# An $\mathrm{L}^p$ spaces-based mixed virtual element method for the two-dimensional Navier-Stokes equations*

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**Abstract**

In this paper we extend the utilization of the Banach spaces-based formulations, usually employed for solving diverse nonlinear problems in continuum mechanics via primal and mixed finite element methods, to the virtual element method (VEM) framework and its respective applications. More precisely, we propose and analyze an $\mathrm{L}^p$ spaces-based mixed virtual element method for a pseudostress-velocity formulation of the two-dimensional Navier-Stokes equations with Dirichlet boundary conditions. To this end, a dual-mixed approach determined by the introduction of a nonlinear tensor linking the usual pseudostress for the Stokes equations with the convective term, is employed. As a consequence, this new tensor, say $\boldsymbol{\sigma}$, and the velocity $\mathbf{u}$ of the fluid constitute the unknowns of the formulation, whereas the pressure is computed via a postprocessing formula. The simplicity of the resulting VEM scheme is reflected by the absence of augmented terms, on the contrary to previous works on this and related models, and by the incorporation in it of only the projector onto the piecewise polynomial tensors and the usual stabilizer depending on the degrees of freedom of the virtual element subspace approximating $\boldsymbol{\sigma}$. In turn, the non-virtual but explicit subspace given by the piecewise polynomial vectors of degree $\leq k$, is employed to approximate $\mathbf{u}$. The corresponding solvability analysis is carried out by using appropriate fixed-point arguments, along with the discrete versions of the Babuška-Brezzi theory and the Banach-Nečas-Babuška theorem, both in subspaces of Banach spaces. A Strang-type lemma is applied to derive the *a priori* error estimates for the virtual element solution as well as for the fully computable approximation of $\boldsymbol{\sigma}$, the postprocessed pressure, and a second postprocessed approximation of $\boldsymbol{\sigma}$. Finally, several numerical results illustrating the performance of the mixed-VEM scheme and confirming the rates of convergence predicted by the theory, are reported.

**Key words:** Navier-Stokes problem, pseudostress-velocity formulation, Banach spaces, mixed virtual element method, high-order approximations

**Mathematical subject classifications (2000):** 65N30, 65N12, 65N15, 65N99, 76M25, 76S05

## 1 Introduction

The numerical solution of diverse linear and nonlinear problems in fluid mechanics via the virtual element method (VEM) is becoming nowadays a very active research area. The models studied

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include, among others, Stokes, Navier-Stokes, Brinkman, Stokes-Darcy, and quasi-Newtonian Stokes flows, whereas the approaches employed usually consider primal and dual-mixed formulations, as well as some variants of them. In particular, regarding the application of a VEM technique to the classical velocity-pressure formulation of the Stokes equations, we refer to [2], [6], [8], [11], [23], [26], [27], [28], and [44], where stream function-based, divergence free, and non-conforming virtual element methods are proposed. In addition, corresponding $p$ and $hp$ versions, associated eigenvalue problems, and the application to the Stokes-Darcy model are analyzed in [29], [43], [24], and [25], respectively. In turn, virtual element methods based on dual-mixed variational formulations have also been developed for the Stokes and related linear models. Indeed, we first recall that a mixed-VEM for the pseudostress-velocity formulation of the Stokes problem, with the pressure being computed via a postprocessing formula, was introduced and analized in [15]. The corresponding extension of this work to the two-dimensional Brinkman problem was presented in [16], in which the pseudostress becomes the only unknown of the resulting dual-mixed formulation.

Furthermore, regarding the applicability of VEM to nonlinear problems, and particularly to the Navier-Stokes equations, we begin by highlighting that [9] constitutes the first work developing an $\mathrm{H}^1$-conforming VEM for the velocity-pressure formulation of this model. In fact, the authors consider there the two-dimensional case and basically extend the approach from [8] to this nonlinear situation, whence pointwise divergence-free discrete velocities are obtained as well. The underlying Stokes complex structure of the virtual element methods introduced in [8] and [9] is addressed later on in [11]. In turn, an $\mathrm{H}^1$ but non-conforming VEM for the Navier-Stokes equations was proposed in [47].

On the other hand, in [38] we considered the same dual-mixed variational formulation from [21] (see also [19], [20]), and developed, up to our knowledge, the first mixed virtual element method for the Navier Stokes equations. More precisely, the approach in [38] is based on the incorporation as unknown of the nonlinear tensor that arises after adding the convective term to the usual pseudostress for Stokes (cf. [15]). In addition, and in order to be able to address the analysis in a Hilbertian framework, Galerkin type consistent terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, all them multiplied by suitable stabilization parameters, are added to the resulting continuous formulation, thus yielding an augmented scheme. As for the discrete setting, the main novelty of [38] lies on the simultaneous use, for the first time, of virtual element subspaces of $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$ approximating the velocity and the nonlinear pseudostress tensor, respectively. The extension of the analysis and results from [38] to the Boussinesq model is provided in [39]. Needless to say, we stress that one of the main advantages of employing the pseudostress and the velocity as main unknowns, lies on the fact that further variables of physical meaning, and hence of wide interest in applications, can be computed by simple postprocessing formulae and without any loss of accuracy. Other contributions dealing with VEM for nonlinear models include [7], [17], [22], [37], and [47]. In particular, a virtual element method for quasilinear elliptic problems is studied in [22], whereas the approaches from [15] and [16] are extended in [17] and [37] to derive mixed-VEM schemes for quasi-Newtonian Stokes flows and for nonlinear Brinkman models of porous media flow.

Going back to [38], we emphasize that the augmented formulation introduced there, and the consequent use of two different types of virtual element subspaces to define the discrete scheme, are originated by the wish of performing the respective solvability analysis within a Hilbertian framework. However, it is well known that the introduction of additional terms into the formulation, while having some advantages, also leads to much more expensive schemes in terms of complexity and computational implementation. In the particular case of the usual mixed finite element method, there is an increasing development in recent years on Banach spaces-based approaches to solve a wide family of nonlinear problems in continuum mechanics (see, e.g. [12], [18], [30], [32], [40], and the references therein). This kind of procedures shows two advantages at least: no augmentation is required, and the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy-Schwarz and Hölder inequalities to the terms resulting from the testing and integration by parts of the equations of the model. As a consequence, simpler and closer to the original physical model formulations are obtained.

According to the previous discussion, our long-term objective is to continue extending the applicability of the Banach spaces-based analysis, but now to address the solvability, via mixed virtual element methods, of diverse nonlinear problems in continuum mechanics. In the present paper we begin to contribute to the achievement of this goal by considering as a model the two-dimensional Navier Stokes equations. The rest of the paper is organized as follows. In Section 2 we resort to [18] to set the model of interest, recall the associated dual-mixed variational formulation with the unknowns $\boldsymbol{\sigma}$ and $\mathbf{u}$ living in suitable Banach spaces, and state the main result establishing its well-posedness. The mixed virtual element scheme is introduced and analyzed in Section 3. Some preliminaries on the VEM methodology, which includes the orthogonal projectors onto polynomial spaces and their associated approximation properties, are provided first. Then, the finite dimensional subspaces to be employed and the VEM scheme itself, are defined. In Section 4 we apply a fixed-point strategy to analyze the solvability of our discrete formulation. Besides the usual estimates concerning the bilinear and trilinear forms involved, a key step of our analysis is a local stability bound for the virtual interpolation operator, thanks to which a fundamental discrete inf-sup condition can be proved. The classical Banach fixed-point theorem allows to conclude the main result. *A priori* error estimates for the full solution of the virtual element scheme, as well as for computable postprocessed approximations of $\boldsymbol{\sigma}$ and the pressure $p$, are derived in Section 5. Finally, several examples examples confirming the predicted performance of the method, are described in Section 6.

We end this section with some notations to be used along the paper, including those already employed above. Firstly, for any vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$ we set the gradient, divergence and tensor product operators as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \qquad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}, \qquad \text{and} \qquad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,2}\,,$$

respectively. In addition, denoting by $\mathbb{I}$ the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathrm{R}^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})\,, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\,, \qquad \text{and} \qquad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij},$$

which corresponds, respectively, to the transpose, the trace, and the deviatoric tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. Next, given a Lipschitz-continuous domain $\mathcal{O}$ with boundary $\Gamma$, we adopt standard notations for Lebesgue spaces $\mathrm{L}^t(\mathcal{O})$ and Sobolev spaces $\mathrm{W}^{\ell,t}(\mathcal{O})$ with $\ell \geq 0$ and $t \in [1, +\infty)$, whose corresponding norms and seminorm, either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\mathcal{O}}$, $\|\cdot\|_{\ell,t;\mathcal{O}}$ and $|\cdot|_{\ell,t;\mathcal{O}}$, respectively. Note that $\mathrm{W}^{0,t}(\mathcal{O}) = \mathrm{L}^t(\mathcal{O})$, and if $t = 2$ we write $\mathrm{H}^\ell(\mathcal{O})$ instead of $\mathrm{W}^{\ell,2}(\mathcal{O})$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{\ell,\mathcal{O}}$ and $|\cdot|_{\ell,\mathcal{O}}$, respectively. Furthermore, given a generic scalar functional space $\mathrm{M}$, we let $\mathbf{M}$ and $\mathbb{M}$ be its vectorial and tensorial counterparts, respectively, with norms and seminorms denoted exactly as those of $\mathrm{M}$, examples of which are $\mathbf{L}^t(\mathcal{O}) := [\mathrm{L}^t(\mathcal{O})]^n$, $\mathbf{W}^{\ell,t}(\mathcal{O}) := [\mathrm{W}^{\ell,t}(\mathcal{O})]^n$, and $\mathbb{H}^t(\mathcal{O}) := [\mathrm{H}^t(\Omega)]^{n\times n}$. On the other hand, letting $\mathbf{div}$ be the usual divergence operator div acting along the rows of a given tensor, and given $p \in (1, +\infty)$, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_p;\Omega) \;:=\; \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \;:\quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^p(\Omega)\right\},$$

endowed with the natural norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_p;\Omega)\,.$$

Finally, we employ C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretisation parameters, which may take different values at different places.

## 2 The model and its continuous formulation

In this section we recall the mixed variational formulation introduced in [18] for the two-dimensional steady-state Navier-Stokes equations with constant viscosity $\mu > 0$ and Dirichlet boundary conditions. To this end, we first let $\Omega$ be a bounded polygonal domain in $\mathbf{R} := \mathrm{R}^2$ with boundary $\Gamma$ and respective unit outward normal denoted by $\boldsymbol{n}$. Then, given a volume force $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$ and a Dirichlet datum $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, we seek a velocity vector field $\mathbf{u}$ and a pressure scalar field $p$ such that

$$\begin{aligned} -\mu\Delta\mathbf{u} \,+\, (\nabla\mathbf{u})\,\mathbf{u} \,+\, \nabla p \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, \qquad \mathrm{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega\,, \\ \mathbf{u} \;=\; \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \text{and} \qquad \int_\Omega p \;=\; 0\,. \end{aligned} \tag{2.1}$$

Note from the incompressibility condition that $\mathbf{u}_D$ is required to satisfy the compatibility condition $\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{n} = 0$. Next, we define the constant

$$c_{\mathbf{u}} \;:=\; -\frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u}) \;=\; -\frac{1}{2|\Omega|}\|\mathbf{u}\|_{0,\Omega}^2\,, \tag{2.2}$$

and the pseudostress tensor

$$\boldsymbol{\sigma} \;:=\; \mu\nabla\mathbf{u} \,-\, \mathbf{u}\otimes\mathbf{u} \,-\, (p + c_{\mathbf{u}})\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{2.3}$$

where $\mathbb{I}$ is the identity matrix of $\mathbb{R} := \mathrm{R}^{2\times 2}$. Taking the matrix trace in (2.3), and then solving for the pressure, we arrive at

$$p \;=\; -\frac{1}{2}\Big\{\mathrm{tr}(\boldsymbol{\sigma}) \,+\, \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\Big\} \,-\, c_{\mathbf{u}} \quad \text{in} \quad \Omega\,, \tag{2.4}$$

which allows us to eliminate the pressure variable from the rest of the formulation. In fact, applying the deviatoric operator to (2.3), and realizing, thanks to the incompressibility condition, that

$$\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mu\Delta\mathbf{u} \,-\, (\nabla\mathbf{u})\,\mathbf{u} \,-\, \nabla p\,,$$

we can rewrite (2.1) as the equivalent system: Find the pseudostress $\boldsymbol{\sigma}$ and the velocity $\mathbf{u}$ such that

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{d}} \;=\; \mu\nabla\mathbf{u} \,-\, (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \;=\; -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathbf{u} \;=\; \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \text{and} \qquad \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;=\; 0\,. \end{aligned} \tag{2.5}$$

In this way, we now introduce the spaces

$$\mathbb{H} \;=\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;=\; 0\right\}, \tag{2.6}$$

and

$$\mathbf{Q} \;:=\; \mathbf{L}^4(\Omega)\,, \tag{2.7}$$

so that, following [18], the variational formulation of (2.5) reads: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}\times\mathbf{Q}$ such that

$$\begin{array}{rcll} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\mathbf{u}) + c(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) & = & F(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau}\in\mathbb{H}\,,\\ b(\boldsymbol{\sigma},\mathbf{v}) & = & G(\mathbf{v}) & \forall\,\mathbf{v}\in\mathbf{Q}\,, \end{array} \tag{2.8}$$

where the bilinear forms $a:\mathbb{H}\times\mathbb{H}\to\mathrm{R}$ and $b:\mathbb{H}\times\mathbf{Q}\to\mathrm{R}$, the trilinear form $c:\mathbf{Q}\times\mathbf{Q}\times\mathbb{H}\to\mathrm{R}$, and the functionals $F:\mathbb{H}\to\mathrm{R}$ and $G:\mathbf{Q}\to\mathrm{R}$ are defined, respectively, as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) \;:=\; \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}\,, \tag{2.9}$$

$$b(\boldsymbol{\tau},\mathbf{v}) \;:=\; \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{2.10}$$

$$c(\mathbf{z};\mathbf{v},\boldsymbol{\tau}) \;:=\; \frac{1}{\mu}\int_\Omega (\mathbf{z}\otimes\mathbf{v})^{\mathrm{d}} : \boldsymbol{\tau}\,, \tag{2.11}$$

$$F(\boldsymbol{\tau}) \;:=\; \langle \boldsymbol{\tau}\boldsymbol{n}, \mathbf{u}_D\rangle_\Gamma\,, \tag{2.12}$$

and

$$G(\mathbf{v}) \;:=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,, \tag{2.13}$$

for all $\boldsymbol{\zeta}$, $\boldsymbol{\tau}\in\mathbb{H}$ and for all $\mathbf{z},\mathbf{v}\in\mathbf{Q}$. Note that $a$ and $b$ are clearly bounded with

$$\|a\| \;=\; \frac{1}{\mu} \qquad\text{and}\qquad \|b\| \;=\; 1\,. \tag{2.14}$$

In addition, we know from [18, Lemma 3.5] that there hold

$$|F(\boldsymbol{\tau})| \;\le\; C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,, \tag{2.15}$$

where $C_F$ is a positive constant depending only on $\Omega$, and

$$|G(\mathbf{v})| \;\le\; \|\mathbf{f}\|_{0,4/3;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,. \tag{2.16}$$

The unique solvability of (2.8) is established as follows.

**Theorem 2.1.** *Let $\gamma$ be the positive constant arising from the global inf-sup condition for the left hand side of* (2.8) (cf. [18, eq. (3.29)])*, define the ball*

$$S := \Big\{\mathbf{z}\in\mathbf{Q}: \quad \|\mathbf{z}\|_{0,4;\Omega} \;\le\; \frac{\gamma\mu}{2}\Big\}\,,$$

*and assume that the data satisfy*

$$\frac{4}{\gamma^2\mu}\Big\{C_F\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\Big\} \;<\; 1\,. \tag{2.17}$$

*Then, there exists a unique $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}\times\mathbf{Q}$ solution of* (2.8) *with $\mathbf{u}\in S$, and there holds*

$$\|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \,+\, \|\mathbf{u}\|_{0,4;\Omega} \;\le\; \frac{2}{\gamma}\Big\{C_F\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,. \tag{2.18}$$

**Proof.** It is a slight modification of the proof of [18, Theorem 3.8]. □

# 3 The virtual element method

In this section we introduce a mixed virtual element scheme for (2.8). The corresponding solvability analysis is provided later on in Section 4. We begin with some preliminary definitions and results to be employed in what follows.

## 3.1 Preliminaries

As usual in the VEM philosophy, we begin by letting $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of $\Omega$ in polygonal elements. Then, given $K \in \mathcal{T}_h$, we denote its barycenter, diameter, and number of edges by $\mathbf{x}_K$, $h_K$, and $d_K$, respectively, and set, as usual, $h := \max\{h_K : K \in \mathcal{T}_h\}$. Additionally, we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

(i) the ratio between the shortest edge and the diameter $h_K$ of $K$ is bigger than $C_{\mathcal{T}}$, and

(ii) $K$ is star-shaped with respect to a ball $B$ of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$, that is , for each $x_0 \in B$, all the line segments joining $x_0$ with any $x \in K$ are contained in $K$, or equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in $K$.

It is not difficult to see that the above hypotheses guarantee that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that $d_K \le N_{\mathcal{T}} \quad \forall K \in \mathcal{T}_h$. On the other hand, given an integer $\ell \ge 0$ and $U \subseteq \mathrm{R}^2$, we let $\mathrm{P}_\ell(U)$ be the space of polynomials on $U$ of degree up to $\ell$, so that, according to the notation introduced at the end of Section 1, we set $\mathbf{P}_\ell(U) := [\mathrm{P}_\ell(U)]^2$ and $\mathbb{P}_\ell(U) := [\mathrm{P}_\ell(U)]^{2\times 2}$. Also, throughout the rest of the paper we use the multi-index notation, that is, given $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathbf{R}$ and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)^{\mathtt{t}}$, with non-negative integers $\alpha_1$, $\alpha_2$, we define $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2$. In this way, for each $K \in \mathcal{T}_h$ and for each edge $e \subset \partial K$ with barycenter $x_e$ and diameter $h_e$, we introduce the sets of normalized monomials on $e$ and $K$ given, with generic vectors $x \in e$ and $\mathbf{x} \in K$, by

$$\mathcal{B}_\ell(e) := \left\{ \left( \frac{x - x_e}{h_e} \right)^j \right\}_{0 \le j \le \ell} \qquad \text{and} \qquad \mathcal{B}_\ell(K) := \left\{ \left( \frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^{\boldsymbol{\alpha}} \right\}_{0 \le |\boldsymbol{\alpha}| \le \ell},$$

which constitute basis of $\mathrm{P}_\ell(e)$ and $\mathrm{P}_\ell(K)$, respectively. In turn, the corresponding vectorial versions are denoted by $\boldsymbol{\mathcal{B}}_\ell(e)$ and $\boldsymbol{\mathcal{B}}_\ell(K)$, that is

$$\boldsymbol{\mathcal{B}}_\ell(e) := \left\{ (q, 0)^{\mathtt{t}} : \quad q \in \mathcal{B}_\ell(e) \right\} \cup \left\{ (0, q)^{\mathtt{t}} : \quad q \in \mathcal{B}_\ell(e) \right\},$$

and

$$\boldsymbol{\mathcal{B}}_\ell(K) := \left\{ (\mathbf{q}, 0)^{\mathtt{t}} : \quad \mathbf{q} \in \mathcal{B}_\ell(K) \right\} \cup \left\{ (0, \mathbf{q})^{\mathtt{t}} : \quad \mathbf{q} \in \mathcal{B}_\ell(K) \right\}.$$

Furthermore, for each integer $\ell \ge 0$ we now let $\mathcal{P}_\ell^K : \mathrm{L}^1(K) \to \mathrm{P}_\ell(K)$ be the usual orthogonal projector with respect to the $\mathrm{L}^2(K)$-inner product, that is, given $v \in \mathrm{L}^1(K)$, $\mathcal{P}_\ell^K(v)$ is the unique element in $\mathrm{P}_\ell(K)$ satisfying

$$\int_K \mathcal{P}_\ell^K(v)\, q = \int_K v\, q \qquad \forall\, q \in \mathrm{P}_\ell(K). \tag{3.1}$$

Similarly, we let $\boldsymbol{\mathcal{P}}_\ell^K : \mathbf{L}^1(K) \to \mathbf{P}_\ell(K)$ and $\boldsymbol{\mathscr{P}}_\ell^K : \mathbb{L}^1(K) \to \mathbb{P}_\ell(K)$ be the vectorial and tensorial versions of $\mathcal{P}_\ell^K$, which are characterized by the analogue identities to (3.1).

