Error analysis of pressure reconstruction from discrete velocities

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Abstract

Magnetic Resonance Imaging allows to measure the three-dimensional velocity field of blood flows. Therefore, several methods have been proposed to reconstruct the pressure field from such measurements using the incompressible Navier-Stokes equations. However, those measurements are obtained at limited spatial resolution given by the voxel dimensions in the image. Therefore, the velocity entering to the right-hand-side corresponds to a piecewise linear interpolation of the exact velocity.
In this work we propose a strategy for convergence analysis of state-of-the-art pressure reconstruction methods. We show that many terms of different convergence order appear. However, numerical results show that linear order terms dominate, even when increasing the polynomial degree of the pressure.

1. Introduction

The pressure difference is an important criteria for the severity diagnosis of blood flow obstruction. The gold standard in clinical practice is invasive catheterization. Given that there are recommendations to avoid its use [19], to compute the pressure difference from measured flow fields is strongly preferred.
Time-resolved 3D velocity encoded magnetic resonance imaging, or *4D flow MRI*, offers measuring the complete 3D velocity field within a region of interest [10, 16]. The measured velocities can then be inserted in the linear momentum balance of the incompressible Navier-Stokes equations (NSE) and the velocity terms laid in the right-hand-side while the pressure holds as an unknown, i.e.,for a given measurement of the velocity $\boldsymbol{u}$, the pressure gradient ∇p is found by solving:

$$\nabla p = -\boldsymbol{f}_{\boldsymbol{u}} \text{ in } \Omega \tag{1}$$

with $\Omega \subset \mathbb{R}^d$ and $\boldsymbol{f}_{\boldsymbol{u}} := (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u}$., where the proper function spaces will be defined for each of the methods throughout the article.
In practice, those measurements are obtained at limited spatial resolution –given by the voxel size in the image – and therefore the velocity entering to the right-hand-side corresponds to an interpolated version of the exact velocity. Therefore, there is not a unique numerical approach to compute the reconstructed pressures. A review and preliminary numerical comparison of methods can be found in [2]. Among those methods, only a few can compute pressure fields and not just averaged pressure differences between two locations.
The first one is the so-called Pressure Poisson Estimator (PPE) [4, 15] and it consists of applying the divergence to the NSE obtaining a pressure Poisson equation, similarly as it is used in projection methods [7]. However, the original PPE method cannot include the viscous contribution to the pressure gradient at the level of accuracy of the measured data. Therefore, recently in [13] the PPE method was modified by adding a boundary term with the viscous contribution.

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Another more modern method corresponds to the Stokes Estimator (STE) was reported in [18]. The STE consists in adding to the NSE the Laplacian of an artificial incompressible velocity field with null trace leading to a linear Stokes problem for both pressure and artificial velocity fields. Such artificial velocity is supposed to be zero for perfect velocity measurements. The STE has shown more accurate results than the PPE in numerically simulated data [18, 2] and in real phantom and patient data [11]. However, the STE method is considerably more expensive computationally than the PPE.
To the best of the authors' knowledge, neither a mathematical convergence analysis of both PPE and STE methods or a comparison among discretization schemes for each of the methods has been reported.
Therefore, the purpose of this work is to propose a strategy for performing a priori error analysis and applied it to the PPE and STE methods. The strategy is based on the splitting of the solution in two components and adding their contributions to the overall error. Moreover, for both methods we studied different discretization strategies in order to verify the theoretical analysis and give insights on the cost-effectiveness of each approach.
The remainder of this work is organized as follows. In Section 2 we present and analyze the PPE method in the standard and modified variants using Continuos Galerkin approaches. Section 3 introduces the STE and analyzes the classical Taylor-Hood and a tailored PSPG discretization. Then, in Section 4 we show numerical results using three known analytical solutions for the NSE, confirming the a priori error analysis. Finally, in Section 5 we draw some conclusions.

2. The Poisson Pressure Estimator

2.1. The continuous problem

The Poisson Pressure Estimator (PPE) consists in obtaining the pressure from the classical Navier-Stokes equation by mean a Poisson equation. That is, by applying the divergence operator on equation (1) one gets

$$\begin{cases} -\Delta q &= \nabla \cdot \boldsymbol{f}_{\boldsymbol{u}}, \quad \text{in } \Omega \\ -\frac{\partial q}{\partial \boldsymbol{n}} &= \boldsymbol{f}_{\boldsymbol{u}} \cdot \boldsymbol{n}, \quad \text{on } \partial\Omega \\ \int_\Omega q\, dx &= 0, \end{cases} \tag{2}$$

with $\boldsymbol{n}$ the outward normal vector of $\partial\Omega$.
In the sequel, we will make use of the following function spaces $\boldsymbol{H} := [H^1(\Omega)]^d, \mathbf{V} := [H_0^1(\Omega)]^d$,

$$\boldsymbol{H}_j := \begin{cases} \left\{ \boldsymbol{v} \in \boldsymbol{H} : \Delta \boldsymbol{v} \in [L^2(\Omega)]^d \right\} & \text{if } j = 1 \\ \\ \left\{ \boldsymbol{v} \in \boldsymbol{H} : \nabla \times (\nabla \times \boldsymbol{u}) \in [L^2(\Omega)]^d \right\} & \text{if } j = 2 \end{cases}$$

and

$$Q_j := \begin{cases} \left\{ r \in H^1(\Omega) : \int_\Omega r dx = 0 \right\} & \text{if } j = 1 \\ \\ \left\{ r \in H^1(\Omega) : \boldsymbol{n} \cdot \operatorname{curl} r \in L^2(\partial\Omega) \text{ and } \int_\Omega r\, dx = 0 \right\} & \text{if } j = 2 \end{cases}$$

Assuming $\boldsymbol{u} \in \boldsymbol{H}_j$, the weak formulation of (2) is given by: *Find $q \in Q_j$ such that*

$$\mathcal{A}(q, r) = F_{\boldsymbol{u}}^j(r), \quad \forall\, r \in Q_j, \tag{3}$$

where $\mathcal{A}(q, r) := (\nabla q, \nabla r)_\Omega$ and

$$F_{\boldsymbol{u}}^j(r) := -((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \nabla r)_\Omega + \delta_{1j}(\nu \Delta \boldsymbol{u}, \nabla r)_\Omega + \delta_{2j} \langle \boldsymbol{n} \times \nabla r, \nu \nabla \times \boldsymbol{u} \rangle_{\partial\Omega}, \tag{4}$$

with δ_{ij} is the Kronecker delta. We refer to Standard-PPE if $j = 1$ and Modified-PPE if $j = 2$ [13].

The uniqueness of the solution of Problem (3) follows from the Lax-Milgram lemma [5]. Indeed, the coercivity of the left-side is straightforward. The continuity of $F^j_{\boldsymbol{u}}$ is obtained thanks to the identity

$$-(\nabla r, \nu\nabla\times(\nabla\times\boldsymbol{u}))_\Omega = \langle \boldsymbol{n}\times\nabla r, \nu\nabla\times\boldsymbol{u}\rangle_{\partial\Omega}, \tag{5}$$

and then

$$|F^j_{\boldsymbol{u}}(r)| \le (\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{0,\Omega} + \delta_{1j}\nu\|\Delta\boldsymbol{u}\|_{0,\Omega} + \delta_{2j}\nu\|\nabla\times(\nabla\times\boldsymbol{u})\|_{0,\Omega})|r|_{1,\Omega}$$

2.2. Continuous Galerkin discretizations

Let us denote by $\{\mathcal{T}_h\}_h$ a regular family of triangulations of $\bar\Omega$ composed of simplex K of diameter h_K. We will denote by h the mesh size, where $h = \max\{h_K : K \in \mathcal{T}_h\}$.

The finite element spaces for the pressure and velocity approximation are:

$$\begin{aligned} Q_{jh} &:= \{q_h \in Q_j : q_h|_K \in \mathbb{P}_k(K) \quad \forall K\in\mathcal{T}_h\}, \\ \boldsymbol{H}_{jh} &:= \left\{\boldsymbol{v}_h \in \boldsymbol{H}_j : \boldsymbol{v}_h|_K \in \mathbb{P}_1(K)^d \quad \forall K\in\mathcal{T}_h\right\}, \end{aligned}$$

where $\mathbb{P}_k$ stands for the space of polynomials of total degree less or equal to an integer $k \ge 1$.
We will also consider the Lagrange interpolation operators $\mathcal{J}_h : Q_j \cap H^{k+1}(\Omega) \to Q_{jh}$ and $\mathcal{L}_h : \boldsymbol{H}_j \cap H^2(\Omega)^d \to \boldsymbol{H}_{jh}$, which satisfy:

$$\begin{aligned} |q - \mathcal{J}_h q|_{m,\Omega} &\le a_k\, h^{k+1-m}|q|_{k+1,\Omega}, \qquad \forall\, q\in H^{k+1}(\Omega),\ 0\le m\le k+1. \\ |\boldsymbol{v} - \mathcal{L}_h\boldsymbol{v}|_{m,\Omega} &\le a_k\, h^{2-m}|\boldsymbol{v}|_{2,\Omega}, \qquad \forall\, \boldsymbol{v}\in H^2(\Omega)^d,\ 0\le m\le 2. \end{aligned} \tag{6}$$

The aim is now to find and approximation of the pressure field $q_h \approx q$ from measurements of $\boldsymbol{u}$ which will be given by $\mathcal{L}_h\boldsymbol{u}$.
Thus, the Galerkin scheme associated with the continuous variational formulation (3) – and replacing $\boldsymbol{u}$ by $\mathcal{L}_h\boldsymbol{u}$ – reads as follows: *Find $q_h \in Q_{jh}$ such that*

$$\mathcal{A}(q_h, r_h) := F^j_{\boldsymbol{u}_h}(r_h) \quad \forall\, r_h \in Q_{jh}, \tag{7}$$

with

$$F^j_{\boldsymbol{u}_h}(r_h) := -((\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u}, \nabla r_h)_\Omega + \delta_{1j}(\nu\Delta\mathcal{L}_h\boldsymbol{u}, \nabla r_h)_\Omega + \delta_{2j}\langle\boldsymbol{n}\times\nabla r_h, \nu\nabla\times\mathcal{L}_h\boldsymbol{u}\rangle_{\partial\Omega}, \tag{8}$$

Lax-Milgram's lemma gives that (7) has a unique solution $q_h \in Q_{jh}$.

Remark 1. *Note that, from the definitions* (4) *and* (8)*, problem* (7) *is not a Galerkin scheme of the continuous problem* (3)*.*

The strategy to prove convergence is to use an auxiliary solution of the discrete problem, but without interpolate the velocity $\boldsymbol{u}$. Indeed, let us consider the auxiliary problem given by: *Find $\tilde q_h \in Q_{jh}$ such that*

$$\mathcal{A}(\tilde q_h, r_h) := F^j_{\boldsymbol{u}}(r_h) \quad \forall\, r_h \in Q_{jh}, \tag{9}$$

with $\boldsymbol{u}\in\boldsymbol{H}_j$.
In order to prove the convergence theorems, we need first to state the next result.

