A note on *a posteriori* error estimates for dual mixed methods

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Abstract

In this paper, we describe a technique to develop an a posteriori error estimator for the dual mixed methods when applied to elliptic partial differential equations, with Dirichlet and mixed boundary conditions. The approach considers conforming finite elements for discrete scheme and a quasi Helmholtz decomposition to deduce an estimator of residual type. We prove its equivalence with the norm of the error, that is, reliability and local efficiency, without requiring the standard additional elliptic regularity on the boundary data. Numerical results are in agreement with the developed theory.

1 Introduction

Nowadays, it is well established that one should apply adaptive mesh refinement based on a posteriori error estimators, for efficient implementation of numerical methods (see [3, 33]). Then, the list of references on the *a posteriori* error analysis of the mixed finite element method is quite extensive, and due to our current interest in conforming dual mixed-FEM, we mention [4] where performing a Helmholtz decomposition, an estimator for the L^2-error of the flux approximated by Raviart-Thomas (RT) or Brezzi-Douglas- Marini (BDM) finite elements, is derived. Simultaneously, in [17] the authors considered the dual mixed method approximated by RT elements for the vectorial

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unknown, and obtained two a posteriori error estimates, both based on a saturation assumption. The first one is deduced for the so-called natural norm, that is, $H(\text{div})$-norm for the flux and L^2-norm for scalar unknown. However, the local efficiency was shown to be suboptimal, with a factor of order h^{-1}. To circumvent this lack of optimality, the second estimator is deduced for a mesh depend norm, which use a weighted L^2-norm for the flux and the classical H^1-seminorm for discontinuous functions, i.e., a nonconforming approach. In [19], an efficient and reliable a posteriori error estimates for the natural norm is presented, circumventing the use of the saturation assumption. Following similar ideas as the given in [4], an estimator including a term that measures the rotational of the residual is included. In addition, a comparison of four different kinds of error estimators for mixed finite element discretizations by Raviart-Thomas elements is presented in [29]. In [24], an a posteriori error estimates for the mixed FEM with Lagrange multipliers, applied to second order elliptic equation with mixed boundary condition, is introduced.

In the next years, estimates expressed in terms of a locally post-processed approximation for the scalar unknown, were developed in [28], [2] and [27]. In particular, it should be noted that in [2], all the constants (reliability, efficiency) in that estimator are explicitly computed.

On the other hand, in the framework of augmented mixed FEM, in [8] an alternative a posteriori error estimator to the one developed in [16] is derived. This approach is based on the Ritz projection of the error (see [15]), and in the case of homogeneous Dirichlet boundary condition, we obtain a reliable and local efficient a posteriori error estimator, that only requires the computation of four residuals per element, which is a low computational cost comparing with the eleven terms included in the estimator obtained in [16] for the same case. A similar result can be seen in [9], where the extension toward linear elasticity with mixed boundary condition is studied. There, an a posteriori error estimator with seven terms per element (touching Neumann boundary) is deduced, which reduced the thirteen terms needed in the estimator obtained previously in [7] for the same finite element spaces. Furthermore, for the interior elements, the reduction is again from eleven to five terms per element. Additionally, this kind of a posteriori error estimator, at least, have been developed satisfactorily in different directions, for example, the Poisson problem is studied in [15], Darcy flow in [12] and [13], the Stokes system in [6] and [10], the Brinkman model in [11], linear elasticity in [8, 9] and the Oseen equations in [14].

The purpose of the present note is to additionally contribute in the direction of the results provided in [10] for augmented mixed method, by extending/relaxing the analysis towards conforming dual mixed method, including

Dirichlet and mixed boundary conditions. In other words, using the approach based on the Ritz projection of the error, our interest is to develop an estimator of residual type, reliable and locally efficient, in the framework of the natural norm, circumventing the saturation assumption and including nonhomogeneous Dirichlet and mixed boundary conditions for conforming dual mixed method. In particular, for mixed boundary condition we apply a homogenisation technique and we follow the ideas describe in [13] for the treatment of the Neumann data. Our approach differs from [24], since they imposed weakly the Neumann boundary conditions via the introduction of a new Lagrange multiplier.

The rest of the article is organised as follow: We begin introducing the model problem in Section 2. After that, the a posteriori error analysis with nonhomogeneous Dirichlet and mixed boundary conditions are included in Sections 3 and 4, respectively. Finally, the numerical examples confirming the theoretical results are reported in Section 5. We end this introduction with some notation to be used throughout the paper. Given any Hilbert space H, we denote by H^2 the space of vectors of order 2 with entries in H. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretisation parameter, that may take different values at different occurrences.

2 Model problem and variational formulations

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ. Then, given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we consider the model problem: Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \quad u = g \quad \text{on}\ \Gamma\,. \tag{1}$$

Since we are interested in dual mixed methods, we rewrite (1) as the first order system: Find $u \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$ such that

$$\boldsymbol{\sigma} = -\nabla u \quad \text{in} \quad \Omega\,, \quad \mathrm{div}(\boldsymbol{\sigma}) = f \quad \text{in} \quad \Omega\,, \quad u = g \quad \text{on}\ \Gamma. \tag{2}$$

Hence, proceeding in the usual way we arrive to the following dual mixed variational formulation of (2): Find $(\boldsymbol{\sigma}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) \quad - b(u,\boldsymbol{\tau}) \;&=\; -\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, g\rangle \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega)\,, \\ -b(w,\boldsymbol{\sigma}) \;&=\; -\int_\Omega f\, w\, dx \qquad \forall\, w \in L^2(\Omega)\,, \end{aligned} \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to $L^2(\Gamma)$- inner product, and the bilinear forms $a : H(\mathrm{div};\Omega) \times$

$H(\mathrm{div};\Omega) \to \mathbb{R}$ and $b : L^2(\Omega) \times H(\mathrm{div};\Omega) \to \mathbb{R}$, are given by

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau}\,dx \quad \text{and} \quad b(w,\boldsymbol{\tau}) := \int_\Omega w\,\mathrm{div}(\boldsymbol{\tau})\,dx\,.$$

Thanks to the Babuška-Brezzi's condition, it can be shown that there exists a unique pair $(\boldsymbol{\sigma},u) \in H(\mathrm{div};\Omega)\times L^2(\Omega)$ solution of (3) (see [23]). In what follows, we assume that Ω is a polygonal region and let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar\Omega$ such that $\bar\Omega = \cup\{\, T \,:\, T\in\mathcal{T}_h\,\}$. Given a triangle $T\in\mathcal{T}_h$, we denote by h_T its diameter and define the mesh size $h := \max\{\, h_T \,:\, T\in\mathcal{T}_h\,\}$. In addition, given an integer $\ell\geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathcal{P}_\ell(S)$ the space of polynomials in two variables defined in S of total degree at most ℓ, and for each $T\in\mathcal{T}_h$, we define the local Raviart-Thomas space of order κ (cf. [30]), $\mathcal{RT}_\kappa(T) := [\mathcal{P}_\kappa(T)]^2 \oplus \mathbf{x}\mathcal{P}_\kappa(T) \subseteq [\mathcal{P}_{\kappa+1}(T)]^2 \quad \forall\mathbf{x}\in T$. Then, given an integer $r\geq 0$, we define the finite element subspaces

$$H^{\boldsymbol{\sigma}}_{h,r} := \{\, \boldsymbol{\tau}_h \in H(\mathrm{div};\Omega) \,:\, \boldsymbol{\tau}_h|_T \in \mathcal{RT}_r(T)\,, \quad \forall T\in\mathcal{T}_h\,\}\,, \tag{4}$$

$$H^{u}_{h,r} := \left\{\, v_h \in [L(\Omega)]^2 \,:\, v_h|_T \in \mathcal{P}_r(T)\,, \quad \forall T\in\mathcal{T}_h\,\right\}, \tag{5}$$

Under these assumptions, the discrete version of (3): Find $(\boldsymbol{\sigma}_h,u_h)\in H^{\boldsymbol{\sigma}}_{h,r}\times H^{u}_{h,r}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) \quad -b(u_h,\boldsymbol{\tau}_h) &= -\langle \boldsymbol{\tau}_h\cdot\boldsymbol{n}, g\rangle \quad \forall\,\boldsymbol{\tau}_h\in H^{\boldsymbol{\sigma}}_{h,r}\,,\\ -b(w_h,\boldsymbol{\sigma}_h) &= -\int_\Omega f\,w_h\,dx \qquad \forall\,w_h\in H^{u}_{h,r}\,, \end{aligned} \tag{6}$$

has a unique solution $(\boldsymbol{\sigma}_h,u_h)\in H^{\boldsymbol{\sigma}}_{h,r}\times H^{u}_{h,r}$ (see [23]).

3 A posteriori error analysis

In this section, we follow [8] (see also [10]), and develop an a posteriori error analysis for the discrete scheme (6), taking into account an appropriate Ritz projection of the error and a quasi Helmholtz decomposition. We first introduce some notations and results, concerning the Clément and Raviart-Thomas interpolation operators.

3.1 Notation and some well known results

Given $T\in\mathcal{T}_h$, we let $E(T)$ be the set of its edges. By E_h we denote the set of all edges induced by the triangulation $\mathcal{T}_h$. Then, we write $E_h = E_I \cup E_\Gamma$, where $E_I := \{e \in E_h \,:\, e \subseteq \Omega\}$ and $E_\Gamma := \{e \in E_h \,:\, e \subseteq \Gamma\}$. Also, for each $T \in \mathcal{T}_h$, we fix a unit normal exterior vector $\boldsymbol{n}_T := (n_1, n_2)^{\mathtt{t}}$, and let $\mathbf{t}_T := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along ∂T. From now on, when no confusion arises, we simply write $\boldsymbol{n}$ and $\mathbf{t}$ instead of $\boldsymbol{n}_T$ and $\mathbf{t}_T$, respectively. In addition, let q and $\boldsymbol{\tau}$ be scalar - and vector -valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(q_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(q_T, \boldsymbol{\tau}_T)$ to e. Then, given $e \in E_I$, we define the jump of q and $\boldsymbol{\tau}$ at $\mathbf{x} \in e$, by

$$[\![q]\!] := q_{T,e} - q_{T',e}\,, \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{T,e} \cdot \mathbf{t}_T + \boldsymbol{\tau}_{T',e} \cdot \mathbf{t}_{T'}\,,$$

where T and T' are the two elements in $\mathcal{T}_h$ sharing the edge $e \in E_I$. The duality pairing between $H^{-1/2}(q)$ and $H^{1/2}(q)$ with respect to $L^2(q)$- inner product, is denoted by $\langle \cdot, \cdot \rangle_q$. Finally, given a smooth scalar field v and a vector $\boldsymbol{\tau} = (\tau_1, \tau_2)^{\mathtt{t}}$, we define

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \mathrm{rot}(\boldsymbol{\tau}) := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}\,.$$

Next, we introduce the Clément interpolation operator $I_h : H^1(\Omega) \to X_h$ (cf. [22]), where $X_h := \{v_h \in H_0^1(\Omega) \,:\, v_h|_T \in \mathcal{P}_1(T)\,,\ \forall T \in \mathcal{T}_h\}$. The following lemma establishes the main local approximation properties of I_h.

