbf{R}_h^{E,b},$$

where the image of the operator $\mathbf{R}_h^{E,b}$ lays in the span of $\{\varphi_{\mathsf{e}} \mathbf{t}_E^{\mathsf{e}}\}_{\mathsf{e} \in \mathcal{E}_h^E}$ with φ_{e} scalar edge bubble functions (which are quadratic along the edge e). Briefly speaking, the coefficients of the bubble part $\mathbf{R}_h^{E,b}$ are chose in order to satisfy (O4); we refer to [11] for the details.

Given the above properties (O'1)-(O'3), following the same proof shown in [11] one immediately obtains that $\mathbf{R}_h^E$ satisfies (O1), (O3), (O4) and the bound

(O"2) $|\mathbf{R}_h^E \boldsymbol{\eta}_h|^2_{1,E} \leq C|||\boldsymbol{\eta}_h|||^2_{H_h,E} \quad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h.$

Furthermore, since the added bubble part is null on linear functions, it still holds that $\mathbf{R}_h^E(\mathbf{p_1})_{\mathbf{I},E} = \mathbf{p_1}$ for all linear vector functions $\mathbf{p_1}$ on E. Let now $A : E \to \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$ be a symmetric matrix field and $B : E \to \mathbb{R}^{n\times n}$ an anti-symmetric matrix field. Then, from the orthogonality with respect to the contraction operator $A : B = 0$, we get

$$||A + B||^2_{0,E} = ||A||^2_{0,E} + ||B||^2_{0,E} \geq ||A||^2_{0,E}\ . \tag{72}$$

First, using definition (10), then property (O"2) and finally that the operator $\mathbf{R}_h^E$ preserves linear vector functions, yields

$$\begin{aligned} \|\boldsymbol{\eta}_h\|^2_{H_h,E} &= \min_{c\in\mathbb{R}} |||\boldsymbol{\eta}_h - c([-\bar{y}, \bar{x}])_{\mathbf{I},E}|||^2_{H_h,E} \\ &\geq C' \min_{c\in\mathbb{R}} \| \nabla \mathbf{R}_h^E(\boldsymbol{\eta}_h - c([-\bar{y}, \bar{x}])_{\mathbf{I},E}) \|^2_{0,E} \\ &= C' \min_{c\in\mathbb{R}} \|\nabla \mathbf{R}_h^E \boldsymbol{\eta}_h - c\nabla[-\bar{y}, \bar{x}]\|^2_{0,E}\ . \end{aligned} \tag{73}$$

for all $\boldsymbol{\eta}_h \in H_h|_E$. Splitting $\nabla \mathbf{R}_h^E \boldsymbol{\eta}_h$ into its symmetric and anti-symmetric part and observing that $\nabla[-\bar{y}, \bar{x}]$ is an anti-symmetric matrix, from (72), (73) we obtain

$$\|\boldsymbol{\eta}_h\|^2_{H_h,E} \geq C' \|\varepsilon(\mathbf{R}_h^E \boldsymbol{\eta}_h)\|^2_{0,E} \qquad \forall \boldsymbol{\eta}_h \in H_h|_E \quad \forall E \in \Omega_h,$$

which is property (O2).

Proof of bound (67). The norm appearing on the left hand side of inequality (67) is a discrete L^2 norm, while that appearing on the right hand side is a $||\boldsymbol{\varepsilon}(\cdot)||_{L^2}$ type norm. Therefore, due to the boundary conditions on H_h, bound (67) is quite natural. Although relation (67) does not involve the lifting operator, but only the degrees of freedom of H_h, for simplicity we will prove it making use of the lifting $\mathbf{R}_h$ appearing above. A more direct proof should involve in particular a "discrete Korn inequality", which is beyond the scopes of the paper.

By definition and due to (M2) it immediately follows

$$||\Pi_h \boldsymbol{\theta}_h||^2_{\Gamma_h} \leq C \sum_{E\in\Omega_h} |E| \Big(\sum_{\mathsf{e}\in\mathcal{E}_h^E} |\theta_E^{\mathsf{e}}|^2 + \sum_{\mathsf{v}\in\mathcal{V}_h^E} ||\boldsymbol{\theta}^{\mathsf{v}}||^2 \Big) \tag{74}$$

$$||\boldsymbol{\theta}_h||^2_{H_h} \geq C \sum_{E\in\Omega_h} \sum_{\mathsf{e}\in\mathcal{E}_h^E} |\theta_E^{\mathsf{e}}|^2 + ||\boldsymbol{\theta}^{\mathsf{v}_1} - \boldsymbol{\theta}^{\mathsf{v}_2}||^2 \tag{75}$$

where v_1 and v_2 are as usual the two vertices of the edge e. Therefore the bound on the bubble part follows immediately from (74) and (75) observing that $|E| \leq |\Omega|$ for all elements E:

$$\sum_{E\in\Omega_h} |E| \sum_{\mathsf{e}\in\mathcal{E}_h^E} |\theta_E^{\mathsf{e}}|^2 \leq C ||\boldsymbol{\theta}_h||^2_{H_h}. \tag{76}$$

From the definition of $\mathbf{R}_h$, for all $E \in \Omega_h$

$$|E| \sum_{\mathsf{v}\in\mathcal{V}_h^E} ||\boldsymbol{\theta}^{\mathsf{v}}||^2 \leq |E| \, ||\mathbf{R}_h^E \boldsymbol{\theta}_h||^2_{L^\infty(E)}. \tag{77}$$

Let now h_E^{min} indicate the diameter of the smaller element of $\mathcal{T}_h|_E$. First applying an inverse inequality (see for instance Lemma 4.15 of [44]), then using that due to (H1)-(H2) the ratio h_E/h_E^{min} is uniformly bounded, we get

$$\begin{aligned} ||\mathbf{R}_h^E \boldsymbol{\theta}_h||^2_{L^\infty(E)} &\leq C\Big(1 + \log\big(\frac{h_E}{h_E^{min}}\big)\Big) \Big(|\mathbf{R}_h^E \boldsymbol{\theta}_h|^2_{1,E} + |E|^{-1} ||\mathbf{R}_h^E \boldsymbol{\theta}_h||^2_{0,E} \Big) \\ &\leq C|E|^{-1} ||\mathbf{R}_h^E \boldsymbol{\theta}_h||^2_{1,E} \ . \end{aligned} \tag{78}$$

Combining (77), (78), summing over the elements, applying the Korn inequality on Ω and finally property (O2) yields

$$\sum_{E\in\Omega_h} |E| \sum_{\mathsf{v}\in\mathcal{V}_h^E} ||\boldsymbol{\theta}^{\mathsf{v}}||^2 \leq C ||\mathbf{R}_h \boldsymbol{\theta}_h||^2_{1,\Omega} \leq C ||\boldsymbol{\varepsilon}(\mathbf{R}_h \boldsymbol{\theta}_h)||^2_{0,\Omega} \leq C ||\boldsymbol{\theta}_h||^2_{H_h}. \tag{79}$$

The result follows from (76) and (79).

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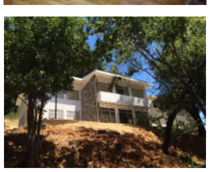

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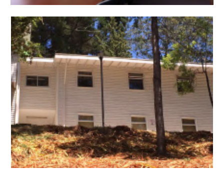
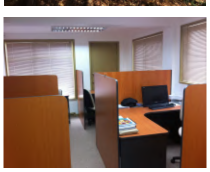

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