UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Augmented mixed finite element methods for a curl-based formulation of the two-dimensional Stokes problem

Gabriel N. Gatica, Luis F. Gatica,
Antonio Marquez

PREPRINT 2009-02

SERIE DE PRE-PUBLICACIONES

Augmented mixed finite element methods for a curl-based formulation of the two-dimensional Stokes problem[†]

Gabriel N. Gatica[1]* Luis F. Gatica[2] Antonio Márquez[3]

[1] *CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email:* ggatica@ing-mat.udec.cl

[2] *Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, email:* lgatica@ucsc.cl

[3] *Departamento de Construcción e Ingeniería de Fabricación, Universidad de Oviedo, Oviedo, España, email:* amarquez@uniovi.es

SUMMARY

In this paper we consider an augmented curl-based mixed formulation of the Stokes problem in the plane, and then introduce and analyze stable mixed finite element methods to solve the associated Galerkin scheme. In this way, we further extend similar procedures applied recently to linear elasticity and to other mixed formulations for incompressible fluid flows. Indeed, our approach is based on the introduction of the Galerkin least-squares type terms arising from the corresponding constitutive and equilibrium equations, and from the Dirichlet boundary condition for the velocity, all of them multiplied by stabilization parameters. Then, we show that these parameters can be suitably chosen so that the resulting operator equation induces a strongly coercive bilinear form, whence the associated Galerkin scheme becomes well posed for any choice of finite element subspaces. In particular, we can use continuous piecewise linear velocities, piecewise constant pressures, and rotated Raviart-Thomas elements for the stresses. Next, we derive reliable and efficient residual-based a posteriori error estimators for the augmented mixed finite element schemes. In addition, several numerical experiments illustrating the performance of the augmented mixed finite element methods, confirming the properties of the a posteriori estimators, and showing the behaviour of the associated adaptive algorithms are reported. The present work should be considered as a first step aiming finally to derive augmented mixed finite element methods for curl-based formulations of the three-dimensional Stokes problem.

KEY WORDS: mixed finite element; incompressible fluids

*Correspondence to: Gabriel N. Gatica

[†]This research was partially supported by FONDAP and BASAL projects CMM, Universidad de Chile, by Centro de Investigación en Ingeniería Matemática (CI²MA), Universidad de Concepción, and by Conicyt-Chile through the Fondecyt project 1080392.

Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

A new class of augmented mixed finite element methods for the dual-mixed variational formulations of several boundary value problems in continuum mechanics has been derived in recent papers (see [3], [7], [8], [10], [11], and [12]). A common feature of these methods is given by the fact that they are all based on the introduction of suitable Galerkin least-squares terms arising from the corresponding constitutive and equilibrium equations, from the relations among the different unknowns, and from the boundary conditions. In addition, the above mentioned works deal with both 2D and 3D problems, and the main results include a priori and a posteriori error estimates.

In the first work of this serie (cf. [10]), the plane linear elasticity problem with homogeneous Dirichlet boundary conditions was considered. In this case the application of the first Korn's inequality (cf. Theorem 10.1 in [14]) allows to show that the bilinear form arising from the augmented formulation becomes strongly coercive, whence arbitrary finite element subspaces can be utilized in the associated Galerkin scheme. In particular, Raviart-Thomas spaces of lowest order for the stress tensor, piecewise linear elements for the displacement, and piecewise constants for the rotation can be used. The extension of the results in [10] to the case of non-homogeneous Dirichlet boundary conditions was provided in [11]. The corresponding analysis requires the incorporation into the augmented formulation of an additional consistency term, which is defined precisely in terms of the Dirichlet boundary condition. As a consequence of it, the strong coerciveness of the resulting bilinear form follows now from a modified Korn's inequality. Then, the results from [10] and [11] were extended in [12] to three-dimensional linear elasticity problems, whereas a residual based a posteriori error analysis yielding a reliable and efficient estimator for the augmented method from [10], was given in [3].

On the other hand, the results from [10], [11], and [3] were generalized recently in [7] and [8] to the case of incompressible fluid flows with symmetric and non-symmetric stress tensors, respectively. More precisely, the a priori and a posteriori error analyses of augmented mixed finite element methods for a velocity-pressure-stress-rotation formulation of the stationary Stokes equations are provided in [7]. Besides the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, the corresponding formulation makes use of the relations defining the pressure in terms of the symmetric stress tensor and the rotation in terms of the displacement, all them multiplied by stabilization parameters. Alternatively, just a velocity-pressure-stress formulation of the Stokes problem, with a non-symmetric stress, is considered in [8]. As a consequence, the rotation is not required as an auxiliary unknown, which simplifies the resulting augmented formulation and constitutes the main advantage of the approach from [8]. On the contrary, the main advantages of the method employed in [7] are, precisely, the symmetry of the stress, which is a more realistic condition, and the possibility of having a direct approximation of the rotation, which is also a tensor of practical interest.

The purpose of the present paper is to further extend the analysis in [7] and [8] to a curl-based mixed variational formulation of the two-dimensional Stokes problem. To this respect, the present work should be considered as a first contribution aiming finally to define augmented mixed finite element methods for curl-based mixed formulations of the Stokes problem in 3D. At the same time, the methods to be proposed here constitute valid alternatives to those presented in [7] and [8]. The rest of the paper is organized as follows. In Section 2 we describe the boundary value problem of interest, introduce the associated curl-based mixed variational formulation, and prove that it is well-posed. Then, in Sections 3 and 4 we introduce and analyze the continuous and discrete augmented formulations, respectively. Next, in Section 5 we develop a residual-based a posteriori error analysis of the augmented mixed finite element methods yielding reliable and efficient estimators. Finally, several numerical results illustrating the performance of the augmented mixed finite element methods, confirming the reliability and efficiency of the a posteriori error estimators, and showing the good behaviour of the associated adaptive algorithms, are reported in Section 6.

We end this section with some notations to be used below. Given any Hilbert space U, we let $[U]^2$ and $[U]^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. When no confusion arises we simply use U^2 and $U^{2\times 2}$ instead of $[U]^2$ and $[U]^{2\times 2}$, respectively. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write as usual $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}$. In addition, we define

$$\mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \quad \boldsymbol{\tau}^r := \boldsymbol{\tau} - \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J})\,\mathbf{J} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{R}^{2\times 2}.$$

Note that $\boldsymbol{\tau}^r : \mathbf{J} = 0$. On the other hand, given scalar, vector, and tensor valued fields v, $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)^{\mathtt{t}}$, and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix}, \quad \text{and } \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Finally, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants, independent of the discretization parameters, which may take different values at different places.

2. THE PROBLEM AND ITS DUAL-MIXED FORMULATION

Let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with boundary Γ. Given a force density $\mathbf{f} \in [L^2(\Omega)]^2$ and a Dirichlet datum $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we seek a vector field $\mathbf{u}$ (*velocity*) and a scalar field p (*pressure*) such that

$$\mu\,\Delta\,\mathbf{u} - \nabla\,p \;=\; -\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{2.1}$$

where μ is the kinematic viscosity of a fluid occupying the region Ω. As required by the incompressibility condition, we assume from now on that $\mathbf{g}$ satisfies the compatibility condition

$$\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu} = 0\,, \tag{2.2}$$

where $\boldsymbol{\nu} := (\nu_1, \nu_2)^{\mathtt{t}}$ denotes the unit outward normal at Γ.

We now introduce the auxiliary unknown given by the tensor

$$\boldsymbol{\sigma} \;:=\; \mu\,\underline{\mathbf{curl}}(\mathbf{u}) \;-\; p\,\mathbf{J} \quad \text{in} \quad \Omega\,. \tag{2.3}$$

Then, using that $\Delta\mathbf{u} = \mathrm{curl}(\underline{\mathbf{curl}}(\mathbf{u}))$ and $\mathrm{curl}(p\,\mathbf{J}) = \nabla p$, we find that the first equation in (2.1) can be stated as

$$\mathrm{curl}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega\,. \tag{2.4}$$

In addition, since $\mathrm{div}(\mathbf{u}) = \underline{\mathbf{curl}}(\mathbf{u}) : \mathbf{J}$, we notice that the incompressibility condition can be rewritten as $\underline{\mathbf{curl}}(\mathbf{u}) : \mathbf{J} = 0$ in Ω. Hence, instead of (2.1), in what follows we consider the curl-based problem: Find a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$, and a scalar field p, such that

$$\begin{aligned} \boldsymbol{\sigma} \,:=\, \mu\,\underline{\mathbf{curl}}(\mathbf{u}) \,-\, p\,\mathbf{J} \quad \text{in} \quad \Omega\,, \qquad \mathrm{curl}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega,\\ \underline{\mathbf{curl}}(\mathbf{u}) : \mathbf{J} \,=\, 0 \quad \text{in} \quad \Omega, \qquad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.5}$$

It is easy to see that the pair of equations given by

$$\boldsymbol{\sigma} \,:=\, \mu\,\underline{\mathbf{curl}}(\mathbf{u}) \,-\, p\,\mathbf{J} \quad \text{in} \quad \Omega \quad \text{and} \quad \underline{\mathbf{curl}}(\mathbf{u}) : \mathbf{J} \,=\, 0 \quad \text{in} \quad \Omega\,, \tag{2.6}$$

is equivalent to

$$\boldsymbol{\sigma} \,:=\, \mu\,\underline{\mathbf{curl}}(\mathbf{u}) \,-\, p\,\mathbf{J} \quad \text{in} \quad \Omega \quad \text{and} \quad \mathrm{p} \,+\, \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J}) \,=\, 0 \quad \text{in} \quad \Omega\,, \tag{2.7}$$

whence (2.5) becomes equivalent to:

$$\begin{aligned} \boldsymbol{\sigma} \,:=\, \mu\,\underline{\mathbf{curl}}(\mathbf{u}) \,-\, p\,\mathbf{J} \quad \text{in} \quad \Omega, \qquad \mathrm{curl}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega\,,\\ p + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J}) \,=\, 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.8}$$

Then, testing the first three equations of (2.8) with $\boldsymbol{\tau} \in H(\mathrm{curl};\Omega)$, $\mathbf{v} \in [L^2(\Omega)]^2$, and $q \in L^2(\Omega)$, respectively, using the Dirichlet boundary condition, and rearranging the resulting terms, we arrive at the variational formulation: Find $(\boldsymbol{\sigma}, p, \mathbf{u}) \in H(\mathrm{curl};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} \frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^r : \boldsymbol{\tau}^r \,+\, \frac{2}{\mu}\int_\Omega \left(p + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})\right)\left(q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J})\right) \,+\, \int_\Omega \mathbf{u}\cdot\mathrm{curl}(\boldsymbol{\tau}) \quad &= \quad \langle \boldsymbol{\tau}\,\mathbf{s}, \mathbf{g}\rangle\,,\\ \int_\Omega \mathbf{v}\cdot\mathrm{curl}(\boldsymbol{\sigma}) \quad &= \quad -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,, \end{aligned} \tag{2.9}$$

for all $(\boldsymbol{\tau}, q, \mathbf{v}) \in H(\mathrm{curl};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2$, where $\mathbf{s} := (-\nu_2, \nu_1)^{\mathtt{t}}$ is the unit tangential vector along Γ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ with respect to the $[L^2(\Gamma)]^2$-inner product. In addition, we recall here that

$$H(\mathrm{curl};\Omega) := \left\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \;\; \mathrm{curl}(\boldsymbol{\tau}) \,\in\, [L^2(\Omega)]^2\right\}$$

is a Hilbert space with the inner product

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{\mathrm{curl},\Omega} \,:=\, \langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{0,\Omega} \,+\, \langle \mathrm{curl}(\boldsymbol{\zeta}), \mathrm{curl}(\boldsymbol{\tau})\rangle_{0,\Omega} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \,\in\, H(\mathrm{curl};\Omega)\,,$$

where

$$\langle \boldsymbol{\zeta}, \boldsymbol{\tau}\rangle_{0,\Omega} \,:=\, \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}\,,$$

and

$$\langle \mathbf{v}, \mathbf{w}\rangle_{0,\Omega} \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{w} \qquad \forall\,\mathbf{v},\,\mathbf{w} \,\in\, [L^2(\Omega)]^2\,.$$

The corresponding induced norms are denoted by $\|\cdot\|_{\mathrm{curl},\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively.

Next, we observe that for any $c \in \mathbb{R}$, $(c\,\mathbf{J}, -c, \mathbf{0})$ is a solution of the homogeneous version of system (2.9). Hence, in order to avoid this non-uniqueness we consider the decomposition

$$H(\mathrm{curl};\Omega) \;=\; H_0 \oplus \mathbb{R}\,\mathbf{J}\,, \tag{2.10}$$

where

$$H_0 \;:=\; \left\{ \boldsymbol{\tau} \in H(\mathrm{curl};\Omega) : \quad \int_\Omega \boldsymbol{\tau} : \mathbf{J} = 0 \right\}, \tag{2.11}$$

and require from now on that $\boldsymbol{\sigma} \in H_0$. Equivalently, in what follows we look for the H_0-component of $\boldsymbol{\sigma}$, which is also denoted by $\boldsymbol{\sigma}$. Moreover, since the test space can also be restricted to H_0, we now let

$$H \;:=\; H_0 \times L^2(\Omega)\,, \quad Q \;:=\; [L^2(\Omega)]^2\,,$$

and introduce a generalized version of (2.9): Find $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ such that

$$\begin{aligned} a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) \,+\, b(\boldsymbol{\tau}, \mathbf{u}) \;&=\; \langle \boldsymbol{\tau}\,\mathbf{s}, \mathbf{g}\rangle \qquad \forall\,(\boldsymbol{\tau}, q) \in H\,,\\ b(\boldsymbol{\sigma}, \mathbf{v}) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in Q\,, \end{aligned} \tag{2.12}$$

where, given a parameter $\kappa > 0$, $a : H \times H \to \mathbb{R}$ and $b : H_0 \times Q \to \mathbb{R}$ are the bounded bilinear forms defined by

$$a((\boldsymbol{\zeta}, r), (\boldsymbol{\tau}, q)) \;:=\; \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^r : \boldsymbol{\tau}^r + \frac{\kappa}{\mu}\int_\Omega \left(r + \frac{1}{2}(\boldsymbol{\zeta} : \mathbf{J})\right)\left(q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J})\right) \quad \forall\,(\boldsymbol{\zeta}, r),\,(\boldsymbol{\tau}, q) \in H\,, \tag{2.13}$$

and

$$b(\boldsymbol{\zeta}, \mathbf{v}) \;:=\; \int_\Omega \mathbf{v}\cdot\mathrm{curl}(\boldsymbol{\zeta}) \qquad \forall\,\boldsymbol{\zeta} \in H_0\,, \quad \forall\,\mathbf{v} \in Q\,. \tag{2.14}$$

Note that (2.9) corresponds to (2.12) with $\kappa = 2$ and $H(\mathrm{curl};\Omega)$ instead of H_0.

In order to show that the formulations (2.12) are independent of $\kappa > 0$, we prove next that they are all equivalent to the simplified version arising after taking $\kappa = 0$ in (2.12), that is: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ such that

$$\begin{aligned} a_0(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \mathbf{u}) \;&=\; \langle \boldsymbol{\tau}\,\mathbf{s}, \mathbf{g}\rangle \qquad \forall\,\boldsymbol{\tau} \in H_0\,,\\ b(\boldsymbol{\sigma}, \mathbf{v}) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in Q\,, \end{aligned} \tag{2.15}$$

where $a_0 : H_0 \times H_0 \to \mathbb{R}$ is the bounded bilinear form defined by

$$a_0(\boldsymbol{\zeta}, \boldsymbol{\tau}) \;:=\; \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^r : \boldsymbol{\tau}^r \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau} \in H_0\,.$$

Lemma 2.1. *Problems* (2.12) *and* (2.15) *are equivalent. Indeed,* $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$ *is a solution of* (2.12) *if and only if* $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$ *is a solution of* (2.15) *and* $p = -\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$.