Lemma 1 *There exist constants $c_1, c_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there holds*

$$\|v - I_h(v)\|_{H^m(T)} \,\leq\, c_1\, h_T^{1-m} \|v\|_{H^1(\omega(T))}\,, \quad \forall\, m \in \{0, 1\}\,, \forall\, T \in \mathcal{T}_h\,,$$

and

$$\|v - I_h(v)\|_{L^2(e)} \,\leq\, c_2\, h_e^{1/2} \|v\|_{H^1(\omega(e))} \quad \forall\, e \in E_h\,,$$

where $\omega(T) := \cup\{T' \in \mathcal{T}_h \,:\, T' \cap T \neq \emptyset\}$, h_e denotes the length of the side $e \in E_h$ and $\omega(e) := \cup\{T' \in \mathcal{T}_h \,:\, T' \cap e \neq \emptyset\}$.

Proof. We refer to [22]. □

On the other hand, we also need to introduce the Raviart-Thomas interpolation operator (see [18, 30]), $\Pi_h^r : [H^1(\Omega)]^2 \to H_h^{\boldsymbol{\sigma}}$, which given $\boldsymbol{\tau} \in [H^1(\Omega)]^2$, $\Pi_h^r \boldsymbol{\tau} \in H_h^{\boldsymbol{\sigma}}$ is characterized by the following conditions:

$$\int_e \Pi_h^r(\boldsymbol{\tau}) \cdot \boldsymbol{n}\, q \,=\, \int_e \boldsymbol{\tau} \cdot \boldsymbol{n}\, q\,, \quad \forall e \in E_h\,, \quad \forall\, q \in \mathcal{P}_r(e)\,, \quad \text{when } r \geq 0\,, \tag{7}$$

and

$$\int_T \Pi_h^r(\boldsymbol{\tau}) \cdot \boldsymbol{\rho} \,=\, \int_T \boldsymbol{\tau} \cdot \boldsymbol{\rho}\,, \quad \forall\, T \in \mathcal{T}_h\,, \quad \forall\, \boldsymbol{\rho} \in [\mathcal{P}_{r-1}(T)]^2\,, \quad \text{when } r \geq 1\,. \tag{8}$$

The operator Π_h^r satisfies the following approximation properties.

Lemma 2 *There exist constants $c_3, c_4, c_5 > 0$, independent of h, such that for all $T \in \mathcal{T}_h$*

$$\|\boldsymbol{\tau} - \Pi_h^r(\boldsymbol{\tau})\|_{[L^2(T)]^2} \,\leq\, c_3\, h_T^m\, |\boldsymbol{\tau}|_{[H^m(T)]^2} \quad \forall\, \boldsymbol{\tau} \in [H^m(\Omega)]^2\,, \quad 1 \leq m \leq r+1\,, \tag{9}$$

and for all $\boldsymbol{\tau} \in [H^{m+1}(\Omega)]^2$ with $\mathrm{div}(\boldsymbol{\tau}) \in H^m(\Omega)$,

$$||\mathrm{div}(\boldsymbol{\tau} - \Pi_h^r(\boldsymbol{\tau}))||_{L^2(T)} \,\leq\, c_4\, h_T^m\, |\mathrm{div}(\boldsymbol{\tau})|_{H^m(T)}\,, \quad 0 \leq m \leq r+1\,, \tag{10}$$

and

$$\|\boldsymbol{\tau}\cdot\boldsymbol{n} - \Pi_h^r(\boldsymbol{\tau})\cdot\boldsymbol{n}\|_{L^2(e)} \,\leq\, c_5\, h_e^{1/2}\, \|\boldsymbol{\tau}\|_{[H^1(T_e)]^2} \quad \forall\, e \in E_h\,, \quad \forall\, \boldsymbol{\tau} \in [H^1(\Omega)]^2\,, \tag{11}$$

where $T_e \in \mathcal{T}_h$ contains e on its boundary.

Proof. See e.g. [18] or [30]. □

In addition, the interpolation operator Π_h^r can also be defined as a bounded linear operator from the larger space $[H^s(\Omega)]^2 \cap H(\mathrm{div};\Omega)$ into $H_h^{\boldsymbol{\sigma}}$, for all $s \in (0,1]$ (see, e.g. Theorem 3.16 in [26]). In this case, there holds the following interpolation error estimate

$$\|\boldsymbol{\tau} - \Pi_h^r(\boldsymbol{\tau})\|_{[L^2(T)]^2} \,\leq\, C\, h_T^s \left\{ \|\boldsymbol{\tau}\|_{[H^s(T)]^2} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{L^2(T)} \right\}, \quad \forall\, T \in \mathcal{T}_h\,. \tag{12}$$

Using (7) and (8), it is easy to show that

$$\mathrm{div}(\Pi_h^r(\boldsymbol{\tau})) \,=\, P_h^r(\mathrm{div}(\boldsymbol{\tau}))\,, \tag{13}$$

where $P_h^r : L^2(\Omega) \to H_h^u$ is the L^2-orthogonal projector. It is well known (see, e.g. [21]) that for each $v \in H^m(\Omega)$, with $0 \leq m \leq r+1$, there holds

$$\|v - P_h^r(v)\|_{L^2(T)} \,\leq\, C\, h_T^m\, |\mathbf{v}|_{H^m(T)}\,, \quad \forall\, T \in \mathcal{T}_h\,. \tag{14}$$

3.2 Reliability of the estimator

Let $(\boldsymbol{\sigma}, u)$ be the unique solution to problem (3) and assume that the Galerkin scheme (6) has a unique solution, $(\boldsymbol{\sigma}_h, u_h)$. We define the Ritz projection of the error with respect to the inner product of $\boldsymbol{\Sigma} := H(\mathrm{div},\Omega) \times L^2(\Omega)$,

$$\langle(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{\boldsymbol{\Sigma}} := (\boldsymbol{\sigma}, \boldsymbol{\tau})_{H(\mathrm{div};\Omega)} + (u, v)_{L^2(\Omega)}\,,$$

as the unique element $(\bar{\boldsymbol{\sigma}}, \bar{u}) \in \boldsymbol{\Sigma}$, such that for all $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}$,

$$\langle(\bar{\boldsymbol{\sigma}}, \bar{u}), (\boldsymbol{\tau}, v)\rangle_{\boldsymbol{\Sigma}} = A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau}, v))\,. \tag{15}$$

where the global bilinear form $A : \mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbb{R}$ arises from the variational formulation (3) after adding its equations, that is

$$A((\boldsymbol{\rho}, w), (\boldsymbol{\tau}, v)) := a(\boldsymbol{\rho}, \boldsymbol{\tau}) - b(w, \boldsymbol{\tau}) - b(v, \boldsymbol{\rho}) \qquad \forall (\boldsymbol{\rho}, w),\, (\boldsymbol{\tau}, v) \in \mathbf{\Sigma}\,.$$

We remark that the existence and uniqueness of $(\bar{\boldsymbol{\sigma}}, \bar{u}) \in \mathbf{\Sigma}$ is guaranteed by the Lax-Milgram Lemma. Moreover, we point out that the properties of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ implies that $A(\cdot,\cdot)$ satisfies a global inf-sup condition, i.e., there exist $\alpha > 0$ such that

$$\alpha ||(\boldsymbol{\zeta}, w)||_{\mathbf{\Sigma}} \leq \sup_{\theta \neq (\boldsymbol{\tau}, v) \in \mathbf{\Sigma}} \frac{A((\boldsymbol{\zeta}, w), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|_{\mathbf{\Sigma}}}\,, \qquad \forall (\boldsymbol{\zeta}, w) \in \mathbf{\Sigma}\,.$$

This particularity allows us to bound the error in terms of the solution of its Ritz projection, as follows:

$$\alpha ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_{\mathbf{\Sigma}} \leq \sup_{\theta \neq (\boldsymbol{\tau}, v) \in \mathbf{\Sigma}} \frac{A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|_{\mathbf{\Sigma}}} = ||(\bar{\boldsymbol{\sigma}}, \bar{u})||_{\mathbf{\Sigma}}\,. \tag{16}$$

Then, according to (16), and with the aim to obtain a reliable a posteriori error estimate for the discrete scheme (6), it is enough to bound from above the Ritz projection of the error. To this aim, we introduce the space $M := \{\boldsymbol{\zeta} \in H(\mathrm{div};\, \Omega) \,:\, \int_\Omega \mathrm{div}(\boldsymbol{\zeta}) = 0\}$. For each $\boldsymbol{\tau} \in H(\mathrm{div};\, \Omega)$, we decompose $\mathrm{div}(\boldsymbol{\tau}) = \mathrm{div}(\boldsymbol{\tau}_0) + d$ where $\boldsymbol{\tau}_0 \in M$ and $d := \frac{1}{|\Omega|}\int_\Omega \mathrm{div}(\boldsymbol{\tau})$. We notice $\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega} = \|\mathrm{div}(\boldsymbol{\tau}_0)\|_{0,\Omega} + |d|$. Now, since $\mathrm{div}(\boldsymbol{\tau}_0) \in L^2_0(\Omega)$, by Corollary I.2.4 in [25], there exists $\mathbf{\Phi} \in [H^1(\Omega)]^2$ such that $\mathrm{div}(\mathbf{\Phi}) = \mathrm{div}(\boldsymbol{\tau}_0)$ in Ω and $\|\mathbf{\Phi}\|_{1,\Omega} \leq c\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}$. This implies that

$$\mathrm{div}\left(\boldsymbol{\tau} - \mathbf{\Phi} - \frac{d}{2}(x_1\ x_2)^t\right) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \left\langle \left(\boldsymbol{\tau} - \mathbf{\Phi} - \frac{d}{2}(x_1\ x_2)^t\right) \cdot \boldsymbol{n}, 1 \right\rangle = 0\,.$$

Hence, by Theorem I.3.1 and its consequences (cf page 39 in [25]), there exists a stream function $\chi \in H^1_0(\Omega)$ such that $\boldsymbol{\tau} - \mathbf{\Phi} - \frac{d}{2}(x_1\ x_2)^t = \mathbf{curl}(\chi)$ in Ω. In addition, there exists a constant $C > 0$, such that

$$\|\chi\|_{H^1(\Omega)} \,+\, \|\mathbf{\Phi}\|_{[H^1(\Omega)]^2} \,\leq\, C\,\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)}\,. \tag{17}$$

Then, we have obtained the quasi Helmholtz decomposition of $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = \mathbf{curl}(\chi) \,+\, \mathbf{\Phi} \,+\, \frac{d}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now, introducingt $\chi_h := I_h(\chi)$, we define

$$\boldsymbol{\tau}_h := \mathbf{curl}(\chi_h) \;+\; \Pi_h^r(\boldsymbol{\Phi}) + \frac{d}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H_h^{\boldsymbol{\sigma}} \,. \tag{18}$$

