An adaptive stabilized finite element method for the Darcy's equations with pressure dependent viscosities

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Abstract

This work aims to introduce and analyze an adaptive stabilized finite element method to solve a nonlinear Darcy equation with a pressure-dependent viscosity and mixed boundary conditions. We stated the discrete problem's well-posedness and optimal error estimates, in natural norms, under standard assumptions. Next, we introduce and analyze a residual-based a posteriori error estimator for the stabilized scheme. Finally, we present some two- and three-dimensional numerical examples which confirm our theoretical results.

Keywords: Nonlinear Darcy's equation, stabilized finite element method, a priori error analysis, a posteriori error analysis.

1. Introduction

In many important applications, as for example, in oil reservoir, contaminant transport, mesoscale blood flows, filter design and water resource problems, it is necessary to study the flow of fluids in a porous medium. The first model to study this phenomenon corresponds to the Darcy model (see [1]) when it is considered that the viscosity of the flow is a function constant and where the pressure is independent of viscosity. Time later, it was shown experimentally that in a wide variety of industrial applications, for certain kind flow, as in the case of organic liquid, the viscosity can depend on the pressure (see [2]). This situation is presented, for example, when the viscosity depends exponentially on the pressure (see [2]), turning the Darcy problem into a nonlinear problem (for details on the derivation, see[3]).

For the classical Darcy equation, there are a large number of numerical schemes that approximate the velocity and pressure of the fluid, starting with mixed methods that consider the stable subspace of $H(\mathrm{div};\Omega)$, as the Raviart-Thomas elements [4] or Brezzi-Douglas-Marini family [5]. For a non complete list of these stable schemes, see [6, 7, 8, 9, 10, 11], and the references therein.
On the other hand, in fluid dynamics simulations the use of equal-order interpolation subspaces for the pressure and the velocity is a desirable property, but, unfortunately this choice does not lead to stable finite element methods in the sense of Babuska-Brezzi-Ladyzenskaya (see [12], and the references therein). In order to overcome this problem, several stabilized finite-element methods have been proposed in the last decades. A common class of this method, adding residual terms to the Galerkin formulation, are SUPG/PSPG or SDFEM methods (see, for instance [13, 14, 15, 16]). A small variation of these schemes is the called Residual Local Projection (RELP) stabilized methods which reintroduced the residues through interpolation operators (fluctuation operators) on finite dimensional spaces through the solution of local problems (see, for example [17] for the Stokes equations, for the Darcy equations [18] and [19] for the Navier–Stokes equations). When the extra terms are not residual-based, then we have, for example, the Orthogonal Subscales method (see [20]), the CIP methods (see [21]), or the Local Projection Stabilization (LPS) method (see [22]). The LPS methods can also be seen as a simplification of the RELP method. This method considers symmetric term-by-term fluctuation terms generating a simpler method, but lacking the consistency of the method (see, for

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instance [23] for the Oseen equations and [24] for the Navier–Stokes equations). Another list of stabilized schemes for the linear Darcy equation are [25, 26, 27, 28, 29, 30]. A different strategy to consider this problem consists of multi-level approximation as the Multiscale Hybrid–Mixed (MHM) method (for details, see [31, 32]).

Concerning the nonlinear Darcy equation when, for instance, the viscosity depends on the pressure, the list of numerical schemes is not long. In [33], an approximation of the nonlinear equation in a circular well domain using a spectral element discretization was propose and analyzed. In [34] this strategy was extended to consider an a posteriori error estimator to improve the performance of a simplified model where the dependence of pressure does not vary too much. In [35], the authors use an Euler's implicit scheme to extend the spectral element discretization to the non-stationary case. A study of the convergence of a stable finite element discretization can be found in [36] for the nonlinear problem when the dependence of the viscosity on the pressure is a bounded function. A mixed finite element method with, a Lagrange multiplier, was introduced and analyzed in [37] for stable subspaces. The authors also introduced an a posteriori error estimator to improve the quality of theirs results. Extending the ideas of [25], a stabilized finite element method was proposed in [16] when the dependence of viscosity on the pressure can occur in several different ways. Recently, a scalable numerical formulation based on variational inequalities was presented in [38] and the convergence of this two last scheme was done in numerical form.

The purpose of this work is to present and analyze a stabilized finite element method for the nonlinear Darcy equations when the viscosity can depend exponentially on the pressure, for example when this dependency satisfies the Barus law [39]. As in [37], we use a change of variable that allows us to transform the nonlinear equation into a linear problem and then, using the ideas of [25], we define a new stabilized finite element method. This new scheme is free of mesh dependent stabilization parameters and allow the use of the classical $\mathbb{P}_k^d \times \mathbb{P}_k$ elements for the velocity and pressure. For the stability of the method, we use some tools presented in [37], as the Banach fixed point Theorem and a generalized Lax–Milgram Theorem. In the convergence analysis we use strategies coming from the analysis of other classical stabilized finite element method (for details, see [40, 41]). Thus, our contribution is the numerical analysis of the discrete scheme, and the definition an analysis of a residual-based a posteriori error estimator.

This work is organized as follows. In Section 2 we introduce the nonlinear Darcy equation together with the variational formulation of the linear problem that is obtained from a change of variable. This section end with some preliminary results that will be needed later. Section 3 is devoted to prove the existence and uniqueness of solution to the variational formulation. In Section 4, we describe the proposed stabilized finite element approximation and include the well-posedness of this scheme. This section also includes a priori error estimates for the elements $\mathbb{P}_k^d \times \mathbb{P}_k$. In Section 5, we present and analyze an a posteriori error estimator related to the new stabilized scheme. We also present the equivalence between the error estimator and the approximate error in natural norms. In Section 6 we present the adaptive procedure joint with numerical results that confirm the a priori error results and the performance of the a posteriori error estimator. Finally, in Appendix A we prove a technical result that will be essential for our adaptive scheme.

2. Model problem and preliminary results

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with polygonal boundary $\partial\Omega$ divided in Γ_D and Γ_N, with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \neq \emptyset$. We focus in to seek the velocity and pressure solution $(\tilde{\boldsymbol{u}}, \tilde{p})$ to the nonlinear Darcy equations, with mixed boundary condition, given by:

$$\left\{\begin{array}{rcll} \alpha(\tilde{p})\,\tilde{\boldsymbol{u}} + \nabla\tilde{p} & = & \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\tilde{\boldsymbol{u}} & = & 0 & \text{in } \Omega, \\ \tilde{p} & = & \tilde{\varphi} & \text{on } \Gamma_D, \\ \tilde{\boldsymbol{u}}\cdot\boldsymbol{n} & = & 0 & \text{on } \Gamma_N, \end{array}\right. \tag{2.1}$$

where $\alpha(\tilde{p})$ is the drag function, $\tilde{\varphi} \in H^{1/2}(\Gamma_D)$ is the prescribed pressure in Γ_D, $\boldsymbol{f} \in L^2(\Omega)^d$ is a given source and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$.

Remark 1. *When $\Gamma_N = \partial\Omega$ it is necessary, for the uniqueness of the solution, to impose the condition $\int_\Omega \tilde{p} = 0$. In this work, we consider $|\Gamma_D| > 0$, which is more complex to analyze.*

In the standard Darcy equation, the drag coefficient α is equal to the ratio of the viscosity μ of the fluid and the permeability κ of the porous media, i.e.

$$\alpha = \frac{\mu}{\kappa}. \tag{2.2}$$

In this work, we follow to Barus [39], who proposed the exponential dependence of pressure on viscosity which is given by the function

$$\mu(s) = \mu_0 e^{\gamma s}, \quad \forall s \in \mathbb{R}, \tag{2.3}$$

where μ_0 is a positive constant and γ is called the Barus coefficient, which can be obtained experimentally (see [2]). Thereby, from (2.2) and (2.3) we get

$$\alpha(s) = \alpha_0 e^{\gamma s}, \quad \forall s \in \mathbb{R}, \tag{2.4}$$

where $\alpha_0 := \dfrac{\mu_0}{\kappa}$.

Now, thanks to the (2.4) and in view of analysis, we will rewrite the problem (2.1) in a more convenient form. To this end, the first equality of (2.1) is reduced to

$$\tilde{\boldsymbol{u}} = \frac{1}{\alpha(\tilde{p})}(\boldsymbol{f} - \nabla \tilde{p}) = \frac{1}{\alpha_0} e^{-\gamma\tilde{p}} \boldsymbol{f} + \frac{1}{\alpha_0 \gamma} \nabla(e^{-\gamma\tilde{p}}).$$

Now, defining $\boldsymbol{u} := \tilde{\boldsymbol{u}}$, and $p := e^{-\gamma\tilde{p}} - 1$, and using (2.1), we define the following Darcy equation

$$\left\{ \begin{array}{rcll} \boldsymbol{u} & = & \dfrac{1}{\alpha_0}(p+1)\boldsymbol{f} + \dfrac{1}{\varepsilon}\nabla p & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} & = & 0 & \text{in } \Omega, \\ p & = & \varphi & \text{on } \Gamma_D, \\ \boldsymbol{u} \cdot \boldsymbol{n} & = & 0 & \text{on } \Gamma_N, \end{array} \right. \tag{2.5}$$

where $\varepsilon := \alpha_0 \gamma > 0$ and $\varphi := e^{-\gamma\tilde{\varphi}} - 1$. This transformation was introduced in [37], where the authors present a similar mixed variational formulation for (2.5), with different Hilbert spaces and using a Lagrange multiplier to weakly impose some boundary conditions.

In the sequel we will use the following Hilbert spaces,

$$\boldsymbol{H} := \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_N\},$$
$$\tilde{Q} := L^2(\Omega),$$

and the norms

$$\|\boldsymbol{v}\|_{\boldsymbol{H}} = \|\boldsymbol{v}\|_{\mathrm{div};\Omega} \qquad \text{and} \qquad \|q\|_{\tilde{Q}} = \|q\|_{0,\Omega},$$

for all $\boldsymbol{v} \in \boldsymbol{H}$ and $q \in \tilde{Q}$.

The variational formulation of problem (2.5) can be written as: *Find $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \tilde{Q}$ such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{\Gamma_D} + \gamma\,(p\boldsymbol{f}, \boldsymbol{v}) + \gamma\,(\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}, \tag{2.6}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall q \in \tilde{Q}, \tag{2.7}$$

where $a : \boldsymbol{H} \times \boldsymbol{H} \longrightarrow \mathbb{R}$ and $b : \boldsymbol{H} \times \tilde{Q} \longrightarrow \mathbb{R}$ are the bilinear forms defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \varepsilon\,(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall (\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{H} \times \boldsymbol{H}, \tag{2.8}$$

$$b(\boldsymbol{v}, q) := (q, \nabla \cdot \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H} \times \tilde{Q}. \tag{2.9}$$

Here $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product, where we use the same notation for vector, or tensor, valued functions, and $\langle \cdot,\cdot \rangle_{\Gamma_D}$ is the duality pairing between $H^{-1/2}(\Gamma_D)$ and $H^{1/2}(\Gamma_D)$.

Also we consider the norm, on $H^{-1/2}(\Gamma_D)$, given by

$$\|\mu\|_{-1/2,\Gamma_D} := \inf_{\substack{\sigma\in H(\mathrm{div};\,\Omega)\\ \sigma\cdot n=\mu \text{ on } \Gamma_D}} \|\boldsymbol{\sigma}\|_{\boldsymbol{H}}, \tag{2.10}$$

for all $\mu \in H^{-1/2}(\Gamma_D)$.

We equip the space $\boldsymbol{H} \times \tilde{Q}$ with the product norm

$$\|(\boldsymbol{w}, r)\|_{\boldsymbol{H}\times\tilde{Q}} = \|\boldsymbol{w}\|_{\boldsymbol{H}} + \|r\|_{0,\Omega}.$$

Throughout this paper C and C_i, $i > 0$ will denote positive constants independent of the mesh size h, but who may depend on the parameter ε.

The next result states some inequalities which will be used in the sequel.

Lemma 1. *Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be the bilinear forms given by* (2.8) *and* (2.9)*, respectively. Then, there exists a positive constant β_b, such that*

$$\begin{aligned}
|a(\boldsymbol{u}, \boldsymbol{v})| &\leq \varepsilon\|\boldsymbol{u}\|_{\boldsymbol{H}}\|\boldsymbol{v}\|_{\boldsymbol{H}} \qquad \forall\ \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}, && (2.11)\\
|b(\boldsymbol{v}, q)| &\leq \|q\|_{\tilde{Q}}\|\boldsymbol{v}\|_{\boldsymbol{H}} \qquad \forall\ \boldsymbol{v} \in \boldsymbol{H},\ \forall q \in \tilde{Q}, && (2.12)\\
\sup_{\boldsymbol{v}\in\boldsymbol{H}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}}} &\geq \beta_b\,\|q\|_{\tilde{Q}} \qquad \forall q \in \tilde{Q}. && (2.13)
\end{aligned}$$

Proof. The proof of (2.11) and (2.12) follows from the norm definitions. To prove (2.13), let $q \in \tilde{Q}$ and define the auxiliary problem:

$$\left\{\begin{array}{rcll}
-\Delta z &=& -q & \text{in } \Omega,\\
z &=& 0 & \text{on } \Gamma_D,\\
\nabla z\cdot \boldsymbol{n} &=& 0 & \text{on } \Gamma_N,
\end{array}\right.$$

with $z \in H^1_{\Gamma_D}(\Omega)$ and $|z|_{1,\Omega} \leq C\|q\|_{\tilde{Q}}$. Now, taking $\tilde{\boldsymbol{v}} := \nabla z$, we get that

$$\nabla\cdot\tilde{\boldsymbol{v}} = q \quad \text{in } \Omega \qquad \text{and} \qquad \tilde{\boldsymbol{v}}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_N,$$

thus $\tilde{\boldsymbol{v}} \in \boldsymbol{H}$, and

$$\|\tilde{\boldsymbol{v}}\|^2_{\boldsymbol{H}} = \|\tilde{\boldsymbol{v}}\|^2_{0,\Omega} + \|\nabla\cdot\tilde{\boldsymbol{v}}\|^2_{0,\Omega} \leq C\|q\|^2_{\tilde{Q}},$$

So, we get

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}}} \geq \frac{b(\tilde{\boldsymbol{v}}, q)}{\|\tilde{\boldsymbol{v}}\|_{\boldsymbol{H}}} \geq \beta_b\|q\|_{\tilde{Q}},$$

which proves the Lemma. □

3. Existence and uniqueness of weak solution

This section is devoted to show the well-posedness of the variational formulation (2.6)-(2.7). To this end, we will write the variational formulation as a fixed point problem to apply the Banach's fixed point theorem.

Let $B : (\boldsymbol{H}\times\tilde{Q}) \times (\boldsymbol{H}\times\tilde{Q}) \longrightarrow \mathbb{R}$, the bilinear form given by

$$B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q),$$

for all $(\boldsymbol{u}, p), (\boldsymbol{v}, q) \in H \times \tilde{Q}$ and let $\boldsymbol{G} \in \boldsymbol{H}'$, arbitrary.

In the sequel we will use extensively the solution of the following kind of problem: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H}\times\tilde{Q}$ *such that*

$$B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{G}(\boldsymbol{v}), \tag{3.14}$$

for all $(\boldsymbol{v}, q) \in H \times \tilde{Q}$. Regarding the solvability of (3.14) we have the following result

Theorem 2. *The problem* (3.14) *is well-posed and the solution satisfies*

$$\|(\boldsymbol{u}, p)\|_{\boldsymbol{H}\times\tilde{Q}} \leq \left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\|\boldsymbol{G}\|_{\boldsymbol{H}'}. \tag{3.15}$$

Proof. Let $\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H} : \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega\}$ be the kernel of $b(\cdot, \cdot)$. Then,

$$a(\boldsymbol{v}, \boldsymbol{v}) = \varepsilon(\boldsymbol{v}, \boldsymbol{v}) = \varepsilon\|\boldsymbol{v}\|_{0,\Omega}^2 = \varepsilon\|\boldsymbol{v}\|_{\boldsymbol{H}}^2, \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}, \tag{3.16}$$

which implies that $a(\cdot, \cdot)$ is coercive in $\boldsymbol{V}$. Thus, using Lemma 1 and (3.16), we get that problem (3.14) is well posed. On the other hand, using [12, Theorem 4.1], it is clear that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}} \leq \frac{1}{\varepsilon}\|\boldsymbol{G}\|_{\boldsymbol{H}'},$$

and

$$\|p\|_{\tilde{Q}} \leq \frac{1}{\beta_b}\ \{\|\boldsymbol{G}\|_{\boldsymbol{H}'} + \varepsilon\|\boldsymbol{u}\|_{\boldsymbol{H}}\},$$

and the result follows. □

Let us assume that $\boldsymbol{f} \in L^\infty(\Omega)^d$ and, for a given $r \in \tilde{Q}$, define the linear functional $\boldsymbol{F}_r : \boldsymbol{H} \to \mathbb{R}$ by

$$\boldsymbol{F}_r(\boldsymbol{v}) := \gamma(r\,\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}.$$

In the special case when $r = 1$, we will use the notation $\boldsymbol{F}$ instead of $\boldsymbol{F}_1$. It is clear that

$$|\boldsymbol{F}_r(\boldsymbol{v})| \leq \gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\|\boldsymbol{v}\|_{0,\Omega} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}, \tag{3.17}$$

which means that $\boldsymbol{F}_r \in \boldsymbol{H}'$.

Note that problem (2.6)–(2.7) can be written as: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \tilde{Q}$ *such that*

$$B((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{F}_p(\boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H} \times \tilde{Q}. \tag{3.18}$$

The next result guarantees the solvability of problem (3.18). The proof is an adaptation of [37, Theorem 3.1].

Theorem 3. *Assume that* $\boldsymbol{f} \in L^\infty(\Omega)^d$ *and*

$$\left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\gamma\|\boldsymbol{f}\|_{\infty,\Omega} < 1. \tag{3.19}$$

Then, problem (3.18) *has a unique solution* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \tilde{Q}$ *and there exist a positive constant* C*, independent of* ε *and* γ*, such that*

$$\|(\boldsymbol{u}, p)\|_{\boldsymbol{H}\times\tilde{Q}} \leq C\left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\|\boldsymbol{f}\|_{\infty,\Omega}. \tag{3.20}$$

Proof. In order to prove the result we introduce the operator $T : \boldsymbol{H} \times \tilde{Q} \to \boldsymbol{H} \times \tilde{Q}$ defined by

$$T(\boldsymbol{w}, r) := (\bar{\boldsymbol{u}}, \bar{p}) \ \in\ \boldsymbol{H} \times \tilde{Q},$$

where $(\bar{\boldsymbol{u}}, \bar{p})$ is the unique solution of the linear problem

$$B((\bar{\boldsymbol{u}}, \bar{p}), (\boldsymbol{v}, q)) = \boldsymbol{F}_r(\boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{v}) \qquad \forall\ (\boldsymbol{v}, q) \in \boldsymbol{H} \times \tilde{Q}.$$

Hence, problem (3.18) is equivalent to the following fix point problem: *Find* $(\boldsymbol{u}, p) \in \boldsymbol{H} \times \tilde{Q}$*, such that*

$$T(\boldsymbol{u}, p) = (\boldsymbol{u}, p).$$

Now, it is clear that for each $(\boldsymbol{w}, r) \in \boldsymbol{H} \times \tilde{Q}$, we have

$$T(\boldsymbol{w}, r) = (\boldsymbol{u}^0, p^0) + S(\boldsymbol{w}, r),$$

where $(\boldsymbol{u}^0, p^0) \in \boldsymbol{H} \times \tilde{Q}$ is the unique solution of the auxiliar problem

$$B((\boldsymbol{u}^0, p^0), (\boldsymbol{v}, q)) = \boldsymbol{F}(\boldsymbol{v}) \qquad \forall\ (\boldsymbol{v}, q) \in \boldsymbol{H} \times \tilde{Q}, \tag{3.21}$$

and $S : \boldsymbol{H} \times Q \to \boldsymbol{H} \times \tilde{Q}$ is the linear operator defined by

$$S(\boldsymbol{w}, r) = (\tilde{\boldsymbol{u}}, \tilde{p}) \in \boldsymbol{H} \times \tilde{Q},$$

with $(\tilde{\boldsymbol{u}}, \tilde{p}) \in \boldsymbol{H} \times \tilde{Q}$ the solution of the problem

$$B((\tilde{\boldsymbol{u}}, \tilde{p}), (\boldsymbol{v}, q)) = \boldsymbol{F}_r(\boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{H} \times \tilde{Q}. \tag{3.22}$$

Using (3.15) and (3.17), we have

$$\|S(\boldsymbol{w}, r)\|_{\boldsymbol{H}\times\tilde{Q}} = \|(\tilde{\boldsymbol{u}}, \tilde{p})\|_{\boldsymbol{H}\times\tilde{Q}} \leq \left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}. \tag{3.23}$$

Let $(\boldsymbol{w}_1, r_1), (\boldsymbol{w}_2, r_2) \in \boldsymbol{H} \times Q$, then, using (3.23), we have

$$\begin{aligned}
\|T(\boldsymbol{w}_1, r_1) - T(\boldsymbol{w}_2, r_2)\|_{\boldsymbol{H}\times\tilde{Q}} &= \|S((\boldsymbol{w}_1, r_1) - (\boldsymbol{w}_2, r_2))\|_{\boldsymbol{H}\times\tilde{Q}} \\
&\leq \left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|r_1 - r_2\|_{0,\Omega} \\
&\leq \left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|(\boldsymbol{w}_1, r_1) - (\boldsymbol{w}_2, r_2)\|_{\boldsymbol{H}\times\tilde{Q}}.
\end{aligned}$$

Thus, from (3.19) and the Banach's fixed point Theorem, we get the well-posedness of the problem (3.14). Finally, by the triangle inequality, it follows that

$$\|(\boldsymbol{u}, p)\|_{\boldsymbol{H}\times\tilde{Q}} = \|T(\boldsymbol{u}, p)\|_{\boldsymbol{H}\times\tilde{Q}} \leq \|(\boldsymbol{u}^0, p^0)\|_{\boldsymbol{H}\times\tilde{Q}} + \|S(\boldsymbol{u}, p)\|_{\boldsymbol{H}\times\tilde{Q}} \leq \left(\frac{1}{\varepsilon} + \frac{2}{\beta_b}\right)\gamma\|\boldsymbol{f}\|_{\infty,\Omega}(|\Omega|^{1/2} + \|p\|_{0,\Omega}),$$

thus, using again (3.19), we conclude the proof. □

In the rest of this work we will replace the space $\tilde{Q} = L^2(\Omega)$ by $Q := H^1(\Omega)$, and over $\boldsymbol{H} \times Q$ we will define the following norm

$$|||(\boldsymbol{w}, r)||| := \varepsilon^{1/2}\,\|\boldsymbol{w}\|_{\boldsymbol{H}} + \|r\|_{1,\Omega},$$

for all $(\boldsymbol{w}, r) \in \boldsymbol{H} \times Q$.

4. The stabilized finite element method

From now on, we denote by $\{\mathcal{T}_h\}_{h>0}$ a regular family of triangulations of $\bar{\Omega}$ composed by simplexes. For a $\mathcal{T}_h$ we will denote by T the elements of the triangulation, and by $\mathcal{E}_h$ the set of all edges (faces) of $\mathcal{T}_h$, with the splitting $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_D \cup \mathcal{E}_N$, where $\mathcal{E}_\Omega$ stands for the edges (faces) lying in the interior of Ω, $\mathcal{E}_D$ and $\mathcal{E}_N$ stands for the edges (faces) on the boundaries Γ_D and Γ_N, respectively. As usual h_T means the diameter of T, $h = \max_{T\in\mathcal{T}_h} h_T$, and $h_F := |F|$ for $F \in \mathcal{E}_h$.

We introduce two finite element subspaces of $\boldsymbol{H}$ and Q, given by

$$\begin{aligned}
\boldsymbol{H}_h :=& \{\boldsymbol{v} \in C(\overline{\Omega})^d \ :\ \boldsymbol{v}|_T \in \mathbb{P}_k(T)^d, \quad \forall T \in \mathcal{T}_h\} \cap \boldsymbol{H}, \\
Q_h :=& \{q \in C(\overline{\Omega}) \ :\ q|_T \in \mathbb{P}_k(T), \quad \forall T \in \mathcal{T}_h\},
\end{aligned}$$

with $k \geq 1$, where $\mathbb{P}_k$ stands for the space of polynomials of total degree less or equal to k.

Next, we consider the following discrete stabilized scheme: *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ such that

$$B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \varphi \rangle_{\Gamma_D} + \gamma((p_h + 1)\boldsymbol{f}, \boldsymbol{v}_h) - \frac{1}{2}(\varepsilon^{-1}\, \gamma\, (p_h + 1)\boldsymbol{f}, \varepsilon \boldsymbol{v}_h + \nabla q_h), \tag{4.24}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$, where

$$B_{\text{stab}}((\boldsymbol{w}_h, r_h), (\boldsymbol{v}_h, q_h)) := B((\boldsymbol{w}_h, r_h), (\boldsymbol{v}_h, q_h)) - \frac{1}{2}(\varepsilon^{-1}(\varepsilon \boldsymbol{w}_h - \nabla r_h), \varepsilon\, \boldsymbol{v}_h + \nabla q_h) + \varepsilon\, (\nabla \cdot \boldsymbol{w}_h, \nabla \cdot \boldsymbol{v}_h). \tag{4.25}$$

Remark 2. *This scheme, as well as the proposed in [16], is based on the stabilized finite element method, for the linear Darcy equation, proposed in [25], however there are two major differences. The first is that our stabilized scheme indirectly approach the nonlinear problem due to the change of variable done in Section 2, and, as second difference, our scheme contains a* $\varepsilon\, (\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)$ *term which improves the numerical results.*

As it is well known, the bilinear form $b(\cdot, \cdot)$ does not satisfies an inf–sup condition, using the subspaces $\boldsymbol{H}_h$ and Q_h, but it satisfies the following weak inf–sup condition.

Lemma 4. *There exist positive constants* β_w *and* λ, *independent of* ε *and* h, *such that*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{H}_h} \frac{b(\boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}}} \geq \beta_w\, \|p_h\|_{0,\Omega} - \lambda\, |p_h|_{1,\Omega} \qquad \forall p_h \in Q_h. \tag{4.26}$$

Proof. The proof of this result uses similar arguments to those used in [40, Lemma 3.3]. Let $p_h \in Q_h$, then there exist $\bar{p}_h \in \mathbb{R}$ and $p_h^* \in L_0^2(\Omega)$, such that $p_h = \bar{p}_h + p_h^*$. Additionally, there exists $\boldsymbol{w} \in H_0^1(\Omega)^d$ (see [12]) such that

$$(\nabla \cdot \boldsymbol{w}, p_h) = (\nabla \cdot \boldsymbol{w}, p_h^*) \geq C_1 \|p_h^*\|_{0,\Omega} \|\boldsymbol{w}\|_{1,\Omega}. \tag{4.27}$$

Furthermore, let $C_h \boldsymbol{w} \in \boldsymbol{H}_h \cap H_0^1(\Omega)^d$ the Clément interpolate of $\boldsymbol{w}$ (see [42]). This interpolation operator satisfies

$$\left\{ \sum_{K \in \mathcal{T}_h} h_K^{-2} \|\boldsymbol{w} - C_h \boldsymbol{w}\|_{0,K}^2 \right\}^{1/2} \leq C_2 \|\boldsymbol{w}\|_{1,\Omega}, \tag{4.28}$$

and

$$\|C_h \boldsymbol{w}\|_{1,\Omega} \leq C_3 \|\boldsymbol{w}\|_{1,\Omega}. \tag{4.29}$$

Using (4.27) and integration by parts, we get that

$$\begin{aligned}
(\nabla \cdot C_h \boldsymbol{w}, p_h) &= (\nabla \cdot C_h \boldsymbol{w}, p_h^*) \\
&= (\nabla \cdot (C_h \boldsymbol{w} - \boldsymbol{w}), p_h^*) + (\nabla \cdot \boldsymbol{w}, p_h^*) \\
&\geq \sum_{K \in \mathcal{T}_h} (C_h \boldsymbol{w} - \boldsymbol{w}, \nabla p_h^*)_K + C_1 \|p_h^*\|_{0,\Omega} \|\boldsymbol{w}\|_{1,\Omega}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and (4.28), we obtain

$$\begin{aligned}
(\nabla \cdot C_h \boldsymbol{w}, p_h) &\geq -\left(\sum_{K \in \mathcal{T}_h} h_K^{-2} \|C_h \boldsymbol{w} - \boldsymbol{w}\|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h^*\|_{0,K}^2 \right)^{1/2} + C_1 \|p_h^*\|_{0,\Omega} \|\boldsymbol{w}\|_{1,\Omega} \\
&\geq \left\{ -C_2 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h^*\|_{0,K}^2 \right)^{1/2} + C_1 \|p_h^*\|_{0,\Omega} \right\} \|\boldsymbol{w}\|_{1,\Omega},
\end{aligned}$$

and, in consequence

$$\frac{(\nabla \cdot C_h \boldsymbol{w}, p_h)}{\|\boldsymbol{w}\|_{1,\Omega}} \geq C_1 \|p_h^*\|_{0,\Omega} - C_2 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h^*\|_{0,K}^2 \right)^{1/2}. \tag{4.30}$$

We can assume (see [40]) that, for a reasonable mesh, there exist $z_h \in \boldsymbol{H}_h$, $z_h \neq \mathbf{0}$, such that

$$\frac{(\nabla \cdot z_h, \bar{p}_h)}{\|z_h\|_{1,\Omega}} \geq C_4 \|\bar{p}_h\|_{0,\Omega}. \tag{4.31}$$

Let $\tilde{\boldsymbol{v}}_h := \|\boldsymbol{w}\|_{1,\Omega}^{-1} C_h \boldsymbol{w} + \delta \|z_h\|_{1,\Omega}^{-1} z_h$, with $\delta > 0$. It is clear that $\tilde{\boldsymbol{v}}_h \in \boldsymbol{H}_h$ and using (4.30), (4.31), we get

$$\begin{aligned}
(\nabla \cdot \tilde{\boldsymbol{v}}_h, p_h) &= \frac{(\nabla \cdot C_h \boldsymbol{w}, p_h)}{\|\boldsymbol{w}\|_{1,\Omega}} + \delta \frac{(\nabla \cdot z_h, \bar{p}_h)}{\|z_h\|_{1,\Omega}} + \delta \frac{(\nabla \cdot z_h, p_h^*)}{\|z_h\|_{1,\Omega}} \\
&\geq C_1 \|p_h^*\|_{0,\Omega} - C_2 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h^*\|_{0,K}^2 \right)^{1/2} + \delta C_4 \|\bar{p}_h\|_{0,\Omega} - \delta C_5 \|p_h^*\|_{0,\Omega} \\
&\geq C_6 \|p_h\|_{0,\Omega} - C_7 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h^*\|_{0,K}^2 \right)^{1/2},
\end{aligned}$$

assuming that $\delta < C_1/C_5$. On the other hand, note that the definition of $\tilde{\boldsymbol{v}}_h$ and (4.29) shows $\|\tilde{\boldsymbol{v}}_h\|_{\boldsymbol{H}} \leq C \|\tilde{\boldsymbol{v}}_h\|_{1,\Omega} \leq C\, C_3 + C\, \delta$, and hence, we have that

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{H}_h} \frac{(\nabla \cdot \boldsymbol{v}_h, p_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}}} \geq C_8 \|p_h\|_{0,\Omega} - C_9 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla p_h\|_{0,K}^2 \right)^{1/2},$$

which conclude the proof. □

The following result will be fundamental to prove the well–posedness of the stabilized finite element scheme. The proof is based on the same arguments used in [41, Lemma 4.1].

Lemma 5. *Let $B_{\text{stab}}(\cdot,\cdot)$ be the bilinear form defined in* (4.25). *Then there is a positive constant β_s, independent of h and ε, such that*

$$\sup_{(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h} \frac{B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h))}{|||(\boldsymbol{v}_h, q_h)|||} \geq \beta_s\, |||(\boldsymbol{u}_h, p_h)|||, \tag{4.32}$$

for all $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$.

Proof. Let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ and let $\boldsymbol{w}_h \in \boldsymbol{H}_h$ be a function for which the supremum in Lemma 4 is attained, and such that $\|\boldsymbol{w}_h\|_{\boldsymbol{H}} = \|p_h\|_{0,\Omega}$. If we consider $\tilde{\boldsymbol{w}}_h = -\boldsymbol{w}_h$, we have

$$\frac{-(p_h, \nabla \cdot \tilde{\boldsymbol{w}}_h)}{\|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}}} = \frac{(p_h, \nabla \cdot \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{\boldsymbol{H}}} \geq \beta_w \|p_h\|_{0,\Omega} - \lambda |p_h|_{1,\Omega},$$

and therefore,

$$-(p_h, \nabla \cdot \tilde{\boldsymbol{w}}_h) \geq \beta_w \|p_h\|_{0,\Omega}^2 - \lambda |p_h|_{1,\Omega} \|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}}. \tag{4.33}$$

Then, for $(\boldsymbol{v}_h, q_h) := (\boldsymbol{u}_h - \delta\tilde{\boldsymbol{w}}_h, p_h)$, with $\delta > 0$, we get that

$$
\begin{aligned}
B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) =& B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) - \delta\, B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\tilde{\boldsymbol{w}}_h, 0)) \\
=& B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) - \delta \left[B_{\text{stab}}((\boldsymbol{u}_h, 0), (\tilde{\boldsymbol{w}}_h, 0)) + B_{\text{stab}}((\boldsymbol{0}, p_h), (\tilde{\boldsymbol{w}}_h, 0)) \right] \\
=& \frac{1}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\varepsilon^{-1}\, |p_h|_{1,\Omega}^2 \\
& - \delta \left[\frac{1}{2}\varepsilon\, (\boldsymbol{u}_h, \tilde{\boldsymbol{w}}_h) + \varepsilon\, (\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\tilde{\boldsymbol{w}}_h) + (p_h, \nabla\cdot\tilde{\boldsymbol{w}}_h) + \frac{1}{2}\, (\nabla p_h, \tilde{\boldsymbol{w}}_h) \right] \\
=& \frac{1}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\varepsilon^{-1}\, |p_h|_{1,\Omega}^2 \\
& - \frac{\delta}{2}\varepsilon\, (\boldsymbol{u}_h, \tilde{\boldsymbol{w}}_h) - \delta\varepsilon\, (\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\tilde{\boldsymbol{w}}_h) - \delta\, (p_h, \nabla\cdot\tilde{\boldsymbol{w}}_h) - \frac{\delta}{2}\, (\nabla p_h, \tilde{\boldsymbol{w}}_h) \\
\geq& \frac{1}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\varepsilon^{-1}\, |p_h|_{1,\Omega}^2 + \delta\beta_w\, \|p_h\|_{0,\Omega}^2 - \delta\lambda |p_h|_{1,\Omega}\, \|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}} \\
& - \frac{\delta}{2}\varepsilon\, (\boldsymbol{u}_h, \tilde{\boldsymbol{w}}_h) - \delta\varepsilon\, (\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\tilde{\boldsymbol{w}}_h) - \frac{\delta}{2}\, (\nabla p_h, \tilde{\boldsymbol{w}}_h) \\
\geq& \frac{1}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\varepsilon^{-1}\, |p_h|_{1,\Omega}^2 + \delta\beta_w\, \|p_h\|_{0,\Omega}^2 \\
& - \frac{\delta}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}\|\tilde{\boldsymbol{w}}_h\|_{0,\Omega} - \delta\varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}\|\nabla\cdot\tilde{\boldsymbol{w}}_h\|_{0,\Omega} - \frac{\delta}{2}|p_h|_{1,\Omega}\|\tilde{\boldsymbol{w}}_h\|_{0,\Omega} - \delta\lambda|p_h|_{1,\Omega}\|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}}.
\end{aligned}
$$

Now, using Young's inequality $2ab \leq \dfrac{a^2}{\gamma} + \gamma\, b^2$, for all $a, b, \gamma > 0$, and the fact that $\|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}} = \|p_h\|_{0,\Omega}$, we get

$$
\begin{aligned}
B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) \geq& \frac{1}{2}\varepsilon\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\varepsilon^{-1}\, |p_h|_{1,\Omega}^2 + \delta\beta_w\, \|p_h\|_{0,\Omega}^2 \\
& - \frac{\delta\varepsilon}{4\gamma_1}\, \|\boldsymbol{u}_h\|_{0,\Omega}^2 - \frac{\delta\varepsilon\gamma_1}{4}\, \|\tilde{\boldsymbol{w}}_h\|_{0,\Omega}^2 \\
& - \frac{\delta\varepsilon}{2\gamma_2}\, \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 - \frac{\delta\varepsilon\gamma_2}{2}\, \|\nabla\cdot\tilde{\boldsymbol{w}}_h\|_{0,\Omega}^2 \\
& - \frac{\delta}{4\gamma_3}|p_h|_{1,\Omega}^2 - \frac{\delta\gamma_3}{4}\|\tilde{\boldsymbol{w}}_h\|_{0,\Omega}^2 \\
& - \frac{\delta\lambda}{2\gamma_4}|p_h|_{1,\Omega}^2 - \frac{\delta\lambda\gamma_4}{2}\|\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}}^2 \\
\geq& \frac{1}{2}\varepsilon\left(1 - \frac{\delta}{2\gamma_1}\right) \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\left(1 - \frac{\delta}{2\gamma_2}\right) \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 \\
& + \frac{1}{2}\left(\varepsilon^{-1} - \frac{\delta}{2\gamma_3} - \frac{\delta\lambda}{2\gamma_4}\right) |p_h|_{1,\Omega}^2 \\
& + \delta\left(\beta_w - \frac{\delta\varepsilon\gamma_1}{4} - \frac{\delta\varepsilon\gamma_2}{2} - \frac{\delta\gamma_3}{4} - \frac{\delta\lambda\gamma_4}{2}\right) \|p_h\|_{0,\Omega}^2 \\
\geq& \frac{1}{2}\varepsilon\left(1 - \frac{\delta}{2\gamma_1}\right) \|\boldsymbol{u}_h\|_{0,\Omega}^2 + \varepsilon\left(1 - \frac{\delta}{2\gamma_2}\right) \|\nabla\cdot\boldsymbol{u}_h\|_{0,\Omega}^2 \\
& + \frac{1}{2}\left(\varepsilon^{-1} - \frac{\delta}{2\gamma_3} - \frac{\delta\lambda}{2\gamma_4}\right) |p_h|_{1,\Omega}^2 + \delta C_{10}\, \|p_h\|_{0,\Omega}^2,
\end{aligned}
$$

with $C_{10} > 0$, if $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are chosen small enough.

Now, if we choose $0 < \delta < \min\left\{2\gamma_1, 2\gamma_2, \dfrac{2\gamma_3\gamma_4\varepsilon^{-1}}{\gamma_4 + \gamma_3\lambda}, \varepsilon^{-1/2}\right\}$, we have

$$
B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) \geq C\, |||(\boldsymbol{u}_h, p_h)|||^2. \tag{4.34}
$$

On the other hand, using the definition of $\boldsymbol{v}_h, q_h, \delta$ and the triangle inequality, we have

$$|||(\boldsymbol{v}_h, q_h)||| \leq \sqrt{\varepsilon}\,\|\boldsymbol{u}_h\|_{\boldsymbol{H}} + \delta\sqrt{\varepsilon}\,\|p_h\|_{0,\Omega} + \|p_h\|_{1,\Omega} \leq C\,|||(\boldsymbol{u}_h, p_h)|||,$$

thus, using (4.34), we complete the proof. □

Remark 3. *This result is also valid for the continuous spaces $\boldsymbol{H}$ and Q and it will be used in the analysis of the a posteriori error estimator proposed in Section 5 (for details, see Lemma 12). On the other hand, if we use stable subspaces of $\boldsymbol{H}$ and Q, as for example, Raviart-Thomas elements of degree k, for the velocity, and piecewise polynomial elements of order k, for the pressure, or the Brezzi-Douglas-Marini spaces of order k, for the velocity, and piecewise polynomial elements of order $k-1$, for the pressure, Lemma 5 is also true (for details on stable subspaces, see [43]). In both cases the proof is similar to that proposed for Lemma 12 and therefore* (4.24) *can be seen as an augmented finite element method when stable subspaces of $\boldsymbol{H}$ and Q are used.*

Concerning the well-posedness of the stabilized discrete problem (4.24), we have the following result.

Theorem 6. *Let $\beta_s > 0$ as in* (4.32) *and $\beta_c > 0$ as in* (A.1). *If*

$$\gamma\,\|\boldsymbol{f}\|_{\infty,\Omega} \leq \frac{\min\{\beta_s, \beta_c\}\,\varepsilon^{1/2}}{3 + \varepsilon^{-1/2}}, \tag{4.35}$$

then the discrete stabilized problem (4.24) *has a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$.*

Proof. As in the continuous case, we write the solution of the problem (4.24), like the solution of a fixed point problem. Thereby, given $r \in L^2(\Omega)$, we define the linear functionals

$$\boldsymbol{F}^s_r : \boldsymbol{H}_h \times Q_h \longrightarrow \mathbb{R} \quad \text{and} \quad \boldsymbol{F}^s : \boldsymbol{H}_h \times Q_h \longrightarrow \mathbb{R},$$

by

$$\begin{aligned}
\boldsymbol{F}^s_r(\boldsymbol{v}_h, q_h) &:= \gamma(r\,\boldsymbol{f}, \boldsymbol{v}_h) - \frac{1}{2}(\varepsilon^{-1}\gamma\, r\,\boldsymbol{f},\ \varepsilon\,\boldsymbol{v}_h + \nabla q_h),\\
\boldsymbol{F}^s(\boldsymbol{v}_h, q_h) &:= \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \varphi\rangle_{\Gamma_D} + \gamma(\boldsymbol{f}, \boldsymbol{v}_h) - \frac{1}{2}(\varepsilon^{-1}\gamma\,\boldsymbol{f},\ \varepsilon\,\boldsymbol{v}_h + \nabla q_h).
\end{aligned}$$

Now, we can write equation (4.24) as

$$B_{\text{stab}}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \boldsymbol{F}^s_{p_h}(\boldsymbol{v}_h, q_h) + \boldsymbol{F}^s(\boldsymbol{v}_h, q_h) \qquad \forall\ (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h.$$

If we assume that $\boldsymbol{f} \in L^\infty(\Omega)^d$, the functional $\boldsymbol{F}^s_r$ satisfy

$$\begin{aligned}
|\boldsymbol{F}^s_r(\boldsymbol{v}_h, q_h)| &\leq \gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\|\boldsymbol{v}_h\|_{0,\Omega} + \frac{1}{2}\varepsilon^{-1}\gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\,\{\varepsilon\|\boldsymbol{v}_h\|_{0,\Omega} + \|\nabla q_h\|_{0,\Omega}\}\\
&\leq \gamma\varepsilon^{-1/2}\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}|||(\boldsymbol{v}_h, q_h)||| + \frac{1}{2}\varepsilon^{-1}\gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\left\{\varepsilon^{1/2} + 1\right\}|||(\boldsymbol{v}_h, q_h)|||\\
&\leq \frac{\varepsilon^{-1/2}}{2}\left\{3 + \varepsilon^{-1/2}\right\}\gamma\|r\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}|||(\boldsymbol{v}_h, q_h)|||.
\end{aligned} \tag{4.36}$$

Moreover, let $T_h : \boldsymbol{H}_h \times Q_h \longrightarrow \boldsymbol{H}_h \times Q_h$ the operator defined, for a given $(\boldsymbol{w}_h, r_h) \in \boldsymbol{H}_h \times Q_h$, by

$$T_h(\boldsymbol{w}_h, r_h) = (\bar{\boldsymbol{u}}_h, \bar{p}_h),$$

where $(\bar{\boldsymbol{u}}_h, \bar{p}_h) \in \boldsymbol{H}_h \times Q_h$ is the unique solution of the linear problem

$$B_{\text{stab}}((\bar{\boldsymbol{u}}_h, \bar{p}_h), (\boldsymbol{v}_h, q_h)) = \boldsymbol{F}^s_{r_h}(\boldsymbol{v}_h, q_h) + \boldsymbol{F}^s(\boldsymbol{v}_h, q_h) \qquad \forall\ (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h.$$

In this way, the discrete problem (4.24) can be written as follows: *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ such that*

$$T_h(\boldsymbol{u}_h, p_h) = (\boldsymbol{u}_h, p_h).$$

Now, we can observe that

$$T_h(\boldsymbol{w}_h, r_h) = (\boldsymbol{u}_h^0, p_h^0) + S_h(\boldsymbol{w}_h, r_h) \qquad \forall\, (\boldsymbol{w}_h, r_h) \in \boldsymbol{H}_h \times Q_h,$$

where $(\boldsymbol{u}_h^0, p_h^0) \in \boldsymbol{H}_h \times Q_h$ is the unique solution of the auxiliar problem

$$B_{\text{stab}}((\boldsymbol{u}_h^0, p_h^0), (\boldsymbol{v}_h, q_h)) = \boldsymbol{F}^s(\boldsymbol{v}_h, q_h), \quad \forall\ (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h, \tag{4.37}$$

and $S_h : \boldsymbol{H}_h \times Q_h \longrightarrow \boldsymbol{H}_h \times Q_h$ is the linear operator defined by

$$S_h(\boldsymbol{w}_h, r_h) = (\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{H}_h \times Q_h,$$

where $(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \in \boldsymbol{H}_h \times Q_h$ satisfies the problem

$$B_{\text{stab}}((\tilde{\boldsymbol{u}}_h, \tilde{p}_h), (\boldsymbol{v}_h, q_h)) = \boldsymbol{F}^s_{r_h}(\boldsymbol{v}_h, q_h), \quad \forall\ (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h. \tag{4.38}$$

Furthermore, using the continuous dependence result and inequality (4.36), we have that

$$|||S_h(\boldsymbol{w}_h, r_h)||| = |||(\tilde{\boldsymbol{u}}_h, \tilde{p}_h)||| \leq \frac{\varepsilon^{-1/2}}{2\beta_s} \left\{3 + \varepsilon^{-1/2}\right\} \gamma \|r_h\|_{0,\Omega} \|\boldsymbol{f}\|_{\infty,\Omega}. \tag{4.39}$$

Let $(\boldsymbol{w}_h^1, r_h^1), (\boldsymbol{w}_h^2, r_h^2) \in \boldsymbol{H}_h \times Q_h$. Then, from (4.39) we have

$$\begin{aligned}
|||T_h(\boldsymbol{w}_h^1, r_h^1) - T_h(\boldsymbol{w}_h^2, r_h^2)||| =& |||S_h(\boldsymbol{w}_h^1, r_h^1) - S_h(\boldsymbol{w}_h^2, r_h^2)||| \\
=& |||S_h(\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2, r_h^1 - r_h^2)||| \\
\leq& \frac{\varepsilon^{-1/2}}{2\beta_s} \left\{3 + \varepsilon^{-1/2}\right\} \gamma \|r_h^1 - r_h^2\|_{0,\Omega} \|\boldsymbol{f}\|_{\infty,\Omega} \\
\leq& \frac{\varepsilon^{-1/2}}{2\beta_s} \left\{3 + \varepsilon^{-1/2}\right\} \gamma \|\boldsymbol{f}\|_{\infty,\Omega} |||(\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2, r_h^1 - r_h^2)|||.
\end{aligned}$$

Thus, using condition (4.35), we have that

$$|||T_h(\boldsymbol{w}_h^1, r_h^1) - T_h(\boldsymbol{w}_h^2, r_h^2)||| \leq \frac{1}{2} |||(\boldsymbol{w}_h^1 - \boldsymbol{w}_h^2, r_h^1 - r_h^2)|||.$$

The result follows using the Banach fixed point theorem. □

We consider the Lagrange interpolants $\boldsymbol{I}_h : H^{k+1}(\Omega)^d \longrightarrow \boldsymbol{H}_h$ and $\mathcal{J}_h : H^{k+1}(\Omega) \longrightarrow Q_h$ such that (see [44] for details):

$$|\boldsymbol{u} - \boldsymbol{I}_h \boldsymbol{u}|_{l,\Omega} \leq C h^{s-l}\, |\boldsymbol{u}|_{s,\Omega}, \tag{4.40}$$

$$|p - \mathcal{J}_h p|_{l,\Omega} \leq C h^{s-l}\, |p|_{s,\Omega}, \tag{4.41}$$

for all $\boldsymbol{u} \in H^s(\Omega)^d$ and all $p \in H^s(\Omega)$ with $l = 0, 1$ and $1 \leq s \leq k+1$.

Lemma 7. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of (3.18) and (4.24), respectively. If $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \cap \boldsymbol{H} \times H^{k+1}(\Omega)$, then it holds*

$$|B_{\text{stab}}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h))| \leq \frac{\varepsilon^{-1/2}}{2} (3 + \varepsilon^{-1/2})\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} \|p - p_h\|_{0,\Omega}\, |||(\boldsymbol{v}_h, q_h)|||, \tag{4.42}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h$.

Proof. Using the regularity of the solution $(\boldsymbol{u}, p)$ of (3.18), we can prove that $\varepsilon\boldsymbol{u} - \gamma(p+1)\boldsymbol{f} - \nabla p = \boldsymbol{0}$, $\nabla\cdot\boldsymbol{u} = 0$ and $p = \varphi$ on Γ_D. Now, using integration by parts and the definition of $B_{\text{stab}}(\cdot,\cdot)$, we have

$$\begin{aligned}
B_{\text{stab}}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\boldsymbol{v}_h,q_h)) &= B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{v}_h,q_h)) - B_{\text{stab}}((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h,q_h))\\
&= \varepsilon(\boldsymbol{u},\boldsymbol{v}_h) + (p,\nabla\cdot\boldsymbol{v}_h) - (q_h,\nabla\cdot\boldsymbol{u})\\
&\quad - \frac{1}{2}(\varepsilon^{-1}(\varepsilon\boldsymbol{u}-\nabla p), \varepsilon\boldsymbol{v}_h + \nabla q_h) + \varepsilon\,(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v}_h)\\
&\quad - \langle \boldsymbol{v}_h\cdot\boldsymbol{n},\varphi\rangle_{\Gamma_D} - \gamma((p_h+1)\boldsymbol{f},\boldsymbol{v}_h) + \frac{1}{2}(\varepsilon^{-1}\gamma(p_h+1)\boldsymbol{f}, \varepsilon\boldsymbol{v}_h+\nabla q_h)\\
&= (\varepsilon\boldsymbol{u} - \gamma(p_h+1)\boldsymbol{f} - \nabla p, \boldsymbol{v}_h)\\
&\quad - \frac{1}{2}(\varepsilon^{-1}[\varepsilon\boldsymbol{u} - \nabla p - \gamma(p_h+1)\boldsymbol{f}], \varepsilon\boldsymbol{v}_h + \nabla q_h)\\
&=(\gamma(p-p_h)\boldsymbol{f},\boldsymbol{v}_h) - \frac{1}{2}(\varepsilon^{-1}\gamma(p-p_h)\boldsymbol{f}, \varepsilon\boldsymbol{v}_h+\nabla q_h)\\
&= \frac{1}{2}(\gamma(p-p_h)\boldsymbol{f},\boldsymbol{v}_h) - \frac{1}{2}(\varepsilon^{-1}\gamma(p-p_h)\boldsymbol{f},\nabla q_h)\\
&\leq \frac{1}{2}\left\{\gamma\|p-p_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\varepsilon^{-1/2} + \gamma\|p-p_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\varepsilon^{-1}\right\} |||(\boldsymbol{v}_h,q_h)|||\\
&= \frac{\varepsilon^{-1/2}}{2}(1+\varepsilon^{-1/2})\,\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|p-p_h\|_{0,\Omega}\,|||(\boldsymbol{v}_h,q_h)|||\\
&\leq \frac{\varepsilon^{-1/2}}{2}(3+\varepsilon^{-1/2})\,\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|p-p_h\|_{0,\Omega}\,|||(\boldsymbol{v}_h,q_h)|||,
\end{aligned}$$

and the result follows. □

Theorem 8 (Main Result)**.** *Let $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \cap \boldsymbol{H} \times H^{k+1}(\Omega)$, be the solution of* (3.18) *and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ solution of* (4.24)*. If we assume* (4.35)*, then it holds*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq C\,h^k\left\{(\varepsilon^{1/2}+1)\,\|\boldsymbol{u}\|_{k+1,\Omega} + (\varepsilon^{-1/2}+\varepsilon^{-1}+1)\,\|p\|_{k+1,\Omega}\right\},$$

with $C > 0$ independent of h and ε.

Proof. Let

$$(\eta^{\boldsymbol{u}}, \eta^p) := (\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, p - \mathcal{J}_h p) \quad\text{and}\quad (e_h^{\boldsymbol{u}}, e_h^p) := (\boldsymbol{u}_h - \boldsymbol{I}_h\boldsymbol{u}, p_h - \mathcal{J}_h p).$$

Using the definition of B_{stab} given in (4.25), and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&B_{\text{stab}}((\eta^{\boldsymbol{u}},\eta^p),(\boldsymbol{v}_h,q_h))\\
&= \varepsilon(\eta^{\boldsymbol{u}},\boldsymbol{v}_h) + (\eta^p,\nabla\cdot\boldsymbol{v}_h) - (q_h,\nabla\cdot\eta^{\boldsymbol{u}}) - \frac{1}{2}(\varepsilon^{-1}(\varepsilon\eta^{\boldsymbol{u}} - \nabla\eta^p), \varepsilon\boldsymbol{v}_h+\nabla q_h) + \varepsilon\,(\nabla\cdot\eta^{\boldsymbol{u}},\nabla\cdot\boldsymbol{v}_h)\\
&\leq \left\{\varepsilon^{1/2}\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \frac{1}{2}\,(\varepsilon^{1/2}+1)[\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1}\,|\eta^p|_{1,\Omega}] + \varepsilon^{1/2}\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega}\right\}|||(\boldsymbol{v}_h,q_h)|||\\
&\leq \left\{\frac{1}{2}\,(3\varepsilon^{1/2}+1)\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + (\varepsilon^{1/2}+1)\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \frac{1}{2}\,\varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega}\right\}|||(\boldsymbol{v}_h,q_h)|||. \qquad (4.43)
\end{aligned}$$

Using lemmas 5 and 7, and inequality (4.43), we get that

$$\begin{aligned}
B_{\text{stab}}((e_h^{\boldsymbol{u}},e_h^p),(\boldsymbol{v}_h,q_h)) &= B_{\text{stab}}((\eta^{\boldsymbol{u}},\eta^p),(\boldsymbol{v}_h,q_h)) - B_{\text{stab}}((\boldsymbol{u}-\boldsymbol{u}_h,p-p_h),(\boldsymbol{v}_h,q_h))\\
&\leq \Big[\frac{1}{2}\,(3\varepsilon^{1/2}+1)\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + (\varepsilon^{1/2}+1)\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} +\\
&\quad \frac{1}{2}\,\varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega}\Big]|||(\boldsymbol{v}_h,q_h)||| +\\
&\quad \frac{\varepsilon^{-1/2}}{2}(3+\varepsilon^{-1/2})\,\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|p-p_h\|_{0,\Omega}\,|||(\boldsymbol{v}_h,q_h)|||.
\end{aligned}$$

Now, using Lemma 5 and (4.35), we have

$$
\begin{aligned}
&\beta_s\, |||(e_h^{\boldsymbol{u}}, e_h^p)||| \\
\leq &\frac{1}{2}\, (3\varepsilon^{1/2}+1)\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + (\varepsilon^{1/2}+1)\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \\
&\frac{1}{2}\,\varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega} + \frac{\varepsilon^{-1/2}}{2}(3+\varepsilon^{-1/2})\,\gamma\|\boldsymbol{f}\|_{\infty,\Omega}\|p-p_h\|_{0,\Omega} \\
\leq &\frac{1}{2}\, (3\varepsilon^{1/2}+1)\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + (\varepsilon^{1/2}+1)\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \\
&\frac{1}{2}\,\varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega} + \frac{\varepsilon^{-1/2}}{2}(3+\varepsilon^{-1/2})\,\gamma\|\boldsymbol{f}\|_{\infty,\Omega}|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \\
\leq &\frac{1}{2}\, (3\varepsilon^{1/2}+1)\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + (\varepsilon^{1/2}+1)\|\nabla\cdot\eta^{\boldsymbol{u}}\|_{0,\Omega} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \\
&\frac{1}{2}\,\varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega} + \frac{\beta_s}{2}\, |||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)|||.
\end{aligned}
\tag{4.44}
$$

Furthermore, using the triangle inequality and (4.44), we obtain

$$
\begin{aligned}
|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq |||(\eta^{\boldsymbol{u}},\eta^p)||| &+ C\left\{(\varepsilon^{1/2}+1)\,\|\eta^{\boldsymbol{u}}\|_{\boldsymbol{H}} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \varepsilon^{-1}(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega}\right\} \\
&+ \frac{1}{2}\, |||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)|||,
\end{aligned}
$$

thus

$$
|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq C\left\{|||(\eta^{\boldsymbol{u}},\eta^p)||| + (\varepsilon^{1/2}+1)\,\|\eta^{\boldsymbol{u}}\|_{\boldsymbol{H}} + \varepsilon^{-1/2}\|\eta^p\|_{0,\Omega} + \varepsilon^{-1}\,(\varepsilon^{1/2}+1)\,|\eta^p|_{1,\Omega}\right\}.
$$

Finally, using the properties of $\boldsymbol{I}_h$ and $\mathcal{J}_h$, we have

$$
|||(\eta^{\boldsymbol{u}},\eta^p)||| \leq C\, h^k\left\{\varepsilon^{1/2}\,\|\boldsymbol{u}\|_{k+1,\Omega} + \|p\|_{k+1,\Omega}\right\},
$$

and the result follows. □

5. A posteriori error analysis

In this section, we present a residual a posteriori error estimator for the stabilized finite element method (4.24). Let $\Gamma_{D,h}$ be the partition of Γ_D inherited from the triangulation $\mathcal{T}_h$, and define the mesh size $h_D := \max\{|F| : F \in \Gamma_{D,h}\}$. For simplicity, we assume that

- $\boldsymbol{f}$ is a piecewise polynomial in Ω; i.e $\boldsymbol{f}\Big|_K \in \mathbb{P}_l(K)^d$, $\forall K \in \mathcal{T}_h$, $l \geq 0$.
- φ is a continuous piecewise polynomial in $\Gamma_{D,h}$; i.e $\varphi \in C^0(\Gamma_{D,h})$, $\varphi\Big|_F \in \mathbb{P}_l(F)$, $\forall F \in \Gamma_{D,h}$, $l \geq 0$.

For each $K \in \mathcal{T}_h$ and each $F \in \mathcal{E}_D$, we define the residuals

$$
\begin{aligned}
\mathcal{R}_K :=& \Big(\gamma\,(p_h+1)\boldsymbol{f} - \varepsilon\,\boldsymbol{u}_h + \nabla p_h\Big)\Big|_K, \\
\mathcal{R}_F :=& (\varphi - p_h)\Big|_F.
\end{aligned}
$$

Thus, our residual–based error estimator is given by

$$\eta := \left\{ \sum_{K\in\mathcal{T}_h} \eta_K^2 \right\}^{1/2}, \tag{5.45}$$

where, for each $K \in \mathcal{T}_h$, we have that

$$\eta_K^2 := \|\mathcal{R}_K\|_{0,K}^2 + \varepsilon^2 \, \|\nabla\cdot \boldsymbol{u}_h\|_{0,K}^2 + \sum_{F\subset\mathcal{E}(K)\cap\mathcal{E}_D} h_F^{-1}\, \|\mathcal{R}_F\|_{0,F}^2. \tag{5.46}$$

Lemma 9. *Let $(\boldsymbol{u}, p) \in \boldsymbol{H}\times Q$ and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$, be the solutions of* (2.5) *and* (4.24)*, respectively. Then, for all $(\boldsymbol{v}, q) \in \boldsymbol{H}\times Q$, we have*

$$\begin{aligned}
&B_{\mathrm{stab}}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\boldsymbol{v},q))\\
&= \langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi - p_h\rangle_{\Gamma_D} + \frac{1}{2}\left(\varepsilon^{-1/2}\gamma(p-p_h)\boldsymbol{f}, \varepsilon^{1/2}\boldsymbol{v} - \varepsilon^{-1/2}\nabla q\right)\\
&\quad + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\varepsilon^{-1/2}(\mathcal{R}_K), \varepsilon^{1/2}\boldsymbol{v} - \varepsilon^{-1/2}\nabla q\right)_K + (q, \nabla\cdot\boldsymbol{u}_h) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}).
\end{aligned}$$

Proof. Using (2.6), (2.7), (4.24) and integration by parts, we get that

$$\begin{aligned}
&B_{\mathrm{stab}}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\boldsymbol{v},q))\\
=&B_{\mathrm{stab}}((\boldsymbol{u}, p),(\boldsymbol{v},q)) - B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h),(\boldsymbol{v},q))\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{\Gamma_D} + \gamma((p+1)\boldsymbol{f}, \boldsymbol{v}) - \frac{1}{2}\left(\varepsilon^{-1}(\gamma\,(p+1)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right)\\
&-\varepsilon\,(\boldsymbol{u}_h, \boldsymbol{v}) - (p_h, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h) + \frac{1}{2}\left(\varepsilon^{-1}(\varepsilon\boldsymbol{u}_h - \nabla p_h), \varepsilon\boldsymbol{v} + \nabla q\right) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{\Gamma_D} + \gamma((p+1)\boldsymbol{f}, \boldsymbol{v}) + \varepsilon\,(\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) - \varepsilon\,(\boldsymbol{u}_h, \boldsymbol{v}) - (p_h, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h)\\
&+\frac{1}{2}\left(\varepsilon^{-1}(\varepsilon\boldsymbol{u}_h - \nabla p_h - \gamma\,(p+1)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi - p_h\rangle_{\Gamma_D} + (\gamma(p+1)\boldsymbol{f} - \varepsilon\,\boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}) + \varepsilon\,(\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h)\\
&+\frac{1}{2}\left(\varepsilon^{-1}(\varepsilon\boldsymbol{u}_h - \nabla p_h - \gamma\,(p+1)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi - p_h\rangle_{\Gamma_D} + (\gamma(p-p_h)\boldsymbol{f}, \boldsymbol{v}) + (\gamma(p_h+1)\boldsymbol{f} - \varepsilon\,\boldsymbol{u}_h + \nabla p_h, \boldsymbol{v}) + \varepsilon\,(\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h)\\
&-\frac{1}{2}\left(\varepsilon^{-1}(\gamma\,(p-p_h)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right) + \frac{1}{2}\left(\varepsilon^{-1}(\varepsilon\boldsymbol{u}_h - \nabla p_h - \gamma\,(p_h+1)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi - p_h\rangle_{\Gamma_D} + (\gamma(p-p_h)\boldsymbol{f}, \boldsymbol{v}) + \sum_{K\in\mathcal{T}_h}(\mathcal{R}_K, \boldsymbol{v})_K + \varepsilon\,(\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) + (q, \nabla\cdot\boldsymbol{u}_h)\\
&-\frac{1}{2}\left(\varepsilon^{-1}(\gamma\,(p-p_h)\boldsymbol{f}), \varepsilon\boldsymbol{v} + \nabla q\right) - \frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\varepsilon^{-1}(\mathcal{R}_K), \varepsilon\boldsymbol{v} + \nabla q\right)_K - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v})\\
=&\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi - p_h\rangle_{\Gamma_D} + \frac{1}{2}(\varepsilon^{-1/2}\gamma(p-p_h)\boldsymbol{f}, \varepsilon^{1/2}\boldsymbol{v} - \varepsilon^{-1/2}\nabla q)\\
&+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\left(\varepsilon^{-1/2}(\mathcal{R}_K), \varepsilon^{1/2}\boldsymbol{v} - \varepsilon^{-1/2}\nabla q\right)_K + (q, \nabla\cdot\boldsymbol{u}_h) - \varepsilon\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}),
\end{aligned}$$

and the result follows. □

To introduce the main result of this section, we need to define the following mesh–dependent norm for the pressure

$$\|p\|_{\omega_F} := \left\{ \sum_{K\in\omega_F}\left[h_K^{-2}\|p\|_{0,K}^2 + |p|_{1,K}^2\right]\right\}^{1/2},$$

for all $p \in Q$, and for all $F \in \mathcal{E}_h$, where ω_F is the set of elements K of $\mathcal{T}_h$ such that $F \in \partial K$.

Lemma 10. *There exists $C > 0$, independent of h, such that*

$$\|\psi\|^2_{0,\partial K} \le C \left\{ h_K^{-1} \|\psi\|^2_{0,K} + h_K |\psi|^2_{1,K} \right\},$$

for all $K \in \mathcal{T}_h$ and all $\psi \in H^1(K)$.

Proof. See [45, Theorem 3.10] or [46, (10.3.8)]. □

We are ready to prove the efficiency and reliability of the error estimator (5.45).

Theorem 11. *Let $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$, be the solutions of* (2.5) *and* (4.24)*, respectively, and suppose valid* (4.35)*. Then, the following holds*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \le C\, \varepsilon^{-1/2} \max\left\{1, \varepsilon^{-1/2}\right\} \eta,$$

where $C > 0$ is independent of ε, γ and h, and

$$\eta_K \le C \left\{ \varepsilon^2 \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathrm{div},K} + \beta_K^2 \|p - p_h\|^2_{1,K} + \sum_{F \subset \mathcal{E}(K) \cap \mathcal{E}_D} \|p - p_h\|^2_{\omega_F} \right\},$$

for all $K \in \mathcal{T}_h$, where $\beta_K := \max\{\gamma \|\boldsymbol{f}\|_{\infty,\Omega}, 1\}$.

Proof. From [47, 48] we have the following inverse estimate $\|\varphi - p_h\|_{1/2,\Gamma_D} \le C h_D^{-1/2} \|\varphi - p_h\|_{0,\Gamma_D}$, thus using Cauchy-Schwarz inequality, the definition of norm $\|\cdot\|_{-1/2,\Gamma_D}$ given in (2.10) and Lemma 9, we get

$$\begin{aligned}
& B_{\mathrm{stab}}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}, q)) \\
\le & C \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{-1/2,\Gamma_D} \|\varphi - p_h\|_{1/2,\Gamma_D} + \frac{1}{2} \varepsilon^{-1/2} \|p - p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} \|\varepsilon^{1/2} \boldsymbol{v} - \varepsilon^{-1/2} \nabla q\|_{0,\Omega} + \\
& \frac{1}{2} \sum_{K \in \mathcal{T}_h} \varepsilon^{-1/2} \|\mathcal{R}_K\|_{0,K} \|\varepsilon^{1/2} \boldsymbol{v} - \varepsilon^{-1/2} \nabla q\|_{0,K} + \|q\|_{0,\Omega} \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} + \varepsilon \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} \|\nabla \cdot \boldsymbol{v}\|_{0,\Omega} \\
\le & C\, h_D^{-1/2} \|\boldsymbol{v}\|_{\boldsymbol{H}} \|\varphi - p_h\|_{0,\Gamma_D} + \frac{1}{2} \varepsilon^{-1/2} \|p - p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} \|\varepsilon^{1/2} \boldsymbol{v} - \varepsilon^{-1/2} \nabla q\|_{0,\Omega} + \\
& \frac{1}{2} \sum_{K \in \mathcal{T}_h} \varepsilon^{-1/2} \|\mathcal{R}_K\|_{0,K} \|\varepsilon^{1/2} \boldsymbol{v} - \varepsilon^{-1/2} \nabla q\|_{0,K} + \|q\|_{0,\Omega} \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} + \varepsilon \|\nabla \cdot \boldsymbol{u}_h\|_{0,\Omega} \|\nabla \cdot \boldsymbol{v}\|_{0,\Omega} \\
\le & C \|\boldsymbol{v}\|_{\boldsymbol{H}} \left\{ \sum_{F \subset \mathcal{E}_D} h_F^{-1} \|\varphi - p_h\|^2_{0,F} \right\}^{1/2} + \frac{1}{2} \varepsilon^{-1/2} \|p - p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} \left\{ \varepsilon^{1/2} \|\boldsymbol{v}\|_{0,\Omega} + \varepsilon^{-1/2} |q|_{1,\Omega} \right\} + \\
& C \left\{ \sum_{K \in \mathcal{T}_h} \left[\varepsilon^{-1} \|\mathcal{R}_K\|^2_{0,K} + \varepsilon \|\nabla \cdot \boldsymbol{u}_h\|^2_{0,K} \right] \right\}^{1/2} \left\{ \sum_{K \in \mathcal{T}_h} \left[\varepsilon \|\boldsymbol{v}\|^2_{0,K} + \varepsilon \|\nabla \cdot \boldsymbol{v}\|^2_{0,K} + \varepsilon^{-1} \|q\|^2_{0,K} + \varepsilon^{-1} |q|^2_{1,K} \right] \right\}^{1/2} \\
\le & \frac{1}{2} \varepsilon^{-1/2} \|p - p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} \left\{ \varepsilon^{1/2} \|\boldsymbol{v}\|_{0,\Omega} + \varepsilon^{-1/2} |q|_{1,\Omega} \right\} + \\
& C \left\{ \sum_{K \in \mathcal{T}_h} \left[\varepsilon^{-1} \|\mathcal{R}_K\|^2_{0,K} + \varepsilon \|\nabla \cdot \boldsymbol{u}_h\|^2_{0,K} + \sum_{F \subset \mathcal{E}(K) \cap \mathcal{E}_D} \varepsilon^{-1} h_F^{-1} \|\mathcal{R}_F\|^2_{0,F} \right] \right\}^{1/2} \left\{ \varepsilon \|\boldsymbol{v}\|^2_{\boldsymbol{H}} + \varepsilon^{-1} \|q\|^2_{1,\Omega} \right\}^{1/2} \\
\le & \varepsilon^{-1/2} \max\left\{1, \varepsilon^{-1/2}\right\} \left\{ \frac{1}{2} \|p - p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} + C\, \eta \right\} |||(\boldsymbol{v}, q)|||.
\end{aligned} \tag{5.47}$$

Additionally, from Lemma 12, (4.35) and (5.47), we arrive at

$$\begin{aligned}
\beta_c \, |||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq & \sup_{(\boldsymbol{v},q)\in \boldsymbol{H}\times Q} \frac{B_{\text{stab}}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \\
\leq & \varepsilon^{-1/2} \max\left\{1, \varepsilon^{-1/2}\right\} \frac{1}{2} \|p-p_h\|_{0,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} + C\,\varepsilon^{-1/2} \max\left\{1,\varepsilon^{-1/2}\right\} \eta \\
\leq & \frac{3+\varepsilon^{-1/2}}{2} \varepsilon^{-1/2} \|p-p_h\|_{1,\Omega}\, \gamma \|\boldsymbol{f}\|_{\infty,\Omega} + C\,\varepsilon^{-1/2} \max\left\{1,\varepsilon^{-1/2}\right\} \eta \\
\leq & \frac{\beta_c}{2} \|p-p_h\|_{1,\Omega} + C\,\varepsilon^{-1/2} \max\left\{1,\varepsilon^{-1/2}\right\} \eta,
\end{aligned}$$

and therefore,

$$|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq C\,\varepsilon^{-1/2} \max\left\{1,\varepsilon^{-1/2}\right\} \eta.$$

On the other hand, using the definition of $\mathcal{R}_K$ and (2.5), we deduce that

$$\begin{aligned}
\|\mathcal{R}_K\|_{0,K} &= \|\gamma\, p_h \boldsymbol{f} + \gamma\, \boldsymbol{f} - \varepsilon\, \boldsymbol{u}_h + \nabla p_h\|_{0,K} \\
&= \|\gamma\, p_h \boldsymbol{f} + \varepsilon\, u - \gamma\, p \boldsymbol{f} - \nabla p - \varepsilon\, \boldsymbol{u}_h + \nabla p_h\|_{0,K} \\
&= \|\gamma\,(p_h - p)\boldsymbol{f} + \varepsilon\,(\boldsymbol{u}-\boldsymbol{u}_h) + \nabla(p_h - p)\|_{0,K} \\
&\leq \|p-p_h\|_{0,K}\, \gamma\, \|\boldsymbol{f}\|_{\infty,K} + \varepsilon\, \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} + |p-p_h|_{1,K}.
\end{aligned} \tag{5.48}$$

In addition, as $\nabla\cdot\boldsymbol{u} = 0$ in Ω, we have

$$\|\nabla\cdot\boldsymbol{u}_h\|_{0,K} = \|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,K}. \tag{5.49}$$

Similarly, as $p = \varphi$ on Γ_D, using the triangle inequality, Lemma 10 and the mesh regularity, we have that

$$h_F^{-1} \|\mathcal{R}_F\|_{0,F}^2 = h_F^{-1} \|p-p_h\|_{0,F}^2 \leq C \sum_{K\in\omega_F} \left[h_K^{-2} \|p-p_h\|_{0,K}^2 + |p-p_h|_{1,K}^2 \right]. \tag{5.50}$$

Finally, using the definition of η_K and (5.48)–(5.50), we get the result.

□

6. Numerical results

In this section we present some numerical tests that illustrate the performance of our adapted stabilized finite element method given in (4.24). In particular, we confirm the results presented in Theorem 8 and the quality of the a posteriori error estimator (5.45) for the Darcy equation (2.5).

The stabilized finite element scheme was implemented using the open source finite element library FEniCS [49].

We will use the following notation for the error in velocity and pressure, respectively

$$e_{\boldsymbol{u}} := \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}}, \qquad \text{and} \qquad e_p := \|p-p_h\|_{1,\Omega},$$

while the convergence rates are denoted by

$$r_m(x) = \frac{\log(e_x^i / e_x^{i-1})}{\log(h^i / h^{i-1})}, \quad \text{with } x \in \{\boldsymbol{u}, p\},$$

where m is the polynomial degree, h^i, h^{i-1}, and, e_x^i, e_x^{i-1} represent two consecutive mesh sizes and two consecutive errors, respectively.

Finally, we define the effectivity index E as follows

$$E := \frac{\eta}{|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)|||}.$$

Our adaptive algorithm is given by

Algorithm 1 Adaptivity procedure

Require: $\theta \in (0,1)$ and a coarse mesh $\mathcal{T}_h$.

1: Solve the stabilized discrete scheme (4.24) on the current mesh.

2: For each $K \in \mathcal{T}_h$, compute the local error indicator η_K given by (5.46).

3: Given $K \in \mathcal{T}_h$ such that $\eta_K \geq \theta \max_{K' \in \mathcal{T}_h} \eta_{K'}$, mark K and generate a new mesh $\mathcal{T}_h$ refining the marked elements.

4: If the stop criterion is not satisfied, go to step 1.

6.1. Analytic solution

In this example, we will test the approximation capability of the stabilized method in a non-convex domain with a nearly singular solution close to the origin of coordinates. We will also show that our error estimator adapts the meshes where it is expected and has a good effectivity index.
In this case our domain is $\Omega := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \backslash (0,1)^2$, with $\Gamma_N := (0,1)\times\{0\} \cup \{0\}\times(0,1)$ and $\Gamma_D := \partial\Omega \backslash \Gamma_N$. The data $\boldsymbol{f}$ and φ are such that the exact solution is given by

$$\boldsymbol{u}(x,y) := [(x-c)^2 + (y-c)^2]^{-1/2}(c-y, x-c), \quad p(x,y) := \frac{1 - x^2 - y^2}{(x-c)^2 + (y-c)^2},$$

where $c = 0.025$. For the drag function (2.4), we take $\alpha_0 = 1.0$, while $\gamma = 1, 10^{-2}, 10^{-4}$.

In Tables 1 – 6 we show the approximation error using $\mathbb{P}_1^2 \times \mathbb{P}_1$ and $\mathbb{P}_2^2 \times \mathbb{P}_2$. We note that the errors on the velocity and pressure have the order predicted by Theorem 8. On the other hand, the error estimator η, given by (5.45), has a quite good quality reflected on the fact that effectivity indexes are close to the unity. Note that there is a small degradation of the effectivity index when ε goes to 0.

h	$\lVert p - p_h \rVert_{1,\Omega}$	$r_1(p)$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\boldsymbol{H}}$	$r_1(u)$	$\lvert\lvert\lvert(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\rvert\rvert\rvert$	η	E
0.079946	2056.657064	-	54.729693	-	2111.386757	1888.309544	0.894346
0.039990	1318.996805	0.641253	28.518113	0.941022	1347.514919	1315.964898	0.976587
0.019998	868.049234	0.603722	13.842753	1.042973	881.891987	854.508058	0.968949
0.010000	463.443573	0.905514	5.588739	1.308725	469.032312	460.407948	0.981612
0.005000	235.075873	0.979267	1.851054	1.594176	236.926927	234.680437	0.990518
0.002500	115.909138	1.020132	0.579464	1.675556	116.488602	115.857322	0.994581

Table 1: $\mathbb{P}_1^2 \times \mathbb{P}_1$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 1$.

h	$\lVert p - p_h \rVert_{1,\Omega}$	$r_1(p)$	$\lVert \boldsymbol{u} - \boldsymbol{u}_h \rVert_{\boldsymbol{H}}$	$r_1(u)$	$\lvert\lvert\lvert(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\rvert\rvert\rvert$	η	E
0.079946	2055.714421	-	5464.968894	-	2602.211310	1884.365482	0.724140
0.039990	1319.171243	0.640401	2851.557217	0.939039	1604.326965	1316.475766	0.820578
0.020000	868.110785	0.603812	1381.329850	1.045919	1006.243770	854.723621	0.849420
0.010000	463.470837	0.905531	558.754010	1.305961	519.346238	460.505925	0.886703
0.005000	235.078411	0.979337	185.072224	1.594125	253.585634	234.689286	0.925483
0.002500	115.909236	1.020147	57.943499	1.675370	121.703586	115.857687	0.951966

Table 2: $\mathbb{P}_1^2 \times \mathbb{P}_1$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 10^{-2}$.

h	$\|p-p_h\|_{1,\Omega}$	$r_1(p)$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}}$	$r_1(u)$	$\|\|\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|\|\|$	η	E
0.079946	2055.705163	-	546489.178588	-	7520.596949	1884.326586	0.250555
0.039990	1319.173081	0.640392	285155.519779	0.939019	4170.728279	1316.481140	0.315648
0.019998	868.112101	0.603811	138413.649790	1.045919	2252.248599	854.727871	0.379500
0.010000	463.471114	0.905533	55875.284002	1.308893	1022.223954	460.506911	0.450495
0.005000	235.078437	0.979337	18507.189808	1.594124	420.150335	234.689375	0.558584
0.002500	115.909237	1.020147	5794.347388	1.675368	173.852711	115.857691	0.666413

Table 3: $\mathbb{P}_1^2 \times \mathbb{P}_1$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 10^{-4}$.

h	$\|p-p_h\|_{1,\Omega}$	$r_2(p)$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}}$	$r_2(u)$	$\|\|\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|\|\|$	η	E
0.0799	1107.895742	-	60.328929	-	1167.701879	967.730613	0.828748
0.0399	470.076480	1.237528	18.287520	1.761370	488.364000	457.382915	0.936561
0.0199	169.449270	1.4726600	3.462993	2.410449	172.912263	170.783398	0.987688
0.0099	51.791285	1.710069	0.686595	2.335218	52.477880	51.906513	0.989112
0.0049	13.031846	1.990669	0.080419	3.094040	13.112265	13.041533	0.994606
0.0024	3.230428	2.012244	0.009300	3.112197	3.239729	3.230962	0.997294

Table 4: $\mathbb{P}_2^2 \times \mathbb{P}_2$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 1$.

h	$\|p-p_h\|_{1,\Omega}$	$r_2(p)$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}}$	$r_2(u)$	$\|\|\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|\|\|$	η	E
0.079946	1107.408226	-	6033.736118	-	1710.781838	967.887247	0.565757
0.039990	470.062955	1.237020	1828.499303	1.723048	652.912885	457.301064	0.700401
0.019998	169.448973	1.472322	346.295943	2.401102	204.078568	170.782099	0.836845
0.010000	51.791298	1.710316	68.659412	2.334812	58.657239	51.906578	0.884913
0.005000	13.031845	1.990668	8.041910	3.093847	13.836036	13.041530	0.942577
0.002500	3.230428	2.012244	0.930030	3.112189	3.323431	3.230962	0.972176

Table 5: $\mathbb{P}_2^2 \times \mathbb{P}_2$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 10^{-2}$.

h	$\|p-p_h\|_{1,\Omega}$	$r_2(p)$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}}$	$r_2(u)$	$\|\|\|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)\|\|\|$	η	E
0.079946	1107.408586	-	603374.475161	-	7141.153338	967.888845	0.135537
0.039990	470.062821	1.237021	182849.680322	1.723453	2298.559625	457.300247	0.198951
0.019998	169.448970	1.472321	34629.590957	2.401101	515.744880	170.782086	0.331137
0.010000	51.791298	1.710316	6865.941117	2.334813	120.450710	51.906579	0.430936
0.005000	13.031845	1.990668	804.191016	3.093847	21.073755	13.041530	0.618852
0.002500	3.230428	2.012244	93.003039	3.112188	4.160459	3.230962	0.776588

Table 6: $\mathbb{P}_2^2 \times \mathbb{P}_2$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 10^{-4}$.

In Figure 1 we show some of the adapted meshes obtained with Algorithm 1. Note that most of the refinement is close to the origin due to the fact that the exact solution has a singularity at the point (c, c) with $c = 0.025$, which is close to $(0, 0)$. Finally, in Figure 2 we compare the approximated solution, obtained by our proposed scheme, and the exact solution. Note that the approximated solution has a good agreement with the exact one.

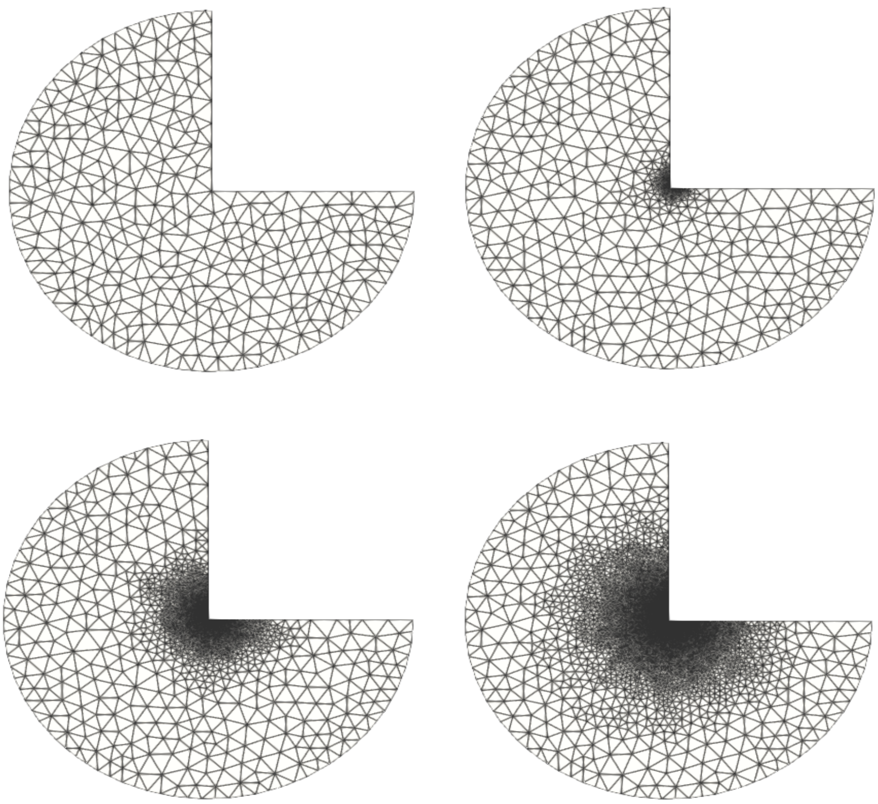

Figure 1: Suite of adaptive meshes: $\mathcal{T}_{h,0}$ (top left), $\mathcal{T}_{h,8}$ (top right), $\mathcal{T}_{h,16}$ (bottom left) and $\mathcal{T}_{h,20}$ (bottom right).

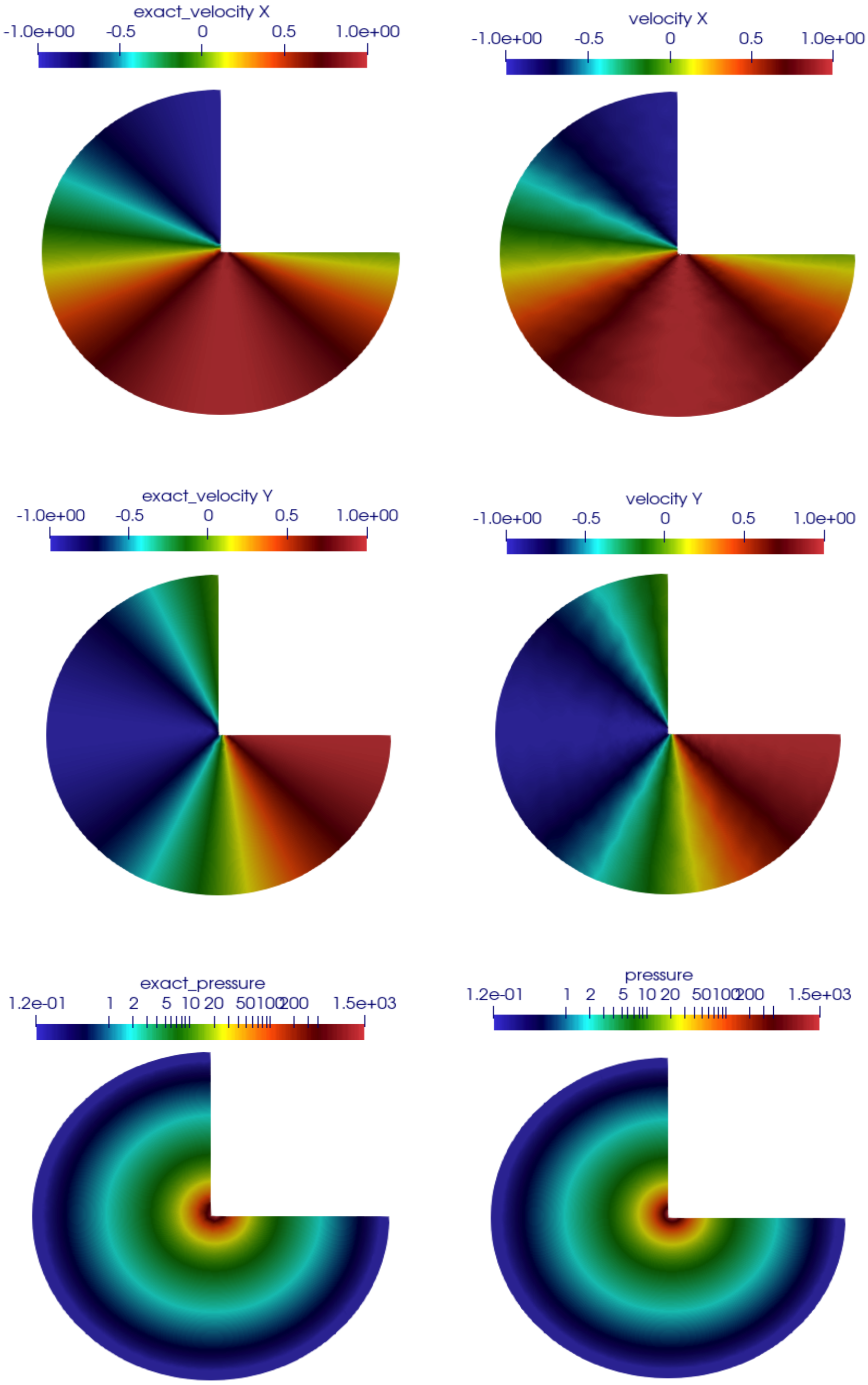

Figure 2: Components of the exact solution (left column) compared with the approximated solution (right column) obtained using $\mathbb{P}_1^2 \times \mathbb{P}_1$ on the final adapted mesh of 178,596 elements.

6.2. *A reservoir simulation*

For our second problem, we have taken a reservoir problem from [16]. Here $\Omega := (0,2) \times (0,1)$, $\alpha_0 = 1$ and $\gamma = 0.005$. On Γ_D^1 we prescribe $p = p_{atm} = 1.0$ and on Γ_D^2, $p = p_{enh} = 5$. On the rest of the boundary we impose

$\boldsymbol{v} \cdot \boldsymbol{n} = 0$.

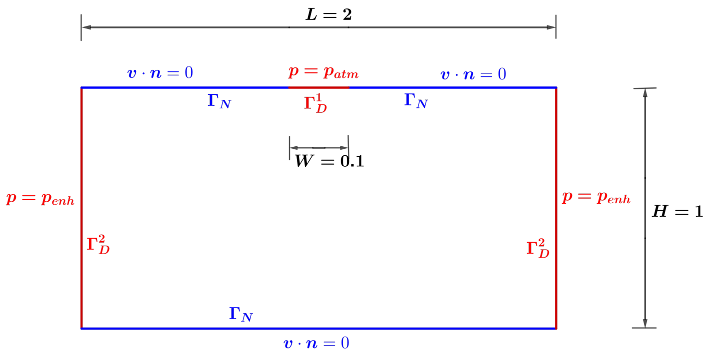

Figure 3: Sketch of Ω with Dirichlet boundary conditions for the pressure in Γ_D^1 and Γ_D^2, and normal trace zero for the velocity in Γ_N.

As in the previous example, we show in Figure 4 some of the adapted meshes obtained with Algorithm 1. Note that most of the refinement is close to the Γ_D^1, which is consistent with the physics of the problem. Finally, in Figures 5 and 6, we compare the approximated solution, obtained by our stabilized scheme, and the exact solution. Note that the approximated solution has a good agreement with the exact one.

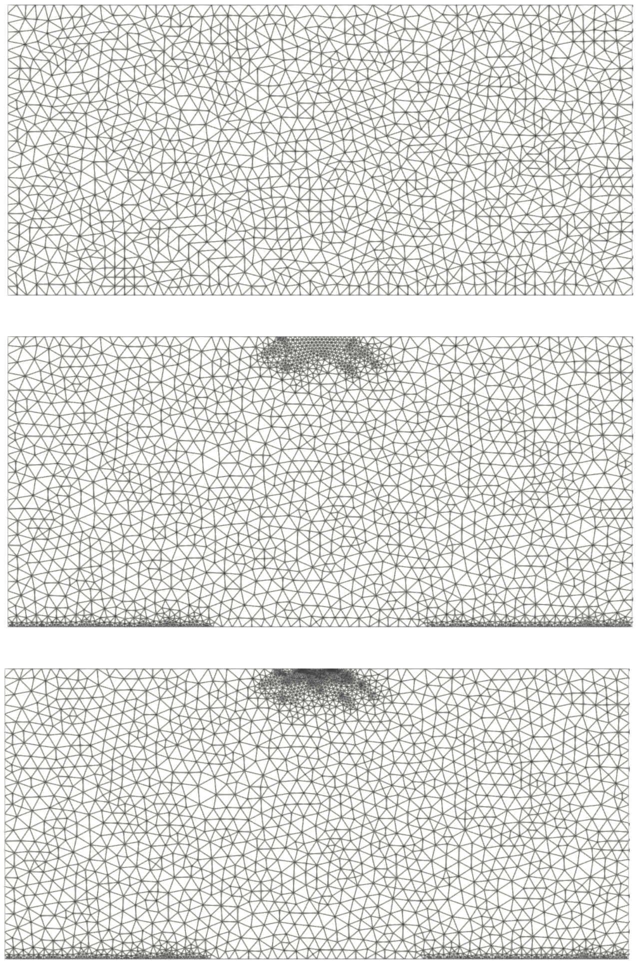

Figure 4: Suite of adaptive meshes: $\mathcal{T}_{h,0}$ (top), $\mathcal{T}_{h,2}$ (middle) and $\mathcal{T}_{h,9}$ (bottom).

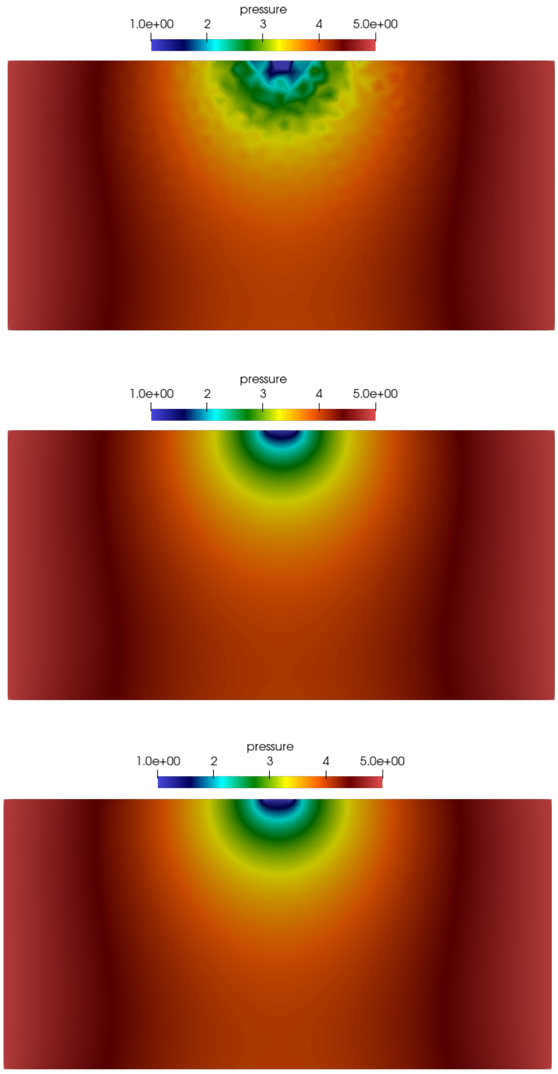

Figure 5: Isolines of the pressure using $\mathbb{P}_1^2 \times \mathbb{P}_1$ finite element spaces, corresponding to the solution with 1,286 elements on the initial mesh (top), the solution with 6,994 elements on the adapted mesh (middle) and the reference solution on a fine uniform mesh with 773,034 elements (bottom).

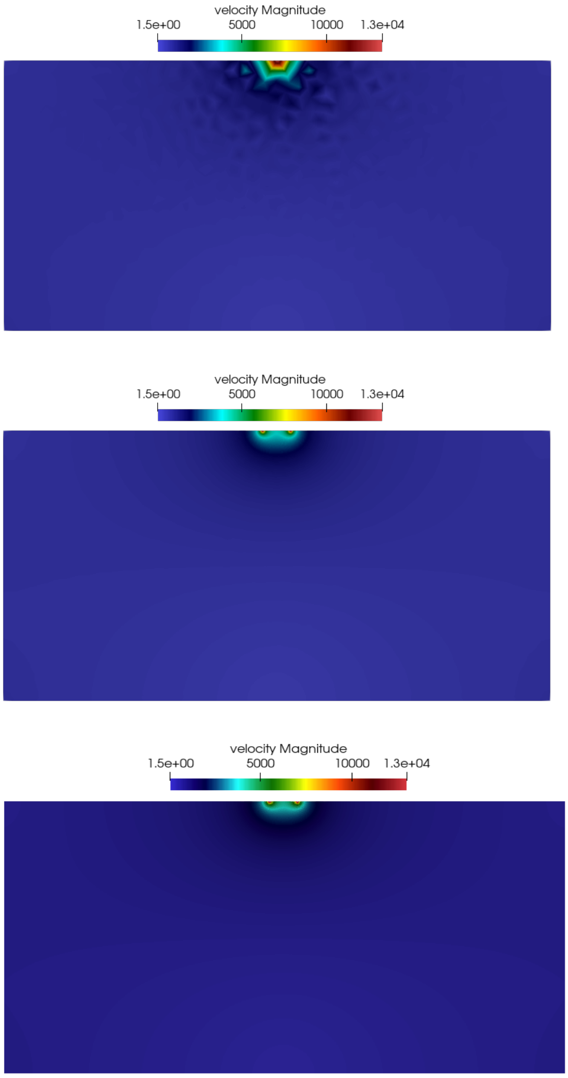

Figure 6: Isolines of the velocity magnitude using $\mathbb{P}_1^2 \times \mathbb{P}_1$ finite element spaces, corresponding to the solution with 1,286 elements on the initial mesh (top), the solution with 6,994 elements on the adapted mesh (middle) and the reference solution on a fine uniform mesh with 773,034 elements (bottom).

6.3. A 3D simulation

Let $\Omega := (0, 1)^3$, and let $\boldsymbol{f}$ such that the exact solution is given by

$$\boldsymbol{u}(x, y, z) := \frac{1}{2}(-y^2, z^2, x^2) \qquad \text{and} \qquad p(x, y, z) := 2 + xyz.$$

We assume that $\Gamma_D := \{0\}\times]0, 1[\times]0, 1[\cup]0, 1[\times\{0\}\times]0, 1[\cup]0, 1[\times]0, 1[\times\{0\}$ with $\varphi := 2$, and $\Gamma_N := \partial\Omega\backslash\Gamma_D$. We choose $\alpha_0 = 1.0$ and $\gamma = 0.25$, i.e. $\varepsilon = 0.25$.

In Table 7 we present the approximation errors and our a posteriori error estimator η. As in the two-dimensional case, we can see that the errors on the velocity $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}}$ and pressure $\|p - p_h\|_{0,\Omega}$ show a perfect agreement with those predicted by the theory. Moreover, the effectivity index for the residual a posteriori error estimator η is close to one.

h	$\|p - p_h\|_{1,\Omega}$	$r_1(p)$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}}$	$r_1(u)$	$\|\|\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|\|\|$	η	E
0.866025	0.264920	-	0.232318	-	0.352355	0.279711	0.793833
0.433013	0.155252	0.770946	0.078652	1.562548	0.174038	0.162626	0.934427
0.216506	0.082301	0.915627	0.023047	1.770898	0.085467	0.086156	1.008067
0.108253	0.041952	0.972170	0.006394	1.849788	0.042436	0.043960	1.035904
0.054127	0.021103	0.991305	0.001730	1.885972	0.021173	0.022123	1.044853
0.027063	0.010570	0.997446	0.000462	1.904757	0.010580	0.011078	1.047037

Table 7: $\mathbb{P}_1^3 \times \mathbb{P}_1$ stabilized scheme with a quasi-uniform refinement and $\varepsilon = 0.25$.

Finally, we present the approximated solutions obtained with the stabilized scheme in a highly uniform refined mesh in Figure 7. Here we used $\mathbb{P}_1^3 \times \mathbb{P}_1$ elements and we observe that the overall results are in accordance with the expected ones.

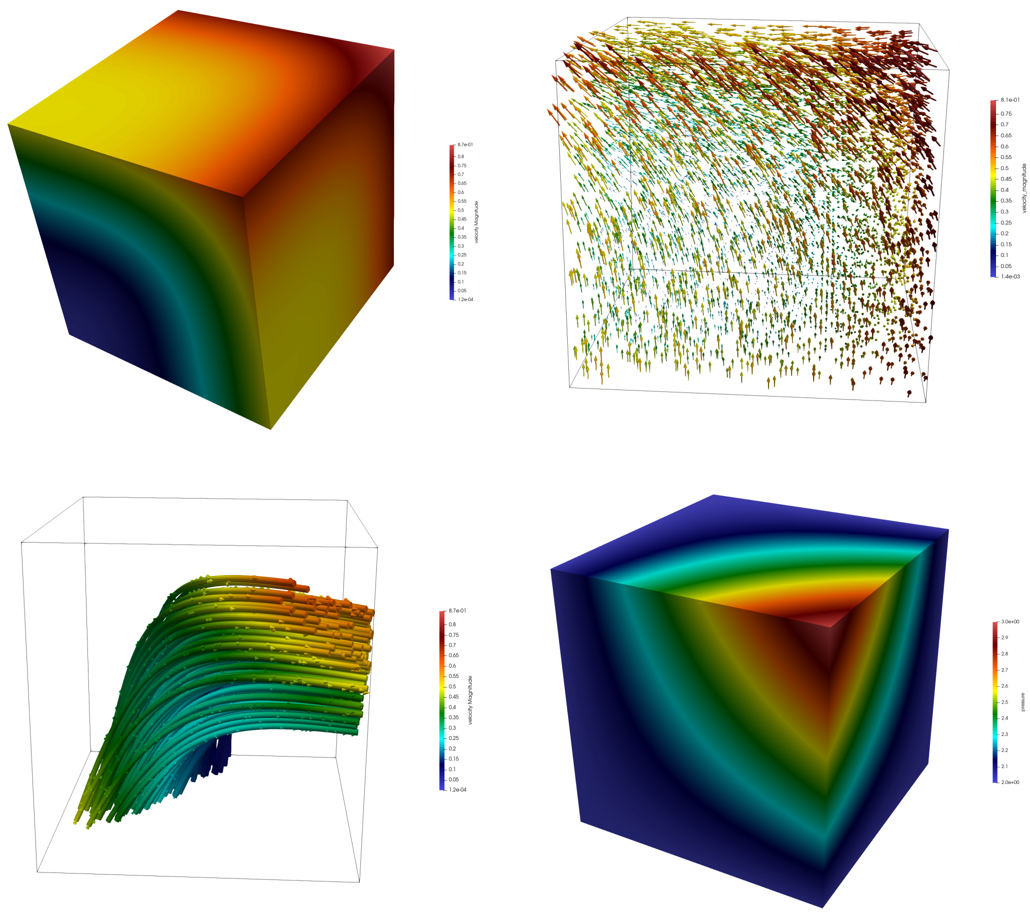

Figure 7: Approximated solution. Velocity magnitude (top left), velocity vectors (top right), velocity streamlines (bottom left) and isovalues of the pressure (bottom right). We use $\mathbb{P}_1^3 \times \mathbb{P}_1$ elements on a uniform mesh of 1,572,864 elements.

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Appendix A.

Lemma 12. *There exists a positive constant β_c, independent of ε, such that*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{H}\times Q} \frac{B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{v},q))}{|||(\boldsymbol{v},q)|||} \geq \beta_c\, |||(\boldsymbol{v},q)|||, \tag{A.1}$$

for all $(\boldsymbol{u},p) \in \boldsymbol{H}\times Q$.

Proof. Given $p \in L^2(\Omega)$, from the proof of Lemma 1, there exists $\boldsymbol{w} \in \boldsymbol{H}$ such that $\nabla\cdot\boldsymbol{w} = -p$ and $\|\boldsymbol{w}\|_{\boldsymbol{H}} \leq C\|p\|_{0,\Omega}$. Then, for $(\boldsymbol{v},q) := (\boldsymbol{u}-\delta\boldsymbol{w},p)$, with $\delta > 0$, we have

$$\begin{aligned}
B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{v},q)) &= B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{u},p)) - \delta\, B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{w},0)) \\
&= B_{\text{stab}}((\boldsymbol{u},p),(\boldsymbol{u},p)) - \delta\left[B_{\text{stab}}((\boldsymbol{u},0),(\boldsymbol{w},0)) + B_{\text{stab}}((\boldsymbol{0},p),(\boldsymbol{w},0))\right] \\
&= \frac{1}{2}\varepsilon\,\|\boldsymbol{u}\|^2_{0,\Omega} + \varepsilon\,\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} + \frac{1}{2}\varepsilon^{-1}\,|p|^2_{1,\Omega} \\
&\quad - \delta\left[\frac{1}{2}\varepsilon\,(\boldsymbol{u},\boldsymbol{w}) + \varepsilon\,(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{w}) + (p,\nabla\cdot\boldsymbol{w}) + \frac{1}{2}\,(\nabla p,\boldsymbol{w})\right] \\
&= \frac{1}{2}\varepsilon\,\|\boldsymbol{u}\|^2_{0,\Omega} + \varepsilon\,\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} + \frac{1}{2}\varepsilon^{-1}\,|p|^2_{1,\Omega} \\
&\quad - \frac{\delta}{2}\varepsilon\,(\boldsymbol{u},\boldsymbol{w}) - \delta\varepsilon\,(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{w}) - \delta\,(p,\nabla\cdot\boldsymbol{w}) - \frac{\delta}{2}\,(\nabla p,\boldsymbol{w}) \\
&= \frac{1}{2}\varepsilon\,\|\boldsymbol{u}\|^2_{0,\Omega} + \varepsilon\,\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} + \frac{1}{2}\varepsilon^{-1}\,|p|^2_{1,\Omega} + \delta\,\|p\|^2_{0,\Omega} \\
&\quad - \frac{\delta}{2}\varepsilon\,(\boldsymbol{u},\boldsymbol{w}) - \delta\varepsilon\,(\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{w}) - \frac{\delta}{2}\,(\nabla p,\boldsymbol{w}) \\
&\geq \frac{1}{2}\varepsilon\,\|\boldsymbol{u}\|^2_{0,\Omega} + \varepsilon\,\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} + \frac{1}{2}\varepsilon^{-1}\,|p|^2_{1,\Omega} + \delta\,\|p\|^2_{0,\Omega} \\
&\quad - \frac{\delta}{2}\varepsilon\,\|\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{w}\|_{0,\Omega} - \delta\varepsilon\,\|\nabla\cdot\boldsymbol{u}\|_{0,\Omega}\|\nabla\cdot\boldsymbol{w}\|_{0,\Omega} - \frac{\delta}{2}|p|_{1,\Omega}\|\boldsymbol{w}\|_{0,\Omega}.
\end{aligned}$$

The result follows using similar arguments as in the proof of the Lemma 5. □

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