Proof. It suffices to take $\boldsymbol{\tau} = \mathbf{0}$ in (2.12) and then use that the products $\boldsymbol{\tau} : \mathbf{J}$ live in $L^2(\Omega)$ for all $\boldsymbol{\tau} \in H(\mathrm{curl};\Omega)$, as the pressure test functions do. □

Another way of seeing the equivalence between (2.12) and (2.15) is the following. We observe that eliminating the pressure unknown from (2.8), that is replacing p by $-\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$ in its first equation, we are lead to the reduced problem:

$$\frac{1}{\mu}\boldsymbol{\sigma}^r = \underline{\mathbf{curl}}(\mathbf{u}) \quad \text{in} \quad \Omega, \qquad \operatorname{curl}(\boldsymbol{\sigma}) = -\mathbf{f} \quad \text{in} \quad \Omega, \qquad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma, \tag{2.16}$$

whose variational formulation is precisely (2.15). Hence, (2.12) can also be considered as the equivalent augmented formulation arising from (2.15) after adding the equation

$$\frac{\kappa}{\mu}\int_\Omega \left(p + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})\right)\left(q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J})\right) = 0 \qquad \forall\, (\boldsymbol{\tau}, q) \in H\,.$$

Certainly, if we had to choose, we would prefer (2.15) since it is simpler than (2.12). However, the interest in (2.12) lies in the corresponding Galerkin scheme, which, as we show below in Section 4, provides more flexibility for choosing the pressure finite element subspace.

The well-posedness of (2.12) and (2.15) is proved next. We need the following lemmas.

Lemma 2.2. *There exists a positive constant β, depending only on Ω, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in H_0 \\ \boldsymbol{\tau}\neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\operatorname{curl}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\operatorname{curl},\Omega}} \geq \beta\|\mathbf{v}\|_Q \qquad \forall\, \mathbf{v} \in Q\,. \tag{2.17}$$

Proof. Given $\mathbf{v} \in Q$, $\mathbf{v} \neq 0$, we let $\boldsymbol{\tau}_0 := -\underline{\mathbf{curl}}(\mathbf{z})$, where $\mathbf{z} \in [H^1(\Omega)]^2$ is the unique weak solution of the boundary value problem:

$$-\Delta\mathbf{z} = \mathbf{v} \qquad \text{in} \quad \Omega\,, \qquad \mathbf{z} = \mathbf{0} \qquad \text{on} \quad \Gamma. \tag{2.18}$$

It is clear that $\boldsymbol{\tau}_0 \in [L^2(\Omega)]^{2\times 2}$, $\operatorname{curl}(\boldsymbol{\tau}_0) = \mathbf{v} \in Q$, and

$$\int_\Omega \boldsymbol{\tau}_0 : \mathbf{J} = -\int_\Omega \underline{\mathbf{curl}}(\mathbf{z}) : \mathbf{J} = -\int_\Omega \operatorname{div}(\mathbf{z}) = -\int_\Gamma \mathbf{z}\cdot\boldsymbol{\nu} = 0\,,$$

whence $\boldsymbol{\tau}_0 \in H_0$. In addition, there holds

$$\|\boldsymbol{\tau}_0\|^2_{\operatorname{curl},\Omega} = \|\underline{\mathbf{curl}}(\mathbf{z})\|^2_{0,\Omega} + \|\mathbf{v}\|^2_{0,\Omega} \leq \|\mathbf{z}\|^2_{1,\Omega} + \|\mathbf{v}\|^2_{0,\Omega}\,,$$

which, thanks to the continuous dependence result for (2.18), yields

$$\|\boldsymbol{\tau}_0\|_{\operatorname{curl},\Omega} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,. \tag{2.19}$$

In this way, we conclude that

$$\sup_{\substack{\boldsymbol{\tau}\in H_0 \\ \boldsymbol{\tau}\neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\operatorname{curl}(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\operatorname{curl},\Omega}} \geq \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\operatorname{curl}(\boldsymbol{\tau}_0)}{\|\boldsymbol{\tau}_0\|_{\operatorname{curl},\Omega}} = \frac{\|\mathbf{v}\|^2_{0,\Omega}}{\|\boldsymbol{\tau}_0\|_{\operatorname{curl},\Omega}} \geq \beta\,\|\mathbf{v}\|_{0,\Omega} = \beta\,\|\mathbf{v}\|_Q\,,$$

where (2.19) has been used in the last inequality. □

Lemma 2.3. *There exists $c_1 > 0$, depending only on Ω, such that*

$$c_1\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \leq \|\boldsymbol{\tau}^r\|^2_{0,\Omega} + \|\operatorname{curl}(\boldsymbol{\tau})\|^2_{0,\Omega} \quad \forall\, \boldsymbol{\tau} \in H_0\,. \tag{2.20}$$

Proof. Given $\boldsymbol{\tau} := \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \in H_0$, we define $\hat{\boldsymbol{\tau}} := \begin{pmatrix} \tau_{12} & -\tau_{11} \\ \tau_{22} & -\tau_{21} \end{pmatrix}$. Since $\mathbf{div}(\hat{\boldsymbol{\tau}}) = \mathrm{curl}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2$, it is clear that $\hat{\boldsymbol{\tau}} \in H(\mathbf{div};\Omega)$. Also, we note that

$$\int_\Omega \mathrm{tr}(\hat{\boldsymbol{\tau}}) = \int_\Omega (\tau_{12} - \tau_{21}) = \int_\Omega \boldsymbol{\tau} : \mathbf{J} = 0\,,$$

which implies that $\hat{\boldsymbol{\tau}} \in \left\{ \mathbf{s} \in H(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\mathbf{s}) = 0 \right\}$. Hence, according to Lemma 3.1 in [1] or Proposition 3.1 of Chapter IV in [4], there exists a constant $c > 0$, depending only on Ω, such that

$$c\,\|\hat{\boldsymbol{\tau}}\|_{0,\Omega}^2 \;\leq\; \|\hat{\boldsymbol{\tau}}^d\|_{0,\Omega}^2 \;+\; \|\mathbf{div}(\hat{\boldsymbol{\tau}})\|_{0,\Omega}^2\,. \tag{2.21}$$

In this way, noting that $\|\hat{\boldsymbol{\tau}}\|_{0,\Omega} = \|\boldsymbol{\tau}\|_{0,\Omega}$, $\|\hat{\boldsymbol{\tau}}^d\|_{0,\Omega} = \|\boldsymbol{\tau}^r\|_{0,\Omega}$, and $\|\mathbf{div}(\hat{\boldsymbol{\tau}})\|_{0,\Omega} = \|\mathrm{curl}(\boldsymbol{\tau})\|_{0,\Omega}$, we see that (2.21) becomes (2.20). □

Theorem 2.4. *Problem* (2.15) *has a unique solution* $(\boldsymbol{\sigma}, \mathbf{u}) \in H_0 \times Q$. *Moreover, there exists a positive constant* C, *depending only on* Ω, *such that*

$$\|\boldsymbol{\sigma}\|_{\mathrm{curl},\Omega} \;+\; \|\mathbf{u}\|_Q \;\leq\; C\left\{ \|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{g}\|_{1/2,\Gamma} \right\}.$$

Proof. It suffices to check that the bilinear forms a_0 and b satisfy the hypotheses of the Babuška-Brezzi theory. Indeed, from Lemma 2.2 we have the continuos inf-sup condition for b. Now, let V be the kernel of the operator induced by b, that is

$$V \;:=\; \left\{ \boldsymbol{\tau} \in H_0 : \quad b(\boldsymbol{\tau}, \mathbf{v}) \;=\; 0 \quad \forall\, \mathbf{v} \,\in\, Q \right\} \;=\; \left\{ \boldsymbol{\tau} \in H_0 : \quad \mathrm{curl}(\boldsymbol{\tau}) \;=\; \mathbf{0} \;\text{ in }\; Q \right\}.$$

It follows, applying Lemma 2.3, that for each $\boldsymbol{\tau} \in V$ there holds

$$a_0(\boldsymbol{\tau}, \boldsymbol{\tau}) \;=\; \frac{1}{\mu}\,\|\boldsymbol{\tau}^r\|_{0,\Omega}^2 \;\geq\; \frac{c_1}{\mu}\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \;=\; \frac{c_1}{\mu}\,\|\boldsymbol{\tau}\|_{\mathrm{curl},\Omega}^2\,,$$

which shows that the bilinear form a_0 is strongly coercive in V. Hence, a straightforward application of the clasical result given by Theorem 1.1 in Chapter II of [4] completes the proof. □

Theorem 2.5. *Problem* (2.12) *has a unique solution* $((\boldsymbol{\sigma}, p), \mathbf{u}) \in H \times Q$, *independent of* κ, *and there holds* $p = -\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$. *Moreover, there exists a constant* $C > 0$, *depending only on* Ω, *such that*

$$\|((\boldsymbol{\sigma}, p), \mathbf{u})\|_{H\times Q} \;\leq\; C\left\{ \|\mathbf{f}\|_{0,\Omega} \;+\; \|\mathbf{g}\|_{1/2,\Gamma} \right\}.$$

Proof. It is a direct consequence of Lemma 2.1, which gives the equivalence between (2.12) and (2.15), and Theorem 2.5, which yields the well-posedness of (2.15). □

3. THE AUGMENTED DUAL-MIXED VARIATIONAL FORMULATIONS

In the following we enrich the formulations (2.12) and (2.15) with residuals arising from the modified constitutive equation, the equilibrium equation, and the Dirichlet boundary condition (cf. (2.16)). More precisely, we substract the second from the first equation in (2.12) and then add the Galerkin least-squares terms given by

$$\kappa_1 \int_\Omega \left(\underline{\mathbf{curl}}(\mathbf{u}) \,-\, \frac{1}{\mu}\boldsymbol{\sigma}^r \right) : \left(\underline{\mathbf{curl}}(\mathbf{v}) \,+\, \frac{1}{\mu}\boldsymbol{\tau}^r \right) \,=\, 0\,, \tag{3.1}$$

$$\kappa_2 \int_\Omega \mathrm{curl}(\boldsymbol{\sigma}) \cdot \mathrm{curl}(\boldsymbol{\tau}) \,=\, -\,\kappa_2 \int_\Omega \mathbf{f} \cdot \mathrm{curl}(\boldsymbol{\tau})\,, \tag{3.2}$$

and

$$\kappa_3 \int_\Gamma \mathbf{u} \cdot \mathbf{v} \,=\, \kappa_3 \int_\Gamma \mathbf{g} \cdot \mathbf{v}\,, \tag{3.3}$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in H_0 \times [H^1(\Omega)]^2$, where $(\kappa_1, \kappa_2, \kappa_3)$ is a vector of positive parameters to be specified later. We notice that the above terms implicitly require now the velocity $\mathbf{u}$ to live in the smaller space $[H^1(\Omega)]^2$.

In other words, instead of (2.12) we propose the following augmented variational formulation: Find $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H} \,:=\, H_0 \times L^2(\Omega) \times [H^1(\Omega)]^2$ such that

$$A((\boldsymbol{\sigma}, p, \mathbf{u}), (\boldsymbol{\tau}, q, \mathbf{v})) \,=\, F(\boldsymbol{\tau}, q, \mathbf{v}) \qquad \forall\, (\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H}\,, \tag{3.4}$$

where the bilinear form $A : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and the functional $F : \mathbf{H} \to \mathbb{R}$ are defined by

$$\begin{aligned} A((\boldsymbol{\sigma}, p, \mathbf{u}), (\boldsymbol{\tau}, q, \mathbf{v})) \,&:=\, a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) \,+\, b(\boldsymbol{\tau}, \mathbf{u}) \,-\, b(\boldsymbol{\sigma}, \mathbf{v}) \\ +\quad \kappa_1 \int_\Omega \left(\underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r \right) &: \left(\underline{\mathbf{curl}}(\mathbf{v}) + \frac{1}{\mu}\boldsymbol{\tau}^r \right) \,+\, \kappa_2 \int_\Omega \mathrm{curl}(\boldsymbol{\sigma}) \cdot \mathrm{curl}(\boldsymbol{\tau}) \,+\, \kappa_3 \int_\Gamma \mathbf{u} \cdot \mathbf{v}\,, \end{aligned} \tag{3.5}$$

and

$$F(\boldsymbol{\tau}, q, \mathbf{v}) \,:=\, \langle \boldsymbol{\tau}\mathbf{s}, \mathbf{g} \rangle \,+\, \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \kappa_2\,\mathrm{curl}(\boldsymbol{\tau})) \,+\, \kappa_3 \int_\Gamma \mathbf{g} \cdot \mathbf{v}\,. \tag{3.6}$$

Similarly, instead of (2.15) we propose: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}_0 \,:=\, H_0 \times [H^1(\Omega)]^2$ such that

$$A_0((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) \,=\, F_0(\boldsymbol{\tau}, \mathbf{v}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_0\,, \tag{3.7}$$

where the bilinear form $A_0 : \mathbf{H}_0 \times \mathbf{H}_0 \to \mathbb{R}$ and the functional $F_0 : \mathbf{H}_0 \to \mathbb{R}$ are defined by

$$\begin{aligned} A_0((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) \,&:=\, a_0(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \mathbf{u}) \,-\, b(\boldsymbol{\sigma}, \mathbf{v}) \\ +\quad \kappa_1 \int_\Omega \left(\underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r \right) &: \left(\underline{\mathbf{curl}}(\mathbf{v}) + \frac{1}{\mu}\boldsymbol{\tau}^r \right) \,+\, \kappa_2 \int_\Omega \mathrm{curl}(\boldsymbol{\sigma}) \cdot \mathrm{curl}(\boldsymbol{\tau}) \,+\, \kappa_3 \int_\Gamma \mathbf{u} \cdot \mathbf{v}\,, \end{aligned} \tag{3.8}$$

and

$$F_0(\boldsymbol{\tau}, \mathbf{v}) \,:=\, \langle \boldsymbol{\tau}\mathbf{s}, \mathbf{g} \rangle \,+\, \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \kappa_2\,\mathrm{curl}(\boldsymbol{\tau})) \,+\, \kappa_3 \int_\Gamma \mathbf{g} \cdot \mathbf{v}\,. \tag{3.9}$$

The analogue of Lemma 2.1 is given now.

Lemma 3.1. *Problems* (3.4) *and* (3.7) *are equivalent. Indeed,* $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H}$ *is a solution of* (3.4) *if and only if* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}_0$ *is a solution of* (3.7) *and* $p \,=\, -\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$.