Lemma 1. *Let us assume that $\boldsymbol{u}\in H^2(\Omega)^d$ and define*

$$T(a_1, a_2, \boldsymbol{u}, h) := a_1\|\boldsymbol{u}\|_{0,\Omega} + a_2 h|\boldsymbol{u}|_{1,\Omega} + a_1\, a_2\, h^2|\boldsymbol{u}|_{2,\Omega}. \tag{10}$$

where a_1, a_2 are the error interpolation constants given in (6)*. Then,*

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega} \le h|\boldsymbol{u}|_{2,\Omega}T(a_1, a_2, \boldsymbol{u}, h). \tag{11}$$

Proof. Using (6) and Young inequality we have

$$\begin{aligned}
\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u})\|_{0,\Omega} &= \|(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u})+((\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u})\cdot\nabla)\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}\\
&\le |\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{u}\|_{0,\Omega}+|\mathcal{L}_h\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}\\
&\le |\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{u}\|_{0,\Omega}+|\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}+|\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}|_{1,\Omega}\|\boldsymbol{u}-\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}\\
&\le a_1h|\boldsymbol{u}|_{2,\Omega}\|\boldsymbol{u}\|_{0,\Omega}+|\boldsymbol{u}|_{1,\Omega}a_2h^2|\boldsymbol{u}|_{2,\Omega}+a_1h|\boldsymbol{u}|_{2,\Omega}a_2h^2|\boldsymbol{u}|_{2,\Omega}\\
&= h|\boldsymbol{u}|_{2,\Omega}\Big(a_1\|\boldsymbol{u}\|_{0,\Omega}+a_2h|\boldsymbol{u}|_{1,\Omega}+a_1a_2h^2|\boldsymbol{u}|_{2,\Omega}\Big)\\
&= h|\boldsymbol{u}|_{2,\Omega}\,T(a_1,a_2,\boldsymbol{u},h)
\end{aligned} \tag{12}$$

■

In the sequel we will use the notation $T_K(a_1,a_2,\boldsymbol{u},h)$, where the norms of $\boldsymbol{u}$ are defined on the element K.

Lemma 2. *Let q and $\tilde{q}_h$ solutions of (3) and (9), respectively, and assume that $q\in Q_j\cap H^{k+1}(\Omega)$ with $k\ge 1$. Then, there exists $a_k>0$, independent of h, such that*

$$|q-\tilde{q}_h|_{1,\Omega}\le\; a_kh^k|q|_{k+1,\Omega}. \tag{13}$$

Proof. See [5]. ■

Lemma 3. *Let q_h and $\tilde{q}_h$ in Q_{jh} be solutions of (7) and (9), respectively, and assume that $\boldsymbol{u}\in\boldsymbol{H}_j\cap H^2(\Omega)^d$ if $j=1$ and $\boldsymbol{u}\in\boldsymbol{H}_j$ with $\boldsymbol{u}\big|_{\partial\Omega}\in H^2(\partial\Omega)^d$ if $j=2$. Then,*

$$|q_h-\tilde{q}_h|_{1,\Omega}\le h\,T(a_1,a_2,\boldsymbol{u},h)|\boldsymbol{u}|_{2,\Omega}+\delta_{1,j}\nu\|\Delta\boldsymbol{u}\|_{0,\Omega}+\delta_{2,j}\tilde{C}\,\nu\,h^{1/2}|\boldsymbol{u}|_{2,\partial\Omega},$$

where $T(a_1,a_2,\boldsymbol{u},h)$ is given as in Lemma 1.

Proof. We consider the coercivity of the bilinear form $\mathcal{A}$ and the Cauchy-Schwarz inequality. Then,

$$\begin{aligned}
|q_h-\tilde{q}_h|_{1,\Omega} &\le \sup_{\substack{r_h\in Q_{jh}\\ r_h\neq 0}}\frac{|\mathcal{A}(q_h-\tilde{q}_h,r_h)|}{|r_h|_{1,\Omega}}\\
&= \sup_{\substack{r_h\in Q_{jh}\\ r_h\neq 0}}\frac{|F^j_{\boldsymbol{u}}(r_h)-F^k_{\boldsymbol{u}_h}(r_h)|}{|r_h|_{1,\Omega}}\\
&\le \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\mathcal{I}_h\boldsymbol{u}\cdot\nabla)\mathcal{I}_h\boldsymbol{u}\|_{0,\Omega}+\delta_{1,j}\|\Delta\boldsymbol{u}\|_{0,\Omega}+\delta_{2,j}\sup_{\substack{r_h\in Q_{jh}\\ r_h\neq 0}}\frac{|\langle\boldsymbol{n}\times\nabla r_h,\nu\,\nabla\times(u-\mathcal{I}_h\boldsymbol{u})\rangle_{\partial\Omega}|}{|r_h|_{1,\Omega}}.
\end{aligned}$$

To bound the first term in above inequality we use Lemma 1, and for the third term we have

$$\frac{|\langle\boldsymbol{n}\times\nabla r_h,\nu\,\nabla\times(u-\mathcal{I}_h\boldsymbol{u})\rangle_{\partial\Omega}|}{|r_h|_{1,\Omega}}\le\frac{\nu\,\|\nabla r_h\|_{0,\partial\Omega}\|\nabla\times(u-\mathcal{I}_h\boldsymbol{u})\|_{0,\partial\Omega}}{|r_h|_{1,\Omega}} \tag{14}$$

Now, thanks to [3, Lemma 1.46], we have

$$\|\nabla r_h\|_{0,\partial\Omega}\le C_{tr}h^{-1/2}|r_h|_{1,\Omega}, \tag{15}$$

with $C_{tr}:=\left(\dfrac{(k+1)(k+2)}{2}\right)^{1/2}$ (see [20, Theorem 3]). Besides, from [17, Lemma 10.8], we get

$$\|\nabla\times(\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u})\|_{0,\partial\Omega}\le|\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}|_{1,\partial\Omega}\le a_{2,I}\,h\,|\boldsymbol{u}|_{2,\partial\Omega}. \tag{16}$$

thus, from (14), (15) and (16), we arrive at

$$\frac{|\langle \boldsymbol{n} \times \nabla r_h, \nu \nabla \times (u - \mathcal{I}_h \boldsymbol{u})\rangle_{\partial\Omega}|}{|r_h|_{1,\Omega}} \leq C_{tr}\, a_{2,I}\, \nu\, h^{1/2} |\boldsymbol{u}|_{2,\partial\Omega}.$$

■

Finally, the next theorem holds.

Theorem 1 (Main Result I). *Let $q \in Q_j \cap H^{k+1}(\Omega)$ and $q_h \in Q_{jh}$ solutions of (3) and (7), respectively. In addition, assume that $\boldsymbol{u} \in \boldsymbol{H}_j \cap H^2(\Omega)^d$ and $\boldsymbol{u} \in \boldsymbol{H}_j$ with $\boldsymbol{u}\big|_{\partial\Omega} \in H^2(\partial\Omega)^d$, for $j = 1$ and $j = 2$, respectively. Then,*

$$|q - q_h|_{1,\Omega} \leq\; a_k h^k |q|_{k+1,\Omega} + \delta_{1,j} \nu \|\boldsymbol{\Delta u}\|_{0,\Omega} + \delta_{2,j}\, C_{tr}\, a_{2,I}\, \nu\, h^{1/2} |\boldsymbol{u}|_{2,\partial\Omega} + hT(a_1, a_2, \boldsymbol{u}, h)|\boldsymbol{u}|_{2,\Omega}.$$

with $k \geq 1$.

Proof. It follows directly from Lemmas 2 and 3. ■

Corollary 1. *Assuming that the hypothesis of Theorem 1 hold. Then,*

$$\|q - q_h\|_{0,\Omega} \leq a_k\, C_4\, C_p\, h^{k+1}\, |q|_{k+1,\Omega} + C_p\, \delta_{1,j}\, \nu \|\boldsymbol{\Delta u}\|_{0,\Omega} + \delta_{2,j} C_p C_{tr}\, a_{2,I}\, \nu\, h^{1/2} |\boldsymbol{u}|_{2,\partial\Omega} + C_p h T(a_1, a_2, \boldsymbol{u}, h)|\boldsymbol{u}|_{2,\Omega},$$

with $\tilde{q}_h$ solution of (9) and $k \geq 1$.

Proof. The proof consists in to apply Poincaré's inequality [1, Theorem 3.2] to obtain

$$\|q - \tilde{q}_h\|_{0,\Omega} \leq C_p |q - \tilde{q}_h|_{1,\Omega}.$$

and to solve the elliptic problem

$$-\Delta r = q - \tilde{q}_h,$$

with the zero Dirichlet boundary conditon, and apply Aubin-Nitsche Lemma [5, Lemma 2.31] and Theorem 3. ■

3. The Stokes Estimator

3.1. The continuous problem

The STE method consists in adding the Laplacian of an incompressible auxiliary velocity $\boldsymbol{w} \in \mathbf{V}$ to the left-hand side of (1):

$$\begin{aligned} -\Delta \boldsymbol{w} + \nabla q &= -\boldsymbol{f}_{\boldsymbol{u}} \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{w} &= 0 \quad \text{in } \Omega. \end{aligned} \tag{17}$$

Let us define the space $P := L_0^2(\Omega)$. The weak problem associated to (17) is given by: *Find $(\boldsymbol{w}, q) \in \mathbf{V} \times P$ such that*

$$\mathcal{B}((\boldsymbol{w}, q), (\boldsymbol{v}, r)) = \mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}, r) \quad \forall (\boldsymbol{v}, r) \in \mathbf{V} \times P, \tag{18}$$

where

$$\mathcal{B}((\boldsymbol{w}, q), (\boldsymbol{v}, r)) := (\nabla \boldsymbol{w}, \nabla \boldsymbol{v})_\Omega - (q, \nabla \cdot \boldsymbol{v})_\Omega + (r, \nabla \cdot \boldsymbol{w})_\Omega \quad \text{and} \quad \mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}, r) := -((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v})_\Omega - \nu (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega. \tag{19}$$

For the analysis, we will use the following norm

$$\|(\boldsymbol{v}, r)\|_{\mathbf{V} \times P} := |\boldsymbol{v}|_{1,\Omega} + \|r\|_{0,\Omega},$$

and if F is an linear functional operator we use the norm

$$\|F\|_{(\mathbf{V}\times P)'} := \sup_{\substack{(\boldsymbol{v},r)\in\mathbf{V}\times P\\(\boldsymbol{v},r)\neq\mathbf{0}}} \frac{|F(\boldsymbol{v},r)|}{\|(\boldsymbol{v},r)\|_{\mathbf{V}\times P}} \tag{20}$$

Note that the problem (18) is well posed thanks to the $\mathbf{V}-$ellipticity of the bilinear form $(\nabla\boldsymbol{w},\nabla\boldsymbol{v})_\Omega$, and that $(q,\nabla\cdot\boldsymbol{w})_\Omega$ satisfy an inf-sup condition (cf. [5, Prop. 2.36]).