Then, resorting to the analysis and results provided in [38, Section 3.4], we are able to establish next the approximation properties of the projectors $\mathcal{P}_\ell^K$, $\boldsymbol{\mathcal{P}}_\ell^K$, and $\mathbb{P}_\ell^K$, with respect to general Sobolev semi-norms.

**Lemma 3.1.** *Let $K \in \mathcal{T}_h$, $p > 1$, and $\ell$, $s$, $m$ be integers such that $\ell \geq 0$ and $0 \leq m \leq s \leq \ell + 1$. Then, there exists a constant $C_\ell > 0$, depending only on $\ell$, and hence independent of $K$, such that*

$$|v - \mathcal{P}_\ell^K(v)|_{m,p;K} \;\leq\; C_\ell\, h_K^{s-m} |v|_{s,p;K} \qquad \forall\, v \in \mathrm{W}^{s,p}(K)\,, \tag{3.2}$$

$$|\mathbf{v} - \boldsymbol{\mathcal{P}}_\ell^K(\mathbf{v})|_{m,p;K} \;\leq\; C_\ell\, h_K^{s-m} |\mathbf{v}|_{s,p;K} \qquad \forall\, \mathbf{v} \in \mathbf{W}^{s,p}(K)\,, \tag{3.3}$$

*and*

$$|\boldsymbol{\tau} - \mathbb{P}_\ell^K(\boldsymbol{\tau})|_{m,p;K} \;\leq\; C_\ell\, h_K^{s-m} |\boldsymbol{\tau}|_{s,p;K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{s,p}(K)\,. \tag{3.4}$$

**Proof.** The proof of (3.2) follows from [38, Lemma 3.7] by noting that the arguments employed there for $p \geq 2$ are valid for $p \in (1,2)$ as well. Then, (3.3) and (3.4) are straightforward consequences of (3.2). □

We remark now that Lemma 3.1 implies the boundedness of $\mathcal{P}_\ell^K$ (cf. [38, Lemma 3.8]), as well as that of $\boldsymbol{\mathcal{P}}_\ell^K$ and $\mathbb{P}_\ell^K$, with respect to the above Sobolev semi-norms. In other words, given $p > 1$, and $\ell$, $s$, $m$ integers such that $\ell \geq 0$ and $0 \leq m \leq s \leq \ell + 1$, there exists a constant $M_\ell \geq 1$, depending only on $\ell$, and hence independent of $K$, such that for each $K \in \mathcal{T}_h$ there hold

$$|\mathcal{P}_\ell^K(v)|_{s,p;K} \;\leq\; M_\ell\, |v|_{s,p;K} \qquad \forall\; v \in \mathrm{W}^{s,p}(K)\,, \tag{3.5}$$

$$|\boldsymbol{\mathcal{P}}_\ell^K(\mathbf{v})|_{s,p;K} \;\leq\; M_\ell\, |\mathbf{v}|_{s,p;K} \qquad \forall\; \mathbf{v} \in \mathbf{W}^{s,p}(K)\,, \tag{3.6}$$

and

$$|\mathbb{P}_\ell^K(\boldsymbol{\tau})|_{s,p;K} \;\leq\; M_\ell\, |\boldsymbol{\tau}|_{s,p;K} \qquad \forall\; \boldsymbol{\tau} \in \mathbb{W}^{s,p}(K)\,. \tag{3.7}$$

We end this section by stressing that all the above properties of $\mathcal{P}_\ell^K$, $\boldsymbol{\mathcal{P}}_\ell^K$ and $\mathbb{P}_\ell^K$, extend to their respective global counterparts

$$\mathcal{P}_\ell^h : \mathrm{L}^1(\Omega) \to \mathrm{P}_\ell(\mathcal{T}_h)\,, \quad \boldsymbol{\mathcal{P}}_\ell^h : \mathbf{L}^1(\Omega) \to \mathbf{P}_\ell(\mathcal{T}_h)\,, \qquad \text{and} \qquad \mathbb{P}_\ell^h : \mathbb{L}^1(\Omega) \to \mathbb{P}_\ell(\mathcal{T}_h)\,,$$

where

$$\mathrm{P}_\ell(\mathcal{T}_h) \;:=\; \Big\{ v \in \mathrm{L}^1(\Omega) : \quad v|_K \in \mathrm{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \tag{3.8}$$

and analogue definitions hold for $\mathbf{P}_\ell(\mathcal{T}_h)$ and $\mathbb{P}_\ell(\mathcal{T}_h)$.

### 3.2 The discrete subspaces

In this section we introduce a suitable virtual element subspace approximating the continuous space $\mathbb{H}$ (cf. (2.6)), and define an explicit (non-virtual) finite element subspace approximating $\mathbf{Q}$ (cf. (2.7)). In fact, given an integer $k \geq 0$ and $K \in \mathcal{T}_h$, for the former we follow [39, Section 3.2] (see also [38, Section 3.3]) and consider the local virtual element subspace of order $k$ (cf. [5]):

$$\begin{aligned} \mathbb{H}_k^K \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; K) \cap \mathbb{H}(\mathbf{rot}; K) \;:\; \quad & \boldsymbol{\tau}\boldsymbol{n}|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \subset \partial K\,, \\ \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{P}_k(K)\,, \qquad \text{and} \qquad & \mathbf{rot}(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(K) \Big\}\,, \end{aligned} \tag{3.9}$$

where $\mathbf{P}_{-1}(K) := \{\mathbf{0}\}$, and $\mathbf{rot}(\boldsymbol{\tau}) := (\partial_{x_1}\tau_{12} - \partial_{x_2}\tau_{11}, \partial_{x_1}\tau_{22} - \partial_{x_2}\tau_{21})^{\mathtt{t}}$. It is well-known (see [4]) that the tensors $\boldsymbol{\tau} \in \mathbb{H}_k^K$ are uniquely determined by the local degrees of freedom given by

$$\begin{aligned} \boldsymbol{m}_{\mathbf{q},\boldsymbol{n}}^K(\boldsymbol{\tau}) &:= \int_e \boldsymbol{\tau}\boldsymbol{n}\cdot\mathbf{q} && \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_k(e)\,, \quad \forall \text{ edge } e \subset \partial K\,, \\ \boldsymbol{m}_{\mathbf{q},\mathbf{div}}^K(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \nabla\mathbf{q} && \forall\, \mathbf{q} \in \boldsymbol{\mathcal{B}}_k(K) \setminus \{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\}\,, \\ \boldsymbol{m}_{\boldsymbol{\rho},\mathbf{rot}}^K(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \boldsymbol{\rho} && \forall\, \boldsymbol{\rho} \in \boldsymbol{\mathcal{G}}_k(K)\,, \end{aligned} \tag{3.10}$$

where $\boldsymbol{\mathcal{G}}_k(K)$ is a basis of $(\nabla\mathbf{P}_{k+1}(K))^{\perp} \cap \mathbb{P}_k(K)$, the $\mathbb{L}^2(K)$-orthogonal of $\nabla\mathbf{P}_{k+1}(K)$ in $\mathbb{P}_k(K)$. Alternatively, it would suffice to choose $\boldsymbol{\mathcal{G}}_k(K)$ as a basis of any space $\widetilde{\mathbb{P}}_k(K)$, not necessarily orthogonal to $\nabla\mathbf{P}_{k+1}(K)$, such that $\mathbb{P}_k(K) = \nabla\mathbf{P}_{k+1}(K) \oplus \widetilde{\mathbb{P}}_k(K)$. In any case, we stress that for each $\boldsymbol{\tau} \in \mathbb{H}_k^K$, the projection $\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})$ is explicitly calculable in terms of the degrees of freedom given by (3.10) (see, e.g. [16, Section 3.3]).

We now denote by $n_k^K$ the amount of local degrees of freedom from (3.10), and gather them in the set $\left\{m_i^K\right\}_{i=1}^{n_k^K}$. Then, proceeding analogously to [38, Section 3.3], we introduce the interpolation operator $\Pi_k^K : \mathbb{W}^{1,1}(K) \to \mathbb{H}_k^K$, which is defined for each $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(K)$ as the unique $\Pi_k^K(\boldsymbol{\tau}) \in \mathbb{H}_k^K$ such that

$$m_i^K\big(\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\big) \,=\, 0 \qquad \forall\, i \in \big\{1, \ldots, n_k^K\big\}\,. \tag{3.11}$$

Regarding the approximation properties of $\Pi_k^K$, we first recall from [5, eq. (3.19)] that for each integer $s \in [1, k+1]$ there exists a constant $C > 0$, independent of $K$, such that

$$\|\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\|_{0,K} \,\leq\, C\, h_K^s\, |\boldsymbol{\tau}|_{s,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s(K)\,. \tag{3.12}$$

In turn, similarly to [16, eq. (3.14)]), and employing the identities given by (3.11), we easily find that

$$\mathbf{div}(\Pi_k^K(\boldsymbol{\tau})) \,=\, \boldsymbol{\mathcal{P}}_k^K(\mathbf{div}(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,1}(K)\,, \tag{3.13}$$

which, together with (3.3), imply that for each integer $s \in [0, k+1]$ there exists a constant $C > 0$, independent of $K$, such that

$$\|\mathbf{div}\big(\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\big)\|_{0,4/3;K} \,=\, \|\mathbf{div}(\boldsymbol{\tau}) - \boldsymbol{\mathcal{P}}_k^K\big(\mathbf{div}(\boldsymbol{\tau})\big)\|_{0,4/3;K} \,\leq\, C\, h_K^s\, |\mathbf{div}(\boldsymbol{\tau})|_{s,4/3;K} \tag{3.14}$$

for all $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(K)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(K)$.

Having established the above, we now introduce the virtual element subspace of $\mathbb{H}$ given by

$$\mathbb{H}_h \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H} : \quad \boldsymbol{\tau}|_K \in \mathbb{H}_k^K \quad \forall\, K \in \mathcal{T}_h\Big\}\,. \tag{3.15}$$

In turn, we consider $\mathbf{P}_k(\mathcal{T}_h)$ (cf. (3.8)) as the finite dimensional subspace of $\mathbf{Q}$, that is

$$\mathbf{Q}_h \,:=\, \Big\{\mathbf{v} \in \mathbf{Q} : \quad \mathbf{v}|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,. \tag{3.16}$$

On the other hand, given integers $s$, $m \geq 0$, and given $p > 1$, we introduce the broken semi-norms

$$|\boldsymbol{\tau}|_{s;\mathsf{b},\Omega} \,:=\, \Big\{\sum_{K\in\mathcal{T}_h} |\boldsymbol{\tau}|_{s,K}^2\Big\}^{1/2} \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \text{ such that } \boldsymbol{\tau}|_K \in \mathbb{H}^s(K) \quad \forall\, K \in \mathcal{T}_h\,, \tag{3.17}$$

and

$$|\mathbf{v}|_{m,p;\mathtt{b},\Omega} \,:=\, \left\{ \sum_{K\in\mathcal{T}_h} |\mathbf{v}|^p_{m,p;K} \right\}^{1/p} \quad \forall\, \mathbf{v}\in \mathbf{L}^p(\Omega) \ \text{ such that } \ \mathbf{v}|_K \in \mathbf{W}^{m,p}(K) \quad \forall\, K\in\mathcal{T}_h\,. \tag{3.18}$$

In this way, according to (3.12), (3.14), and (3.3), the approximation properties of $\mathbb{H}_h$ and $\mathbf{Q}_h$ reduce, respectively, to:

($\mathbf{AP}^{\boldsymbol{\sigma}}_h$) for each integer $s\in[1,k+1]$ there exists $C>0$, independent of $h$, such that

$$\mathrm{dist}(\boldsymbol{\tau},\mathbb{H}_h) \,:=\, \inf_{\boldsymbol{\tau}_h\in\mathbb{H}_h} \|\boldsymbol{\tau}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,\le\, C\,h^s \left\{ |\boldsymbol{\tau}|_{s;\mathtt{b},\Omega} \,+\, |\mathbf{div}(\boldsymbol{\tau})|_{s,4/3;\mathtt{b},\Omega} \right\},$$

for all $\boldsymbol{\tau}\in\mathbb{H}$ such that $\boldsymbol{\tau}|_K\in\mathbb{H}^s(K)$ and $\mathbf{div}(\boldsymbol{\tau})|_K\in\mathbf{W}^{s,4/3}(K)$ for all $K\in\mathcal{T}_h$, and

($\mathbf{AP}^{\mathbf{u}}_h$) for each integer $s\in[0,k+1]$ there exists $C>0$, independent of $h$, such that

$$\mathrm{dist}(\mathbf{v},\mathbf{Q}_h) \,:=\, \inf_{\mathbf{v}_h\in\mathbf{Q}_h} \|\mathbf{v}-\mathbf{v}_h\|_{0,4;\Omega} \,\le\, C\,h^s\,|\mathbf{v}|_{s,4;\mathtt{b},\Omega}\,,$$

for all $\mathbf{v}\in\mathbf{L}^4(\Omega)$ such that $\mathbf{v}|_K\in\mathbf{W}^{s,4}(K)$ for all $K\in\mathcal{T}_h$.

### 3.3 The virtual element scheme

We begin by observing, according to the definitions of the discrete spaces $\mathbb{H}_h$ (cf. (3.9), (3.15)) and $\mathbf{Q}_h$ (cf. (3.16)) that the bilinear form $b$ (cf. (2.10) ) is explicitly calculable for each $(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_h\times\mathbf{Q}_h$ as

$$b(\boldsymbol{\tau},\mathbf{v}) \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \sum_{K\in\mathcal{T}_h}\int_K \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,.$$

On the contrary, and since $\boldsymbol{\tau}\in\mathbb{H}_h$ is unknown on each $K\in\mathcal{T}_h$, the bilinear form $a$ (cf. (2.9)) and the trilinear form $c$ (cf. (2.11)) are not explicitly calculable due to both terms in the former and the third term only in the latter. According to it, we now define a calculable discrete version of $a$ depending on the local degrees of freedom $\left\{m_i^K\right\}_{i=1}^{n_k^K}$ (cf. (3.11)) and the projectors $\mathcal{P}_k^K$, $K\in\mathcal{T}_h$. Indeed, we first let $\mathcal{S}_h^K:\mathbb{H}_k^K\times\mathbb{H}_k^K\to\mathrm{R}$ be the bilinear form associated to the identity matrix in $\mathrm{R}^{n_k^K\times n_k^K}$ with respect to the canonical basis of $\mathbb{H}_k^K$ determined by the aforementioned degrees of freedom, that is

$$\mathcal{S}_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \sum_{i=1}^{n_k^K} m_i^K(\boldsymbol{\zeta})\,m_i^K(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\zeta},\boldsymbol{\tau}\in\mathbb{H}_k^K\,. \tag{3.19}$$

Then, we introduce for each $K\in\mathcal{T}_h$ the calculable local discrete version of $a$ as

$$a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \frac{1}{\mu}\left\{ \int_K \left(\mathcal{P}_k^K(\boldsymbol{\zeta})\right)^{\mathtt{d}} : \left(\mathcal{P}_k^K(\boldsymbol{\tau})\right)^{\mathtt{d}} + \mathcal{S}_h^K(\boldsymbol{\zeta}-\mathcal{P}_k^K(\boldsymbol{\zeta}),\boldsymbol{\tau}-\mathcal{P}_k^K(\boldsymbol{\tau})) \right\} \qquad \forall\,\boldsymbol{\zeta},\boldsymbol{\tau}\in\mathbb{H}_k^K\,, \tag{3.20}$$

and set the calculable discrete version of $a$ as the bilinear form $a_h:\mathbb{H}_h\times\mathbb{H}_h\to\mathrm{R}$ defined by

$$a_h(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \sum_{K\in\mathcal{T}_h} a_h^K(\boldsymbol{\zeta}_K,\boldsymbol{\tau}_K) \qquad \forall\,\boldsymbol{\zeta},\boldsymbol{\tau}\in\mathbb{H}_h\,, \tag{3.21}$$

where, given $\boldsymbol{\zeta} \in \mathbb{H}_h$ and $K \in \mathcal{T}_h$, $\boldsymbol{\zeta}_K \in \mathbb{H}_k^K$ denotes the restriction of $\boldsymbol{\zeta}$ to $K$. Similarly, we let $c_h : \mathbf{Q}_h \times \big(\mathbf{Q}_h \times \mathbb{H}_h\big) \to \mathrm{R}$ be the trilinear form defining the calculable discrete version of $c$, that is

$$c_h(\mathbf{z};\mathbf{v},\boldsymbol{\tau}) \;:=\; \frac{1}{\mu}\sum_{K\in\mathcal{T}_h}\int_K (\mathbf{z}\otimes\mathbf{v})^{\mathrm{d}} : \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau}) \qquad \forall\,\big(\mathbf{z},(\mathbf{v},\boldsymbol{\tau})\big) \in \mathbf{Q}_h \times \big(\mathbf{Q}_h \times \mathbb{H}_h\big)\,. \tag{3.22}$$

Note that the discrete form $c_h$ is also defined in $\mathbf{Q}\times\big(\mathbf{Q}\times\mathbb{H}\big)$, which will be employed below in Lemmas 4.5 and 4.6. Finally, since the functionals $F$ (cf. (2.12)) and $G$ (cf. (2.13)) are calculable as well on $\mathbb{H}_h$ and $\mathbf{Q}_h$, respectively, which follows again from the definitions of these discrete spaces (cf. (3.9), (3.15), (3.16)), we propose the following virtual element scheme for (2.8): Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{lcll} a_h(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) \,+\, b(\boldsymbol{\tau}_h,\mathbf{u}_h) \,+\, c_h(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h) & = & F(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h\,, \\ b(\boldsymbol{\sigma}_h,\mathbf{v}_h) & = & G(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,. \end{array} \tag{3.23}$$

We end this section by remarking that our virtual element scheme (3.23) presents two important advantages as compared with the previous scheme proposed in [38]. Indeed, on one hand, augmented terms increasing the complexity of the method are not needed anymore, and on the other hand, only one virtual element subspace is required, which significantly simplifies the approach from [38], in which a virtual element subspace for $\mathbf{H}^1$-conforming elements is additionally employed. As a consequence, we now obtain a much cleaner and easier computational implementation.

# 4 Solvability analysis

In this section we follow a similar fixed-point strategy to the one employed in [18] (see also [20], [21], [30], and [31]) to analyze the solvability of our discrete formulation (3.23). We begin by collecting some useful results concerning the bilinear forms $a_h$ and $c_h$.

## 4.1 Preliminaries on the discrete bilinear forms

We first recall from [38, Lemma 4.1] a key estimate on $\mathcal{S}_h^K$.

**Lemma 4.1.** *There exist constants $\widehat{c}_0, \widehat{c}_1 > 0$, depending only on $C_{\mathcal{T}}$, such that*

$$\widehat{c}_0\,\|\boldsymbol{\zeta}\|_{0,K}^2 \;\le\; \mathcal{S}_h^K(\boldsymbol{\zeta},\boldsymbol{\zeta}) \;\le\; \widehat{c}_1\,\|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{H}_k^K\,,\quad \forall\,K\in\mathcal{T}_h\,. \tag{4.1}$$

*Proof.* See [5, eqs. (3.36) and (6.2)] (see also [14, eq. (5.8)] and [15, Lemma 4.5]). □

The estimate (4.1) and the well-known boundedness properties of the $\mathbb{L}^2(K)$-orthogonal projector $\boldsymbol{\mathcal{P}}_k^K$, namely

$$\|\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K} \;\le\; \|\boldsymbol{\tau}\|_{0,K} \qquad \text{and} \qquad \|\boldsymbol{\tau} - \boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K} \;\le\; \|\boldsymbol{\tau}\|_{0,K} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{L}^2(K)\,, \tag{4.2}$$

are utilized in what follows. Note that we could also employ the bounds arising from (3.7) with $s = 0$ and $p = 2$, but the ones in the foregoing equation are certainly sharper.

We begin with the following lemma concerning $a_h^K$ (cf. (3.20)).

**Lemma 4.2.** *There exist constants $\alpha_1$, $\alpha_2 > 0$, independent of $h$, such that*

$$|a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau})| \;\le\; \alpha_2\,\|\boldsymbol{\zeta}\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} \qquad \forall\,\boldsymbol{\zeta},\boldsymbol{\tau}\in\mathbb{H}_k^K,\quad \forall\,K\in\mathcal{T}_h\,, \tag{4.3}$$

*and*

$$\alpha_1\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,K}^2 \;\le\; a_h^K(\boldsymbol{\zeta},\boldsymbol{\zeta}) \;\le\; \alpha_2\,\|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{H}_k^K,\quad \forall\,K\in\mathcal{T}_h\,. \tag{4.4}$$

*Proof.* While this proof is standard (see [15, Lemma 4.6]), we provide it below for sake of completeness. In fact, applying the Cauchy-Schwarz inequality to $\mathcal{S}_h^K$, and then employing the upper bound from (4.1), we first obtain

$$\mathcal{S}_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\leq\, \Big\{\mathcal{S}_h^K(\boldsymbol{\zeta},\boldsymbol{\zeta})\Big\}^{1/2}\Big\{\mathcal{S}_h^K(\boldsymbol{\tau},\boldsymbol{\tau})\Big\}^{1/2} \,\leq\, \widehat{c}_1\,\|\boldsymbol{\zeta}\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{H}_k^K\,. \tag{4.5}$$

Hence, according to the definition of $a_h^K$ (cf. (3.20)), we utilize the Cauchy-Schwarz inequality again, and the estimate (4.5), to deduce that

$$|a_h^K(\boldsymbol{\zeta},\boldsymbol{\tau})| \,\leq\, \frac{1}{\mu}\Big\{\|\boldsymbol{\zeta}\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} \,+\, \widehat{c}_1\|\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\|_{0,K}\Big\} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{H}_k^K\,,$$

which, taking into account (4.2), gives (4.3) with $\alpha_2 := \frac{1}{\mu}\big(1+\widehat{c}_1\big) > 0$. Next, concerning (4.4), it is clear that the corresponding upper bound follows straightforwardly from (4.3). In turn, adding and subtracting $\big(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathtt{d}}$, and applying the lower estimate from (4.1), we find that

$$\begin{aligned}
\|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,K}^2 \,&\leq\, 2\,\big\|\big(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathtt{d}}\big\|_{0,K}^2 \,+\, 2\,\big\|\big(\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathtt{d}}\big\|_{0,K}^2 \\
&\leq 2\,\big\|\big(\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})\big)^{\mathtt{d}}\big\|_{0,K}^2 \,+\, \frac{2}{\widehat{c}_0}\,\mathcal{S}_h^K(\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta}),\boldsymbol{\zeta}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\zeta})) \\
&\leq\, 2\mu\,\max\Big\{1,\frac{1}{\widehat{c}_0}\Big\}\,a_h^K(\boldsymbol{\zeta},\boldsymbol{\zeta}) \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{H}_k^K\,,
\end{aligned}$$

which yields the lower bound of (4.4) with $\alpha_1 := \frac{1}{2\mu}\,\min\big\{1,\widehat{c}_0\big\}$. □

As a consequence of (3.21) and (4.3), we conclude the boundedness of the bilinear form $a_h$, that is

$$|a_h(\boldsymbol{\zeta},\boldsymbol{\tau})| \,\leq\, \alpha_2\,\|\boldsymbol{\zeta}\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \alpha_2\,\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbb{H}_h\,. \tag{4.6}$$

We now aim to establish the ellipticity of $a_h$ on the discrete kernel $\mathbb{V}_h$ of the bilinear form $b$, that is

$$\mathbb{V}_h \,:=\, \Big\{\boldsymbol{\tau}\in\mathbb{H}_h:\quad b(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) = 0 \quad \forall\,\mathbf{v}\in\mathbf{Q}_h\Big\}\,.$$

To this end, we first observe from the definitions of $\mathbb{H}_h$ (cf. (3.9), (3.15)) and $\mathbf{Q}_h$ (cf. (3.16)) that there holds $\mathbf{div}(\mathbb{H}_h) \subseteq \mathbf{Q}_h$, which implies that

$$\mathbb{V}_h \,=\, \Big\{\boldsymbol{\tau}\in\mathbb{H}_h:\quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega\Big\}\,. \tag{4.7}$$

Then, the announced result on $a_h$ is established as follows.

**Lemma 4.3.** *There exists a constant $\alpha_{\mathtt{d}} > 0$, independent of $h$, such that*

$$a_h(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\geq\, \alpha_{\mathtt{d}}\,\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}^2 \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{V}_h\,. \tag{4.8}$$

*Proof.* Given $\boldsymbol{\zeta}\in\mathbb{V}_h$, and bearing in mind the definitions of $a_h^K$ (cf. (3.20)) and $a_h$ (cf. (3.21)), a direct application of the lower bound from (4.4) yields

$$a_h(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\geq\, \alpha_1\,\|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,\Omega}^2\,.$$

On the other hand, the estimate given by [18, Lemma 3.2] (see also [30, eq. (3.43)]), which is a slight generalization of [35, Lemma 2.3], establishes the existence of a constant $c_1 > 0$, depending only on $\Omega$, such that

$$\|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}^2 \geq c_1 \|\boldsymbol{\tau}\|_{0,\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,.$$

Hence, the foregoing two equations and the fact that $\boldsymbol{\zeta}$ is divergence free, imply the required estimate (4.8) with $\alpha_{\mathtt{d}} = \alpha_1\, c_1$. □

In order to state the next result, we now recall from Section 3.1 that $\mathcal{P}_k^h : \mathbb{L}^1(\Omega) \to \mathbb{P}_k(\mathcal{T}_h)$ is the global counterpart of $\mathcal{P}_k^K : \mathbb{L}^1(K) \to \mathbb{P}_k(K)$, which means that

$$\mathcal{P}_k^h(\boldsymbol{\tau})|_K := \mathcal{P}_k^K(\boldsymbol{\tau}|_K) \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^1(\Omega)\,.$$

Then, we have the following lemma establishing a stability estimate for the difference between the bilinear forms $a$ and $a_h$.

**Lemma 4.4.** *There exist a constant $C_a > 0$, independent of $h$, such that*

$$|a(\boldsymbol{\zeta},\boldsymbol{\tau}) - a_h(\boldsymbol{\zeta},\boldsymbol{\tau})| \leq C_a \|\boldsymbol{\zeta} - \mathcal{P}_k^h(\boldsymbol{\zeta})\|_{0,\Omega} \|\boldsymbol{\tau}\|_{0,\Omega} \qquad \forall\, \boldsymbol{\zeta},\boldsymbol{\tau} \in \mathbb{H}_h\,. \tag{4.9}$$

*Proof.* Given $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in \mathbb{H}_h$, we first observe, thanks to the orthogonality property satisfied by $\mathcal{P}_k^h$, which follows from those of the local projectors $\mathcal{P}_k^K$, that

$$\int_\Omega \big(\mathcal{P}_k^h(\boldsymbol{\zeta})\big)^{\mathtt{d}} : \big(\mathcal{P}_k^h(\boldsymbol{\tau})\big)^{\mathtt{d}} = \int_\Omega \big(\mathcal{P}_k^h(\boldsymbol{\zeta})\big)^{\mathtt{d}} : \mathcal{P}_k^h(\boldsymbol{\tau}) = \int_\Omega \big(\mathcal{P}_k^h(\boldsymbol{\zeta})\big)^{\mathtt{d}} : \boldsymbol{\tau}\,,$$

and then, according to the definitions of $a$ (cf. (2.9)) and $a_h$ (cf. (3.20), (3.21)), we find that

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) - a_h(\boldsymbol{\zeta},\boldsymbol{\tau}) = \frac{1}{\mu}\int_\Omega \Big(\boldsymbol{\zeta} - \mathcal{P}_k^h(\boldsymbol{\zeta})\Big)^{\mathtt{d}} : \boldsymbol{\tau} - \frac{1}{\mu}\sum_{K\in\mathcal{T}_h} \mathcal{S}_h^K\big(\boldsymbol{\zeta} - \mathcal{P}_k^K(\boldsymbol{\zeta}), \boldsymbol{\tau} - \mathcal{P}_k^K(\boldsymbol{\tau})\big)\,.$$

In this way, employing the triangle and Cauchy-Schwarz inequalities, and the estimate (4.5), we obtain

$$|a(\boldsymbol{\zeta},\boldsymbol{\tau}) - a_h(\boldsymbol{\zeta},\boldsymbol{\tau})| \leq \frac{1}{\mu}\Big\{\|\boldsymbol{\zeta} - \mathcal{P}_k^h(\boldsymbol{\zeta})\|_{0,\Omega}\|\boldsymbol{\tau}\|_{0,\Omega} + \widehat{c}_1\|\boldsymbol{\zeta} - \mathcal{P}_k^h(\boldsymbol{\zeta})\|_{0,\Omega}\|\boldsymbol{\tau} - \mathcal{P}_k^h(\boldsymbol{\tau})\|_{0,\Omega}\Big\}\,,$$

which, thanks to (4.2), gives (4.9) with $C_a := \frac{1}{\mu}(1+\widehat{c}_1)$. □

We end this section with a couple of simple estimates concerning the trilinear form $c_h$ (cf. (3.22)). In particular, its boundedness is established as follows.

**Lemma 4.5.** *There holds*

$$|c_h(\mathbf{z};\mathbf{v},\boldsymbol{\tau})| \leq \frac{1}{\mu}\|\mathbf{z}\|_{0,4;\Omega}\|\mathbf{v}\|_{0,4;\Omega}\|\boldsymbol{\tau}\|_{0,\Omega} \qquad \forall\, \big(\mathbf{z},(\mathbf{v},\boldsymbol{\tau})\big) \in \mathbf{Q}\times\big(\mathbf{Q}\times\mathbb{H}\big)\,. \tag{4.10}$$

*Proof.* It follows from the definition of $c_h$ (cf. (3.22)), the Cauchy-Schwarz and Hölder inequalities, and (4.2). □

In turn, a stability estimate for the difference between $c$ and $c_h$ is provided next.

**Lemma 4.6.** *There holds*

$$|c(\mathbf{z};\mathbf{v},\boldsymbol{\tau}) - c_h(\mathbf{z};\mathbf{v},\boldsymbol{\tau})| \leq \frac{1}{\mu}\|(\mathbf{z}\otimes\mathbf{v}) - \mathcal{P}_k^h(\mathbf{z}\otimes\mathbf{v})\|_{0,\Omega}\|\boldsymbol{\tau}\|_{0,\Omega} \quad \forall\, \big(\mathbf{z},(\mathbf{v},\boldsymbol{\tau})\big) \in \mathbf{Q}\times\big(\mathbf{Q}\times\mathbb{H}\big)\,. \tag{4.11}$$

*Proof.* Let $\big(\mathbf{z},(\mathbf{v},\boldsymbol{\tau})\big)\in\mathbf{Q}\times\big(\mathbf{Q}\times\mathbb{H}\big)$. Then, having in mind the definitions of $c$ (cf. (2.11)) and $c_h$ (cf. (3.22)), and employing the orthogonality property satisfied by $\boldsymbol{\mathcal{P}}_k^K$, we deduce that

$$\begin{aligned} c(\mathbf{z};\mathbf{v},\boldsymbol{\tau})\,-\,c_h(\mathbf{z};\mathbf{v},\boldsymbol{\tau}) \,&=\, \frac{1}{\mu}\sum_{K\in\mathcal{T}_h}\int_K(\mathbf{z}\otimes\mathbf{v})^{\mathtt{d}}:\big\{\boldsymbol{\tau}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big\}\\ &=\,\frac{1}{\mu}\sum_{K\in\mathcal{T}_h}\int_K\Big\{(\mathbf{z}\otimes\mathbf{v})^{\mathtt{d}}-\big(\boldsymbol{\mathcal{P}}_k^K(\mathbf{z}\otimes\mathbf{v})\big)^{\mathtt{d}}\Big\}:\big\{\boldsymbol{\tau}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big\}\\ &=\,\frac{1}{\mu}\sum_{K\in\mathcal{T}_h}\int_K\Big\{(\mathbf{z}\otimes\mathbf{v})-\boldsymbol{\mathcal{P}}_k^K(\mathbf{z}\otimes\mathbf{v})\Big\}^{\mathtt{d}}:\big\{\boldsymbol{\tau}-\boldsymbol{\mathcal{P}}_k^K(\boldsymbol{\tau})\big\}\,, \end{aligned}\tag{4.12}$$

from which, applying Cauchy-Schwarz's inequality and (4.2), we arrive at (4.11) and end the proof. □

## 4.2 The discrete inf-sup condition for $b$

In order to establish the discrete inf-sup condition for the bilinear form $b$ we need two preliminary results, the second being consequence of the first as well as the one to be finally employed for the aforementioned purpose. Indeed, we begin with a stability estimate for the local interpolation operator $\Pi_k^K$ when applied to the space

$$\widetilde{\mathbb{W}}^{1,1}(K)\,:=\,\Big\{\boldsymbol{\tau}\in\mathbb{W}^{1,1}(K):\quad\int_K\boldsymbol{\tau}=\mathbf{0}\Big\}\,,\tag{4.13}$$

for which we follow basically [3] and make use of some techniques and results from [10].

**Lemma 4.7.** *There exists a constant $\widetilde{C}>0$ such that*

$$\|\Pi_k^K(\boldsymbol{\tau})\|_{0,K}\,\le\,\widetilde{C}\,|\boldsymbol{\tau}|_{1,1;K}\qquad\forall\,\boldsymbol{\tau}\in\widetilde{\mathbb{W}}^{1,1}(K),\quad\forall\,K\in\mathcal{T}_h\,.\tag{4.14}$$

*Proof.* Given $\boldsymbol{\tau}\in\widetilde{\mathbb{W}}^{1,1}(K)$, we denote $\boldsymbol{\tau}_h:=\Pi_k^K(\boldsymbol{\tau})\in\mathbb{H}_k^K$, and let $\boldsymbol{\rho}:=(\rho_1,\rho_2)\in\mathbf{H}^1(K)$ be the unique solution of the boundary value problem

$$\Delta\boldsymbol{\rho}\,=\,\mathbf{div}(\boldsymbol{\tau}_h)\quad\text{in}\quad K,\quad\nabla\boldsymbol{\rho}\,\boldsymbol{n}=\boldsymbol{\tau}_h\,\boldsymbol{n}\quad\text{on}\quad\partial K,\quad\int_K\boldsymbol{\rho}=\mathbf{0}\,.\tag{4.15}$$

It follows that $\mathbf{div}(\boldsymbol{\tau}_h-\nabla\boldsymbol{\rho})=\mathbf{0}$ in $K$, and hence a straightforward application of [41, Theorem 3.1, Section 3.1, Chapter I] implies the existence of $\boldsymbol{\psi}:=(\psi_1,\psi_2)\in\mathbf{H}^1(K)$ such that there holds the Helmholtz decomposition

$$\boldsymbol{\tau}_h\,=\,\mathbf{curl}(\boldsymbol{\psi})\,+\,\nabla\boldsymbol{\rho}\quad\text{in}\quad K\,,\tag{4.16}$$

where

$$\mathbf{curl}(\boldsymbol{\psi}):=\begin{pmatrix}\frac{\partial\psi_1}{\partial x_2} & -\frac{\partial\psi_1}{\partial x_1}\\ \frac{\partial\psi_2}{\partial x_2} & -\frac{\partial\psi_2}{\partial x_1}\end{pmatrix}.$$

Applying $\mathbf{rot}$ to (4.16), and making use of the Neumann boundary condition satisfied by $\boldsymbol{\rho}$, we deduce, respectively, that $\mathbf{rot}(\boldsymbol{\tau}_h)=\mathbf{rot}\big(\mathbf{curl}(\boldsymbol{\psi})\big)=\Delta\boldsymbol{\psi}$ in $K$ and $\mathbf{curl}(\boldsymbol{\psi})\,\boldsymbol{n}=\mathbf{0}$ on $\partial K$. Thus, since the latter indicates that $\boldsymbol{\psi}$ is a constant vector on $\partial K$, we can assume without loss of generality that $\boldsymbol{\psi}$ vanishes on $\partial K$, whence $\boldsymbol{\psi}$ becomes the unique solution of the boundary value problem

$$\Delta\boldsymbol{\psi}\,=\,\mathbf{rot}(\boldsymbol{\tau}_h)\quad\text{in}\quad K,\quad\boldsymbol{\psi}=\mathbf{0}\quad\text{on}\quad\partial K\,.\tag{4.17}$$

Next, integrating by parts, and denoting by $\langle\cdot,\cdot\rangle_{\partial K}$ the duality pairing between $\mathbf{H}^{-1/2}(\partial K)$ and $\mathbf{H}^{1/2}(\partial K)$, we find that

$$\int_K \mathbf{curl}(\boldsymbol{\psi}) : \nabla\boldsymbol{\rho} = -\int_K \boldsymbol{\rho}\cdot\mathbf{div}\big(\mathbf{curl}(\boldsymbol{\psi})\big) + \langle \mathbf{curl}(\boldsymbol{\psi})\,\boldsymbol{n},\boldsymbol{\rho}\rangle_{\partial K} = 0\,,$$

which shows that $\mathbf{curl}(\boldsymbol{\psi})$ and $\nabla\boldsymbol{\rho}$ are $\mathbb{L}^2(K)$-orthogonal, and thus

$$\|\boldsymbol{\tau}_h\|^2_{0,K} = \|\mathbf{curl}(\boldsymbol{\psi})\|^2_{0,K} + \|\nabla\boldsymbol{\rho}\|^2_{0,K}\,. \tag{4.18}$$

Now, integrating by parts again, denoting by $\mathbf{s}$ the unit tangential vector to $\partial K$, employing (4.17), and applying the Cauchy-Schwarz and Poincaré inequalities, we get

$$\begin{aligned}\|\mathbf{curl}(\boldsymbol{\psi})\|^2_{0,K} &= \int_K \mathbf{curl}(\boldsymbol{\psi}) : \mathbf{curl}(\boldsymbol{\psi}) = -\int_K \boldsymbol{\psi}\cdot\mathbf{rot}\big(\mathbf{curl}(\boldsymbol{\psi})\big) \;+\; \langle \mathbf{curl}(\boldsymbol{\psi})\cdot\mathbf{s},\boldsymbol{\psi}\rangle_{\partial K}\\ &= -\int_K \boldsymbol{\psi}\cdot\Delta\boldsymbol{\psi} \;=\; -\int_K \boldsymbol{\psi}\cdot\mathbf{rot}(\boldsymbol{\tau}_h) \;\le\; \|\boldsymbol{\psi}\|_{0,K}\,\|\mathbf{rot}(\boldsymbol{\tau}_h)\|_{0,K}\\ &\le C\,h_K\,|\boldsymbol{\psi}|_{1,K}\,\|\mathbf{rot}(\boldsymbol{\tau}_h)\|_{0,K} \;=\; C\,h_K\,\|\mathbf{curl}(\boldsymbol{\psi})\|_{0,K}\,\|\mathbf{rot}(\boldsymbol{\tau}_h)\|_{0,K}\,,\end{aligned} \tag{4.19}$$

from which it follows

$$\|\mathbf{curl}(\boldsymbol{\psi})\|^2_{0,K} \;\le\; C\,h_K^2\,\|\mathbf{rot}(\boldsymbol{\tau}_h)\|^2_{0,K}\,. \tag{4.20}$$

Similarly, but using now (4.15) instead of (4.17), we obtain

$$\begin{aligned}\|\nabla\boldsymbol{\rho}\|^2_{0,K} \;&=\; -\int_K \boldsymbol{\rho}\cdot\Delta\boldsymbol{\rho} + \langle\nabla\boldsymbol{\rho}\,\boldsymbol{n},\boldsymbol{\rho}\rangle_{\partial K} \;=\; -\int_K \boldsymbol{\rho}\cdot\mathbf{div}(\boldsymbol{\tau}_h) + \int_{\partial K}\boldsymbol{\tau}_h\,\boldsymbol{n}\,\boldsymbol{\rho}\\ &\le\; \|\boldsymbol{\rho}\|_{0,K}\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{0,K} + \|\boldsymbol{\tau}_h\,\boldsymbol{n}\|_{0,\partial K}\,\|\boldsymbol{\rho}\|_{0,\partial K}\\ &\le\; C\,|\boldsymbol{\rho}|_{1,K}\Big\{h_K\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{0,K} + h_K^{1/2}\,\|\boldsymbol{\tau}_h\,\boldsymbol{n}\|_{0,\partial K}\Big\}\,,\end{aligned} \tag{4.21}$$

where, besides the Poincaré inequality, the last estimate makes use of the fact (cf. [10, Lemma 2.1, eq. (2)]) that $\|\boldsymbol{\rho}\|_{0,\partial K} \le C\,h_K^{1/2}\,|\boldsymbol{\rho}|_{1,K}$, which holds precisely because $\boldsymbol{\rho}\in\mathbf{H}^1(K)$ has zero average on $K$. In this way, we easily conclude from (4.21) that

$$\|\nabla\boldsymbol{\rho}\|^2_{0,K} \;\le\; C\Big\{h_K^2\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{0,K} + h_K\,\|\boldsymbol{\tau}_h\,\boldsymbol{n}\|^2_{0,\partial K}\Big\}\,, \tag{4.22}$$

which, together with (4.18) and (4.20), yield

$$\|\boldsymbol{\tau}_h\|^2_{0,K} \;\le\; C\Big\{h_K^2\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{0,K} + h_K^2\,\|\mathbf{rot}(\boldsymbol{\tau}_h)\|^2_{0,K} + h_K\,\|\boldsymbol{\tau}_h\,\boldsymbol{n}\|^2_{0,\partial K}\Big\}\,. \tag{4.23}$$

We now proceed to estimate each one of the terms on the right hand side of (4.23) by using that the degrees of freedom defined by (3.10) obviously coincide for $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_h = \Pi_k^K(\boldsymbol{\tau})$. Indeed, as a consequence of this fact, we first observe that

$$h_K^2\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{0,K} = h_K^2\int_K \mathbf{div}(\boldsymbol{\tau}_h)\cdot\mathbf{div}(\boldsymbol{\tau}) \;\le\; h_K^2\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{0,\infty;K}\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,1;K}\,,$$

so that, applying a polynomial inverse inequality to $\mathbf{div}(\boldsymbol{\tau}_h)$, we get

$$h_K^2\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{0,K} \;\le\; C\,h_K\|\mathbf{div}(\boldsymbol{\tau}_h)\|_{0,K}\,|\boldsymbol{\tau}|_{1,1;K}\,,$$

which leads to

$$h_K^2\,\|\mathbf{div}(\boldsymbol{\tau}_h)\|^2_{0,K} \;\le\; C\,|\boldsymbol{\tau}|^2_{1,1;K}\,. \tag{4.24}$$

An analogous reasoning allows to prove that

$$h_K^2 \, \|\mathbf{rot}(\boldsymbol{\tau}_h)\|_{0,K}^2 \,\le\, C\, |\boldsymbol{\tau}|_{1,1;K}^2 \,. \tag{4.25}$$

In turn, for the boundary term we have

$$h_K \, \|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\partial K}^2 = h_K \int_{\partial K} \boldsymbol{\tau}_h \, \boldsymbol{n} \cdot \boldsymbol{\tau} \, \boldsymbol{n} \,\le\, h_K \, \|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\infty;\partial K} \, \|\boldsymbol{\tau}\|_{0,1;\partial K} \,,$$

from which, employing that $\|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\infty;\partial K} \,\le\, C\, h_K^{-1/2} \, \|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\partial K}$, invoking the scaled trace inequality $\|\boldsymbol{\tau}\|_{0,1;\partial K} \,\le\, C \left\{ |\boldsymbol{\tau}|_{1,1;K} + h_K^{-1} \, |\boldsymbol{\tau}|_{0,1;K} \right\}$, and performing some algebraic manipulations, we arrive at

$$h_K \, \|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\partial K}^2 \,\le\, C \left\{ |\boldsymbol{\tau}|_{1,1;K}^2 + h_K^{-2} \, |\boldsymbol{\tau}|_{0,1;K}^2 \right\}.$$

Hence, using that for $\boldsymbol{\tau} \in \widetilde{\mathbb{W}}^{1,1}(K)$ the Poincaré inequality establishes that $|\boldsymbol{\tau}|_{0,1;K} \,\le\, C\, h_K \, |\boldsymbol{\tau}|_{1,1;K}$, we conclude from the foregoing equation that

$$h_K \, \|\boldsymbol{\tau}_h \, \boldsymbol{n}\|_{0,\partial K}^2 \,\le\, C\, |\boldsymbol{\tau}|_{1,1;K}^2 \,. \tag{4.26}$$

Finally, replacing (4.24), (4.25), and (4.26) back into (4.23), we get (4.14) and end the proof. □

We now let $\Pi_k^h : \mathbb{W}^{1,1}(\Omega) \to \mathbb{H}_h$ be the global counterpart of $\Pi_k^K : \mathbb{W}^{1,1}(K) \to \mathbb{H}_k^K$, that is

$$\Pi_k^h(\boldsymbol{\tau})|_K \,=\, \Pi_k^K(\boldsymbol{\tau}|_K) \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,1}(\Omega) \,.$$

Then, as a consequence of Lemma 4.7, we can prove the following stability estimate for $\Pi_k^h$.

**Lemma 4.8.** *For each $p \in (1,2)$ there exists a constant $C_{\mathtt{sta}} > 0$ such that*

$$\|\Pi_k^h(\boldsymbol{\tau})\|_{0,\Omega} \,\le\, C_{\mathtt{sta}} \, \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,p}(\Omega) \,. \tag{4.27}$$

*Proof.* We begin by recalling that the Sobolev embedding Theorem (cf. [1, Theorem 4.12], [33, Corollary B.43], [46, Theorem 1.3.4]) guarantees the continuous injection of $\mathbb{W}^{1,p}(\Omega)$ into $\mathbb{L}^2(\Omega)$, which means that there exists $C_p > 0$ such that

$$\|\boldsymbol{\tau}\|_{0,\Omega} \,\le\, C_p \, \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,p}(\Omega) \,. \tag{4.28}$$

Next, given $\boldsymbol{\tau} \in \mathbb{W}^{1,p}(\Omega)$, we consider the local decompositions

$$\boldsymbol{\tau}|_K \,=\, \bar{\boldsymbol{\tau}}_K \,+\, \widetilde{\boldsymbol{\tau}}_K \qquad \forall\, K \in \mathcal{T}_h \,, \tag{4.29}$$

where

$$\bar{\boldsymbol{\tau}}_K \,:=\, \frac{1}{|K|} \int_K \boldsymbol{\tau} \,\in\, \mathbb{P}_0(K) \qquad \text{and} \qquad \widetilde{\boldsymbol{\tau}}_K \in \widetilde{\mathbb{W}}^{1,p}(K) := \left\{ \boldsymbol{\zeta} \in \mathbb{W}^{1,p}(K) : \quad \int_K \boldsymbol{\zeta} = \mathbf{0} \right\}.$$

Note that $\|\boldsymbol{\tau}\|_{0,K}^2 = \|\bar{\boldsymbol{\tau}}_K\|_{0,K}^2 + \|\widetilde{\boldsymbol{\tau}}_K\|_{0,K}^2$. Then, using that $\Pi_k^K$ preserves tensors in $\mathbb{P}_0(K)$, applying the estimate (4.14), and observing that $|\widetilde{\boldsymbol{\tau}}_K|_{1,1;K} = |\boldsymbol{\tau}|_{1,1;K} \le |K|^{1-1/p} |\boldsymbol{\tau}|_{1,p;K} \le c |\boldsymbol{\tau}|_{1,p;K}$, we find

$$\begin{aligned} \|\Pi_k^K(\boldsymbol{\tau})\|_{0,K}^2 \,&=\, \|\bar{\boldsymbol{\tau}}_K + \Pi_k^K(\widetilde{\boldsymbol{\tau}}_K)\|_{0,K}^2 \,\le\, 2\, \|\bar{\boldsymbol{\tau}}_K\|_{0,K}^2 + 2\, \|\Pi_k^K(\widetilde{\boldsymbol{\tau}}_K)\|_{0,K}^2 \\ &\le\, 2\, \|\boldsymbol{\tau}\|_{0,K}^2 + 2\, \widetilde{C}^2 \, |\widetilde{\boldsymbol{\tau}}_K|_{1,1;K}^2 \,\le\, 2\, \|\boldsymbol{\tau}\|_{0,K}^2 + 2\, \widetilde{C}^2 \, c^2 \, |\boldsymbol{\tau}|_{1,p;K}^2 \,, \end{aligned}$$

from which, summing up over all $K \in \mathcal{T}_h$, and employing (4.28), we obtain

$$\|\Pi_k^h\|_{0,\Omega}^2 \,\le\, 2\, C_p^2 \, \|\boldsymbol{\tau}\|_{1,p;\Omega}^2 + 2\, \widetilde{C}^2\, c^2 \sum_{K\in\mathcal{T}_h} |\boldsymbol{\tau}|_{1,p;K}^2 \,. \tag{4.30}$$

Finally, invoking the sub-additive property with exponent $\frac{p}{2} \in (0,1)$, we get

$$\sum_{K\in\mathcal{T}_h} |\boldsymbol{\tau}|_{1,p;K}^2 = \left( \left\{ \sum_{K\in\mathcal{T}_h} |\boldsymbol{\tau}|_{1,p;K}^2 \right\}^{p/2} \right)^{2/p} \le\, |\boldsymbol{\tau}|_{1,p;\Omega}^2 \,,$$

which replaced back into (4.30) yields (4.27) and ends the proof. □

We are now in a position to establish the discrete inf-sup condition for the bilinear from $b$. More precisely, we have the following lemma.

**Lemma 4.9.** *There exists $\beta_{\mathtt{d}} > 0$, independent of $h$, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau}_h,\,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \beta_{\mathtt{d}}\, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \,. \tag{4.31}$$

*Proof.* It proceeds analogously to the proof of [15, Lemma 5.3] by employing some tools from [30, Lemma 5.5]. In fact, since it was already shown in [18, Lemma 3.4] that $b$ satisfies the continuous inf-sup condition, that is that there exists a constant $\beta > 0$ such that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau},\,\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \beta\, \|\mathbf{v}\|_{0,4;\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{Q} \,, \tag{4.32}$$

it suffices to apply Fortin's Lemma (cf. [35, Lemma 2.6]), which is valid in Banach spaces as well, to conclude that $b$ verifies the discrete version of (4.32). This means that we need to construct a sequence of uniformly bounded operators $\left\{\Theta_k^h\right\}_{h>0} \subseteq \mathcal{L}(\mathbb{H},\mathbb{H}_h)$, such that $b\big(\boldsymbol{\tau} - \Theta_k^h(\boldsymbol{\tau}), \mathbf{v}_h\big) \,=\, 0$ for all $\boldsymbol{\tau} \in \mathbb{H}$, and for all $\mathbf{v}_h \in \mathbf{Q}_h$. To this end, we now let $\mathcal{O}$ be a convex bounded domain containing $\bar{\Omega}$, so that, given $\boldsymbol{\tau} \in \mathbb{H}$, we set

$$\mathbf{g} \,:=\, \begin{cases} \mathbf{div}(\boldsymbol{\tau}) & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{in} \quad \mathcal{O}\setminus\bar{\Omega}\,, \end{cases}$$

which certainly belongs to $\mathbf{L}^{4/3}(\mathcal{O})$ and satisfies $\|\mathbf{g}\|_{0,4/3;\mathcal{O}} \,=\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}$. It follows from [34, Corollary 1] that there exists a unique $\mathbf{z} \in \mathbf{W}_0^{1,4/3}(\mathcal{O}) \cap \mathbf{W}^{2,4/3}(\mathcal{O})$ solution of

$$\Delta\mathbf{z} \,=\, \mathbf{g} \quad \text{in} \quad \mathcal{O}, \qquad \mathbf{z} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\,, \tag{4.33}$$

and there exists a constant $C_{\mathtt{reg}} > 0$, depending only on $\mathcal{O}$, such that

$$\|\mathbf{z}\|_{2,4/3;\mathcal{O}} \,\le\, C_{\mathtt{reg}}\, \|\mathbf{g}\|_{0,4/3;\mathcal{O}} \,=\, C_{\mathtt{reg}}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \,. \tag{4.34}$$

Next, we let $\boldsymbol{\zeta} := \nabla\mathbf{z}|_{\Omega} \in \mathbb{W}^{1,4/3}(\Omega)$ and observe from (4.33) and (4.34) that

$$\mathbf{div}(\boldsymbol{\zeta}) \,=\, \mathbf{div}(\boldsymbol{\tau}) \quad \text{in} \quad \Omega \qquad \text{and} \qquad \|\boldsymbol{\zeta}\|_{1,4/3;\Omega} \,\le\, C_{\mathtt{reg}}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \,. \tag{4.35}$$

Then, recalling that $\mathbb{I}$ is the identity matrix in $\mathbb{R}$, we define our Fortin's operator by

$$\Theta_k^h(\boldsymbol{\tau}) \,:=\, \Pi_k^h(\boldsymbol{\zeta}) \,-\, \left\{ \frac{1}{2|\Omega|} \int_\Omega \Pi_k^h(\boldsymbol{\zeta}) \right\} \mathbb{I}\,, \tag{4.36}$$

which clearly belongs to $\mathbb{H}_h$, and notice, thanks to (4.27) (with $p = 4/3$) and the inequality from (4.35), that

$$\|\Theta_k^h(\boldsymbol{\tau})\|_{0,\Omega} \,\le\, \|\Pi_k^h(\boldsymbol{\zeta})\|_{0,\Omega} \,\le\, C_{\mathtt{sta}}\,\|\boldsymbol{\zeta}\|_{1,4/3;\Omega} \,\le\, C_{\mathtt{sta}}\,C_{\mathtt{reg}}\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}\,. \tag{4.37}$$

In turn, recalling from Section 3.1 that $\boldsymbol{\mathcal{P}}_k^h$ is the global counterpart of $\boldsymbol{\mathcal{P}}_k^K$, applying the respective global version of (3.13), and using the identity from (4.35), we obtain

$$\mathbf{div}\big(\Theta_k^h(\boldsymbol{\tau})\big) \,=\, \mathbf{div}\big(\Pi_k^h(\boldsymbol{\zeta})\big) \,=\, \boldsymbol{\mathcal{P}}_k^h\big(\mathbf{div}(\boldsymbol{\zeta})\big) \,=\, \boldsymbol{\mathcal{P}}_k^h\big(\mathbf{div}(\boldsymbol{\tau})\big)\,. \tag{4.38}$$

In this way, making use of the boundedness property given by (3.6) (with $s = 0$ and $p = 4/3$), we get

$$\|\mathbf{div}\big(\Theta_k^h(\boldsymbol{\tau})\big)\|_{0,4/3;\Omega} \,=\, \|\boldsymbol{\mathcal{P}}_k^h\big(\mathbf{div}(\boldsymbol{\tau})\big)\|_{0,4/3;\Omega} \,\le\, M_k\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}\,,$$

which, together with (4.37), confirms the uniform boundedness of $\Theta_k^h$. Finally, according to (4.38) and the fact that $\boldsymbol{\mathcal{P}}_k^h$ projects precisely into $\mathbf{Q}_h$, we find that for each $\mathbf{v}_h \in \mathbf{Q}_h$ there holds

$$b\big(\Theta_k^h(\boldsymbol{\tau}),\mathbf{v}_h\big) = \int_\Omega \mathbf{v}_h\cdot\boldsymbol{\mathcal{P}}_k^h\big(\mathbf{div}(\boldsymbol{\tau})\big) = \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}) = b(\boldsymbol{\tau},\mathbf{v}_h)\,,$$

which completes the proof. □

### 4.3 The fixed-point strategy

In this section we study the solvability of the virtual element scheme (3.23) by means of an equivalent fixed-point operator equation. Indeed, we let $\mathbf{T}_h : \mathbf{Q}_h \to \mathbf{Q}_h$ be the operator defined for each $\mathbf{z}_h \in \mathbf{Q}_h$ as $\mathbf{T}_h(\mathbf{z}_h) \,:=\, \widetilde{\mathbf{u}}_h$, where $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$ is the unique solution (to be confirmed below) of (3.23) with $\mathbf{z}_h$ instead of $\mathbf{u}_h$ in the first component of the trilinear form $c_h$, that is

$$\begin{array}{rcll} a_h(\widetilde{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h) \,+\, b(\boldsymbol{\tau}_h,\widetilde{\mathbf{u}}_h) \,+\, c_h(\mathbf{z}_h;\widetilde{\mathbf{u}}_h,\boldsymbol{\tau}_h) & = & F(\boldsymbol{\tau}_h) & \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h\,,\\ b(\widetilde{\boldsymbol{\sigma}}_h,\mathbf{v}_h) & = & G(\mathbf{v}_h) & \forall\,\mathbf{v}_h\in\mathbf{Q}_h\,. \end{array} \tag{4.39}$$

Then, it is easy to see that solving (3.23) reduces to seeking $\mathbf{u}_h \in \mathbf{Q}_h$ such that

$$\mathbf{T}_h(\mathbf{u}_h) \,=\, \mathbf{u}_h\,. \tag{4.40}$$

In order to analyze the solvability of this fixed-point equation, we first address in what follows the well-posedness of (4.39), equivalently the well-definedness of the operator $\mathbf{T}_h$, by employing the discrete versions of the Babuška-Brezzi theorem (cf. [13, Corollary 2.2] and [33, Proposition 2.42]) and the Banach-Nečas-Babuška theorem (cf. [33, Theorem 2.22]), both with finite dimensional subspaces of Banach spaces. The respective continuous versions can be found in [13, Theorem 2.1, Corollary 2.1, Section 2.1] and [33, Theorem 2.34] for the former, and in [33, Theorem 2.6] for the latter.

We begin by letting $A_h : (\mathbb{H}_h\times\mathbf{Q}_h)\times(\mathbb{H}_h\times\mathbf{Q}_h) \to \mathrm{R}$ be the bounded bilinear form arising after adding the left-hand sides of the equations of (4.39), but without including the form $c_h$, that is

$$A_h\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,:=\, a_h(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) \,+\, b(\boldsymbol{\tau}_h,\mathbf{w}_h) \,+\, b(\boldsymbol{\zeta}_h,\mathbf{v}_h) \tag{4.41}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)$, $(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$. Note that the boundedness of $A$ is consequence of those of $a_h$ with $\|a_h\| = \alpha_2$ (cf. (4.6)), and $b$ with $\|b\| = 1$ (cf. (2.14)). Hence, bearing additionally in mind the ellipticity of $a_h$ in the discrete kernel $\mathbb{V}_h$ of $b$ (cf. Lemma 4.3), and the discrete inf-sup condition satisfied by $b$ (cf. Lemma 4.32), a straightforward application of [13, Corollary 2.2] (or [33, Proposition

2.42]) yields the inf-sup condition for $A_h$ in $\mathbb{H}_h \times \mathbf{Q}_h$. More precisely, there exists a positive constant $\widetilde{\alpha}$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, and $\|a_h\|$, such that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\(\boldsymbol{\tau}_h,\mathbf{v}_h)\neq\mathbf{0}}} \frac{A_h\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \,\geq\, \widetilde{\alpha}\,\|(\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathbb{H}\times\mathbf{Q}} \qquad \forall\,(\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\,. \tag{4.42}$$

Now, we notice that (4.39) can be reformulated as: Find $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$ such that

$$B_{\mathbf{z}_h}\big((\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) \,=\, F(\boldsymbol{\tau}_h)+G(\mathbf{v}_h) \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\,, \tag{4.43}$$

where

$$B_{\mathbf{z}_h}\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) := A_h\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big) + c_h(\mathbf{z}_h;\mathbf{w}_h,\boldsymbol{\tau}_h) \tag{4.44}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)$, $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$. Hence, employing (4.42) and the boundedness estimate (4.10) for $c_h$, we readily obtain that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\(\boldsymbol{\tau}_h,\mathbf{v}_h)\neq\mathbf{0}}} \frac{B_{\mathbf{z}_h}\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \,\geq\, \left(\widetilde{\alpha} \,-\, \frac{1}{\mu}\,\|\mathbf{z}_h\|_{0,4;\Omega}\right)\|(\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathbb{H}\times\mathbf{Q}} \tag{4.45}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$, from which we conclude that for each $\mathbf{z}_h\in\mathbf{Q}_h$ such that $\|\mathbf{z}_h\|_{0,4;\Omega}\,\leq\,\frac{\widetilde{\alpha}\mu}{2}$, there holds

$$\sup_{\substack{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\(\boldsymbol{\tau}_h,\mathbf{v}_h)\neq\mathbf{0}}} \frac{B_{\mathbf{z}_h}\big((\boldsymbol{\zeta}_h,\mathbf{w}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \,\geq\, \frac{\widetilde{\alpha}}{2}\,\|(\boldsymbol{\zeta}_h,\mathbf{w}_h)\|_{\mathbb{H}\times\mathbf{Q}} \tag{4.46}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$. We stress here that we could have also chosen any $\delta\in(0,1)$ and imposed the condition $\|\mathbf{z}_h\|_{0,4;\Omega}\,\leq\,\delta\,\widetilde{\alpha}\mu$. In this case, the closer $\delta$ to 1, the larger the range for $\mathbf{z}_h$, but then the constant on the right hand side of (4.46) becomes much smaller. Conversely, the closer $\delta$ to 0, the larger the aforementioned constant, but then the range for $\mathbf{z}_h$ is too restrictive. According to this, it seems more reasonable to simply choose the midpoint of the range of $\delta$, as we just did.

In this way, we are now able to prove the following lemma establishing the well-posedness of (4.39), which, as already mentioned, is equivalent to the well-definedness of $\mathbf{T}_h$.

**Lemma 4.10.** *For each $\mathbf{z}_h\in\mathbf{Q}_h$ such that $\|\mathbf{z}_h\|_{0,4;\Omega}\,\leq\,\frac{\widetilde{\alpha}\mu}{2}$, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$ solution to (4.39). In addition, there holds*

$$\|\mathbf{T}_h(\mathbf{z}_h)\|_{0,4;\Omega} \,=\, \|\widetilde{\mathbf{u}}_h\|_{0,4;\Omega} \,\leq\, \|(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \,\leq\, \frac{2}{\widetilde{\alpha}}\left\{C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\}. \tag{4.47}$$

*Proof.* In virtue of the inf-sup condition (4.46) satisfied by $B_{\mathbf{z}_h}$ for each $\mathbf{z}_h\in\mathbf{Q}_h$ as stated, the unique solvability of (4.39) follows from a straightforward application of the discrete Banach-Nečas-Babuška theorem (cf. [33, Theorem 2.22]). In turn, the corresponding continuous dependence result reads

$$\|(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \,\leq\, \frac{2}{\widetilde{\alpha}}\left\{\|F\|_{\mathbb{H}'} \,+\, \|G\|_{\mathbf{Q}'}\right\},$$

which, along with (2.15) and (2.16), yields (4.47). □

Having proved that $\mathbf{T}_h$ is well-defined, we now analyze the solvability of the fixed-point equation (4.40) by means of the classical Banach theorem. We begin by identifying a sufficient condition under which $\mathbf{T}_h$ maps a closed ball of $\mathbf{Q}_h$ into itself. In fact, we now define

$$S_h \,:=\, \left\{\mathbf{z}_h\in\mathbf{Q}_h \,:\quad \|\mathbf{z}_h\|_{0,4;\Omega}\,\leq\,\frac{\widetilde{\alpha}\mu}{2}\right\}, \tag{4.48}$$

and prove the following result.

**Lemma 4.11.** *Assume that the data satisfy*

$$\frac{4}{\widetilde{\alpha}^2\mu}\Big\{C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\le\,1\,. \tag{4.49}$$

*Then* $\mathbf{T}_h(S_h)\subseteq S_h$.

*Proof.* It follows directly from Lemma 4.10 and the *a priori* estimate provided by (4.47). □

Next, we establish the Lipschitz-continuity of $\mathbf{T}_h$.

**Lemma 4.12.** *There holds*

$$\|\mathbf{T}_h(\mathbf{z}_h)-\mathbf{T}_h(\mathbf{y}_h)\|_{0,4;\Omega}\,\le\,\frac{4}{\widetilde{\alpha}^2\mu}\Big\{C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\|\mathbf{z}_h-\mathbf{y}_h\|_{0,4;\Omega}\quad\forall\,\mathbf{z}_h,\,\mathbf{y}_h\in S_h\,. \tag{4.50}$$

*Proof.* Given $\mathbf{z}_h,\,\mathbf{y}_h\in S_h$, we let $\mathbf{T}_h(\mathbf{z}_h):=\widetilde{\mathbf{u}}_h\in\mathbf{Q}_h$ and $\mathbf{T}_h(\mathbf{y}_h):=\bar{\mathbf{u}}_h\in\mathbf{Q}_h$, where $(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)$ and $(\bar{\boldsymbol{\sigma}}_h,\bar{\mathbf{u}}_h)$, both in $\mathbb{H}_h\times\mathbf{Q}_h$, are the unique solutions of (4.39) (equivalently (4.43)) with $\mathbf{z}_h$ itself and with $\mathbf{z}_h=\mathbf{y}_h$, respectively. It follows from (4.43) that

$$B_{\mathbf{z}_h}\big((\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)\,=\,B_{\mathbf{y}_h}\big((\bar{\boldsymbol{\sigma}}_h,\bar{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)$$

for all $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$, which, according to the definitions of $B_{\mathbf{z}_h}$ and $B_{\mathbf{y}_h}$ (cf. (4.44)), becomes

$$A_h\big((\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)\,=\,c_h(\mathbf{y}_h;\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)-c_h(\mathbf{z}_h;\widetilde{\mathbf{u}}_h,\boldsymbol{\tau}_h)\,,$$

and hence

$$\begin{aligned}B_{\mathbf{z}_h}\big((\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)&:=A_h\big((\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)\big)+c_h(\mathbf{z}_h;\widetilde{\mathbf{u}}_h-\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)\\&=\,c_h(\mathbf{y}_h;\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)-c_h(\mathbf{z}_h;\widetilde{\mathbf{u}}_h,\boldsymbol{\tau}_h)+c_h(\mathbf{z}_h;\widetilde{\mathbf{u}}_h-\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)=c_h(\mathbf{y}_h-\mathbf{z}_h;\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)\end{aligned}$$

for all $(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$. Therefore, applying (4.46) to $(\boldsymbol{\zeta}_h,\mathbf{w}_h)=(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h)$, and then employing the foregoing identity and the estimate (4.10) for $c_h$, we arrive at

$$\frac{\widetilde{\alpha}}{2}\,\|(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\,\le\,\sup_{\substack{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\(\boldsymbol{\tau}_h,\mathbf{v}_h)\neq\mathbf{0}}}\frac{c_h(\mathbf{y}_h-\mathbf{z}_h;\bar{\mathbf{u}}_h,\boldsymbol{\tau}_h)}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbb{H}\times\mathbf{Q}}}\,\le\,\frac{1}{\mu}\,\|\mathbf{z}_h-\mathbf{y}_h\|_{0,4;\Omega}\,\|\bar{\mathbf{u}}_h\|_{0,4;\Omega}\,,$$

whence, using the *a priori* bound (4.47) for $\|\bar{\mathbf{u}}_h\|_{0,4;\Omega}\,=\,\|\mathbf{T}_h(\mathbf{y}_h)\|_{0,4;\Omega}$, and observing that certainly $\|\mathbf{T}_h(\mathbf{z}_h)-\mathbf{T}_h(\mathbf{y}_h)\|_{0,4;\Omega}\,\le\,\|(\widetilde{\boldsymbol{\sigma}}_h,\widetilde{\mathbf{u}}_h)-(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}_h)\|_{\mathbb{H}\times\mathbf{Q}}$, we arrive at (4.50) and end the proof. □

Consequently, we are now in position to state the main result of this section.

**Theorem 4.1.** *Assume that the data satisfy*

$$\frac{4}{\widetilde{\alpha}^2\mu}\Big\{C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,<\,1\,. \tag{4.51}$$

*Then, the mixed virtual element scheme* (3.23) *has a unique solution* $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$ *with* $\mathbf{u}_h\in S_h$, *and there holds*

$$\|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}}\,\le\,\frac{2}{\widetilde{\alpha}}\,\Big\{C_F\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,. \tag{4.52}$$

*Proof.* We first notice from (4.49) (cf. Lemma 4.11) and (4.50) (cf. Lemma 4.12) that the assumption (4.51) guarantees both that $\mathbf{T}_h$ maps $S_h$ into itself and that $\mathbf{T}_h$ is a contraction. Hence, the equivalence between (3.23) and (4.40), and a direct application of the Banach fixed-point theorem, imply the existence of a unique solution $(\boldsymbol{\sigma}_h,\mathbf{u}_h)$ of (3.23) with $\mathbf{u}_h\in S_h$. Finally, the stability result (4.52) follows directly from (4.47). □

# 5 A priori error analysis

In this section we derive *a priori* error estimates for the solution of the virtual element scheme (3.23), for computable approximations of the pseudostress $\boldsymbol{\sigma}$ and the pressure $p$, and for a postprocessed approximation of $\boldsymbol{\sigma}$.

## 5.1 The main error estimate

We begin by establishing a Céa type estimate for the error

$$\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}} := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,$$

where, under the assumptions of Theorems 2.1 and 4.1, $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}\times\mathbf{Q}$ and $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$ are the unique solutions of (2.8) (with $\mathbf{u}\in S$) and (3.23) (with $\mathbf{u}_h\in S_h$), respectively. To this end, and aiming to employ next a suitable Strang estimate, we rewrite (2.8) and (3.23) as the following pair of a continuous formulation and its associated discrete one, that is

$$\begin{array}{llll} a(\boldsymbol{\sigma},\boldsymbol{\tau})\,+\,b(\boldsymbol{\tau},\mathbf{u}) & = & F_{\mathbf{u}}(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau}\in\mathbb{H}\,,\\ b(\boldsymbol{\sigma},\mathbf{v}) & = & G(\mathbf{v}) & \forall\,\mathbf{v}\in\mathbf{Q}\,,\\ a_h(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)\,+\,b(\boldsymbol{\tau}_h,\mathbf{u}_h) & = & F_{\mathbf{u}_h}(\boldsymbol{\tau}_h) & \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h\,,\\ b(\boldsymbol{\sigma}_h,\mathbf{v}_h) & = & G(\mathbf{v}_h) & \forall\,\mathbf{v}_h\in\mathbf{Q}_h\,, \end{array} \tag{5.1}$$

where

$$F_{\mathbf{u}}(\boldsymbol{\tau}) \,:=\, F(\boldsymbol{\tau})-c(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}\,, \tag{5.2}$$

and

$$F_{\mathbf{u}_h}(\boldsymbol{\tau}_h) \,:=\, F(\boldsymbol{\tau}_h)-c_h(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h\in\mathbb{H}_h\,. \tag{5.3}$$

In what follows, given a subspace $X_h$ of a generic Banach space $(X,\|\cdot\|_X)$, we set for each $x\in X$

$$\mathrm{dist}(x,X_h) \,:=\, \inf_{x_h\in X_h}\|x-x_h\|_X\,.$$

Then, applying the Strang *a priori* error estimate for dual mixed formulations in Banach spaces (see, e.g. [12, Lemma 5.2] or [33, Lemma 2.27] for a more general case) to the context given by (5.1), we deduce that there exists a constant $C_{\mathtt{st}}>0$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|a_h\|=\alpha_2$, and $\|b\|=1$, such that

$$\begin{aligned} \|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}} \,\le\, C_{\mathtt{st}}\Bigg\{ & \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}}\frac{F_{\mathbf{u}}(\boldsymbol{\tau}_h)-F_{\mathbf{u}_h}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}}\,+\,\mathrm{dist}(\mathbf{u},\mathbf{Q}_h)\\ & +\,\inf_{\boldsymbol{\zeta}_h\in\mathbb{H}_h}\left(\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbb{H}}\,+\,\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}}\frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)-a_h(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}}\right)\Bigg\}\,. \end{aligned} \tag{5.4}$$

We now proceed to estimate the terms on the right hand side of (5.4). In fact, adding and subtracting suitable evaluations of the form $c_h$, and then employing the estimates (4.11) (cf. Lemma 4.6) and (4.10) (cf. Lemma 4.5), we obtain

$$\begin{aligned} |F_{\mathbf{u}}(\boldsymbol{\tau}_h)-F_{\mathbf{u}_h}(\boldsymbol{\tau}_h)| \,&=\, |c(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)-c_h(\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h)|\\ &\le\, |c(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)-c_h(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h)|+|c_h(\mathbf{u};\mathbf{u}-\mathbf{u}_h,\boldsymbol{\tau}_h)|+|c_h(\mathbf{u}-\mathbf{u}_h;\mathbf{u}_h,\boldsymbol{\tau}_h)|\\ &\le\, \frac{1}{\mu}\Big\{\|(\mathbf{u}\otimes\mathbf{u})-\mathcal{P}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega}+\big(\|\mathbf{u}\|_{0,4;\Omega}+\|\mathbf{u}_h\|_{0,4;\Omega}\big)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,\|\boldsymbol{\tau}_h\|_{0,\Omega}\,, \end{aligned}$$

from which, using the *a priori* estimates for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$ provided by (2.18) and (4.52), respectively, and defining the constant $\widetilde{C}_{\mathtt{st}} := \frac{2}{\mu}\big(\frac{1}{\gamma} + \frac{1}{\widetilde{\alpha}}\big) \max\big\{C_F, 1\big\}$, we conclude that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{F_{\mathbf{u}}(\boldsymbol{\tau}_h) - F_{\mathbf{u}_h}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \le \frac{1}{\mu}\,\|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} + \widetilde{C}_{\mathtt{st}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,. \tag{5.5}$$

In turn, applying (4.9) (cf. Lemma 4.4), we find that

$$|a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) - a_h(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)| \le C_a\,\|\boldsymbol{\zeta}_h - \mathcal{P}_k^h(\boldsymbol{\zeta}_h)\|_{0,\Omega}\,\|\boldsymbol{\tau}_h\|_{0,\Omega}\,,$$

from which, adding and subtracting $\boldsymbol{\sigma}$ and $\mathcal{P}_k^h(\boldsymbol{\sigma})$ in the first factor, and using (4.2), we arrive at

$$\begin{aligned}|a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) - a_h(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)| &\le C_a\Big\{\|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega} + \|\mathcal{P}_k^h(\boldsymbol{\sigma}-\boldsymbol{\zeta}_h)\|_{0,\Omega}\Big\}\|\boldsymbol{\tau}_h\|_{0,\Omega}\\ &\le C_a\Big\{\|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + 2\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega}\Big\}\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}\,.\end{aligned}$$

Then, replacing this bound into the supremum within the infimum of (5.4), and noting that certainly $\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{0,\Omega} \le \|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}$, we get

$$\inf_{\boldsymbol{\zeta}_h\in\mathbb{H}_h}\left(\|\boldsymbol{\sigma}-\boldsymbol{\zeta}_h\|_{\mathbb{H}} + \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) - a_h(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}}\right) \le (1+2C_a)\,\mathrm{dist}(\boldsymbol{\sigma},\mathbb{H}_h) + C_a\,\|\boldsymbol{\sigma}-\mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,. \tag{5.6}$$

Hence, employing the upper bounds provided by (5.5) and (5.6) in (5.4), and reordering the resulting terms, we get

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}} \le C_{\mathtt{st}}\bigg\{&C_a\,\|\boldsymbol{\sigma}-\mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \frac{1}{\mu}\,\|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega}\\ &+ (1+2C_a)\,\mathrm{dist}(\boldsymbol{\sigma},\mathbb{H}_h) + \mathrm{dist}(\mathbf{u},\mathbf{Q}_h) + \widetilde{C}_{\mathtt{st}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\bigg\}.\end{aligned} \tag{5.7}$$

Consequently, we are now in a position to state the announced Céa type estimate.

**Theorem 5.1.** *Assume that the data satisfy*

$$C_{\mathtt{st}}\,\widetilde{C}_{\mathtt{st}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\} \le \frac{1}{2}\,. \tag{5.8}$$

*Then there exists a constant $C > 0$, depending only on $C_{\mathtt{st}}$, $C_a$, and $\mu$, such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}} &= \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} + \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\\ &\le C\Big\{\|\boldsymbol{\sigma}-\mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|(\mathbf{u}\otimes\mathbf{u}) - \mathcal{P}_k^h(\mathbf{u}\otimes\mathbf{u})\|_{0,\Omega} + \mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h\times\mathbf{Q}_h\big)\Big\}\,.\end{aligned} \tag{5.9}$$

*Proof.* It suffices to use the assumption (5.8) in (5.7), bound $\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}$ by $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}\times\mathbf{Q}}$, and then subtract this resulting expression from the left-hand side. □

Having established Theorem 5.1, and recalling the definitions of the broken seminorms given by (3.17) and (3.18), we now provide the corresponding rates of convergence.

**Theorem 5.2.** *Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H} \times \mathbf{Q}$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ be the unique solutions of the continuous and discrete schemes (2.8) and (3.23), respectively. Assume that for integers $r \in [1, k+1]$ and $s \in [0, k+1]$ there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$, $(\mathbf{u} \otimes \mathbf{u})|_K \in \mathbb{H}^s(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{W}^{r,4/3}(K)$, and $\mathbf{u}|_K \in \mathbf{W}^{s,4}(K)$, for each $K \in \mathcal{T}_h$. Then, there exists a positive constant $C$, independent of $h$, such that*

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H} \times \mathbf{Q}} \\ &\qquad \le C\, h^{\min\{r,s\}} \Big\{ |\boldsymbol{\sigma}|_{r;\mathtt{b},\Omega} + |\mathbf{u} \otimes \mathbf{u}|_{s;\mathtt{b},\Omega} + |\mathbf{div}(\boldsymbol{\sigma})|_{r,4/3;\mathtt{b},\Omega} + |\mathbf{u}|_{s,4;\mathtt{b},\Omega} \Big\}. \end{aligned} \tag{5.10}$$

*Proof.* It reduces to apply Theorem 5.1, for which we need to bound the terms on the right hand side of (5.9). Indeed, thanks to the global version of (3.4) we readily obtain

$$\|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \le C\, h^r\, |\boldsymbol{\sigma}|_{r;\mathtt{b},\Omega}$$

and

$$\|(\mathbf{u} \otimes \mathbf{u}) - \mathcal{P}_k^h(\mathbf{u} \otimes \mathbf{u})\|_{0,\Omega} \le C\, h^s\, |\mathbf{u} \otimes \mathbf{u}|_{s;\mathtt{b},\Omega}.$$

The foregoing estimates and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ and $(\mathbf{AP}_h^{\mathbf{u}})$ complete the proof. □

### 5.2 Computable approximations of $\boldsymbol{\sigma}$ and $p$

We now propose computable approximations $\widehat{\boldsymbol{\sigma}}$ and $\widehat{p}$ of the pseudostress tensor $\boldsymbol{\sigma}$ and the pressure $p$ of the fluid, respectively, and provide the corresponding *a priori* error estimates, as well as the resulting rates of convergence. In fact, proceeding as in [38, Section 5.3, eq. (5.32)], we define

$$\widehat{\boldsymbol{\sigma}}_h := \mathcal{P}_k^h(\boldsymbol{\sigma}_h) \tag{5.11}$$

and

$$\widehat{p}_h := -\frac{1}{2}\Big\{ \operatorname{tr}(\widehat{\boldsymbol{\sigma}}_h) + \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h) \Big\} + \frac{1}{2|\Omega|} \int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h). \tag{5.12}$$

Note, in particular, that (5.12) is suggested by (2.2) and (2.4).

Next, adding and subtracting $\mathcal{P}_k^h(\boldsymbol{\sigma})$, and employing the triangle inequality and the global version of (4.2), we get

$$\begin{aligned} \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} = \|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega} &\le \|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\mathcal{P}_k^h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \\ &\le \|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}. \end{aligned} \tag{5.13}$$

In turn, proceeding analogously to [38, Theorem 5.5, eqs. (5.38) and (5.39)], we deduce the existence of a constant $C > 0$, depending on the data, but independent of $h$, such that

$$\|p - \widehat{p}_h\|_{0,\Omega} \le C \Big\{ \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \Big\}. \tag{5.14}$$

In this way, as a direct consequence of (5.13) and (5.14), we are able to state the following result.

**Theorem 5.3.** *There exists a positive constant $C > 0$, independent of $h$, such that*

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|p - \widehat{p}_h\|_{0,\Omega} \le C \Big\{ \|\boldsymbol{\sigma} - \mathcal{P}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H} \times \mathbf{Q}} \Big\}. \tag{5.15}$$

The corresponding rates of convergence are established as follows.

**Theorem 5.4.** *Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H} \times \mathbf{Q}$ be the unique solution of the continuous scheme (2.8), and let $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{p}_h$ be the discrete approximations introduced in (5.11) and (5.12), respectively. Assume that for integers $r \in [1, k+1]$ and $s \in [0, k+1]$ there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$, $(\mathbf{u} \otimes \mathbf{u})|_K \in \mathbb{H}^s(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{W}^{r,4/3}(K)$, and $\mathbf{u}|_K \in \mathbf{W}^{s,4}(K)$, for each $K \in \mathcal{T}_h$. Then, there exists a positive constant $C$, independent of $h$, such that*

$$\begin{aligned} &\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|p - \widehat{p}_h\|_{0,\Omega} \\ &\quad \leq C\, h^{\min\{r,s\}} \Big\{ |\boldsymbol{\sigma}|_{r;\mathtt{b},\Omega} + |\mathbf{u} \otimes \mathbf{u}|_{s;\mathtt{b},\Omega} + |\mathbf{div}(\boldsymbol{\sigma})|_{r,4/3;\mathtt{b},\Omega} + |\mathbf{u}|_{s,4;\mathtt{b},\Omega} \Big\}. \end{aligned} \tag{5.16}$$

*Proof.* It follows from Theorem 5.3, the rates of convergence provided by Theorem 5.2, and the approximation property of $\mathcal{P}_k^h$ (global version of (3.4)). □

### 5.3 A second postprocessed approximation of $\boldsymbol{\sigma}$

Here we assume that $\boldsymbol{\sigma}|_K \in \mathbb{H}(\mathbf{div}; K)$ for all $K \in \mathcal{T}_h$, and adopt a similar approach to that in [36] and [38] for introducing a second approximation $\boldsymbol{\sigma}_h^\star$, defined in terms of $\widehat{\boldsymbol{\sigma}}_h$, of the pseudostress tensor $\boldsymbol{\sigma}$. In addition, we show that $\boldsymbol{\sigma}_h^\star$ yields an optimal rate of convergence in the broken norm of $\mathbb{H}(\mathbf{div}; \Omega) = \mathbb{H}(\mathbf{div}_2; \Omega)$ given by

$$\|\boldsymbol{\tau}\|_{\mathbf{div};\mathtt{b},\Omega} := \Big\{ \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\tau}\|^2_{\mathbf{div};K} \Big\}^{1/2} \quad \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \text{ such that } \boldsymbol{\tau}|_K \in \mathbb{H}(\mathbf{div}; K) \quad \forall\, K \in \mathcal{T}_h\,. \tag{5.17}$$

More precisely, following [36, eq. (3.7)], for each $K \in \mathcal{T}_h$ we let $(\cdot,\cdot)_{\mathbf{div};K}$ be the usual inner product of $\mathbb{H}(\mathbf{div}; K)$ with induced norm $\|\cdot\|_{\mathbf{div};K}$, and set $\boldsymbol{\sigma}_h^\star|_K := \boldsymbol{\sigma}_{h,K}^\star \in \mathbb{P}_{k+1}(K)$, where $\boldsymbol{\sigma}_{h,K}^\star$ is the unique solution of the local problem:

$$(\boldsymbol{\sigma}_{h,K}^\star, \boldsymbol{\tau}_h)_{\mathbf{div};K} = \int_K \widehat{\boldsymbol{\sigma}}_h : \boldsymbol{\tau}_h - \int_K \mathbf{f} \cdot \mathbf{div}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{P}_{k+1}(K)\,. \tag{5.18}$$

We stress that $\boldsymbol{\sigma}_{h,K}^\star$ can be explicitly (and efficiently) calculated for each $K \in \mathcal{T}_h$ independently.

The following result establishes an *a priori* error estimate for $\boldsymbol{\sigma}_{h,K}^\star$.

**Theorem 5.5.** *There holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,K}^\star\|_{\mathbf{div};K} \leq \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,K} + \|\boldsymbol{\sigma} - \mathcal{P}_{k+1}^K(\boldsymbol{\sigma})\|_{\mathbf{div};K} \qquad \forall\, K \in \mathcal{T}_h\,. \tag{5.19}$$

*Proof.* It is an adaptation of the proof of [36, Lemma 3.1]. We first let $\Pi_{\mathbf{div}}^K : \mathbb{H}(\mathbf{div}; K) \to \mathbb{P}_{k+1}(K)$ be the orthogonal projector with respect to $(\cdot,\cdot)_{\mathbf{div};K}$, which, given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}; K)$, is characterized by the orthogonality condition

$$(\boldsymbol{\zeta} - \Pi_{\mathbf{div}}^K(\boldsymbol{\zeta}), \boldsymbol{\tau}_h)_{\mathbf{div};K} = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{P}_{k+1}(K)\,. \tag{5.20}$$

Then, using (5.18) and recalling from (2.5) that $\mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f}$, we find that

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,K}^\star, \boldsymbol{\tau}_h)_{\mathbf{div};K} = \int_K (\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h) : \boldsymbol{\tau}_h \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{P}_{k+1}(K)\,,$$

which, according to (5.20) with $\boldsymbol{\zeta} = \boldsymbol{\sigma}$, becomes

$$(\Pi_{\mathbf{div}}^K(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_{h,K}^\star, \boldsymbol{\tau}_h)_{\mathbf{div};K} = \int_K (\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h) : \boldsymbol{\tau}_h \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{P}_{k+1}(K)\,. \tag{5.21}$$

Next, taking $\boldsymbol{\tau}_h := \Pi^K_{\mathbf{div}}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}^\star_{h,K} \in \mathbb{P}_{k+1}(K)$ in (5.21), and using the Cauchy-Schwarz inequality, we get

$$\|\Pi^K_{\mathbf{div}}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}^\star_{h,K}\|_{\mathbf{div};K} \;\leq\; \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,K}\,,$$

which, along with the triangle inequality, yields

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\star_{h,K}\|_{\mathbf{div};K} \;&\leq\; \|\boldsymbol{\sigma} - \Pi^K_{\mathbf{div}}(\boldsymbol{\sigma})\|_{\mathbf{div};K} \,+\, \|\Pi^K_{\mathbf{div}}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}^\star_{h,K}\|_{\mathbf{div};K}\\ &\leq\; \|\boldsymbol{\sigma} - \Pi^K_{\mathbf{div}}(\boldsymbol{\sigma})\|_{\mathbf{div};K} \,+\, \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,K}\,.\end{aligned}$$

The foregoing inequality and the fact that $\|\boldsymbol{\sigma} - \Pi^K_{\mathbf{div}}(\boldsymbol{\sigma})\|_{\mathbf{div};K} \leq \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})\|_{\mathbf{div};K}$ give (5.19) and finish the proof. □

The rate of convergence for $\boldsymbol{\sigma}^\star_h$ is stated as follows.

**Theorem 5.6.** *Let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H} \times \mathbf{Q}$ be the unique solution of the continuous scheme (2.8), and let $\widehat{\boldsymbol{\sigma}}_h$ and $\boldsymbol{\sigma}^\star_h$ be the discrete approximations of $\boldsymbol{\sigma}$ introduced in (5.11) and (5.18), respectively. Assume that for integers $r \in [1, k+1]$ and $s \in [0, k+1]$ there hold $\boldsymbol{\sigma}|_K \in \mathbb{H}^{r+1}(K)$, $(\mathbf{u}\otimes\mathbf{u})|_K \in \mathbb{H}^s(K)$, $\mathbf{f}|_K = -\mathbf{div}(\boldsymbol{\sigma})|_K \in \mathbf{W}^{r,4/3}(K)$, and $\mathbf{u}|_K \in \mathbf{W}^{s,4}(K)$, for each $K \in \mathcal{T}_h$. Then, there exists a positive constant $C$, independent of $h$, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\star_h\|_{\mathbf{div};\mathrm{b},\Omega} \;\leq\; C\, h^{\min\{r,s\}} \Big\{ |\boldsymbol{\sigma}|_{r+1;\mathrm{b},\Omega} + |\mathbf{u}\otimes\mathbf{u}|_{s;\mathrm{b},\Omega} + |\mathbf{div}(\boldsymbol{\sigma})|_{r,4/3;\mathrm{b},\Omega} + |\mathbf{u}|_{s,4;\mathrm{b},\Omega} \Big\}\,. \tag{5.22}$$

*Proof.* According to (5.19) (cf. Theorem 5.5), the rates of convergence for the terms $\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,K}$ and $\|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})\|_{\mathbf{div};K}$ imply that of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\star_h\|_{\mathbf{div};\mathrm{b},\Omega}$. Those for the former are provided by (5.16) (cf. Theorem 5.4), for which it suffices to assume that $\boldsymbol{\sigma}|_K \in \mathbb{H}^r(K)$ for each $K \in \mathcal{T}_h$, and keep the rest of the present regularities for the other unknowns. In turn, for the latter we first notice that

$$\|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})\|_{\mathbf{div};K} \;\leq\; C\Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})\|_{0,K} \,+\, |\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})|_{1,K} \Big\}\,,$$

and then apply the approximation property of $\boldsymbol{\mathcal{P}}^K_{k+1}$ (cf. (3.4)). Note that in order to maintain an $O(h^r)$ for $|\boldsymbol{\sigma} - \boldsymbol{\mathcal{P}}^K_{k+1}(\boldsymbol{\sigma})|_{1,K}$ we need to assume now that $\boldsymbol{\sigma}|_K \in \mathbb{H}^{r+1}(K)$ for each $K \in \mathcal{T}_h$. Further details are omitted. □

## 6 Numerical results

In this section we present three numerical experiments illustrating the performance of the mixed virtual element scheme (3.23) introduced and analyzed in Sections 3, 4, and 5. More precisely, in all the computations we consider the specific virtual element subspaces $\mathbb{H}_h$ and $\mathbf{Q}_h$ (cf. (3.15)-(3.16)) with $k \in \{0, 1, 2\}$. Furthermore, as it is suggested in [16, Section 6], the zero mean condition for tensors in the space $\mathbb{H}_h$ is imposed via a real Lagrange multiplier, which means that, instead of (3.23), we actually solve the modified discrete scheme given by: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \xi_h) \in \widetilde{\mathbb{H}}_h \times \mathbf{Q}_h \times \mathrm{R}$ such that

$$\left\{\begin{array}{lcll} a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, b(\boldsymbol{\tau}_h, \mathbf{u}_h) \,+\, c_h(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) \,+\, \xi_h \displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) & = & F(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h\,,\\ b(\boldsymbol{\sigma}_h, \mathbf{v}_h) & = & G(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,,\\ \eta_h \displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h) & = & 0 & \forall\, \eta_h \in \mathrm{R}\,, \end{array}\right. \tag{6.1}$$

where

$$\widetilde{\mathbb{H}}_h := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbb{H}_k^K \quad \forall\, K \in \mathcal{T}_h \Big\}$$

and $\xi_h$ is an artificial null unknown introduced just to make (6.1) symmetric. Concerning the decompositions of $\Omega$ employed in our computations, we consider quasi-uniform triangles, distorted squares and distorted hexagons. We refer to Figures 6.1 up to 6.4 below for visualizing in advance the kind of meshes to be utilized.

We now introduce additional notations. In what follows, $N$ stands for the total number of degrees of freedom (unknowns) of (6.1), that is

$$\begin{aligned} N \,:=\, & 2(k+1) \times \{\text{number of edges } e \in \mathcal{T}_h\} \\ & + (k+2)(3k+1) \times \{\text{number of elements } K \in \mathcal{T}_h\} \,+\, 1\,. \end{aligned}$$

Also, the individual errors are defined by

$$\mathtt{e}(\boldsymbol{\sigma}) \,:=\, \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,, \quad \mathtt{e}(\mathbf{u}) \,:=\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \quad \mathtt{e}(p) \,:=\, \|p - \widehat{p}_h\|_{0,\Omega}\,,$$

and

$$\mathtt{e}(\boldsymbol{\sigma}^\star) \,:=\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^\star\|_{\mathbf{div};\mathtt{b},\Omega}\,,$$

where $\widehat{\boldsymbol{\sigma}}_h$, $\widehat{p}_h$, and $\boldsymbol{\sigma}_h^\star$ are computed according to (5.11), (5.12) and (5.18), respectively. In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}(\%) \,:=\, \frac{\log\big(\mathtt{e}(\%)\,/\,\mathtt{e}'(\%)\big)}{\log(h\,/\,h')} \qquad \forall\, \% \in \big\{\boldsymbol{\sigma}, \mathbf{u}, p, \boldsymbol{\sigma}^\star\big\}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the corresponding errors for two consecutive meshes with sizes $h$ and $h'$, respectively.

The nonlinear algebraic system arising from (6.1) is solved by the Newton method with a tolerance of $10^{-6}$ and taking as initial iteration the solution of the associated linear Stokes problem. The latter is obtained by eliminating the convective term $(\nabla\mathbf{u})\mathbf{u}$ in (2.1), which turns out to the removal of the trilinear form $c_h$ in (6.1). We stress that the well-possessedness of the resulting linear discrete formulation is guaranteed by the global discrete inf-sup condition satisfied by the bilinear form $\mathbf{A}_h$ (cf. (4.42)). In turn, we notice in advance that four iterations are required to achieve the given tolerance in Examples 1 and 2, whereas two iterations are required in Example 3, all them described next.

In Example 1, we consider $\Omega := (0,1)^2$, $\mu = \frac{1}{2}$, and choose the data $\mathbf{f}$ and $\mathbf{u}_D$ such that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} x_1^2\exp(-x_1)(1+x_2)\big(2\sin(1+x_2) \,+\, (1+x_2)\cos(1+x_2)\big) \\ x_1(x_1-2)\exp(-x_1)(1+x_2)^2\sin(1+x_2) \end{pmatrix}$$

and

$$p(\mathbf{x}) \;=\; \sin(2\pi x_1)\sin(2\pi x_2)\,,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$.

In Example 2 we consider $\Omega := (-0.5, 1.5) \times (0,2)$, $\mu = \frac{1}{10}$, and adequately manufacture the data so that the exact solution is given by the flow from [42], that is

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} 1 - \exp(\lambda x_1)\cos(2\pi x_2) \\ \dfrac{\lambda}{2\pi}\exp(\lambda x_1)\sin(2\pi x_2) \end{pmatrix}$$

and

$$p(\mathbf{x}) \;=\; \frac{1}{2}\exp(2\lambda x_1) \;-\; \frac{1}{8\lambda}\big\{\exp(3\lambda) - \exp(-\lambda)\big\}\,,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, where $\lambda := \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ and $Re := \mu^{-1} = 10$ is the Reynolds number.

In Example 3 we follow [15] and [16], and consider the $L$-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$, $\mu = 1$, and the terms on the right-hand sides are adjusted so that the exact solution is given by the functions

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} x_2^2 \\ -x_1^2 \end{pmatrix} \qquad \text{and} \qquad p(\mathbf{x}) \;=\; (x_1^2 + x_2^2)^{1/3} - p_0\,,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, where $p_0 \in \mathrm{R}$ is such that $\int_\Omega p = 0$ holds. Observe in this example that the partial derivatives of $p$, and hence, in particular $\mathbf{div}(\boldsymbol{\sigma})$, are singular at the origin. More precisely, because of the power $1/3$, there holds $\boldsymbol{\sigma} \in \mathbb{H}^{5/3-\varepsilon}(\Omega)$ and $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{H}^{2/3-\varepsilon}(\Omega)$ for each $\varepsilon > 0$.

In Tables 6.1 up to 6.3, and Tables 6.4 up to 6.6, we summarize the convergence history of the mixed virtual element scheme (3.23) as applied to Examples 1 and 2, respectively. In both cases we observe that the theoretical rates of convergence $O(h^{k+1})$ predicted by Theorems 5.4 and 5.6 with $r = s = k+1$, are attained by all the unknowns, for triangular as well as for quadrilateral and hexagonal meshes. On the other hand, in Tables 6.7 up to 6.9 we display the corresponding convergence history for Example 3, where, as suggested by the regularity of the exact solution, we note that the orders $O(h^{\min\{k+1,5/3\}})$ and $O(h^{2/3})$ are attained by $(\widehat{\boldsymbol{\sigma}}_h, \widehat{p}_h)$ and $\boldsymbol{\sigma}_h^\star$, respectively. In turn, we see here that $\mathbf{u}_h$ shows a convergence rate of $O(h^{\min\{k,7/6\}+1})$. These sub-optimal rates of convergence suggest the need of incorporating an adaptive strategy based on a proper *a posteriori* error estimator (as done for instance in [45]), which we plan to address in a separate work. We end this paper by displaying some components of the approximate solutions for Examples 2 and 3, in Figures 6.1 to 6.4. They all correspond to those obtained with the second mesh of each kind (triangles, quadrilaterals and hexagons, respectively) and for the polynomial degree $k = 2$.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0643 | 4929 | 1.11e-01 | −− | 3.40e-02 | −− | 5.19e-02 | −− | 4.51e-01 | −− |
| | 0.0488 | 8527 | 8.38e-02 | 1.02 | 2.57e-02 | 1.02 | 3.84e-02 | 1.09 | 3.42e-01 | 1.00 |
| 0 | 0.0248 | 32719 | 4.22e-02 | 1.01 | 1.30e-02 | 1.01 | 1.89e-02 | 1.05 | 1.74e-01 | 1.00 |
| | 0.0166 | 72591 | 2.83e-02 | 1.01 | 8.72e-03 | 1.00 | 1.25e-02 | 1.02 | 1.17e-01 | 1.00 |
| | 0.0129 | 121441 | 2.18e-02 | 1.00 | 6.74e-03 | 1.00 | 9.65e-03 | 1.01 | 9.03e-02 | 1.00 |
| | 0.0643 | 17601 | 4.79e-03 | −− | 9.16e-04 | −− | 2.67e-03 | −− | 2.38e-02 | −− |
| | 0.0488 | 30509 | 2.79e-03 | 1.97 | 5.25e-04 | 2.01 | 1.53e-03 | 2.01 | 1.37e-02 | 1.99 |
| 1 | 0.0248 | 117421 | 7.30e-04 | 1.98 | 1.35e-04 | 2.01 | 3.96e-04 | 2.00 | 3.55e-03 | 2.00 |
| | 0.0166 | 260781 | 3.29e-04 | 1.99 | 6.09e-05 | 2.00 | 1.78e-04 | 2.00 | 1.60e-03 | 2.00 |
| | 0.0129 | 436481 | 1.97e-04 | 1.99 | 3.63e-05 | 2.00 | 1.06e-04 | 2.00 | 9.55e-04 | 2.00 |
| | 0.0643 | 36081 | 1.75e-04 | −− | 1.20e-05 | −− | 1.10e-04 | −− | 9.55e-04 | −− |
| | 0.0488 | 62583 | 7.62e-05 | 3.00 | 5.03e-06 | 3.14 | 4.81e-05 | 3.00 | 4.17e-04 | 3.00 |
| 2 | 0.0248 | 241111 | 1.00e-05 | 3.00 | 6.37e-07 | 3.06 | 6.32e-06 | 3.00 | 5.50e-05 | 3.00 |
| | 0.0166 | 535671 | 3.02e-06 | 3.00 | 1.91e-07 | 3.02 | 1.91e-06 | 3.00 | 1.66e-05 | 3.00 |
| | 0.0129 | 896721 | 1.39e-06 | 3.00 | 8.77e-08 | 3.01 | 8.79e-07 | 3.00 | 7.65e-06 | 3.00 |

Table 6.1: Example 1, history of convergence using triangles.

**Acknowledgements**. We express our deep gratitude to Professor Lourenco Beirão da Veiga for providing through [3] and [10] most details regarding the local stability result given by Lemma 4.7.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0538 | 5521 | 7.96e-02 | −− | 3.19e-02 | −− | 4.66e-02 | −− | 4.02e-01 | −− |
| | 0.0404 | 9761 | 5.83e-02 | 1.08 | 2.38e-02 | 1.01 | 3.42e-02 | 1.07 | 3.02e-01 | 1.00 |
| 0 | 0.0215 | 34051 | 3.02e-02 | 1.05 | 1.26e-02 | 1.01 | 1.77e-02 | 1.05 | 1.61e-01 | 1.00 |
| | 0.0147 | 73041 | 2.04e-02 | 1.02 | 8.57e-03 | 1.01 | 1.20e-02 | 1.02 | 1.10e-01 | 1.00 |
| | 0.0111 | 126731 | 1.54e-02 | 1.01 | 6.51e-03 | 1.00 | 9.05e-03 | 1.01 | 8.32e-02 | 1.00 |
| | 0.0538 | 18241 | 3.27e-03 | −− | 6.11e-04 | −− | 2.23e-03 | −− | 1.98e-02 | −− |
| | 0.0404 | 32321 | 1.84e-03 | 2.00 | 3.43e-04 | 2.01 | 1.26e-03 | 1.99 | 1.12e-02 | 1.99 |
| 1 | 0.0215 | 113101 | 5.24e-04 | 2.00 | 9.85e-05 | 1.98 | 3.58e-04 | 2.00 | 3.18e-03 | 2.00 |
| | 0.0147 | 242881 | 2.43e-04 | 2.00 | 4.56e-05 | 2.01 | 1.66e-04 | 2.00 | 1.48e-03 | 2.00 |
| | 0.0111 | 421661 | 1.40e-04 | 2.00 | 2.63e-05 | 2.00 | 9.56e-05 | 2.00 | 8.52e-04 | 2.00 |
| | 0.0538 | 36361 | 1.11e-04 | −− | 6.95e-06 | −− | 7.80e-05 | −− | 6.79e-04 | −− |
| | 0.0404 | 64481 | 4.67e-05 | 3.02 | 2.93e-06 | 3.01 | 3.27e-05 | 3.02 | 2.88e-04 | 2.98 |
| 2 | 0.0215 | 225901 | 7.04e-06 | 3.01 | 4.43e-07 | 3.00 | 4.93e-06 | 3.01 | 4.38e-05 | 3.00 |
| | 0.0147 | 485321 | 2.23e-06 | 3.00 | 1.40e-07 | 3.01 | 1.56e-06 | 3.00 | 1.39e-05 | 3.00 |
| | 0.0111 | 842741 | 9.73e-07 | 3.00 | 6.11e-08 | 3.00 | 6.81e-07 | 3.00 | 6.05e-06 | 3.00 |

Table 6.2: Example 1, history of convergence using quadrilaterals.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0488 | 8147 | 7.06e-02 | −− | 2.94e-02 | −− | 4.19e-02 | −− | 3.79e-01 | −− |
| | 0.0377 | 13563 | 5.44e-02 | 1.01 | 2.28e-02 | 0.97 | 3.23e-02 | 1.01 | 2.94e-01 | 0.98 |
| 0 | 0.0277 | 24579 | 4.01e-02 | 1.00 | 1.70e-02 | 0.97 | 2.38e-02 | 1.00 | 2.17e-01 | 1.00 |
| | 0.0197 | 48603 | 2.83e-02 | 1.01 | 1.20e-02 | 1.01 | 1.68e-02 | 1.01 | 1.54e-01 | 1.00 |
| | 0.0146 | 88637 | 2.09e-02 | 1.00 | 8.95e-03 | 0.98 | 1.24e-02 | 1.00 | 1.14e-01 | 1.00 |
| | 0.0488 | 24437 | 2.50e-03 | −− | 5.43e-04 | −− | 1.72e-03 | −− | 1.54e-02 | −− |
| | 0.0377 | 40757 | 1.50e-03 | 1.97 | 3.30e-04 | 1.92 | 1.03e-03 | 1.97 | 9.24e-03 | 1.97 |
| 1 | 0.0277 | 73733 | 8.16e-04 | 1.99 | 1.80e-04 | 1.99 | 5.61e-04 | 1.99 | 5.02e-03 | 2.00 |
| | 0.0197 | 145805 | 4.11e-04 | 1.99 | 9.00e-05 | 2.02 | 2.83e-04 | 1.99 | 2.53e-03 | 2.00 |
| | 0.0146 | 266089 | 2.26e-04 | 1.99 | 4.99e-05 | 1.96 | 1.55e-04 | 1.99 | 1.39e-03 | 1.99 |
| | 0.0488 | 46835 | 8.36e-05 | −− | 5.37e-06 | −− | 5.60e-05 | −− | 4.50e-04 | −− |
| | 0.0377 | 78175 | 3.88e-05 | 2.96 | 2.53e-06 | 2.89 | 2.60e-05 | 2.95 | 2.08e-04 | 2.97 |
| 2 | 0.0277 | 141319 | 1.55e-05 | 3.00 | 1.01e-06 | 3.03 | 1.04e-05 | 3.01 | 8.34e-05 | 2.99 |
| | 0.0197 | 279457 | 5.53e-06 | 3.00 | 3.56e-07 | 3.02 | 3.70e-06 | 3.00 | 2.99e-05 | 2.99 |
| | 0.0146 | 510153 | 2.25e-06 | 2.99 | 1.46e-07 | 2.97 | 1.51e-06 | 2.99 | 1.22e-05 | 2.99 |

Table 6.3: Example 1, history of convergence using hexagons.

# References


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| $k$ | $h$ | $N$ | $\texttt{e}(\boldsymbol{\sigma})$ | $\texttt{r}(\boldsymbol{\sigma})$ | $\texttt{e}(\mathbf{u})$ | $\texttt{r}(\mathbf{u})$ | $\texttt{e}(p)$ | $\texttt{r}(p)$ | $\texttt{e}(\boldsymbol{\sigma}^\star)$ | $\texttt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1230 | 5383 | 5.13e+00 | $--$ | 1.36e+00 | $--$ | 3.23e+00 | $--$ | 8.04e+00 | $--$ |
| | 0.0943 | 9121 | 3.57e+00 | 1.36 | 8.50e-01 | 1.76 | 2.27e+00 | 1.32 | 5.97e+00 | 1.12 |
| 0 | 0.0488 | 33873 | 1.35e+00 | 1.48 | 2.74e-01 | 1.72 | 7.72e-01 | 1.64 | 2.83e+00 | 1.13 |
| | 0.0354 | 64321 | 8.66e-01 | 1.38 | 1.64e-01 | 1.59 | 4.46e-01 | 1.70 | 2.00e+00 | 1.08 |
| | 0.0283 | 100401 | 6.50e-01 | 1.29 | 1.18e-01 | 1.49 | 3.09e-01 | 1.65 | 1.59e+00 | 1.05 |
| | 0.1230 | 19229 | 2.94e-01 | $--$ | 6.03e-02 | $--$ | 2.04e-01 | $--$ | 4.95e-01 | $--$ |
| | 0.0943 | 32641 | 1.57e-01 | 2.36 | 2.95e-02 | 2.69 | 9.92e-02 | 2.71 | 2.83e-01 | 2.10 |
| 1 | 0.0488 | 121569 | 3.72e-02 | 2.18 | 5.59e-03 | 2.53 | 1.92e-02 | 2.49 | 7.37e-02 | 2.04 |
| | 0.0354 | 231041 | 1.92e-02 | 2.06 | 2.63e-03 | 2.34 | 9.26e-03 | 2.27 | 3.86e-02 | 2.01 |
| | 0.0283 | 360801 | 1.22e-02 | 2.03 | 1.60e-03 | 2.24 | 5.70e-03 | 2.17 | 2.46e-02 | 2.01 |
| | 0.1230 | 39423 | 1.99e-02 | $--$ | 2.75e-03 | $--$ | 9.56e-03 | $--$ | 2.62e-02 | $--$ |
| | 0.0943 | 66961 | 8.75e-03 | 3.09 | 1.03e-03 | 3.68 | 4.02e-03 | 3.26 | 1.17e-02 | 3.04 |
| 2 | 0.0488 | 249633 | 1.18e-03 | 3.04 | 1.04e-04 | 3.49 | 5.04e-04 | 3.15 | 1.60e-03 | 3.02 |
| | 0.0354 | 474561 | 4.46e-04 | 3.02 | 3.62e-05 | 3.27 | 1.88e-04 | 3.07 | 6.09e-04 | 3.01 |
| | 0.0283 | 741201 | 2.28e-04 | 3.01 | 1.78e-05 | 3.18 | 9.50e-05 | 3.05 | 3.11e-04 | 3.01 |

Table 6.4: Example 2, history of convergence using triangles.

| $k$ | $h$ | $N$ | $\texttt{e}(\boldsymbol{\sigma})$ | $\texttt{r}(\boldsymbol{\sigma})$ | $\texttt{e}(\mathbf{u})$ | $\texttt{r}(\mathbf{u})$ | $\texttt{e}(p)$ | $\texttt{r}(p)$ | $\texttt{e}(\boldsymbol{\sigma}^\star)$ | $\texttt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1008 | 6273 | 5.27e+00 | $--$ | 1.55e+00 | $--$ | 3.53e+00 | $--$ | 7.63e+00 | $--$ |
| | 0.0787 | 10251 | 3.89e+00 | 1.23 | 1.01e+00 | 1.74 | 2.64e+00 | 1.17 | 5.83e+00 | 1.08 |
| 0 | 0.0404 | 38721 | 1.35e+00 | 1.58 | 3.03e-01 | 1.79 | 8.94e-01 | 1.62 | 2.61e+00 | 1.20 |
| | 0.0307 | 66571 | 8.54e-01 | 1.69 | 1.89e-01 | 1.74 | 5.41e-01 | 1.84 | 1.90e+00 | 1.16 |
| | 0.0229 | 119851 | 5.35e-01 | 1.58 | 1.17e-01 | 1.63 | 3.16e-01 | 1.82 | 1.38e+00 | 1.10 |
| | 0.1008 | 20737 | 3.19e-01 | $--$ | 7.38e-02 | $--$ | 2.60e-01 | $--$ | 4.20e-01 | $--$ |
| | 0.0787 | 33949 | 1.68e-01 | 2.58 | 3.79e-02 | 2.69 | 1.30e-01 | 2.79 | 2.39e-01 | 2.28 |
| 1 | 0.0404 | 128641 | 3.14e-02 | 2.51 | 6.48e-03 | 2.64 | 1.93e-02 | 2.86 | 5.47e-02 | 2.21 |
| | 0.0307 | 221341 | 1.67e-02 | 2.32 | 3.29e-03 | 2.50 | 9.12e-03 | 2.75 | 3.09e-02 | 2.10 |
| | 0.0229 | 398749 | 8.72e-03 | 2.21 | 1.62e-03 | 2.40 | 4.20e-03 | 2.63 | 1.68e-02 | 2.06 |
| | 0.1008 | 41345 | 1.62e-02 | $--$ | 2.75e-03 | $--$ | 8.41e-03 | $--$ | 1.84e-02 | $--$ |
| | 0.0787 | 67733 | 7.39e-03 | 3.18 | 1.17e-03 | 3.44 | 3.53e-03 | 3.50 | 8.52e-03 | 3.11 |
| 2 | 0.0404 | 256961 | 8.98e-04 | 3.15 | 1.16e-04 | 3.46 | 3.43e-04 | 3.49 | 1.07e-03 | 3.11 |
| | 0.0307 | 442261 | 3.89e-04 | 3.08 | 4.65e-05 | 3.36 | 1.40e-04 | 3.29 | 4.64e-04 | 3.06 |
| | 0.0229 | 796933 | 1.58e-04 | 3.06 | 1.75e-05 | 3.32 | 5.42e-05 | 3.22 | 1.89e-04 | 3.04 |

Table 6.5: Example 2, history of convergence using quadrilaterals.


[4] L. Beirão da Veiga, F. Brezzi, L. Marini, G. Manzini and A. Russo, *$H(div)$ and $H(curl)$-conforming virtual element method.* Numer. Math., 133 (2016), no. 2, 303–332.

[5] L. Beirão da Veiga, F. Brezzi, L. Marini and A. Russo, *Mixed virtual element methods for general second order elliptic problems on polygonal meshes.* ESAIM Math. Model. Numer. Anal., 50 (2016), no. 3, 727–747.

[6] L. Beirão da Veiga, F. Dassi and G. Vacca, *The Stokes complex for virtual elements in three dimensions.* Math. Models Methods Appl. Sci. 30 (2020), no. 3, 477–512.

[7] L. Beirão da Veiga, C. Lovadina and D. Mora, *A virtual element method for elastic and inelastic problems on polytope meshes.* Comput. Methods Appl. Mech. Engrg. 295 (2015), 327–346.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0959 | 8459 | 3.40e+00 | −− | 8.88e-01 | −− | 2.04e+00 | −− | 5.75e+00 | −− |
| | 0.0732 | 14315 | 2.44e+00 | 1.24 | 5.94e-01 | 1.49 | 1.49e+00 | 1.15 | 4.37e+00 | 1.02 |
| 0 | 0.0527 | 27373 | 1.46e+00 | 1.56 | 3.43e-01 | 1.67 | 8.52e-01 | 1.71 | 3.05e+00 | 1.09 |
| | 0.0390 | 49507 | 9.07e-01 | 1.59 | 2.13e-01 | 1.60 | 4.88e-01 | 1.85 | 2.22e+00 | 1.07 |
| | 0.0301 | 82899 | 6.34e-01 | 1.38 | 1.46e-01 | 1.44 | 3.12e-01 | 1.73 | 1.69e+00 | 1.05 |
| | 0.0959 | 25429 | 1.66e-01 | −− | 3.87e-02 | −− | 7.80e-02 | −− | 2.90e-01 | −− |
| | 0.0732 | 42941 | 9.61e-02 | 2.03 | 2.04e-02 | 2.37 | 4.28e-02 | 2.23 | 1.71e-01 | 1.96 |
| 1 | 0.0527 | 82217 | 4.88e-02 | 2.06 | 9.72e-03 | 2.26 | 1.91e-02 | 2.45 | 9.07e-02 | 1.93 |
| | 0.0390 | 148517 | 2.68e-02 | 2.00 | 5.06e-03 | 2.18 | 9.78e-03 | 2.23 | 5.04e-02 | 1.95 |
| | 0.0301 | 248869 | 1.58e-02 | 2.04 | 2.90e-03 | 2.14 | 5.43e-03 | 2.26 | 2.99e-02 | 2.01 |
| | 0.0959 | 48783 | 1.57e-02 | −− | 1.71e-03 | −− | 6.20e-03 | −− | 1.80e-02 | −− |
| | 0.0732 | 82301 | 7.23e-03 | 2.86 | 7.13e-04 | 3.24 | 2.84e-03 | 2.89 | 8.25e-03 | 2.89 |
| 2 | 0.0527 | 157665 | 2.76e-03 | 2.93 | 2.57e-04 | 3.10 | 1.06e-03 | 3.01 | 3.17e-03 | 2.91 |
| | 0.0390 | 284655 | 1.14e-03 | 2.95 | 1.02e-04 | 3.09 | 4.39e-04 | 2.93 | 1.31e-03 | 2.95 |
| | 0.0301 | 477143 | 5.26e-04 | 2.98 | 4.64e-05 | 3.03 | 2.02e-04 | 2.98 | 6.03e-04 | 2.99 |

Table 6.6: Example 2, history of convergence using hexagons.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0832 | 8807 | 1.33e-01 | −− | 3.42e-02 | −− | 3.28e-02 | −− | 1.60e-01 | −− |
| | 0.0589 | 17473 | 9.34e-02 | 1.03 | 2.42e-02 | 1.00 | 2.08e-02 | 1.32 | 1.17e-01 | 0.91 |
| | 0.0471 | 27241 | 7.44e-02 | 1.02 | 1.94e-02 | 1.00 | 1.58e-02 | 1.23 | 9.61e-02 | 0.89 |
| 0 | 0.0404 | 37031 | 6.37e-02 | 1.01 | 1.66e-02 | 1.00 | 1.32e-02 | 1.18 | 8.40e-02 | 0.87 |
| | 0.0363 | 45943 | 5.71e-02 | 1.01 | 1.49e-02 | 1.00 | 1.17e-02 | 1.15 | 7.65e-02 | 0.86 |
| | 0.0329 | 55815 | 5.17e-02 | 1.01 | 1.35e-02 | 1.00 | 1.05e-02 | 1.12 | 7.04e-02 | 0.86 |
| | 0.0307 | 63849 | 4.83e-02 | 1.01 | 1.26e-02 | 1.00 | 9.70e-03 | 1.11 | 6.64e-02 | 0.85 |
| | 0.0289 | 72423 | 4.54e-02 | 1.01 | 1.19e-02 | 1.00 | 9.05e-03 | 1.10 | 6.30e-02 | 0.85 |
| | 0.0832 | 31485 | 2.03e-03 | −− | 3.93e-04 | −− | 8.89e-04 | −− | 2.97e-02 | −− |
| | 0.0589 | 62593 | 1.06e-03 | 1.89 | 1.97e-04 | 2.00 | 4.78e-04 | 1.80 | 2.36e-02 | 0.67 |
| | 0.0471 | 97681 | 6.95e-04 | 1.88 | 1.26e-04 | 2.00 | 3.21e-04 | 1.78 | 2.03e-02 | 0.67 |
| 1 | 0.0404 | 132861 | 5.21e-04 | 1.87 | 9.26e-05 | 2.00 | 2.44e-04 | 1.78 | 1.83e-02 | 0.67 |
| | 0.0363 | 164893 | 4.26e-04 | 1.86 | 7.46e-05 | 2.00 | 2.02e-04 | 1.77 | 1.70e-02 | 0.67 |
| | 0.0329 | 200381 | 3.55e-04 | 1.86 | 6.13e-05 | 2.00 | 1.70e-04 | 1.76 | 1.60e-02 | 0.67 |
| | 0.0307 | 229265 | 3.13e-04 | 1.86 | 5.36e-05 | 2.00 | 1.51e-04 | 1.76 | 1.53e-02 | 0.67 |
| | 0.0289 | 260093 | 2.79e-04 | 1.85 | 4.72e-05 | 2.00 | 1.35e-04 | 1.76 | 1.46e-02 | 0.67 |
| | 0.0832 | 64567 | 2.02e-04 | −− | 9.58e-06 | −− | 1.39e-04 | −− | 1.70e-02 | −− |
| | 0.0589 | 128449 | 1.13e-04 | 1.68 | 4.54e-06 | 2.17 | 7.81e-05 | 1.67 | 1.35e-02 | 0.67 |
| | 0.0471 | 200521 | 7.81e-05 | 1.67 | 2.80e-06 | 2.17 | 5.38e-05 | 1.67 | 1.16e-02 | 0.67 |
| 2 | 0.0404 | 272791 | 6.04e-05 | 1.67 | 2.00e-06 | 2.17 | 4.16e-05 | 1.67 | 1.05e-02 | 0.67 |
| | 0.0363 | 338599 | 5.04e-05 | 1.67 | 1.59e-06 | 2.17 | 3.47e-05 | 1.67 | 9.76e-03 | 0.67 |
| | 0.0307 | 470857 | 3.83e-05 | 1.67 | 1.11e-06 | 2.17 | 2.64e-05 | 1.67 | 8.75e-03 | 0.67 |
| | 0.0329 | 411511 | 4.28e-05 | 1.67 | 1.28e-06 | 2.17 | 2.95e-05 | 1.67 | 9.15e-03 | 0.67 |
| | 0.0289 | 534199 | 3.45e-05 | 1.67 | 9.67e-07 | 2.17 | 2.38e-05 | 1.67 | 8.38e-03 | 0.67 |

Table 6.7: Example 3, history of convergence using triangles.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0589 | 10561 | 9.49e-02 | −− | 2.94e-02 | −− | 3.13e-02 | −− | 1.31e-01 | −− |
| | 0.0404 | 22331 | 6.25e-02 | 1.11 | 2.01e-02 | 1.00 | 1.76e-02 | 1.53 | 9.45e-02 | 0.87 |
| | 0.0329 | 33627 | 5.01e-02 | 1.07 | 1.64e-02 | 1.00 | 1.30e-02 | 1.45 | 7.97e-02 | 0.83 |
| | 0.0289 | 43611 | 4.36e-02 | 1.06 | 1.44e-02 | 1.00 | 1.08e-02 | 1.40 | 7.16e-02 | 0.81 |
| | 0.0257 | 54891 | 3.87e-02 | 1.05 | 1.28e-02 | 1.00 | 9.27e-03 | 1.36 | 6.53e-02 | 0.80 |
| | 0.0236 | 65281 | 3.53e-02 | 1.04 | 1.17e-02 | 1.00 | 8.26e-03 | 1.33 | 6.10e-02 | 0.79 |
| | 0.0218 | 76571 | 3.25e-02 | 1.04 | 1.08e-02 | 1.00 | 7.44e-03 | 1.30 | 5.73e-02 | 0.78 |
| | 0.0205 | 86251 | 3.06e-02 | 1.03 | 1.02e-02 | 1.00 | 6.90e-03 | 1.28 | 5.47e-02 | 0.78 |
| 1 | 0.0589 | 34945 | 1.37e-03 | −− | 2.55e-04 | −− | 7.16e-04 | −− | 3.86e-02 | −− |
| | 0.0404 | 74061 | 6.88e-04 | 1.83 | 1.20e-04 | 2.00 | 3.76e-04 | 1.71 | 3.00e-02 | 0.67 |
| | 0.0329 | 111629 | 4.74e-04 | 1.81 | 7.92e-05 | 2.00 | 2.65e-04 | 1.70 | 2.62e-02 | 0.67 |
| | 0.0289 | 144845 | 3.74e-04 | 1.80 | 6.10e-05 | 2.00 | 2.12e-04 | 1.70 | 2.40e-02 | 0.67 |
| | 0.0257 | 182381 | 3.04e-04 | 1.80 | 4.84e-05 | 2.00 | 1.75e-04 | 1.69 | 2.22e-02 | 0.67 |
| | 0.0236 | 216961 | 2.60e-04 | 1.79 | 4.07e-05 | 2.00 | 1.51e-04 | 1.69 | 2.10e-02 | 0.67 |
| | 0.0218 | 254541 | 2.26e-04 | 1.79 | 3.46e-05 | 2.00 | 1.32e-04 | 1.69 | 1.99e-02 | 0.67 |
| | 0.0205 | 286765 | 2.03e-04 | 1.78 | 3.07e-05 | 2.00 | 1.19e-04 | 1.69 | 1.91e-02 | 0.67 |
| 2 | 0.0589 | 69697 | 2.51e-04 | −− | 7.63e-06 | −− | 1.71e-04 | −− | 2.66e-02 | −− |
| | 0.0404 | 147841 | 1.34e-04 | 1.67 | 3.37e-06 | 2.17 | 9.13e-05 | 1.67 | 2.07e-02 | 0.67 |
| | 0.0329 | 222913 | 9.50e-05 | 1.67 | 2.16e-06 | 2.17 | 6.48e-05 | 1.67 | 1.81e-02 | 0.67 |
| | 0.0289 | 289297 | 7.64e-05 | 1.67 | 1.63e-06 | 2.16 | 5.21e-05 | 1.67 | 1.65e-02 | 0.67 |
| | 0.0257 | 364321 | 6.31e-05 | 1.67 | 1.26e-06 | 2.17 | 4.30e-05 | 1.67 | 1.53e-02 | 0.67 |
| | 0.0236 | 433441 | 5.45e-05 | 1.67 | 1.05e-06 | 2.16 | 3.72e-05 | 1.67 | 1.45e-02 | 0.67 |
| | 0.0218 | 508561 | 4.77e-05 | 1.67 | 8.81e-07 | 2.17 | 3.25e-05 | 1.67 | 1.37e-02 | 0.67 |
| | 0.0205 | 572977 | 4.32e-05 | 1.67 | 7.74e-07 | 2.17 | 2.94e-05 | 1.67 | 1.32e-02 | 0.67 |

Table 6.8: Example 3, history of convergence using quadrilaterals.


[8] L. Beirão da Veiga, C. Lovadina and G. Vacca, *Divergence free virtual elements for the Stokes problem on polygonal meshes.* ESAIM Math. Model. Numer. Anal. 51 (2017), no. 2, 509–535.

[9] L. Beirão da Veiga, C. Lovadina and G. Vacca, *Virtual elements for the Navier-Stokes problem on polygonal meshes.* SIAM J. Numer. Anal. 56 (2018), no. 3, 1210–1242.

[10] L. Beirão da Veiga and L. Mascotto, *Interpolation and stability properties of low order face and edge virtual element spaces.* arXiv:2011.12834, (2020).

[11] L. Beirão da Veiga, D. Mora and G. Vacca, *The Stokes complex for virtual elements with application to Navier-Stokes flows.* J. Sci. Comput. 81 (2019), no. 2, 990–1018.

[12] G.A. Benavides, S. Caucao, G.N. Gatica and A. Hopper, *A Banach spaces-based analysis of a new mixed-primal finite element method for a coupled flow-transport problem.* Comput. Methods Appl. Mech. Engrg. 371 (2020), 113285, 29 pp.

[13] C. Bernardi, C. Canuto and Y. Maday, *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem.* SIAM J. Numer. Anal. 25 (1988), no. 6, 1237–1271.

[14] F. Brezzi, R.S. Falk and L.D. Marini, *Basic principles of mixed virtual element methods.* ESAIM Math. Model. Numer. Anal. 48 (2014), no. 4, 1227–1240.

| $k$ | $h$ | $N$ | $\mathtt{e}(\boldsymbol{\sigma})$ | $\mathtt{r}(\boldsymbol{\sigma})$ | $\mathtt{e}(\mathbf{u})$ | $\mathtt{r}(\mathbf{u})$ | $\mathtt{e}(p)$ | $\mathtt{r}(p)$ | $\mathtt{e}(\boldsymbol{\sigma}^\star)$ | $\mathtt{r}(\boldsymbol{\sigma}^\star)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0621 | 10867 | 1.01e-01 | −− | 3.50e-02 | −− | 2.57e-02 | −− | 1.48e-01 | −− |
| | 0.0438 | 21501 | 7.04e-02 | 1.03 | 2.48e-02 | 0.99 | 1.66e-02 | 1.26 | 1.08e-01 | 0.91 |
| | 0.0361 | 31523 | 5.78e-02 | 1.01 | 2.05e-02 | 0.98 | 1.32e-02 | 1.17 | 8.93e-02 | 0.96 |
| 0 | 0.0311 | 42451 | 4.95e-02 | 1.04 | 1.76e-02 | 1.00 | 1.10e-02 | 1.24 | 7.95e-02 | 0.78 |
| | 0.0280 | 52193 | 4.46e-02 | 0.99 | 1.59e-02 | 0.98 | 9.76e-03 | 1.10 | 7.32e-02 | 0.79 |
| | 0.0257 | 61723 | 4.09e-02 | 1.04 | 1.46e-02 | 1.00 | 8.82e-03 | 1.21 | 6.86e-02 | 0.77 |
| | 0.0238 | 71853 | 3.78e-02 | 1.01 | 1.35e-02 | 0.99 | 8.05e-03 | 1.18 | 6.46e-02 | 0.78 |
| | 0.0221 | 82869 | 3.52e-02 | 1.00 | 1.26e-02 | 0.99 | 7.46e-03 | 1.07 | 6.11e-02 | 0.77 |
| | 0.0621 | 32597 | 1.77e-03 | −− | 4.14e-04 | −− | 8.80e-04 | −− | 4.83e-02 | −− |
| | 0.0438 | 64497 | 9.28e-04 | 1.86 | 2.07e-04 | 1.99 | 4.76e-04 | 1.76 | 3.84e-02 | 0.66 |
| | 0.0361 | 94629 | 6.45e-04 | 1.86 | 1.39e-04 | 2.03 | 3.38e-04 | 1.75 | 3.36e-02 | 0.68 |
| 1 | 0.0311 | 127349 | 4.88e-04 | 1.86 | 1.04e-04 | 1.95 | 2.59e-04 | 1.78 | 3.04e-02 | 0.66 |
| | 0.0280 | 156657 | 4.01e-04 | 1.86 | 8.38e-05 | 2.04 | 2.16e-04 | 1.74 | 2.84e-02 | 0.65 |
| | 0.0257 | 185165 | 3.44e-04 | 1.84 | 7.09e-05 | 2.00 | 1.87e-04 | 1.73 | 2.69e-02 | 0.66 |
| | 0.0238 | 215649 | 2.97e-04 | 1.87 | 6.07e-05 | 2.00 | 1.62e-04 | 1.81 | 2.55e-02 | 0.70 |
| | 0.0221 | 248705 | 2.61e-04 | 1.82 | 5.26e-05 | 2.01 | 1.44e-04 | 1.68 | 2.43e-02 | 0.66 |
| | 0.0621 | 62475 | 2.95e-04 | −− | 5.28e-06 | −− | 2.03e-04 | −− | 3.77e-02 | −− |
| | 0.0438 | 123615 | 1.64e-04 | 1.68 | 2.48e-06 | 2.17 | 1.13e-04 | 1.67 | 2.99e-02 | 0.67 |
| | 0.0361 | 181423 | 1.18e-04 | 1.68 | 1.63e-06 | 2.17 | 8.19e-05 | 1.67 | 2.62e-02 | 0.67 |
| 2 | 0.0311 | 244083 | 9.22e-05 | 1.67 | 1.18e-06 | 2.17 | 6.39e-05 | 1.67 | 2.37e-02 | 0.67 |
| | 0.0280 | 300325 | 7.74e-05 | 1.67 | 9.36e-07 | 2.17 | 5.36e-05 | 1.67 | 2.21e-02 | 0.67 |
| | 0.0257 | 354897 | 6.73e-05 | 1.67 | 7.81e-07 | 2.17 | 4.66e-05 | 1.67 | 2.09e-02 | 0.67 |
| | 0.0238 | 413403 | 5.91e-05 | 1.67 | 6.60e-07 | 2.17 | 4.09e-05 | 1.67 | 1.99e-02 | 0.67 |
| | 0.0221 | 476767 | 5.25e-05 | 1.67 | 5.65e-07 | 2.17 | 3.63e-05 | 1.67 | 1.89e-02 | 0.67 |

Table 6.9: Example 3, history of convergence using hexagons.


[15] E. Cáceres and G.N. Gatica, *A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem.* IMA J. Numer. Anal. 37 (2017), no. 1, 296–331.

[16] E. Cáceres, G.N. Gatica and F.A. Sequeira, *A mixed virtual element method for the Brinkman problem.* Math. Models Methods Appl. Sci., 27 (2017), no. 4, 707–743.

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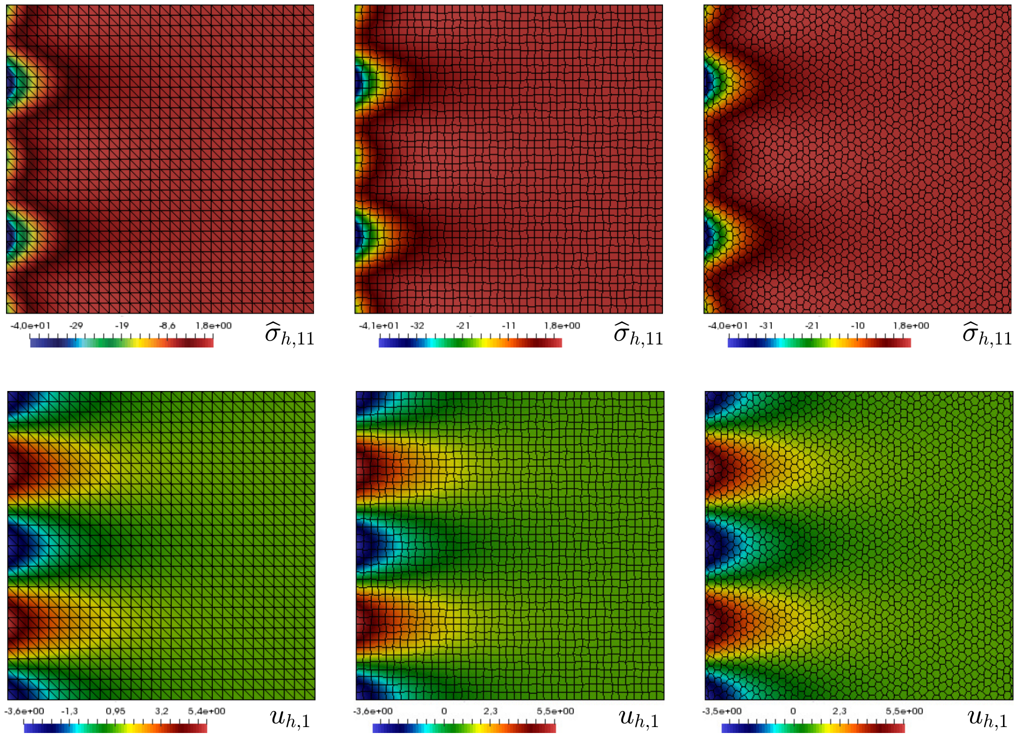


Figure 6.1: Example 2, $\widehat{\sigma}_{h,11}$ (top), and $u_{h,1}$ (bottom), using $k = 2$ and the second mesh of each kind (columns).

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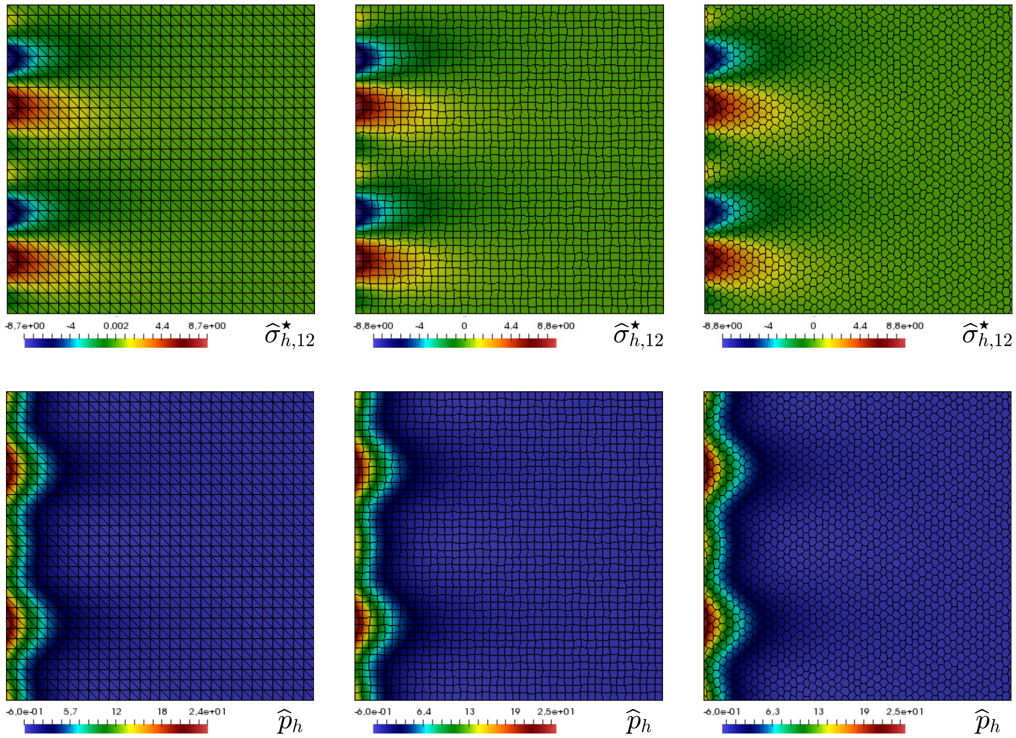


Figure 6.2: Example 2, $\sigma^{\star}_{h,12}$ (top) and $p_h$ (bottom), using $k = 2$ and the second mesh of each kind (columns).

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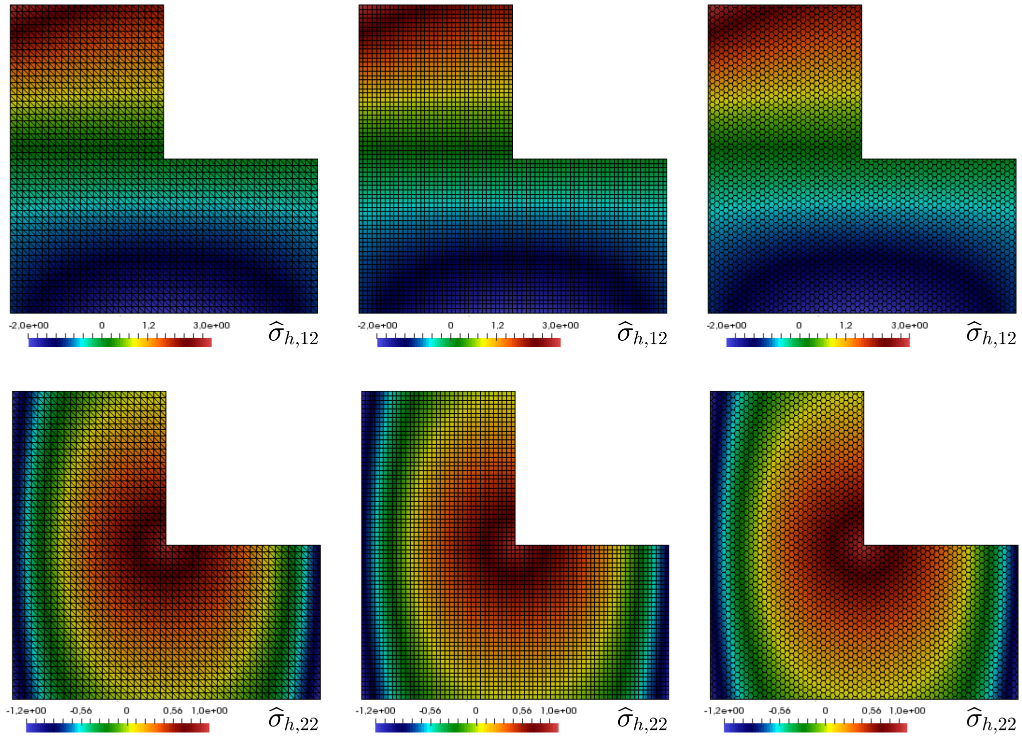


Figure 6.3: Example 3, $\widehat{\sigma}_{h,12}$ (top) and $\widehat{\sigma}_{h,22}$ (bottom), using $k = 2$ and the second mesh of each kind (columns).


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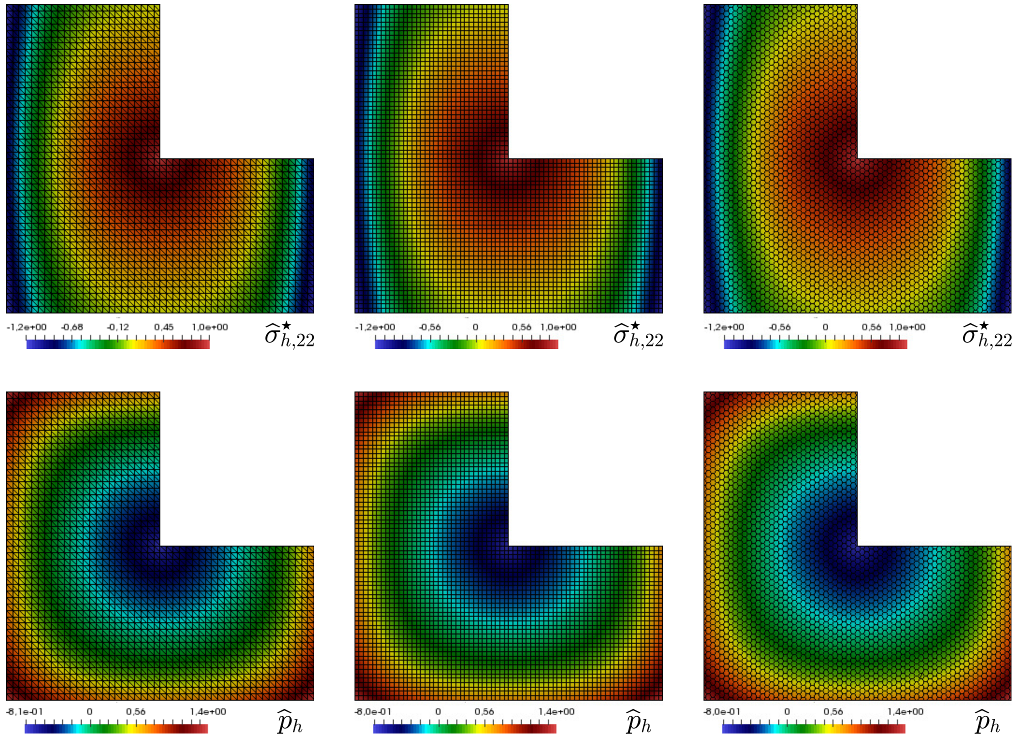


Figure 6.4: Example 3, $\sigma^{\star}_{h,22}$ (top) and $\widehat{p}_h$ (bottom), using $k = 2$ and the second mesh of each kind (columns).

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