Proof. It suffices to take $(\boldsymbol{\tau}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$ in (3.4) and then use again that the products $\boldsymbol{\tau} : \mathbf{J}$ live in $L^2(\Omega)$ for all $\boldsymbol{\tau} \in H(\mathrm{curl}; \Omega)$, as the pressure test functions do. $\square$

In what follows we aim to show the well-posedness of (3.7). The main idea is to choose the vector of parameters $(\kappa_1, \kappa_2, \kappa_3)$ in a way such that A_0 becomes strongly coercive on $\mathbf{H}_0$ with respect to the norm $\|\cdot\|_{\mathbf{H}_0}$ defined by

$$\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathbf{H}_0} := \left\{ \|\boldsymbol{\tau}\|^2_{\mathrm{curl},\Omega} + \|\mathbf{v}\|^2_{1,\Omega} \right\}^{1/2} .$$

We first notice, after simple computations, that

$$\int_\Omega \left(\underline{\mathbf{curl}}(\mathbf{v}) - \frac{1}{\mu} \boldsymbol{\tau}^r \right) : \left(\underline{\mathbf{curl}}(\mathbf{v}) + \frac{1}{\mu} \boldsymbol{\tau}^r \right) = \|\underline{\mathbf{curl}}(\mathbf{v})\|^2_{0,\Omega} - \frac{1}{\mu^2} \|\boldsymbol{\tau}^r\|^2_{0,\Omega} .$$

On the other hand, using Peetre-Tartar Lemma (see, e.g., [13], Chapter I, Theorem 2.1) and the generalized Poincaré inequality, one can prove that there exists $C_1 > 0$, such that

$$|\mathbf{v}|^2_{1,\Omega} + \|\mathbf{v}\|^2_{0,\Gamma} \geq C_1 \|\mathbf{v}\|^2_{1,\Omega} \qquad \forall\, \mathbf{v} \in [H^1(\Omega)]^2 . \tag{3.10}$$

Hence, assuming that $0 < \kappa_1 < \mu$ and $0 < \kappa_2, \kappa_3$, noting that $\|\underline{\mathbf{curl}}(\mathbf{v})\|_{0,\Omega} = |\mathbf{v}|_{1,\Omega}$, and applying Lemma 2.3 and (3.10), we deduce that

$$\begin{aligned} A_0((\boldsymbol{\tau}, \mathbf{v}), (\boldsymbol{\tau}, \mathbf{v})) &= \frac{1}{\mu} \left(1 - \frac{\kappa_1}{\mu} \right) \|\boldsymbol{\tau}^r\|^2_{0,\Omega} + \kappa_1 \|\underline{\mathbf{curl}}(\mathbf{v})\|^2_{0,\Omega} + \kappa_2 \|\mathrm{curl}(\boldsymbol{\tau})\|^2_{0,\Omega} + \kappa_3 \|\mathbf{v}\|^2_{0,\Gamma} \\ &\geq c_1\, \alpha_1 \|\boldsymbol{\tau}\|^2_{0,\Omega} + \frac{\kappa_2}{2} \|\mathrm{curl}(\boldsymbol{\tau})\|^2_{0,\Omega} + \alpha_2 \left(|\mathbf{v}|^2_{1,\Omega} + \|\mathbf{v}\|^2_{0,\Gamma} \right) \\ &\geq \alpha_3 \|\boldsymbol{\tau}\|^2_{\mathrm{curl},\Omega} + C_1\, \alpha_2 \|\mathbf{v}\|^2_{1,\Omega} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_0 , \end{aligned}$$

where

$$\alpha_1 := \min \left\{ \frac{1}{\mu} \left(1 - \frac{\kappa_1}{\mu} \right), \frac{\kappa_2}{2} \right\} , \quad \alpha_2 := \min \{\kappa_1, \kappa_3\} , \quad \text{and} \quad \alpha_3 := \min \left\{ c_1\, \alpha_1, \frac{\kappa_2}{2} \right\} .$$

In this way, we find that

$$A_0((\boldsymbol{\tau}, \mathbf{v}), (\boldsymbol{\tau}, \mathbf{v})) \geq \alpha \|(\boldsymbol{\tau}, \mathbf{v})\|^2_{\mathbf{H}_0} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_0 , \tag{3.11}$$

where $\alpha := \min \{\alpha_3, C_1\, \alpha_2\}$. In particular, taking

$$\kappa_2 = \frac{2}{\mu} \left(1 - \frac{\kappa_1}{\mu} \right) \quad \text{and} \quad \kappa_3 = \kappa_1 , \tag{3.12}$$

we obtain $\alpha_1 = \frac{\kappa_2}{2}$, $\alpha_2 = \kappa_1$, and $\alpha = \min \left\{ c_1 \frac{\kappa_2}{2}, \frac{\kappa_2}{2}, C_1\, \kappa_1 \right\}$. For instance, $\kappa_1 = \frac{\mu}{2}$ yields $\kappa_2 = \frac{1}{\mu}$ and $\kappa_3 = \frac{\mu}{2}$.

As a consequence of the above analysis, we obtain the following main results.

Theorem 3.2. *Assume that there hold $0 < \kappa_1 < \mu$ and $0 < \kappa_2, \kappa_3$. Then, the augmented formulation* (3.7) *has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}_0$. Moreover, there exists a positive constant C, depending only on μ, $(\kappa_1, \kappa_2, \kappa_3)$, c_1, and C_1, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathbf{H}_0} \leq C \|F_0\|_{\mathbf{H}_0'} \leq C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\} .$$

Proof. It is clear from (3.8) and (3.11) that A_0 is bounded and strongly coercive on $\mathbf{H}_0$ with constants depending on μ, $(\kappa_1, \kappa_2, \kappa_3)$, c_1, and C_1. Also, the linear functional F_0 (cf. (3.9)) is clearly bounded. Therefore, the assertion is a simple consequence of the Lax-Milgram Lemma. □

Theorem 3.3. *Assume that there hold $0 < \kappa_1 < \mu$ and $0 < \kappa_2, \kappa_3$. Then, the augmented formulation* (3.4) *has a unique solution $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H}$, independent of κ, and there holds $p = -\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$. Moreover, there exists a positive constant C, depending only on μ, $(\kappa_1, \kappa_2, \kappa_3)$, c_1, and C_1, such that*

$$\|(\boldsymbol{\sigma}, p, \mathbf{u})\|_{\mathbf{H}} \leq C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\} .$$

Proof. It is a direct consequence of Lemma 3.1 and Theorem 3.2. □

4. THE AUGMENTED MIXED FINITE ELEMENT METHODS

We now let $H_{0,h}^{\boldsymbol{\sigma}}$, H_h^p, $H_h^{\mathbf{u}}$ be arbitrary finite element subspaces of H_0, $L^2(\Omega)$ and $[H^1(\Omega)]^2$, respectively, and define

$$\mathbf{H}_h := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^p \times H_h^{\mathbf{u}} \quad \text{and} \quad \mathbf{H}_{0,h} := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}} .$$

In addition, let κ, κ_1, κ_2 and κ_3 be given positive parameters. Then, the Galerkin schemes associated with (3.4) and (3.7) read: Find $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ such that

$$A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)) = F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \quad \forall (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in \mathbf{H}_h , \tag{4.1}$$

and: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_{0,h}$ such that

$$A_0((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) = F_0(\boldsymbol{\tau}_h, \mathbf{v}_h) \quad \forall (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{H}_{0,h} . \tag{4.2}$$

The following theorem provides the unique solvability, stability, and convergence of (4.2).

Theorem 4.1. *Assume that the parameters κ_1, κ_2 and κ_3 satisfy the assumptions of Theorem 3.2 and let $\mathbf{H}_{0,h}$ be any finite element subspace of $\mathbf{H}_0$. Then, the Galerkin scheme* (4.2) *has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_{0,h}$, and there exist positive constants C, $\tilde{C}$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}_0} \leq C \sup_{\substack{(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{H}_{0,h} \\ (\boldsymbol{\tau}_h, \mathbf{v}_h) \neq \mathbf{0}}} \frac{|F_0(\boldsymbol{\tau}_h, \mathbf{v}_h)|}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathbf{H}_0}} \leq C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \right\} ,$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}_0} \leq \tilde{C} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{H}_{0,h}} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathbf{H}_0} . \tag{4.3}$$

Proof. Since A_0 is bounded and strongly coercive on $\mathbf{H}_0$ (cf. (3.10)) with constants depending on μ, $(\kappa_1, \kappa_2, \kappa_3)$, c_1, and C_1, the proof follows from a straightforward application of the Lax-Milgram Lemma and Cea's estimate. □

In order to define an explicit finite element subspace of $\mathbf{H}_0$, we now let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T such that $\bar{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $l \geq 0$ and a subset

S of $\mathbb{R}^2$, we denote by $\mathbb{P}_l(S)$ the space of polynomials of total degree at most l defined on S. Also, for each $T \in \mathcal{T}_h$ we define the local rotated Raviart-Thomas space of order zero

$$W(T) := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} : \quad a,\, b,\, c \in \mathrm{I\!R} \right\} \subseteq [\mathbb{P}_1(T)]^2 ,$$

and let $\tilde{H}_h^{\boldsymbol{\sigma}}$ be the corresponding global space, that is

$$\tilde{H}_h^{\boldsymbol{\sigma}} := \{ \boldsymbol{\tau}_h \in H(\mathrm{curl};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_T \in W(T)^{\mathtt{t}} \quad \forall\, i \in \{1,2\}\,, \quad \forall\, T \in \mathcal{T}_h \}\,, \tag{4.4}$$

where $\boldsymbol{\tau}_{h,i}$ denotes the i-th row of $\boldsymbol{\tau}_h$. Then we let

$$\tilde{\mathbf{H}}_{0,h} := \tilde{H}_{0,h}^{\boldsymbol{\sigma}} \times \tilde{H}_h^{\mathbf{u}}\,, \tag{4.5}$$

where

$$\tilde{H}_{0,h}^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in \tilde{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{J} = 0 \right\}\,, \tag{4.6}$$

and

$$\tilde{H}_h^{\mathbf{u}} := \{ \mathbf{v}_h \in [C(\bar{\Omega})]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_1(T)]^2 \quad \forall\, T \in \mathcal{T}_h \}\,. \tag{4.7}$$

As in [10], it is easy to see that the number of degrees of freedom defining $\tilde{\mathbf{H}}_{0,h}$ behaves asymptoticaly as 4 times the number of triangles of $\mathcal{T}_h$. In addition, the approximation properties of these subspaces are given as follows (see [4], [5]):

($\mathrm{AP}_{0,h}^{\boldsymbol{\sigma}}$) For each $\boldsymbol{\tau} \in [H^1(\Omega)]^{2\times 2} \cap H_0$ with $\mathrm{curl}(\boldsymbol{\tau}) \in [H^1(\Omega)]^2$ there exists $\boldsymbol{\tau}_h \in \tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathrm{curl},\Omega} \le C\, h \left\{ \|\boldsymbol{\tau}\|_{1,\Omega} + \|\mathrm{curl}(\boldsymbol{\tau})\|_{1,\Omega} \right\}.$$

($\mathrm{AP}_h^{\mathbf{u}}$) For each $\mathbf{v} \in [H^2(\Omega)]^2$ there exists $\mathbf{v}_h \in \tilde{H}_h^{\mathbf{u}}$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \le C\, h\, \|\mathbf{v}\|_{2,\Omega}\,.$$

Then, we have the following result providing the rate of convergence of (4.2) with $\mathbf{H}_{0,h} = \tilde{\mathbf{H}}_{0,h}$.

Theorem 4.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H}_0$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \tilde{\mathbf{H}}_{0,h}$ *be the unique solutions of the continuous and discrete augmented formulations* (3.7) *and* (4.2)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathrm{curl}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, *and* $\mathbf{u} \in [H^{1+r}(\Omega)]^2$, *for some* $r \in (0,1]$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}_0} \le C\, h^r \left\{ \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathrm{curl}(\boldsymbol{\sigma})\|_{r,\Omega} + \|\mathbf{u}\|_{1+r,\Omega} \right\}.$$

Proof. It follows from the Cea estimate (4.3), the above approximation properties, and the interpolation theorems in the corresponding function spaces. $\square$

We now state the discrete analogue of Lemma 3.1, which gives a sufficient condition for the equivalence between (4.1) and (4.2) with arbitrary finite element subspaces $\mathbf{H}_h := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^p \times H_h^{\mathbf{u}}$ and $\mathbf{H}_{0,h} := H_{0,h}^{\boldsymbol{\sigma}} \times H_h^{\mathbf{u}}$, respectively.

Lemma 4.3. *Assume that*

$$(\boldsymbol{\tau}_h : \mathbf{J}) \in H_h^p \qquad \forall\, \boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}} . \tag{4.8}$$

Then, problems (4.1) *and* (4.2) *are equivalent*: $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ *is a solution of* (4.1) *if and only if* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_{0,h}$ *is a solution of* (4.2) *and* $p_h = -\frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})$.

Proof. Let $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ be a solution of (4.1). It is clear from (4.8) that $p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})$ belongs to H_h^p. Then, taking $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) = (\mathbf{0}, p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J}), \mathbf{0}) \in \mathbf{H}_h$, we find from (4.1) that

$$\frac{\kappa}{\mu} \int_\Omega \left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J}) \right)^2 = 0 ,$$

which yields $p_h = -\frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})$. Conversely, given $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_{0,h}$ a solution of (4.2), we let $p_h := -\frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})$ and see that $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ becomes a solution of (4.1). □

It is important to emphasize from Lemma 4.3 that the augmented scheme (4.1) makes sense only for pressure finite element subspaces not satisfying the condition (4.8). According to the above, we now aim to show that (4.1) is well-posed when an arbitrary finite element subspace $\mathbf{H}_h$ of $\mathbf{H}$ is considered. The idea, similarly as for A_0, is to choose κ, κ_1, κ_2 and κ_3 in such a way that A becomes strongly coercive on $\mathbf{H}$ with respect to the norm $\|\cdot\|_{\mathbf{H}}$ defined by

$$\|(\boldsymbol{\tau}, q, \mathbf{v})\|_{\mathbf{H}} := \left\{ \|\boldsymbol{\tau}\|_{\mathrm{curl},\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\mathbf{v}\|_{1,\Omega}^2 \right\}^{1/2} .$$

In fact, we first notice from (3.5) and the definition of a (cf. (2.13)) that

$$\begin{aligned} A((\boldsymbol{\tau}, q, \mathbf{v}), (\boldsymbol{\tau}, q, \mathbf{v})) \,=\, & \frac{1}{\mu}\left(1 - \frac{\kappa_1}{\mu}\right) \|\boldsymbol{\tau}^r\|_{0,\Omega}^2 \,+\, \frac{\kappa}{\mu} \left\| q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J}) \right\|_{0,\Omega}^2 \\ & + \kappa_1 \, |\mathbf{v}|_{1,\Omega}^2 \,+\, \kappa_2 \, \|\mathrm{curl}(\boldsymbol{\tau})\|_{0,\Omega}^2 \,+\, \kappa_3 \, \|\mathbf{v}\|_{0,\Gamma}^2 , \end{aligned}$$

which, employing the estimate

$$\left\| q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J}) \right\|_{0,\Omega}^2 \geq \frac{1}{2} \|q\|_{0,\Omega}^2 - \frac{1}{2} \|\boldsymbol{\tau}\|_{0,\Omega}^2 ,$$

and taking $\kappa > 0$, yields

$$\begin{aligned} A((\boldsymbol{\tau}, q, \mathbf{v}), (\boldsymbol{\tau}, q, \mathbf{v})) \,\geq\, & \frac{1}{\mu}\left(1 - \frac{\kappa_1}{\mu}\right) \|\boldsymbol{\tau}^r\|_{0,\Omega}^2 - \frac{\kappa}{2\mu} \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \kappa_2 \, \|\mathrm{curl}(\boldsymbol{\tau})\|_{0,\Omega}^2 \\ & + \frac{\kappa}{2\mu} \|q\|_{0,\Omega}^2 + \kappa_1 \, |\mathbf{v}|_{1,\Omega}^2 + \kappa_3 \, \|\mathbf{v}\|_{0,\Gamma}^2 \qquad \forall\, (\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H} . \end{aligned}$$

Then, assuming that $0 < \kappa_1 < \mu$ and $0 < \kappa_2, \kappa_3$, and applying Lemma 2.4 and (3.10), we deduce that

$$\begin{aligned} A((\boldsymbol{\tau}, q, \mathbf{v}), (\boldsymbol{\tau}, q, \mathbf{v})) \,\geq\, & \left(c_1 \, \alpha_1 - \frac{\kappa}{2\mu} \right) \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{\kappa_2}{2} \|\mathrm{curl}(\boldsymbol{\tau})\|_{0,\Omega}^2 \\ & + \frac{\kappa}{2\mu} \|q\|_{0,\Omega}^2 + C_1 \, \alpha_2 \, \|\mathbf{v}\|_{1,\Omega}^2 \qquad \forall\, (\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H} , \end{aligned}$$

where c_1 and C_1 are the constants from Lemma 2.4 and (3.10), respectively, and the constants α_1 and α_2 are given by

$$\alpha_1 := \min\left\{\frac{1}{\mu}\left(1-\frac{\kappa_1}{\mu}\right), \frac{\kappa_2}{2}\right\} \quad \text{and} \quad \alpha_2 := \min\{\kappa_1, \kappa_3\}.$$

Hence, choosing the parameter κ such that $0 < \kappa < 2\,c_1\,\mu\,\alpha_1$, we find that

$$A((\boldsymbol{\tau}, q, \mathbf{v}), (\boldsymbol{\tau}, q, \mathbf{v})) \geq \alpha\,\|(\boldsymbol{\tau}, q, \mathbf{v})\|^2_{\mathbf{H}} \qquad \forall\,(\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H}, \tag{4.9}$$

where $\alpha := \min\left\{\alpha_3, \frac{\kappa}{2\mu}, C_1\,\alpha_2\right\}$ and $\alpha_3 := \min\left\{c_1\,\alpha_1 - \frac{\kappa}{2\mu}, \frac{\kappa_2}{2}\right\}$.

We are now in a position to establish the following result.