Lemma 4. *There exists a positive constant $C_{\mathcal{B}}$ such that*

$$\|\mathcal{B}\| \leq C_{\mathcal{B}}.$$

Proof. Using triangular inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|\mathcal{B}((\boldsymbol{w},q),(\boldsymbol{v},r))| &\leq |\boldsymbol{w}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega} + \sqrt{d}\|q\|_{0,\Omega}|\boldsymbol{v}|_{1,\Omega} + \sqrt{d}\|r\|_{0,\Omega}|\boldsymbol{w}|_{1,\Omega}\\
&\leq \sqrt{d}|\boldsymbol{w}|_{1,\Omega}|\boldsymbol{v}|_{1,\Omega} + \sqrt{d}\|q\|_{0,\Omega}|\boldsymbol{v}|_{1,\Omega} + \sqrt{d}\|r\|_{0,\Omega}|\boldsymbol{w}|_{1,\Omega}\\
&\leq \sqrt{d}(|\boldsymbol{w}|^2_{1,\Omega} + |\boldsymbol{w}|^2_{1,\Omega} + \|q\|_{0,\Omega})^{1/2}(|\boldsymbol{v}|^2_{1,\Omega} + |\boldsymbol{v}|^2_{1,\Omega} + \|r\|_{0,\Omega})^{1/2}\\
&\leq C_{\mathcal{B}}(|\boldsymbol{w}|^2_{1,\Omega} + \|q\|^2_{0,\Omega})^{1/2}(|\boldsymbol{v}|^2_{1,\Omega} + \|r\|^2_{0,\Omega})^{1/2}\\
&\leq C_{\mathcal{B}}\|(\boldsymbol{w},q)\|_{\mathbf{V}\times P}\|(\boldsymbol{v},r)\|_{\mathbf{V}\times P},
\end{aligned}$$

and the result is obtained directly, with $C_{\mathcal{B}} := 2\sqrt{d}$. ∎

3.2. Discrete spaces

Now, we define the following finite-dimensional spaces:

$$\begin{aligned}
\mathbf{W}_h &:= \left\{\boldsymbol{v}_h \in [H^1(\Omega)]^d : \boldsymbol{v}_h|_K \in \mathbb{P}_l(K)^d \quad \forall K \in \mathcal{T}_h\right\},\\
P_h &:= \{q_h \in P : q_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h\},\\
\boldsymbol{H}_h &:= \left\{\boldsymbol{w}_h \in \boldsymbol{H} : \boldsymbol{w}_h|_K \in \mathbb{P}_1(K)^d \quad \forall K \in \mathcal{T}_h\right\}.
\end{aligned} \tag{21}$$

For our error analysis we will need to make use of some known results.

Theorem 2. *For all $\boldsymbol{w} \in \mathbb{P}_l(K)^d$ there holds,*

$$\|\nabla\boldsymbol{w}\|_{0,K} \leq \sqrt{\kappa_l}\frac{|\partial K|}{|K|}\|\boldsymbol{w}\|_{0,K}. \tag{22}$$

For $d = 2$, it holds $\kappa_1 = 6$ and $\kappa_2 = \dfrac{45}{2}$.

Proof. See [12, Theorem 2] ∎

Corollary 2. *Assuming in Theorem 2 that $d = 2$ and $l = 1$, there holds*

$$\|\nabla\boldsymbol{w}\|_{0,K} \leq C_I h_K^{-1}\|\boldsymbol{w}\|_{0,K}, \tag{23}$$

where $C_I = \dfrac{6\sqrt{\kappa_1}}{\sin^2(\theta)}$, being θ the minimun angle of the element K.

Proof. The proof is a direct consequence of the minimum angle condition and Theorem 2. ∎

3.3. Taylor–Hood discretization

For the discrete STE we set $\mathbf{V}_h = \mathbf{W}_h \cap \mathbf{V}$. Note that the inf–sup conditions requires the use of different discrete spaces to approximate the velocity and the pressure. In our case, we will use the so-called Taylor-Hood spaces, where $l = k + 1$. Otherwise, it is not possible to use conforming spaces of lowest order for the discrete velocity.
Let us consider the property of interpolation operator $\mathcal{I}_h : \mathbf{V} \cap [H^{k+1}(\Omega)]^d \to \mathbf{V}_h$:

$$|\boldsymbol{w} - \mathcal{I}_h \boldsymbol{w}|_{m,\Omega} \leq a_k\, h^{k+1-m} |\boldsymbol{w}|_{k+1,\Omega}, \qquad \forall \boldsymbol{w} \in [H^{k+1}(\Omega)]^d,\ 0 \leq m \leq k+1, \tag{24}$$

Thereby, the discrete version of the problem (18) reads as follows: *Find* $(\boldsymbol{w}_h, q_h) \in \mathbf{V}_h \times P_h$ *such that*

$$\mathcal{B}((\boldsymbol{w}_h, q_h), (\boldsymbol{v}_h, r_h)) = \mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h) \qquad \forall (\boldsymbol{v}_h, r_h) \in \mathbf{V}_h \times P_h, \tag{25}$$

where the bilinear form $\mathcal{B}$ is like in the continuous case, and

$$\mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h) := -((\mathcal{L}_h \boldsymbol{u} \cdot \nabla)\mathcal{L}_h \boldsymbol{u}, \boldsymbol{v}_h)_\Omega - \nu(\nabla \mathcal{L}_h \boldsymbol{u}, \nabla \boldsymbol{v}_h)_\Omega, \tag{26}$$

with $\mathcal{L}_h : [H^2(\Omega)]^d \longrightarrow \boldsymbol{H}_h$ a Lagrange interpolant.

Lemma 5. *There exists a constant $\beta_1 > 0$, independent of h, such that*

$$\sup_{\substack{(\boldsymbol{v}_h, r_h) \in \mathbf{V}_h \times P_h \\ (\boldsymbol{v}_h, r_h) \neq \mathbf{0}}} \frac{\mathcal{B}((\boldsymbol{w}_h, q_h), (\boldsymbol{v}_h, r_h))}{\|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V} \times P}} \geq \beta_1 \|(\boldsymbol{w}_h, q_h)\|_{\mathbf{V} \times P}, \qquad \forall\ (\boldsymbol{w}_h, q_h)\ \in\ \mathbf{V}_h \times P_h.$$

Proof. See equation [6, (1.39)] and [6, Corollary 4.1] ■
For the next result, we consider the pair $(\tilde{\boldsymbol{w}}_h, \tilde{q}_h) \in \mathbf{V}_h \times P_h$, such that

$$\mathcal{B}((\tilde{\boldsymbol{w}}_h, \tilde{q}_h), (\boldsymbol{v}_h, r_h)) = \mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h, r_h) \qquad \forall (\boldsymbol{v}_h, r_h) \in \mathbf{V}_h \times P_h, \tag{27}$$

where $\mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h, r_h) := -((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}_h)_{0,\Omega} - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h)_\Omega$, with the continuous velocity $\boldsymbol{u}$. Let us recall the following convergence result.

Lemma 6. *Let $(\boldsymbol{w}, q)$ and $(\tilde{\boldsymbol{w}}_h, \tilde{q}_h)$ solutions of (18) and (27), respectively. Assume that $(\boldsymbol{w}, q) \in [H_0^1(\Omega) \cap H^{k+1}(\Omega)]^d \times [L_0^2(\Omega) \cap H^k(\Omega)]$, with $k \geq 1$. Then, there exists $C > 0$, independent of h, such that*

$$\|(\boldsymbol{w} - \tilde{\boldsymbol{w}}_h, q - \tilde{q}_h)\|_{\mathbf{V} \times P} \leq C_1 C_2 h^k \big(|\boldsymbol{w}|_{k+1,\Omega} + |q|_{k,\Omega}\big),$$

with $C_2 = 1 + \dfrac{C_{\mathcal{B}}}{\beta_1}$ and β_1 being the constant given in Lemma 5.

Proof. See [5, Lemma 2.44]

■

In order to show the convergence of q_h (Main Result I, see later Theorem 3), we set the following Lemma.

Lemma 7. *Let $(\boldsymbol{w}_h, q_h), (\tilde{\boldsymbol{w}}_h, \tilde{q}_h) \in \mathbf{V}_h \times P_h$, solutions of (25) and (27) respectively, and β_1 the constant given in Lemma 5. Then,*

$$\|(\tilde{\boldsymbol{w}}_h - \boldsymbol{w}_h, \tilde{q}_h - q_h)\|_{\mathbf{V} \times P} \leq \beta_1^{-1} \|\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h}\|_{(\mathbf{V} \times P)'}$$

with $(\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h})(\boldsymbol{v}_h, r_h) := -((\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\mathcal{L}_h \boldsymbol{u} \cdot \nabla)\mathcal{L}_h \boldsymbol{u}, \boldsymbol{v}_h)_\Omega - \nu(\nabla \boldsymbol{u} - \nabla \mathcal{L}_h \boldsymbol{u}, \nabla \boldsymbol{v}_h)_\Omega.$

Proof. By Lemma 5 together with the Cauchy - Schwarz inequality, we arrive to the inequality

$$\begin{aligned}
\beta_1 \|(\tilde{\boldsymbol{w}}_h - \boldsymbol{w}_h, \tilde{q}_h - q_h)\|_{\mathbf{V}\times P} &\leq \sup_{\substack{(\boldsymbol{v}_h, r_h)\in \mathbf{V}_h\times P_h \\ (\boldsymbol{v}_h, r_h)\neq \mathbf{0}}} \frac{\mathcal{B}((\tilde{\boldsymbol{w}}_h - \boldsymbol{w}_h, \tilde{q}_h - q_h), (\boldsymbol{v}_h, r_h))}{\|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P}} \\
&= \sup_{\substack{(\boldsymbol{v}_h, r_h)\in \mathbf{V}_h\times P_h \\ (\boldsymbol{v}_h, r_h)\neq \mathbf{0}}} \frac{\mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h, r_h) - \mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h)}{\|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P}} \\
&= \sup_{\substack{(\boldsymbol{v}_h, r_h)\in \mathbf{V}_h\times P_h \\ (\boldsymbol{v}_h, r_h)\neq \mathbf{0}}} \frac{(\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h})(\boldsymbol{v}_h, r_h)}{\|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P}} \\
&\leq \|\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h}\|_{(\mathbf{V}\times P)'}.
\end{aligned}$$

■

Lemma 8. *Let $\mathcal{G}_{\boldsymbol{u}}$ and $\mathcal{G}_{\boldsymbol{u}_h}$ be as in* (19) *and* (26) *respectively and denote by $(\mathbf{V}\times P)'$ the dual space of the product space $\mathbf{V}\times P$. In addition, we assume that $\boldsymbol{u}\in [H^2(\Omega)]^d$. Then*

$$\|\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h}\|_{(\mathbf{V}\times P)'} \leq h|\boldsymbol{u}|_{2,\Omega}\Big[C_p T(a_1, a_2, \boldsymbol{u}, h) + \nu\, a_1\Big].$$

Proof.

$$\|\mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h, r_h) - \mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h)\|_{0,\Omega} \leq \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u})\|_{0,\Omega}\|\boldsymbol{v}_h\|_{0,\Omega} + \nu\|\nabla\boldsymbol{u} - \nabla\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{v}_h\|_{0,\Omega}.$$

For the first term of the right-hand side, we use Lemma 1.
For the second term of the right-hand-side, we have from the interpolation bounds:

$$\|\nabla(\boldsymbol{u} - \mathcal{L}_h\boldsymbol{u})\|_{0,\Omega} \leq a_1 h|\boldsymbol{u}|_{2,\Omega}.$$

Finally, using the above inequalities and Poincaré inequality we get

$$\begin{aligned}
\|(\mathcal{G}_{\boldsymbol{u}} - \mathcal{G}_{\boldsymbol{u}_h})(\boldsymbol{v}_h, r_h)\|_{0,\Omega} &\leq h|\boldsymbol{u}|_{2,\Omega}\Big[C_p\, T(a_1, a_2, \boldsymbol{u}, h) + \nu\, a_1\Big]|\boldsymbol{v}_h|_{1,\Omega} \\
&\leq h|\boldsymbol{u}|_{2,\Omega}\Big[C_p\, T(a_1, a_2, \boldsymbol{u}, h) + \nu\, a_1\Big]\|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P},
\end{aligned}$$

Thereby, we arrive straight to the result of the lemma.

■

Finally, we can derive the first main convergence result.

