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On Completely Generalized Multi-valued Co-variational Inequalities Involving Strongly Accretive Operators

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Abstract

In this paper, we consider the completely generalized multi-valued co-variational inequality problems in Banach spaces and construct an iterative algorithm. We prove the existence of solutions for our problems involving strongly accerative operators and convergence of iterative sequences generated by the algorithm.

Key words: Co-variational inequality, algorithm, accretive map, m-accretive map, retraction map

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1. Introduction

The theory of variational inequalities provides us an unified frame work to deal with a wide class of problems arising in elasticity, structural analysis, economics, optimization, operations research, physical and engineering sciences, etc; see for example [1-2,5,9] and references therein.

In this paper, we consider a more general form of multi-valued variational inequalities problems in Banach spaces, called *completely generalized multi-valued co-variational inequality problem*. By extending the technique of Alber and Yao [4], we suggest an iterative algorithm for finding the approximate solution of our problem. The convergence of iterative sequences generated by our algorithm is studied. We also prove the existence of a solution of our problem.The main result of this paper(Theorem 4.1)is stated and established for uniformly smooth Banach spaces in section 4. Several special cases are also considered.

2. Preliminaries

Let B be a real Banach space with its dual B^* and $\langle x, f\rangle$ a pairing between $x \in B$ and $f \in B^*$. We denote by $C(B)$ and 2^B the family of non empty compact subsets of B and the family of nonempty subsets of B, respectively. Let $N(.,.) : B \times B \rightarrow B$, $G : B \rightarrow B$ be the nonlinear mappings, $T, A : B \rightarrow C(B)$ be the multivalued mappings, $K : B \rightarrow 2^B$ be a multivalued mapping such that $K(x)$ is a nonempty, closed and convex set for all $x \in B$. We consider the following *completely generalized multi-valued co-variational inequality problem* :

$$\text{(CGMCVIP)} \quad \begin{cases} \text{Find } x \in B,\ u \in T(x),\ \text{and } v \in A(x) \\ \text{ such that } G(x) \in K(x) \text{ and} \\ \langle N(u,v), J(z - G(x))\rangle \geq 0, \quad \forall\, z \in K(x), \end{cases}$$

where $J : B \rightarrow B^*$ is the normalized duality operator.

As an application of (CGMCVIP), we consider an elastoplasticity problem, which is mainly due to Panagiotopoulos and Stavroulakis [11].

Example 2.1. Let a general hyperelastic material law holds for the elastic behaviour of the elastoplastic material under consideration. Let us assume the decomposition

$$E = E^e + E^p,$$

where E^e denotes the elastic and E^p denotes the plastic deformation of three-dimensional elastoplastic body. We write the complementary virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma\rangle + \langle E^p, \tau - \sigma\rangle = \langle f, \tau - \sigma\rangle, \quad \text{for all } \ \tau \in Z.$$

Here we have assumed that the body on a part Γ_U of its boundary Γ has given displacements, that is , $\mu_i = U_i$ on Γ_U and that on the rest of its boundary $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on Γ_F, where the following energy inner products appear:

$$\langle E, \sigma\rangle = \int_\Omega \varepsilon_{ij}\sigma_{ij} d\Omega$$

$$\langle f, \sigma\rangle = \int_{\Gamma_U} U_i S_i d\Gamma$$

$$Z = \{\tau : \tau_{i_j,j} + f_i = 0 \ \text{ on } \Omega, \ \ i,j = 1,2,3, \ \ T_i = F_i \ \text{ on } \Gamma_F, \ \ i = 1,2,3\},$$

is the set of statically admissible stresses and Ω is the structure of the body.

Let us assume that the material of the structure Ω is hyperelastic such that

$$\langle E^e(\sigma), \tau - \sigma\rangle \leq \langle W_m'(\sigma), \tau - \sigma\rangle, \qquad \text{for all} \;\; \tau \in \Re^6,$$

where W_m is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasidifferentiable, that is, there exist convex and compact subsets $\underline{\partial}' W_m$ amd $\bar{\partial}' W_m$ such that

$$\langle W_m'(\sigma), \tau - \sigma\rangle = \max_{W_1^e \in \underline{\partial}' W_m} \langle W_1^e, \tau - \sigma\rangle + \max_{W_2^e \in \; \bar{\partial}' W_m} \langle W_2^e, \tau - \sigma\rangle.$$

We also introduce the generally nonconvex yield function $P \subset Z$, which is defined by means of general quasidifferentiable function $F(\sigma)$, that is,

$$P = \{\sigma \in Z; F(\sigma) \leq 0.$$

Here W_m is a generally nonconvex and nonsmooth, but quasidifferentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining these facts, Panagiotopoulos and Stavroulakis [11] have obtained the following multivalued variational inequality problem:

Find $\sigma \in Z$, $W_1^e \in \underline{\partial}' W_m(\sigma)$, $W_2^e \in \bar{\partial}' W_m(\sigma)$ such that

$$\langle W_1^e + W_2^e, \tau - \sigma\rangle \geq \langle f, \tau - \sigma\rangle, \qquad \text{for all} \;\; \tau \in Z,$$

which is exactly the problem (CGMCVIP), with $u = W_1^e$, $v = W_2^e$, $N(u,v) = \underline{\partial}' W_m(\sigma) + \bar{\partial}' W_m(\sigma)$, $J = I$, $f = 0$, $K(x) = K$, $G = I$, $T(x) = \underline{\partial}' W_m(\sigma)$, $A(x) = \bar{\partial}' W_m(\sigma)$ and $B = Z$.

Recall that the normalized duality operator $J : B \to B^*$ is defined for arbitrary Banach space by the condition

$$\| Jx \|_{B^*} = \| x \| \;\text{ and }\; \langle x, Jx\rangle = \| x \|^2, \quad \forall\, x \in B.$$

Some examples and properties of the mapping J can be found in [3].

Special Cases

(I) If T is a single-valued nonlinear operator and $N(u,v) = Tx + Av$, then (CGMCVIP) is equivalent to find $x \in B$, $v \in A(x)$ such that $G(x) \in K(x)$ and

$$\langle Tx + Av, \; J(z - G(x))\rangle \;\; \geq 0, \quad \text{for all } z \in K(x) \tag{2.1}$$

Problem (2.1) is called *generalized multi-valued co-variational inequality*, recently considered and studied by Alber and Yao [4].

(II) When B is a Hibert space, J reduces to the identity mapping. Consequently, problem (2.1) reduces to the following problem: Find $x \in B$, $v \in A(x)$ such that $G(x) \in K(x)$ and

$$\langle Tx + Av,\ z - G(x)\rangle \ \geq 0, \quad \forall\, z \in K(x) \tag{2.2}$$

Problem (2.2) is called *generalized multi-valued variational inequality* introduced and studied by Jou and Yao [10].

It is clear, from these special cases that our problem (2.1) is more general than the problem considered in [4] and generalizes many problems in the literature. See, e.g., [8], [13].

We first recall that the uniform convexity of the space B means that for any given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B$, $\| x \| \leq 1$, $\| y \| \leq 1$, $\| x - y \| = \epsilon$, the following inequality

$$\| x + y \| \ \leq 2(1 - \delta)$$

holds. The function

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{\| x + y \|}{2} : \ \| x \| = 1, \ \| y \| = 1, \ \| x - y \| = \epsilon\right\}$$

is called the modulus of the convexity of the space B.

The uniform smoothness of the space B means that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\| x + y \| + \| x - y \|}{2} - 1 \leq \epsilon \| y \|$$

holds. The function

$$\rho_B(t) = \sup\left\{\frac{\| x + y \| + \| x - y \|}{2} - 1 : \ \| x \| = 1, \ \| y \| = t\right\}$$

is called the modulus of the smoothness of the space B.

We observe that the space B is a uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$ and it is uniformly smooth if and only if $\lim\limits_{t \to 0} t^{-1}\rho_B(t) = 0$.

Remark 2.1. All Hilbert spaces, L_p (or l_p) spaces $(p \geq 2)$ and the Sobolev spaces $W_m^p (p \geq 2)$ are two uniformly smooth, while, for $1 < p \leq 2$, L_p (or l_p) and $W_m^p (p \geq 2)$

spaces are p-uniformly smooth.

The following inequalities will be used in the proof of our main result and the proof of these inequalities can be found, e.g. in [3], and hence, we omit it.

Proposition 2.1. Let B be a uniformly smooth Banach space and J the normalized duality mapping from B to B^*. Then, for all $x, y \in B$, we have

(i) $\| x + y \|^2 \leq \| x \|^2 + 2\langle y, J(x+y)\rangle$,
(ii) $\langle x - y, Jx - Jy\rangle \leq 2d^2 \rho_B(4 \| x - y \| / d)$, where $d = \sqrt{(\| x \|^2 + \| y \|^2)/2}$.

Let us recall the following definitions.

Definition 2.1. The mapping $G : B \to B$ is said to be strongly accretive if there exist a constant $\gamma > 0$ such that

$$\langle Gx - Gy, J(x-y)\rangle \geq \gamma \| x - y \|^2, \quad \text{for all} \quad x, y \in B.$$

Definition 2.2. Let $T, A : B \to C(B)$ be two multivalued mappings, $N(.,.) : B \times B \to B$ be a nonlinear mapping.

The mapping $u \mapsto N(u, v)$ is said to be strongly accretive with respect to the mapping T, if for any $x_1, x_2 \in B$ there exists a constant $t > 0$ such that for any $u_1 \in T(x_1),\ u_2 \in T(x_2)$ and any $v \in A(x)$,

$$\langle N(u_1, v) - N(u_2, v), J(x_1 - x_2)\rangle \geq t \| x_1 - x_2 \|^2 .$$

Remark 2.2. If T, A are single-valued mappings and $N(T(x), A(x)) = G(x)$, then Definition 2.2 reduces to Definition 2.1.

Definition 2.3. The mapping $N(.,.) : B \times B \to B$ is said to be Lipschitz continuous with respect to first argument, if there exists a constant $\beta > 0$ such that

$$\| N(u_1, .) - N(u_2, .) \| \leq \beta \| u_1 - u_2 \| .$$

Definition 2.4. The mapping $A : B \to C(B)$ is said to be H-Lipschitz continuous if there exists a positive constant η such that

$$H(A(x), A(y)) \leq \eta \| x - y \|, \quad \forall \ x, y \in B.$$

where $H(.,.)$ is the Hausdorff metric on $C(B)$.

Let B be a real Banach space and Ω a nonempty closed convex subset of B.

Definition 2.5.[6,7,12] A mapping $Q_\Omega : B \to \Omega$ is said to be

(i) retraction on Ω if $Q_\Omega^2 = Q_\Omega$;
(ii) nonexpansive retraction on Ω if it satisfies the inequality

$$\| Q_\Omega x - Q_\Omega y \| \leq \| x - y \|, \ \forall \ x, y \in B$$

(iii) sunny retraction on Ω if for all $x \in B$ and for all $-\infty < t < \infty$

$$Q_\Omega(Q_\Omega x + t(x - Q_\Omega x)) = Q_\Omega x$$

We have the following characterization of a sunny nonexpensive retraction mapping.

Proposition 2.2.[7] Q_Ω is a sunny nonexpansive retraction if and only if for all $x \in B$ and for all $y \in \Omega$

$$\langle x - Q_\Omega x, \ J(Q_\Omega x - y)\rangle \geq 0.$$

Proposition 2.3.[4] Let B be a Banach space, Ω a nonempty closed and convex subset of B, $m = m(x) : B \to B$ and $Q_\Omega : B \to \Omega$ be a sunny nonexpensive retraction. Then for all $x \in B$, we have

$$Q_{\Omega + m(x)} x = m(x) + Q_\Omega(x - m(x))$$

3. Iterative Algorithm

In this section, we first give some characterizations of solutions of (CGMCVIP).

Theorem 3.1. Let B be a Banach space, $T, A : B \to C(B)$, $N(.,.) : B \times B \to B$, $G : B \to B$, $Q_\Omega : B \to \Omega$ be a sunny nonexpensive retraction and $K : B \to 2^B$ such that $K(x)$ is nonempty closed convex subset for all $x \in B$.

Then the following statements are equivalent:
(i) $x \in B, \ u \in T(x), \ v \in A(x)$ are solutions of (CGMCVIP);
(ii) $x \in B, \ u \in T(x), \ v \in A(x)$ and $Gx = Q_{K(x)}(Gx - \tau(N(u, v)))$ for any $\tau > 0$.

Proof. For the proof, we refer to [4] and references mentioned therein.

By combining Proposition 2.3 and Theorem 3.1, we have the following theorem.

Theorem 3.2. Let B be a Banach space, X a nonempty closed convex subset of B. Let $T, A : B \to C(B)$, $N(.,.) : B \times B \to B$, $G : B \to B$, $Q_\Omega : B \to \Omega$ be a sunny nonexpensive retraction and $K : B \to 2^B$ such that $K(x) = m(x) + X$ for all $x \in B$. Then $x \in B,\ u \in T(x),\ v \in A(x)$ are solutions of (CGMCVIP) if and only if

$$x = x - Gx + m(x) + Q_X(Gx - \tau(N(u,v)) - m(x)), \quad \text{for any} \quad \tau > 0.$$

Algorithm 3.1. We now construct the algorithm for finding approximate solutions of (CGMCVIP). Let $K(x) = m(x) + X$, where X is a nonempty closed convex subset of B and $\tau > 0$ be fixed.

Given $x_0 \in B$, take any $u_0 \in T(x_0),\ v_0 \in A(x_0)$ and let

$$x_1 = x_0 - Gx_0 + m(x_0) + Q_X(Gx_0 - \tau(N(u_0, v_0)) - m(x_0)).$$

since $T(x_0)$ and $A(x_0)$ are nonempty and compact sets, there exist $u_1 \in T(x_1),\ v_1 \in A(x_1)$ such that

$$\| u_0 - u_1 \| \leq H(T(x_0), T(x_1))$$

$$\| v_0 - v_1 \| \leq H(A(x_0), A(x_1))$$

Let

$$x_2 = x_1 - Gx_1 + m(x_1) + Q_X(Gx_1 - \tau(N(u_1, v_1)) - m(x_1))$$

By induction, we can obtain sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$

and

$$x_{n+1} = x_n - Gx_n + m(x_n) + Q_X(Gx_n - \tau(N(u_n, v_n)) - m(x_n)), \tag{3.1}$$

$$\begin{aligned} u_n \in T(x_n),\ \| u_n - u_{n+1} \| &\leq H(T(x_n), T(x_{n+1})) \\ v_n \in A(x_n),\ \| v_n - v_{n+1} \| &\leq H(A(x_n), A(x_{n+1})), \end{aligned}$$

$n = 0, 1, 2 \cdots$

4. Convergence Theory

We apply the Algorithm 3.1 to prove the following convergence and existence result.

Theorem 4.1. Let B be a uniformly smooth Banach space with the module of smoothness $\rho_B(t) \leq Ct^2$ for some $C > 0$. Let X be a closed convex subset of B, $N(.,.) : B \times B \to B$ be a bifunction, $T, A : B \to C(B)$ be the multivalued mappings, $G, m : B \to B$

be single-valued mappings. Let $Q_\Omega : B \to \Omega$ be a sunny nonexpensive retraction, $K : B \to 2^B$ be a multivalued mapping such that $K(x) = m(x) + X$ for all $x \in B$. Suppose that the following conditions are satisfied:

(i) $N(.,.)$ is strongly accretive with respect to mappings T and A with corresponding constants $t > 0$, $s > 0$; Lipschitz continuous in both the arguments with corresponding constants $\beta > 0$ and $\alpha > 0$,

(ii) G is both strongly accretive with constant $\gamma > 0$ and Lipschitz continuous with constant $\delta > 0$,

(iii) m is Lipschitz continuous with constant $\theta > 0$,

(iv) T and A are H-Lipschitz continuous with constant $\xi > 0$ and $\eta > 0$, respectively,

(v) $0 < 2(1 - 2\gamma + 64C\delta^2)^{\frac{1}{2}} + 2\theta + (1 - 2\tau(t + s) + 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))^{\frac{1}{2}} < 1.$

Then there exist $x \in B$, $u \in T(x)$ and $v \in A(x)$ which are solutions of (CGMCVIP) and the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by the algorithm 3.1 converge strongly to x, u and v, respectively i.e. $x_n \to x$, $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Proof. By the iterative scheme (3.1) and Proposition 2.3, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|x_n - Gx_n + m(x_n) + Q_X(Gx_n - \tau(N(u_n, v_n)) - m(x_n)) - (x_{n-1} \\
&\quad -Gx_{n-1} + m(x_{n-1}) - Q_X(Gx_{n-1} - \tau(N(u_{n-1}, v_{n-1})) - m(x_{n-1})))\| \\
&\leq \|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| + \|x_n - x_{n-1} \\
&\quad -(Gx_n - Gx_{n-1})\| + \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\| \\
&= 2\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\| + 2\|m(x_n) - m(x_{n-1})\| \\
&\quad +\|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\| \qquad (4.1)
\end{aligned}$$

By proposition (2.1), we have

$$\begin{aligned}
&\|x_n - x_{n-1} - (Gx_n - Gx_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\langle Gx_n - Gx_{n-1}, J(x_n - x_{n-1} - (Gx_n - Gx_{n-1}))\rangle \\
&= \|x_n - x_{n-1}\|^2 - 2\langle Gx_n - Gx_{n-1}, J(x_n - x_{n-1})\rangle - 2\langle Gx_n - Gx_{n-1}, \\
&\qquad J(x_n - x_{n-1} - (Gx_n - Gx_{n-1})) - J(x_n - x_{n-1})\rangle \\
&\leq \|x_n - x_{n-1}\|^2 - 2\gamma\|x_n - x_{n-1}\|^2 + 4d^2\rho_B\left(\frac{4\|Gx_n - Gx_{n-1}\|}{d}\right) \\
&\leq \|x_n - x_{n-1}\|^2 - 2\gamma\|x_n - x_{n-1}\|^2 + 64C\|Gx_n - Gx_{n-1}\|^2 \\
&\leq (1 - 2\gamma + 64C\delta^2)\|x_n - x_{n-1}\|^2. \qquad (4.2).
\end{aligned}$$

By Proposition 2.1, we have

$$\|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\|^2$$

$$\begin{aligned}
\leq\ & \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), \\
& J(x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1})))\rangle \\
=\ & \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), \\
& J(x_n - x_{n-1})\rangle - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1} \\
& -\tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))) - J(x_n - x_{n-1})\rangle \\
=\ & \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n) \\
& +N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1})\rangle \\
& -2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1} \\
& -\tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))) - J(x_n - x_{n-1})\rangle \\
=\ & \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n), \\
& J(x_n - x_{n-1})\rangle - 2\tau\langle N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), \\
& J(x_n - x_{n-1})\rangle - 2\tau\langle (N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1} \\
& -\tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))) - J(x_n - x_{n-1})\rangle. \qquad (4.3)
\end{aligned}$$

Since N is strongly accretive with respect to the mappings T and A, we have

$$\begin{aligned}
& \langle N(u_n, v_n) - N(u_{n-1}, v_n), J(x_n - x_{n-1})\rangle + \langle N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1})\rangle \\
\geq\ & (t + s)\|x_n - x_{n-1}\|^2. \qquad (4.4)
\end{aligned}$$

Using (4.4) and (ii) of Proposition 2.1, (4.3) becomes

$$\begin{aligned}
& \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\tau(t + s)\|x_n - x_{n-1}\|^2 \\
& +4d^2\rho_B\left(\frac{4\tau^2\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|}{d}\right). \qquad (4.5)
\end{aligned}$$

Using Lipschitz continuity of N in both the arguments and Algorithm 3.1, we estimate the following

$$\begin{aligned}
& 4d^2\rho_B\left(\frac{4\tau^2\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|}{d}\right) \\
=\ & 4d^2\rho_B\left(\frac{4\tau^2}{d}(\|N(u_n, v_n) - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|)\right) \\
\leq\ & 4d^2\rho_B\left(\frac{4\tau^2}{d}(\|N(u_n, v_n) - N(u_n, v_{n-1})\| + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|)\right) \\
\leq\ & 64C\tau^3(\|N(u_n, v_n) - N(u_n, v_{n-1})\|^2 + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|^2) \\
\leq\ & 64C\tau^3(\alpha^2\|v_n - v_{n-1}\|^2 + \beta^2\|u_n - u_{n-1}\|^2)
\end{aligned}$$

$$\le 64C\tau^3(\alpha^2 H^2(A(x_n), A(x_{n-1})) + \beta^2 H^2(T(x_n), T(x_{n-1})))$$
$$\le 64C\tau^3(\alpha^2\eta^2\|x_n - x_{n-1}\|^2 + \beta^2\xi^2\|x_n - x_{n-1}\|^2)$$
$$= 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))\|x_n - x_{n-1}\|^2. \quad (4.6)$$

It is clear from the Lipschitz continuity of m that

$$\| m(x_n) - m(x_{n-1}) \| \le \theta \| x_n - x_{n-1} \| \quad (4.7)$$

From (4.2)-(4.7), we have the following inequality:

$$\| x_{n+1} - x_n \| \le k \| x_n - x_{n-1} \|$$

where $k = 2(1-2\gamma+64C\delta^2)^{\frac{1}{2}} + 2\theta + (1-2\tau(t+s)+64C\tau^3(\alpha^2\eta^2+\beta^2\xi^2))^{\frac{1}{2}}$ and $0 < k < 1$ by (v).

Consequently, $\{x_n\}$ is a Cauchy sequence, and thus, converges to some $x \in B$. Now we prove that $u_n \to u \in T(x)$ and $v_n \to v \in A(x)$. From Algorithm 3.1, we have

$$\| u_{n+1} - u_n \| \le H(T(x_{n+1}), T(x_n)) \le \xi \| x_{n+1} - x_n \|$$

and

$$\| v_{n+1} - v_n \| \le H(A(x_{n+1}), A(x_n)) \le \eta \| x_{n+1} - x_n \|$$

which imply that the sequence $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in B. Let $u_n \to u$ and $v_n \to v$. Since Q_X, G, T, A, $N(.,.)$ and m are continuous in B, we have

$$x = x - Gx + m(x) + Q_X(G_X - \tau(N(u,v)) - m(x)).$$

It remains to show that $u \in T(x)$ and $v \in A(x)$. In fact,

$$\begin{aligned} d(u, T(x)) &= \inf\{\| u - w \| : w \in T(x)\} \\ &\le \| u - u_n \| + d(u_n, T(x)) \\ &\le \| u - u_n \| + H(T(x_n), T(x)) \\ &\le \| u - u_n \| + \xi \| x_n - x \| \to 0 \end{aligned}$$

Hence $d(u, T(x)) = 0$ and therefore $u \in T(x)$. Similarly, we can prove that $v \in A(x)$. The result then follows from Theorem 3.2.

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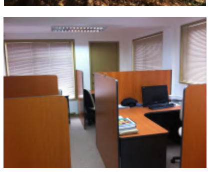

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