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Centro de Investigación en Ingeniería Matemática ( $\mathrm{CI}^{2} \mathrm{MA}$ )



## Stabilization of a second order scheme for a GKdV-4 equation modelling surface water waves

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# Stabilization of a second order scheme for a GKdV-4 equation modelling surface water waves 

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#### Abstract

SUMMARY This work is devoted to the study of a second order numerical scheme for the critical generalized Korteweg-de Vries equation (GKdV with $p=4$ ) in a bounded domain. The KdV equation and some of its generalizations as the GKdV type equations appear in Physics, for example in the study of waves on shallow water. Based on the analysis of stability of the first order scheme introduced by Pazoto et al. [1], we add a vanishing numerical viscosity term to a semi-discrete scheme of second order in space so as to preserve similar properties of stability, and thus able to prove the convergence in $L^{4}$-strong. The semi-discretization of the spatial structure via second-order central finite difference method yields a stiff system of ODE. Hence, for the temporal discretization, we resort to the two-stage implicit Runge-Kutta scheme of the Gauss-Legendre type. The resulting system is unconditionally stable and possesses favorable nonlinear properties. On the other hand, despite the formation of blow up for the critical case of GKdV, it is known that a localized damping term added to the GKdV-4 equation leads to the exponential decay of the energy for small enough initial conditions, which is interesting from the standpoint of the Control Theory. Then, combining the result of convergence in $L^{4}$-strong with discrete multipliers and a contradiction argument, we show that the presence of the vanishing numerical viscosity term allows the uniform (with respect to the mesh size) exponential decay of the total energy associated to the the semi-discrete scheme of higher-order in space with the localized damping term. Numerical experiments are provided to illustrate the performance of the method and to confirm the theoretical results.


KEy words: Higher order scheme; Generalized Korteweg-de Vries equation; Critical case; Implicit Runge-Kutta scheme
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## 1. INTRODUCTION

The Korteweg-de Vries equation and some of its generalizations play an important role in the development of the soliton theory (see Korteweg and de Vries [2]). They arise in many physical contexts such as surface water waves where dispersion and nonlinearity dominate, while dissipative effects are small enough to be neglected in the lowest-order approximation. In many real situations, however, one cannot neglect energy dissipation mechanisms and external

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excitation, especially for the long-time behavior. In this context, several energy dissipation mechanisms were derived and, depending on the physical situation, they must be taken into account, at least, as a perturbation. In this context, a variation of the KdV equation is known in the literature as the generalized Korteweg-de Vries (GKdV) equation. Historically, these types of equations first arose in the study of 2D shallow wave propagation, but have since appeared as limiting cases of many dispersive models. There is a large body of literature for the analysis of the GKdV equation on the line, the half-line, the torus, and boundary domains (see for example, $[3,4,5]$ and the references therein). The GKdV-4 equation or quintic (masscritical) GKdV equation are of special interest: the problem becomes degenerated and presents a lot of similarities with the critical nonlinear Schrödinger equation (see [6]).
In this work we study a second order scheme for the critical generalized Korteweg-de Vries equation (GKdV-4 equation) in a bounded domain $(0, L)$ with or without the effect of a possible localized damping term:

$$
\begin{align*}
& u_{t}+u_{x x x}+u^{4} u_{x}+u_{x}+a(x) u=0, \quad(0, L) \times(0,+\infty),  \tag{1}\\
& u(0, t)=u(L, t)=0, \quad t \in(0,+\infty),  \tag{2}\\
& u_{x}(L, t)=0, \quad t \in(0,+\infty)  \tag{3}\\
& u(x, 0)=u_{0}(x), \quad x \in(0, L) . \tag{4}
\end{align*}
$$

The nonlinearity $u^{4} u_{x}$ in (1) is particularly interesting due to its mass-critical nature $[3,6,7,8,9]$. The drift term $u_{x}$ in (1) is not necessary here, and could be not considered, but it is convenient both theoretically and practically to have the extra flexibility inherent in formulation (1)-(4) (see for example, [4]). The function $a=a(x)$ of the damping term in (1), must be non-negative $(a(x) \geqslant 0$, a.e. in $(0, L))$. It can be eventually identically to zero $(a(x) \equiv 0)$, in which case (1) is the pure GKdV-4 dispersive equation without damping term. On the other hand, in order to obtain the asymptotic behavior of the exponential decay of the total energy associated to the model given by

$$
\begin{equation*}
E(t)=\frac{1}{2} \int_{0}^{L}|u(x, t)|^{2} d x \tag{5}
\end{equation*}
$$

the function $a=a(x)$ must satisfy the following additional assumption characterizing the localized damping term

$$
\begin{cases}a \in L^{\infty}(0, L) \text { and } a(x) \geqslant a_{0}>0 & \text { a.e. in } \Omega  \tag{6}\\ \text { where } \Omega \text { is a nonempty open subset of }(0, L) .\end{cases}
$$

This system (1)-(4) was studied first in [3], where the authors prove the exponential decay of the energy, under the hypotheses (6) and for a small initial data $u_{0}$ such that $\left\|u_{0}\right\|_{L^{2}(0, L)}<\sqrt{3 \pi / 2}$. The same system was studied in a numerical point of view in [1], where the authors prove a uniform stabilization and convergence of a first order numerical schemes for the system (1)-(4) under the same assumptions. In this paper, we give an improvement in the order of this finite difference scheme maintaining the properties of uniform stability and convergence of [1].
There are several numerical schemes for the KdV and GKdV type equations in the literature with stability and conservative properties (see $[1,4,7,10,11,12]$ and the reference therein). However, in the case of GKdV-4 equation, we had to make a special treatment of the nonlinearity $u^{4} u_{x}$, rewriting it in a particular and a bit sophisticated way, taking into account
the invariance of the mass scaling in the $L^{2}$-norm. The numerical schemes proposed here, is the generalization to the second order of the first order scheme introduced by Pazoto et al. [1] where such treatment of the nonlinearity is taken into account. In the case of the first-order scheme, the approximation of the dispersive term by decentered finite differences, provides a natural numerical viscosity which makes the numerical approximation of $\delta x^{1 / 2} u_{x x}$ be bounded in $L^{2}$. In contrast, the second-order approximation of the dispersive term has not such natural diffusive term and therefore we must add one artificially. The vanishing numerical viscosity term added to obtain the required stability corresponds to a numerical approximation of $\delta x^{\theta} u_{x x x x}$ with $1<\theta<10 / 9$. That is, the numerical scheme for which we prove the uniform stability may be only of order least than $10 / 9$. It is not a second-order scheme, but at least it is a scheme of higher-order (greater than 1). This uniform stability allows us to obtain the convergence in $L^{4}$ in space-time of the solution of the scheme, and the convergence in $L^{10 / 9}$ in space-time of the discretization of the nonlinearity. Therefore, using discrete multiplier techniques, the contradiction argument and the Holmgren's Uniqueness Theorem leads to have the uniform exponential decay rate of the discrete energy associated to the numerical approximation when de assumption (6) is verified. Additionally, to ensure unconditional stability, we choose for our numerical examples, a full implicit scheme discretization. The resulting system of equations can be approximately solved using the Newton method or a fixed point argument.

The outline of this paper is organized as follows. In Section 2, we briefly describe the known results of well-posedness, estimates and exponential decay of the energy for the solutions of the critical GKdV equation with damping (the continuous case). In Section 3 we describe the numerical methods, introducing the semi-discrete scheme of second-order in space, and a perturbation of this scheme by a vanishing numerical viscosity term of 4 th order. We present also some previous lemmas which we need for the next sections. The well-posedness, convergence, and uniform exponential decay of the energy, for the semi-discrete scheme is proved in Section 4. Finally, in Section 5 we present the full discrete scheme of second-order in space and time, and we give some numerical tests and illustrative examples.