Theorem 3 (Main Result II). *Let $(\boldsymbol{w}_h, q_h), (\tilde{\boldsymbol{w}}_h, \tilde{q}_h)\in \mathbf{V}_h\times P_h$, solutions of* (25) *and* (27) *respectively and $\boldsymbol{u}$ the exact velocity. Also, assume that $(\boldsymbol{w}, q)\in [H_0^1(\Omega)\cap H^{k+1}(\Omega)]^d\times[L_0^2(\Omega)\cap H^k(\Omega)]$ and $\boldsymbol{u}\in [H^2(\Omega)]^d$. Then,*

$$|\boldsymbol{w} - \boldsymbol{w}_h|_{1,\Omega} + \|q - q_h\|_{0,\Omega} \leq C_1 C_2 h^k\big(|\boldsymbol{w}|_{k+1,\Omega} + |q|_{k,\Omega}\big) + \beta_1^{-1} h|\boldsymbol{u}|_{2,\Omega}\Big(\rho\, C_p\, T(a_1, a_2, \boldsymbol{u}, h) + \mu\, a_1\Big).$$

Proof. The proof follows from Lemmas 7, 8 and 10. ■

3.4. Stabilized PSPG discretization

Let us consider again the Stokes problem given in (17) and its respective variational formulation (18). We will now analyze the PSPG Stabilization [8] with the end of comparing the error of convergence between the pressure obtained with both schemes.
Let us now consider the spaces

$$\tilde{P} := \left\{ r \in L_0^2(\Omega) : r|_K \in H^1(K) \right\}$$
$$\tilde{P}_h := \left\{ q_h \in \tilde{P} : q_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\},$$

and to use the same space for the auxiliar velocity $\mathbf{W}_h$ defined in (21).
The goal now is to use the same polynomial order for both unknowns, i.e., we wil hand $k = l$. The stabilized formulation that we will use is given by

$$\mathcal{B}^s((\boldsymbol{w}_h, q_h)(\boldsymbol{v}_h, r_h)) = \mathcal{G}^s_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h) \tag{28}$$

where

$$\mathcal{B}^s((\boldsymbol{w}_h, q_h)(\boldsymbol{v}_h, r_h)) := \mathcal{B}((\boldsymbol{w}_h, q_h)(\boldsymbol{v}_h, r_h)) + \sum_{K \in \mathcal{T}_h} \delta\, h_K^2 (\nabla q_h, \nabla r_h)_K$$
$$\mathcal{G}^s_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h) := \mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h, r_h) + \sum_{K \in \mathcal{T}_h} \delta\, h_K^2 (-\boldsymbol{f}_{\boldsymbol{u}_h}, \nabla r_h)_K$$

with $\mathcal{B}((\cdot,\cdot),(\cdot,\cdot))$ and $\mathcal{G}_{\boldsymbol{u}_h}(\cdot,\cdot)$ defined as in (26), and

$$\boldsymbol{f}_{\boldsymbol{u}_h} := (\mathcal{L}_h \boldsymbol{u} \cdot \nabla)\mathcal{L}_h \boldsymbol{u}. \tag{29}$$

Remark 2. *Note that the term $\Delta \boldsymbol{w}_h$ is not included in the stabilization. This is possible to do while keeping strong consistency since $\boldsymbol{w} = \mathbf{0}$. Our choice allows also to avoid conditional well-posedness of the discrete solution as in standard PSPG stabilized formulations.*

Let us define the mesh-dependent norm on the product space $\mathbf{V} \times \tilde{P}$

$$\|(\boldsymbol{v}, r)\|_h^2 := \mathcal{B}^s((\boldsymbol{v}, r), (\boldsymbol{v}, r)) = \|\nabla \boldsymbol{v}\|_{0,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \delta h_K^2 \|\nabla r\|_{0,K}^2. \tag{30}$$

Remark 3. *It is possible to prove that $\|(\boldsymbol{v}_h, r_h)\|_h \le C_{eq}\, \|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P}$ for all $(\boldsymbol{v}_h, r_h) \in \mathbf{V}_h \times \tilde{P}_h$. Indeed, applying the inequality (23) and the previous assumptions we get*

$$\|(\boldsymbol{v}_h, r_h)\|_h^2 := \|\nabla \boldsymbol{v}_h\|_{0,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \delta h_K^2 \|\nabla r_h\|_{0,K}^2 \le \|\nabla \boldsymbol{v}_h\|_{0,\Omega}^2 + \delta C_I^2 \|r_h\|_{0,\Omega}^2 \le \max\left\{1, \delta C_I^2\right\} (\|\nabla \boldsymbol{v}_h\|_{0,\Omega}^2 + \|r_h\|_{0,\Omega}^2),$$

and then,

$$\|(\boldsymbol{v}_h, r_h)\|_h \le C_{eq} \|(\boldsymbol{v}_h, r_h)\|_{\mathbf{V}\times P}, \tag{31}$$

with

$$C_{eq} := \left[\max\left\{1, \delta C_I^2\right\} \right]^{1/2}$$

Lemma 9.

$$\|\mathcal{B}^s\| \le C_{\mathcal{B}^s} := \max\left\{ C_{\mathcal{B}}, \sqrt{\delta} C_{eq}\, C_I \right\} \tag{32}$$

Proof. Using the inequalities (23) and (31), Theorem 4 and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
|\mathcal{B}^s((\boldsymbol{w}_h,q_h),(\boldsymbol{v}_h,r_h))| &\le \|\mathcal{B}\| \|(\boldsymbol{w}_h,q_h)\|_{\mathbf{V}\times P}\|(\boldsymbol{v}_h,r_h)\|_{\mathbf{V}\times P} + \sum_{K\in\mathcal{T}_h} \delta h_K^2 \|\nabla q_h\|_{0,K}\|\nabla r_h\|_{0,K} \\
&\le C_{\mathcal{B}}\|(\boldsymbol{w}_h,q_h)\|_{\mathbf{V}\times P}\|(\boldsymbol{v}_h,r_h)\|_{\mathbf{V}\times P} + \left(\sum_{K\in\mathcal{T}_h} \delta h_K^2 \|\nabla q_h\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} \delta h_K^2 \|\nabla r_h\|_{0,K}^2\right)^{1/2} \\
&\le C_{\mathcal{B}}\|(\boldsymbol{w}_h,q_h)\|_{\mathbf{V}\times P}\|(\boldsymbol{v}_h,r_h)\|_{\mathbf{V}\times P} + \left(\sum_{K\in\mathcal{T}_h} \delta C_I^2 \|q_h\|_{0,K}^2\right)^{1/2} \|(\boldsymbol{v}_h,r_h)\|_h \\
&\le C_{\mathcal{B}^s}\|(\boldsymbol{w}_h,q_h)\|_{\mathbf{V}\times P}\|(\boldsymbol{v}_h,r_h)\|_{\mathbf{V}\times P},
\end{aligned}$$

and then the result follows. ∎

In the next lemmas we will consider the pair $(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)\in\mathbf{V}_h\times\tilde{P}_h$ which is the solution of

$$\mathcal{B}^s((\boldsymbol{w}_h,q_h)(\boldsymbol{v}_h,r_h)) = \mathcal{G}^s_{\boldsymbol{u}}(\boldsymbol{v}_h,r_h), \qquad \forall\ (\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times\tilde{P}_h \tag{33}$$

where

$$\mathcal{G}^s_{\boldsymbol{u}}(\boldsymbol{v}_h,r_h) := \mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h,r_h) + \sum_{K\in\mathcal{T}_h} \delta\, h_K^2(\boldsymbol{f}_{\boldsymbol{u}},\nabla r_h)_K.$$

We highlight that the solvability of the problem (33) has been guaranteed in [8].

Lemma 10. *Let $(\boldsymbol{w},q)$ and $(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)$ solutions of (18) and (33), respectively. Assume that $(\boldsymbol{w},q)\in[H_0^1(\Omega)\cap H^{k+1}(\Omega)]^d\times[L_0^2(\Omega)\cap H^k(\Omega)]$. Then, there is $C>0$, independent of h, such that*

$$|\boldsymbol{w}-\tilde{\boldsymbol{w}}_h|_{1,\Omega} + \|q-\tilde{q}_h\|_{0,\Omega} \le C_1\, C_3\, h^k\big(|\boldsymbol{w}|_{k+1,\Omega}+|q|_{k,\Omega}\big)$$

with $C_3 := 1+\|\mathcal{B}^s\|$.

Proof. We note that $(\boldsymbol{w},q)$ and $(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)$ satisfy the orthogonality property

$$\mathcal{B}^s((\boldsymbol{w}-\tilde{\boldsymbol{w}}_h,q-\tilde{q}_h),(\boldsymbol{v}_h,r_h)) = 0 \qquad \forall\ (\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times\tilde{P}_h.$$

Indeed, thanks to the consistency of the scheme we get

$$\begin{aligned}
\mathcal{B}^s((\boldsymbol{w}-\tilde{\boldsymbol{w}}_h,q-\tilde{q}_h),(\boldsymbol{v}_h,r_h)) &= \mathcal{B}((\boldsymbol{w}-\tilde{\boldsymbol{w}}_h,q-\tilde{q}_h),(\boldsymbol{v}_h,r_h)) + \sum_{K\in\mathcal{T}_h} \delta h_K^2(\nabla q,\nabla r_h)_K \\
&\quad - \sum_{K\in\mathcal{T}_h} \delta h_K^2(\nabla\tilde{q}_h,\nabla r_h)_K \\
&= \sum_{K\in\mathcal{T}_h} \delta h_K^2(\boldsymbol{f}_{\boldsymbol{u}},\nabla r_h)_K - \sum_{K\in\mathcal{T}_h} \delta h_K^2(\boldsymbol{f}_{\boldsymbol{u}},\nabla r_h)_K \\
&= 0.
\end{aligned}$$

By the triangle inequality we can obtain,

$$\begin{aligned}
\|(\boldsymbol{w}-\tilde{\boldsymbol{w}}_h,q-\tilde{q}_h)\|_{\mathbf{V}\times P} &= |\boldsymbol{w}-\mathcal{I}_h\boldsymbol{w}+\mathcal{I}_h\boldsymbol{w}-\boldsymbol{w}_h|_{1,\Omega} + \|q-\mathcal{J}_hq+\mathcal{J}_hq-q_h\|_{0,\Omega} \\
&\le \|(\boldsymbol{w}-\mathcal{I}_h\boldsymbol{w},q-\mathcal{J}_hq)\|_{\mathbf{V}\times P} + \|(\tilde{\boldsymbol{w}}_h-\mathcal{I}_h\boldsymbol{w},\tilde{q}_h-\mathcal{J}_hq)\|_{\mathbf{V}\times P}.
\end{aligned} \tag{34}$$

For the second term of the right-hand side, we use [8, Lemma 4.4], to obtain

$$\|(\tilde{\boldsymbol{w}}_h-\mathcal{I}_h\boldsymbol{w},\tilde{q}_h-\mathcal{J}_hq)\|_{\mathbf{V}\times P} \le \sup_{\substack{(\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times P_h\\(\boldsymbol{v}_h,r_h)\neq\mathbf{0}}} \frac{\mathcal{B}^s((\tilde{\boldsymbol{w}}_h-\mathcal{I}_h\boldsymbol{w},\tilde{q}_h-\mathcal{J}_hq),(\boldsymbol{v}_h,r_h))}{\|(\boldsymbol{v}_h,r_h)\|_h}$$

$$
\begin{aligned}
&= \sup_{\substack{(\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times P_h\\(\boldsymbol{v}_h,r_h)\neq\mathbf{0}}} \frac{\mathcal{B}^s((\boldsymbol{w}-\mathcal{I}_h\boldsymbol{w},q-\mathcal{J}_hq),(\boldsymbol{v}_h,r_h))}{\|(\boldsymbol{v}_h,r_h)\|_h}\\
&\leq \|\mathcal{B}^s\|\|(\boldsymbol{w}-\mathcal{I}_h\boldsymbol{w},q-\mathcal{J}_hq)\|_{\mathbf{V}\times P},
\end{aligned}
$$

thus using (34), we get

$$
\|(\boldsymbol{w}-\tilde{\boldsymbol{w}}_h,q-\tilde{q}_h)\|_{\mathbf{V}\times P} \leq \big(1+\|\mathcal{B}^s\|\big)\|(\boldsymbol{w}-\mathcal{I}_h\boldsymbol{w},q-\mathcal{J}_hq)\|_{\mathbf{V}\times P}
$$

and thereby we arrive to

$$
|\boldsymbol{w}-\tilde{\boldsymbol{w}}_h|_{1,\Omega}+\|q-\tilde{q}_h\|_{0,\Omega} \leq C_1C_3h^k\big(|\boldsymbol{w}|_{k+1,\Omega}+|q|_{k,\Omega}\big).
$$

■

Lemma 11. *Let $(\tilde{\boldsymbol{w}}_h,\tilde{q}_h)$ and $(\boldsymbol{w}_h,q_h)$ be solutions of (33) and (28), respectively. Additionally, we assume that $\boldsymbol{u}\in H^2(\Omega)^d$. Then, the following bound is satisfied:*

$$
\|(\tilde{\boldsymbol{w}}_h-\boldsymbol{w}_h,\tilde{q}_h-q_h)\|_h \leq \|\mathcal{G}_{\boldsymbol{u}}-\mathcal{G}_{\boldsymbol{u}_h}\|_{(\mathbf{V}\times P)'}+\sqrt{\delta}h^2T(a_1,a_2,\boldsymbol{u},h)|\boldsymbol{u}|_{2,\Omega}+\sqrt{\delta}\nu h\|\Delta\boldsymbol{u}\|_{0,\Omega},
$$

Proof. Let $\boldsymbol{e}_h^{\boldsymbol{w}}:=\tilde{\boldsymbol{w}}_h-\boldsymbol{w}_h$ and $e_h^q:=\tilde{q}_h-q_h$. Then, thanks to the stability of $\mathcal{B}^s$ given in (30), we have

$$
\begin{aligned}
\|(\boldsymbol{e}_h^{\boldsymbol{w}},e_h^q)\|_h = \frac{\mathcal{B}^s((\boldsymbol{e}_h^{\boldsymbol{w}},e_h^q)(\boldsymbol{e}_h^{\boldsymbol{w}},e_h^q))}{\|(\boldsymbol{e}_h^{\boldsymbol{w}},e_h^q)\|_h} &\leq \sup_{\substack{(\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times P_h\\(\boldsymbol{v}_h,r_h)\neq\mathbf{0}}} \frac{\mathcal{B}^s((\boldsymbol{e}_h^{\boldsymbol{w}},e_h^q)(\boldsymbol{v}_h,q_h))}{\|(\boldsymbol{v}_h,q_h)\|_h}\\
&= \sup_{\substack{(\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times P_h\\(\boldsymbol{v}_h,r_h)\neq\mathbf{0}}} \frac{\mathcal{G}^s_{\boldsymbol{u}}(\boldsymbol{v}_h,r_h)-\mathcal{G}^s_{\boldsymbol{u}_h}(\boldsymbol{v}_h,r_h)}{\|(\boldsymbol{v}_h,q_h)\|_h}\\
&= \sup_{\substack{(\boldsymbol{v}_h,r_h)\in\mathbf{V}_h\times P_h\\(\boldsymbol{v}_h,r_h)\neq\mathbf{0}}} \frac{\mathcal{G}_{\boldsymbol{u}}(\boldsymbol{v}_h,r_h)-\mathcal{G}_{\boldsymbol{u}_h}(\boldsymbol{v}_h,r_h)-\sum_{K\in\mathcal{T}_h}\delta h_K^2(\boldsymbol{f}_{\boldsymbol{u}}-\boldsymbol{f}_{\boldsymbol{u}_h},\nabla r_h)_K}{\|(\boldsymbol{v}_h,q_h)\|_h}
\end{aligned}
$$

We take the term within sum, making use of the Cauchy-Schwarz inequality and proceeding similarly as in (12), we obtain

$$
\begin{aligned}
-\sum_{K\in\mathcal{T}_h}\delta h_K^2(\boldsymbol{f}_{\boldsymbol{u}}-\boldsymbol{f}_{\boldsymbol{u}_h},\nabla r_h)_K &= \sum_{K\in\mathcal{T}_h}\delta h_K^2((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u},\nabla r_h)_K\\
&\quad -\nu\sum_{K\in\mathcal{T}_h}\delta h_K^2(\Delta\boldsymbol{u},\nabla r_h)_K\\
&\leq \sum_{K\in\mathcal{T}_h}\delta h_K^2\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u}\|_{0,\Omega}\|\nabla r_h\|_{0,K}\\
&\quad +\nu\sum_{K\in\mathcal{T}_h}\delta h_K^2\|\Delta\boldsymbol{u}\|_{0,K}\|\nabla r_h\|_{0,K}\\
&\leq \Bigg(\sum_{K\in\mathcal{T}_h}\delta h_K^2\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\mathcal{L}_h\boldsymbol{u}\cdot\nabla)\mathcal{L}_h\boldsymbol{u}\|_{0,K}^2\Bigg)^{1/2}\Bigg(\sum_{K\in\mathcal{T}_h}\delta h_K^2\|\nabla r_h\|_{0,K}^2\Bigg)^{1/2}\\
&\quad +\nu\Bigg(\sum_{K\in\mathcal{T}_h}\delta h_K^2\|\Delta\boldsymbol{u}\|_{0,K}^2\Bigg)^{1/2}\Bigg(\sum_{K\in\mathcal{T}_h}\delta h_K^2\|\nabla r_h\|_{0,K}^2\Bigg)^{1/2}\\
&\leq \sqrt{\delta}\Bigg[\sum_{K\in\mathcal{T}_h}h_K^4T_K^2(a_1,a_2,\boldsymbol{u},h)|\boldsymbol{u}|_{2,K}^2\Bigg]^{1/2}\|(\boldsymbol{v}_h,r_h)\|_h
\end{aligned}
$$

$$+\nu\sqrt{\delta}\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\Delta\boldsymbol{u}\|_{0,K}^2\Big)^{1/2}\|(\boldsymbol{v}_h,r_h)\|_h$$
$$\leq \sqrt{\delta}h^2T(a_1,a_2,\boldsymbol{u},h)|\boldsymbol{u}|_{2,\Omega}\|(\boldsymbol{v}_h,r_h)\|_h+\nu\sqrt{\delta}h\|\Delta\boldsymbol{u}\|_{0,\Omega}\|(\boldsymbol{v}_h,r_h)\|_h.$$

■

The main result of this section is given in the following Theorem

Theorem 4 (Main Result III). *Assume that the hypothesis of Theorem 3 hold. Then,*

$$|\boldsymbol{w}-\boldsymbol{w}_h|_{1,\Omega}+\|q-q_h\|_{0,\Omega}\leq C_1C_3h^k\big(|\boldsymbol{w}|_{k+1,\Omega}+|q|_{k,\Omega}\big)$$
$$+h|\boldsymbol{u}|_{2,\Omega}\Big[C_pT(a_1,a_2,\boldsymbol{u},h)+\nu\, a_1+\sqrt{\delta}T(a_1,a_2,\boldsymbol{u},h)\Big]+\nu\sqrt{\delta}h\|\Delta\boldsymbol{u}\|_{0,\Omega}.$$

Proof. The result follows from Lemmas 8, 10 and 11. ■

4. Numerical Results

In this section we present some numerical examples to illustrate the theoretical results previously described. We also provide the error bounds, except for the Modified-PPE since to the best of our knowledge values for the interpolation constant $a_{2,I}$ have not been reported. The legends in the plots follow the notation:

- $b1$: Bound performed with $\mathbb{P}_1$.
- $b2$: Bound performed with $\mathbb{P}_2$.
- $e_1(q)$: Pressure error in L^2-norm with $\mathbb{P}_1$
- $e_2(q)$: Pressure error in L^2-norm with $\mathbb{P}_2$,

where b_i corresponds to the right-side of every main result, and

$$e_i(q)=\frac{\|q-q_h\|_{0,\Omega}}{\|q\|_{0,\Omega}},$$

with $i=1,2$. For the STE, the bounds correspond to the ones from the theorems minus the $|\boldsymbol{w}_h|_{1,\Omega}$, since $\boldsymbol{w}=\boldsymbol{0}$.
We consider C_p as the Poincaré constant, with $C_p=\dfrac{D_\Omega}{\pi}$ if the Ω is convex. Moreover, in [9, Theorem 2] the authors proved that $a_1=0.4923\,\phi(\theta)$ and $a_2=\sqrt{12}(1+|\cos\theta|)$, with $\phi(\theta):=\frac{1+|\cos\theta|}{\sqrt{1-|\cos\theta|}}$ where θ is the maximum angle of the mesh.

Example 1. *For the first example, we consider the exact solution of the two dimensional Kovasznay flow*

$$\boldsymbol{u}(x,y)=\begin{pmatrix}1-e^{\lambda x}\cos(2\pi y)\\ \frac{\lambda}{2\pi}e^{\lambda x}\sin(2\pi y)\end{pmatrix},\quad p(x,y)=\frac{1}{2}e^{\lambda x}-(e^{3\lambda}-e^{-\lambda}),$$

where $\Omega=\Big(-\dfrac{1}{2},\dfrac{3}{2}\Big)\times\big(0,2\big)$ *and the parameter* λ *is given by* $\lambda=\dfrac{1}{2\nu}-\sqrt{\dfrac{1}{4\nu^2}+4\pi^2}$. *For this ilustration we have taken the Reynold number as in [14] which is given by* $Re=\dfrac{1}{\nu}$.
The convergence results are shown in figure 1 and the isovalues in figures 2, 3, 4 and 5.

Example 2. *Next we turn to the testing the scheme, where the computational domain is the rectangle* $\Omega=[0,1]^2$ *and we consider the exact solution of the Navier-Stokes equation given by*

$$u(x,y)=(\frac{\nu}{4}e^x\sin(\nu y),\frac{1}{4}e^x\cos(\nu y))\quad and\quad p(x,y)=-\frac{\nu}{2}e^{2x}+\frac{\nu}{4}(e^2-1)\tag{35}$$

The convergence results are shown in figure 6 and the isovalues in figures 7, 8, 9 and 10.

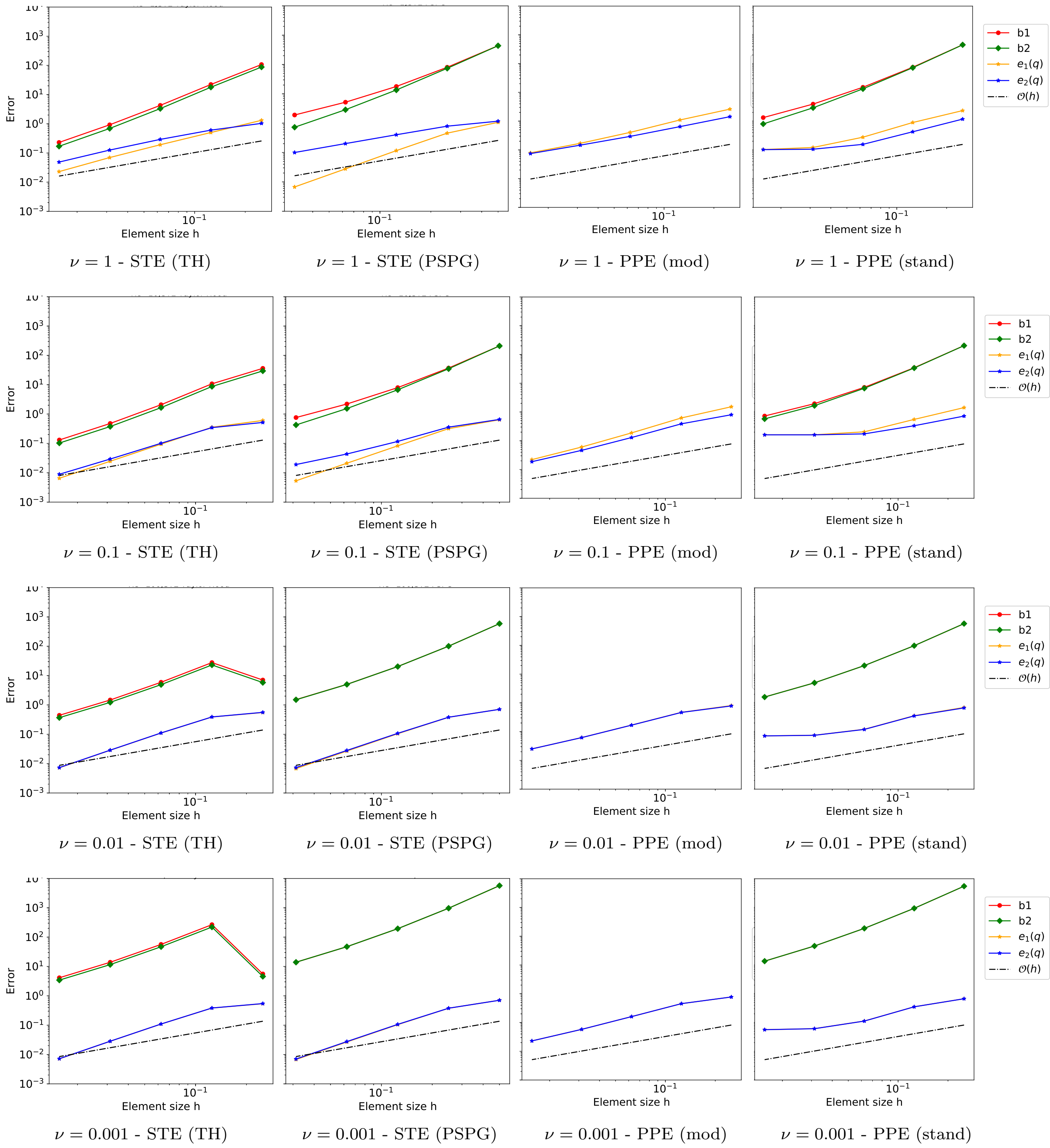


Figure 1: Pressure error curves and bounds for viscosities values $1, 10^{-1}, 10^{-2}$ and 10^{-3} of Example 1 (Kovaznay flow). The figures for the modified PPE do not present the error bound as explained in the beginning of Section 4.

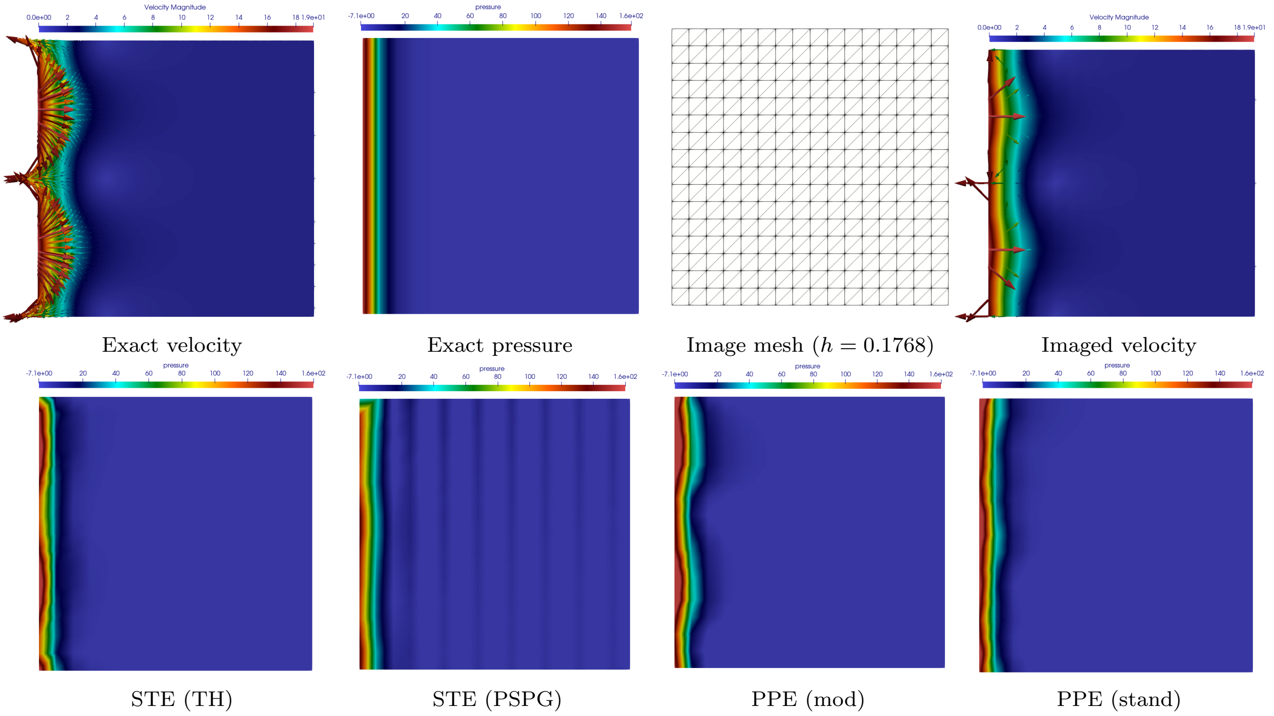


Figure 2: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = 1$ in Example 1 (Kovaznay flow).

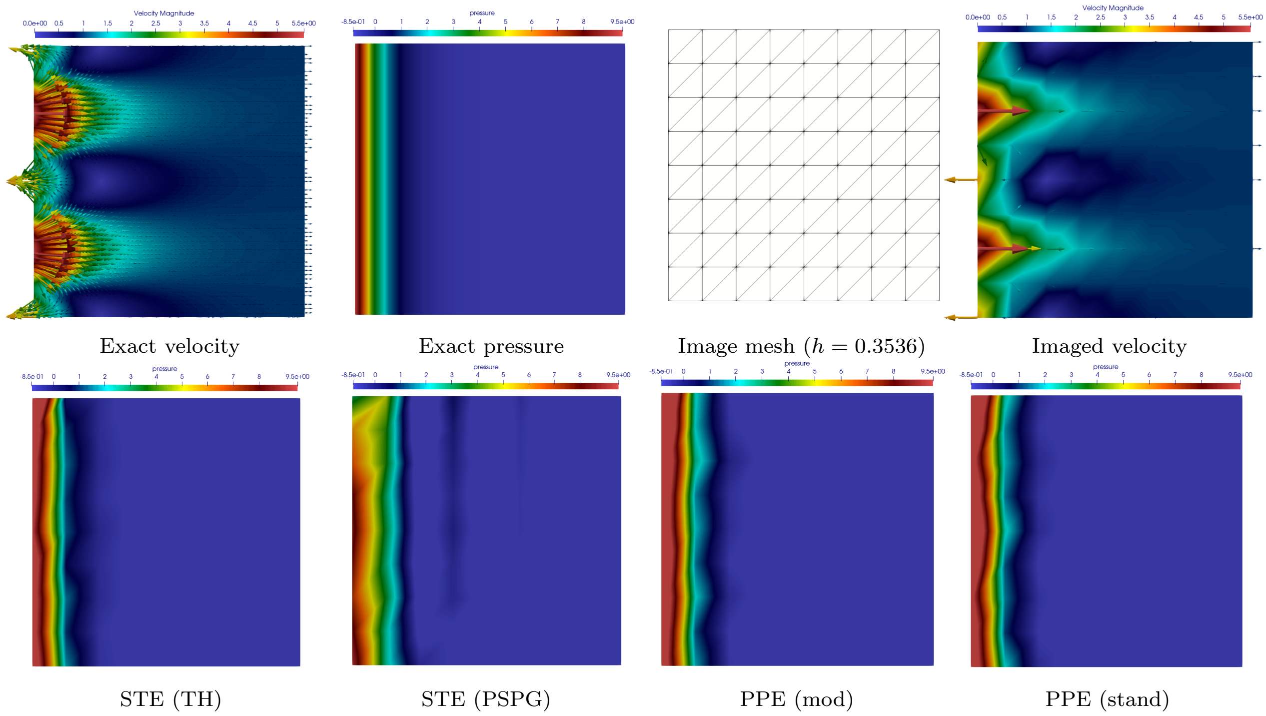


Figure 3: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = 0.1$ in Example 1 (Kovaznay flow).

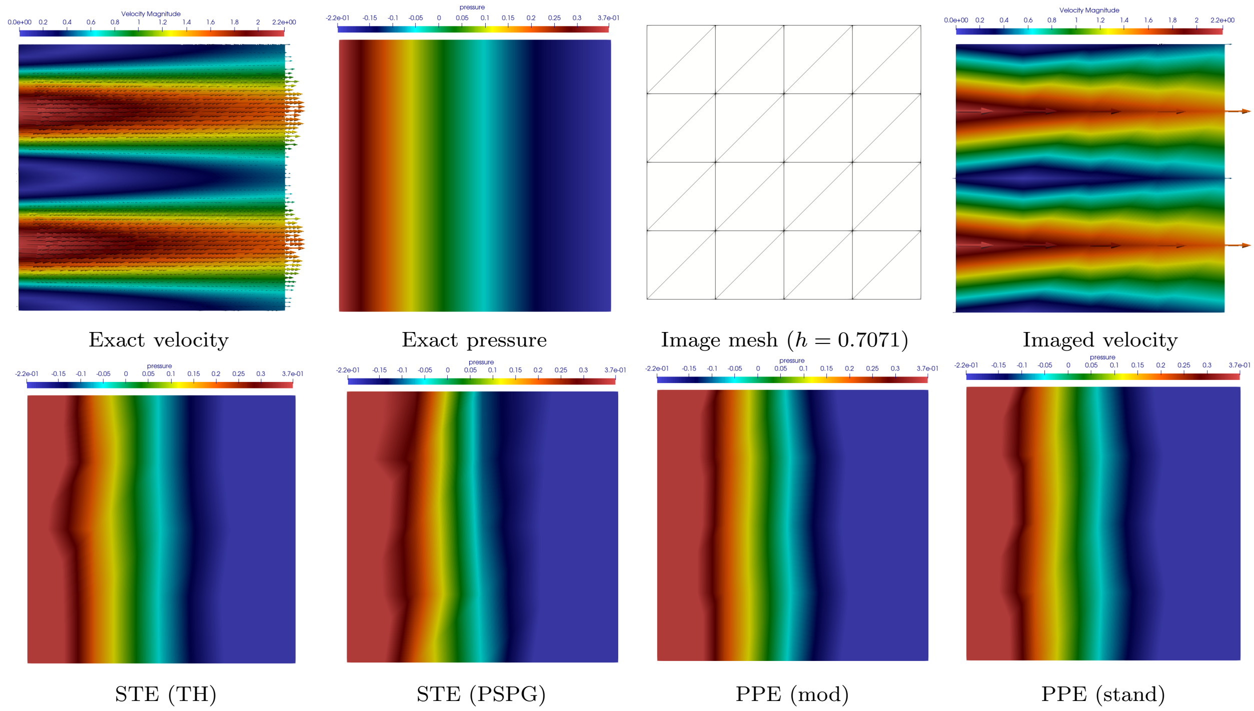


Figure 4: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = 0.01$ in Example 1 (Kovaznay flow).

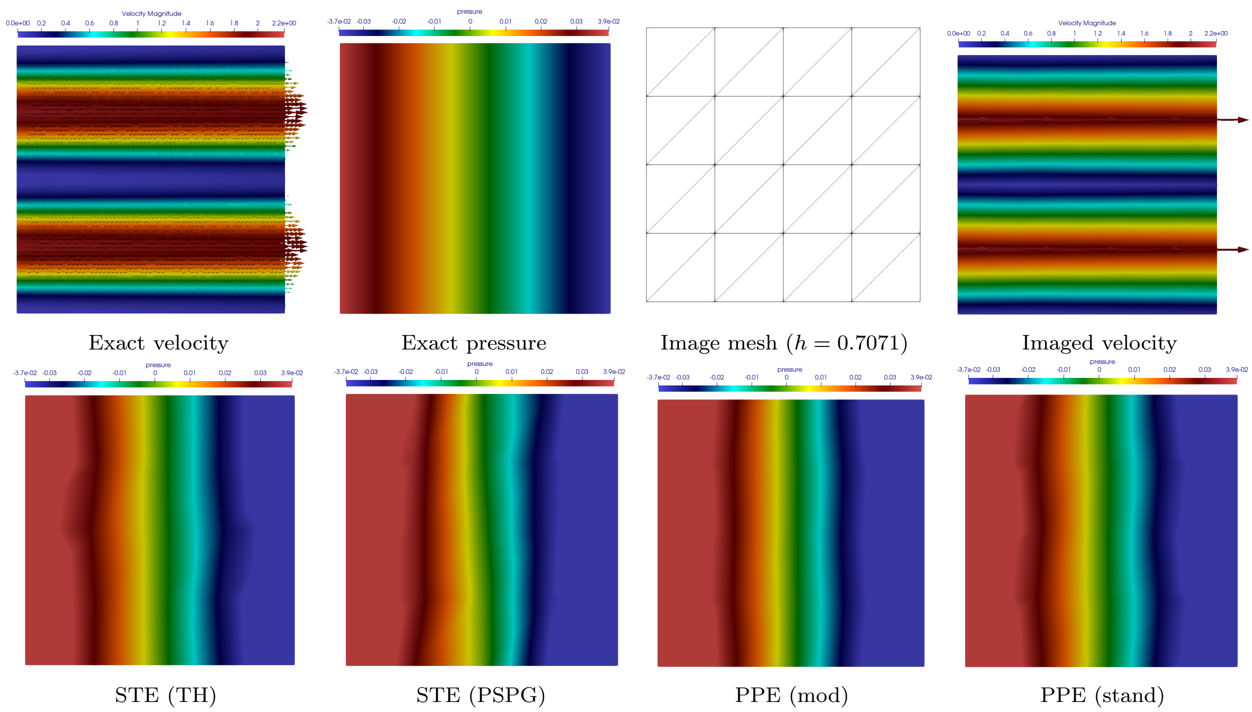
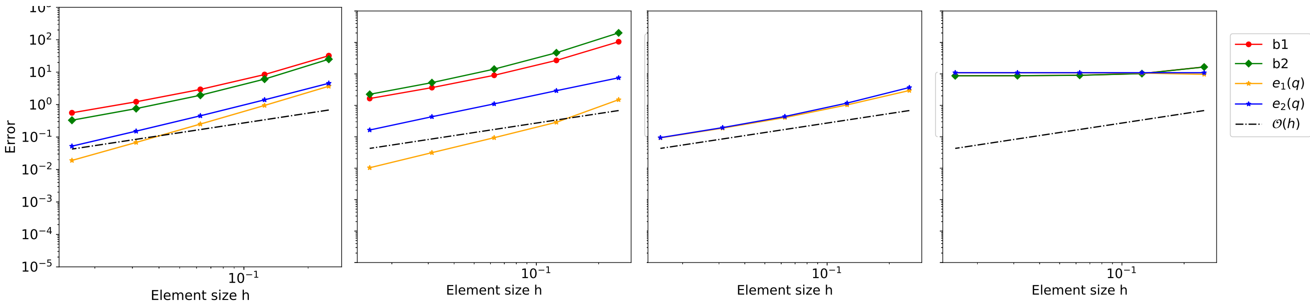
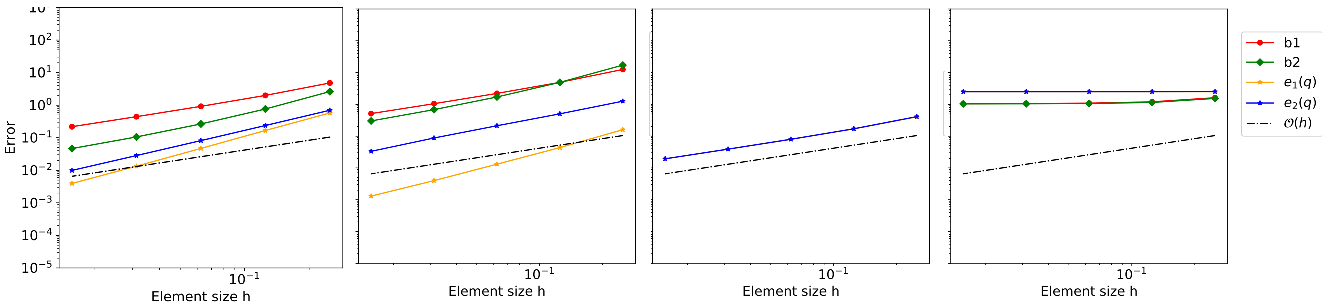


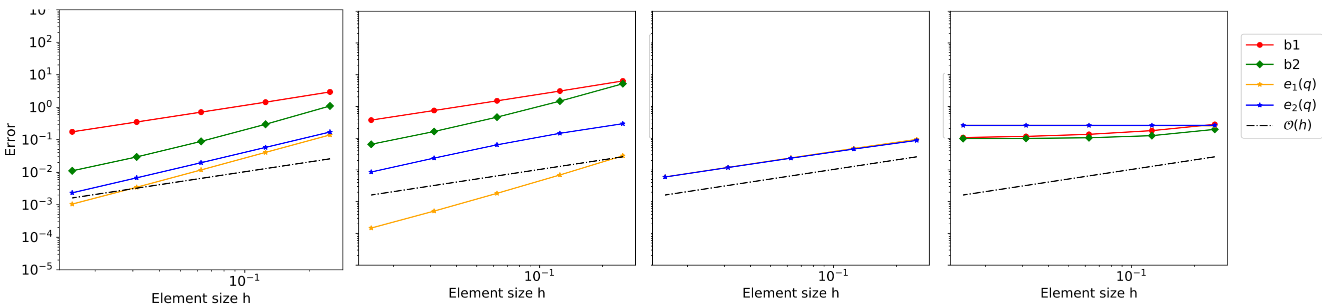
Figure 5: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = 0.001$ in Example 1 (Kovaznay flow).



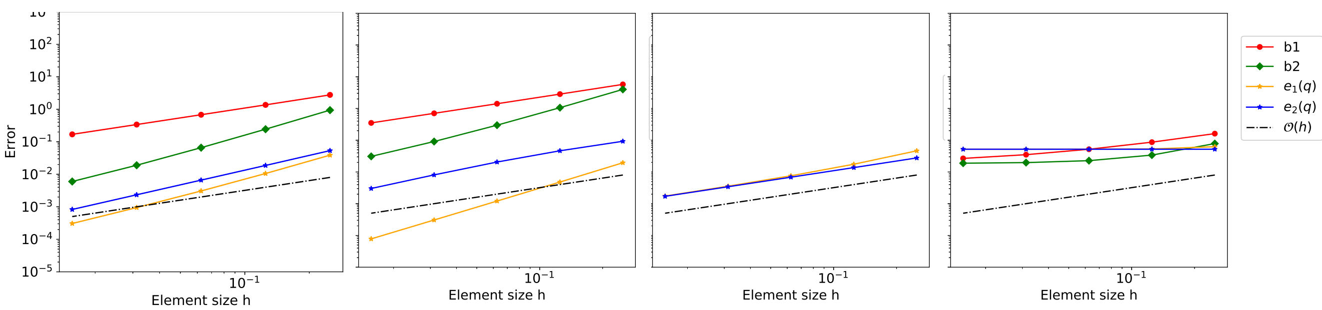
$\nu = 2\pi$ - STE (TH) $\nu = 2\pi$ - STE (PSPG) $\nu = 2\pi$ - PPE (mod) $\nu = 2\pi$ - PPE (stand)



$\nu = \pi$ - STE (TH) $\nu = \pi$ - STE (PSPG) $\nu = \pi$ - PPE (mod) $\nu = \pi$ - PPE (stand)



$\nu = \pi/2$ - STE (TH) $\nu = \pi/2$ - STE (PSPG) $\nu = \pi/2$ - PPE (mod) $\nu = \pi/2$ - PPE (stand)



$\nu = \pi/4$ - STE (TH) $\nu = \pi/4$ - STE (PSPG) $\nu = \pi/4$ - PPE (mod) $\nu = \pi/4$ - PPE (stand)

Figure 6: Pressure error curves and bounds for viscosities values $\pi/4, \pi/2, \pi$ and 2π of Example 2. The figures for the modified PPE do not present the error bound as explained in the beginning of Section 4.

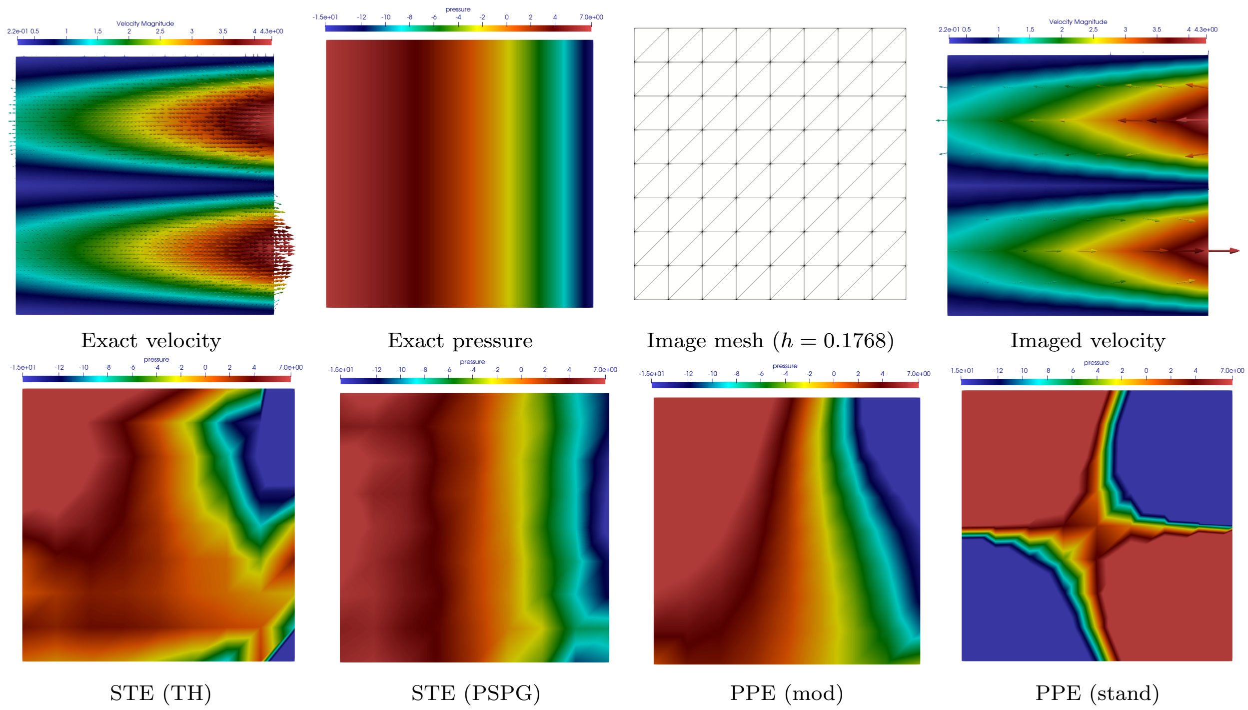


Figure 7: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = 2\pi$ in Example 2.

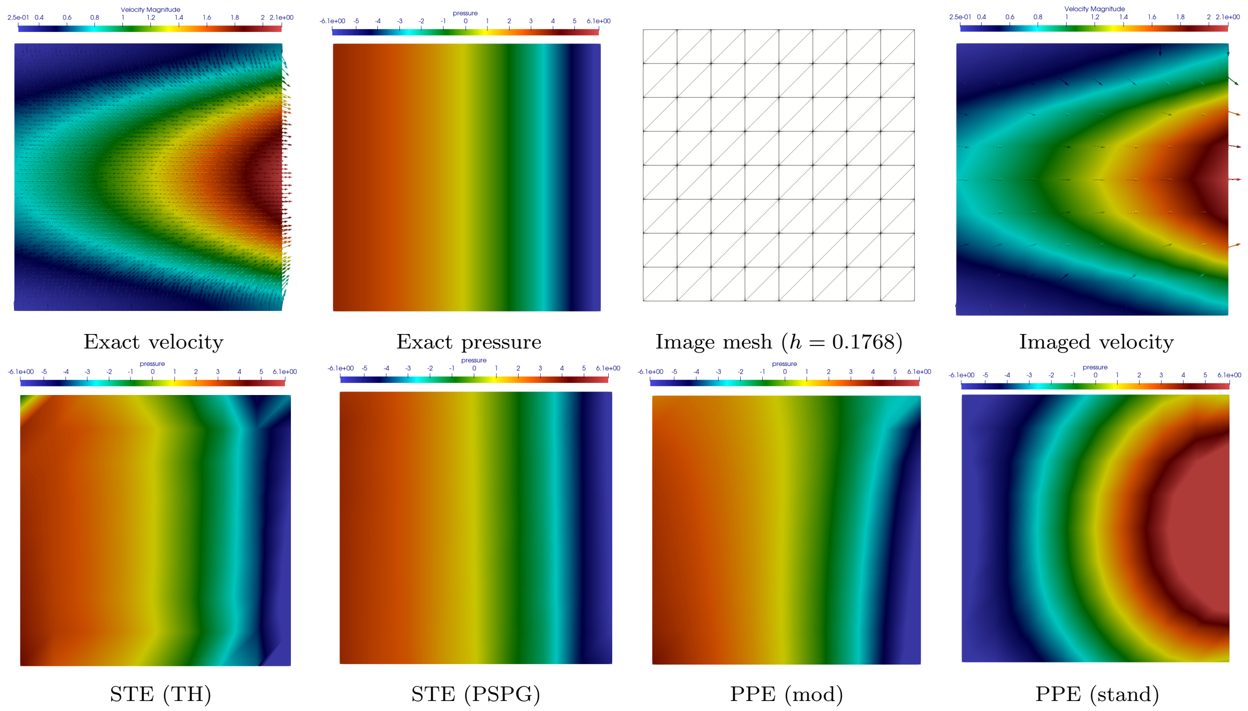


Figure 8: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = \pi$ in Example 2.

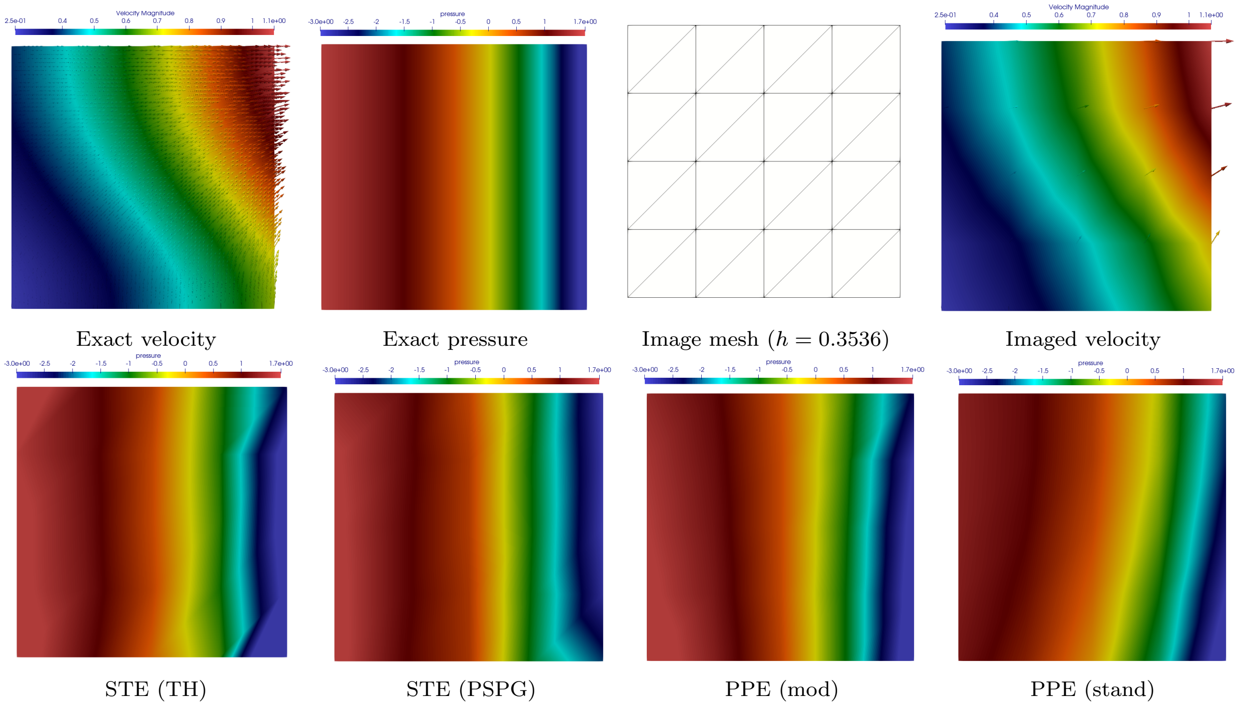


Figure 9: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = \pi/2$ in Example 2.

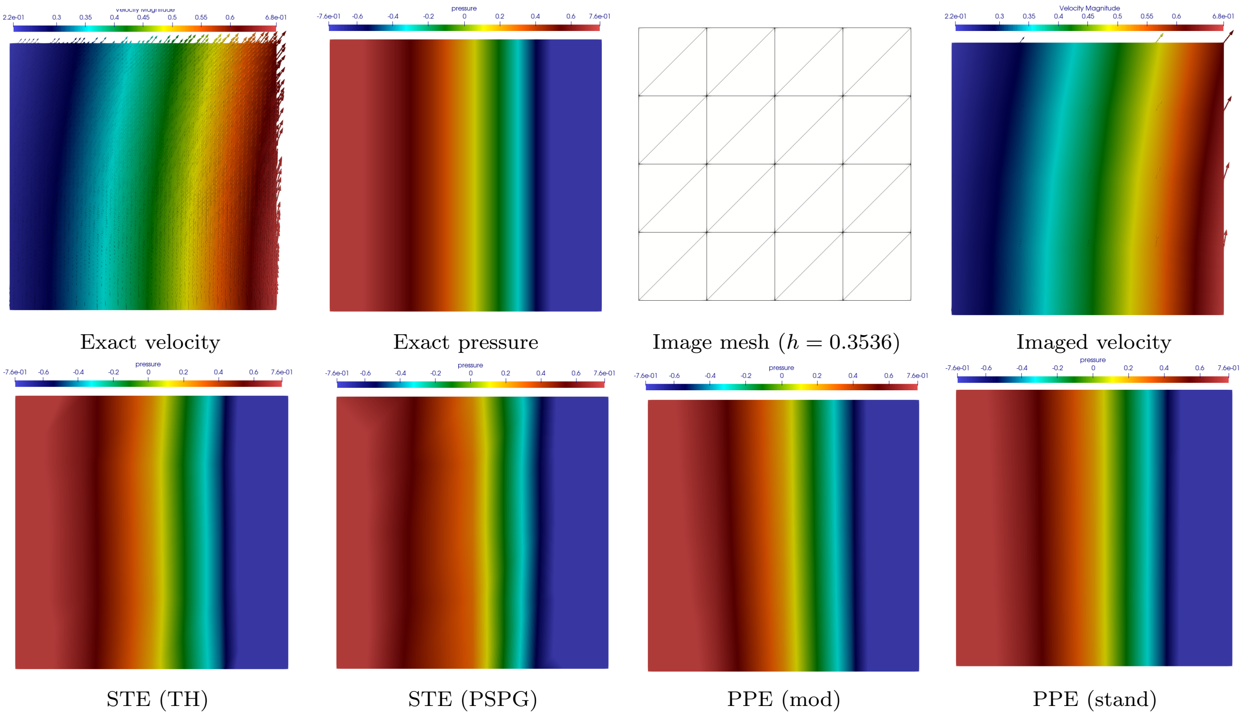


Figure 10: $\mathbb{P}_1$-interpolated reference velocity and pressure fields (top) and reconstructed pressure fields with order $k = 1$ (bottom) for $\nu = \pi/4$ in Example 2.

5. Conclusions

This article is devoted to the error analysis of methods for approximation of the pressure from discrete velocity fields, STE and PPE. The STE is analyzed using the classical Taylor-Hood finite element spaces and pressure-stabilizing Petrov-Galerkin (PSPG). The PPE is analyzed using a continuous Galerkin method, where two versions of the PPE have been considered, the classical approach without the viscous term and a new approach with a boundary viscous term as proposed in [13].
Due to the piecewise linear nature of the measured velocities, the error bounds show terms of different convergence orders. However, for the numerical examples with polynomial order 1 and 2 for the pressure mainly convergence order 1 is observed. Therefore, in contrast to a classical problem solved by finite elements with analytical right-hand-side, here we can see that it is not worth to increment of the polynomial order.
A result that draws attention is that in each example the relative error decreases when the Reynolds number increases possibly due to the reduced viscous contribution which appear to be more difficult to retrieve for all methods.
The numerical results show that the methods of choice are the STE-PSPG and the Modified-PPE leading to the best error-cost relations. However, it remains to be clarify if the modification of the PPE achieve comparable accuracies using measured MRI data in real arteries, where the viscous contribution appears to be negligible and data has larger errors in the boundaries due to imaging artifacts.

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