We refer to (18) as a *discrete quasi Helmholtz decomposition* of $\boldsymbol{\tau}_h$. Therefore, we can write

$$\boldsymbol{\tau} - \boldsymbol{\tau}_h \;=\; \mathbf{curl}(\chi - \chi_h) \;+\; \boldsymbol{\Phi} - \Pi_h^r(\boldsymbol{\Phi})\,, \tag{19}$$

which yields

$$\operatorname{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \;=\; \operatorname{div}(\boldsymbol{\Phi} - \Pi_h^r(\boldsymbol{\Phi})) \tag{20}$$

On the other hand, it is not difficult to see the following orthogonality relation

$$A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\zeta}_h, v_h)) = 0\,, \quad \forall\, (\boldsymbol{\zeta}_h, v_h) \in \boldsymbol{\Sigma}_h := H_h^{\boldsymbol{\sigma}} \times H_h^u\,. \tag{21}$$

From now on, for each $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}$, we denote its induced discrete pair by $(\boldsymbol{\tau}_h, 0) \in \boldsymbol{\Sigma}_h$, where $\boldsymbol{\tau}_h$ is defined as in 18. Hence, considering 21 with $(\boldsymbol{\zeta}_h, v_h) := (\boldsymbol{\tau}_h, 0)$, and knowing that $(\boldsymbol{\sigma}, u)$ is the unique solution of problem 3, we obtain

$$\begin{aligned}
\langle(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}), (\boldsymbol{\tau}, \mathbf{v})\rangle_{\boldsymbol{\Sigma}} \;&=\; A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_h, v)) \\
&=\; -\langle(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\cdot\boldsymbol{n}, g\rangle - \int_\Omega f\, v - A((\boldsymbol{\sigma}_h, u_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_h, v))
\end{aligned}$$

Equivalently,

$$\begin{aligned}
(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})_{H(\mathrm{div};\,\Omega)} \;&=\; F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,, \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\,\Omega)\,, \\
(\bar{u}, v)_{L^2(\Omega)} \;&=\; F_2(v)\,, \qquad\qquad \forall\, v \in L^2(\Omega)\,.[1.1ex]
\end{aligned}$$

where $F_1 : H(\mathrm{div};\,\Omega) \to \mathbb{R}$ and $F_2 : L^2(\Omega) \to \mathbb{R}$ are the bounded linear functionals defined as

$$F_1(\boldsymbol{\rho}) := -\langle \boldsymbol{\rho}\cdot\boldsymbol{n}, g\rangle \;-\; \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\rho} + \int_\Omega u_h \operatorname{div}(\boldsymbol{\rho})\,, \quad \forall\, \boldsymbol{\rho} \in H(\mathrm{div};\,\Omega)\,,$$

$$F_2(w) \;:=\; -\int_\Omega (f - \operatorname{div}(\boldsymbol{\sigma}_h))\, w\,, \quad \forall w \in L^2(\Omega)\,.$$

Hence, taking into account (19) and (20) we can rewrite $F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)$ as follows

$$F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \;=\; R_1(\boldsymbol{\Phi}) \;+\; R_2(\chi)\,,$$

where

$$R_1(\mathbf{\Phi}) \,:=\, -\langle(\mathbf{\Phi}-\Pi_h^k(\mathbf{\Phi}))\cdot\boldsymbol{n}, g-u_h\rangle \,-\, \int_\Omega(\boldsymbol{\sigma}_h+\nabla u_h)\,\cdot\,(\mathbf{\Phi}-\Pi_h^k(\mathbf{\Phi}))$$

$$+\sum_{T\in\mathcal{T}_h}\int_{\partial T\cap E_I} u_h\,(\mathbf{\Phi}-\Pi_h^k(\mathbf{\Phi}))\cdot\boldsymbol{n}\,,$$

and

$$R_2(\chi) \,:=\, -\int_\Omega \boldsymbol{\sigma}_h\cdot\,\mathbf{curl}(\chi-\chi_h)\,.$$

Our aim now is to obtain upper bounds for each one of the terms $F_2(v)$, $R_1(\mathbf{\Phi})$ and $R_2(\chi)$.

Lemma 3 *For any $v\in L^2(\Omega)$ there holds*

$$|F_2(v)| \,\leq\, \Big(\sum_{T\in\mathcal{T}_h}\|f-\mathrm{div}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)}\Big)^{1/2}\,\|v\|_{L^2(\Omega)}\,.$$

Proof. The proof follows from a straightforward application of Cauchy-Schwarz inequality. □

Lemma 4 *There exists $C>0$, independent of h, such that*

$$|R_1(\mathbf{\Phi})|\leq C\Big(\sum_{e\in E_\Gamma}h_e\,\|g-u_h\|^2_{L^2(e)}+\sum_{e\in E_I}h_e\,\|[\![u_h]\!]\|^2_{[L^2(e)]^2}$$
$$+\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla u_h+\boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2}\Big)^{1/2}\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)}\,.$$

Proof. It is a slight modification of Lemma 3.5 in [10]. We omit further details. □

Lemma 5 *There exists $C>0$, independent of h, such that*

$$|R_2(\chi)| \,\leq\, C\,\Big(\sum_{T\in\mathcal{T}_h}h_T^2\|\mathrm{rot}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)}+\sum_{e\in E(T)}h_e\|[\![\boldsymbol{\sigma}_h]\!]\|^2_{L^2(e\cap E_I)}\Big)^{1/2}\,\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)}\,.$$

Proof. Integrating by parts, we deduce

$$\int_\Omega\boldsymbol{\sigma}_h\cdot\,\mathbf{curl}(\chi-\chi_h)=\sum_{T\in\mathcal{T}_h}\int_T\boldsymbol{\sigma}_h\cdot\,\mathbf{curl}(\chi-\chi_h)$$

$$=\sum_{T\in\mathcal{T}_h}\Big\{\int_T\mathrm{rot}(\boldsymbol{\sigma}_h)\,(\chi-\chi_h)+\,\langle\chi-\chi_h,\boldsymbol{\sigma}_h\cdot\mathbf{t}\rangle_{\partial T}\Big\}$$

$$\leq \sum_{T\in\mathcal{T}_h} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} \|\chi - \chi_h\|_{L^2(T)} + \int_{E_I} (\chi - \chi_h) [\![\boldsymbol{\sigma}_h]\!]$$

$$\leq \sum_{T\in\mathcal{T}_h} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} \|\chi - \chi_h\|_{L^2(T)} + \sum_{e\in E_I} \|\chi - \chi_h\|_{L^2(e)} \|[\![\boldsymbol{\sigma}_h]\!]\|_{L^2(e)} \,.$$

Therefore, the proof is completed using Lemma 1, the Cauchy-Schwarz inequality, the regularity of the mesh and (17). □

The previous results suggest the definition of the following residual estimator

$$\eta \,:=\, \Big(\sum_{T\in\mathcal{T}_h} \eta_T^2 \Big)^{1/2} , \tag{22}$$

where

$$\eta_T^2 \,:=\, \|f - \mathrm{div}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)} \,+\, h_T^2 \|\boldsymbol{\sigma}_h + \nabla u_h\|^2_{[L^2(T)]^2} \,+\, h_T^2 \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)}$$

$$+ \sum_{e\in E(T)} h_e \left\{ \|[\![u_h]\!]\|^2_{L^2(e\cap E_I)} \,+\, \|[\![\boldsymbol{\sigma}_h]\!]\|^2_{L^2(e\cap E_I)} \,+\, \|g - u_h\|^2_{L^2(e\cap E_\Gamma)} \right\}$$

An upper bound for $||(\bar{\boldsymbol{\sigma}}, \bar{u})||_{\boldsymbol{\Sigma}}$ is established in the next lemma, in terms of (22).

Lemma 6 *There exists a constant $C > 0$, independent of h, such that*

$$||(\bar{\boldsymbol{\sigma}}, \bar{u})||_{\boldsymbol{\Sigma}} \,\leq\, C\,\eta\,, \tag{23}$$

where η is given in (22).

Proof. Invoking Lemmas 4 and 5, we deduce that there exists $c > 0$, independent of h, such that

$$c|F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)| \leq \left(\sum_{T\in\mathcal{T}_h} \left\{ h_T^2 \|\boldsymbol{\sigma}_h + \nabla u_h\|^2_{[L^2(T)]^2} \right\} + \sum_{e\in E_h} h_e \left\{ ||g - u_h||^2_{L^2(e\cap E_\Gamma)} \right.\right.$$

$$\left.\left. + \|[\![u_h]\!]\|^2_{L^2(e\cap E_I)} \,+\, \|[\![\boldsymbol{\sigma}_h]\!]\|^2_{L^2(e\cap E_I)} \right\} \right)^{1/2} \|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} \,.$$

Hence, (23) follows from the above bound, Lemma 3 and a discrete Cauchy-Schwarz inequality. □

The following theorem exhibits the main result of this section, which establishes the reliability and efficiency of the estimator η.

Theorem 7 *There exists a positive constant $C_{\tt rel}$, independent of h, such that*

$$||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_{\boldsymbol{\Sigma}} \leq C_{\tt rel}\, \eta\,.$$

Additionally, there exists $C_{\tt eff} > 0$, independent of h, such that

$$\eta_T^2 \leq C_{\tt eff} ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_T \tag{24}$$

with $||(\boldsymbol{\tau}, \mathbf{v})||_T^2 := \|\boldsymbol{\tau}\|^2_{H(\mathrm{div};T)} + \|\mathbf{v}\|^2_{L^2(T)}$.

Proof. The reliability of η (first inequality) follows from (16) and Lemma 6. The efficiency of η (second inequality) is treated in the next subsection. We omit further details. □

3.3 Efficiency of the estimator

In this subsection we prove the local efficiency of the estimator (22) (cf. (59)). We begin by introducing some notations and preliminary results. Given $T \in \mathcal{T}_h$ and $e \in E(T)$, we let ψ_T and ψ_e be the standard triangle-bubble and edge-bubble functions, respectively. In particular, ψ_T satisfies $\psi_T \in \mathcal{P}_3(T)$, $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on ∂T, and $0 \leq \psi_T \leq 1$ in T. Similarly, $\psi_e|_T \in \mathcal{P}_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e := \cup\{T' \in \mathcal{T}_h : \quad e \in E(T')\}$, $\psi_e = 0$ on $\partial\omega_e$, and $0 \leq \psi_e \leq 1$ in ω_e. We also recall from [32] that, given $k \in \mathbb{N} \cup \{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathcal{P}_k(T)$ and $L(p)|_e = p \;\forall\, p \in \mathcal{P}_k(e)$. Additional properties of ψ_T, ψ_e, and L are collected in the following lemma.