Theorem 4.4. *Assume that there hold*

$$0 < \kappa < 2\,c_1\,\mu\,\alpha_1, \quad 0 < \kappa_1 < \mu, \quad \textit{and} \quad 0 < \kappa_2, \kappa_3.$$

In addition, let $\mathbf{H}_h$ be any finite element subspace of $\mathbf{H}$. Then, the Galerkin scheme (4.1) *has a unique solution $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$, and there exist positive constants C, $\tilde{C}$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h)\|_{\mathbf{H}} \leq C \sup_{\substack{(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in \mathbf{H}_h \\ (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \neq 0}} \frac{|F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)|}{\|(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)\|_{\mathbf{H}}} \leq C\left\{\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\right\},$$

and

$$\|(\boldsymbol{\sigma}, p, \mathbf{u}) - (\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h)\|_{\mathbf{H}} \leq \tilde{C} \inf_{(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in \mathbf{H}_h} \|(\boldsymbol{\sigma}, p, \mathbf{u}) - (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)\|_{\mathbf{H}}.$$

Proof. Since A is bounded and strongly coercive on $\mathbf{H}$ (cf. (3.5) and (4.9)) with constants depending on μ, $(\kappa, \kappa_1, \kappa_2, \kappa_3)$, c_1, and C_1, the proof follows from a straightforward application of the Lax-Milgram Lemma, and Cea's estimate. □

An explicit finite element subspace of $\mathbf{H}$, not satifying (4.8), is given by

$$\tilde{\mathbf{H}}_h := \tilde{H}^{\boldsymbol{\sigma}}_{0,h} \times \tilde{H}^p_h \times \tilde{H}^{\mathbf{u}}_h, \tag{4.10}$$

where $\tilde{H}^{\boldsymbol{\sigma}}_{0,h}$ and $\tilde{H}^{\mathbf{u}}_h$ are defined by (4.6) and (4.7), respectively, and

$$\tilde{H}^p_h := \left\{q_h \in L^2(\Omega) : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h\right\}.$$

The approximation property of $\tilde{H}^p_h$ is given as follows (see [4], [5]):

(AP^p_h) For each $q \in H^1(\Omega)$ there exists $q_h \in \tilde{H}^p_h$ such that

$$\|q - q_h\|_{0,\Omega} \leq C\,h\,\|q\|_{1,\Omega}.$$

Then, we have the following theorem providing the rate of convergence of (4.1) with $\mathbf{H}_h = \tilde{\mathbf{H}}_h$. In this case the number of degrees of freedom defining $\tilde{\mathbf{H}}_h$ behaves asymptoticaly as 5 times the number of triangles of $\mathcal{T}_h$.

Theorem 4.5. *Let* $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H}$ *and* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \tilde{\mathbf{H}}_h$ *be the unique solutions of the continuous and discrete augmented formulations* (3.4) *and* (4.1)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathrm{curl}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, *and* $\mathbf{u} \in [H^{r+1}(\Omega)]^2$, *for some* $r \in (0,1]$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\|(\boldsymbol{\sigma}, p, \mathbf{u}) - (\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h)\|_{\mathbf{H}} \,\le\, C\, h^r \left\{ \|\boldsymbol{\sigma}\|_{r,\Omega} \,+\, \|\mathrm{curl}(\boldsymbol{\sigma})\|_{r,\Omega} \,+\, \|\mathbf{u}\|_{r+1,\Omega} \right\}.$$

Proof. We first notice, according to Theorem 3.3 and the hypothesis on $\boldsymbol{\sigma}$, that $p = \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$ belongs to $H^r(\Omega)$ and that $\|p\|_{H^r(\Omega)} \le C\,\|\boldsymbol{\sigma}\|_{r,\Omega}$. Then, the proof follows from Cea's estimate (cf. Theorem 4.4), the approximation properties $(\mathrm{AP}^{\boldsymbol{\sigma}}_{0,h})$, (AP^{p}_{h}), and $(\mathrm{AP}^{\mathbf{u}}_{h})$, and the interpolation theorems in the corresponding function spaces. □

At this point we remark that the mean value condition $\displaystyle\int_\Omega \boldsymbol{\tau} : \mathbf{J} = 0$ required by the elements of $\tilde{H}^{\boldsymbol{\sigma}}_{0,h}$ (cf. (4.6)), is incorporated into the augmented schemes by means of a Lagrange multiplier $\phi_h \in \mathrm{I\!R}$. In particular, instead of (4.1), we consider the equivalent problem: Find $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \phi_h) \in \tilde{H}^{\boldsymbol{\sigma}}_h \times \tilde{H}^p_h \times \tilde{H}^{\mathbf{u}}_h \times \mathbb{R}$ such that

$$\begin{aligned} A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)) \,+\, \phi_h \int_\Omega \boldsymbol{\tau}_h : \mathbf{J} \quad &= \quad F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)\,, \\ \varphi_h \int_\Omega \boldsymbol{\sigma}_h : \mathbf{J} \quad &= \quad 0\,, \end{aligned} \tag{4.11}$$

for all $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \varphi_h) \in \tilde{H}^{\boldsymbol{\sigma}}_h \times \tilde{H}^p_h \times \tilde{H}^{\mathbf{u}}_h \times \mathrm{I\!R}$. We omit further details here and refer to [10] and [11] for a similar analysis.

On the other hand, if we assume that the Dirichlet datum $\mathbf{g} = \mathbf{0}$, then it follows from our analysis in Section 2 that the tensor $\boldsymbol{\sigma}$ (cf. (2.3)) belongs to H_0. In addition, the velocity $\mathbf{u}$ lives in $[H^1_0(\Omega)]^2$, and the semi-norm $|\cdot|_{1,\Omega}$ is a norm on $[H^1_0(\Omega)]^2$, equivalent to the usual $[H^1(\Omega)]^2$-norm. Consequently, there is no need of introducing the boundary consistent term, whence our augmented dual-mixed variational formulation reduces to: Find $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \tilde{\mathbf{H}} := H_0 \times L^2(\Omega) \times [H^1_0(\Omega)]^2$ such that

$$\tilde{A}((\boldsymbol{\sigma}, p, \mathbf{u}), (\boldsymbol{\tau}, q, \mathbf{v})) \,=\, \tilde{F}(\boldsymbol{\tau}, q, \mathbf{v}) \qquad \forall\, (\boldsymbol{\tau}, q, \mathbf{v}) \in \tilde{\mathbf{H}}\,, \tag{4.12}$$

where the bilinear form $\tilde{A} : \tilde{\mathbf{H}} \times \tilde{\mathbf{H}} \to \mathbb{R}$ and the functional $\tilde{F} : \tilde{\mathbf{H}} \to \mathbb{R}$ are defined by

$$\begin{aligned} \tilde{A}((\boldsymbol{\sigma}, p, \mathbf{u}), (\boldsymbol{\tau}, q, \mathbf{v})) \,:=\, & \frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^r : \boldsymbol{\tau}^r \,+\, \frac{\kappa}{\mu}\int_\Omega \left(p + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})\right)\left(q + \frac{1}{2}(\boldsymbol{\tau} : \mathbf{J})\right) \\ + \int_\Omega \mathbf{u} \cdot \mathrm{curl}(\boldsymbol{\tau}) \,-\, & \int_\Omega \mathbf{v} \cdot \mathrm{curl}(\boldsymbol{\sigma}) \,+\, \kappa_1 \int_\Omega (\underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r) : (\underline{\mathbf{curl}}(\mathbf{v}) + \frac{1}{\mu}\boldsymbol{\tau}^r) \\ & +\, \kappa_2 \int_\Omega \mathrm{curl}(\boldsymbol{\sigma}) \cdot \mathrm{curl}(\boldsymbol{\tau})\,, \end{aligned} \tag{4.13}$$

and

$$\tilde{F}(\boldsymbol{\tau}, q, \mathbf{v}) \,:=\, \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \kappa_2\, \mathrm{curl}(\boldsymbol{\tau}))\,. \tag{4.14}$$

In this way, following the same procedure of Section 3, we can replace Theorem 3.3 by the following.

Theorem 4.6. *Assume that there hold $0 < \kappa_1 < \mu$ and $0 < \kappa_2$. Then, the augmented variational formulation (4.12) has a unique solution $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \tilde{\mathbf{H}}$, independent of κ, and there holds $p = -\frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})$. Moreover, there exists a positive constant C, depending only on μ, (κ_1, κ_2), c_1, and C_1, such that*

$$\|(\boldsymbol{\sigma}, p, \mathbf{u})\|_{\tilde{\mathbf{H}}} \le C \|\mathbf{f}\|_{0,\Omega},$$

where

$$\|(\boldsymbol{\tau}, q, \mathbf{v})\|_{\tilde{\mathbf{H}}} := \left\{ \|\boldsymbol{\tau}\|_{\mathrm{curl},\Omega}^2 + \|q\|_{0,\Omega}^2 + |\mathbf{v}|_{1,\Omega}^2 \right\}^{1/2} \qquad \forall (\boldsymbol{\tau}, q, \mathbf{v}) \in \tilde{\mathbf{H}}.$$

Next, given an arbitrary finite element subspace $\tilde{\mathbf{H}}_h \subseteq \tilde{\mathbf{H}}$, the Galerkin scheme associated with (4.12) reads: Find $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \tilde{\mathbf{H}}_h$ such that

$$\tilde{A}((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h)) = \tilde{F}(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \qquad \forall (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in \tilde{\mathbf{H}}_h. \tag{4.15}$$

In particular, we consider

$$\tilde{\mathbf{H}}_h := \tilde{H}_{0,h}^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}},$$

where $\tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ is defined by (4.6),

$$\tilde{H}_h^p := \left\{ q_h \in L^2(\Omega) : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h \right\},$$

and

$$\tilde{H}_{0,h}^{\mathbf{u}} := \left\{ \mathbf{v}_h \in \tilde{H}_h^{\mathbf{u}} : \quad \mathbf{v}_h = \mathbf{0} \quad \text{on} \quad \Gamma \right\}, \tag{4.16}$$

where $\tilde{H}_h^{\mathbf{u}}$ is defined by (4.7).

The rest of the analysis, including the well-posedness of (4.12) and (4.15), the corresponding a priori error estimates, and the rates of convergences, follows exactly as in Sections 3 and 4. We omit further details.

5. A POSTERIORI ERROR ANALYSIS

In this section we follow the approach from [3] (see also [7]) and derive residual based a posteriori error estimators for (4.1) and (4.2). Actually, the analysis focuses in (4.1) and the corresponding estimator for (4.2) follows as a particular case.

First we introduce several notations. Given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges, and let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $\mathcal{E}_h := \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\Gamma}$, where $\mathcal{E}_{h,\Omega} := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$, and $\mathcal{E}_{h,\Gamma} := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the length of the edge e. Further, given $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in C(T)$ on each $T \in \mathcal{T}_h$, an edge $e \in \mathcal{E}_{h,\Omega}$, and the unit tangential vector $\mathbf{s}_T$ along e, we let $J[\boldsymbol{\tau}\, \mathbf{s}_T]$ be the corresponding jump across e, that is, $J[\boldsymbol{\tau}\, \mathbf{s}_T] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\, \mathbf{s}_T$, where T' is the other triangle of $\mathcal{T}_h$ having e as an edge. Abusing notation, when $e \in \mathcal{E}_{h,\Gamma}$, we also write $J[\boldsymbol{\tau}\, \mathbf{s}_T] := \boldsymbol{\tau}|_e\, \mathbf{s}_T$. We recall here that $\mathbf{s}_T := (-\nu_2, \nu_1)^{\mathbf{t}}$, where $\boldsymbol{\nu}_T := (\nu_1, \nu_2)^{\mathbf{t}}$ is the unit outward vector normal to ∂T. Analogously, we define the normal jumps $J[\boldsymbol{\tau}\, \boldsymbol{\nu}_T]$.

Then, letting $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H}$ and $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ be the unique solutions of the continuous and discrete augmented formulations (3.4) and (4.1), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$
\begin{aligned}
\theta_T^2 \;:=\; & \|\mathbf{f} + \operatorname{curl}(\boldsymbol{\sigma}_h)\|_{0,T}^2 + \left\| p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J}) \right\|_{0,T}^2 + h_T^2 \left\| \mathbf{div}\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \right\|_{0,T}^2 \\
& + \; h_T^2 \left\| \mathbf{div}\left\{ \left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J}) \right) \mathbf{J} \right\} \right\|_{0,T}^2 + h_T^2 \left\| \mathbf{div}\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \right\|_{0,T}^2 \\
& + \; h_T^2 \left\| \operatorname{curl}\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \right\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T)} h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T)} h_e \left\| J\left[\left\{ \left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J}) \right) \mathbf{J} \right\} \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T)} h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Gamma}} h_e \|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Gamma}} h_e \left\| \frac{d\mathbf{g}}{d\mathbf{s}_T} - \frac{d\mathbf{u}_h}{d\mathbf{s}_T} \right\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Gamma}} h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \mathbf{s}_T \right] \right\|_{0,e}^2 . \qquad (5.1)
\end{aligned}
$$

The residual character of each term on the right hand side of (5.1) is quite clear. Next, we let

$$
\boldsymbol{\theta} \;:=\; \left\{ \sum_{T \in \mathcal{T}_h} \theta_T^2 \right\}^{1/2} \qquad (5.2)
$$

be the global residual error estimator. Then, the following theorem is the main result of this section.

Theorem 5.1. *Let* $(\boldsymbol{\sigma}, p, \mathbf{u}) \in \mathbf{H}$ *and* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h) \in \mathbf{H}_h$ *be the unique solutions of* (3.4) *and* (4.1)*, respectively. Then there exist positive constants* C_{eff} *and* C_{rel}*, independent of* h*, such that*

$$
C_{\text{eff}}\, \boldsymbol{\theta} \;\le\; \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}} \;\le\; C_{\text{rel}}\, \boldsymbol{\theta} . \qquad (5.3)
$$

The upper and lower bounds in (5.3), which are called reliability and efficiency, respectively, of the global estimator $\boldsymbol{\theta}$, are proved next in Sections 5.1 and 5.2.

5.1. Reliability

We begin with the following preliminary estimate.

Lemma 5.2. *There exists $C > 0$, independent of h, such that*

$$C \,\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}} \;\le\; \sup_{\substack{(\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H}\setminus\{\mathbf{0}\} \\ \mathrm{curl}(\boldsymbol{\tau}) = \mathbf{0}}} \frac{A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h), (\boldsymbol{\tau}, q, \mathbf{v}))}{\|(\boldsymbol{\tau}, q, \mathbf{v})\|_{\mathbf{H}}} \;+\; \|\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,. \tag{5.4}$$

Proof. Let us define $\boldsymbol{\sigma}^* \;=\; \underline{\mathbf{curl}}(\mathbf{z})$, where $\mathbf{z} \,\in\, [H_0^1(\Omega)]^2$ is the unique solution of the boundary value problem: $-\Delta \mathbf{z} \;=\; \mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h)$ in Ω, $\mathbf{z} \;=\; \mathbf{0}$ on Γ. It is easy to see that $\boldsymbol{\sigma}^* \,\in\, H_0$ (cf. (2.11)) and the corresponding continuous dependence result establishes the existence of $c > 0$ such that

$$\|\boldsymbol{\sigma}^*\|_{\mathrm{curl},\Omega} \;\le\; c\, \|\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,. \tag{5.5}$$

In addition, since $\mathrm{curl}(\underline{\mathbf{curl}}(\mathbf{z})) \;=\; \Delta \mathbf{z}$, we find that

$$\mathrm{curl}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*) \;=\; -\,\mathbf{f} - \mathrm{curl}(\boldsymbol{\sigma}_h) + (\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h)) \;=\; \mathbf{0} \quad \text{in} \quad \Omega \,.$$

Now, let α and M be the coercivity and boundedness constants of A. Then, there holds

$$\begin{aligned}
&\alpha \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}}^2 \\
&\quad\le\; A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h)) \\
&\quad\le\; A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h)) \\
&\quad-\; A((\boldsymbol{\sigma}^*, 0, \mathbf{0}), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h)) \,,
\end{aligned}$$

which, employing the boundedness of A, yields

$$\begin{aligned}
&\alpha \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}} \\
&\le \sup_{\substack{(\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H}\setminus\{\mathbf{0}\} \\ \mathrm{curl}(\boldsymbol{\tau}) = \mathbf{0}}} \frac{A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h), (\boldsymbol{\tau}, q, \mathbf{v}))}{\|(\boldsymbol{\tau}, q, \mathbf{v})\|_{\mathbf{H}}} \;+\; M\, \|\boldsymbol{\sigma}^*\|_{\mathrm{curl},\Omega} \,.
\end{aligned} \tag{5.6}$$

Hence (5.4) follows straightforwardly from the triangle inequality, (5.5) and (5.6). □

In order to bound the first term on the right hand side of (5.4), we will make use of the Clément interpolation operator $I_h : H^1(\Omega) \to X_h$ (cf. [6]), where X_h is given by

$$X_h \,:=\, \left\{ v_h \in C(\bar{\Omega}) : \quad v_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h \right\} \,.$$

It is well known that I_h satisfies the following local approximation properties.