## 2. PRELIMINARIES

Let us recall in this section, the main results of [3], detailing and commenting on some inaccuracies of this work. We start by stating the following existence result due to Rosier and Zhang [14].
Theorem 2.1 (See [14], Theorem 2.13)
Let $u_{0} \in L^{2}(0, L)$ and $T>0$ be given. Then, there exists $T^{*} \in(0, T]$ such that the problem (1)-(4) admits a unique solution $u \in C\left(\left[0, T^{*}\right] ; L^{2}(0, L)\right) \cap L^{2}\left(\left[0, T^{*}\right) ; H_{0}^{1}(0, L)\right)$.

In the next result we establish an a priori estimate obtained by Pazoto et al. [1].
Proposition 2.1 (see [1], Proposition 2.1)
Let $u$ be a solution of problem (1)-(4) obtained in Theorem 2.1. If the initial data satisfies $\left\|u_{0}\right\|_{L^{2}(0, L)}<\sqrt{\frac{3 \pi}{2}}$, then

$$
\begin{equation*}
\|u\|_{L^{2}\left(0, T ; H_{0}^{1}(0, L)\right)}^{2} \leqslant c \frac{\left\|u_{0}\right\|_{L^{2}(0, L)}^{2}}{\left[1-\frac{4}{9 \pi^{2}}\left\|u_{0}\right\|_{L^{2}(0, L)}^{4}\right]} \tag{7}
\end{equation*}
$$

where $c=c(T, L)$. Furthermore,

$$
\begin{equation*}
u_{t} \in L^{6 / 5}\left(0, T ; H^{-2}(0, L)\right) \tag{8}
\end{equation*}
$$

For the sake of completeness we state the stabilization result obtained in [3].
Theorem 2.2 (See [3], Theorem 3.3)
Let $u$ be the solution of problem (1)-(4) given by Theorem 2.1 and let $\Omega$ and $a=a(x)$ be as in (6). Then, for any $0<R<\sqrt{3 \pi / 2}$ and $T>0$, there exist positive constants $c=c(R, T)$ and $\mu=\mu(R)$, such that

$$
\begin{equation*}
E(t) \leqslant c\left\|u_{0}\right\|_{L^{2}(0, L)}^{2} e^{-\mu t} \tag{9}
\end{equation*}
$$

holds for all $t>0$ and $u_{0}$ satisfying $\left\|u_{0}\right\|_{L^{2}(0, L)} \leqslant R$.
From the numerical point of view, it is possible to find several numerical scheme with stabilization properties. Here we are interested in establishing a stabilization result for solutions of a high-order numerical scheme of (1)-(4) with an exponential decay rate uniform with respect to the mesh size.

## 3. DESCRIPTION OF THE SEMI-DISCRETE NUMERICAL SCHEME

This section is devoted to give a description of a family of conservative numerical scheme for the GKdV-4 equation. We introduce a semi-discrete scheme of second-order in space based on centered finite differences, and a variation of it by a vanishing numerical viscosity in order to obtain the desired stability and convergence similar to the case of first-order [1]. Consequently, we take here also recall some lemmata related to stability, introducing some variations and new ones adapted to second-order scheme. Let us introduce the discrete space

$$
X_{J}^{1}=\left\{u=\left(u_{0}, \ldots, u_{J}\right) \in \mathbb{R}^{J+1} \mid \text { with } u_{0}=0 \text { and } u_{J}=u_{J-1}=0\right\}
$$

$\left(D^{+} u\right)_{j}=\frac{u_{j+1}-u_{j}}{\delta x},\left(D^{-} u\right)_{j}=\frac{u_{j}-u_{j-1}}{\delta x}$, for $j=1, \ldots, J-1$, and $D=\frac{1}{2}\left(D^{+}+D^{-}\right)$the classical difference operators, where $\delta x$ is the space-step and $\delta t$ is the time-step, for $j=0, \ldots, J$, and $n=0, \ldots, N$. We also introduce the following inner products in $X_{J}$

$$
\begin{equation*}
(z, w)=\sum_{j=1}^{J-1} \delta x z_{j} w_{j}, \quad(z, w)_{x}=(z, x w)=\sum_{j=1}^{J-1} j \delta x^{2} z_{j} w_{j} \tag{10}
\end{equation*}
$$

for all $z, w \in \mathbb{R}^{J+1}$, and the norms $|z|=\sqrt{(z, z)}, \quad|z|_{x}=\sqrt{(z, z)_{x}}$, for all $z \in \mathbb{R}^{J+1}$. Additionally, we introduce the following $p-$ norms in $X_{J}$

$$
\begin{equation*}
|z|_{p} \doteq\left(\sum_{j=1}^{J-1} \delta x\left|z_{j}\right|^{p}\right)^{1 / p} \quad \text { and } \quad|z|_{\infty} \doteq \max _{j=1, \ldots, J-1}\left|z_{j}\right| \tag{11}
\end{equation*}
$$

for all $z, w \in \mathbb{R}^{J+1}$.

We describe now the semi-discrete numerical scheme. Denoting by $u_{j}(t)$ the approximate value of $u(j \delta x, t)$, solutions of the nonlinear problem (1)-(4), the approximation of the nonlinear problem (1)-(4) reads as the following system of ODEs :

$$
\begin{align*}
& \frac{d}{d t}\left[u_{j}\right]+\left(A^{(\theta)} u\right)_{j}+F(u)_{j}+a_{\delta} u_{j}=0, \quad j=1, \ldots, J-1  \tag{12}\\
& u(t) \in X_{J} \quad(\forall t>0), \quad u^{0}=\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_{0}(x) d x \quad(j=1, \ldots, J-1), \tag{13}
\end{align*}
$$

where $x_{j+\frac{1}{2}}=j+\frac{1}{2} \delta x$ and $x_{j}=j \delta x$. The matrix $A^{(\theta)} \in \mathbb{R}^{(J-1) \times(J-1)}$ is an approximation of order $\theta$ of the dispersive term $u_{x x x}$ and the linear convective term $u_{x}$. For instance, the first-order approximation in space considered by [1] was given by

$$
\begin{equation*}
A^{(1)}=D^{+} D^{+} D^{-}+D \tag{14}
\end{equation*}
$$

In our case where we are most interested in the scheme of second order, we take the central differences approximation as follows

$$
\begin{equation*}
A^{(2)}=D^{+} D D^{-}+D \tag{15}
\end{equation*}
$$

However, in a point of view of the analysis, we can not argue with this second-order approximation (15) in the same way as it was do it for the first order scheme [1]. Nevertheless, there is a way to correct the approximation (15), adding a vanishing numerical term of 4th order:

$$
\begin{equation*}
A^{(\theta)}=\delta x^{\theta} D^{-} D^{+} D^{+} D^{-}+D^{+} D D^{-}+D \tag{16}
\end{equation*}
$$

with $1<\theta<2$. Thus, we obtain a slightly more precise numerical scheme, and it is possible to prove with similar argument to those used by Pazoto et al. [1]. This idea of adding a vanishing numerical term to ensure the property of uniform exponential decay is not new. It was suggested for the wave equation with a vanishing numerical term of 2 nd order (see for example $[13,15]$ ).

The approximation of the damping function $a=a(x)$ is given by $a_{\delta}=\left(a_{j}\right)_{j=1}^{J-1} \in \mathbb{R}^{J-1}$, where each component $a_{j}$ is given by $a_{j}=\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} a(x) d x$.