Lemma 8 *For any triangle T there exist positive constants c_1, c_2, c_3 and c_4, depending only on k and the shape of T, such that for all $q \in \mathcal{P}_k(T)$ and $p \in \mathcal{P}_k(e)$, there hold*

$$||\psi_T\, q||^2_{L^2(T)} \leq ||q||^2_{L^2(T)} \leq c_1\, ||\psi_T^{1/2}\, q||^2_{L^2(T)}\,, \tag{25}$$

$$||\psi_e\, p||^2_{L^2(e)} \leq ||p||^2_{L^2(e)} \leq c_2\, ||\psi_e^{1/2}\, p||^2_{L^2(e)}\,, \tag{26}$$

$$c_4\, h_e\, ||p||^2_{L^2(e)} \leq ||\psi_e^{1/2}\, L(p)||^2_{L^2(T)} \leq c_3\, h_e\, ||p||^2_{L^2(e)}\,, \tag{27}$$

Proof. See Lemma 4.1 in [32]. □

The following inverse estimate will also be useful.

Lemma 9 *Let $l, m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, for any triangle T, there exists $c > 0$, depending only on k, l, m and the shape of T, such that*

$$|q|_{H^m(T)} \leq c\, h_T^{l-m} |q|_{H^l(T)} \quad \forall\, q \in \mathcal{P}_k(T)\,. \tag{28}$$

Proof. See Theorem 3.2.6 in [21]. $\square$

Since $f = \mathrm{div}(\boldsymbol{\sigma})$ in Ω, we have that

$$||f - \mathrm{div}(\boldsymbol{\sigma}_h)||_{L^2(T)} \;=\; ||\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_{L^2(T)} \,. \tag{29}$$

Lemma 10 *There exists $C_1 > 0$, independent of the meshsize, such that for any $T \in \mathcal{T}_h$*

$$h_T\, ||\boldsymbol{\sigma}_h + \nabla u_h||_{[L^2(T)]^2} \leq C_1 \Big(\|u - u_h\|_{L^2(T)} \;+\; h_T\, ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} \Big) \,. \tag{30}$$

Proof. We introduce $\boldsymbol{\rho}_h := \boldsymbol{\sigma}_h + \nabla u_h$ in T. Then, using the property (25) and integrating by parts, we have

$$c_1^{-1}\, ||\boldsymbol{\rho}_h||^2_{[L^2(T)]^2} \;\leq\; ||\psi_T^{1/2} \boldsymbol{\rho}_h||^2_{[L^2(T)]^2} \;=\; \int_T (\boldsymbol{\sigma}_h + \nabla u_h) \cdot \psi_T \boldsymbol{\rho}_h$$

$$= \int_T \boldsymbol{\sigma}_h \cdot \psi_T \boldsymbol{\rho}_h + \int_T \nabla u_h \cdot \psi_T \boldsymbol{\rho}_h \;=\; \int_T \boldsymbol{\sigma}_h \cdot \psi_T \boldsymbol{\rho}_h - \int_T u_h \mathrm{div}(\psi_T \boldsymbol{\rho}_h)$$

$$= \int_T (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \psi_T \boldsymbol{\rho}_h + \int_T (u - u_h) \mathrm{div}(\psi_T \boldsymbol{\rho}_h) \,.$$

Now, applying Cauchy-Schwarz inequality as well as inverse inequality (28) and property $0 \leq \psi_T \leq 1$, we derive

$$\begin{aligned} c_1^{-1}\, ||\boldsymbol{\rho}_h||^2_{[L^2(T)]^2} \;\leq\; & \Big\{ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} ||\psi_T^{1/2} \boldsymbol{\rho}_h||_{[L^2(T)]^2} \\ & + ||u - u_h||_{L^2(T)} \|\mathrm{div}(\psi_T \boldsymbol{\rho}_h)\|_{[L^2(T)]^2} \Big\} \\ \leq\; & ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\, ||\boldsymbol{\rho}_h||_{[L^2(T)]^2} \;+ \sqrt{2} ||u - u_h||_{L^2(T)}\, ||\nabla(\psi_T \boldsymbol{\rho}_h)||_{[L^2(T)]^{2\times 2}} \\ \leq\; & C \Big\{ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} \;+ \sqrt{2} h_T^{-1} ||u - u_h||_{L^2(T)} \Big\}\, ||\boldsymbol{\rho}_h||_{[L^2(T)]^2} \,. \end{aligned}$$

Hence, simplifying $||\boldsymbol{\rho}_h||_{[L^2(T)]^2}$ and multiplying by the factor h_T, we complete the proof of the lemma. $\square$

In the following lemma, we bound the jump of u_h,

Lemma 11 *There exists $C_2 > 0$, independent of the mesh size, such that for any $e \in \mathcal{E}_I$*

$$h_e ||[\![u_h]\!]||^2_{L^2(e)} \;\leq\; C_2 \left\{ ||u - u_h||^2_{L^2(\omega_e)} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L(\omega_e)]^2} \right\} \,. \tag{31}$$

Proof. First, given $e \in E_I$ we denote $\omega_e = T \cup T'$, with $T, T' \in \mathcal{T}_h$. Introducing $w_h := [\![u_h]\!]$ on e and $\boldsymbol{\rho}_e = \psi_e \, L(w_h)\boldsymbol{n}_{T,e}$ in ω_e, we notice $\boldsymbol{\rho}_e \in H(\mathrm{div}, \omega_e)$. Using (26), knowing that $[\![u]\!] = 0$ on E_I, and integrating by parts, it follows that

$$c_2^{-1}||w_h||^2_{L^2(e)} \leq ||\psi_e^{1/2} w_h||^2_{L^2(e)} = \int_e \psi_e L(w_h)[\![u_h - u]\!] = \int_e [\![u_h - u]\!]\boldsymbol{\rho}_e \cdot \boldsymbol{n}_T$$

$$= \int_{\omega_e} (u_h - u)\mathrm{div}(\boldsymbol{\rho}_e) + \int_{\omega_e} \nabla_h(u_h - u) \cdot \boldsymbol{\rho}_e$$

$$= \int_{\omega_e} (u_h - u)\mathrm{div}(\boldsymbol{\rho}_e) + \int_{\omega_e} (\boldsymbol{\sigma}_h + \nabla_h u_h) \cdot \boldsymbol{\rho}_e + \int_{\omega_e} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{\rho}_e \, .$$

Using that $\int_{\omega_e} = \int_T + \int_{T'}$ and applying Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} & c_2^{-1}||w_h||^2_{L^2(e)} \\ & \leq \|u - u_h\|_{L^2(T)}\|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T)} + \|u - u_h\|_{L^2(T')}\|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T')} \\ & + \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T)]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T')]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T')]^2} \\ & + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T')]^2} \, . \end{aligned} \tag{32}$$

Now, the inverse inequality (28), $0 \leq \psi_e \leq 1$ together with (27), implies for each $T \in \mathcal{T}_h$

$$\|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T)} \leq \sqrt{2}\|\nabla\boldsymbol{\rho}_e\|_{[L^2(T)]^{2\times2}} \leq c\sqrt{2}h_T^{-1}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2}$$

$$= c\sqrt{2}h_T^{-1}\|\psi_e^{1/2}\, L(w_h)\|_{L^2(T)} \leq c\, c_3\sqrt{2}h_T^{-1/2}\|w_h\|_{L^2(e)} \, .$$

This inequality, together with (27), allow us to rewrite (32) as follows: There exists $c > 0$ independent of mesh size, such that

$$\begin{aligned} c||w_h||^2_{L^2(e)} \leq \Big\{ & h_T^{-1/2}\|u - u_h\|_{L^2(T)} + h_{T'}^{-1/2}\|u - u_h\|_{L^2(T')} \\ & + h_T\, \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T)]^2} + h_{T'}\|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T')]^2} \\ & + h_T\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} + h_{T'}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2}\Big\}\|w_h\|_{L^2(e)} \, . \end{aligned}$$

Then the proof follows after multiplying by h_e, and applying Lemma 10. □

Lemma 12 *There exists $C_3 > 0$, independent of the meshsize, such that for any $T \in \mathcal{T}_h$*

$$h_T \, ||\mathrm{rot}(\boldsymbol{\sigma}_h)||_{L^2(T)} \leq C_1 \Big(||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} \Big) \,. \tag{33}$$

Proof. We introduce $\rho_h := \mathrm{rot}(\boldsymbol{\sigma}_h)$ in T. Then, invoking the property (25), $\mathrm{rot}(\boldsymbol{\sigma}) = 0$ in T, and integrating by parts, we have

$$c_1^{-1} \, ||\rho_h||^2_{L^2(T)} \;\leq\; ||\psi_T^{1/2} \rho_h||^2_{[L^2(T)]^2} \;=\; \int_T \mathrm{rot}(\boldsymbol{\sigma}_h) \psi_T \rho_h$$

$$= \int_T \mathrm{rot}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \psi_T \rho_h \;=\; \int_T (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{curl}(\psi_T \rho_h) \,.$$

Now, applying Cauchy-Schwarz inequality, as well as inverse inequality (28) and property $0 \leq \psi_T \leq 1$ in T, we derive

$$\begin{aligned} c_1^{-1} \, ||\rho_h||^2_{[L^2(T)]^2} \;&\leq\; ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} || \, \mathbf{curl}(\psi_T \rho_h)||_{[L^2(T)]^2} \\ &=\; ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} \, ||\nabla(\psi_T \rho_h)||_{[L^2(T)]^2} \\ &\leq\; C ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} h_T^{-1} ||\psi_T \rho_h||_{L^2(T)} . \\ &\leq\; C h_T^{-1} ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} \, ||\rho_h||_{L^2(T)} \,. \end{aligned}$$

Hence, simplifying $||\rho_h||_{[L^2(T)]^2}$ and multiplying by the factor h_T, we complete the proof of the lemma. □

The tangential component jump is treated in the next lemma.