Lemma 5.3. *There exist constants C_1, $C_2 > 0$, independent of h, such that for all $v \,\in\, H^1(\Omega)$ there holds*

$$\|v - I_h(v)\|_{0,T} \;\le\; C_1\, h_T\, \|v\|_{1,\tilde{\omega}_T} \qquad \forall\, T \in \mathcal{T}_h \,,$$

and

$$\|v - I_h(v)\|_{0,e} \;\le\; C_2\, h_e^{1/2}\, \|v\|_{1,\tilde{\omega}_e} \qquad \forall\, e \in \mathcal{T}_h \,,$$

where $\tilde{\omega}_T$ and $\tilde{\omega}_e$ are the union of all elements sharing at least one point with T and e, respectively.

Proof. See [6]. □

We now let $(\boldsymbol{\tau}, q, \mathbf{v}) \in \mathbf{H}, (\boldsymbol{\tau}, q, \mathbf{v}) \neq \mathbf{0}$, such that $\mathrm{curl}(\boldsymbol{\tau}) = \mathbf{0}$ in Ω. Since Ω is simply connected, there exists a function $\boldsymbol{\varphi} := (\varphi_1, \varphi_2) \in [H^1(\Omega)]^2$ such that $\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$ and $\boldsymbol{\tau} = \nabla \boldsymbol{\varphi}$. Note that

$$\|\boldsymbol{\varphi}\|_{1,\Omega} \le C\,|\boldsymbol{\varphi}|_{1,\Omega} = \|\boldsymbol{\tau}\|_{0,\Omega} = \|\boldsymbol{\tau}\|_{\mathrm{curl},\Omega}\,. \tag{5.7}$$

Then, we let $\boldsymbol{\varphi}_h := (I_h(\varphi_1), I_h(\varphi_2))$ and define $\boldsymbol{\tau}_h := \nabla \boldsymbol{\varphi}_h$. It is easy to see that $\boldsymbol{\tau}_h$ belongs to $\tilde{H}_h^{\boldsymbol{\sigma}}$ (cf. (4.4)), and there holds the decomposition $\boldsymbol{\tau}_h = \boldsymbol{\tau}_{h,0} + c_h\,\mathbf{J}$, where $\boldsymbol{\tau}_{h,0} \in \tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ (cf. (4.6)) and $c_h := \frac{1}{2|\Omega|}\int_\Omega \boldsymbol{\tau}_h : \mathbf{J} \in \mathbb{R}$.

Next, we define $\mathbf{v}_h := (I_h(v_1), I_h(v_2)) \in \tilde{H}_h^{\mathbf{u}}$, the vector Clément interpolant of $\mathbf{v} := (v_1, v_2) \in [H^1(\Omega)]^2$ (cf. (4.7)), and deduce, according to the Galerkin orthogonality and the definition of the bilinear form A (cf. (3.5)), that

$$A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h),(\boldsymbol{\tau}, q, \mathbf{v})) = A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h),(\boldsymbol{\tau}-\boldsymbol{\tau}_{h,0}, q, \mathbf{v}-\mathbf{v}_h)) \tag{5.8}$$

and

$$A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h),(c_h\mathbf{J}, 0, \mathbf{0})) = A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h),(\mathbf{0}, c_h, \mathbf{0})) = 0\,. \tag{5.9}$$

We have assumed here, without los of generality, that $\mathbb{P}_0(\Omega) \subseteq H_h^p$. Hence, it follows from (5.8), (5.9), and (4.1), after some algebraic manipulations, that

$$\begin{aligned}
&A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h),(\boldsymbol{\tau}, q, \mathbf{v}))\\
&=\int_\Omega \left\{\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r\right) - \frac{\kappa}{2\mu}\left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\right)\mathbf{J} - \frac{\kappa_1}{\mu}\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\right\} : (\boldsymbol{\tau}-\boldsymbol{\tau}_h)\\
&+ \int_\Gamma (\mathbf{g}-\mathbf{u}_h)\cdot(\boldsymbol{\tau}-\boldsymbol{\tau}_h)\,\mathbf{s} - \kappa_1 \int_\Omega \left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r\right) : \underline{\mathbf{curl}}(\mathbf{v}-\mathbf{v}_h)\\
&+ \int_\Omega (\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h))\cdot(\mathbf{v}-\mathbf{v}_h) + \kappa_3 \int_\Gamma (\mathbf{g}-\mathbf{u}_h)\cdot(\mathbf{v}-\mathbf{v}_h) - \frac{\kappa}{\mu}\int_\Omega \left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\right) q.
\end{aligned} \tag{5.10}$$

The rest of reliability consists in deriving suitable upper bounds for each one of the terms appearing on the right hand side of (5.10). We begin with those terms involving $\boldsymbol{\tau}-\boldsymbol{\tau}_h := \nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)$. In fact, given $\boldsymbol{\xi} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\xi}|_T \in [H^1(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h$, we find that

$$\begin{aligned}
\int_\Omega \boldsymbol{\xi} : (\boldsymbol{\tau}-\boldsymbol{\tau}_h) &= \int_\Omega \boldsymbol{\xi} : \nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) = \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\xi} : \nabla(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\\
&= -\sum_{T\in\mathcal{T}_h}\int_T \mathbf{div}(\boldsymbol{\xi})\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) + \sum_{e\in\mathcal{E}_h}\langle J[\boldsymbol{\xi}\,\boldsymbol{\nu}_T], \boldsymbol{\varphi}-\boldsymbol{\varphi}_h\rangle_{0,e}\,,
\end{aligned}$$

and hence, applying the Cauchy-Schwarz inequality, Lemma 5.4, the fact that the number of triangles of $\tilde{\omega}_T$ and $\tilde{\omega}_e$ are bounded independently of h, and inequality (5.7), we get

$$\begin{aligned}
&\left|\int_\Omega \boldsymbol{\xi} : (\boldsymbol{\tau}-\boldsymbol{\tau}_h)\right|\\
&\le C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\|\mathbf{div}(\boldsymbol{\xi})\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e\,\|J[\boldsymbol{\xi}\,\boldsymbol{\nu}_T]\|_{0,e}^2\right\}^{1/2}\|\boldsymbol{\tau}\|_{\mathrm{curl},\Omega}\,.
\end{aligned} \tag{5.11}$$

In this way, applying (5.11) to $\boldsymbol{\xi} = \left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)$, then to $\boldsymbol{\xi} := \frac{\kappa}{2\mu}\left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\right)\mathbf{J}$, and finally to $\boldsymbol{\xi} := \frac{\kappa_1}{\mu}\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)$, we obtain the reliability estimates for the terms appearing in the first row of the right hand side of (5.10).

Similarly, for the term involving $(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,\mathbf{s} := \frac{\mathbf{d}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)}{d\mathbf{s}}$, we obtain

$$\int_\Gamma (\mathbf{g} - \mathbf{u}_h)\cdot(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,\mathbf{s} = \int_\Gamma (\mathbf{g} - \mathbf{u}_h)\cdot\frac{\mathbf{d}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)}{d\mathbf{s}} = -\sum_{e\in\mathcal{E}_{h,\Gamma}}\int_e \left(\frac{d\mathbf{g}}{d\mathbf{s}} - \frac{d\mathbf{u}_h}{d\mathbf{s}}\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\,,$$

which yields

$$\left|\int_\Gamma (\mathbf{g} - \mathbf{u}_h)\cdot(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,\mathbf{s}\right| \le C\left\{\sum_{e\in\mathcal{E}_{h,\Gamma}} h_e \left\|\frac{d\mathbf{g}}{d\mathbf{s}} - \frac{d\mathbf{u}_h}{d\mathbf{s}}\right\|_{0,e}^2\right\}^{1/2} \|\boldsymbol{\varphi}\|_{1,\Omega}\,. \tag{5.12}$$

On the other hand, for the terms in (5.10) containing the expression $\mathbf{v} - \mathbf{v}_h$, we first notice that

$$\left|\int_\Omega (\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h))\cdot(\mathbf{v} - \mathbf{v}_h)\right| \le C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\|\mathbf{f} + \mathrm{curl}(\boldsymbol{\sigma}_h)\|_{0,T}^2\right\}^{1/2} \|\mathbf{v}\|_{1,\Omega}\,, \tag{5.13}$$

and

$$\left|\int_\Gamma (\mathbf{g} - \mathbf{u}_h)\cdot(\mathbf{v} - \mathbf{v}_h)\right| \le C\left\{\sum_{e\in\mathcal{E}_{h,\Gamma}} h_e\,\|\mathbf{g} - \mathbf{u}_h\|_{0,e}^2\right\}^{1/2} \|\mathbf{v}\|_{1,\Omega}\,. \tag{5.14}$$

In addition, letting $\boldsymbol{\xi}_h := \underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$, we find that

$$\begin{aligned}\int_\Omega \boldsymbol{\xi}_h : \underline{\mathbf{curl}}(\mathbf{v} - \mathbf{v}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\xi}_h : \underline{\mathbf{curl}}(\mathbf{v} - \mathbf{v}_h)\\ &= -\sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}(\boldsymbol{\xi}_h)\cdot(\mathbf{v} - \mathbf{v}_h) + \sum_{e\in\mathcal{E}_h}\langle J[\boldsymbol{\xi}_h\,\mathbf{s}_T], \mathbf{v} - \mathbf{v}_h\rangle_{0,e}\,,\end{aligned}$$

which gives

$$\begin{aligned}&\left|\int_\Omega \boldsymbol{\xi}_h : \underline{\mathbf{curl}}(\mathbf{v} - \mathbf{v}_h)\right|\\ &\le C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\,\|\mathrm{curl}(\boldsymbol{\xi}_h)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e\,\|J[\boldsymbol{\xi}_h\,\mathbf{s}_T]\|_{0,e}^2\right\}^{1/2}\|\mathbf{v}\|_{1,\Omega}\,.\end{aligned} \tag{5.15}$$

Certainly, the arguments yielding (5.11) have also been employed to derive (5.12) - (5.15).

Finally, we apply again the Cauchy-Schwarz inequality and obtain

$$\left|\int_\Omega \left(p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\right) q\right| \le \left\|p_h + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\right\|_{0,\Omega} \|q\|_{0,\Omega}. \tag{5.16}$$

In this way, we conclude from (5.10) - (5.16) that

$$\sup_{\substack{(\boldsymbol{\tau},q,\mathbf{v})\in\mathbf{H}\setminus\{\mathbf{0}\}\\ \mathrm{curl}(\boldsymbol{\tau})=\mathbf{0}}} \frac{A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,p-p_h,\mathbf{u}-\mathbf{u}_h),(\boldsymbol{\tau},q,\mathbf{v}))}{\|(\boldsymbol{\tau},q,\mathbf{v})\|_{\mathbf{H}}} \le C\,\boldsymbol{\theta}\,,$$

which, together with Lemma 5.2, completes the proof of reliability of $\boldsymbol{\theta}$ (cf. (5.1) and (5.2)).

At this point we remark that when the finite element subspace $\mathbf{H}_h$ is given by (4.10), that is, when $\boldsymbol{\sigma}_h|_T \in [W(T)^{\mathrm{t}}]^2$, $p_h|_T \in \mathbb{P}_0(T)$, and $\mathbf{u}_h|_T \in [\mathbb{P}_1(T)]^2$, then the expression (5.1) for θ_T^2 simplifies to

$$\begin{aligned}
\theta_T^2 \; := \;& \|\mathbf{f}+\mathrm{curl}(\boldsymbol{\sigma}_h)\|_{0,T}^2 + \left\|p_h+\tfrac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})\right\|_{0,T}^2 + h_T^2\left\|\mathbf{div}\left(\tfrac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\right\|_{0,T}^2\\
+\;& h_T^2\left\|\mathbf{div}\left(\tfrac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})\mathbf{J}\right)\right\|_{0,T}^2 + h_T^2\left\|\mathrm{curl}\left(\tfrac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\right\|_{0,T}^2\\
+\;& \sum_{e\in\mathcal{E}(T)} h_e\left\|J\left[\left(\mathbf{curl}(\mathbf{u}_h)-\frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\boldsymbol{\nu}_T\right]\right\|_{0,e}^2\\
+\;& \sum_{e\in\mathcal{E}(T)} h_e\left\|J\left[\left\{\left(p_h+\frac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})\right)\mathbf{J}\right\}\boldsymbol{\nu}_T\right]\right\|_{0,e}^2\\
+\;& \sum_{e\in\mathcal{E}(T)} h_e\left\|J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r-\frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\boldsymbol{\nu}_T\right]\right\|_{0,e}^2\\
+\;& \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Gamma}} h_e\,\|\mathbf{g}-\mathbf{u}_h\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Gamma}} h_e\left\|\frac{d\mathbf{g}}{d\mathbf{s}_T}-\frac{d\mathbf{u}_h}{d\mathbf{s}_T}\right\|_{0,e}^2\\
+\;& \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Gamma}} h_e\left\|J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h)-\frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\mathbf{s}_T\right]\right\|_{0,e}^2 .
\end{aligned} \tag{5.17}$$

We end this section by mentioning that the local error indicator θ_T^2 for the Galerkin scheme (4.2) arises from (5.1) after eliminating the terms containing the discrete pressure p_h. Indeed, this follows easily from the fact that $p = -\frac{1}{2}(\boldsymbol{\sigma}:\mathbf{J})$ and that $p_h = -\frac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})$ when $(\boldsymbol{\tau}_h:\mathbf{J}) \in H_h^p \quad \forall\,\boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}}$.

5.2. *Efficiency of the a posteriori error estimators*

In this section, we apply inverse inequalities (see [5]) and the localization technique based on bubble functions (see [15]), to prove the lower bound of the estimate (5.3). More precisely, in what follows we estimate each one of the 12 terms defining the error indicator θ_T^2 (cf. (5.1)).

