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Approximation of the buckling problem for Reissner-Mindlin  
plates

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# APPROXIMATION OF THE BUCKLING PROBLEM FOR REISSNER-MINDLIN PLATES

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**Abstract.** This paper deals with the approximation of the buckling coefficients and modes of a clamped plate modeled by the Reissner-Mindlin equations. These coefficients are related with the eigenvalues of a non-compact operator. We give a spectral characterization of this operator and show that the relevant buckling coefficients correspond to isolated nondefective eigenvalues. Then we consider the numerical solution of the buckling problem. For the finite element approximation of Reissner-Mindlin equations, it is well known that some kind of reduced integration or mixed interpolation has to be used to avoid locking. In particular we consider Duran-Liberman elements, which have been already proved to be locking-free for load and vibration problems. We adapt the classical approximation theory for non-compact operators to obtain optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. These estimates are valid with constants independent of the plate thickness. We report some numerical experiments confirming the theoretical results. Finally, we refine the analysis in the case of a uniformly compressed plate.

**Key words.** Buckling, Reissner-Mindlin plates, finite elements, non-compact spectral problems.

**AMS subject classifications.** 65N25, 65N30, 74S05, 74K20

**1. Introduction.** This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest, since it is frequently encountered in engineering applications, such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit elastic stability of the plate (i.e., eigenvalues-critical loads and eigenfunctions-buckling modes).

This problem has been studied for years by many researchers, being the Kirchhoff-Love and Reissner-Mindlin plate theories the most used. For the Kirchhoff-Love theory, there exists a thorough mathematical analysis; let us mention, for instance, [5, 13, 16, 17, 18]. This is not the case for the Reissner-Mindlin theory, for which only numerical experiments (cf. [15, 22]) or analytical solutions in particular cases (cf. [23]) have been reported so far. Recently, Dauge and Suri gave in [7] a mathematical spectral analysis of a problem of this kind based on three-dimensional elasticity which we will adapt to our case.

The Reissner-Mindlin theory is the most used model to approximate the deformation of an elastic thin or moderately thick plate. It is very well understood that standard finite element methods applied to this model lead to wrong results when the thickness is small with respect to the other dimensions of the plate, due to the *locking* phenomenon. Several families of methods have been rigorously shown to be free from locking and optimally convergent. We mention the recent monograph by Falk [12] for a thorough description of the state of the art and further references.

The aim of this paper is to analyze one of these methods applied to compute the critical load and buckling modes of a clamped plate. We choose the low-order,

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nonconforming finite elements introduced by Duran and Liberman in [11] (see also [10] for the analysis of this method applied to the plate vibration problem). However, the developed framework could be useful to analyze other methods, as well.

One disadvantage of the Reissner-Mindlin formulation for plate buckling is the fact that the corresponding resolvent operator is non-compact. This is the reason why the essential spectrum no longer reduces to zero (as is the case for compact operators). This means that the spectrum may now contain nonzero eigenvalues of infinite multiplicity, accumulation points, continuous spectrum, etc. Thus, our first task is to prove that the eigenvalue corresponding to the critical load can be isolated from the essential spectrum, at least for sufficiently thin plates.

On the other hand, the abstract spectral theory for non-compact operators introduced by Descloux, Nassif, and Rappaz in [8, 9] cannot be directly applied to analyze the numerical method, because we look for error estimates valid uniformly in the plate thickness. However, using the optimal order of convergence for the Duran-Liberman elements (cf. [10, 11]), the standard theory used to prove regularity results for Reissner-Mindlin equations (cf. [1]), and assuming that the family of meshes is *quasi-uniform*, we can adapt the theory from [8, 9] to obtain optimal order error estimates for the approximation of the buckling modes and a double order for the critical load. Moreover, these estimates are shown to be valid with constants independent of the plate thickness.

An outline of the paper is as follows. In the next section we recall the buckling problem and introduce the non-compact linear operator whose spectrum is related with the solution of this problem. In Section 3 we provide a thorough spectral characterization of this operator. In Section 4 we introduce a finite element discretization of the problem based on Duran-Liberman elements. In Section 5 we prove that the proposed numerical scheme is free of spurious modes and that optimal order error estimates hold true. In Section 6 we report some numerical tests which confirm the theoretical results. We include in this section a benchmark with a known analytical solution for a simply supported plate, which shows the efficiency of the method under other kind of boundary conditions, as well. Finally, in an appendix, we show that the results of Sections 3, 4, and 5 can be refined when considering a uniformly compressed plate.

Throughout the paper we will use standard notations for Sobolev norms and seminorms. Moreover, we will denote with  $C$  a generic constant independent of the mesh parameter  $h$  and the plate thickness  $t$ , which may take different values in different occurrences.

**2. The buckling problem.** Consider an elastic plate of thickness  $t$  with reference configuration  $\Omega \times (-\frac{t}{2}, \frac{t}{2})$ , where  $\Omega$  is a convex polygonal domain of  $\mathbb{R}^2$  occupied by the midsection of the plate. The deformation of the plate is described by means of the Reissner-Mindlin model in terms of the rotations  $\beta = (\beta_1, \beta_2)$  of the fibers initially normal to the plate midsurface and the transverse displacement  $w$ . Assuming that the plate is clamped on its whole boundary  $\partial\Omega$  and subjected to a plane stress tensor field  $\sigma : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ , the buckling problem reads as follows (see for instance [15, 22]):

$$(2.1) \quad \text{Find } \lambda \in \mathbb{R} \text{ and } 0 \neq (\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \text{ such that} \\ \alpha(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda (\sigma \nabla w, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega).$$

Above,  $\kappa := Ek/(2(1+\nu))$  is the shear modulus, with  $E$  being the Young modulus,  $\nu$  the Poisson ratio, and  $k$  a correction factor usually taken as  $5/6$  for clamped plates.

The applied stress tensor field is assumed to satisfy

$$(2.2) \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^t \quad \text{in } \Omega,$$

$$(2.3) \quad \boldsymbol{\sigma} \in W^{1,\infty}(\Omega)^{2 \times 2}.$$

Finally,  $a$  is the  $H_0^1(\Omega)^2$  elliptic bilinear form defined by

$$a(\beta, \eta) := \frac{E}{12(1-\nu^2)} \int_{\Omega} [(1-\nu) \boldsymbol{\varepsilon}(\beta) : \boldsymbol{\varepsilon}(\eta) + \nu \operatorname{div}(\beta) \operatorname{div}(\eta)],$$

with  $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1 \leq i,j \leq 2}$  being the standard strain tensor with components  $\varepsilon_{ij}(\beta) := \frac{1}{2} (\partial_i \beta_j + \partial_j \beta_i)$ ,  $1 \leq i, j \leq 2$ . Moreover,  $(\cdot, \cdot)_{0,\Omega}$  denotes the usual  $L^2$  inner product.

*Remark 2.1.* Problem (2.1) is obtained by *scaling* the *physical* buckling problem:

$$t^3 a(\beta, \eta) + t\kappa (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda_b t (\boldsymbol{\sigma} \nabla w, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega).$$

Accordingly, the buckling coefficients are related with the eigenvalues of problem (2.1) by  $\lambda_b = t^2 \lambda$ , while the eigenfunctions  $(\beta, w)$  are exactly the same.

Introducing the shear strain  $\gamma := \frac{\kappa}{t^2} (\nabla w - \beta)$ , problem (2.1) can be written as follows:

$$(2.4) \quad \begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ and } 0 \neq (\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \text{ such that} \\ a(\beta, \eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = \lambda (\boldsymbol{\sigma} \nabla w, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma = \frac{\kappa}{t^2} (\nabla w - \beta). \end{cases}$$

The *source* problem associated with the problem above reads:

Given  $f \in H_0^1(\Omega)$ , find  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  such that

$$(2.5) \quad \begin{cases} a(\beta, \eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma = \frac{\kappa}{t^2} (\nabla w - \beta). \end{cases}$$

Using the Helmholtz decomposition

$$(2.6) \quad \gamma = \nabla \psi + \operatorname{curl} p,$$

with  $\psi \in H_0^1(\Omega)$  and  $p \in H^1(\Omega)/\mathbb{R}$ , we have that problem (2.5) is equivalent to the following one (see [1]):

Given  $f \in H_0^1(\Omega)$ , find  $(\psi, \beta, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H^1(\Omega)/\mathbb{R} \times H_0^1(\Omega)$  such that

$$(2.7) \quad \begin{cases} (\nabla \psi, \nabla v)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} \quad \forall v \in H_0^1(\Omega), \\ a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla \psi, \eta)_{0,\Omega} \quad \forall \eta \in H_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1} t^2 (\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 \quad \forall q \in H^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1} t^2 (\nabla \psi, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega). \end{cases}$$

We recall the following result for the solution of problem (2.7) (see [1]):

**THEOREM 2.1.** *Let  $\Omega$  be a convex polygon or a smoothly bounded domain in the plane. For any  $t \in (0, 1]$ ,  $\boldsymbol{\sigma} \in L^\infty(\Omega)^{2 \times 2}$ , and  $f \in H_0^1(\Omega)$ , there exists a unique*

solution of problem (2.7). Moreover,  $\beta \in H^2(\Omega)^2$ ,  $p \in H^2(\Omega)$  and there exists a constant  $C$ , independent of  $t$  and  $f$ , such that

$$\|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|p\|_{1,\Omega} + t\|p\|_{2,\Omega} + \|w\|_{1,\Omega} \leq C\|f\|_{1,\Omega}.$$

As a consequence of Theorem 2.1, by virtue of (2.6) and the equivalence between problems (2.5) and (2.7), we have that problem (2.5) is well-posed and there exists a constant  $C$ , independent of  $t$  and  $f$ , such that

$$(2.8) \quad \|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|\gamma\|_{0,\Omega} \leq C\|f\|_{1,\Omega}.$$

Let  $T_t$  be the following bounded linear operator:

$$(2.9) \quad \begin{aligned} T_t : H_0^1(\Omega) &\rightarrow H_0^1(\Omega), \\ f &\mapsto w, \end{aligned}$$

where  $(\beta, w)$  is the solution of problem (2.5). It is easy to see that  $(\mu, w)$ , with  $\mu \neq 0$ , is an eigenpair of  $T_t$  (i.e.  $T_t w = \mu w$ ,  $w \neq 0$ ) if and only if  $(\lambda, \beta, w)$  is a solution of problem (2.4), with  $\lambda = 1/\mu$  and a suitable  $\beta \in H_0^1(\Omega)^2$ .

**3. Spectral properties.** The aim of this section is to prove a spectral characterization for the operator  $T_t$  defined above, to study the convergence of  $T_t$  and the behavior of its spectrum as  $t$  goes to zero, and to prove additional regularity for the eigenfunctions of  $T_t$ .

**3.1. Spectral characterization.** Given a generic linear bounded operator  $T : X \rightarrow X$ , defined on a Hilbert space  $X$ , we denote the spectrum of  $T$  by  $\text{Sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$  and by  $\rho(T) := \mathbb{C} \setminus \text{Sp}(T)$  the resolvent set of  $T$ . Moreover, for any  $z \in \rho(T)$ ,  $R_z(T) := (zI - T)^{-1} : X \rightarrow X$  denotes the resolvent operator of  $T$  corresponding to  $z$ .

We recall the definitions of the following components of the spectrum.

- *Discrete spectrum:*

$$\text{Sp}_d(T) := \{z \in \mathbb{C} : \text{Ker}(zI - T) \neq \{0\} \text{ and } (zI - T) : X \rightarrow X \text{ is Fredholm}\}.$$

- *Essential spectrum:*

$$\text{Sp}_e(T) := \{z \in \mathbb{C} : (zI - T) : X \rightarrow X \text{ is not Fredholm}\}.$$

The main result of this section is the following theorem which provides a suitable spectral characterization for the operator  $T_t$  defined in (2.9).

**THEOREM 3.1.** *The spectrum  $\text{Sp}(T_t)$  decomposes into:*

- $\text{Sp}_d(T_t)$ , which consists of real isolated eigenvalues of finite multiplicity and ascent one,
- $\text{Sp}_e(T_t)$ , essential spectrum.

Moreover,  $\text{Sp}_e(T_t) \subset \left\{ z \in \mathbb{C} : |z| \leq \kappa^{-1} t^2 \|\sigma\|_{\infty, \Omega} \right\}$ .

Here and thereafter, we denote  $\|\sigma\|_{\infty, \Omega} := \max_{x \in \bar{\Omega}} |\sigma(x)|$ , with  $|\cdot|$  being the matrix norm induced by the standard Euclidean norm in  $\mathbb{R}^2$ . Notice that the maximum above is well defined because of (2.3) and the fact that  $W^{1,\infty}(\Omega) \subset \mathcal{C}(\bar{\Omega})$ .

As a consequence of this theorem we know that, although  $T_t$  may have essential spectrum, all the points of  $\text{Sp}(T_t)$  outside a ball centered at the origin of the complex

plane are nondefective isolated eigenvalues. Moreover, the thinner the plate, the smaller the ball containing the essential spectrum.

The proof of Theorem 3.1 will be an immediate consequence of the results that follow. Consider the continuous bilinear forms defined in  $H_0^1(\Omega)^2 \times H_0^1(\Omega)$

$$(3.1) \quad A((\beta, w), (\eta, v)) := a(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega},$$

$$(3.2) \quad B((g, f), (\eta, v)) := (\sigma \nabla f, \nabla v)_{0,\Omega}.$$

We notice that  $A(\cdot, \cdot)$  is symmetric and elliptic (cf. [4]). Moreover, from the symmetry of  $\sigma$  (cf. (2.2)), it follows that  $B(\cdot, \cdot)$  is symmetric too. Consider the bounded linear operator

$$(3.3) \quad \begin{aligned} \tilde{T}_t : H_0^1(\Omega)^2 \times H_0^1(\Omega) &\rightarrow H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ (g, f) &\mapsto (\beta, w), \end{aligned}$$

where  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  is the solution of

$$A((\beta, w), (\eta, v)) = B((g, f), (\eta, v)) \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega).$$

By virtue of the symmetry of  $A(\cdot, \cdot)$  and  $B(\cdot, \cdot)$ , we have

$$A(\tilde{T}_t(g, f), (\eta, v)) = B((g, f), (\eta, v)) = B((\eta, v), (g, f)) = A((g, f), \tilde{T}_t(\eta, v)),$$

for every  $(g, f), (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ . Therefore,  $\tilde{T}_t$  is self-adjoint with respect to the inner product  $A(\cdot, \cdot)$ . As a consequence, we have the following theorem (see, for instance, [7, Theorem 3.3]).

**THEOREM 3.2.** *There holds  $\text{Sp}(\tilde{T}_t) \subset \mathbb{R}$ . Moreover, the spectrum of  $\tilde{T}_t$  decomposes as follows:  $\text{Sp}(\tilde{T}_t) = \text{Sp}_d(\tilde{T}_t) \cup \text{Sp}_e(\tilde{T}_t)$ . Finally, if  $\mu \in \text{Sp}_d(\tilde{T}_t)$ , then  $\mu$  is an isolated eigenvalue of finite multiplicity.*

The following result shows that the essential spectrum of  $\tilde{T}_t$  is confined in a neighborhood of the origin of diameter proportional to  $t^2$ .

**PROPOSITION 3.3.** *Let  $\mu \in \mathbb{R}$  be such that  $\mu \in \text{Sp}(\tilde{T}_t)$  and  $|\mu| > \kappa^{-1}t^2 \|\sigma\|_{\infty,\Omega}$ . Then  $\mu \in \text{Sp}_d(\tilde{T}_t)$ .*

*Proof.* Let  $\mu \in \mathbb{R}$  be such that  $\mu \in \text{Sp}(\tilde{T}_t)$  and  $|\mu| > \kappa^{-1}t^2 \|\sigma\|_{\infty,\Omega}$ . By virtue of Theorem 3.2, we only have to prove that  $(\mu \tilde{I} - \tilde{T}_t)$  is a Fredholm operator. To this end, it is enough to show that there exists a compact operator  $\tilde{G}$  such that  $(\mu \tilde{I} - \tilde{T}_t + \tilde{G})$  is invertible. Let us introduce the operator  $S$  as follows:

$$\begin{aligned} S : H_0^1(\Omega) &\rightarrow H_0^1(\Omega)^2, \\ f &\mapsto \beta, \end{aligned}$$

where  $(\beta, w)$  is the unique solution of problem (2.5). Notice that

$$(3.4) \quad \tilde{T}_t(g, f) = (Sf, T_t f).$$

According to (2.8), we have that  $\beta \in H^2(\Omega)^2$  and hence  $S$  is compact. Let us now define the operator  $G$  as follows:

$$(3.5) \quad \begin{aligned} G : H_0^1(\Omega) &\rightarrow H_0^1(\Omega), \\ f &\mapsto u, \end{aligned}$$

where  $u \in H_0^1(\Omega)$  is the unique solution of

$$(\nabla u, \nabla \xi)_{0,\Omega} = (Sf, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega).$$

The operator  $G$  is compact as a consequence of the compactness of  $S$ . Next, we define  $\tilde{G}$  as follows:

$$\begin{aligned} \tilde{G} : H_0^1(\Omega)^2 \times H_0^1(\Omega) &\rightarrow H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ (g, f) &\mapsto (Sf, Gf). \end{aligned}$$

Since  $S$  and  $G$  are compact,  $\tilde{G}$  is compact, too. In addition,

$$(\mu \tilde{I} - \tilde{T}_t + \tilde{G})(g, f) = ((\mu g - Sf + Sf), (\mu I - T_t + G)f) = (\mu g, (\mu I - T_t + G)f).$$

From the fourth equation in (2.7), we notice that  $v := (\mu I - T_t + G)f$  satisfies

$$\begin{aligned} (\nabla v, \nabla \xi)_{0,\Omega} &= \mu (\nabla f, \nabla \xi)_{0,\Omega} - (\nabla w, \nabla \xi)_{0,\Omega} + (\beta, \nabla \xi)_{0,\Omega} \\ &= ((\mu I - \kappa^{-1}t^2\sigma) \nabla f, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega). \end{aligned}$$

Consequently, the operator  $(\mu I - T_t + G)$  will be invertible if and only if, given  $v \in H_0^1(\Omega)$ , there exists a unique  $f \in H_0^1(\Omega)$  solution of

$$(3.6) \quad ((\mu I - \kappa^{-1}t^2\sigma) \nabla f, \nabla \xi)_{0,\Omega} = (\nabla v, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega).$$

Now, because of the symmetry of  $\sigma(x)$ , there exists an orthogonal tensor  $P(x)$  such that  $\sigma(x) = P(x)D(x)P^t(x)$ , where

$$D(x) := \begin{bmatrix} \bar{\omega}(x) & 0 \\ 0 & \underline{\omega}(x) \end{bmatrix},$$

with  $\underline{\omega}(x) \leq \bar{\omega}(x)$  being the eigenvalues of  $\sigma(x)$ . Hence, we write

$$(\mu I - \kappa^{-1}t^2\sigma) = P(x) \begin{bmatrix} \mu - \kappa^{-1}t^2\bar{\omega}(x) & 0 \\ 0 & \mu - \kappa^{-1}t^2\underline{\omega}(x) \end{bmatrix} P(x)^t.$$

Let us denote  $\omega_{\max} := \max_{x \in \bar{\Omega}} \bar{\omega}(x)$  and  $\omega_{\min} := \min_{x \in \bar{\Omega}} \underline{\omega}(x)$ . Since  $\|\sigma\|_{\infty,\Omega} = \max_{x \in \bar{\Omega}} |\sigma(x)| = \max\{|\omega_{\max}|, |\omega_{\min}|\}$ , if  $|\mu| > \kappa^{-1}t^2\|\sigma\|_{\infty,\Omega}$ , then there holds either  $\mu > \kappa^{-1}t^2\omega_{\max}$  or  $\mu < \kappa^{-1}t^2\omega_{\min}$ . Hence,  $(\mu I - \kappa^{-1}t^2\sigma)$  is uniformly positive definite in the first case or uniformly negative definite in the second one. Therefore, in both cases, there exists a unique solution  $f \in H_0^1(\Omega)$  of (3.6). Consequently,  $(\mu I - T_t + G)$  is invertible. Consequently,  $(\mu \tilde{I} - \tilde{T}_t)$  is Fredholm and  $\mu \in \text{Sp}_d(\tilde{T}_t)$ .  $\square$

The following result shows that  $T_t$  and  $\tilde{T}_t$  have the same spectrum.

LEMMA 3.4. *If  $T_t$  and  $\tilde{T}_t$  are the operators defined in (2.9) and (3.3), respectively, then  $\text{Sp}(\tilde{T}_t) = \text{Sp}(T_t)$ .*

*Proof.* We will prove that  $\rho(\tilde{T}_t) = \rho(T_t)$ . Let  $z$  be such that  $(z\tilde{I} - \tilde{T}_t)$  is invertible. We will prove that  $(zI - T_t)$  is invertible, too. By hypothesis, for every  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  there exists a unique  $(g, f) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  such that

$$(3.7) \quad (z\tilde{I} - \tilde{T}_t)(g, f) = (\beta, w).$$

Recalling (3.4), we infer that there is a unique  $(g, f)$  such that  $zg - Sf = \beta$  and  $(zI - T_t)f = w$ . Hence, we deduce that the operator  $(zI - T_t) : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$



is onto. Now, let us assume that there exists another  $\hat{f}$  such that  $(zI - T_t)\hat{f} = w$ . Taking  $\hat{g} = \frac{1}{z}(S\hat{f} + \beta)$ , we see that  $(z\tilde{I} - \tilde{T}_t)(\hat{g}, \hat{f}) = (\beta, w)$ . Since, by assumption,  $(z\tilde{I} - \tilde{T}_t)$  is invertible, from (3.7) it follows that  $f = \hat{f}$ . Therefore,  $(zI - T_t)$  is also one-to-one and thus it is invertible.

Conversely, let  $z$  be such that  $(zI - T_t)$  is invertible. We will prove that  $(z\tilde{I} - \tilde{T}_t)$  is invertible, too. Recalling (3.4) again, we have to show that for every  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$ , there exists a unique  $(g, f) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  such that

$$\begin{cases} zg - Sf = \beta, \\ zf - T_t f = w. \end{cases}$$

Let  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  be given. There exists a unique  $f \in H_0^1(\Omega)$  such that  $(zI - T_t)f = w$ . Therefore, taking  $g := \frac{1}{z}(Sf + \beta)$ , we obtain  $(z\tilde{I} - \tilde{T}_t)(g, f) = (\beta, w)$ . The uniqueness of  $g$  follows immediately from the uniqueness of  $f$  and the first equation of the system above. The proof is complete.  $\square$

The following result shows that the eigenvalues of  $T_t$  are nondefective.

LEMMA 3.5. *Suppose that  $\mu \neq 0$  is an isolated eigenvalue of  $T_t$ . Then its ascent is one.*

*Proof.* By contradiction. Let  $(\mu, w)$  be an eigenpair of  $T_t$ ,  $\mu \neq 0$ , and let us assume that  $T_t$  has a corresponding generalized eigenfunction, namely,  $\exists \hat{w} \neq 0$  such that  $T_t \hat{w} = \mu \hat{w} + w$ . Since  $(\mu, w)$  is an eigenpair of  $T_t$ , there exists  $\beta \in H_0^1(\Omega)^2$  such that (cf. (2.9) and (2.4))

$$a(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \frac{1}{\mu} (\sigma \nabla w, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega). \quad (3.8)$$

On the other hand, since  $T_t \hat{w} = \mu \hat{w} + w$ , the definition of  $T_t$  implies the existence of  $\hat{\beta} \in H_0^1(\Omega)^2$  such that

$$a(\hat{\beta}, \eta) + \frac{\kappa}{t^2} (\nabla(w + \mu \hat{w}) - \hat{\beta}, \nabla v - \eta)_{0,\Omega} = (\sigma \nabla \hat{w}, \nabla v)_{0,\Omega} \quad \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega).$$

Defining  $\bar{\beta} := (\hat{\beta} - \beta)/\mu$ , the equation above can be written as follows:

$$\mu a(\bar{\beta}, \eta) + a(\beta, \eta) + \frac{\kappa \mu}{t^2} (\nabla \hat{w} - \bar{\beta}, \nabla v - \eta)_{0,\Omega} + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = (\sigma \nabla \hat{w}, \nabla v)_{0,\Omega}.$$

We now take  $(\eta, v) = \mu(\bar{\beta}, \hat{w})$  in (3.8) and  $(\eta, v) = (\beta, w)$  in the equation above and we subtract the resulting equations. Using also the symmetry of  $a(\cdot, \cdot)$  and  $\sigma$ , we obtain

$$a(\beta, \beta) + \frac{\kappa}{t^2} \|\nabla w - \beta\|_{0,\Omega}^2 = 0.$$

Thus, from the ellipticity of  $a(\cdot, \cdot)$ , we infer  $\beta = 0$  and hence  $w = 0$ , which is a contradiction since  $w$  is an eigenfunction of  $T_t$ . The proof is complete.  $\square$

We are now in a position to prove Theorem 3.1.

*Proof of Theorem 3.1.* The proof follows easily by combining Lemma 3.4 with Theorem 3.2, Proposition 3.3, and Lemma 3.5.  $\square$

**3.2. Limit problem.** In this section we study the convergence properties of the operator  $T_t$  as  $t$  goes to zero. First, let us recall that it is well-known (see [4]) that, when  $t$  goes to zero, the solution  $(\beta, w, \gamma)$  of problem (2.5) converges to the solution  $(\beta_0, w_0, \gamma_0) \in H_0^1(\Omega)^2 \times H_0^1(\Omega) \times H_0(\text{rot}; \Omega)'$  of

$$(3.9) \quad \begin{cases} a(\beta_0, \eta) + \langle \gamma_0, \nabla v - \eta \rangle = (\sigma \nabla f, \nabla v)_{0,\Omega} & \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \nabla w_0 - \beta_0 = 0, \end{cases}$$

where,  $\langle \cdot, \cdot \rangle$  stands for the duality pairing in  $H_0(\text{rot}; \Omega)$ . Problem (3.9) is a mixed formulation for the following well-posed problem:

Find  $w_0 \in H_0^2(\Omega)$  such that

$$\frac{E}{12(1-\nu^2)} (\Delta w_0, \Delta v)_{0,\Omega} = (\sigma \nabla f, \nabla v)_{0,\Omega} \quad \forall v \in H_0^2(\Omega).$$

Let  $T_0$  be the bounded linear operator defined by

$$\begin{aligned} T_0 : H_0^1(\Omega) &\rightarrow H_0^1(\Omega), \\ f &\mapsto w_0, \end{aligned}$$

where  $(\beta_0, w_0, \gamma_0)$  is the solution of problem (3.9). Since  $w_0 \in H_0^2(\Omega)$ , the operator  $T_0$  is compact. Hence, apart from  $\mu_0 = 0$ , the spectrum of  $T_0$  consists of a sequence of finite multiplicity isolated eigenvalues converging to zero. The following lemma, which yields the convergence in norm of  $T_t$  to  $T_0$  has been essentially proved in [10, Lemma 3.1].

LEMMA 3.6. *There exists a constant  $C$ , independent of  $t$ , such that*

$$\|(T_t - T_0)f\|_{1,\Omega} \leq Ct \|f\|_{1,\Omega} \quad \forall f \in H_0^1(\Omega).$$

As a consequence of this lemma, standard properties about the separation of isolated parts of the spectrum (see [14], for instance) yield the following result.

LEMMA 3.7. *Let  $\mu_0$  be an eigenvalue of  $T_0$  of multiplicity  $m$ . Let  $D$  be any disc in the complex plane centered at  $\mu_0$  and containing no other element of the spectrum of  $T_0$ . Then there exists  $t_0 > 0$  such that,  $\forall t < t_0$ ,  $D$  contains exactly  $m$  isolated eigenvalues of  $T_t$  (repeated according to their respective multiplicities). Consequently, each eigenvalue  $\mu_0$  of  $T_0$  is a limit of isolated eigenvalues  $\mu_t$  of  $T_t$ , as  $t$  goes to zero.*

Our next goal is to show that the largest eigenvalues of  $T_t$  converge to the largest eigenvalues of  $T_0$  as  $t$  goes to zero. With this aim, we prove first the following lemma. Here and thereafter, we will use  $\|\cdot\|$  to denote the operator norm induced by the  $H^1(\Omega)$  norm.

LEMMA 3.8. *Let  $F \subset \mathbb{C}$  be a closed set such that  $F \cap \text{Sp}(T_0) = \emptyset$ . Then there exist strictly positive constants  $t_0$  and  $C$  such that,  $\forall t < t_0$ ,  $F \cap \text{Sp}(T_t) = \emptyset$  and*

$$\|R_z(T_t)\| := \sup_{\substack{w \in H_0^1(\Omega) \\ w \neq 0}} \frac{\|R_z(T_t)w\|_{1,\Omega}}{\|w\|_{1,\Omega}} \leq C \quad \forall z \in F.$$

*Proof.* The mapping  $z \mapsto \|(zI - T_0)^{-1}\|$  is continuous for all  $z \in \rho(T_0)$  and goes to zero as  $|z| \rightarrow \infty$ . Consequently, it attains its maximum on any closed subset  $F \subset \rho(T_0)$ . Let  $C_1 := 1/\max_{z \in F} \|(zI - T_0)^{-1}\|$ ; there holds

$$\|(zI - T_0)w\|_{1,\Omega} \geq \frac{1}{C_1} \|w\|_{1,\Omega} \quad \forall w \in H_0^1(\Omega) \quad \forall z \in F.$$

Now, according to Lemma 3.6, there exists  $t_1 > 0$  such that, for all  $t < t_1$ ,

$$\|(T_t - T_0)w\|_{1,\Omega} \leq \frac{1}{2C_1} \|w\|_{1,\Omega} \quad \forall w \in H_0^1(\Omega).$$

Therefore, for all  $w \in H_0^1(\Omega)$ , for all  $z \in F$ , and for all  $t < t_1$ ,

$$(3.10) \quad \|(zI - T_t)w\|_{1,\Omega} \geq \|(zI - T_0)w\|_{1,\Omega} - \|(T_t - T_0)w\|_{1,\Omega} \geq \frac{1}{2C_1} \|w\|_{1,\Omega}$$

and, consequently,  $z \notin \text{Sp}_d(T_t)$ .

On the other hand, since  $\text{Sp}(T_0) \ni 0$ ,  $F \cap \text{Sp}(T_0) = \emptyset$ , and  $F$  is closed, we have that  $d := \min_{z \in F} |z|$  is strictly positive. Let  $t_2 > 0$  be such  $\kappa^{-1}t_2^2 \|\sigma\|_{\infty,\Omega} < d$ . Hence, for all  $z \in F$  and for all  $t < t_2$ ,  $|z| > \kappa^{-1}t^2 \|\sigma\|_{\infty,\Omega}$  and, by virtue of Theorem 3.1, either  $z \in \text{Sp}_d(T_t)$  or  $z \notin \text{Sp}(T_t)$ .

Altogether, if  $t_0 := \min\{t_1, t_2\}$ , then  $(zI - T_t)$  is invertible for all  $t < t_0$  and all  $z \in F$ . Moreover, because of (3.10),

$$\|R_z(T_t)\| = \|(zI - T_t)^{-1}\| \leq 2C_1$$

and we conclude the proof.  $\square$

Since  $T_0$  is a compact operator, its nonzero eigenvalues are isolated and we can order them as follows:

$$|\mu_0^{(1)}| \geq |\mu_0^{(2)}| \geq \dots \geq |\mu_0^{(k)}| \geq \dots$$

where each eigenvalue is repeated as many times as its corresponding multiplicity. According to Lemma 3.7, for  $t$  sufficiently small there exist eigenvalues of  $T_t$  close to each  $\mu_0^{(k)}$ . On the other hand, according to Theorem 3.1, the essential spectrum of  $T_t$  is confined in a ball centered at the origin of the complex plane with radius proportional to  $t^2$ . Therefore, at least for  $t$  sufficiently small, the points of the spectrum of  $T_t$  largest in modulus have to be isolated eigenvalues. Hence we order them as we did with those of  $T_0$ :

$$|\mu_t^{(1)}| \geq |\mu_t^{(2)}| \geq \dots \geq |\mu_t^{(k)}| \geq \dots$$

The following theorem shows that the  $k$ -th eigenvalue of  $T_t$  converge to the  $k$ -th eigenvalue of  $T_0$  as  $t$  goes to zero.

**THEOREM 3.9.** *Let  $\mu_t^{(k)}$ ,  $k \in \mathbb{N}$ ,  $t \geq 0$ , be as defined above. For all  $k \in \mathbb{N}$ ,  $\mu_t^{(k)} \rightarrow \mu_0^{(k)}$  as  $t \rightarrow 0$ .*

*Proof.* We will prove the result for the first eigenvalue  $\mu_t^{(1)}$ . The proof for the others is a straightforward modification of this one.

Let  $D_0$  be an open disk in the complex plane centered at  $\mu_0^{(1)}$  with radius  $r_0 < (|\mu_0^{(1)}| - |\mu_0^{(k)}|)/2$ , where  $\mu_0^{(k)}$  is the first eigenvalue of  $T_0$  such that  $|\mu_0^{(1)}| > |\mu_0^{(k)}|$ .

Let  $D_1$  be another open disk in the complex plane centered at the origin with radius  $r_1 := |\mu_0^{(1)}| - r_0$ . Therefore  $\text{Sp}(T_0) \subset D_0 \cup D_1$ . Let  $F := \mathbb{C} \setminus (D_0 \cup D_1)$ .  $F$  is a closed set and  $F \cap \text{Sp}(T_0) = \emptyset$ . Hence, according to Lemma 3.8, there exists  $t_0 > 0$  such that, for all  $t < t_0$ ,  $F \cap \text{Sp}(T_t) = \emptyset$ , too, and hence  $\text{Sp}(T_t) \subset D_0 \cup D_1$ , as well.

On the other hand, because of Lemma 3.7, there exists  $t_1 > 0$  such that, for all  $t < t_1$ ,  $D_0$  contains eigenvalues of  $T_t$ . Therefore, for all  $t < \min\{t_0, t_1\}$ , the eigenvalue of  $T_t$  largest in modulus,  $\mu_t^{(1)}$ , has to lie in  $D_0$ . Since  $D_0$  can be taken arbitrarily small, we conclude that  $\mu_t^{(1)}$  converges to  $\mu_0^{(1)}$  as  $t$  goes to zero. Thus, we conclude the proof.  $\square$

**3.3. Additional regularity of the eigenfunctions.** The aim of this section is to prove a regularity result for the eigenfunctions of problem (2.4). More precisely, we have the following proposition.

**PROPOSITION 3.10.** *Let  $\mu_t^{(k)}$ ,  $k \in \mathbb{N}$ ,  $t \geq 0$ , be as in Theorem 3.9. Let  $(\lambda, \beta, w, \gamma)$  be a solution of problem (2.4) with  $\lambda = 1/\mu_t^{(k)}$ . Then there exists  $t_0 > 0$  such that, for all  $t < t_0$ ,  $\beta \in H^2(\Omega)^2$ ,  $w \in H^2(\Omega)$ ,  $\text{div } \gamma \in L^2(\Omega)$ , and there holds*

$$(3.11) \quad \|\beta\|_{2,\Omega} \leq C |\lambda| \|w\|_{1,\Omega},$$

$$(3.12) \quad \|w\|_{2,\Omega} \leq C |\lambda| \|w\|_{1,\Omega},$$

$$(3.13) \quad \|\text{div } \gamma\|_{0,\Omega} \leq C |\lambda| \|w\|_{2,\Omega},$$

with  $C$  a positive constant independent of  $t$ .

*Proof.* Using the Helmholtz decomposition (2.6), problem (2.4) is equivalent to the following one:

Find  $\lambda \in \mathbb{R}$  and  $0 \neq (\psi, \beta, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times H^1(\Omega)/\mathbb{R} \times H_0^1(\Omega)$  such that

$$\begin{cases} (\nabla \psi, \nabla v)_{0,\Omega} = \lambda (\sigma \nabla w, \nabla v)_{0,\Omega} & \forall v \in H_0^1(\Omega), \\ a(\beta, \eta) - (\text{curl } p, \eta)_{0,\Omega} = (\nabla \psi, \eta)_{0,\Omega} & \forall \eta \in H_0^1(\Omega)^2, \\ -(\beta, \text{curl } q)_{0,\Omega} - \kappa^{-1} t^2 (\text{curl } p, \text{curl } q)_{0,\Omega} = 0 & \forall q \in H^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1} t^2 (\nabla \psi, \nabla \xi)_{0,\Omega} & \forall \xi \in H_0^1(\Omega). \end{cases}$$

From Theorem 2.1 applied to the problem above, we immediately obtain that  $\beta \in H^2(\Omega)^2$  and the estimate (3.11).

On the other hand, the first and the last equations from the system above lead to

$$((\mathbf{I} - \lambda \kappa^{-1} t^2 \sigma) \nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega).$$

Since  $\mu_t^{(k)} \rightarrow \mu_0^{(k)} \neq 0$  as  $t \rightarrow 0$ , there exists  $t_1 > 0$  such that  $|\mu_t^{(k)}| > |\mu_0^{(k)}|/2 \quad \forall t < t_1$ . Hence  $\lambda = 1/\mu_t^{(k)} < 2/|\mu_0^{(k)}|$ . We take  $t_0 < t_1$  such that  $\kappa^{-1} t_0^2 \|\sigma\|_{\infty,\Omega} < |\mu_0^{(k)}|/2$ . Therefore, for all  $t < t_0$ ,  $(\mathbf{I} - \lambda \kappa^{-1} t^2 \sigma)$  is uniformly positive definite. Thus, since  $w$  is the solution of the problem

$$\begin{cases} \text{div} [(\mathbf{I} - \lambda \kappa^{-1} t^2 \sigma) \nabla w] = \text{div } \beta & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

using a standard regularity result (see [21]), we have that  $w \in H^2(\Omega)$  and

$$\|w\|_{2,\Omega} \leq C \|\text{div } \beta\|_{0,\Omega} \leq C \|\beta\|_{1,\Omega} \leq C |\lambda| \|w\|_{1,\Omega},$$

the last inequality because of (3.11).

Furthermore, taking  $\eta = 0$  in (2.4), using the estimate above and (2.3), it follows that

$$\text{div } \gamma = \lambda \text{div}(\sigma \nabla w) \in L^2(\Omega).$$

and

$$\|\text{div } \gamma\|_{0,\Omega} \leq C |\lambda| \|w\|_{2,\Omega}.$$

The proof is complete.  $\square$

**4. Spectral approximation.** For the numerical approximation, we focus on the finite element method proposed and studied in [11]. In what follows we introduce briefly this method (see this reference for further details). Let  $\{\mathcal{T}_h\}_{h>0}$  be a regular family of triangular meshes of  $\bar{\Omega}$ . We will define finite element spaces  $H_h$ ,  $W_h$ , and  $\Gamma_h$  for the rotations, the transverse displacement, and the shear strain, respectively.

For  $K \in \mathcal{T}_h$ , let  $\alpha_1, \alpha_2, \alpha_3$  be its barycentric coordinates. We denote by  $\tau_i$  a unit vector tangent to the edge  $\alpha_i = 0$  and define

$$p_1^K = \alpha_2 \alpha_3 \tau_1, \quad p_2^K = \alpha_1 \alpha_3 \tau_2, \quad p_3^K = \alpha_1 \alpha_2 \tau_3.$$

The finite element space for the rotations is defined by

$$H_h := \left\{ \eta_h \in H_0^1(\Omega)^2 : \eta_h|_K \in \mathbb{P}_1^2 \oplus \langle p_1^K, p_2^K, p_3^K \rangle \quad \forall K \in \mathcal{T}_h \right\}.$$

To approximate the transverse displacements, we use the usual piecewise-linear continuous finite element space:

$$W_h := \left\{ v_h \in H_0^1(\Omega) : v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

Finally, for the shear strain, we use the lowest-order rotated Raviart-Thomas space:

$$\Gamma_h := \left\{ \phi \in H_0(\text{rot}; \Omega) : \phi|_K \in \mathbb{P}_0^2 \oplus (x_2, -x_1)\mathbb{P}_0 \quad \forall K \in \mathcal{T}_h \right\}.$$

We consider the reduction operator

$$R : H^1(\Omega)^2 \cap H_0(\text{rot}; \Omega) \rightarrow \Gamma_h,$$

which is uniquely determined by

$$\int_{\ell} R\phi \cdot \tau_{\ell} = \int_{\ell} \phi \cdot \tau_{\ell},$$

for every edge  $\ell$  of the triangulation. Above,  $\tau_{\ell}$  denotes a unit vector tangent to  $\ell$ . It is well-known that

$$(4.1) \quad \|R\phi\|_{0,\Omega} \leq C \|\phi\|_{1,\Omega} \quad \forall \phi \in H^1(\Omega)^2,$$

$$(4.2) \quad \|\phi - R\phi\|_{0,\Omega} \leq Ch \|\phi\|_{1,\Omega} \quad \forall \phi \in H^1(\Omega)^2.$$

Moreover, the operator  $R$  can be extended continuously to  $H^s(\Omega)^2 \cap H_0(\text{rot}; \Omega)$  for any  $s > 0$  and it is also well known that, for all  $v \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ ,

$$(4.3) \quad R(\nabla v) = \nabla v_I,$$

where  $v_I \in W_h$  is the standard piecewise-linear Lagrange interpolant of  $v$  (which is well defined because  $H^{1+s}(\Omega) \subset \mathcal{C}(\bar{\Omega})$  for all  $s > 0$ ).

The discretization of problem (2.4) is as follows:

Find  $\lambda_h \in \mathbb{R}$  and  $0 \neq (\beta_h, w_h) \in H_h \times W_h$  such that

$$(4.4) \quad \begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h (\sigma \nabla w_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h). \end{cases}$$

The corresponding discrete source problem reads:

Given  $f \in H_0^1(\Omega)$ , find  $(\beta_h, w_h) \in H_h \times W_h$  such that

$$(4.5) \quad \begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\sigma \nabla f, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h). \end{cases}$$

The method is nonconforming, since consistency terms arise because of the reduction operator  $R$ . Existence and uniqueness of the solution to problem (4.5) follow easily (see [11]).

*Remark 4.1.* Given a generic  $f \in H_0^1(\Omega)$ , the usual convergence rate in terms of positive powers of the mesh-size  $h$  does not hold for problems (2.5) and (4.5). Indeed, the right-hand sides of both problems are not regular enough, since  $\text{div}(\sigma \nabla f) \notin L^2(\Omega)$ . However, whenever  $f$  is more regular, for instance assuming  $f \in H^2(\Omega)$  and taking into account the regularity of  $\sigma$  (cf. (2.3)), the convergence results of [11] can be applied, to obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} + \|w - w_h\|_{1,\Omega} \leq Ch \|f\|_{2,\Omega}.$$

As for the continuous case, we introduce the operator

$$T_{th} : H_0^1(\Omega) \rightarrow W_h \hookrightarrow H_0^1(\Omega), \\ f \mapsto w_h,$$

where  $(\beta_h, w_h)$  is the solution of problem (4.5). The following lemma shows that this operator is bounded uniformly in  $t$  and  $h$ .

LEMMA 4.1. *There exists  $C > 0$  such that  $\|T_{th}\| \leq C$  for all  $t > 0$  and all  $h > 0$ .*

*Proof.* Let  $f \in H_0^1(\Omega)$  and  $(\beta_h, w_h)$  be the solution of problem (4.5). Taking  $(\eta_h, v_h) = (\beta_h, w_h)$  as test function in (4.5), we obtain

$$a(\beta_h, \beta_h) + \kappa^{-1} t^2 \|\gamma_h\|_{0,\Omega}^2 \leq \|\sigma\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega},$$

where  $\gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h)$  as in (4.5). Hence, from the ellipticity of  $a(\cdot, \cdot)$ ,

$$\|\beta_h\|_{1,\Omega}^2 + \kappa^{-1} t^2 \|\gamma_h\|_{0,\Omega}^2 \leq C \|\sigma\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega}.$$

Therefore, using (4.1),

$$\|\nabla w_h\|_{0,\Omega}^2 = \|\kappa^{-1} t^2 \gamma_h + R\beta_h\|_{0,\Omega}^2 \leq C \|\sigma\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega},$$

which allows us to conclude the proof.  $\square$

Clearly, the nonzero eigenvalues of  $T_{th}$  are given by  $\mu_h := 1/\lambda_h$ , with  $\lambda_h$  being the eigenvalues of (4.4), and the corresponding eigenfunctions coincide. In what follows, we will prove a characterization of the discrete spectral problem (4.4).

LEMMA 4.2. *Let  $Y_h := \{w_h \in W_h : (\sigma \nabla w_h, \nabla v_h)_{0,\Omega} = 0 \quad \forall v_h \in W_h\}$ . Then problem (4.4) has exactly  $\dim W_h - \dim Y_h$  eigenvalues, repeated according to their respective multiplicities. All of them are real and nonzero.*

*Proof.* We eliminate  $\gamma_h$  in problem (4.4) to write it as follows:

$$(4.6) \quad a(\beta_h, \eta_h) + \frac{\kappa}{t^2} (\nabla w_h - R\beta_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h (\sigma \nabla w_h, \nabla v_h)_{0,\Omega} \\ \forall (\eta_h, v_h) \in H_h \times W_h.$$

Taking particular bases of  $H_h$  and  $W_h$ , this problem can be written in matrix form as follows:

$$(4.7) \quad \mathcal{A} \begin{bmatrix} \beta_h \\ w_h \end{bmatrix} = \lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \beta_h \\ w_h \end{bmatrix},$$

where  $\beta_h$  and  $w_h$  denote the vectors whose entries are the components in those basis of  $\beta_h$  and  $w_h$ , respectively. The matrix  $\mathcal{A}$  is symmetric and positive definite because the bilinear form on the left-hand side of (4.6) is elliptic on  $H_0^1(\Omega)^2 \times H_0^1(\Omega)$  (cf. [11]). Consequently,  $\lambda_h \neq 0$  and, since  $\mathbf{E}$  is also symmetric,  $\lambda_h \in \mathbb{R}$ . Now, (4.7) holds true if and only if

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \beta_h \\ w_h \end{bmatrix} = \mu_h \mathcal{A} \begin{bmatrix} \beta_h \\ w_h \end{bmatrix}$$

with  $\lambda_h = 1/\mu_h$  and  $\mu_h \neq 0$ . The latter is a well-posed generalized eigenvalue problem with  $\dim W_h - \dim \text{Ker}(\mathbf{E})$  nonzero eigenvalues. Thus, we conclude the lemma by noting that  $\mathbf{E}w_h = \mathbf{0}$  if and only if  $w_h \in Y_h$ .  $\square$

*Remark 4.2.* If  $(\lambda_h, \beta_h, w_h)$  is a solution of problem (4.4), then

$$w_h^T \mathbf{E} w_h = (\sigma \nabla w_h, \nabla w_h)_{0,\Omega} \neq 0.$$

In fact, this follows by left multiplying both sides of (4.7) by  $(\beta_h, w_h)$  and using the positive definiteness of  $\mathcal{A}$ .

We will adapt the theory in [8, 9] to prove convergence of our spectral approximation and to obtain error estimates. To this end, we will use the following properties:

$$\begin{aligned} \text{P1.} \quad & \|T_0 - T_{th}\|_h := \sup_{\substack{f_h \in W_h \\ f_h \neq 0}} \frac{\|(T_0 - T_{th})f_h\|_{1,\Omega}}{\|f_h\|_{1,\Omega}} \rightarrow 0, \quad \text{as } (h, t) \rightarrow (0, 0); \\ \text{P2.} \quad & \forall u \in H_0^1(\Omega), \quad \inf_{v_h \in W_h} \|u - v_h\|_{1,\Omega} \rightarrow 0, \quad \text{as } h \rightarrow 0. \end{aligned}$$

From now on, we will use the operator norm  $\|\cdot\|_h$  as defined in property P1.

Since property P2 follows from standard approximation results, in the sequel we focus on property P1. We first notice that

$$(4.8) \quad \|T_0 - T_{th}\|_h \leq \|T_0 - T_t\|_h + \|T_t - T_{th}\|_h,$$

where  $T_t$  is the operator defined in (2.9). Since  $W_h \subset H_0^1(\Omega)$ , from Lemma 3.6 we deduce that for all  $h > 0$

$$(4.9) \quad \|T_0 - T_t\|_h \leq Ct.$$

Regarding the other term in the right-hand side of (4.8), we aim at proving the following result.

**PROPOSITION 4.3.** *Suppose that the family  $\{\mathcal{T}_h\}_{h>0}$  is quasi-uniform. Then we have*

$$\|T_t - T_{th}\|_h \leq C(h + t).$$

To give the proof of Proposition 4.3, we consider problems (2.5) and (4.5) with source term in  $W_h$ , namely:

(4.10) *Given  $f_h \in W_h$ , find  $(\beta, w) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)$  such that*

$$\begin{cases} a(\beta, \eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\sigma \nabla f_h, \nabla v)_{0,\Omega} & \forall (\eta, v) \in H_0^1(\Omega)^2 \times H_0^1(\Omega), \\ \gamma = \frac{\kappa}{t^2} (\nabla w - \beta). \end{cases}$$

(4.11) *Given  $f_h \in W_h$ , find  $(\beta_h, w_h) \in H_h \times W_h$  such that*

$$\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\sigma \nabla f_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \frac{\kappa}{t^2} (\nabla w_h - R\beta_h). \end{cases}$$

We need some results concerning the solutions of problems (4.10) and (4.11). First, we apply the Helmholtz decomposition (2.6) to the term  $\gamma$  from (4.10):

$$(4.12) \quad \gamma = \nabla \psi + \operatorname{curl} p, \quad \psi \in H_0^1(\Omega), \quad p \in H^1(\Omega)/\mathbb{R}.$$

Then we apply Theorem 2.1 and (2.8), to obtain the following a priori estimate for the solution to problem (4.10):

$$(4.13) \quad \|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|p\|_{1,\Omega} + t \|p\|_{2,\Omega} + \|\gamma\|_{0,\Omega} \leq C \|f_h\|_{1,\Omega}.$$

The following result shows that  $w$  and  $\psi$  are actually smoother, because  $f_h \in W_h$ , and an inverse estimate which will be used to prove Proposition 4.3.

LEMMA 4.4. *Let  $w$  be defined by problem (4.10) and  $\psi$  as in (4.12). Then  $w, \psi \in H^{1+s}(\Omega)$  for all  $s \in (0, \frac{1}{2})$ . Moreover, if the family  $\{\mathcal{T}_h\}_{h>0}$  is quasi-uniform, then*

$$\|\psi\|_{1+s,\Omega} \leq Ch^{-s} \|f_h\|_{1,\Omega}.$$

*Proof.* Recalling the equivalence between problems (4.10) and (2.7), the latter with source term  $f_h$  instead of  $f$ , from the first equation of (2.7) we have that  $\psi$  is the weak solution of

$$(4.14) \quad \begin{cases} \Delta \psi = \operatorname{div}(\sigma \nabla f_h), \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $f_h$  is a continuous piecewise linear function, we have that  $f_h \in H^{1+s}(\Omega) \quad \forall s \in (0, \frac{1}{2})$ . Therefore, the assumption (2.3) implies  $\sigma \nabla f_h \in H^s(\Omega)^2$ . Hence,  $\operatorname{div}(\sigma \nabla f_h) \in H^{s-1}(\Omega)$ . Then, from standard regularity results for problem (4.14),  $\psi \in H^{1+s}(\Omega) \quad \forall s \in (0, \frac{1}{2})$  and

$$\|\psi\|_{1+s,\Omega} \leq C \|\operatorname{div}(\sigma \nabla f_h)\|_{s-1,\Omega} \leq C \|f_h\|_{1+s,\Omega}.$$

If the family of meshes is quasi-uniform, then the inverse inequality  $\|f_h\|_{1+s,\Omega} \leq Ch^{-s} \|f_h\|_{1,\Omega}$  holds true and from this and the estimate above we obtain

$$\|\psi\|_{1+s,\Omega} \leq Ch^{-s} \|f_h\|_{1,\Omega}.$$

On the other hand, from the last equation of (2.7) we have that

$$(\nabla(w - \kappa^{-1}t^2\psi), \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega).$$



Therefore,  $(w - \kappa^{-1}t^2\psi)$  is the weak solution to the problem

$$\begin{cases} \Delta(w - \kappa^{-1}t^2\psi) = \operatorname{div} \beta \in L^2(\Omega), \\ (w - \kappa^{-1}t^2\psi) = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Hence,  $(w - \kappa^{-1}t^2\psi) \in H^2(\Omega)$  (recall  $\Omega$  is convex) and  $w = (w - \kappa^{-1}t^2\psi) + \kappa^{-1}t^2\psi \in H^{1+s}(\Omega)$  for all  $s \in (0, \frac{1}{2})$ . Thus the proof is complete.  $\square$

The following lemma is the key point to prove Proposition 4.3.

LEMMA 4.5. *If  $(\beta, w)$  and  $(\beta_h, w_h)$  are the solutions of (4.10) and (4.11), respectively, then*

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \leq C(h+t) \|f_h\|_{1,\Omega}.$$

*Proof.* It has been proved in [11] (see Example 4.1 from this reference) that there exists  $\tilde{\beta} \in H_h$  such that

$$\begin{aligned} R\tilde{\beta} &= R\beta, \\ \|\beta - \tilde{\beta}\|_{1,\Omega} &\leq Ch \|\beta\|_{2,\Omega}. \end{aligned}$$

Let

$$\tilde{\gamma} := \frac{\kappa}{t^2}(\nabla w_I - R\tilde{\beta}),$$

where the Lagrange interpolant  $w_I \in W_h$  is well defined because of Lemma 4.4. Because of (4.3) and the equation above,

$$\tilde{\gamma} = R\gamma.$$

It has also been proved in [11] that

$$\|\tilde{\beta} - \beta_h\|_{1,\Omega} + t \|\tilde{\gamma} - \gamma_h\|_{0,\Omega} \leq C \left( \|\tilde{\beta} - \beta\|_{1,\Omega} + t \|\tilde{\gamma} - \gamma\|_{0,\Omega} + h \|\gamma\|_{0,\Omega} \right).$$

Hence, by adding and subtracting  $\tilde{\beta}$  and  $\tilde{\gamma} = R\gamma$ , we obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \leq C \left( \|\beta - \tilde{\beta}\|_{1,\Omega} + t \|\gamma - R\gamma\|_{0,\Omega} + h \|\gamma\|_{0,\Omega} \right).$$

The first and last term in the right-hand side above are already bounded. To estimate the second one, we use (4.12), Lemma 4.4, and (4.3), to obtain

$$(4.15) \quad \|\gamma - R\gamma\|_{0,\Omega} \leq \|\nabla \psi - \nabla \psi_I\|_{0,\Omega} + \|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega}.$$

Next, from standard error estimates for the Lagrange interpolant, we have

$$\|\nabla \psi - \nabla \psi_I\|_{0,\Omega} \leq Ch^s \|\psi\|_{1+s,\Omega},$$

whereas from (4.2) and the fact that  $p \in H^2(\Omega)$  (cf. (4.13))

$$\|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega} \leq Ch \|p\|_{2,\Omega}.$$

Thus, by using Lemma 4.4, we conclude

$$\begin{aligned} \|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} &\leq C \left( h \|\beta\|_{2,\Omega} + t \|f_h\|_{1,\Omega} + th \|p\|_{2,\Omega} + h \|\gamma\|_{0,\Omega} \right) \\ &\leq C (h + t) \|f_h\|_{1,\Omega}, \end{aligned}$$

where we have used (4.13) for the last inequality. The proof is complete.  $\square$

We are now in a position to prove Proposition 4.3.

*Proof of Proposition 4.3.* Under the same assumptions of Lemma 4.5, we need to prove that

$$\|w - w_h\|_{1,\Omega} \leq C (h + t) \|f_h\|_{1,\Omega}.$$

From problems (4.10) and (4.11), we have

$$\nabla w - \nabla w_h = \kappa^{-1} t^2 (\gamma - \gamma_h) + (\beta - R\beta_h).$$

Adding and subtracting  $R\beta$ , we obtain

$$(4.16) \quad \|\nabla w - \nabla w_h\|_{0,\Omega} \leq \kappa^{-1} t^2 \|\gamma - \gamma_h\|_{0,\Omega} + \|\beta - R\beta\|_{0,\Omega} + \|R(\beta - \beta_h)\|_{0,\Omega}.$$

Hence, using Poincaré inequality, (4.1), Lemma 4.5, (4.2), and (4.13), we have

$$\|w - w_h\|_{1,\Omega} \leq C (h + t) \|f_h\|_{1,\Omega}.$$

The proof is complete.  $\square$

We end this section by proving property P1.

LEMMA 4.6. *Suppose that the family  $\{\mathcal{T}_h\}_{h>0}$  is quasi-uniform. Then we have*

$$\|T_0 - T_{th}\|_h \leq C (h + t).$$

*Proof.* The assertion follows immediately from estimate (4.8), by using (4.9) and Proposition 4.3.  $\square$

**5. Error estimates.** In this section we will adapt the arguments from [9] to prove error estimates for the approximate eigenvalues and eigenfunctions. Throughout this section, we will assume that the family of meshes  $\{\mathcal{T}_h\}_{h>0}$  is quasi-uniform, so that property P1 holds true, although such assumption is not actually necessary in some particular cases (see the appendix below).

Our first goal is to prove that, provided the plate is sufficiently thin, the numerical method does not introduce spurious modes with eigenvalues interspersed among the relevant ones of  $T_t$  (namely, around  $\mu_t^{(k)}$  for small  $k$ ). Let us remark that such a spectral pollution could be in principle expected from the fact that  $T_t$  has a nontrivial essential spectrum. However, that this is not the case is an immediate consequence of the following theorem, which is essentially identical to Lemma 1 from [8].

THEOREM 5.1. *Let  $F \subset \mathbb{C}$  be a closed set such that  $F \cap \text{Sp}(T_0) = \emptyset$ . There exist strictly positive constants  $h_0$ ,  $t_0$ , and  $C$  such that,  $\forall h < h_0$  and  $\forall t < t_0$ , there holds  $F \cap \text{Sp}(T_{th}) = \emptyset$  and*

$$\|R_z(T_{th})\|_h \leq C \quad \forall z \in F.$$

*Proof.* The same arguments used to prove Lemma 3.8 (but using Lemma 4.6 instead of Lemma 3.6) allow us to show an estimate analogous to (3.10), namely, for all  $w_h \in W_h$  and all  $z \in F$ ,

$$\|(zI - T_{th})w_h\|_{1,\Omega} \geq \|(zI - T_0)w_h\|_{1,\Omega} - \|(T_0 - T_{th})w_h\|_{1,\Omega} \geq \frac{1}{2C_1} \|w_h\|_{1,\Omega},$$

provided  $h$  and  $t$  are small enough. Since  $W_h$  is finite dimensional, we deduce from above that  $(zI - T_{th})|_{W_h}$  is invertible and, hence,  $z \notin \text{Sp}(T_{th}|_{W_h})$ . Now,  $\text{Sp}(T_{th}) = \text{Sp}(T_{th}|_{W_h}) \cup \{0\}$  (see, for instance, [3, Lemma 4.1]) and, for  $z \in F$ ,  $z \neq 0$ . Thus,  $z \notin \text{Sp}(T_{th})$  either. Then  $(zI - T_{th})$  is invertible too and

$$\|R_z(T_{th})\|_h = \|(zI - T_{th})^{-1}\|_h \leq 2C_1 \quad \forall z \in F.$$

The proof is complete.  $\square$

We have already proved in Theorem 3.1 that the essential spectrum of  $T_t$  is confined to the real interval  $(-\kappa^{-1}t^2\|\sigma\|_{\infty,\Omega}, \kappa^{-1}t^2\|\sigma\|_{\infty,\Omega})$ . The spectrum of  $T_t$  outside this interval consists of finite multiplicity isolated eigenvalues of ascent one, which converge to eigenvalues of  $T_0$ , as  $t$  goes to zero (cf. Theorem 3.9). The eigenvalue of  $T_t$  with physical significance is the largest in modulus,  $\mu_t^{(1)}$ , which corresponds to the critical load that leads to buckling effects. This eigenvalue is typically simple and converges to a simple eigenvalue of  $T_0$ , as  $t$  tends to zero. Because of this, for simplicity, from now on we restrict our analysis to simple eigenvalues.

Let  $\mu_0 \neq 0$  be an eigenvalue of  $T_0$  with multiplicity  $m = 1$ . Let  $D$  be a closed disk centered at  $\mu_0$  with boundary  $\Gamma$  such that  $0 \notin D$  and  $D \cap \text{Sp}(T_0) = \{\mu_0\}$ . Let  $t_0 > 0$  be small enough, so that for all  $t < t_0$ :

- $D$  contains only one eigenvalue of  $T_t$ , which we already know is simple (cf. Lemma 3.7) and
- $D$  does not intersect the real interval  $(-\kappa^{-1}t^2\|\sigma\|_{\infty,\Omega}, \kappa^{-1}t^2\|\sigma\|_{\infty,\Omega})$ , which contains the essential spectrum of  $T_t$ .

According to Theorem 5.1 there exist  $t_0 > 0$  and  $h_0 > 0$  such that  $\forall t < t_0$  and  $\forall h < h_0$ ,  $\Gamma \subset \rho(T_{th})$ . Moreover, proceeding as in [8, Section 2], from properties P1 and P2 it follows that, for  $h$  small enough,  $T_{th}$  has exactly one eigenvalue  $\mu_{th} \in D$ . The theory in [9] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of  $T_{th}$  to those of  $T_0$  as  $h$  and  $t$  go to zero. However, our goal is not this, but to prove that  $\mu_{th}$  converges to  $\mu_t$  as  $h$  goes to zero, with  $t < t_0$  fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [9].

Let  $\Pi_h : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  be the projector with range  $W_h$  defined for all  $u \in H_0^1(\Omega)$  by

$$(\nabla(\Pi_h u - u), \nabla v_h)_{0,\Omega} = 0 \quad \forall v_h \in W_h.$$

The projector  $\Pi_h$  is bounded uniformly on  $h$ , namely,  $\|\Pi_h u\|_{1,\Omega} \leq \|u\|_{1,\Omega}$ , and the following error estimate is well known:

$$(5.1) \quad \|\Pi_h u - u\|_{1,\Omega} \leq Ch \|u\|_{2,\Omega} \quad \forall u \in H^2(\Omega).$$

Let us define

$$B_{th} := T_{th}\Pi_h : H_0^1(\Omega) \rightarrow W_h \hookrightarrow H_0^1(\Omega).$$

It is clear that  $T_{th}$  and  $B_{th}$  have the same nonzero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result (cf. [9, Lemma 1]).

LEMMA 5.2. *There exist  $h_0$ ,  $t_0$ , and  $C$  such that*

$$\|R_z(B_{th})\| \leq C \quad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

*Proof.* Since  $B_{th}$  is compact it suffices to verify that  $\|(zI - B_{th})u\|_{1,\Omega} \geq C \|u\|_{1,\Omega}$  for all  $u \in H_0^1(\Omega)$  and  $z \in \Gamma$ . Taking into account that  $0 \notin \Gamma$  and using Theorem 5.1, we have

$$\|u\|_{1,\Omega} \leq \|\Pi_h u\|_{1,\Omega} + \|u - \Pi_h u\|_{1,\Omega} \leq C \|(zI - T_{th})\Pi_h u\|_{1,\Omega} + |z|^{-1} \|z(u - \Pi_h u)\|_{1,\Omega}.$$

By using properties of the projector  $\Pi_h$ , we obtain

$$\begin{aligned} \|u\|_{1,\Omega} &\leq C \|(zI - B_{th})\Pi_h u\|_{1,\Omega} + |z|^{-1} \|z(u - \Pi_h u) - B_{th}(u - \Pi_h u)\|_{1,\Omega} \\ &= C \|\Pi_h(zI - B_{th})u\|_{1,\Omega} + |z|^{-1} \|(I - \Pi_h)(zI - B_{th})u\|_{1,\Omega} \\ &\leq C \|(zI - B_{th})u\|_{1,\Omega}. \end{aligned}$$

Thus we end the proof.  $\square$

Next, we introduce:

- $E_t : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ , the spectral projector of  $T_t$  corresponding to the isolated eigenvalue  $\mu_t$ , namely,

$$E_t := \frac{1}{2\pi i} \int_{\Gamma} R_z(T_t) dz;$$

- $F_{th} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ , the spectral projector of  $B_{th}$  corresponding to the eigenvalue  $\mu_{th}$ , namely,

$$F_{th} := \frac{1}{2\pi i} \int_{\Gamma} R_z(B_{th}) dz.$$

As a consequence of Lemma 5.2, the spectral projectors  $F_{th}$  are bounded uniformly in  $h$  and  $t$ , for  $h$  and  $t$  small enough. Notice that, under our assumptions,  $E_t(H_0^1(\Omega))$  and  $F_{th}(H_0^1(\Omega))$  are both one dimensional. We have the following error estimate (cf. [9, Lemma 3]).

LEMMA 5.3. *There exist positive constants  $h_0$ ,  $t_1$ , and  $C$ , such that for all  $h < h_0$  and for all  $t < t_1$ ,*

$$\|(E_t - F_{th})|_{E_t(H_0^1(\Omega))}\| \leq C \|(T_t - B_{th})|_{E_t(H_0^1(\Omega))}\| \leq Ch.$$

*Proof.* The first inequality is proved using the same arguments of [9, Lemma 3] and Lemmas 3.8 and 5.2. For the other estimate, fix  $w \in E_t(H_0^1(\Omega))$ . From Proposition 3.10, Remark 4.1, Lemma 4.1, and (5.1), we have

$$\begin{aligned} \|(T_t - B_{th})w\|_{1,\Omega} &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|(T_{th} - B_{th})w\|_{1,\Omega} \\ &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|T_{th}\| \|(I - \Pi_h)w\|_{1,\Omega} \\ &\leq Ch \|w\|_{2,\Omega}. \end{aligned}$$

Therefore, by using (3.12), we conclude the proof.  $\square$

Our next goal is to prove an optimal order error estimate for the eigenfunctions. With this aim, we first need a preliminary result.

LEMMA 5.4. *Let*

$$\Lambda_{th} := F_{th}|_{E_t(H_0^1(\Omega))} : E_t(H_0^1(\Omega)) \rightarrow F_{th}(H_0^1(\Omega)).$$

For  $h$  and  $t$  small enough, the operator  $\Lambda_{th}$  is invertible and

$$\|\Lambda_{th}^{-1}\| \leq C,$$

with  $C$  independent of  $h$  and  $t$ .

*Proof.* See the proof of Theorem 1 in [9].  $\square$

We recall the definition of the gap  $\hat{\delta}$  between two closed subspaces  $Y$  and  $Z$  of  $H_0^1(\Omega)$ :

$$\hat{\delta}(Y, Z) := \max\{\delta(Y, Z), \delta(Z, Y)\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y \in Y \\ \|y\|_{1,\Omega}=1}} \left( \inf_{z \in Z} \|y - z\|_{1,\Omega} \right).$$

THEOREM 5.5. *There exist constants  $h_0$ ,  $t_1$ , and  $C$ , such that, for all  $h < h_0$  and for all  $t < t_1$ , there holds*

$$\hat{\delta}(F_{th}(H_0^1(\Omega)), E_t(H_0^1(\Omega))) \leq Ch.$$

*Proof.* It follows by arguing exactly as in the proof of Theorem 1 from [9], and using Lemmas 5.3 and 5.4.  $\square$

Next, we prove a preliminary sub-optimal error estimate for  $|\mu_t - \mu_{th}|$ , which will be improved below (cf. Theorem 5.8).

LEMMA 5.6. *There exists a positive constant  $C$  such that, for  $h$  and  $t$  small enough,*

$$|\mu_t - \mu_{th}| \leq Ch.$$

*Proof.* We define the following operators:

$$\begin{aligned} \hat{T}_t &:= T_t|_{E_t(H_0^1(\Omega))} : E_t(H_0^1(\Omega)) \rightarrow E_t(H_0^1(\Omega)), \\ \hat{B}_{th} &:= \Lambda_{th}^{-1} B_{th} \Lambda_{th} : E_t(H_0^1(\Omega)) \rightarrow E_t(H_0^1(\Omega)). \end{aligned}$$

The operator  $\hat{T}_t$  has a unique eigenvalue  $\mu_t$  of multiplicity  $m = 1$ , while the unique eigenvalue of  $\hat{B}_{th}$  is  $\mu_{th}$ .

Let  $v \in E_t(H_0^1(\Omega))$ . Since  $(\Lambda_{th}^{-1} F_{th} - I) T_t|_{E_t(H_0^1(\Omega))} = 0$  and  $B_{th}$  commutes with its spectral projector  $F_{th}$ , we have

$$(\hat{T}_t - \hat{B}_{th})v = (T_t - B_{th})v + (\Lambda_{th}^{-1} F_{th} - I)(T_t - B_{th})v.$$

Therefore, using Lemmas 5.3 and 5.4 and the fact that  $\|F_{th}\|$  is bounded uniformly in  $h$  and  $t$ , for  $h$  and  $t$  small enough, we obtain

$$\|(\widehat{T}_t - \widehat{B}_{th})v\|_{1,\Omega} \leq \|(T_t - B_{th})v\|_{1,\Omega} + \|(\Lambda_{th}^{-1}F_{th} - I)(T_t - B_{th})v\|_{1,\Omega} \leq Ch\|v\|_{1,\Omega}.$$

Hence, the lemma follows from the fact that  $\widehat{T}_t = \mu_t I$  and  $\widehat{B}_{th} = \mu_{th} I$ .  $\square$

Since the eigenvalue  $\mu_t \neq 0$  of  $T_t$  corresponds to an eigenvalue  $\lambda = 1/\mu_t$  of problem (2.4), Lemma 5.6 leads to a sub-optimal error estimate for the approximation of  $\lambda$  as well. We now aim at improving that result. Let  $\lambda_h := 1/\mu_{th}$ ,  $w_h$ ,  $\beta_h$  and  $\gamma_h$  be such that  $(\lambda_h, w_h, \beta_h, \gamma_h)$  is a solution of problem (4.4), with  $\|w_h\|_{1,\Omega} = 1$ . According to Theorem 5.5, there exists a solution  $(\lambda, w, \beta, \gamma)$  of problem (2.4), with  $\|w\|_{1,\Omega} = 1$ , such that

$$\|w - w_h\|_{1,\Omega} \leq Ch.$$

The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues.

**LEMMA 5.7.** *Let  $(\lambda, w, \beta, \gamma)$  be a solution of (2.4), with  $\|w\|_{1,\Omega} = 1$ , and let  $(\lambda_h, w_h, \beta_h, \gamma_h)$  be a solution of (4.4), with  $\|w_h\|_{1,\Omega} = 1$ , such that*

$$(5.2) \quad \|w - w_h\|_{1,\Omega} \leq Ch.$$

*Then for  $h$  and  $t$  small enough there holds*

$$(5.3) \quad \|\beta - \beta_h\|_{1,\Omega} + t\|\gamma - \gamma_h\|_{0,\Omega} \leq Ch.$$

*Proof.* Let  $\hat{w}_h \in W_h$ ,  $\hat{\beta}_h \in H_h$  and  $\hat{\gamma}_h$  be the solution of the auxiliary problem:

$$\begin{cases} a(\hat{\beta}_h, \eta_h) + (\hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda(\sigma \nabla w, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \hat{\gamma}_h = \frac{\kappa}{t^2}(\nabla \hat{w}_h - R\hat{\beta}_h). \end{cases}$$

This problem is the finite element discretization of (2.4), with source term  $f = \lambda w \in H^2(\Omega) \cap H_0^1(\Omega)$ . Then from Remark 4.1, (3.12), and the fact that  $\|w_h\|_{1,\Omega} = 1$ , we obtain the following error estimate:

$$(5.4) \quad \|\beta - \hat{\beta}_h\|_{1,\Omega} + t\|\gamma - \hat{\gamma}_h\|_{0,\Omega} + \|w - \hat{w}_h\|_{1,\Omega} \leq Ch|\lambda|\|w\|_{2,\Omega} \leq Ch|\lambda|.$$

On the other hand, from (4.4), we have that  $(\beta_h - \hat{\beta}_h, w_h - \hat{w}_h) \in H_h \times W_h$  satisfies

$$\begin{cases} a(\beta_h - \hat{\beta}_h, \eta_h) + (\gamma_h - \hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\sigma \nabla (\lambda_h w_h - \lambda w), \nabla v_h)_{0,\Omega} \\ \gamma_h - \hat{\gamma}_h = \frac{\kappa}{t^2}(\nabla (w_h - \hat{w}_h) - R(\beta_h - \hat{\beta}_h)). \end{cases} \quad \forall (\eta_h, v_h) \in H_h \times W_h,$$

Taking  $\eta_h = \beta_h - \hat{\beta}_h$  and  $v_h = w_h - \hat{w}_h$  in the system above, from the ellipticity of  $a(\cdot, \cdot)$ , we obtain

$$\begin{aligned} & \|\beta_h - \hat{\beta}_h\|_{1,\Omega}^2 + \kappa^{-1}t^2\|\gamma_h - \hat{\gamma}_h\|_{0,\Omega}^2 \\ & \leq C\|\lambda_h w_h - \lambda w\|_{1,\Omega}\|w_h - \hat{w}_h\|_{1,\Omega} \\ & \leq C\left(|\lambda_h|\|w - w_h\|_{1,\Omega} + |\lambda - \lambda_h|\|w\|_{1,\Omega}\right)\left(\|w - w_h\|_{1,\Omega} + \|w - \hat{w}_h\|_{1,\Omega}\right) \\ & \leq Ch^2, \end{aligned}$$

where we have used Lemma 5.6 and estimates (5.2) and (5.4) for the last inequality. Therefore, we have

$$\|\beta_h - \hat{\beta}_h\|_{1,\Omega} + t \|\gamma_h - \hat{\gamma}_h\|_{0,\Omega} \leq Ch.$$

Thus, the lemma follows from this estimate and (5.4).  $\square$

We are now in a position to prove an optimal double-order error estimate for the eigenvalues.

**THEOREM 5.8.** *There exist positive constants  $h_0$ ,  $t_1$ , and  $C$  such that,  $\forall h < h_0$  and  $\forall t < t_1$ ,*

$$|\lambda - \lambda_h| \leq Ch^2.$$

*Proof.* We adapt to our case a standard argument for eigenvalue problems (see [2, Lemma 9.1]). Let  $(\lambda, \beta, w, \gamma)$  and  $(\lambda_h, \beta_h, w_h, \gamma_h)$  be as in Lemma 5.7. We will use the bilinear forms  $A$  and  $B$  defined in (3.1) and (3.2), respectively, as well as the bilinear form  $A_h$  defined in  $H_h \times W_h$  as follows:

$$A_h((\beta_h, w_h), (\eta_h, v_h)) := a(\beta_h, \eta_h) + \frac{\kappa}{t^2} (\nabla w_h - R\beta_h, \nabla v_h - R\eta_h)_{0,\Omega}.$$

With this notation, problems (2.4) and (4.4) can be written as follows:

$$\begin{aligned} A((\beta, w), (\eta, v)) &= \lambda B((\beta, w), (\eta, v)), \\ A_h((\beta_h, w_h), (\eta_h, v_h)) &= \lambda_h B((\beta_h, w_h), (\eta_h, v_h)). \end{aligned}$$

From these equations, straightforward computations lead to

$$\begin{aligned} (5.5) \quad (\lambda_h - \lambda) B((\beta_h, w_h), (\beta_h, w_h)) &= A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ &\quad - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ &\quad + [A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h))]. \end{aligned}$$

Next, we define  $\bar{\gamma}_h := \frac{\kappa}{t^2} (\nabla w_h - \beta_h)$ . Recalling that  $R\nabla w_h = \nabla w_h$  (cf. (4.3)), from the definition of  $\gamma_h$  (cf. (4.4)) we have that  $\gamma_h = R\bar{\gamma}_h$ . On the other hand, from the definition of  $A$  and  $A_h$  we write

$$\begin{aligned} A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) &= a(\beta - \beta_h, \beta - \beta_h) + \kappa^{-1} t^2 \|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2, \\ A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h)) &= \kappa^{-1} t^2 \left( \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} (\lambda_h - \lambda) B((\beta_h, w_h), (\beta_h, w_h)) &= a(\beta - \beta_h, \beta - \beta_h) \\ &\quad + \kappa^{-1} t^2 \left( \|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right) \\ &\quad - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)). \end{aligned}$$

The first and the third term in the right-hand side above are easily bounded by virtue of (5.2) and (5.3). For the second term, we write

$$\begin{aligned} (5.6) \quad \|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 &= \|\gamma - R\bar{\gamma}_h\|_{0,\Omega}^2 - 2(\gamma, \bar{\gamma}_h - R\bar{\gamma}_h)_{0,\Omega} \\ &= \|\gamma - \gamma_h\|_{0,\Omega}^2 + \frac{2\kappa}{t^2} (\gamma, \beta_h - R\beta_h)_{0,\Omega}. \end{aligned}$$

For  $\beta \in H^2(\Omega)^2 \cap H_0^1(\Omega)$ , we denote by  $\beta^I \in H_h$  the standard Clément interpolant of  $\beta$ , which satisfies

$$(5.7) \quad \|\beta^I\|_{1,\Omega} \leq C \|\beta\|_{1,\Omega} \quad \text{and} \quad \|\beta - \beta^I\|_{1,\Omega} \leq Ch \|\beta\|_{2,\Omega}.$$

It follows that

$$\begin{aligned} (\gamma, \beta_h - R\beta_h)_{0,\Omega} &= (\gamma, (\beta_h - \beta^I) - R(\beta_h - \beta^I))_{0,\Omega} + (\gamma, \beta^I - R\beta^I)_{0,\Omega} \\ &\leq \|\gamma\|_{0,\Omega} \|(\beta_h - \beta^I) - R(\beta_h - \beta^I)\|_{0,\Omega} + (\gamma, \beta^I - R\beta^I)_{0,\Omega}. \end{aligned}$$

Thus, using (4.2) and Lemma 3.3 from [10], we obtain

$$\begin{aligned} (\gamma, \beta_h - R\beta_h)_{0,\Omega} &\leq Ch \|\gamma\|_{0,\Omega} \|\beta_h - \beta^I\|_{1,\Omega} + Ch^2 \|\operatorname{div} \gamma\|_{0,\Omega} \|\beta\|_{1,\Omega} \\ &\leq Ch \|\gamma\|_{0,\Omega} (\|\beta - \beta_h\|_{1,\Omega} + \|\beta - \beta^I\|_{1,\Omega}) + Ch^2 \|\operatorname{div} \gamma\|_{0,\Omega} \|\beta\|_{1,\Omega}, \end{aligned}$$

and from Lemma 5.7, (5.7), and Proposition 3.10, we have

$$(\gamma, \beta_h - R\beta_h)_{0,\Omega} \leq Ch \|\gamma\|_{0,\Omega} (Ch + Ch \|\beta\|_{2,\Omega}) + Ch^2 |\lambda| \|w\|_{2,\Omega} \|\beta\|_{1,\Omega} \leq Ch^2 |\lambda|.$$

Finally, we use this estimate, (5.5), (5.6), and the fact that  $B((\beta_h, w_h), (\beta_h, w_h)) = (\sigma \nabla w_h, \nabla w_h)_{0,\Omega} \neq 0$  (cf. Remark 4.2) to obtain

$$|\lambda - \lambda_h| \leq C \frac{\|\beta - \beta_h\|_{1,\Omega}^2 + \|w - w_h\|_{1,\Omega}^2 + \kappa^{-1} t^2 \|\gamma - \gamma_h\|_{0,\Omega}^2 + Ch^2 |\lambda|}{|B((\beta_h, w_h), (\beta_h, w_h))|}.$$

Consequently, from Lemma 5.7,

$$|\lambda - \lambda_h| \leq Ch^2$$

and we conclude the proof.  $\square$

**6. Numerical results.** In this section we report some numerical experiments carried out with our method applied to problem (2.4).

For all the computations we have taken  $\Omega := (0, 6) \times (0, 4)$  and typical parameters of steel: the Young modulus has been chosen  $E = 1.44 \times 10^{11}$  Pa and the Poisson ratio  $\nu = 0.30$ . All the lengths are measured in meters and the shear correction factor has been taken  $k = 5/6$ . We recall that the buckling coefficients  $\lambda_b$  may be easily computed from the eigenvalues of (2.4) (cf. Remark 2.1).

We have used uniform meshes as shown in Fig. 6.1; the meaning of the refinement parameter  $N$  can be easily deduced from this figure. Notice that  $h \sim N^{-1}$ .

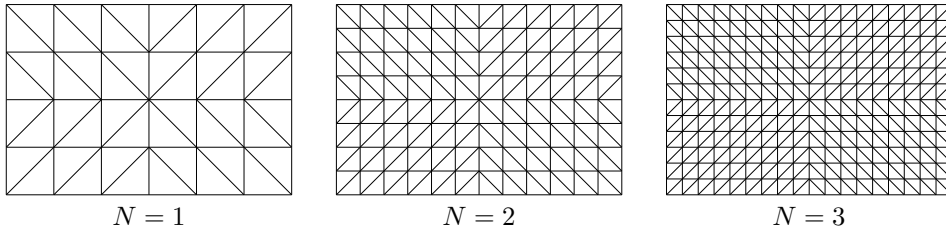


FIG. 6.1. Rectangular plate. Uniform meshes.



**6.1. Uniformly compressed rectangular plate.** For this test we have used  $\sigma = I$ , which corresponds to a uniformly compressed plate.

**6.1.1. Simply supported plate.** First, we have considered a simply supported plate, since analytical solutions are available in that case (see [19, 20]). Even though our theoretical analysis has been developed only for clamped plates, we think that the results of Sections 4 and 5 should hold true for more general boundary conditions, as well. The results that follow give some numerical evidence of this.

In Table 6.1 we report the four lowest eigenvalues ( $\lambda_i$ ,  $i = 1, 2, 3, 4$ ) for a thickness  $t = 0.001$ , computed by our method with four different meshes ( $N = 2, 4, 8, 16$ ). The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. Furthermore, the last column shows the exact eigenvalues.

TABLE 6.1  
Lowest eigenvalues  $\lambda_i$  (multiplied by  $10^{-10}$ ) of a uniformly compressed simply supported plate with thickness  $t = 0.001$ .

Eigenvalue	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated	Exact
$\lambda_1$	1.1793	1.1759	1.1752	1.1750	2.14	1.1750	1.1749
$\lambda_2$	2.2638	2.2602	2.2596	2.2595	2.68	2.2595	2.2595
$\lambda_3$	3.7293	3.6441	3.6224	3.6170	1.98	3.6151	3.6152
$\lambda_4$	4.1573	4.0892	4.0726	4.0685	2.03	4.0672	4.0671

It can be seen from Table 6.1 that the method converges to the exact values with an optimal quadratic order.

Figure 6.2 shows the transverse displacements for the principal buckling mode computed with the finest mesh ( $N = 16$ ).

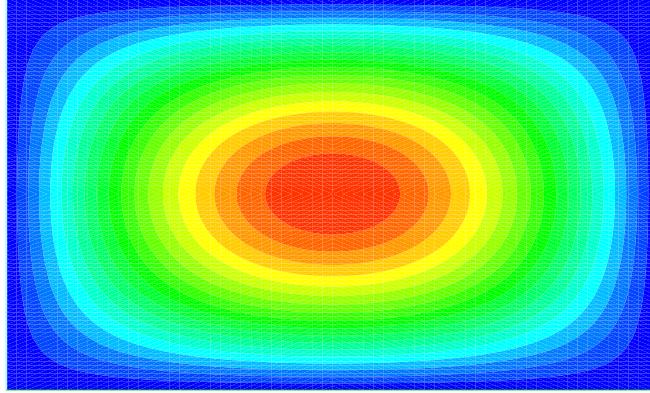


FIG. 6.2. Uniformly compressed simply supported plate; principal buckling mode.

**6.1.2. Clamped plate.** In Table 6.2 we present the results for the lowest eigenvalue of a uniformly compressed clamped rectangular plate, with varying thickness. We have used the same meshes as in the previous test. Again, we have computed the orders of convergence, and more accurate values obtained by a least-squares procedure. Furthermore, in the last row we also report for each mesh the limit values as  $t$  goes to zero obtained by extrapolation.

TABLE 6.2  
 Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of uniformly compressed clamped plates with varying thickness.

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	3.4031	3.3440	3.3293	3.3258	2.02	3.3246
$t = 0.01$	3.4324	3.3723	3.3571	3.3533	1.99	3.3520
$t = 0.001$	3.4327	3.3726	3.3574	3.3536	1.99	3.3522
$t = 0.0001$	3.4327	3.3726	3.3574	3.3536	1.98	3.3522
$t = 0$ (extrap.)	3.4327	3.3726	3.3574	3.3536	1.99	3.3523

Figure 6.3 shows the transverse displacements for the principal buckling mode, for  $t = 0.1$ , and the finest mesh ( $N = 16$ ).

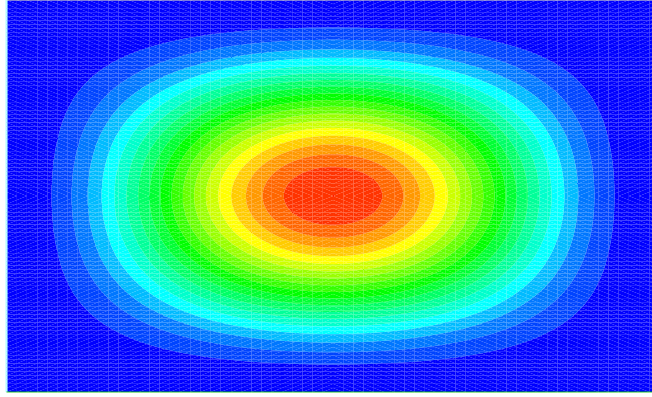


FIG. 6.3. Uniformly compressed clamped plate; principal buckling mode.

According to Lemma 3.7, the values on the last row of Table 6.2 should correspond to the lowest eigenvalues of a Kirchhoff-Love uniformly compressed clamped plate with thickness  $t = 1$ . As a further test, we have also computed the latter, by using the methods analyzed in [6] and [17]. We show the obtained results in Table 6.3, where an excellent agreement with the last row of Table 6.2 can be appreciated.

TABLE 6.3  
 Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of a uniformly compressed clamped thin plate (Kirchhoff-Love model) computed with the methods from [6] and [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	3.3718	3.3611	3.3573	3.3555	1.97	3.3523
[17]	3.3514	3.3519	3.3521	3.3522	1.95	3.3523

It is clear that the results from the Reissner-Mindlin model do not deteriorate as the plate thickness become smaller, which confirms that our method is *locking-free*.

**6.2. Clamped plate uniformly compressed in one direction.** We have used for this test

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which corresponds to a plate uniformly compressed in one direction. Notice that in this test  $\sigma$  is only positive semi-definite. Table 6.4 shows the same quantities as Table 6.2 in this case.

TABLE 6.4  
Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of clamped plates with varying thickness, uniformly compressed in one direction.

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	6.7969	6.7274	6.7104	6.7066	2.05	6.7052
$t = 0.01$	6.8825	6.8143	6.7971	6.7930	2.00	6.7915
$t = 0.001$	6.8834	6.8151	6.7980	6.7939	2.00	6.7924
$t = 0.0001$	6.8834	6.8152	6.7980	6.7939	2.00	6.7924
$t = 0$ (extrap.)	6.8834	6.8152	6.7980	6.7939	2.00	6.7924

Figure 6.4 shows the principal buckling mode for  $t = 0.1$  and the finest mesh ( $N = 16$ ).

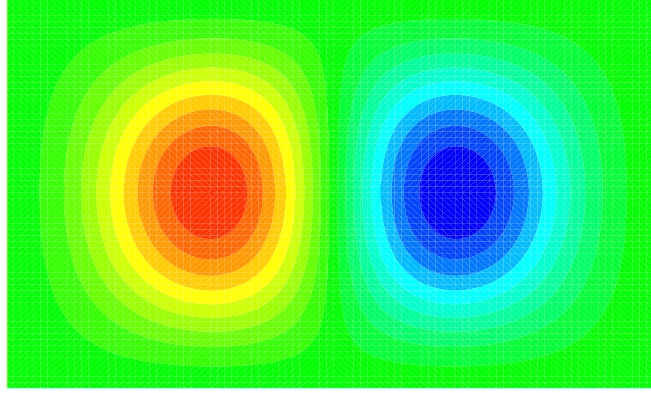


FIG. 6.4. Clamped plate uniformly compressed in one direction; principal buckling mode.

Finally, Table 6.5 shows the same quantities as Table 6.3 in this case. Once more, an excellent agreement with the values extrapolated from the Reissner-Mindlin model (last row of Table 6.4) can be clearly appreciated.

TABLE 6.5  
Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of a clamped thin plate (Kirchhoff-Love model) uniformly compressed in one direction, computed with the methods from [6] and [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	6.8450	6.8158	6.8056	6.8009	2.00	6.7925
[17]	6.7904	6.7913	6.7917	6.7920	1.92	6.7926

**6.3. Shear loaded clamped plate.** In this case we have used

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which corresponds to a uniform shear load. Notice that  $\sigma$  is indefinite in this test. The numerical results are reported in Table 6.6, Figure 6.5, and Table 6.7, using the

same pattern as the previous tests.

TABLE 6.6  
Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of shear loaded clamped plates with varying thickness.

Thickness	$N = 4$	$N = 8$	$N = 12$	$N = 16$	Order	Extrapolated
$t = 0.1$	9.4306	9.2179	9.1783	9.1645	1.99	9.1464
$t = 0.01$	9.6098	9.3923	9.3514	9.3371	1.98	9.3184
$t = 0.001$	9.6116	9.3942	9.3533	9.3389	1.98	9.3202
$t = 0.0001$	9.6117	9.3942	9.3533	9.3389	1.98	9.3202
$t = 0$ (extrap.)	9.6117	9.3942	9.3533	9.3389	1.98	9.3202

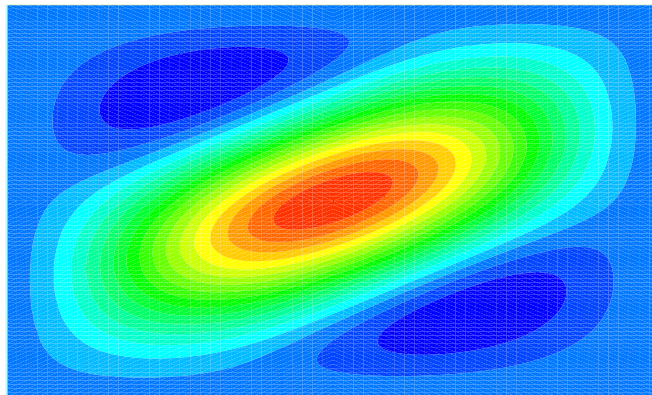


FIG. 6.5. Shear loaded clamped plate; principal buckling mode.

TABLE 6.7  
Lowest eigenvalue  $\lambda_1$  (multiplied by  $10^{-10}$ ) of a shear loaded clamped thin plate (Kirchhoff-Love model) computed with the methods from [6] and [17].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[6]	9.4625	9.3840	9.3563	9.3435	1.98	9.3203
[17]	9.3660	9.3408	9.3319	9.3278	1.99	9.3204

In all cases, an excellent agreement between the numerical experiments and the theoretical results detailed in Section 5 can be noticed and the method appears thoroughly locking-free.

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### Appendix. Uniformly compressed plates.

The aim of this appendix is to show that the results of Sections 3, 4, and 5 can be refined when  $\sigma = \mathbf{I}$ , which corresponds to a uniformly compressed plate. In this case, we are able to give a better characterization of the spectrum of  $T_t$  and to prove the spectral approximation without assuming that the family of meshes is *quasi-uniform*.

**A.1. Spectral characterization.** We have the following counterpart of Theorem 3.1, showing that the spectrum of  $T_t$  is simply a shift of the spectrum of a compact operator.

**THEOREM A.1.** *Suppose that  $\sigma = \mathbf{I}$ . For all  $t > 0$ , the spectrum of  $T_t$  satisfies*

$$\mathrm{Sp}(T_t) = \mathrm{Sp}(G) + \kappa^{-1}t^2,$$

where  $G$  is the compact operator defined in (3.5).

*Proof.* The first equation of (2.7) leads in this case to  $\psi = f$ , due to the fact that  $\sigma = \mathbf{I}$ . Therefore, (2.7) reduces to

$$(A.1) \quad \begin{cases} a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla f, \eta)_{0,\Omega} & \forall \eta \in H_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1} t^2 (\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in H^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1} t^2 (\nabla f, \nabla \xi)_{0,\Omega} & \forall \xi \in H_0^1(\Omega). \end{cases}$$

Next, recall that  $G$  is defined in (3.5) as the operator mapping  $f \mapsto u$ , with  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \quad \forall \xi \in H_0^1(\Omega),$$

where  $\beta \in H_0^1(\Omega)^2$  is determined in this case by the first two equations from (A.1). Therefore, the third equation from (A.1) yields  $T_t = G + \kappa^{-1} t^2 I$ . Since  $G$  has been already shown to be compact, this allows us to conclude the theorem.  $\square$

As a consequence of this theorem,  $\operatorname{Sp}(T_t) = \{\kappa^{-1} t^2\} \cup \{\mu_n : n \in \mathbb{N}\}$ , with  $\mu_n$  being a sequence of finite-multiplicity eigenvalues converging to  $\kappa^{-1} t^2$ . Therefore, in this particular case, the *essential spectrum* of  $T_t$  reduces to a unique point:  $\kappa^{-1} t^2$ .

**A.2. Spectral approximation.** In this particular case, we will improve the error estimate shown in Section 4 in that we will not need to assume quasi-uniformity of the meshes. Indeed, this property was used above only to prove Proposition 4.3. Instead, we have now the following result.

**PROPOSITION A.2.** *Suppose that  $\sigma = \mathbf{I}$ . Then for any regular family of triangular meshes  $\{\mathcal{T}_h\}_{h>0}$ , there exists  $C > 0$  such that, for all  $t > 0$ ,*

$$\|T_t - T_{th}\|_h \leq Ch.$$

*Proof.* We will simply sketch the proof, since it follows exactly the same steps as that of Proposition 4.3. First, we notice that in the decomposition (4.12) we have  $\psi = f_h \in W_h$  (cf. problem (4.14) with  $\sigma = \mathbf{I}$ ).

As a consequence, we infer that the term  $\|\nabla \psi - \nabla \psi_I\|_{0,\Omega}$  in (4.15) vanishes. Hence, the last estimate in the proof of Lemma 4.5 changes into

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} \leq C \left( h \|\beta\|_{2,\Omega} + th \|p\|_{2,\Omega} + h \|\gamma\|_{0,\Omega} \right) \leq Ch \|f_h\|_{1,\Omega}.$$

By using the above estimate in the proof of Proposition 4.3 (in particular in (4.16)), we obtain

$$\|(T_t - T_{th}) f_h\|_{1,\Omega} = \|w - w_h\|_{1,\Omega} \leq Ch \|f_h\|_{1,\Omega},$$

from which we conclude the proof.  $\square$

As a consequence of Proposition A.2, we can improve Lemma 4.6. In fact, now for  $t$  small enough there holds directly

$$\|T_t - T_{th}\|_h \leq Ch,$$

with a constant  $C$  independent of  $h$  and  $t$ . By using this instead of property P1, we could give somewhat simpler proofs for the error estimates from Section 5. However the final results, Theorems 5.1, 5.5, and 5.8 are the same, although now valid for any regular family of triangular meshes, without the need of being quasi-uniform.

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