Lemma 13 *There exists $C_3 > 0$, independent of the mesh size, such that for any $e \in \mathcal{E}_I$*

$$h_e ||[\![\boldsymbol{\sigma}_h]\!]||^2_{L^2(e)} \;\leq\; C_3 \left\{ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L(\omega_e)]^2} \right\} \,. \tag{34}$$

Proof. Given $e \in E_I$, let $T, T' \in \mathcal{T}_h$ such that $\omega_e = T \cup T'$ and $\bar{T} \cap \bar{T}' = e$. Denoting by $w_h := [\![\boldsymbol{\sigma}_h]\!]$ on e, and using (26), it follows that

$$\begin{aligned} c_2^{-1} ||w_h||^2_{L^2(e)} \leq ||\psi_e^{1/2} w_h||^2_{L^2(e)} &= \int_e \psi_e L(w_h) [\![\boldsymbol{\sigma}_h]\!] \\ &= \int_e \psi_e L(w_h) \boldsymbol{\sigma}_h \cdot \mathbf{t}_T + \int_e \psi_e L(w_h) \boldsymbol{\sigma}_h \cdot \mathbf{t}_{T'} \\ &= - \int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \boldsymbol{\sigma}_h \;+\; \int_{\omega_e} \psi_e L(w_h) \mathrm{rot}(\boldsymbol{\sigma}_h) \\ &= \int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \;+\; \int_{\omega_e} \psi_e L(w_h) \mathrm{rot}_h(\boldsymbol{\sigma}_h) \,, \end{aligned} \tag{35}$$

where in the last equality we take into account

$$\int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \boldsymbol{\sigma} = - \int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \nabla u = \int_{\partial\omega_e} \psi_e L(w_h)\, \nabla u \cdot \mathbf{t} = 0\,.$$

In addition, using that $\int_{\omega_e} = \int_T + \int_{T'}$ and applying Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} c_2^{-1} \|w_h\|_{L^2(e)}^2 \;\leq\; & \| \mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T)]^2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \\ & + \|\psi_e L(w_h)\|_{L^2(T)} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} \\ & + \| \mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T')]^2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2} \\ & + \|\psi_e L(w_h)\|_{L^2(T')} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T')}\,. \end{aligned} \tag{36}$$

Straightforwardly, using $0 \leq \psi_e^{1/2} \leq 1$ in joint with (27), for each $T \in \mathcal{T}_h$, we deduce

$$\|\psi_e L(w_h)\|_{L^2(T)} \leq c_3 h_T^{1/2} \|w_h\|_{L^2(e)}\,. \tag{37}$$

Now, the inverse inequality (28), $0 \leq \psi_e^{1/2} \leq 1$, together with (27), implies for each $T \in \mathcal{T}_h$

$$\begin{aligned} & \| \mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T)]^2} = \|\nabla(\psi_e L(w_h))\|_{[L^2(T)]^2} \\ & \leq c h_T^{-1} \|\psi_e L(w_h)\|_{L^2(T)} \leq c h_T^{-1} \|\psi_e^{1/2}\, L(w_h)\|_{L^2(T)} \\ & \leq c\, c_3 h_T^{-1/2} \|w_h\|_{L^2(e)}\,. \end{aligned} \tag{38}$$

Inequalities (37) and (38) allow us to rewrite (36) as follows: There exists $c > 0$ independent of meshsize, such that

$$\begin{aligned} c\|w_h\|_{L^2(e)}^2 \;\leq\; & \Big\{ h_T^{-1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} + h_{T'}^{-1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2} \\ & + h_T^{1/2}\, \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} \,+\, h_{T'}^{1/2} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T')} \Big\} \|w_h\|_{L^2(e)}\,. \end{aligned}$$

Then, (34) follows simplifying $\|w_h\|_{L^2(e)}$, multiplying by $h_e^{1/2}$ and invoking Lemma 12. □

Now, in order to bound the boundary terms $h_e\|\mathbf{g} - \mathbf{u}_h\|_{[L^2(e)]^2}^2$, $e \in E_\Gamma$, we need to recall a discrete trace inequality. Indeed, as established in Theorem 3.10 in [1] (see also equation (2.4) in [5]), there exists $c > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h$ and $e \in E(T)$, there holds

$$||v||^2_{L^2(e)} \le c\left\{h_e^{-1}\,||v||^2_{L^2(T)} + h_e\,\|\nabla v\|^2_{[L^2(T)]^2}\right\}, \quad \forall\, v \in H^1(T)\,. \tag{39}$$

Then, a straightforward application of (39) and Lemma 10, knowing that $u = g$ on Γ, help us to establish.

Lemma 14 *There exists $C > 0$, independent of h, such that for each $e \in E_\Gamma$ there holds*

$$h_e||g - u_h||^2_{L^2(e)} \le C\Big(||u - u_h||^2_{L^2(T_e)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(T_e)]^2}\Big)\,, \tag{40}$$

where T_e is the triangle having e as an edge.

4 Mixed Boundary conditions

We assume that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D is a closed part of Γ with positive measure and $\Gamma_N = \Gamma \setminus \Gamma_D$. We introduce the spaces $H^1_{\Gamma_D} := \{v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_D\}$ and $H^{1/2}_{00}(\Gamma_N) = \{v|_{\Gamma_N} : \ v \in H^1_{\Gamma_D}(\Omega)\}$. Given $f \in L^2(\Omega)$ and $g_N \in H^{-1/2}_{00}(\Gamma_N)$, we consider the first order system: find the flux $\tilde{\boldsymbol{\sigma}}$ and scalar unknown u such that

$$\left\{\begin{array}{rcll} \tilde{\boldsymbol{\sigma}} &=& -\nabla u & \text{in } \Omega, \\ \text{div}(\tilde{\boldsymbol{\sigma}}) &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma_D, \\ \tilde{\boldsymbol{\sigma}}\cdot\mathbf{n} &=& -g_N & \text{on } \Gamma_N, \end{array}\right. \tag{41}$$

where $\mathbf{n}$ denote the exterior normal unitary vector. Let $\boldsymbol{\sigma}_{g_N} \in H(\text{div};\Omega)$ such that $\boldsymbol{\sigma}_{g_N}\cdot\mathbf{n} = -g_N$ on Γ_N. Then, we can write $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}_{g_N}$, with $\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0}$ on Γ_N. We remark that problem (41) is equivalent to the following problem:

$$\left\{\begin{array}{rcll} \text{div}(\boldsymbol{\sigma}) &=& \tilde{f} & \text{in } \Omega\,, \\ \boldsymbol{\sigma} &=& -\nabla u - \boldsymbol{\sigma}_{g_N} & \text{in } \Omega\,, \\ u &=& 0 & \text{on } \Gamma_D\,, \\ \boldsymbol{\sigma}\cdot\mathbf{n} &=& 0 & \text{on } \Gamma_N\,, \end{array}\right. \tag{42}$$

with $\tilde{f} := f - \textbf{div}(\boldsymbol{\sigma}_{g_N})$. From now on, we define the spaces $H_0 := \{\boldsymbol{\tau} \in H(\text{div};\Omega) : \ \boldsymbol{\tau}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N\}$ and $\mathcal{L} := \{v \in L^2(\Omega) : \ v = 0 \quad \text{on } \Gamma_D\}$.

Therefore, we have the following dual-mixed variational formulation of problem (42): find $\boldsymbol{\sigma} \in H_{\mathbf{0}}$ and $u \in L^2(\Omega)$ such that

$$\left\{\begin{array}{rcll} \displaystyle\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} - \int_\Omega u\,\mathrm{div}(\boldsymbol{\tau}) & = & \displaystyle -\int_\Omega \boldsymbol{\sigma}_{g_N}\cdot\boldsymbol{\tau}\,, & \forall\,\boldsymbol{\tau}\in H_{\mathbf{0}}\,, \\ \displaystyle -\int_\Omega \mathrm{div}(\boldsymbol{\sigma})\,v & = & \displaystyle -\int_\Omega \tilde{f}\,v\,, & \forall\,v\in L^2(\Omega)\,. \end{array}\right. \tag{43}$$

Let us define the bilinear forms $\tilde{a}(\cdot,\cdot) : H_0 \times H_0 \to \mathbb{R}$ and $\tilde{b}(\cdot,\cdot) : L^2(\Omega)\times H_0 \to \mathbb{R}$ as follows:

$$\tilde{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau}\,, \qquad \tilde{b}(v,\boldsymbol{\tau}) := \int_\Omega v\,\mathrm{div}(\boldsymbol{\tau})\,,$$

for any $\boldsymbol{\zeta},\boldsymbol{\tau}\in H_0$ and $v\in L^2(\Omega)$.

We also define the linear functionals $l : L^2(\Omega)\to\mathbb{R}$ by $l(v) := -\int_\Omega \tilde{f}\,v$, $\forall\,v\in L^2(\Omega)$ and $m : H_0\to\mathbb{R}$ by $m(\boldsymbol{\tau}) := -\int_\Omega \boldsymbol{\sigma}_{g_N}\cdot\boldsymbol{\tau}$, $\forall\,\boldsymbol{\tau}\in H_0$.

Then, the dual-mixed variational formulation (43) can be written in the saddle-point general framework: find $\boldsymbol{\sigma}\in H_{\mathbf{0}}$ and $u\in L^2(\Omega)$ such that

$$\left\{\begin{array}{rcll} \tilde{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) - \tilde{b}(u,\boldsymbol{\tau}) & = & m(\boldsymbol{\tau})\,, & \forall\,\boldsymbol{\tau}\in H_{\mathbf{0}}\,, \\ -\tilde{b}(v,\boldsymbol{\sigma}) & = & l(v)\,, & \forall\,v\in L^2(\Omega)\,. \end{array}\right. \tag{44}$$

According to [23] (also see [18]), we can ensure that problem (44) has a unique solution.

4.1 The discrete scheme

In order to introduce the discrete scheme, given an integer $r\geq 0$, we define the finite element subspaces

$$H^{\boldsymbol{\sigma}}_{0,h} := \left\{\,\boldsymbol{\tau}_h\in H^{\boldsymbol{\sigma}}_{h,r} : \boldsymbol{\tau}_h\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N\right\}, \tag{45}$$

$$H^u_{\Gamma_D,h} := \left\{\,v_h\in H^u_{h,r} : v_h = 0 \quad \text{on } \Gamma_D\,\right\}, \tag{46}$$

where $H^{\boldsymbol{\sigma}}_{h,r}$ and $H^u_{h,r}$ are given by (4) and (5), respectively. The discrete version of (44) reads as follows: Find $(\boldsymbol{\sigma}_h,u_h)\in\boldsymbol{\Sigma}_{0,h} := H^{\boldsymbol{\sigma}}_{0,h}\times H^u_{\Gamma_D,h}$ such that

$$\begin{array}{rcll} \tilde{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) \; -\tilde{b}(u_h,\boldsymbol{\tau}_h) & = & m(\boldsymbol{\tau}_h) & \forall\,\boldsymbol{\tau}_h\in H^{\boldsymbol{\sigma}}_{0,h}\,, \\ -\tilde{b}(w_h,\boldsymbol{\sigma}_h) & = & l(w_h) & \forall\,w_h\in H^u_{\Gamma_D,h}\,. \end{array} \tag{47}$$

It is not difficult to check that this scheme has a unique solution $(\boldsymbol{\sigma}_h,u_h)\in H^{\boldsymbol{\sigma}}_{0,h}\times H^u_{\Gamma_D,h}$ (see [23]).

4.2 Reliability of the estimator

In what follows, let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h)$ be the unique solutions to problem (44) and (47), respectively. We define the Ritz projection of the error, as the unique element $(\bar{\boldsymbol{\sigma}}, \bar{u}) \in \boldsymbol{\Sigma}_0 := H_0 \times L^2(\Omega)$ such that for all $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_0$,

$$\langle(\bar{\boldsymbol{\sigma}}, \bar{u}), (\boldsymbol{\tau}, v)\rangle_{\boldsymbol{\Sigma}} = \tilde{A}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau}, v)) \,, \tag{48}$$

where the global bilinear form $\tilde{A} : \boldsymbol{\Sigma}_0 \times \boldsymbol{\Sigma}_0 \to \mathbb{R}$ arises from the variational formulation (44), after adding its equations.