We first employ the equilibrium equation $\mathbf{f} = -\mathrm{curl}(\boldsymbol{\sigma})$ in Ω, the incompressibility condition $p+\frac{1}{2}(\boldsymbol{\sigma}:\mathbf{J}) = 0$ in Ω, and the Dirichlet boundary condition $\mathbf{u}=\mathbf{g}$ on Γ. Indeed, it follows that

$$\|\mathbf{f}+\mathrm{curl}(\boldsymbol{\sigma}_h)\|_{0,T}^2 = \|\mathrm{curl}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,T}^2 \quad \forall\, T\in\mathcal{T}_h\,, \tag{5.18}$$

$$\left\|p_h+\frac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})\right\|_{0,T}^2 \le \|p-p_h\|_{0,T}^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2 \quad \forall\, T\in\mathcal{T}_h\,, \tag{5.19}$$

and

$$\sum_{e\in\mathcal{E}_{h,\Gamma}} h_e \,\|\mathbf{g}-\mathbf{u}_h\|_{0,e}^2 \;\le\; h\,\|\mathbf{g}-\mathbf{u}_h\|_{0,\Gamma}^2 \;=\; h\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Gamma}^2 \;\le\; C\,h\,\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}^2\,. \tag{5.20}$$

In order to derive the upper bounds of the remaining terms, we will make use of Lemmata 5.4 - 5.7 below. More precisely Lemma 5.4 is required for the terms involving the **div** operator, Lemma 5.5 handles the terms containing the curl operator, Lemma 5.6 is required for the terms involving the normal jumps across the edges of $\mathcal{T}_h$, and Lemma 5.7 is used to take care of the terms encompassing tangential jumps across the edges of $\mathcal{T}_h$. For the proofs of these lemmas we refer to [3] and references therein. In what follows, we denote

$$\omega_e \;:=\; \cup\{T'\in\mathcal{T}_h : \quad e\in\mathcal{E}(T')\}\,.$$

Lemma 5.4. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polinomial of degree $k\ge 0$ on each $T\in\mathcal{T}_h$. Then, there exists $c>0$, independent of h, such that for any $T\in\mathcal{T}_h$*

$$\|\mathbf{div}(\boldsymbol{\rho}_h)\|_{0,T} \;\le\; c\,h_T^{-1}\,\|\boldsymbol{\rho}_h\|_{0,T}\,. \tag{5.21}$$

Lemma 5.5. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polinomial of degree $k\ge 0$ on each $T\in\mathcal{T}_h$. In addition, let $\boldsymbol{\rho}\in[L^2(\Omega)]^{2\times 2}$ be such that $\mathrm{curl}(\boldsymbol{\rho})=\mathbf{0}$ on each $T\in\mathcal{T}_h$. Then, there exists $c>0$, independent of h, such that for any $T\in\mathcal{T}_h$*

$$\|\mathrm{curl}(\boldsymbol{\rho}_h)\|_{0,T} \;\le\; c\,h_T^{-1}\,\|\boldsymbol{\rho}-\boldsymbol{\rho}_h\|_{0,T}\,. \tag{5.22}$$

Lemma 5.6. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polinomial of degree $k\ge 0$ on each $T\in\mathcal{T}_h$. Then, there exists $c>0$, independent of h, such that for any $e\in\mathcal{E}_h$*

$$\|J[\boldsymbol{\rho}_h\boldsymbol{\nu}_T]\|_{0,e} \;\le\; c\,h_e^{-1/2}\,\|\boldsymbol{\rho}_h\|_{0,\omega_e}\,. \tag{5.23}$$

Lemma 5.7. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polinomial of degree $k\ge 0$ on each $T\in\mathcal{T}_h$. Then, there exists $c>0$, independent of h, such that for any $e\in\mathcal{E}_h$*

$$\|J[\boldsymbol{\rho}_h\mathbf{s}_T]\|_{0,e} \;\le\; c\,h_e^{-1/2}\,\|\boldsymbol{\rho}_h\|_{0,\omega_e}\,. \tag{5.24}$$

We are now in a position to complete the proof of efficiency of $\boldsymbol{\theta}$ by conveniently applying Lemmata 5.4 - 5.7 to the corresponding terms defining θ_T^2. In fact, we have the following estimates.

Lemma 5.8. *There exist C_1, C_2, $C_3>0$, independent of h, such that for any $T\in\mathcal{T}_h$ there hold*

$$h_T^2\,\left\|\mathbf{div}\left(\underline{\mathbf{curl}}(\mathbf{u}_h)-\frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\right\|_{0,T}^2 \;\le\; C_1\,\left\{|\mathbf{u}-\mathbf{u}_h|_{1,T}^2 \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2\right\}, \tag{5.25}$$

$$h_T^2\,\left\|\mathbf{div}\left(p_h\mathbf{J}+\frac{1}{2}(\boldsymbol{\sigma}_h:\mathbf{J})\mathbf{J}\right)\right\|_{0,T}^2 \;\le\; C_2\,\left\{\|p-p_h\|_{0,T}^2 \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2\right\}, \tag{5.26}$$

and

$$h_T^2\,\left\|\mathbf{div}\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r-\frac{1}{\mu}\boldsymbol{\sigma}_h^r\right)\right\|_{0,T}^2 \;\le\; C_3\,\left\{|\mathbf{u}-\mathbf{u}_h|_{1,T}^2 \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,T}^2\right\}. \tag{5.27}$$

Proof. The estimate (5.25) follows from Lemma 5.4 defining $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$, introducing $\mathbf{0} = \underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r$ (cf. (2.16)), and then using the triangle inequality and the continuity of the operators $\underline{\mathbf{curl}}$ and $\boldsymbol{\tau} \to \boldsymbol{\tau}^r$. Similarly, (5.26) and (5.27) follow from Lemma 5.4 with $\boldsymbol{\rho}_h := p_h\mathbf{J} + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\mathbf{J}$ and $\mathbf{0} = p\mathbf{J} + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})\mathbf{J}$ (cf. (2.8)), and $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$ and $\mathbf{0} = \underline{\mathbf{curl}}(\mathbf{u})^r - \frac{1}{\mu}\boldsymbol{\sigma}^r$ (cf. (2.16)), respectively. □

Lemma 5.9. *There exists $C_4 > 0$, independent of h, such that for any $T \in \mathcal{T}_h$ there holds*

$$h_T^2 \left\| \operatorname{curl}\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \right\|_{0,T}^2 \leq C_4 \left\{ |\mathbf{u} - \mathbf{u}_h|_{1,T}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T}^2 \right\} . \tag{5.28}$$

Proof. It follows by applying Lemma 5.5 with $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$ and $\boldsymbol{\rho} := \underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r = \mathbf{0}$, and then using the triangle inequality and the continuity of the linear operators $\underline{\mathbf{curl}}$ and $\boldsymbol{\tau} \to \boldsymbol{\tau}^r$. □

Lemma 5.10. *There exists C_5, C_6, $C_7 > 0$, independent of h, such that for any $e \in \mathcal{E}_h$ there hold*

$$h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \leq C_5 \left\{ |\mathbf{u} - \mathbf{u}_h|_{1,\omega_e}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \right\} , \tag{5.29}$$

$$h_e \left\| J\left[\left(p_h\mathbf{J} + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\mathbf{J} \right) \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \leq C_6 \left\{ \|p - p_h\|_{0,\omega_e}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \right\} , \tag{5.30}$$

and

$$h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \boldsymbol{\nu}_T \right] \right\|_{0,e}^2 \leq C_7 \left\{ |\mathbf{u} - \mathbf{u}_h|_{1,\omega_e}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \right\} . \tag{5.31}$$

Proof. The estimates (5.29), (5.30) and (5.31) follow all from Lemma 5.6 with $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$, $\boldsymbol{\rho}_h := p_h\mathbf{J} + \frac{1}{2}(\boldsymbol{\sigma}_h : \mathbf{J})\mathbf{J}$, and $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h)^r - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$ respectively, and then employing that $\mathbf{0} = \underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r$, $\mathbf{0} = p\mathbf{J} + \frac{1}{2}(\boldsymbol{\sigma} : \mathbf{J})\mathbf{J}$, and $\mathbf{0} = \underline{\mathbf{curl}}(\mathbf{u})^r - \frac{1}{\mu}\boldsymbol{\sigma}^r$, respectively. □

Lemma 5.11. *There exists $C_8 > 0$, independent of h, such that*

$$\sum_{e \in \mathcal{E}_{h,\Gamma}} h_e \left\| \frac{d\mathbf{g}}{d\mathbf{s}_T} - \frac{d\mathbf{u}_h}{d\mathbf{s}_T} \right\|_{0,e}^2 \leq C_8 \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2 . \tag{5.32}$$

Proof. The proof is similar to Lemma 5.7 in [9] and Lemma 4.5 in [2]. We omit further details. □

Lemma 5.12. *There exists $C_9 > 0$, independent of h, such that for any $e \in \mathcal{E}_h$*

$$h_e \left\| J\left[\left(\underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r \right) \mathbf{s}_T \right] \right\|_{0,e}^2 \leq C_9 \left\{ |\mathbf{u} - \mathbf{u}_h|_{1,\omega_e}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e}^2 \right\} . \tag{5.33}$$

Proof. The estimate (5.33) arises from a direct application of Lemma 5.7 to $\boldsymbol{\rho}_h := \underline{\mathbf{curl}}(\mathbf{u}_h) - \frac{1}{\mu}\boldsymbol{\sigma}_h^r$, introducing $\mathbf{0} = \underline{\mathbf{curl}}(\mathbf{u}) - \frac{1}{\mu}\boldsymbol{\sigma}^r$(cf. (2.16)), and then employing again the triangle inequality and the continuity of the operators $\underline{\mathbf{curl}}$ and $\boldsymbol{\tau} \to \boldsymbol{\tau}^r$. □

In this way, the efficiency of $\boldsymbol{\theta}$ follows straightforwardly from the estimates (5.18) - (5.20), and (5.25) - (5.33) after summing over all $T \in \mathcal{T}_h$ and using that the number of triangles on each domain ω_e is bounded by two.

6. NUMERICAL RESULTS

In this section we present several examples illustrating the performance of the augmented mixed finite element schemes (4.1) and (4.2), and confirming the reliability and efficiency of the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 5. For the computations we consider the specific finite element subspaces given by $\tilde{\mathbf{H}}_h$ (cf. (4.10)) and $\tilde{\mathbf{H}}_{0,h}$ (cf. (4.5)), respectively.

We now introduce some notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (4.1) and (4.2), which, as mentioned in Section 4, behaves asymptotically as 5 and 4 times, respectively, the number of elements of each triangulation. Also, the individual and total errors are given by

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{curl},\Omega} \,, \quad e(p) := \|p - p_h\|_{0,\Omega} \,,$$

$$e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \,, \quad e(\boldsymbol{\sigma}, \mathbf{u}) := \left\{ (e(\boldsymbol{\sigma}))^2 + (e(\mathbf{u}))^2 \right\}^{1/2} \,,$$

and

$$e(\boldsymbol{\sigma}, p, \mathbf{u}) := \left\{ (e(\boldsymbol{\sigma}))^2 + (e(p))^2 + (e(\mathbf{u}))^2 \right\}^{1/2} \,,$$

whereas the effectivity index with respect to $\boldsymbol{\theta}$ is defined either by

$$eff(\boldsymbol{\theta}) := e(\boldsymbol{\sigma}, p, \mathbf{u})/\boldsymbol{\theta} \quad \text{or} \quad eff(\boldsymbol{\theta}) := e(\boldsymbol{\sigma}, \mathbf{u})/\boldsymbol{\theta} \,.$$

In addition, we define the experimental rates of convergence

$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')} \,, \; r(p) := \frac{\log(e(p)/e'(p))}{\log(h/h')} \,, \; r(\mathbf{u}) := \frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')} \,,$$

$$r(\boldsymbol{\sigma}, \mathbf{u}) := \frac{\log(e(\boldsymbol{\sigma}, \mathbf{u})/e'(\boldsymbol{\sigma}, \mathbf{u}))}{\log(h/h')} \,, \text{ and } r(\boldsymbol{\sigma}, p, \mathbf{u}) := \frac{\log(e(\boldsymbol{\sigma}, p, \mathbf{u})/e'(\boldsymbol{\sigma}, p, \mathbf{u}))}{\log(h/h')} \,,$$

where e and e' denote the corresponding errors at two consecutive triangulations with mesh sizes h and h', respectively. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. The first example is employed to illustrate the performance of the augmented mixed finite element schemes and to confirm the properties of the a posteriori error estimator $\boldsymbol{\theta}$. Then, Examples 2 and 3 are utilized to show the behaviour of the adaptive algorithm for the scheme (4.1), which applies the following procedure from [15]:

1) Start with a coarse mesh $\mathcal{T}_h$.
2) Solve the discrete problem (4.1) for the actual mesh $\mathcal{T}_h$.
3) Compute θ_T (cf. (5.1)) for each triangle $T \in \mathcal{T}_h$.
4) Evaluate stopping criterion and decide to finish or go to next step.
5) Use *blue-green* procedure to refine each $T' \in \mathcal{T}_h$ whose indicator $\theta_{T'}$ satisfies

$$\theta_{T'} \;\geq\; \frac{1}{2} \max\{\theta_T : \quad T \in \mathcal{T}_h \} \,.$$

6) Define resulting mesh as actual mesh $\mathcal{T}_h$ and go to step 2.

In Example 1 we consider $\Omega\ =]0,1[^2$, $\mu\ =\ 1$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x})\ :=\ \frac{1}{4\pi\mu}\left\{-\log\mathtt{r}\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{\mathtt{r}^2}\begin{pmatrix}(x_1-2)^2\\(x_1-2)(x_2-2)\end{pmatrix}\right\}$$

and

$$p(\mathbf{x})\ =\ \frac{(x_1-2)}{2\pi\mathtt{r}^2}\ -\ p_0\,,$$

with $\mathtt{r}=\sqrt{(x_1-2)^2+(x_2-2)^2}$, for all $\mathbf{x}\ :=\ (x_1,x_2)\ \in\ \Omega$, where $p_0\in\mathbb{R}$ is such that $\int_\Omega p=0$ holds. At this point we recall from (2.7) and the fact that $\boldsymbol{\sigma}\ \in\ H_0$, that an admissible solution p must satisfy $\int_\Omega p=0$. Note that $(\mathbf{u},p)$ corresponds to the fundamental solution located at the point $(2,2)$. Hence, $\mathbf{f}=\mathbf{0}$, $\mathbf{u}$ is curl free, and $(\mathbf{u},p)$ is regular in the whole domain Ω.

In Example 2 we take Ω as the L-shaped domain $]-1,1[^2-]0,1[^2$, $\mu\ =\ 2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\mathbf{u}(\mathbf{x})\ :=\ \left((x_1-0.1)^2+(x_2-0.1)^2\right)^{-1/2}\begin{pmatrix}0.1-x_2\\x_1-0.1\end{pmatrix}$$

and

$$p(\mathbf{x})\ =\ \frac{1}{x_1+1.1}\ -\ p_0\,,$$

for all $\mathbf{x}\ :=\ (x_1,x_2)\ \in\ \Omega$. We note that $\mathbf{u}$ is curl free in Ω. In addition, it is clear that $\mathbf{u}$ and p are singular at $(0.1,0.1)$ and along the line $x_1\ =\ -1.1$, respectively. Hence, we should expect regions of high gradients around the origin and along the line $x_1\ =\ -1$.

In Example 3 we consider the standard test case given by a driven cavity. More precisely, we take $\Omega\ =]0,1[^2$, $\mu\ =\ 2$, and choose the data

$$\mathbf{f}\ =\ \mathbf{0}\quad\text{in}\quad\Omega\quad\text{and}\quad\mathbf{g}(x_1,x_2)\ :=\ \begin{cases}(\sin(\pi\,x_1),0) & \text{if}\quad 0<x_1<1,\ x_2=1\\ \qquad 0 & \text{otherwise}\,.\end{cases}$$

The numerical results shown below were obtained using a MATLAB code. In order to emphasize the robustness of (4.2) with respect to the parameters κ_1, κ_2, and κ_3, we follow (3.12) and consider $(\kappa_1,\kappa_2,\kappa_3)\ =\ \left(\frac{\mu}{2},\frac{1}{\mu},\frac{\mu}{2}\right)$ and $(\kappa_1,\kappa_2,\kappa_3)\ =\ \left(\frac{3\mu}{4},\frac{1}{2\mu},\frac{3\mu}{4}\right)$, which certainly satisfy the assumptions of Theorem 3.2. On the other hand, since the choice of κ in (4.1) depends on the unknown constant c_1 from Lemma 2.3 (see Theorem 4.4), we simply take $\kappa\ =\ \frac{\mu}{4}$ and $\kappa\ =\ \frac{\mu}{2}$. As we will see below, these choices worked out well in the examples considered.

In Tables 6.1 - 6.3 and Figure 6.1, we summarize the convergence history of the augmented mixed finite element methods (4.2) and (4.1) as applied to Example 1 for sequences of quasi-uniform triangulations of the domain. We observe there that the rate of convergence $O(h)$ predicted by Theorems 4.2 and 4.5 (when $r\ =\ 1$) is attained in all the unknowns for both schemes. In addition, the results displayed in Tables 6.1 and 6.2, showing almost no difference between $(\kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $(\kappa_1, \kappa_2, \kappa_3) = (\frac{3\mu}{4}, \frac{1}{2\mu}, \frac{3\mu}{4})$, illustrate the robustness of scheme (4.2) with respect to the choice of these parameters. Similarly, though we only display results with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ in Table 6.3, it is also possible to confirm the robustness of (4.1) with respect to κ, κ_1, κ_2, and κ_3. In particular, the hypothesis on κ established by Theorem 4.4, that is $0 < \kappa < 2\, c_1\, \mu\, \alpha_1$, seems to be more technical than truly necessary for practical computations. Next, we remark the good behaviour of the a posteriori error estimator $\boldsymbol{\theta}$ for a sequence of quasi-uniform meshes in Example 1. Indeed, we notice from Table 6.3 that the effectivity index $eff(\boldsymbol{\theta})$ remains always in a neighborhood of 0.222, which illustrates the reliability and efficiency of $\boldsymbol{\theta}$. Finally, in order to emphasize the good performance of our augmented schemes, in Figures 6.2 and 6.3 we display two components of the approximate and exact solutions for Example 1.