The nonlinearity $u^{4} u_{x}$ in the equation (1) will be approximated by a nonlinear function $F(u)$, where $F: \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1}$ is chosen such that, it verifies the following conservation properties:

$$
\begin{align*}
(u, F(u)) & =0  \tag{17}\\
(u, F(u))_{x} & =-\frac{1}{6}|u|_{6}^{6}, \tag{18}
\end{align*}
$$

where the discrete inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{x}$ are defined in (10) and the discrete 6 -norm $|\cdot|_{6}$ is defined in (11). For this we choose the same function F introduced by Pazoto et al. [1] and characterized in the following Lemma:

## Lemma 3.1 (see [1], Lemma 3.1)

The function $F: \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1}$ defined by the following expression

$$
\begin{align*}
& F(u)_{j}=u_{j}^{4}(D u)_{j}-\frac{5}{2} u_{j}^{3}\left(D u^{2}\right)_{j} \\
& \quad+\frac{10}{3} u_{j}^{2}\left(D u^{3}\right)_{j}-\frac{5}{2} u_{j}\left(D u^{4}\right)_{j}+\left(D u^{5}\right)_{j} \tag{19}
\end{align*}
$$

for all $j=1, \ldots, J-1$, verifies the conservativity properties (17) and (18), for all $u \in X_{J}$.
On the other hand, we state the following identities associated to central differences:

## Lemma 3.2

For all $z \in X_{J}$, the following identities

$$
\begin{align*}
(D z, z)= & 0  \tag{20}\\
\left(D^{+} D D^{-} z, z\right)= & \frac{1}{2}\left(\left[\left(D^{-} z\right)_{1}\right]^{2}+\left[\left(D^{-} z\right)_{J}\right]^{2}\right)  \tag{21}\\
(D z, z)_{x}= & \frac{\delta x^{2}}{4}\left|D^{-} z\right|^{2}-\frac{1}{2}|z|^{2}+\frac{\delta x^{3}}{4}\left[\left(D^{-} z\right)_{J}\right]^{2}  \tag{22}\\
\left(D^{+} D D^{-} z, z\right)_{x}= & \frac{\delta x}{2}\left(\left[\left(D^{-} z\right)_{1}\right]^{2}+(J+1)\left[\left(D^{-} z\right)_{J}\right]^{2}\right) \\
& +\frac{1}{2}\left|D^{-} z\right|^{2}+|D z|^{2}-\frac{\delta x^{2}}{4}\left|D^{+} D^{-} z\right|^{2}  \tag{23}\\
\left(D^{-} D^{+} D^{+} D^{-} z, z\right)= & \frac{2}{\delta x}\left[\left(D^{-} z\right)_{J}\right]^{2}+\left|D^{+} D^{-} z\right|^{2}  \tag{24}\\
\left(D^{-} D^{+} D^{+} D^{-} z, z\right)_{x}= & -\left[\left(D^{-} z\right)_{1}\right]^{2}+(2 J+1)\left[\left(D^{-} z\right)_{J}\right]^{2}+\left|D^{+} D^{-} z\right|_{x}^{2} \tag{25}
\end{align*}
$$

Proof. Using the identity $\left(a^{2}-b^{2}\right)+(a-b)^{2}=2(a-b) a$ with $a=z_{j}, w_{j}$ and $b=z_{j-1}, w_{j-1}$, multiplying by 1 and $j \delta x$, and summing by parts over $j=1, \ldots, J-1$, we obtain (20)-(25).

## Remark 3.1

1. Let be the discrete sub-space of $X_{J}$ defined by:

$$
\begin{equation*}
\widetilde{X_{J}}=\left\{u \in X_{J} \mid \text { with } u_{2}=2 u_{1}\right\} \tag{26}
\end{equation*}
$$

the identity (24) and (25) can be reduced to

$$
\begin{align*}
\left(D^{-} D^{+} D^{+} D^{-} z, z\right) & =\left|D^{+} D^{-} z\right|_{2}^{2}  \tag{27}\\
\left(D^{-} D^{+} D^{+} D^{-} z, z\right)_{x} & =-\left[\left(D^{-} z\right)_{1}\right]^{2}+\left|D^{+} D^{-} z\right|_{x}^{2} \tag{28}
\end{align*}
$$

for all $z \in \widetilde{X_{J}}$.
2. The condition $u_{2}=2 u_{1}$ is equivalent to $\left[D^{+} D^{-} u\right]_{1}=0$, for all $u \in X_{J}$. In this sense, numerical scheme (12)-(13), with (16), (19), and $u^{n} \in \widetilde{X_{J}}$ is an approximation of

$$
\begin{aligned}
& u_{t}+\varepsilon u_{x x x x}+u_{x x x}+u^{4} u_{x}+u_{x}+a(x) u=0, \quad(0, L) \times(0,+\infty) \\
& u(0, t)=u(L, t)=0, \quad t \in(0,+\infty) \\
& u_{x}(L, t)=0, \quad t \in(0,+\infty) \\
& u_{x x}(0, t)=0, \quad t \in(0,+\infty) \\
& u(x, 0)=u_{0}(x), \quad x \in(0, L)
\end{aligned}
$$

with $\varepsilon=\delta x^{\theta}$.
3. Due to vanishing numerical viscosity term, the spatial approximation is not yet of secondorder, but it is of $\theta$-order, with $1<\theta<2$. Furthermore, we also note that the boundary conditions imposed by the discrete space $X_{J}$ are approximations of first-order. While there are second-order approximations for the boundary conditions required, the choice of $X_{J}$ space is needed to ensure the stability of the method. Whereupon, the method proposed here is a real improvement on the first-order scheme introduced in [1], to the extent that the boundary conditions are not relevant. This is justified for example if the solution is a soliton that it propagates in a domain large enough so that it rarely touch the boundaries.

To finish with this series of lemmata, we recall some inequalities corresponding to a discrete version of some well known Gagliardo-Nirenberg type inequalities (see [1]):

## Lemma 3.3 (see [1], Lemma 3.3)

For all $u \in X_{J}$,

$$
\begin{align*}
|u|_{\infty}^{2} & \leqslant 2|u|_{2}|D u|_{2}  \tag{29}\\
|u|_{4}^{4} & \leqslant 2|u|_{2}^{3}|D u|_{2}  \tag{30}\\
|u|_{\infty}^{2} & \leqslant 2|u|_{2}^{3 / 2}\left|D^{+} D^{-} u\right|_{2}^{1 / 2}  \tag{31}\\
\left|D^{-} u\right|_{p}^{p} & \leqslant 2^{\frac{p-1}{2}}|u|_{2}^{\frac{p+1}{4}}\left|D^{+} D^{-} u\right|_{2}^{\frac{3 p-1}{4}}, \tag{32}
\end{align*}
$$

for all $p \geqslant 2$.

## 4. CONVERGENCE AND WELL-POSEDNESS

Before stating the convergence results we recall the definition of the extension operator (see $[1,15])$ : for all $v \in \mathbb{R}^{J+1}$ with $v=\left(v_{j}\right)_{j=0}^{J}$, we define

$$
\begin{gathered}
p_{\delta} v_{\delta}=\left\{\begin{array}{l}
\text { the continuous function, linear in each interval } \\
{[j \delta x,(j+1) \delta x]} \\
\text { such that } p_{\delta} v_{\delta}(j h)=v_{j}, \quad j=0, \ldots, J .
\end{array}\right. \\
q_{\delta} v_{\delta}=\left\{\begin{array}{l}
\text { the step function defined in each interval } \\
{\left[\left(j-\frac{1}{2}\right) \delta x,\left(j+\frac{1}{2}\right) \delta x\right] \cap(0, L)} \\
\text { such that } \quad p_{\delta} v_{\delta}(j h)=v_{j}, \quad j=0, \ldots, J .
\end{array}\right.
\end{gathered}
$$

First, we give a well-posedness result for the solution of the numerical scheme (12)-(13).

## Proposition 4.1

Let $\boldsymbol{u}^{0}$ in $X_{J}$, for $\delta x>0$ fixed. Then there exists a unique solution $\boldsymbol{u}_{\delta} \in X_{J}$ of the numerical scheme (12)-(13), for some time interval ( $0, T$ ).
Proof. The numerical scheme (12)-(13) is a system of ordinary differential equations verifying all the hypotheses for the (local) existence and uniqueness: for a fixed $\delta x$, we have that
$u_{\delta} \mapsto A^{(\theta)} u_{\delta}+F\left(u_{\delta}\right)-a_{\delta} u_{\delta}$, is locally Lipschitz continuous. Thus, there is a time interval $(0, T)$ where we can ensure existence and uniqueness of the numerical solution.

Then, we give the following a priori estimates result

## Proposition 4.2

Let $t \mapsto u_{\delta}(t) \in X_{J}$ built by the numerical scheme (12)-(13), with $A^{(\theta)}$ defined by (16) and $F\left(u^{n}\right)$ defined by (19). If $\left|u^{0}\right|_{2} \leq \sqrt{\frac{3}{2}}$, then there exists a constant $C>0$ independent of $\delta x$ and $T$, such that

$$
\begin{align*}
\left\|q_{\delta} u_{\delta}\right\|_{L^{\infty}\left(0, T ; L^{2}(0, L)\right)} & \leqslant\left|u^{0}\right|_{2},  \tag{33}\\
\left\|p_{\delta} u_{\delta}\right\|_{L^{2}\left(0, T ; H_{0}^{1}(0, L)\right)}^{2} & \leqslant \frac{(T+L+\delta x)}{3\left(1-\frac{4}{9}\left|u^{0}\right|_{2}^{4}\right)}\left|u^{0}\right|_{2}^{2},  \tag{34}\\
\left\|q_{\delta} D^{+} D^{-} u_{\delta}\right\|_{L^{2}((0, T) \times(0, L))} & \leqslant \frac{\delta x^{-\theta / 2}}{\sqrt{2}}\left|u^{0}\right|_{2},  \tag{35}\\
\left\|\partial_{t}\left(p_{\delta} u_{\delta}\right)\right\|_{L^{10 / 9}\left(0, T ; H^{-2}(0, L)\right)} & \leqslant C . \tag{36}
\end{align*}
$$

Proof.

1) Estimates for $p_{\delta} u_{\delta}$ and $q_{\delta} u_{\delta}$.

First, we multiply the equation (12) by $u_{\delta}$, and sum over $j=1, \ldots, J-1$. Using (20), (21) and (24), we obtain

$$
\begin{aligned}
& \frac{d}{d t}\left|u_{\delta}\right|_{2}^{2}+\left[\left(D^{-} u_{\delta}\right)_{1}\right]^{2}+\delta x^{\theta}\left|D^{+} D^{-} u_{\delta}\right|_{2}^{2} \\
& +\underbrace{2\left(F\left(u_{\delta}\right), u_{\delta}\right)}_{=0}+2\left(a_{\delta} u_{\delta}, u_{\delta}\right)=0, \\
& \quad \begin{array}{c}
\text { from Lemma 3.1 } \\
\text { and (17) }
\end{array}
\end{aligned}
$$

Next, integrating over $(0, T)$, we obtain

$$
\begin{equation*}
\left|u_{\delta}(T)\right|_{2}^{2}+\int_{0}^{T}\left[\left(D^{-} u\right)_{1}\right]^{2} d t+\delta x^{\theta} \int_{0}^{T}\left|D^{+} D^{-} u_{\delta}\right|_{2}^{2} d t+2 \int_{0}^{T}\left(a_{\delta} u_{\delta}, u_{\delta}\right) d t=\left|u^{0}\right|_{2}^{2} \tag{37}
\end{equation*}
$$

then (33) follows. In addition, we deduce the following estimate for the 3rd term on the left hand side of (37)

$$
\begin{equation*}
\int_{0}^{T}\left|D^{+} D^{-} u_{\delta}\right|_{2}^{2} \leqslant \frac{\delta x^{-\theta}}{2}\left|u^{0}\right|_{2}^{2} \tag{38}
\end{equation*}
$$

which give us (35). In order to prove (34), we follow the same reasoning above. First, we multiply the equation (12) by $j \delta x u_{\delta}$ and sum over $j=1, \ldots, J-1$. Using (22), (23) and (25),
we obtain

$$
\begin{gathered}
\frac{d}{d t}\left|u_{\delta}\right|_{x}^{2}+\left|D^{-} u_{\delta}\right|_{2}^{2}+2\left|D u_{\delta}\right|_{2}^{2}+\delta x^{\theta}\left|D^{+} D^{-} u_{\delta}\right|_{x}^{2}-\delta x^{2}\left|D^{+} D^{-} u_{\delta}\right|_{2}^{2} \\
-\left|u_{\delta}\right|_{2}^{2}-\delta x\left[\left(D^{-} u_{\delta}\right)_{1}\right]^{2}+\underbrace{2\left(F\left(u^{n+1}\right), u^{n+1}\right)_{x}}_{=-\frac{1}{3}\left|u_{\delta}\right|_{6}^{6}}+2\left(a_{\delta} u_{\delta}, u_{\delta}\right)_{x}=0 \\
\text { from Lemma 3.1 } \\
\text { and (18) }
\end{gathered}
$$

Then, integrating over $(0, T)$, we get

$$
\begin{align*}
& \left|u_{\delta}(T)\right|_{x}^{2}+\int_{0}^{T}\left|D^{-} u_{\delta}\right|_{2}^{2} d t+2 \int_{0}^{T}\left|D u_{\delta}\right|_{2}^{2} d t \\
& \quad+\delta x^{\theta} \int_{0}^{T}\left|D^{+} D^{-} u_{\delta}\right|_{x}^{2} d t-\delta x^{2} \int_{0}^{T}\left|D^{+} D^{-} u_{\delta}\right|_{2}^{2} d t+2 \int_{0}^{T}\left(a_{\delta} u_{\delta}, u_{\delta}\right)_{x} d t \\
& \quad=\int_{0}^{T}\left|u_{\delta}\right|_{2}^{2} d t+\delta x \int_{0}^{T}\left[\left(D^{-} u_{\delta}\right)_{1}\right]^{2} d t+\frac{1}{3} \int_{0}^{T}\left|u_{\delta}\right|_{6}^{6} d t+\left|u^{0}\right|_{x}^{2} \tag{39}
\end{align*}
$$

Now, using (37) we can estimate the third term on the left hand side of (39) as follows

$$
\begin{align*}
& \int_{0}^{T}\left|D u_{\delta}\right|_{2}^{2} d t \leqslant \frac{1}{3} \int_{0}^{T}\left|D^{-} u_{\delta}\right|_{2}^{2} d t+\frac{2}{3} \int_{0}^{T}\left|D u_{\delta}\right|_{2}^{2} d t \\
& \quad \leqslant \frac{1}{3} \int_{0}^{T}\left|u_{\delta}\right|_{2}^{2} d t+\frac{\delta x}{3} \int_{0}^{T}\left[\left(D^{-} u_{\delta}\right)_{1}\right]^{2} d t+\frac{1}{9} \int_{0}^{T}\left|u_{\delta}\right|_{6}^{6} d t+\frac{1}{3}\left|u^{0}\right|_{x}^{2} \\
& \quad \leqslant \frac{(T+L+\delta x)}{3}\left|u^{0}\right|_{x}^{2}+\frac{1}{9} \int_{0}^{T}\left|u^{k}\right|_{6}^{6} d t \tag{40}
\end{align*}
$$

Moreover, combining (29), (30), and (33) we deduce that

$$
\begin{equation*}
\int_{0}^{T}\left|u_{\delta}\right|_{6}^{6} d t \leqslant 4\left|u_{0}\right|^{4} \int_{0}^{T}\left|D u_{\delta}\right|_{2}^{2} \tag{41}
\end{equation*}
$$

Replacing (41) in (40), and using the fact that $\left|D u_{\delta}\right|_{2}^{2} \leqslant\left|D^{-} u_{\delta}\right|_{2}^{2}$ the inequality (34) follows.
2) Estimate of the nonlinearity. Here, we treat the nonlinearity $F\left(u_{\delta}\right)$ as same as in [1]. Thus, due to the definition (19), we can write $F\left(u_{\delta}\right)_{j}$ as

$$
\begin{equation*}
F\left(u_{\delta}\right)=\frac{1}{5} D\left[\left(u_{\delta}\right)^{5}\right]+(N L)^{+}(t)+(N L)^{-}(t) \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
(N L)^{ \pm}(t)=\frac{\delta x}{60}\left(4 u_{\delta}^{3}-22 u_{\delta}^{2} \tau_{\delta}^{ \pm} u_{\delta}+27 u_{\delta}\left(\tau_{\delta}^{ \pm} u_{\delta}\right)^{2}-24\left(\tau_{\delta}^{ \pm} u_{\delta}\right)^{3}\right)\left(D^{ \pm}\left[u_{\delta}\right]\right)^{2} \tag{43}
\end{equation*}
$$

for all $u_{\delta} \in X_{J}$, where $\tau_{\delta}^{ \pm}$are the left and right shift operators: $\left(\tau_{\delta}^{+} u_{\delta}\right)_{j}=u_{j+1}$ and $\left(\tau_{\delta}^{-} u_{\delta}\right)_{j}=u_{j-1}$, for $j=1, \ldots, J-1$. On the other hand, using (41) we have

$$
\begin{equation*}
\int_{0}^{T} \sum_{j=1}^{J} \delta x\left|\left(u_{j}\right)^{5}\right|^{6 / 5} d t \leqslant 4\left|u_{0}\right|^{4} \int_{0}^{T}\left|D u^{k}\right|_{2}^{2} d t \tag{44}
\end{equation*}
$$

Therefore, from the estimates (33) and (34) we have that

$$
\left(q_{\delta} u_{\delta}\right)^{5} \quad \text { is bounded in } \quad L^{6 / 5}((0, T) \times(0, L))
$$

independent of $\delta x$. Moreover, since $L^{6 / 5}(0, L) \hookrightarrow H^{-1}(0, L)$ we deduce for the first term on the right hand side of (42) that

$$
\begin{equation*}
\frac{1}{5} q_{\delta} D\left[u_{\delta}^{5}\right] \quad \text { is bounded in } \quad L^{6 / 5}\left(0, T ; H^{-2}(0, L)\right) \tag{45}
\end{equation*}
$$

independently of $\delta x$.
Now, to estimate the nonlinearities (43), we use the discrete Gagliardo-Nirenberg type inequality (31) and the Hölder inequality to obtain

$$
\begin{align*}
\left(4 u_{\delta}^{3}\right. & \left.-22 u_{\delta}^{2} \tau_{\delta}^{ \pm} u_{\delta}+27 u_{\delta}\left(\tau_{\delta}^{ \pm} u_{\delta}\right)^{2}-24\left(\tau_{\delta}^{ \pm} u_{\delta}\right)^{3}\right)^{s} \\
& \leqslant 27^{s}\left|u_{\delta}(t)\right|_{\infty}^{3 s} \leqslant\left(2^{3 / 4} \times 27\right)^{s}\left|u_{\delta}(t)\right|_{2}^{9 s / 4}\left|D^{+} D^{-} u_{\delta}(t)\right|_{2}^{3 s / 4} \tag{46}
\end{align*}
$$

for $s \geqslant 1$. Then, using the discrete Gagliardo-Nirenberg type inequality (32), the following holds

$$
\begin{equation*}
\left|D^{ \pm} u_{\delta}(t)\right|_{2 s}^{2 s} \leqslant 2^{s-1}\left|u_{\delta}(t)\right|_{2}^{\frac{s+1}{2}}\left|D^{+} D^{-} u_{\delta}(t)\right|_{2}^{\frac{3 s-1}{2}} \tag{47}
\end{equation*}
$$

for $s \geqslant 1$. Replacing (46) and (47) in (43), summing over $j=1, \ldots, J$ and integrating over ( $0, T$ ), we deduce that

$$
\int_{0}^{T} \sum_{j=1}^{J} \delta x\left|(N L)_{j}^{ \pm}(t)\right|^{s} d t \leqslant C \delta x^{s} \max _{t \leqslant T}\left(\left|u_{\delta}(t)\right|_{2}^{\frac{11 s+2}{4}}\right) \int_{0}^{T}\left|D^{+} D^{-} u_{\delta}(t)\right|_{2}^{\frac{9 s-2}{4}} d t
$$

with $C=\frac{1}{2}\left(\frac{9}{5 \times 2^{1 / 4}}\right)^{s}$, independent of $\delta x$. Then, taking $s=10 / 9$ in the above inequality, we obtain

$$
\int_{0}^{T} \sum_{j=1}^{J} \delta x\left|(N L)_{j}^{ \pm}(t)\right|^{10 / 9} d t \leqslant C \delta x^{\frac{10}{9}}\left(\left|u^{0}\right|_{2}^{\frac{32}{9}}\right) \int_{0}^{T}\left|D^{+} D^{-} u_{\delta}(t)\right|_{2}^{2} d t
$$

Finally, using estimate (38), we conclude that

$$
\begin{equation*}
q_{\delta}\left(N L^{k}\right)_{\delta}^{ \pm} \quad \text { is in a compact of } \quad L^{10 / 9}((0, T) \times(0, L)) \tag{48}
\end{equation*}
$$

3) Estimate of $\left(p_{\delta} u_{\delta}\right)_{t}$. Now, we can obtain a bound for $\frac{\partial}{\partial t} P_{\delta} u_{\delta}$. Indeed, from (12) we have

$$
\frac{\partial}{\partial t} p_{\delta} u_{\delta}=-p_{\delta}\left(A^{(\theta)} u_{\delta}+F\left(u_{\delta}\right)-a_{\delta} u_{\delta}\right)
$$

estimates (34), (45) and (48) allow to conclude that

$$
\frac{\partial}{\partial t} p_{\delta} u_{\delta} \quad \text { is bounded in } \quad L^{10 / 9}\left(0, T: H^{-2}(0, L)\right)
$$

This complete the proof of Proposition 4.2.
Now, with the previously proven propositions 4.1 and 4.2 , the convergence result is as follows:

## Theorem 4.1

Let $t \mapsto u_{\delta}(t) \in X_{J}$ built by the numerical scheme (12)-(13), where $A^{(\theta)}$ defined by (16) is chosen such that $1<\theta<\frac{10}{9}$, and $F\left(u_{\delta}\right)$ is defined by (19). If $\left|u^{0}\right|_{2} \leq \sqrt{\frac{3}{2}}$, then, there exists a subsequence of $u_{\delta}$, not relabeled, such that

$$
\begin{equation*}
q_{\delta} u_{\delta} \rightarrow u \text { strongly in } L^{4}\left(0, T ; L^{4}(0, L)\right) \tag{49}
\end{equation*}
$$

as $\delta t, \delta x \rightarrow 0$, where $u$ is the weak solution of (1)-(4).

## Proof.

For the compactness result, thanks to the a priori estimates of Proposition 4.2, we can follow the same outlines of Pazoto et al. (see [1], proof of Theorem 4.1), in order to prove the strong convergence (49): since $\left\{u_{\delta}\right\}$ is bounded in $L^{2}\left(0, T ; H_{0}^{1}(0, L)\right) \cap L^{\infty}\left(0, T ; L^{2}(0, L)\right)$, by interpolation we can deduce that $\left\{u_{\delta}\right\}$ is bounded in

$$
\left[L^{q}\left(0, T ; L^{2}(0, L)\right), L^{2}\left(0, T ; H_{0}^{1}(0, L)\right)\right]_{\zeta}=L^{p}\left(0, T ;\left[L^{2}\left(0, T ; H_{0}^{1}(0, L)\right)\right]_{\zeta}\right)
$$

where $\frac{1}{p}=\frac{1-\zeta}{q}+\frac{\zeta}{2}$ and $0<\zeta<1$. Thus, choosing $q=\infty, \zeta=\frac{1}{2}$, we have $p=4$, and $\left[L^{2}(0, L), H_{0}^{1}(0, L)\right]_{\frac{1}{2}}=H^{1 / 2}(0, L)$. On the other hand, $H^{1 / 2}(0, L) \hookrightarrow L^{4}(0, L)$ is compact. Then, due to the estimate (36) and classical compactness results ([18], Corollary 4) we can extract a subsequence of $\left\{Q_{\delta^{\prime}} u_{\delta^{\prime}}\right\}$, still denoted by $\left\{Q_{\delta} u_{\delta}\right\}$, such that

$$
\begin{equation*}
q_{\delta} u_{\delta} \longrightarrow u \quad \text { strongly on } \quad L^{4}\left(0, T ; L^{4}(0, L)\right) \tag{50}
\end{equation*}
$$

thus (49) follows.
To show that $u$ is solution of (1)-(2) we need to pass to the limit. For that, let $\phi \in$ $C_{0}^{3}((0, T) \times(0, L))$ be a test function and set $\phi_{j}(t):=\phi\left(x_{j}, t\right), 0 \leqslant j \leqslant J$ and $0 \leqslant t \leqslant T$. We multiply the numerical scheme (12)-(13) by $\delta x \phi_{j}$, summing and integrating by parts, in order to obtain

$$
\begin{align*}
& \int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j} \partial_{t} \phi_{j} d t-\delta x^{\theta} \int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j} D^{-} D^{+} D^{+} D^{-} \phi_{j} d t \\
&+\int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j}\left(D^{+} D D^{-}+D\right) \phi_{j} d t \\
& \quad+\int_{0}^{T} \sum_{j=1}^{J-1} \delta x F(u)_{j} \phi_{j} d t+\int_{0}^{T} \sum_{j=1}^{J-1} \delta x a_{\delta} u_{j} \phi_{j} d t=0 \tag{51}
\end{align*}
$$

Due to the convergence result (50), it is easy now to pass the limit in the previous terms,
indeed,

$$
\begin{align*}
\int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j} \partial_{t} \phi_{j} d t- & \rightarrow \int_{0}^{T} \int_{0}^{L} u \partial_{t} \phi d x d t  \tag{52}\\
\delta x^{\theta} \int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j} D^{-} D^{+} D^{+} D^{-} \phi_{j} d t & \rightarrow 0  \tag{53}\\
\int_{0}^{T} \sum_{j=1}^{J-1} \delta x u_{j}\left(D^{+} D^{-} D^{-}+D\right) \phi_{j} d t & \rightarrow \int_{0}^{T} \int_{0}^{L} u\left(\phi_{x x x}+\phi_{x}\right) d x d t  \tag{54}\\
\int_{0}^{T} \sum_{j=1}^{J-1} \delta x a_{\delta} u_{j} \phi_{j} d t & \rightarrow \int_{0}^{T} \int_{0}^{L} a(x) u \phi d x d t  \tag{55}\\
\int_{0}^{T} \sum_{j=1}^{J-1} \delta x F\left(u_{\delta}\right)_{j} \phi_{j} d t & \rightarrow<\chi, \phi> \tag{56}
\end{align*}
$$

as $\delta x \rightarrow 0$, where $\chi \in \mathcal{D}^{\prime}((0, T) \times(0, L))$. Moreover, (45) allows us to take the limit of the first term on the right hand side of (42) to the limit as follow

$$
\begin{align*}
& \int_{0}^{T} \sum_{j=1}^{J-1} \delta x \frac{1}{5} D\left[u_{\delta}^{5}\right]_{j} \phi_{j} d t= \\
& \quad-\int_{0}^{T} \sum_{j=1}^{J-1} \delta x \frac{1}{5}\left(u_{\delta}\right)_{j}^{5} D\left[\phi_{j}\right]_{j} d t \rightarrow-\int_{0}^{T} \int_{0}^{L} u^{5} \phi_{x} d x d t \tag{57}
\end{align*}
$$

as $\delta x \rightarrow 0$. For the remaining terms in (42), we can use (48) to get

$$
\begin{align*}
\left|\int_{0}^{T} \sum_{j=1}^{J-1} \delta x\left(N L^{k}\right)_{j}^{ \pm} \phi_{j}^{k} d t\right| \leqslant & C \delta x^{\frac{10}{9}}\|\phi\|_{L^{10}\left(0, T ; L^{10}(0, L)\right)} \\
& \longrightarrow 0 \tag{58}
\end{align*}
$$

as $\delta x \rightarrow 0$. Replacing (57) and (58) in (56) we deduce that $\chi=\frac{1}{5}\left(u^{5}\right)_{x}$ in the sense of distribution, which is formally equivalent to $u^{4} u_{x}$. Due to (52)-(56), we have that $u$ verifies (1)-(4) in a weak sense (boundary and the initial condition are verified by sample arguments), and, consequently, $q_{\delta} u_{\delta}$ converges to the unique solution of (1)-(4).

To conclude this section, we give now a result of uniform exponential decay of energy, i.e. a decay of the energy independent of the size of the discretization $\delta x$. The energy associated to the system (12)-(13) is defined by

$$
\begin{equation*}
E_{\delta}(t)=\left|u_{\delta}\right|_{2}^{2}, \quad \text { for all } 0 \leqslant t \leqslant T \tag{59}
\end{equation*}
$$

This result on the uniform exponential decay of the energy for the solution of the numerical scheme reads as follows:

## Theorem 4.2

Let $u_{\delta}$ be the sequence in $X_{J}$ built by the numerical scheme (12)-(13) and let $\Omega$ and $a=a(x)$ be as in (6). Then, for any $0<R<\sqrt{3 / 2}$ and $T>0$, there exist positive constants $c=c(R, T)$ and $\mu=\mu(R)$, but both independent of $\delta x$, such that

$$
\begin{equation*}
E_{\delta}(t) \leqslant c\left|u^{0}\right|_{2}^{2} e^{-\mu t} \tag{60}
\end{equation*}
$$

holds for all $t>0$ and $u_{0}$ satisfying $\left|u^{0}\right|_{2} \leqslant R$.
Proof. The proof of this theorem is done using a classical argument of contradiction and the Holmgren's Uniqueness Theorem. Thus, thanks to the results of a priori estimates and strong convergence in $L^{4}$ previously proven (see Proposition 4.2, and Theorem 4.1), we can argue similar to Pazoto et al. (see [1], proof of Theorem 5.1). Then forward the details of the proof in this paper.

## 5. NUMERICAL EXAMPLES.

Now we will see the full discretization in space and time. For both the first-order scheme and the second-order scheme, we use an implicit method so as to maintain the unconditional stability property.

### 5.1. Full discrete implicit numerical schemes.

In order to discretize the system (12)-(13) in the temporal variable. We will denote by $u_{j}^{n}$ the approximate value of $u_{j}(n \delta t)$, solutions of the nonlinear system (12)-(13) for $n=0, \ldots, N$.
5.1.1. Implicit Euler. The simplest implicit numerical scheme is given by the implicit Euler of order one and reads as follows:

$$
\begin{align*}
& \frac{u_{j}^{n+1}-u_{j}^{n}}{\delta t}+\left(A^{(1)} u^{n+1}\right)_{j}+F\left(u^{n+1}\right)_{j}+a_{\delta} u_{j}^{n+1}=0, \quad j=1, \ldots, J-1  \tag{61}\\
& u_{0}^{n}=u_{J}^{n}=u_{J-1}^{n}=0,  \tag{62}\\
& u^{0}=\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_{0}(x) d x, \quad j=1, \ldots, J-1 \tag{63}
\end{align*}
$$

where $A^{(1)}$ is defined in (14) and $F(\cdot)$ and $a_{\delta}$ are defined in (19). Pazoto et al. [1] prove that this full discrete scheme have a solution which converges to the weak solution of (1)-(4). Moreover, these authors prove in [1], the stability of the method, the $L^{4}$-strong convergence of the numerical solution, and the uniform exponential decay of energy for this first order scheme.
5.1.2. Implicit Runge-Kutta of order 2. We consider here a temporal discretization by the 2stage Gauss-Legendre implicit Runge-Kutta method, which correspond to the table (for more details on definitions and deductions of these tables, see Butcher [17])


The numerical scheme is now specified, step by step, for $n=0,1, \ldots, N$. We seek $u_{j}^{n}$, by way of the intermediate stages $u_{j}^{n, \ell}$, for $\ell=1,2$, which are solution of the $2(J-1) \times 2(J-1)$ system of nonlinear equations

$$
\begin{align*}
& \frac{u_{j}^{n, \ell}-u_{j}^{n}}{\delta t}+\sum_{m=1}^{2} a_{\ell, m}\left[\left(A^{(\theta)} u^{n, m}\right)_{j}+F\left(u^{n, m}\right)_{j}+a_{\delta} u_{j}^{n, m}\right]=0, \\
& u_{0}^{n, \ell}=u_{J}^{n, \ell}=u_{J-1}^{n, \ell}=0, \quad \ell=1,2 \tag{66}
\end{align*}
$$

using the formula

$$
\begin{equation*}
u_{j}^{n+1}=u_{j}^{n}-\delta t \sum_{\ell=1}^{2} b_{\ell}\left[\left(A^{(\theta)} u^{n, \ell}\right)_{j}+F\left(u^{n, \ell}\right)_{j}+a_{\delta} u_{j}^{n, \ell}\right] \tag{67}
\end{equation*}
$$

where $A^{(\theta)}$ is defined in (16) and $F(\cdot)$ and $a_{\delta}$ are defined in (19). The application of this full implicit Runge-Kutta Method for the temporal discretization of KdV type equations is not new. It was applied before by Bona et al. in [4] using Finite Element Method for the space discretization. In our case, the nonlinear system (65) can be approximately solved using Newton method or a fixed point method. In this section we show some numerical examples to describe a helpful strategy to solve this nonlinear system.

The choice of full implicit schemes (implicit Euler or implicit Runge-Kutta scheme) is important in order to obtain unconditional stability of the method.

### 5.2. Computing strategy

The operators $A^{(\theta)}$ defined in (14), (15) and (16) are well defined as a linear application $X_{J} \rightarrow \mathbb{R}^{J+1}$, in the sense that we do not need additional points on the outside of $[0, L]$ to compute $A^{(\theta)} u$. Thus $A^{(\theta)}$ is represented by a penta-diagonal matrix of order $(J+1) \times(J+1)$ :

$$
A^{(\theta)} u=\left(\begin{array}{ccccccc}
\gamma_{1} & \varepsilon_{1} & \zeta_{1} & & & &  \tag{68}\\
\beta_{2} & \gamma_{2} & \varepsilon_{2} & \zeta_{2} & & 0 & \\
\alpha_{3} & \beta_{3} & \gamma_{3} & \varepsilon_{3} & \ddots & & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
& & & \ddots & \ddots & \ddots & \zeta_{n-2} \\
& 0 & & \ddots & \beta_{J-1} & \gamma_{J-1} & \varepsilon_{J-1} \\
& & & & \alpha_{J} & \beta_{J} & \gamma_{J}
\end{array}\right)
$$

where $\alpha_{i}, \beta_{i}, \gamma_{i}, \varepsilon_{i}$ and $\zeta_{i}$, for $i=1, \ldots, J$ are defined in table 5.2 for first-order and secondorder approximations.

In the case of the implicit Euler approximation, the nonlinear system (61) can be written as

$$
\begin{equation*}
(I+\mathcal{A}) u^{n+1}=u^{n}-\delta t F\left(u^{n+1}\right) \tag{69}
\end{equation*}
$$

where $\mathcal{A}=\delta t\left(\operatorname{diag}\left(a_{\delta}\right)+A^{(\theta)}\right)$. The nonlinear system (69) can be approximately solved using the Newton method or a fixed point method. The number of iterations is determined by a stop

| $\alpha_{i}$ | $\beta_{i}$ | $\gamma_{i}$ | $\varepsilon_{j}$ | $\zeta_{i}$ |
| :---: | :---: | :---: | :---: | :---: |
| $A^{(1)}$ | 0 | $-\frac{1}{\delta x^{3}}-\frac{1}{\delta x}$ | $\frac{3}{\delta x^{3}}(i \neq J)$ | $-\frac{3}{\delta x^{3}}+\frac{1}{\delta x}(i \neq J-1)$ |
|  |  |  | $\frac{1}{\delta x^{3}}-\frac{1}{\delta x}(i \neq J)$ | $-\frac{2}{\delta x^{3}}+\frac{1}{\delta x}(i=J-1)$ |
| $A^{(2)}$ | $-\frac{1}{2 \delta x^{3}}$ | $\frac{1}{2 \delta x^{3}}-\frac{1}{\delta x}(i=J)$ | 0 | $-\frac{1}{\delta x^{3}}+\frac{1}{\delta x}(i \neq J-1)$ |

Table I: Coefficients of the matrix $\mathcal{A}=A^{(1)}$ for the first-order approximation (14) and $\mathcal{A}=A^{(2)}$ for the second-order approximation (15).
criterion, within a tolerance of the error generally less than $10^{-6}$. Obviously, the number of iterations will depend on the example and on the size of $\delta t$. For the examples in this paper and this tolerance of the error, we observed that the number of iterations is not more than 4 or 5 .

In both cases, we have in each iteration to solve a linear system with a positive definite penta-diagonal matrix. Taking into account the structure of the matrix $(I+\mathcal{A})$ it is easy to apply a $L U$ decomposition based on a simple modification of the Thomas algorithm for a penta-diagonal matrix [19, 20].

In the case of the 2-stage Gauss-Legendre implicit Runge-Kutta method, the nonlinear system (65) can be written as

$$
\left\{\begin{array}{l}
\left(I+a_{11} \mathcal{A}\right) u^{n, 1}+a_{12} \mathcal{A} u^{n, 2}=u^{n}-\delta t a_{11} F\left(u^{n, 1}\right)-\delta t a_{12} F\left(u^{n, 2}\right) \\
a_{21} \mathcal{A} u^{n, 1}+\left(I+a_{22} \mathcal{A}\right) u^{n, 2}=u^{n}-\delta t a_{21} F\left(u^{n, 1}\right)-\delta t a_{22} F\left(u^{n, 2}\right)
\end{array}\right.
$$

where $a_{i j}$ are defined in the Butcher table (64), for $i, j=1,2$. We can rewrite this nonlinear system as an uncoupled system in its linear part as

$$
\left\{\begin{array}{l}
\left(12 I+6 \mathcal{A}+\mathcal{A}^{2}\right) u^{n, 1}=(12 I+2 \sqrt{3} \mathcal{A}) u^{n}-\delta t(3 I+\mathcal{A}) F\left(u^{n, 1}\right)-\delta t(3-2 \sqrt{3}) F\left(u^{n, 2}\right)  \tag{70}\\
\left(12 I+6 \mathcal{A}+\mathcal{A}^{2}\right) u^{n, 2}=(12 I-2 \sqrt{3} \mathcal{A}) u^{n}-\delta t(3+2 \sqrt{3}) F\left(u^{n, 1}\right)-\delta t(3 I+\mathcal{A}) F\left(u^{n, 2}\right)
\end{array}\right.
$$

Using now Newton method or a fixed point method, we have in each iteration to solve a linear system with an 9 -diagonal matrix $\left(12 I+6 \mathcal{A}+\mathcal{A}^{2}\right)$. Taking into account the structure of the 9 -diagonal matrix it is easy to apply again, a simple modification of the Thomas algorithm.

### 5.3. Example 1. Comparison with an exact soliton solution

An exact solution for the generalized KdV equation

$$
u_{t}+u_{x x x}+u^{p} u_{x}+u_{x}=0, \quad(0, L) \times(0,+\infty)
$$

with $x \in \mathbb{R}$ can be write as a traveling-wave solution (soliton) of the form

$$
\begin{equation*}
u(x, t)=\frac{\alpha}{\cosh ^{2 / p}\left[\beta p\left(x-\left(4 \beta^{2}+1\right) t-x_{0}\right)\right]} \tag{71}
\end{equation*}
$$



Figure 1: The exact solution and the numerical simulation for $t=100[$ sec. $], p=4$, without damping $(J=100000$ and $n=1000000)$. Left: Implicit Euler method; Right: 2-step GaussLegendre Implicit Runge-Kutta Method of order 2.

| $\delta t$ | $\delta x$ | $\left\\|u_{\delta}-u_{\text {exact }}\right\\|_{L^{\infty}\left(0, T, L^{2}(0, L)\right)}$ |  | $u_{\text {exact }} \\|_{L^{\infty}\left(0, T, L^{2}(0, L)\right)}$ |  | CPU |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Euler | RK-1 | RK-2 | Euler | RK-1 | RK-2 |  |
| $10^{-3}$ | $3.10^{-2}$ | 1.06 | 1.08 | 1.11 | 0.038 | 0.108 | 0.110 |  |
| $10^{-4}$ | $3.10^{-3}$ | $2.53 \times 10^{-1}$ | $2.44 \times 10^{-2}$ | $1.09 \times 10^{-1}$ | 2.33 | 7.39 | 9.66 |  |
| $10^{-5}$ | $3.10^{-4}$ | $4.00 \times 10^{-2}$ | $3.70 \times 10^{-2}$ | $3.21 \times 10^{-3}$ | 149.02 | 697.98 | 701.06 |  |
| $10^{-6}$ | $3.10^{-5}$ | $4.35 \times 10^{-3}$ | $3.99 \times 10^{-3}$ | $9.47 \times 10^{-5}$ | 18215. | 101294. | 102304. |  |

Table II: Comparison between numerical and exact solution for the Implicit Euler method, the Implicit Runge-Kutta method with first-order space discretization (RK-1), and the Implicit Runge-Kutta method with second-order space discretization (RK-2); relative error in norm $L^{\infty}\left(0, T, L^{2}(0, L)\right)$.
where $\alpha$ and $x_{0}$ are arbitrary constants and $\beta=\left[\frac{\alpha^{p}}{2(p+1)(p+2)}\right]^{1 / 2}$ (see for instance, Ablowitz and Segur [21]).

We consider the initial condition of the traveling wave solution (71) with $p=4$, that is

$$
\begin{equation*}
u_{0}(x)=\frac{\alpha}{\cosh ^{1 / 2}\left[4 \beta\left(x-x_{0}\right)\right]} \tag{72}
\end{equation*}
$$

and we make a simulation for $L=300.0 ; T=100 ; \alpha=1.0$ and $x_{0}=50 ; \delta t=T / n$ and $\delta x=L / J$ with different values of $J$ and $n$ (see Figure 1).

## Remark 5.1



Figure 2: Example 2. Perturbations of a soliton by homothetic transformation: (a) The function $T \mapsto \frac{1}{\sqrt{T+L}}\left\|P_{\delta} u_{\delta}\right\|_{L^{2}\left(0, T: H_{0}^{1}(0, L)\right)}$ for different $\eta$ values in formula (72); (b) Homothetic reduction of the initial condition; (c) Soliton without homothetic transformation; (d) Homothetic amplification of the initial condition.

We note here that the mass of the soliton is accumulated practically throughout the interval [ $-350,250]$, and is almost negligible outside it. Thus, the energy of $u$ on $[-350,250]$ can be reasonably approximated by the energy on the entire real line which results to be a calculable
integral. That is, the norm $L^{2}$ of the initial condition of a soliton is given by

$$
\begin{aligned}
\left\|u_{0}\right\|_{L^{2}(-350,250)} & =\left(\int_{-350}^{250} \frac{\alpha^{2} d x}{\cosh \left[\frac{2 \alpha^{2}}{\sqrt{15}}\left(x-x_{0}\right)\right]}\right)^{1 / 2} \\
& =15^{1 / 4} \sqrt{\frac{\pi}{2}}+\mathcal{O}\left(10^{-6}\right) \approx 2.4665
\end{aligned}
$$

which is greater than $\sqrt{3 \pi / 2} \approx 2.1708$, the critical level for the estimates in $H^{1}$ for the solution given by (7)) and consequently it is greater than $\sqrt{3 / 2} \approx 1.2247$, the critical level for the estimate of the numerical approximation given by (34). In this sense, this numerical example gives a successful rate of convergence, even outside the theoretical range that we needed in the previous sections to prove the uniform stability and convergence. Obviously, this does not rule out that may have examples where the uniform convergence and stability are not met.

### 5.4. Example 2. Perturbations of a soliton by homothetic transformation

In this example, we study the numerical behavior of the solution for a perturbation of the soliton like (71). Because we want to study the propagation of waves not touching the boundaries of the interval $(0, L)$ avoiding reflection numerical effects, and on the other hand, we do not want to use too large intervals in order to loose accuracy, we simplify our equation (1) as

$$
u_{t}+u_{x x x}+u^{4} u_{x}=0
$$

eliminating the linear convective term and taking $a(x) \equiv 0$, focusing on the effect of nonlinear convective term and dispersive term on the behavior of the solution under perturbation of the soliton. For that, we multiply the initial condition (72) by a parameter $\eta$, that is

$$
\begin{equation*}
u_{0}(x)=\frac{\eta \alpha}{\cosh ^{1 / 2}\left[4 \beta\left(x-x_{0}\right)\right]} \tag{73}
\end{equation*}
$$

and we make a simulation for $L=200.0 ; T=240 ; \alpha=1.0, \beta=\frac{1}{2 \sqrt{15}}$ and $x_{0}=100 ; \delta t=T / n$ and $\delta x=L / J$, with $J=10000, n=240000$, and different values of $\eta$. In Figure 2, we see the numerical results of these perturbations. Figure 2(a) shows the graphics of the function

$$
\begin{equation*}
T \mapsto \frac{1}{\sqrt{T+L}}\left\|p_{\delta} u_{\delta}\right\|_{L^{2}\left(0, T: H_{0}^{1}(0, L)\right)} \tag{74}
\end{equation*}
$$

for different $\eta$ values, where $p_{\delta}$ is defined in (33). The idea of dividing the norm $L^{2}(0, T$ : $\left.H_{0}^{1}(0, L)\right)$ of the numerical solution by $\sqrt{T+L}$ in $(74)$, is to ensure that the constant of the estimate (34) does not depend on $T$ or $L$, when the initial condition verify $\left|u^{0}\right|_{2} \leq \sqrt{3 / 2}$. We remark that this last condition is verified only when $\eta \leq \frac{(3 / 5)^{1 / 4}}{\sqrt{\pi}} \approx 0.49655$. On the other hand, Figure 2(a) shows that the assumption on small initial condition is not needed in this case. In fact, we obtain in our numerical tests that functions defined in (74) are bounded for $0 \leq \eta \leq 3.745$.

Figure 2 (b), (c) and (d) show the evolution of the waves solution for $\eta=0.1, \eta=1.0$ and $\eta=3.745$, respectively. In these 3 cases and also in all the intermediary cases, there is not blow up.

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