We note that the Lax-Milgram Lemma ensures existence and uniqueness of $(\bar{\boldsymbol{\sigma}}, \bar{u}) \in \boldsymbol{\Sigma}_0$. Additionally, $\tilde{A}(\cdot, \cdot)$ satisfies a global inf-sup condition, i.e. there exists $\tilde{\alpha} > 0$, such that

$$\tilde{\alpha}||(\boldsymbol{\zeta}, w)||_{\boldsymbol{\Sigma}} \leq \sup_{\theta \neq (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_0} \frac{\tilde{A}((\boldsymbol{\zeta}, w), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|_{\boldsymbol{\Sigma}}} \qquad \forall (\boldsymbol{\zeta}, w) \in \boldsymbol{\Sigma}_0 \,.$$

This property allows us to bound the error in terms of the solution of its Ritz projection, indeed:

$$\tilde{\alpha}||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_{\boldsymbol{\Sigma}} \leq \sup_{\theta \neq (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_0} \frac{\tilde{A}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau}, v))}{\|(\boldsymbol{\tau}, v)\|_{\boldsymbol{\Sigma}}} = ||(\bar{\boldsymbol{\sigma}}, \bar{u})||_{\boldsymbol{\Sigma}} \,. \tag{49}$$

To bound the supremum on the right hand side of (49), we need a suitable choice of $\boldsymbol{\tau}_h \in H_{0,h}^{\boldsymbol{\sigma}}$. Next, for each $\boldsymbol{\tau} \in H_0$ we consider its quasi-Helmholtz decomposition (see Lemma 5.1 in [20])

$$\boldsymbol{\tau} = \operatorname{curl}(\chi) \, + \, \boldsymbol{\Phi} \,,$$

where $\chi \in H^1(\Omega)$ and $\boldsymbol{\Phi} \in [H^1_{\Gamma_N}(\Omega)]^2$ satisfies $\operatorname{div}(\boldsymbol{\Phi}) = \operatorname{div}(\boldsymbol{\tau})$ in Ω and $\operatorname{curl}(\chi) \cdot \boldsymbol{n} = 0$ on Γ_N. Moreover, there exists $C > 0$, such that

$$||\chi||_{H^1(\Omega)} \, + \, ||\boldsymbol{\Phi}||_{[H^1(\Omega)]^2} \, \leq \, C \, ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \,. \tag{50}$$

Since $\frac{d\chi}{d\mathbf{t}} = \operatorname{curl}(\chi) \cdot \boldsymbol{n} = 0$ on Γ_N, it follows that $\chi = c$ (a constant) on Γ_N. Then, in order to preserve the boundary value of χ on Γ_N, we proceed as in Lemma 5.2 in [20], and we introduce $\chi_h := \mathcal{SZ}_h(\chi)$, the Scott-Zhang interpolation of χ (see [31]). We notice that $\chi_h = \chi = c$ on Γ_N, thus $\operatorname{curl}(\chi_h) \cdot \boldsymbol{n} = \frac{d}{d\mathbf{t}}(\chi_h) = \frac{d}{d\mathbf{t}}(c) = 0$ on Γ_N, which ensures that $\operatorname{curl}(\chi_h) \in H_{0,h}^{\boldsymbol{\sigma}}$. After that, we define

$$\boldsymbol{\tau}_h := \operatorname{curl}(\chi_h) \, + \, \tilde{\Pi}_h^k(\Phi) \in H_{0,h}^{\boldsymbol{\sigma}} \,, \tag{51}$$

where $\tilde{\Pi}_h^k : H^1(\Omega) \to H_{0,h}^{\boldsymbol{\sigma}}$ denotes the Raviart-Thomas interpolation. Therefore, we can write

$$\boldsymbol{\tau} - \boldsymbol{\tau}_h \;=\; \mathrm{curl}(\chi - \chi_h) \;+\; \boldsymbol{\Phi} - \tilde{\Pi}_h^k(\boldsymbol{\Phi})\,, \tag{52}$$

which yields to

$$\mathrm{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \;=\; \mathrm{div}(\boldsymbol{\Phi} - \tilde{\Pi}_h^r(\boldsymbol{\Phi}))\,. \tag{53}$$

On the other hand, it is not difficult to see the following orthogonality relation

$$\tilde{A}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\zeta}_h, v_h)) = 0\,, \quad \forall\, (\boldsymbol{\zeta}_h, v_h) \in \boldsymbol{\Sigma}_{0,h}\,. \tag{54}$$

From now on, for each $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_0$, we introduce its induced discrete pair by $(\boldsymbol{\tau}_h, 0) \in \boldsymbol{\Sigma}_h$, where $\boldsymbol{\tau}_h$ is defined in (51). Hence, we use (54) with $(\boldsymbol{\zeta}_h, v_h) := (\boldsymbol{\tau}_h, 0)$, and the fact that $(\boldsymbol{\sigma}, u)$ is the unique solution of (47), to obtain

$$\begin{aligned} \langle(\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}), (\boldsymbol{\tau}, \mathbf{v})\rangle_{\boldsymbol{\Sigma}} \;&=\; \tilde{A}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_h, v)) \\ &=\; -\int_\Omega \boldsymbol{\sigma}_{g_N} \cdot (\boldsymbol{\tau} - \boldsymbol{\tau}_h) - \int_\Omega \tilde{f}\, v - \tilde{A}((\boldsymbol{\sigma}_h, u_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_h, v))\,. \end{aligned}$$

Equivalently,

$$\begin{aligned} (\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})_{H(\mathrm{div};\,\Omega)} \;&=\; \tilde{F}_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\,, \qquad \forall\, \boldsymbol{\tau} \in H_0\,, \\ (\bar{u}, v)_{L^2(\Omega)} \;&=\; \tilde{F}_2(v)\,, \qquad\qquad \forall\, v \in L^2(\Omega)\,, \end{aligned}$$

where $\tilde{F}_1 : H_0 \to \mathbb{R}$ and $\tilde{F}_2 : L^2(\Omega) \to \mathbb{R}$ are the bounded linear functionals defined as

$$\begin{aligned} \tilde{F}_1(\boldsymbol{\rho}) \;&:=\; -\int_\Omega (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}) \cdot \boldsymbol{\rho} + \int_\Omega u_h\, \mathrm{div}(\boldsymbol{\rho})\,, \quad \forall\, \boldsymbol{\rho} \in H_0\,, \\ \tilde{F}_2(v) \;&:=\; -\int_\Omega (\tilde{f} - \mathrm{div}(\boldsymbol{\sigma}_h))\, v\,, \quad \forall v \in L^2(\Omega)\,. \end{aligned}$$

Hence, taking into account (52) and (53), we can rewrite $\tilde{F}_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)$, as follows

$$\tilde{F}_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \;=\; \tilde{R}_1(\boldsymbol{\Phi}) \;+\; \tilde{R}_2(\chi)\,,$$

where

$$\begin{aligned} \tilde{R}_1(\boldsymbol{\Phi}) \;:=\; &-\int_\Omega (\boldsymbol{\sigma}_h + \nabla u_h + \boldsymbol{\sigma}_{g_N}) \,\cdot\, (\boldsymbol{\Phi} - \tilde{\Pi}_h^k(\boldsymbol{\Phi})) \\ &+ \sum_{T \in \mathcal{T}_h} \int_{\partial T \cap E_I} u_h\, (\boldsymbol{\Phi} - \tilde{\Pi}_h^k(\boldsymbol{\Phi})) \cdot \boldsymbol{n}\,, \end{aligned}$$

and

$$\tilde{R}_2(\chi) \,:=\, -\int_\Omega (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}) \cdot \,\mathbf{curl}(\chi - \chi_h)\,.$$

Our aim now is to obtain upper bounds for each one of the terms $\tilde{F}_2(v)$, $\tilde{R}_1(\boldsymbol{\Phi})$ and $\tilde{R}_2(\chi)$.

Lemma 15 *For any $v \in L^2(\Omega)$, there holds*

$$|\tilde{F}_2(v)| \,\leq\, \Big(\sum_{T\in\mathcal{T}_h} \|f - \mathrm{div}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\|^2_{L^2(T)} \Big)^{1/2} \|v\|_{L^2(\Omega)}\,.$$

Proof. Noting that $\tilde{f} = f - \mathrm{div}(\boldsymbol{\sigma}_{g_N})$ in Ω, the proof follows from a straightforward application of Cauchy-Schwarz inequality. □

Lemma 16 *There exists $C > 0$, independent of h, such that*

$$|R_1(\boldsymbol{\Phi})| \leq C\Big(\sum_{e\in E_I} h_e \,\|[\![u_h]\!]\|^2_{[L^2(e)]^2} + \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{\sigma}_h + \nabla u_h + \boldsymbol{\sigma}_{g_N}\|^2_{[L^2(T)]^2} \Big)^{1/2} \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,.$$

Proof. It is a consequence of Cauchy-Schwarz inequality together with approximation properties of $\tilde{\Pi}_h^k(\boldsymbol{\Phi})$. We omit further details. □

Lemma 17 *There exists $C > 0$, independent of h, such that*

$$|R_2(\chi)| \,\leq\, C\Bigg(\sum_{T\in\mathcal{T}_h} \Big\{ h_T^2 \|\mathrm{rot}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\|^2_{L^2(T)} + \sum_{e\in E(T)} h_e \|[\![\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}]\!]\|^2_{L^2(e\cap(E_I))} + \sum_{e\in E(T)} h_e \|(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\cdot \mathbf{t}\|^2_{L^2(e\cap E_D)} \Big\} \Bigg)^{1/2} \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,.$$

Proof. It follows slight modifications of the Lemma 5. We omit further details. □

The previous results induce us to define the following residual estimator

$$\bar{\eta} \,:=\, \Bigg(\sum_{T\in\mathcal{T}_h} \bar{\eta}_T^2 \Bigg)^{1/2}\,, \tag{55}$$

where

$$\bar{\eta}_T^2 \,:=\, \|f - \mathrm{div}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\|^2_{L^2(T)} \,+\, h_T^2 \|\boldsymbol{\sigma}_h + \nabla u_h + \boldsymbol{\sigma}_{g_N}\|^2_{[L^2(T)]^2} + h_T^2 \|\mathrm{rot}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\|^2_{L^2(T)} \,+\, \sum_{e\in E(T)} h_e \Big\{ \|[\![u_h]\!]\|^2_{L^2(e\cap E_I)} + \|[\![\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}]\!]\|^2_{L^2(e\cap E_I)} \,+\, \|(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\cdot \mathbf{t}\|^2_{L^2(e\cap E_D)} \Big\}$$

Remark 18 *Since $\boldsymbol{\sigma} = -\nabla u - \boldsymbol{\sigma}_g$ in Ω and $u = 0$ on Γ_D, we deduce that for any given $e \in E_D$, $(\boldsymbol{\sigma} + \boldsymbol{\sigma}_{g_N}) \cdot \mathbf{t} = -\nabla u \cdot \mathbf{t} = -\frac{d}{d\mathtt{t}}(u) = -\frac{d}{d\mathtt{t}}(0) = 0$ on e. This allows us to conclude that each term defining (55), is residual.*

Properties of $\bar{\eta}$ are collected in the next theorem.