Next, in Tables 6.4 and 6.5, we provide the convergence history of the uniform and adaptive schemes (4.1), with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$, as applied to Example 2. We observe that the errors of the adaptive procedure decrease faster than those obtained by the quasi-uniform one, which is confirmed by the experimental rates of convergence provided there. This fact is also illustrated in Figure 6.4 where we display the total errors $e(\boldsymbol{\sigma}, p, \mathbf{u})$ vs. the degrees of freedom N for both refinements. As shown by the values of $r(\boldsymbol{\sigma}, p, \mathbf{u})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain again bounded from above and below, which confirms the reliability and efficiency of $\boldsymbol{\theta}$. Some intermediate meshes obtained with the adaptive refinement are displayed in Figure 6.5. Note here that the method is able to recognize the singularities and the regions with high gradients of the solution. In fact, the adapted meshes concentrate the refinements around the origin and the line $x_1 = -1$. In order to illustrate the good quality of the solutions provided by the adaptive scheme, in Figures 6.6 and 6.7 we display two components of the approximate and exact solutions.

Finally, in Table 6.6 we provide the *convergence* history of the adaptive scheme (4.1), with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$, as applied to the driven cavity (Example 3). The errors and experimental rates of convergence shown there are computed by considering the discrete solution obtained with the finest mesh ($N = 327710$) as the *exact solution*. Two intermediate meshes obtained with the adaptive refinement are displayed in Figure 6.8, and some components of the approximate solution are provided in Figures 6.9 and 6.10.

Summarizing, the numerical results presented here constitute enough support to consider our curl-based augmented mixed finite element schemes and the associated adaptive algorithms, as valid and competitive alternatives to solve the stationary Stokes equations. However, as mentioned in the Introduction, it remains to further extend the present approach to the three-dimensional case, which should be the goal of a separate work.

Table 6.1: EXAMPLE 1, uniform scheme (4.2) with $(\kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

N	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\sigma}, \mathbf{u})$	$r(\boldsymbol{\sigma}, \mathbf{u})$
2179	0.0625	1.705E-03	–	1.368E-03	–	2.186E-03	–
2739	0.0556	1.516E-03	0.999	1.217E-03	0.994	1.944E-03	0.997
3363	0.0500	1.364E-03	0.999	1.096E-03	0.995	1.750E-03	0.998
4051	0.0455	1.240E-03	0.999	9.969E-04	0.996	1.591E-03	0.998
4803	0.0417	1.137E-03	1.000	9.141E-04	0.996	1.459E-03	0.998
5619	0.0385	1.049E-03	1.000	8.440E-04	0.997	1.347E-03	0.999
6499	0.0357	9.744E-04	1.000	7.839E-04	0.997	1.251E-03	0.999
7443	0.0333	9.095E-04	1.000	7.318E-04	0.998	1.167E-03	0.999
8451	0.0313	8.526E-04	1.000	6.861E-04	0.998	1.094E-03	0.999
9523	0.0294	8.025E-04	1.000	6.458E-04	0.998	1.030E-03	0.999
10659	0.0278	7.579E-04	1.000	6.100E-04	0.998	9.729E-04	0.999
13123	0.0250	6.821E-04	1.000	5.491E-04	0.998	8.757E-04	0.999
18819	0.0208	5.684E-04	1.000	4.577E-04	0.999	7.298E-04	1.000
25539	0.0179	4.872E-04	1.000	3.924E-04	0.999	6.255E-04	1.000
33283	0.0156	4.263E-04	1.000	3.433E-04	0.999	5.474E-04	1.000
51843	0.0125	3.410E-04	1.000	2.747E-04	1.000	4.379E-04	1.000
74499	0.0104	2.842E-04	1.000	2.289E-04	1.000	3.649E-04	1.000
101251	0.0089	2.436E-04	1.000	1.962E-04	1.000	3.128E-04	1.000
132099	0.0078	2.131E-04	1.000	1.717E-04	1.000	2.737E-04	1.000
167043	0.0069	1.894E-04	1.000	1.526E-04	1.000	2.433E-04	1.000

Table 6.2: EXAMPLE 1, uniform scheme (4.2) with $(\kappa_1, \kappa_2, \kappa_3) = (\frac{3\mu}{4}, \frac{1}{2\mu}, \frac{3\mu}{4})$

N	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\sigma}, \mathbf{u})$	$r(\boldsymbol{\sigma}, \mathbf{u})$
2179	0.0625	1.717E-03	–	1.369E-03	–	2.196E-03	–
2739	0.0556	1.525E-03	1.009	1.218E-03	0.994	1.951E-03	1.004
3363	0.0500	1.371E-03	1.008	1.096E-03	0.995	1.756E-03	1.003
4051	0.0455	1.246E-03	1.007	9.971E-04	0.996	1.596E-03	1.003
4803	0.0417	1.141E-03	1.006	9.143E-04	0.997	1.462E-03	1.003
5619	0.0385	1.053E-03	1.006	8.441E-04	0.997	1.350E-03	1.002
6499	0.0357	9.774E-04	1.005	7.840E-04	0.997	1.253E-03	1.002
7443	0.0333	9.119E-04	1.005	7.318E-04	0.998	1.169E-03	1.002
8451	0.0313	8.547E-04	1.004	6.862E-04	0.998	1.096E-03	1.002
9523	0.0294	8.042E-04	1.004	6.459E-04	0.998	1.031E-03	1.002
10659	0.0278	7.594E-04	1.004	6.101E-04	0.998	9.741E-04	1.002
13123	0.0250	6.832E-04	1.003	5.491E-04	0.999	8.766E-04	1.001
18819	0.0208	5.691E-04	1.003	4.577E-04	0.999	7.303E-04	1.001
25539	0.0179	4.876E-04	1.002	3.924E-04	0.999	6.259E-04	1.001
33283	0.0156	4.266E-04	1.002	3.433E-04	0.999	5.476E-04	1.001
51843	0.0125	3.412E-04	1.001	2.747E-04	1.000	4.380E-04	1.001
74499	0.0104	2.843E-04	1.001	2.289E-04	1.000	3.650E-04	1.000
101251	0.0089	2.436E-04	1.001	1.962E-04	1.000	3.128E-04	1.000
132099	0.0078	2.132E-04	1.001	1.717E-04	1.000	2.737E-04	1.000
167043	0.0069	1.895E-04	1.000	1.526E-04	1.000	2.433E-04	1.000

Table 6.3: EXAMPLE 1, uniform scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

N	h	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\theta})$
2691	0.0625	1.705E-03	1.368E-03	6.882E-04	2.186E-03	–	0.224
3387	0.0556	1.516E-03	1.217E-03	6.102E-04	1.944E-03	0.997	0.224
4163	0.0500	1.364E-03	1.096E-03	5.480E-04	1.750E-03	0.998	0.223
5019	0.0455	1.240E-03	9.969E-04	4.974E-04	1.591E-03	0.998	0.223
5955	0.0417	1.137E-03	9.141E-04	4.554E-04	1.459E-03	0.998	0.222
6971	0.0385	1.049E-03	8.440E-04	4.200E-04	1.347E-03	0.999	0.222
8067	0.0357	9.744E-04	7.839E-04	3.896E-04	1.251E-03	0.999	0.222
9243	0.0333	9.095E-04	7.318E-04	3.634E-04	1.167E-03	0.999	0.222
10499	0.0313	8.526E-04	6.861E-04	3.405E-04	1.094E-03	0.999	0.222
11835	0.0294	8.025E-04	6.458E-04	3.203E-04	1.030E-03	0.999	0.221
13251	0.0278	7.579E-04	6.100E-04	3.024E-04	9.729E-04	0.999	0.221
16323	0.0250	6.821E-04	5.491E-04	2.719E-04	8.757E-04	0.999	0.221
23427	0.0208	5.684E-04	4.577E-04	2.264E-04	7.298E-04	1.000	0.221
31811	0.0179	4.872E-04	3.924E-04	1.939E-04	6.255E-04	1.000	0.221
41475	0.0156	4.263E-04	3.433E-04	1.696E-04	5.474E-04	1.000	0.221
64643	0.0125	3.410E-04	2.747E-04	1.356E-04	4.379E-04	1.000	0.220
92931	0.0104	2.842E-04	2.289E-04	1.130E-04	3.649E-04	1.000	0.220
126339	0.0089	2.436E-04	1.962E-04	9.681E-05	3.128E-04	1.000	0.220
164867	0.0078	2.131E-04	1.717E-04	8.470E-05	2.737E-04	1.000	0.220
208515	0.0069	1.894E-04	1.526E-04	7.528E-05	2.433E-04	1.000	0.220

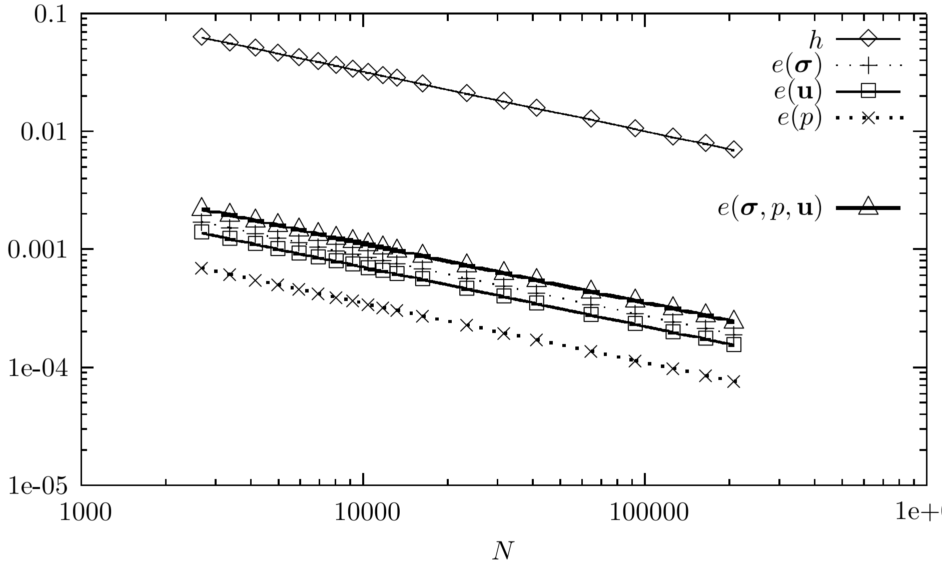


Fig. 6.1: EXAMPLE 1, uniform scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

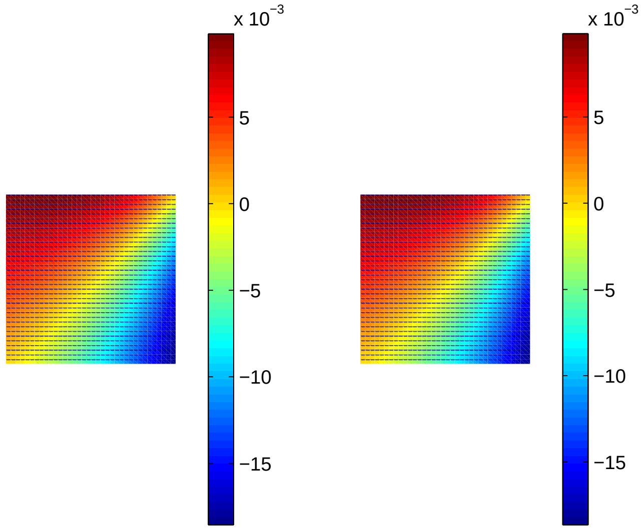


Fig. 6.2: Example 1, approximate (left) and exact σ_{22} for uniform scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 41475$

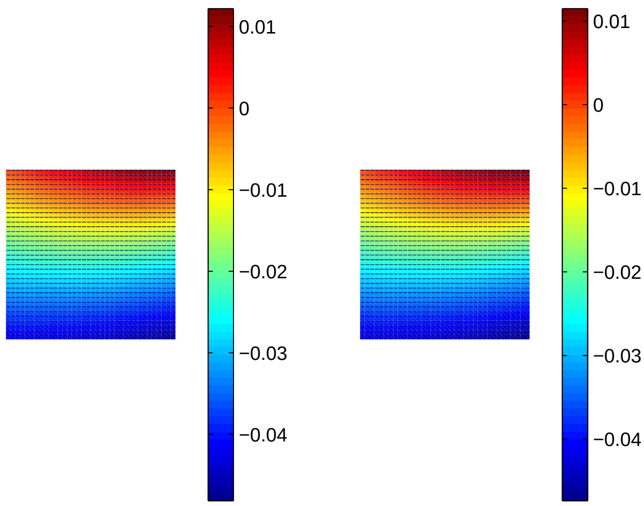


Fig. 6.3: Example 1, approximate (left) and exact u_1 for uniform scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 41475$

Table 6.4: EXAMPLE 2, uniform scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

N	h	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\theta})$
225	0.5000	1.706E+01	1.706E-00	1.621E-00	1.722E+01	–	0.882
481	0.3333	1.566E+01	1.359E-00	1.421E-00	1.578E+01	0.215	0.915
867	0.2500	1.381E+01	1.078E-00	1.140E-00	1.390E+01	0.442	0.922
1263	0.2000	1.259E+01	9.265E-01	9.242E-01	1.266E+01	0.419	0.932
1859	0.1666	1.154E+01	7.982E-01	8.152E-01	1.159E+01	0.482	0.933
2615	0.1429	1.058E+01	7.123E-01	7.214E-01	1.062E+01	0.566	0.934
3441	0.1250	9.603E+00	6.130E-01	6.659E-01	9.646E+00	0.723	0.941
4167	0.1111	8.885E+00	5.696E-01	6.221E-01	8.925E+00	0.659	0.940
5003	0.1000	8.286E+00	5.021E-01	5.823E-01	8.322E+00	0.664	0.943
6299	0.0909	7.925E+00	4.723E-01	5.224E-01	7.956E+00	0.471	0.946
7515	0.0833	7.377E+00	4.381E-01	4.766E-01	7.405E+00	0.825	0.941
8921	0.0769	6.685E+00	4.015E-01	4.267E-01	6.711E+00	1.231	0.941
10117	0.0714	6.403E+00	3.792E-01	4.076E-01	6.427E+00	0.583	0.946
11583	0.0667	6.083E+00	3.505E-01	3.892E-01	6.106E+00	0.743	0.949
13039	0.0625	5.701E+00	3.262E-01	3.564E-01	5.722E+00	1.006	0.945
14935	0.0588	5.383E+00	3.038E-01	3.388E-01	5.402E+00	0.947	0.948
16781	0.0556	5.086E+00	2.823E-01	3.174E-01	5.104E+00	0.996	0.947
20963	0.0500	4.678E+00	2.585E-01	2.896E-01	4.695E+00	0.793	0.948
24975	0.0455	4.220E+00	2.316E-01	2.566E-01	4.235E+00	1.082	0.947
32913	0.0400	3.700E+00	2.038E-01	2.210E-01	3.712E+00	1.029	0.945
43127	0.0345	3.191E+00	1.756E-01	1.971E-01	3.202E+00	0.997	0.948
63383	0.0286	2.751E+00	1.472E-01	1.641E-01	2.760E+00	0.790	0.949
91565	0.0238	2.283E+00	1.210E-01	1.362E-01	2.290E+00	1.024	0.950
127983	0.0200	1.929E+00	1.031E-01	1.130E-01	1.935E+00	0.965	0.948
161639	0.0179	1.750E+00	9.258E-02	1.016E-01	1.756E+00	0.859	0.949
206031	0.0159	1.496E+00	7.980E-02	8.877E-02	1.501E+00	1.332	0.947