Theorem 19 *There exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_{\boldsymbol{\Sigma}} \le C_{\mathtt{rel}}\, \bar{\eta}\,.$$

Additionally, there exists $C_{\mathtt{eff}} > 0$, independent of h, such that

$$\bar{\eta}_T^2 \le C_{\mathtt{eff}} ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)||_T\,, \tag{56}$$

with $||(\boldsymbol{\tau}, \mathbf{v})||_T^2 := \|\boldsymbol{\tau}\|^2_{H(\mathrm{div};T)} + \|\mathbf{v}\|^2_{L^2(T)}$.

Proof. The reliability of $\bar{\eta}$ follows from (49) and Lemma 6. The proof of the efficiency of $\bar{\eta}$ is similar to the developed in Section 3.3. We omit further details. □

In order to deduce an a posteriori error estimator for the main problem (41), we proceed as in [13]. To this end, given an integer $l \ge 0$, we introduce $P_l(\Gamma_{N,h}) := \{p \in L^2(\Gamma_N) : \ p|_e \in P_l(e)\,,\ \forall e \in E_N\}$ where $P_l(e)$ denotes the space of polynomials of total degree at mots l on e, and $E_N := \{e \in E_\Gamma : e \subset \Gamma_N\}$. We define the $L^2(\Gamma_N)$ orthogonal projection $\pi_l := L^2(\Gamma_N) \to P_l(\Gamma_{N,h})$ by

$$\int_{\Gamma_N} \pi_l(\xi)\, q = \int_{\Gamma_N} \xi\, q \quad \forall q \in P_l(\Gamma_{N,h})\,.$$

In what follows, we assume datum $g_N \in L^2(\Gamma_N)$ and we choose $\boldsymbol{\sigma}_{g_N} \in H^{\boldsymbol{\sigma}}_{h,r}$ such that $\boldsymbol{\sigma}_{g_N} \cdot \boldsymbol{n} = \pi_r(g_N)$ on Γ_N. Denoting by $(\boldsymbol{\sigma}, \bar{u}) \in \boldsymbol{\Sigma}_0$ the unique solution of (42) using this $\boldsymbol{\sigma}_{g_N} \in H(\mathrm{div};\Omega)$, we deduce that

$$\begin{cases} \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma} - \boldsymbol{\sigma}_{g_N} = -\nabla(u - \bar{u}) & \text{in } \Omega, \\ \mathrm{div}(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma} - \boldsymbol{\sigma}_{g_N}) = 0 & \text{in } \Omega, \\ u - \bar{u} = 0 & \text{on } \Gamma_D, \\ (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma} - \boldsymbol{\sigma}_{g_N}) \cdot \mathbf{n} = -g_N + \pi_r(g_N) & \text{on } \Gamma_N\,. \end{cases}$$

In addition, the continuous dependence of the solution, with respect to the data, implies that

$$||(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma} + \boldsymbol{\sigma}_{g_N}), u - \bar{u})||_{\boldsymbol{\Sigma}} \le C\, \|g_N - \pi_r(g_N)\|_{H^{-1/2}(\Gamma_N)}.$$

Furthermore, in [13] via a duality argument, is established that

$$\|g_N - \pi_r(g_N)\|_{H^{-1/2}(\Gamma_N)} \le C \left(\sum_{e \in E_N} h_e \|g_N - \boldsymbol{\sigma}_{g_N} \cdot \boldsymbol{n}\|^2_{L^2(e)} \right)^{1/2}. \tag{57}$$

In this way, problem (41) is provided with the following a posteriori error estimator

$$\tilde{\eta} := \Big(\sum_{T\in\mathcal{T}_h} \tilde{\eta}_T^2 \Big)^{1/2}, \tag{58}$$

where

$$\tilde{\eta}_T^2 := \bar{\eta}_T + \sum_{e\in E(T)} h_e \|g_N - \boldsymbol{\sigma}_{g_N}\cdot \boldsymbol{n}\|^2_{L^2(e\cap E_N)}\,.$$

The following theorem exhibits the main result of this section, which establishes the reliability and efficiency of the estimator $\tilde{\eta}$.

Theorem 20 *Assuming $g_N \in L^2(\Gamma_N)$, we let $(\tilde{\boldsymbol{\sigma}}, u)$ be the unique solution of problem (41). We also let $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_{0,h}$ be the solution of the discrete problem (47), with $\boldsymbol{\sigma}_{g_N} \in H^{\boldsymbol{\sigma}}_{h,r}$ such that $\boldsymbol{\sigma}_{g_N}\cdot\boldsymbol{n} = \pi_r(g_N)$ on Γ_N. Then, there exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$||(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}), u - u_h)||_{\boldsymbol{\Sigma}} \le C_{\mathtt{rel}}\,\tilde{\eta}\,.$$

Moreover, there exists $C_{\mathtt{eff}} > 0$, independent of h, such that for each $T\in\mathcal{T}_h$ with $\partial T\cap\Gamma_N = \emptyset$, we have

$$\tilde{\eta}_T^2 \le C_{\mathtt{eff}} ||(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}), u - u_h)||_T\,, \tag{59}$$

with $||(\boldsymbol{\tau}, \mathbf{v})||_T^2 := \|\boldsymbol{\tau}\|^2_{H(\mathrm{div};T)} + \|\mathbf{v}\|^2_{L^2(T)}$.

Proof. Denoting by $(\boldsymbol{\sigma}, \bar{u}) \in \boldsymbol{\Sigma}_0$ the unique solution of (42), knowing that $\boldsymbol{\sigma}_{g_N} \in H(\mathrm{div};\Omega)$, and applying triangle inequality, we deduce

$$||(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}), u - u_h)||_{\boldsymbol{\Sigma}} \le \|(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma} + \boldsymbol{\sigma}_{g_N}), u - \bar{u})\|_{\boldsymbol{\Sigma}} + \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \bar{u} - u_h)\|_{\boldsymbol{\Sigma}}\,.$$

Then, the reliability follows by bounding the first term in the right hand by (57), and the second one by Theorem 19. The efficiency of $\tilde{\eta}$ is established as for $\bar{\eta}$, in Theorem 19. We omit further details. □

Remark 21 *Under the assumptions that g_N is piecewise polynomial, and after invoking Lemma 13 in [24], we can establish that*

$$h_e\|g_N - \boldsymbol{\sigma}_{g_N}\cdot\mathbf{t}\|^2_{L^2(e)} \le c\{\|\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N})\|^2_{L^2(w_e)} + h^2\|\mathrm{div}(\tilde{\boldsymbol{\sigma}} - (\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_{g_N}))\|^2_{L^2(w_e)}\}\,,$$

which implies the local efficiency of $\tilde{\eta}_T$, for all $T\in\mathcal{T}_h$.

Remark 22 *Under the assumptions that g_N is more regular, for example $g_N \in H^1(e)$, for all $e\in E_N$, we can deduce that for each $e\in E_N$, there holds*

$$h_e\|g_N - \boldsymbol{\sigma}_{g_N}\cdot\mathbf{t}\|^2_{L^2(e)} = h_e\|g_N - \pi_r(g_N)\|^2_{L^2(e)} \le c h_e^3 \|g_N\|^2_{H^1(e)}\,.$$

This allows us to consider this residual term as high order one, which implies the local efficiency for all $T\in\mathcal{T}_h$.

5 Numerical examples

In this section, we present numerical examples illustrating the performance of the dual mixed method with Dirichlet and mixed boundary conditions, as well as of the corresponding adaptive procedure. We consider the lowest finite element $\mathcal{RT}_0(T) - \mathcal{P}_0(T)$, while the computational implementational has been done using a MATLAB toolbox.

Hereafter, N is the number of degrees of freedom (unknowns), that is, N = total number of edges + total number of elements.

In what follows, we introduce some useful notations: the individual and total errors are defined as

$$\boldsymbol{e}_0(u) := \|u - u_h\|_{L^2(\Omega)}, \quad \boldsymbol{e}(\boldsymbol{\sigma}) := (\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} + \|\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)})^{1/2},$$

$$\boldsymbol{e} := (\boldsymbol{e}_0(u)^2 + \boldsymbol{e}(\boldsymbol{\sigma})^2)^{1/2},$$

where $(\boldsymbol{\sigma}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ and $(\boldsymbol{\sigma}_h, u_h) \in H^{\boldsymbol{\sigma}}_{h,r} \times H^u_{h,r}$ are the respective unique solutions of the continuous (3) and discrete (6) formulations. Additionally, if $\boldsymbol{e}$ and $\boldsymbol{e}'$ stand for the errors at two consecutive triangulations with N and N' number of degrees of freedom, respectively, we set the experimental rate of convergence of the global error $\boldsymbol{e}$ as

$$r := -2\frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')}.$$

In analogous way, we define $r_0(u)$ and $r(\boldsymbol{\sigma})$.

We present two examples. The data f and g are chosen so that the exact solution of u and the domain are shown in Table 1. The aim of Example 1 is to exhibit the optimal behavior for a smooth solution. The Example 2 is given in polar coordinates, and has a singularity at $(0,0)$ which implies that $u \in H^{1+2/3}(\Omega)$. Then, our goal here, is to show the performance of the following adaptive algorithm: Given an estimator $\gamma := \sum_{T \in \mathcal{T}_h} \gamma_T^2$

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the Galerkin scheme for the current mesh $\mathcal{T}_h$.
3. Compute γ_T for each triangle $T \in \mathcal{T}_h$.
4. Consider stopping criterion and decide to finish or go to the next step.
5. Apply *Blue-green* procedure to refine each element $T' \in \mathcal{T}_h$ such that
$$\gamma_{T'} \geq \frac{1}{2} \max\{\gamma_T : T \in \mathcal{T}_h\}.$$
6. Define the resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

Table 1: Summary of data for the two examples.

Example	Ω	u
1	$(0,1)^2$	$(1-x)(1-y)\mathrm{e}^{-10(x^2+y^2)}$
2	$(-1,1)^2 \setminus [0,1]\times[-1,0]$	$r^{2/3}\sin(\frac{2}{3}\theta)$

5.1 Dirichlet Boundary conditions

In Tables 2 and 3 we give the individual and total errors and the corresponding experimental rates of convergence for the uniform refinements as applied to all examples considered in this paper. We observe in Table 2 that $\boldsymbol{e}_0(u)$ behaves as $\mathcal{O}(h^2)$, which should be consequence of a standard duality argument, while $\boldsymbol{e}(\boldsymbol{\sigma})$ and $\boldsymbol{e}$ behave as $\mathcal{O}(h)$, in accordance with the well-known (optimal) a-priori error estimate. Furthermore, we observe that the effectivity indices $\boldsymbol{e}/\eta$ remain bounded.

On the other hand, due to the singularity of Example 2, the rate of convergence of $\boldsymbol{e}(\boldsymbol{\sigma})$ is of order $\mathcal{O}(h^{2/3})$ for uniform refinement, as expected. The numerical results of the adaptive procedure applied to Example 2 is presented in Table 4, from which we observe that the rate of convergence of $\boldsymbol{e}(\boldsymbol{\sigma})$ improves to $\mathcal{O}(h)$.

5.2 Mixed Boundary conditions

We consider the same exact solutions given in Table 1, where the boundary of Ω is decomposed in two disjoint parts, which are $\Gamma_N := [0,1]\times\{0\}\cup\{0\}\times[0,1]$, $\Gamma_D := \partial\Omega\backslash\Gamma_N$ for Example 1, and $\Gamma_D := [0,1]\times\{0\}\cup\{0\}\times[-1,0]$, $\Gamma_N := \partial\Omega\backslash\Gamma_D$ for the L-shape domain in Example 2. The results are very similar to those obtained in the case of pure Dirichlet boundary condition, and are resumed in Tables 5, 6 and 7.

Table 2: History of convergence of Example 1, with Dirichlet boundary condition (uniform refinement).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	η	$\boldsymbol{e}/\eta$
44	3.6771e-02	–	1.3899e+00	–	1.3904e+00	–	1.6474	0.8440
168	1.0858e-02	1.8209	3.3947e-01	2.1043	3.3965e-01	2.1040	0.9544	0.3559
656	2.8510e-03	1.9633	1.2327e-01	1.4873	1.2331e-01	1.4877	0.5313	0.2321
2592	7.2231e-04	1.9985	5.1749e-02	1.2634	5.1754e-02	1.2637	0.2761	0.1875
10304	1.8120e-04	2.0040	2.4431e-02	1.0877	2.4432e-02	1.0878	0.1402	0.1742
41088	4.5338e-05	2.0033	1.2027e-02	1.0248	1.2027e-02	1.0248	0.0706	0.1704
164096	1.1337e-05	2.0020	5.9895e-03	1.0069	5.9895e-03	1.0069	0.0354	0.1692

Table 3: History of convergence of Example 2, with Dirichlet boundary condition (uniform refinement).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	η	$\boldsymbol{e}/\eta$
34	1.0743e-01	–	1.1604e+00	–	1.1653e+00	–	2.5728	0.4530
128	7.0151e-02	0.6430	6.6600e-01	0.8377	6.6968e-01	0.8358	1.8306	0.3658
496	3.5523e-02	1.0047	3.6812e-01	0.8754	3.6983e-01	0.8767	1.2531	0.2951
1952	1.6177e-02	1.1483	1.9830e-01	0.9031	1.9897e-01	0.9050	0.7944	0.2505
7744	6.9776e-03	1.2204	1.0769e-01	0.8861	1.0791e-01	0.8879	0.4968	0.2172
30848	2.9195e-03	1.2608	5.9743e-02	0.8526	5.9814e-02	0.8539	0.3102	0.1928
123136	1.1993e-03	1.2854	3.4003e-02	0.8143	3.4024e-02	0.8152	0.1940	0.1754

Table 4: History of convergence of Example 2, with Dirichlet boundary condition (adaptive refinement).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	η	$\boldsymbol{e}/\eta$
34	1.0743e-01	–	1.1604e+00	–	1.1654e+00	–	2.5728	0.4530
128	7.0151e-02	0.6430	6.6600e-01	0.8377	6.6968e-01	0.8358	1.8306	0.3658
468	3.5667e-02	1.0436	3.8290e-01	0.8540	3.8455e-01	0.8558	1.2726	0.3022
799	1.6442e-02	2.8954	3.0453e-01	0.8563	3.0497e-01	0.8667	1.0581	0.2882
1826	7.4712e-03	1.9085	2.0925e-01	0.9079	2.0939e-01	0.9099	0.7605	0.2753
2544	3.1312e-03	5.2454	1.7140e-01	1.2037	1.7142e-01	1.2066	0.6477	0.2647
3987	1.3905e-03	3.6136	1.4070e-01	0.8784	1.4071e-01	0.8790	0.5392	0.2609
7651	7.2582e-04	1.9947	1.0258e-01	0.9696	1.0258e-01	0.9697	0.3949	0.2597
10749	3.3356e-04	4.5737	8.2378e-02	1.2901	8.2379e-02	1.2902	0.3300	0.2497
16546	2.0437e-04	2.2715	6.8730e-02	0.8398	6.8731e-02	0.8399	0.2725	0.2522
29744	1.4573e-04	1.1533	5.2045e-02	0.9483	5.2045e-02	0.9483	0.2060	0.2526
43259	7.2708e-05	3.7125	4.1051e-02	1.2670	4.1051e-02	1.2670	0.1683	0.2439
67863	4.7552e-05	1.8861	3.3655e-02	0.8823	3.3655e-02	0.8823	0.1368	0.2461
117979	3.6236e-05	0.9828	2.6197e-02	0.9060	2.6197e-02	0.9060	0.1046	0.2504
173452	1.8459e-05	3.5003	2.0581e-02	1.2522	2.0581e-02	1.2522	0.0851	0.2418
270853	1.1989e-05	1.9369	1.6773e-02	0.9182	1.6773e-02	0.9182	0.0691	0.2429

Table 5: History of convergence of Example 1, with mixed boundary condition (uniform refinement).

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	η	$\boldsymbol{e}/\eta$
44	4.0570e-02	–	6.0993e-01	–	6.1128e-01	–	1.4704	0.4157
168	6.6826e-03	2.6923	2.7431e-01	1.1929	2.7439e-01	1.1957	0.9549	0.2873
656	1.7529e-03	1.9648	1.3181e-01	1.0760	1.3182e-01	1.0763	0.5310	0.2483
2592	4.4756e-04	1.9872	6.5365e-02	1.0209	6.5366e-02	1.0210	0.2756	0.2372
10304	1.1249e-04	2.0012	3.2617e-02	1.0074	3.2617e-02	1.0074	0.1400	0.2329
41088	2.8161e-05	2.0026	1.6300e-02	1.0030	1.6300e-02	1.0030	0.0705	0.2311
164096	7.0427e-06	2.0018	8.1490e-03	1.0013	8.1490e-03	1.0013	0.0354	0.2302

Table 6: History of convergence of Example 2, with mixed boundary condition (uniform refinement).

N	$\mathbf{e}_0(u)$	$r_0(u)$	$\mathbf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathbf{e}$	r	η	$\mathbf{e}/\eta$
34	2.0335e-01	–	1.444e+00	–	1.4583e+00	–	3.9026	0.3737
128	7.2445e-02	1.5571	7.1580e-01	1.0588	7.1945e-01	1.0659	2.6433	0.2722
496	2.6082e-02	1.5084	3.5873e-01	1.0200	3.5968e-01	1.0236	1.6650	0.2160
1952	9.6760e-03	1.4476	1.8257e-01	0.9861	1.8282e-01	0.9879	1.0367	0.1764
7744	3.6888e-03	1.3996	9.4502e-02	0.9557	9.4574e-02	0.9566	0.6461	0.1464
30848	1.4331e-03	1.3681	4.9876e-02	0.9248	4.9896e-02	0.9253	0.4038	0.1236
123136	5.6315e-04	1.3496	2.6929e-02	0.8906	2.6935e-02	0.8908	0.2531	0.1064

Table 7: History of convergence of Example 2, with mixed boundary condition (adaptive refinement).

N	$\mathbf{e}_0(u)$	$r_0(u)$	$\mathbf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathbf{e}$	r	η	$\mathbf{e}/\eta$
34	2.0335e-01	–	1.4440e+00	–	1.4583e+00	–	3.9026	0.3737
128	7.2445e-02	1.5571	7.1580e-01	1.0588	7.1945e-01	1.0659	2.6433	0.2722
295	2.4981e-02	2.5503	5.1533e-01	0.7871	5.1593e-01	0.7965	1.8938	0.2724
540	1.1705e-02	2.5078	3.9560e-01	0.8747	3.9577e-01	0.8771	1.4371	0.2754
850	4.5026e-03	4.2118	3.1063e-01	1.0660	3.1067e-01	1.0674	1.1321	0.2744
1327	2.6913e-03	2.3105	2.5595e-01	0.8693	2.5596e-01	0.8696	0.9229	0.2773
2207	1.8016e-03	1.5781	1.8868e-01	1.1987	1.8869e-01	1.1988	0.7189	0.2625
3446	9.0777e-04	3.0765	1.5561e-01	0.8652	1.5561e-01	0.8653	0.5751	0.2706
5141	6.7315e-04	1.4950	1.3059e-01	0.8761	1.3060e-01	0.8761	0.4785	0.2729
9418	3.5101e-04	2.1512	9.1248e-02	1.1844	9.1249e-02	1.1844	0.3553	0.2568
13668	2.2523e-04	2.3827	7.7280e-02	0.8922	7.7280e-02	0.8923	0.2946	0.2623
20764	1.6689e-04	1.4340	6.5280e-02	0.8071	6.5280e-02	0.8071	0.2443	0.2673
37538	8.8941e-05	2.1257	4.5284e-02	1.2353	4.5284e-02	1.2353	0.1816	0.2493
55955	5.4029e-05	2.4973	3.8146e-02	0.8595	3.8146e-02	0.8595	0.1481	0.2575
83406	4.1762e-05	1.2903	3.2288e-02	0.8354	3.2288e-02	0.8354	0.1234	0.2616
150490	2.2442e-05	2.1047	2.2520e-02	1.2210	2.2520e-02	1.2210	0.0919	0.2451

Conclusions

Our contribution in this paper points in the direction of the results provided in [10] (for augmented mixed method), by extending/relaxing the a posteriori error analysis towards conforming dual mixed finite element method of Poisson equation, with Dirichlet and mixed boundary conditions. In other words, considering the approach based on the Ritz projection of the error, we have derived a posteriori error estimators of residual type, which are reliable and locally efficient, in the framework of the natural norm, circumventing the saturation assumption and including non-homogeneous Dirichlet and mixed boundary conditions.

Specifically, for Dirichlet boundary conditions the reliablity and locally efficiency of the a posteriori error estimator we deduced, do not require any additional regularity of the Dirichlet data, i.e., is enough that it belongs to $H^{1/2}(\Gamma)$. For mixed boundary condition, we have applied a type of homogenization technique, following the ideas described in [13] for the treatment of the Neumann boundary data. This approach differs from [24], since the authors in this work, weakly imposed the Neumann boundary condition via the introduction of a new Lagrange multiplier. Then, the method described can be seen as an alternative procedure to the one given in [24], in order to approximate the solution of the model problem. Furthermore, the a posteriori error estimator introduced for our approach, is reliable and locally efficient, without the additional regularity required in [24]. Indeed, in this work we only need that the Neumann data lives in $L^2(\Gamma_N)$, whereas in [24], the authors need this assumption also, and that the Lagrange multiplier belongs to $H^1(E_N)$. Numerical examples we have shown in this paper, are in agreement with the theoretical results we have deduced.

Acknowledgements

This research was partially supported by ANID through FONDECYT grant No. 1160578; by Direcciones de Investigación y de Postgrado de la Universidad Católica de la Santísima Concepción (Chile), through Incentivo Mensual and Becas de Mantención programs; by CONICYT / PIA / Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal AFB170001, by Vicerrectoría de Investigación y Desarrollo de la Universidad de Concepción (Chile) through project VRID-Enlace No. 218.013.044-1.0, and by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción (Chile).

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