Table 6.5: EXAMPLE 2, adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$	$eff(\boldsymbol{\theta})$
134	1.603E+01	1.870E-00	1.829E-00	1.625E+01	–	0.841
310	1.464E+01	1.370E-00	1.343E-00	1.476E+01	0.228	0.882
572	1.241E+01	1.019E-00	1.076E-00	1.250E+01	0.544	0.879
1090	9.391E+00	6.911E-01	7.851E-01	9.449E+00	0.867	0.896
2202	6.296E+00	5.839E-01	5.647E-01	6.348E+00	1.131	0.874
4730	3.995E+00	4.032E-01	3.842E-01	4.033E+00	1.187	0.852
10029	2.788E+00	2.796E-01	2.780E-01	2.816E+00	0.956	0.846
17373	2.124E+00	2.027E-01	1.929E-01	2.142E+00	0.995	0.851
37497	1.454E+00	1.405E-01	1.327E-01	1.466E+00	0.985	0.849
72351	1.075E+00	9.949E-02	9.683E-02	1.084E+00	0.921	0.849
146971	7.480E-01	6.990E-02	6.838E-02	7.544E-01	1.022	0.847
282166	5.531E-01	5.010E-02	4.948E-02	5.576E-01	0.927	0.848

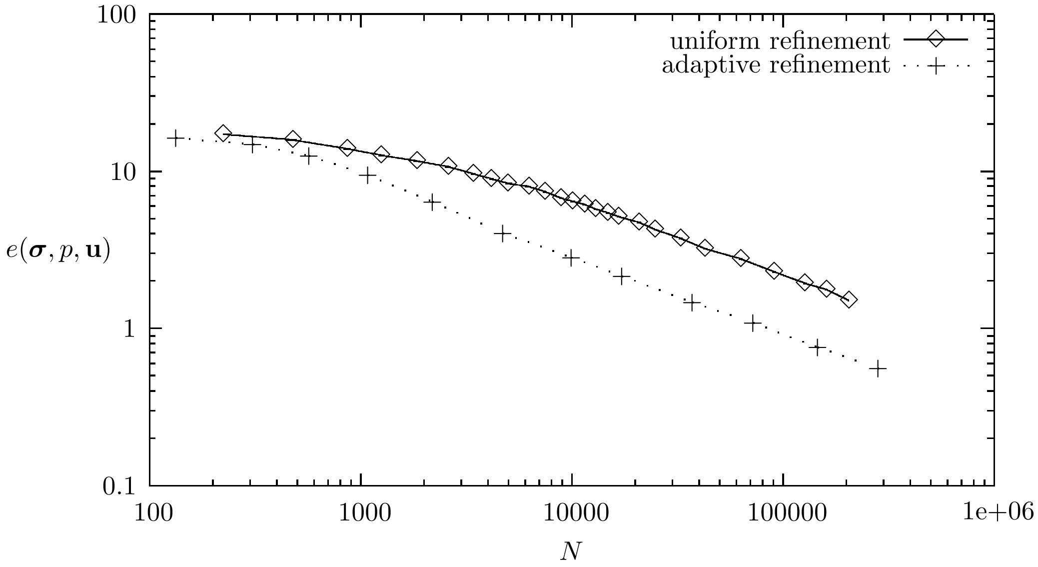


Fig. 6.4: Example 2, uniform/adaptive schemes (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

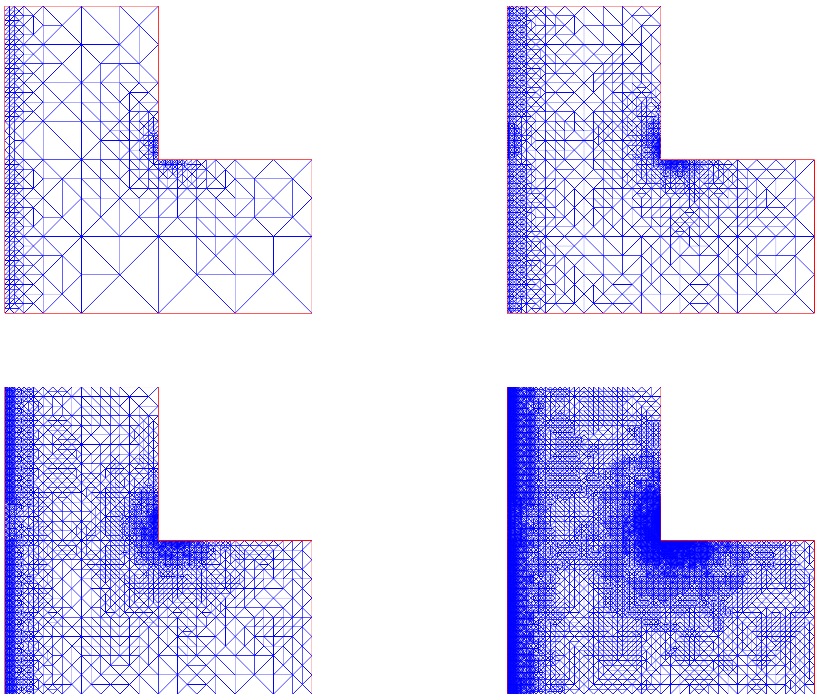


Fig. 6.5: Example 2, adapted intermediate meshes with 4730, 17373, 37497, and 146971 degrees of freedom for scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

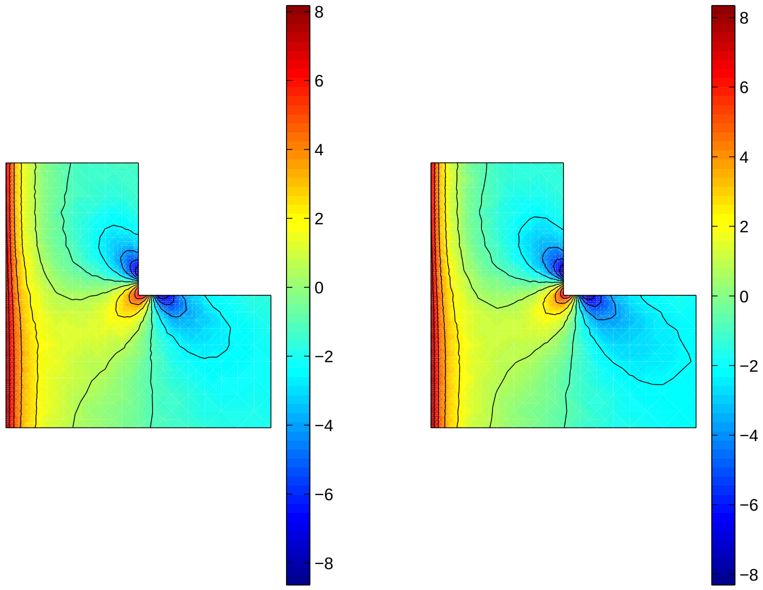


Fig. 6.6: EXAMPLE 2, approximate (left) and exact σ_{21} for adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 17373$

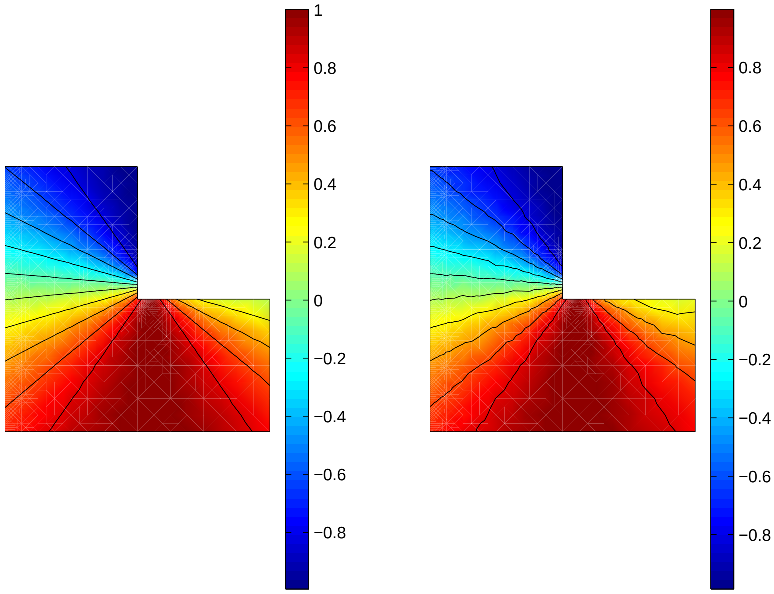


Fig. 6.7: EXAMPLE 2, approximate (left) and exact u_1 for adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{4}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 17373$

Table 6.6: EXAMPLE 3, adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\sigma}, p, \mathbf{u})$	$r(\boldsymbol{\sigma}, p, \mathbf{u})$
59	8.037E-00	5.091E-00	1.737E-00	9.671E-00	–
157	5.272E-00	3.116E-00	1.233E-00	6.247E-00	0.893
328	3.749E-00	2.191E-00	8.082E-01	4.417E-00	0.941
805	2.329E-00	1.292E-00	5.210E-01	2.714E-00	1.085
1358	1.584E-00	8.150E-01	3.839E-01	1.822E-00	1.524
3369	1.016E-00	5.004E-01	2.647E-01	1.163E-00	0.987
6300	7.241E-01	3.357E-01	1.912E-01	8.207E-01	1.115
11539	4.954E-01	2.216E-01	1.332E-01	5.588E-01	1.270
19922	3.611E-01	1.547E-01	1.017E-01	4.058E-01	1.172
47010	2.255E-01	9.466E-02	6.296E-02	2.526E-01	1.105
83819	1.561E-01	6.396E-02	4.560E-02	1.747E-01	1.274
184497	8.261E-02	3.358E-02	2.300E-02	9.209E-02	1.624
327710	–	–	–	–	–

Fig. 6.8: EXAMPLE 3, adapted intermediate meshes with 11539 and 47010 degrees of freedom for scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$

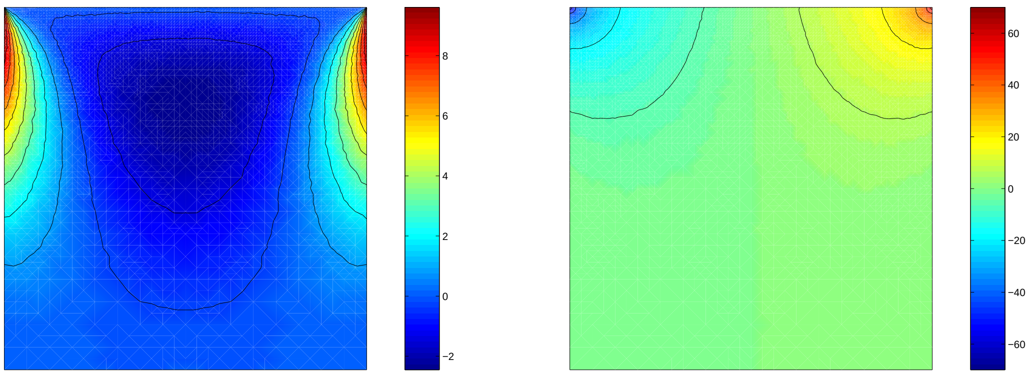


Fig. 6.9: EXAMPLE 3, approximate σ_{22} (left) and p (right) for adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 47010$

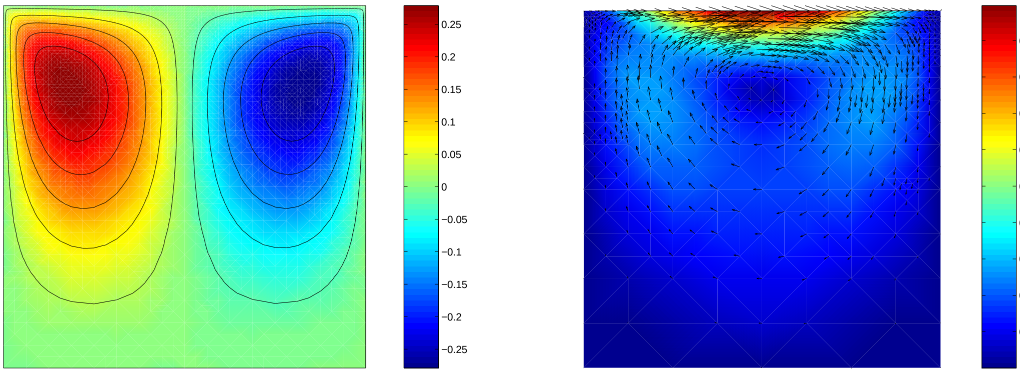


Fig. 6.10: EXAMPLE 3, approximate u_2 (left) and $\mathbf{u}$ (right) for adaptive scheme (4.1) with $(\kappa, \kappa_1, \kappa_2, \kappa_3) = (\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{\mu}, \frac{\mu}{2})$ and $N = 47010$

REFERENCES

1. D.N. ARNOLD, J. DOUGLAS AND CH.P. GUPTA, *A family of higher order mixed finite element methods for plane elasticity.* Numerische Mathematik, vol. 45, pp. 1-22, (1984).
2. T.P. BARRIOS AND G.N. GATICA, *An augmented mixed finite element method with Lagrange multipliers: a-priori and a-posteriori error analyses.* Journal of Computational and Applied Mathematics, vol. 200, pp. 653-676, (2007).
3. T.P. BARRIOS, G.N. GATICA, M. GONZÁLEZ AND N. HEUER, *A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity.* ESAIM Mathematical Modelling and Numerical Analysis, vol. 40, 5 (2006).
4. F. BREZZI AND M. FORTIN, Mixed and Hybrid Finite Element Methods. Springer Verlag, 1991.
5. P.G. CIARLET, The Finite Element Method for Elliptic Problems. North-Holland, 1978.
6. P. CLÉMENT, *Approximation by finite element functions using local regularisation.* RAIRO Modélisation Mathématique et Analyse Numérique, vol. 9, pp. 77-84, (1975).
7. L. FIGUEROA, G.N. GATICA AND N. HEUER, *A priori and a posteriori error analysis of an augmented mixed finite element method for incompressible fluid flows.* Computer Methods in Applied Mechanics and Engineering, vol. 198, 2, pp. 280-291, (2008).
8. L. FIGUEROA, G.N. GATICA AND A. MÁRQUEZ, *Augmented mixed finite element methods for the stationary Stokes equations.* SIAM Journal on Scientific Computing, vol. 31, 2, pp. 1082-1119, (2008).
9. G.N. GATICA, *A note on the efficiency of residual-based a-posteriori error estimators for some mixed finite element methods.* Electronic Transactions on Numerical Analysis, vol. 17, pp. 218-233, (2004).
10. G.N. GATICA, *Analysis of a new augmented mixed finite element method for linear elasticity allowing* $\mathbb{RT}_0 - \mathbb{P}_1 - \mathbb{P}_0$ *approximations.* ESAIM Mathematical Modelling and Numerical Analysis, vol. 40, 1, pp. 1-28, (2006).
11. G.N. GATICA, *An augmented mixed finite element method for linear elasticity with non homogeneous Dirichlet conditions.* Electronic Transactions on Numerical Analysis, vol. 26, pp. 421-438, (2007).
12. G.N. GATICA, A. MÁRQUEZ AND S. MEDDAHI, *An augmented mixed finite element method of low cost for three-dimensional linear elasticity problems.* Preprint 2007-21, Departamento de Ingeniería Matemática, Universidad de Concepcion.
13. V. GIRAULT AND P.A. RAVIART, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Springer Series in Computational Mathematics, Vol. 5, Springer, New York, 1986.
14. W. MCLEAN, Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.
15. R. VERFÜRTH, A Review of A Posteriori Error Estimation and Adaptive-Mesh-Refinement Techniques. Wiley-Teubner (Chichester), 1996.

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

PRE-PUBLICACIONES 2008 - 2009

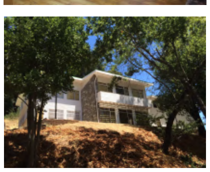
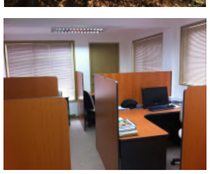
2008-01 RODOLFO ARAYA, ABNER POZA, FREDERIC VALENTIN: *On a hierarchical estimator driven by a stabilized method for the reactive incompressible Navier-Stokes equations*

2009-01 CARLO LOVADINA, DAVID MORA, RODOLFO RODRÍGUEZ: *Approximation of the buckling problem for Reissner-Mindlin plates*

2009-02 GABRIEL N. GATICA, LUIS F. GATICA, ANTONIO MARQUEZ: *Augmented mixed finite element methods for a curl-based formulation of the two-dimensional Stokes problem*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: DIRECTOR, CENTRO DE INVESTIGACIÓN EN INGENIERÍA MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN, CHILE, TEL.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



CENTRO DE INVESTIGACIÓN EN INGENIERÍA MATEMÁTICA (CI^{2}MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl