

A priori and *a posteriori* error analysis of an unfitted HDG method for semi-linear elliptic problems

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Abstract

We present *a priori* and *a posteriori* error analysis of a high order hybridizable discontinuous Galerkin (HDG) method applied to a semi-linear elliptic problem posed on a piecewise curved, non polygonal domain. We approximate Ω by a polygonal subdomain Ω_h and propose an HDG discretization, which is shown to be optimal under mild assumptions related to the non-linear source term and the distance between the boundaries of the polygonal subdomain Ω_h and the true domain Ω . Moreover, a local non-linear post-processing of the scalar unknown is proposed and shown to provide an additional order of convergence. A reliable and locally efficient *a posteriori* error estimator that takes into account the error in the approximation of the boundary data of Ω_h is also provided.

Key words: Hybridizable discontinuous Galerkin (HDG), curved boundary, semi-linear elliptic equations, *a posteriori* error estimates.

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1 Introduction

In this paper, we carry out *a priori* and *a posteriori* error analyses of a hybridizable discontinuous Galerkin (HDG) method [1] applied to semi-linear elliptic problems of the form

$$-\nabla \cdot (\kappa \nabla u) = \mathcal{F}(u) \quad \text{in } \Omega, \quad (1.1a)$$

$$u = g \quad \text{on } \Gamma := \partial\Omega, \quad (1.1b)$$

where the domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is not necessarily polygonal/polyhedral, κ is a positive function in Ω , \mathcal{F} is a source term that depends on the solution u and g is the Dirichlet boundary data on Γ . To avoid the trivial solution, we will assume that if the boundary conditions are homogeneous, the source term will not vanish for $u = 0$.

The authors's original motivation to study this type of problems comes from an application to plasma physics, where the magnetic equilibrium in axisymmetric fusion reactors can be described in terms of the solution of an equation of this type, known in the literature as the *Grad-Shafranov* equation [2, 3]. Due to the symmetry of the device, the equation is posed in a two-dimensional domain,

corresponding to a cross section of the reactor at a constant toroidal angle. The plasma confinement region is the domain enclosed by the zero level set of the solution, which is a piecewise smooth curve that in theoretical studies is often considered given and does not contain the vertical axis [4]. In fact, the Grad-Shafranov equation is nothing but (1.1) where $g = 0$, $\kappa(x, y) = 1/x$ and the source term is a case-dependent function related to the current density in the toroidal direction and the pressure profile in the plasma. Note that in plasma applications, the domain Ω does not include the y axis.

In the present study, the source term \mathcal{F} will be assumed to be Lipschitz-continuous in Ω , i.e, there exists $L_\Omega > 0$ such that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_\Omega \leq L_\Omega \|u_1 - u_2\|_\Omega \quad \forall u_1, u_2 \in L^2(\Omega). \quad (1.2)$$

In addition, we assume that there exist positive constants $\underline{\kappa}$ and $\bar{\kappa}$ such that

$$\underline{\kappa} \leq \kappa(\mathbf{x}) \leq \bar{\kappa} \quad \forall \mathbf{x} \in \Omega.$$

An HDG discretization requires us to formulate the problem in mixed form through the introduction of the flux $\mathbf{q} := -\kappa \nabla u$ as an additional unknown. This choice makes it possible to write (1.1) as the equivalent first order system

$$\mathbf{q} + \kappa \nabla u = 0 \quad \text{in } \Omega, \quad (1.3a)$$

$$\nabla \cdot \mathbf{q} = \mathcal{F}(u) \quad \text{in } \Omega, \quad (1.3b)$$

$$u = g \quad \text{on } \partial\Omega. \quad (1.3c)$$

HDG schemes, as many other discretization methods, are based on a triangulation of the domain. In our case, Ω has a piecewise curved boundary which complicates the use high order methods, since the boundary must be properly interpolated by ‘‘curved’’ triangles or tetrahedra in order to preserve high order convergence. An alternative is to approximate Ω by a polygonal/polyhedral subdomain $\Omega_h \subset \Omega$, that can be easily discretized by a uniform triangulation of size $h > 0$. Then, the system (1.3) can be restricted to Ω_h :

$$\mathbf{q} + \kappa \nabla u = 0 \quad \text{in } \Omega_h, \quad (1.4a)$$

$$\nabla \cdot \mathbf{q} = \mathcal{F}(u) \quad \text{in } \Omega_h, \quad (1.4b)$$

$$u = \varphi \quad \text{on } \Gamma_h := \partial\Omega_h, \quad (1.4c)$$

where the unknown φ is the Dirichlet data on the computational boundary Γ_h . A clever way to determine φ was proposed for one dimension in [5] and then extended to higher dimensions by [6]. The method consists of using the definition of the flux to transfer the Dirichlet data from Γ to Γ_h along segments called *transferring paths*. In fact, given $\mathbf{x} \in \Gamma_h$ and $\bar{\mathbf{x}} \in \Gamma$, one can integrate (1.3a) along a segment of length $l(\mathbf{x})$ with unit tangent vector $\mathbf{t}(\mathbf{x})$ connecting them to obtain the following representation for φ :

$$\varphi(\mathbf{x}) = g(\bar{\mathbf{x}}) + \int_0^{l(\mathbf{x})} (\kappa^{-1} \mathbf{q})(\mathbf{x} + \mathbf{t}(\mathbf{x})s) \cdot \mathbf{t}(\mathbf{x}) ds. \quad (1.5)$$

Above, we have considered that $u(\bar{\mathbf{x}}) = g(\bar{\mathbf{x}})$. At the end of Section 2.1 we will describe a way to pick $\bar{\mathbf{x}}$ in such a way that the transfer will preserve the order of approximation of the underlying discretization. Notice that the assumption (1.2) implies that \mathcal{F} is also Lipschitz continuous in Ω_h with constant $L \leq L_\Omega$; this observation will be useful in the analysis to follow.

In previous works the authors had applied this transfer technique in combination with an iterative HDG discretization to deal with the nonlinear system (1.4) arising from the Grad-Shafranov equation [4] and explored an h-adaptive HDG scheme for the solution of the problem [7]. The adaptive

strategy was powered by a residual-based error estimator first proposed by Cockburn and Zhang [8], albeit for polygonal domains—therefore not requiring the transfer of the boundary data—and linear problems. The goal of this work is to provide a rigorous justification for the numerical results obtained previously by the authors when applying these techniques for semi-linear problems in curved geometries. The present communication is mainly theoretical and we refer the reader interested on numerical experiments to [4, 7] where plenty of experiments are provided within the context of plasma equilibrium. The results presented here, however, are not limited to plasma applications and remain valid for general semi-linear elliptic equations .

2 The discrete scheme

2.1 Basic Notation.

The Computational Domain. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of simplicial triangulations of Ω_h that will be assumed to be shape-regular. i.e., there exists $\beta > 0$ such that for all elements $T \in \mathcal{T}_h$ and all $h > 0$, $h_T/\rho_T \leq \beta$, where h_T is the diameter of T and ρ_T is the diameter of the largest ball contained in T . For every element T , we will denote by \mathbf{n}_T the outward unit normal vector to T , writing \mathbf{n} instead of \mathbf{n}_T when there is no confusion. We will follow the standard convention and denote $h := \max_{T \in \mathcal{T}_h} h_T$. For the sake of simplicity we assume $h < 1$.

We will denote by e any face of a simplex and will call it an *interior face* if there are two elements T^+ and T^- in \mathcal{T}_h such that $e = \partial T^+ \cap \partial T^-$. The set of all such faces will be denoted by \mathcal{E}_h° . Also, we say that a face is a *boundary face* if there is an element $T \in \mathcal{T}_h$ such that $e = \partial T \cap \Gamma_h$ and will denote the set of boundary faces by \mathcal{E}_h^∂ . The entirety of the faces of the triangulation, \mathcal{E}_h , can then be decomposed as $\mathcal{E}_h = \mathcal{E}_h^\circ \cup \mathcal{E}_h^\partial$. The length of a face e will be denoted by h_e .

The jump of a scalar-valued function across interior faces will be denoted by $[[w]] := w^+ - w^-$. At the boundary faces we set $[[w]] := w - \varphi_h$, where φ_h is the approximation of the boundary data at Γ_h that will be defined later in (2.2f). For a vector-valued function \mathbf{v} , its jump across interior faces will be denoted by $[[\mathbf{v}]] := \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-$.

Spaces and norms. We utilize standard terminology for Sobolev spaces and norms, where vector-valued functions and their corresponding spaces are denoted in bold face. In particular, if \mathcal{O} is a domain in \mathbb{R}^d , Σ is an open or closed Lipschitz curve ($d = 2$) or surface ($d = 3$), and $s \in \mathbb{R}$, we define $\mathbf{H}^s(\mathcal{O}) := [\mathbf{H}^s(\mathcal{O})]^d$ and $\mathbf{H}^s(\Sigma) := [\mathbf{H}^s(\Sigma)]^d$. However, when $s = 0$ we write $\mathbf{L}^2(\mathcal{O})$ and $\mathbf{L}^2(\Sigma)$ instead of $\mathbf{H}^0(\mathcal{O})$ and $\mathbf{H}^0(\Sigma)$, respectively. The associated norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\Sigma}$, writing simply $\|\cdot\|_{\mathcal{O}}$ and $\|\cdot\|_{\Sigma}$ when $s = 0$, and the corresponding L^2 inner products will be denoted by $(\cdot, \cdot)_{\mathcal{O}}$ and $\langle \cdot, \cdot \rangle_{\partial\mathcal{O}}$. For $s \geq 0$, we write $|\cdot|_{s,\mathcal{O}}$ for the \mathbf{H}^s -semi norm and \mathbf{H}^s -semi norm.

The mesh-dependent inner products for a triangulation \mathcal{T}_h are given by

$$(\cdot, \cdot)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} (\cdot, \cdot)_T, \quad \langle \cdot, \cdot \rangle_{\partial\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial T} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\Gamma_h} := \sum_{e \in \mathcal{E}_h^\partial} \langle \cdot, \cdot \rangle_e,$$

and their corresponding norms will be denoted, respectively, by

$$\|\cdot\|_{\Omega_h} := \left(\sum_{T \in \mathcal{T}_h} \|\cdot\|_T^2 \right)^{1/2}, \quad \|\cdot\|_{\partial\mathcal{T}_h} := \left(\sum_{T \in \mathcal{T}_h} \|\cdot\|_{\partial T}^2 \right)^{1/2} \quad \text{and} \quad \|\cdot\|_{\Gamma_h} := \left(\sum_{e \in \mathcal{E}_h^\partial} \|\cdot\|_e^2 \right)^{1/2}.$$

To avoid proliferation of unimportant constants, we will use the terminology $a \lesssim b$ whenever $a \leq Cb$ and C is a positive constant independent of h .

The extended domain. Given a triangulation \mathcal{T}_h and a boundary face $e \in \mathcal{E}_h^\partial$ we will denote by T^e the unique element of \mathcal{T}_h having e as a face. To a point $\mathbf{x} \in e$, we associate a point $\bar{\mathbf{x}} \in \Gamma_h$ and set $l(\mathbf{x}) = |\mathbf{x} - \bar{\mathbf{x}}|$. If we let $\mathbf{t} = \mathbf{t}(\mathbf{x})$ be a normalized vector in the direction connecting \mathbf{x} to $\bar{\mathbf{x}}$ then we can parameterize the line segment between them by

$$\sigma_{\mathbf{t}}(\mathbf{x}) := \{\mathbf{x} + s\mathbf{t}, s \in [0, l(\mathbf{x})]\},$$

and define the *extension patch* as

$$T_{ext}^e := \{\mathbf{x} + s\mathbf{t} : 0 \leq s \leq l(\mathbf{x}), \mathbf{x} \in e\}.$$

In principle, $\bar{\mathbf{x}}$ can be specified in several ways. For instance, it can be a point that minimizes the distance between \mathbf{x} and Γ_h . However, in that case $\bar{\mathbf{x}}$ might be not unique and also the union of all such extension patches T_{ext}^e may not cover the set $\Omega_h^c := \Omega \setminus \bar{\Omega}_h$ entirely if Ω is not convex. A second possibility is to set $\bar{\mathbf{x}}$ to be the closest intersection between Γ and the ray starting at \mathbf{x} having tangent vector \mathbf{n} , the normal to the face e where \mathbf{x} belongs. In that case, $\mathbf{t} = \mathbf{n}$ and T_{ext}^e may not cover Ω_h^c . Moreover, $l(\mathbf{x})$ could be extremely large compared to the mesh size. To define the numerical method, we consider the algorithm proposed by [6] that constructs $\bar{\mathbf{x}}$ in such a way that three conditions are satisfied: $\bar{\mathbf{x}}$ is unique, two different line segments $\sigma_{\mathbf{t}}$ in T_{ext}^e do not intersect each other inside T_{ext}^e and the line $\sigma_{\mathbf{t}}$ does not intersect the interior of Ω_h . In this case, the union of T_{ext}^e completely covers Ω_h^c .

The extension operator. Now that an extension patch T_{ext}^e has been defined so that for every $T_{ext}^e \in \Omega \setminus \bar{\Omega}_h$ there corresponds a single $T^e \in \mathcal{T}_h$, we can define a way to extend polynomial functions defined only in the computational domain. This will be needed when transferring the boundary condition to the computational domain Γ_h .

Let $p : T^e \rightarrow \mathbb{R}$ be a polynomial function. We will define its extension to T_{ext}^e as

$$\mathbf{E}_h(p)(\mathbf{y}) := p|_{T^e}(\mathbf{y}) \quad \forall \mathbf{y} \in T_{ext}^e. \quad (2.1)$$

We will keep notation simple and a polynomial function p should be understood as its extrapolation $\mathbf{E}_h(p)$ whenever an evaluation outside of Ω_h is required. This should be clear from the context. For vector-valued polynomial functions, the extension is defined similarly component by component.

2.2 The HDG method

We consider the finite dimensional spaces of piecewise polynomials

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v} \in \mathbf{L}^2(\mathcal{T}_h) : \mathbf{v}|_T \in [\mathbb{P}_k(T)]^2, \forall T \in \mathcal{T}_h\}, \\ W_h &:= \{w \in L^2(\mathcal{T}_h) : w|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}_h\}, \\ M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_T \in \mathbb{P}_k(e), \forall e \in \mathcal{E}_h\}, \end{aligned}$$

where, $\mathbb{P}_k(T)$ denotes the space of polynomials of degree at most k defined in $T \in \mathcal{T}_h$ and the space $\mathbb{P}_k(e)$, for faces $e \in \mathcal{E}_h$, is similarly defined. The HDG scheme associated to (1.3) reads: Find

$(\mathbf{q}_h, u_h, \hat{u}_h) \in \mathbf{V}_h \times W_h \times M_h$, such that

$$(\kappa^{-1} \mathbf{q}_h, \mathbf{v})_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{v})_{\mathcal{T}_h} + \langle \hat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0, \quad (2.2a)$$

$$-(\mathbf{q}_h, \nabla w)_{\mathcal{T}_h} + \langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (\mathcal{F}(u_h), w)_{\mathcal{T}_h}, \quad (2.2b)$$

$$\langle \hat{u}_h, \mu \rangle_{\Gamma_h} = \langle \varphi_h, \mu \rangle_{\Gamma_h}, \quad (2.2c)$$

$$\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = 0, \quad (2.2d)$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$. Here

$$\hat{\mathbf{q}}_h \cdot \mathbf{n} := \mathbf{q}_h \cdot \mathbf{n} + \tau(u_h - \hat{u}_h) \quad \text{on} \quad \partial \mathcal{T}_h, \quad (2.2e)$$

with τ being a positive stabilization function, whose maximum will be denoted by $\bar{\tau}$, and the approximate boundary condition, motivated by (1.5), is given by

$$\varphi_h(\mathbf{x}) := g(\bar{\mathbf{x}}) + \int_0^{l(\mathbf{x})} (\kappa^{-1} \mathbf{q}_h)(\mathbf{x} + \mathbf{t}(\mathbf{x})s) \cdot \mathbf{t}(\mathbf{x}) ds, \quad \text{for} \quad \mathbf{x} \in \Gamma_h. \quad (2.2f)$$

Note that the function κ is defined by (1.3) in the full domain Ω , while the flux \mathbf{q}_h is extended from Ω_h to Ω as defined in (2.1).

2.3 Local post processing of the scalar solution

Our a posteriori error estimator will be obtained in terms of a local post processing u_h^* , which approximates the scalar unknown u with enhanced accuracy. We seek for u_h^* in the space

$$W_h^* := \{w \in L^2(\mathcal{T}_h) : w|_T \in \mathbb{P}_{k+1}(T), \forall T \in \mathcal{T}_h\},$$

such that, in each element $T \in \mathcal{T}_h$, satisfies:

$$(\kappa \nabla u_h^*, \nabla w)_T + (\mathcal{F}(u_h^*), w)_T = -(\mathbf{q}_h, \nabla w)_T + (\mathcal{F}(u_h), w)_T \quad \forall w \in \mathbb{P}_{k+1}(T), \quad (2.3a)$$

$$(u_h^*, w)_T = (u_h, w)_T \quad \forall w \in \mathbb{P}_0(T). \quad (2.3b)$$

In the case where \mathcal{F} is independent of u , it is well known (Section 5.2 in [9]) that u_h^* is well defined and converges to u with order h^{k+2} when the solution has enough regularity. It is also known that there is a variety of choices to construct u_h^* . In fact, we could consider a simpler choice and use $(\kappa \nabla u_h^*, \nabla w)_T = -(\mathbf{q}_h, \nabla w)_T$ instead of (2.3a). However, as we will see in Section 5, the term involving \mathcal{F} plays a key role in deriving the error estimator.

Consider real numbers $l_u, l_q \in [0, k]$ and assume that $u \in H^{l_u+2}(\mathcal{T}_h)$ and $\mathbf{q} \in H^{l_q+1}(\mathcal{T}_h)$. We can state the following result on the well posedness and convergence rate of the post-processing. The proof of this statement makes use of some results derived in the forthcoming *a priori* error analysis and will be postponed to Appendix B.

Lemma 1. *The local post processing u_h^* is well defined for L small enough. Moreover, if $Lh^2 < 1$ and $k \geq 1$, then*

$$\|u - u_h^*\|_{\Omega_h} \lesssim (Rh)^{1/2} (h^{l_u+1} |u|_{l_u+2, \Omega_h} + h^{l_q+1} |\mathbf{q}|_{l_q+2, \Omega_h}) + h^{l_u+2} |u|_{l_u+2, \Omega_h} + Lh^{l_u+1} |u|_{l_u+2, \Omega_h}, \quad (2.4a)$$

$$|u - u_h^*|_{1,T} \lesssim h_T^{l_u+1} |u|_{l_u+1, T} + Lh_T \|\varepsilon^u\|_{0,T} + \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + Lh_T \|u - u_h\|_{0,T} \quad (2.4b)$$

and

$$\sum_{e \in \mathcal{E}_h^\partial} h_e^{1/2} \|\llbracket u_h^* \rrbracket\|_e \lesssim \|u - u_h^*\|_{\Omega_h}^{1/2} \left(\|u - u_h^*\|_{\Omega_h}^2 + h^2 |u - u_h^*|_{1, \Omega_h}^2 \right)^{1/4}. \quad (2.4c)$$

Here, R —which will be defined properly in the following section—is proportional to the product $h^{-1} \text{dist}(\Gamma_h, \Gamma)$. When $\text{dist}(\Gamma_h, \Gamma)$ is of order h , then R is of order one. This result guaranties a superconvergence of h^{k+2} if $L < h$ and R is of order h . If R is of order one, it only ensures a convergence of order $h^{k+3/2}$. However, for the linear case, [6, 10] reported numerical experiments suggesting that the order is indeed h^{k+2} even when R is of order one.

3 Well-posedness

In this section we employ a Banach fixed-point argument to ensure the well-posedness of (2.2). To that end, we define the operator $\mathcal{J} : W_h \rightarrow W_h$ that maps ζ to the second component of the triplet $(\mathbf{q}, u, \hat{u}) \in \mathbf{V}_h \times W_h \times M_h$ satisfying the linearized HDG system (2.2) where the source has been evaluated at ζ , namely

$$(\kappa^{-1} \mathbf{q}, \mathbf{v})_{\mathcal{T}_h} - (u, \nabla \cdot \mathbf{v})_{\mathcal{T}_h} + \langle \hat{u}, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0, \quad (3.1a)$$

$$-(\mathbf{q}, \nabla w)_{\mathcal{T}_h} + \langle \hat{\mathbf{q}} \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (\mathcal{F}(\zeta), w)_{\mathcal{T}_h}, \quad (3.1b)$$

$$\langle \hat{u}, \mu \rangle_{\Gamma_h} = \langle \varphi_{\mathbf{q}}, \mu \rangle_{\Gamma_h}, \quad (3.1c)$$

$$\langle \hat{\mathbf{q}} \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = 0, \quad (3.1d)$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where $\hat{\mathbf{q}} \cdot \mathbf{n} := \mathbf{q} \cdot \mathbf{n} + \tau(u - \hat{u})$ and

$$\varphi_{\mathbf{q}}(\mathbf{x}) := g(\bar{\mathbf{x}}) + \int_0^{l(\mathbf{x})} (\kappa^{-1} \mathbf{q})(\mathbf{x} + \mathbf{t}(\mathbf{x})s) \cdot \mathbf{t}(\mathbf{x}) ds, \quad (3.1e)$$

for $\mathbf{x} \in \Gamma_h$. We stress the difference between the non-linear mapping \mathcal{J} , which maps *arguments* of the source $\mathcal{F}(\cdot)$ to solutions of the corresponding HDG system, and the linear solution operator $\mathcal{S} : W_h \rightarrow W_h$ that maps the source term \mathcal{F} itself to the solution of the corresponding HDG system.

As we can expect, assumptions on the line segments σ_t and distance between Γ_h and Γ will be needed to ensure the well-posedness of (3.1) and the contraction property of the fixed-point operator. In order to specify these assumptions we need to define some additional quantities. Given a boundary face $e \in \mathcal{E}_h^\partial$, its corresponding element T^e , and the extension patch T_{ext}^e , we denote by h_e^\perp (resp. H_e^\perp) the largest distance of a point inside T_e (resp. T_{ext}^e) to the the plane determined by the face e , set $r_e := H_e^\perp / h_e^\perp$ and $R := \max_{e \in \mathcal{E}_h^\partial} r_e$.

For an extended patch T_{ext}^e and element $T^e \in \mathcal{T}_h$ sharing a face e , we define

$$\mathcal{V}^k := \left\{ \mathbf{p} \in [\mathbb{P}_k(T_{ext}^e \cup T^e)]^2, \mathbf{p} \neq \mathbf{0} \right\},$$

and we denote by \mathbf{n}_e the *interior* normal vector to T_{ext}^e along the face e —i.e. the exterior normal vector to T^e pointing in the direction of T_{ext}^e . We can then introduce the constants

$$C_{ext}^e := \frac{1}{\sqrt{r_e}} \sup_{\chi \in \mathcal{V}^k} \frac{\|\chi \cdot \mathbf{n}_e\|_{T_{ext}^e}}{\|\chi \cdot \mathbf{n}_e\|_{T^e}} \quad \text{and} \quad C_{inv}^e := h_e^\perp \sup_{\chi \in \mathcal{V}^k} \frac{\|\nabla \chi \cdot \mathbf{n}_e\|_{T^e}}{\|\chi \cdot \mathbf{n}_e\|_{T^e}},$$

where, in abuse of notation, \mathbf{n}_e is a constant vector field in T_{ext}^e that coincides with the normal vector (pointing outwards T^e) associated to the face e . In [10] it was shown that the constants C_{ext}^e and C_{inv}^e are independent of h , but depend on the polynomial degree; in particular, the supremum appearing in the definition of C_{inv}^e is proportional to $(h_e^\perp)^{-1}$.

With these definitions in place, we can now state the following set of geometric assumptions on the boundary faces of the triangulation.

Assumptions. For each $e \in \mathcal{E}_h^\partial$ we will require the following to hold:

$$\mathbf{t}(\mathbf{x}) = \mathbf{n} \text{ for all } \mathbf{x} \in e, \quad (3.2a) \quad r_e \leq C, \quad (3.2c)$$

$$\bar{\tau} H_e^\perp \underline{\kappa}^{-1} \leq \frac{1}{3}, \quad (3.2b) \quad \bar{\kappa} \underline{\kappa}^{-1} r_e^3 (C_{ext}^e C_{inv}^e)^2 \leq 1. \quad (3.2d)$$

Before proceeding, let us comment on this set of assumptions. As mentioned at the end of Section 2.1, the vector $\mathbf{t}(\mathbf{x})$ does not necessarily have to be normal to the face e . Therefore, the results presented in what follows still hold if (3.2a) is not satisfied as long as the difference $1 - \mathbf{t}(\mathbf{x}) \cdot \mathbf{n}$ is positive and small enough. However, this assumption helps us to facilitate the presentation of the ideas behind the proofs. On the other hand, (3.2b) imposes the geometric constraint that the family of triangulations $\{\mathcal{T}_h\}$ should be such that the distance between the computational boundary Γ_h and the true boundary Γ remains locally of the same order of magnitude as the face mesh parameter h_e . Moreover, it guarantees that as long as $H_e^\perp > 0$, then $H_e^\perp \sim h_e^\perp$; if $H_e^\perp = 0$ for some e , then no transfer of boundary conditions is needed on that particular face—as this would only happen if $e \cap \Omega = e$. Given that the stabilization parameter τ is of order one, (3.2c) states that the minimum value of the diffusion coefficient κ imposes a restriction on how far apart Γ_h and Γ are allowed to be. Due to the proportionality guaranteed by (3.2b), then (3.2c) will hold whenever the mesh size—and therefore the distance between the boundaries—is small enough. Assumption (3.2d) is the most demanding of all. By requiring r_e to be small enough compared to the product $\bar{\kappa} \underline{\kappa}^{-1}$, the condition effectively limits the range of values of κ that the method is able to resolve for a given, fixed, distance between the computational and physical boundaries, as measured by H_e^\perp . Not surprisingly, the closer to zero the diffusion coefficient gets, the smaller H_e^\perp must be with respect to the mesh size near the computational boundary.

The main result of this section, Theorem 1, is that under suitable assumptions \mathcal{J} is a contraction and therefore the solution of (2.2) can be obtained by applying it iteratively. The proof of this fact is almost a straightforward consequence of the linearity of the solution map and of a key stability bound established in Lemma 2 that estimates the norm of u in terms of those of the sources and the boundary conditions. However, the proof of the latter follows from a lengthy series of estimates. In order to prioritize clarity of exposition, we first present this estimate without proving it and show how the main result follows from it. The technical details of the proof of Lemma 2 are then presented afterwards.

Lemma 2. *Suppose that Assumptions (3.2a) throughout (3.2d) and the elliptic regularity of the auxiliary problem (3.9) are satisfied. Then, there exists $\tilde{c} > 0$, independent of h such that*

$$\|u\|_{\Omega_h} \leq 4 \max\{\tilde{c}^2 h, 1\} \|\mathcal{F}(\zeta)\|_{\Omega_h} + 2\tilde{c}(\sqrt{3} + 1) h^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{g}\|_{\Gamma_h}, \quad (3.3)$$

where $\bar{g}(\mathbf{x}) = g(\bar{\mathbf{x}}(\mathbf{x})) \forall \mathbf{x} \in \Gamma_h$.

Thanks to this estimate the main result, from which well-posedness of the problem follows, can be proved in a very compact way, as we now demonstrate.

Theorem 1. *If Assumptions (3.2a) throughout (3.2d) and the elliptic regularity of the auxiliary problem (3.9) hold, then \mathcal{J} is well-defined. Furthermore, if we assume $4L \max\{\tilde{c}^2 h, 1\} < 1$, then \mathcal{J} is a contraction operator.*

Proof. The system in (3.1) is linear and has a unique solution under Assumption S (see [10]), therefore the operator \mathcal{J} is well-defined.

Let $\zeta_1, \zeta_2 \in W_h$ and consider $u_1 = \mathcal{J}(\zeta_1)$ and $u_2 = \mathcal{J}(\zeta_2)$. Then, u_1 and u_2 are the second component of the solution of (3.1) with right hand sides $\mathcal{F}(\zeta_1)$ and $\mathcal{F}(\zeta_2)$, respectively. Hence, the

difference $u_1 - u_2$ satisfies equations (3.1), with source term $\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)$ and homogeneous boundary conditions on Γ . By the stability estimate in Lemma 2 and Lipschitz continuity assumption, we have

$$\|\mathcal{J}(\zeta_1) - \mathcal{J}(\zeta_2)\|_{\Omega_h} = \|u_1 - u_2\|_{\Omega_h} \leq 4 \max\{\tilde{c}^2 h, 1\} \|\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)\|_{\Omega_h} \leq 4L \max\{\tilde{c}^2 h, 1\} \|\zeta_1 - \zeta_2\|_{\Omega_h}.$$

The result follows due to $4L \max\{\tilde{c}^2 h, 1\} < 1$. \square

As a consequence of the above result, system (2.2) subject to the hypotheses of the theorem has a unique solution that depends continuously on the problem data.

We now present the arguments that lead to the proof of Lemma 2. We start by establishing a connection between the norm of the transferred boundary conditions φ_q , the magnitude of the flux \mathbf{q} and the length of the transfer path taken. In order to do so, we will make use of a tool introduced in [10]. For any smooth enough function \mathbf{v} defined in $T^e \cup T_{ext}^e$ and $\mathbf{x} \in \Gamma_h$ we set

$$\delta_v(\mathbf{x}) := \frac{1}{l(\mathbf{x})} \int_0^{l(\mathbf{x})} [\mathbf{v}(\mathbf{x} + \mathbf{n}s) - \mathbf{v}(\mathbf{x})] \cdot \mathbf{n} ds. \quad (3.4)$$

The significance of this function is that it will allow us to separate the contributions to the boundary conditions coming from the flux, from the diffusivity, and from the length of the transfer path. Observe that due to Assumption (3.2a) and (3.1e), we have

$$\begin{aligned} \varphi_q(\mathbf{x}) - g(\bar{\mathbf{x}}(\mathbf{x})) &= \int_0^{l(\mathbf{x})} \kappa^{-1}(\mathbf{x}) \mathbf{q}(\mathbf{x} + \mathbf{n}s) \cdot \mathbf{n} ds \\ &= \kappa^{-1}(\mathbf{x}) \int_0^{l(\mathbf{x})} [\mathbf{q}(\mathbf{x} + \mathbf{n}s) - \mathbf{q}(\mathbf{x})] \cdot \mathbf{n} ds + l(\mathbf{x}) \kappa^{-1}(\mathbf{x}) \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}, \end{aligned}$$

with $\mathbf{q} \in \mathbf{V}_h$, and using the definition of δ_q , given in (3.4), we can rewrite the above as

$$\varphi_q(\mathbf{x}) - g(\bar{\mathbf{x}}(\mathbf{x})) = \kappa^{-1}(\mathbf{x}) l(\mathbf{x}) \delta_q(\mathbf{x}) + \kappa^{-1}(\mathbf{x}) l(\mathbf{x}) \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}. \quad (3.5)$$

This expression, combined with the bounds that we will derive in Lemma 3 below, will enable us to estimate the approximate solution in terms of the sources, as will become evident in Lemma 4.

In proving the next result, we will have to make use of the following properties of δ_v , which hold for each $e \in \mathcal{E}_h^\partial$ [10, Lemma 5.2]:

$$\|l^{1/2} \delta_v\|_e \leq \frac{1}{\sqrt{3}} r_e^{3/2} C_{ext}^e C_{inv}^e \|\mathbf{v}\|_{T^e} \quad \forall \mathbf{v} \in [\mathbf{P}_k(T)]^d, \quad (3.6a)$$

$$\|l^{1/2} \delta_v\|_e \leq \frac{1}{\sqrt{3}} r_e \|h_e^{-1} \partial_n \mathbf{v} \cdot \mathbf{n}\|_{T_{ext}^e} \quad \forall \mathbf{v} \in [H^1(T)]^d. \quad (3.6b)$$

The following three inequalities follow readily from estimate (3.6a), assumptions (3.2d) and (3.2c), and Young's inequalities.

Lemma 3. *Let φ_q be the transferred boundary condition defined in (3.1e) and suppose that Assumptions S are satisfied. It holds*

$$\begin{aligned} |\langle \varphi_q, \delta_q \rangle_{\Gamma_h}| &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} \varphi_q\|_{\Gamma_h}^2 + \frac{1}{2} \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h}^2 \\ |\langle \varphi_q, \tau(u - \hat{u}) \rangle_{\Gamma_h}| &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} \varphi_q\|_{\Gamma_h}^2 + \frac{1}{2} \|\tau^{1/2} (u - \hat{u})\|_{\partial\mathcal{T}_h}^2 \\ |\langle \varphi_q, \kappa l^{-1} \bar{g} \rangle_{\Gamma_h}| &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} \varphi_q\|_{\Gamma_h}^2 + \frac{3}{2} \|\kappa^{1/2} l^{-1/2} \bar{g}\|_{\Gamma_h}^2 \end{aligned}$$

\square

The expression for $\varphi_{\mathbf{q}}$ in (3.5) implies that $\mathbf{q}(\mathbf{x}) \cdot \mathbf{n} = \kappa(\mathbf{x})l^{-1}(\mathbf{x})(\varphi_{\mathbf{q}}(\mathbf{x}) - \bar{\mathbf{g}}) - \delta_{\mathbf{q}}(\mathbf{x})$. Thus, thanks to the definition of $\hat{\mathbf{q}} \cdot \mathbf{n}$, it follows that

$$\hat{\mathbf{q}} \cdot \mathbf{n} = \kappa l^{-1}(\varphi_{\mathbf{q}} - \bar{\mathbf{g}}) - \delta_{\mathbf{q}} + \tau(u - \hat{u}) \quad \text{on } \Gamma_h. \quad (3.7)$$

The above expression can now be combined with the estimates from Lemma 3 to produce a bound for the norm of $(\mathbf{q}, u - \hat{u}, \varphi_{\mathbf{q}})$ in terms of the source $\mathcal{F}(\zeta)$ and the boundary data $\bar{\mathbf{g}}$ as we will show next.

Lemma 4. *If Assumptions S hold, then*

$$\|(\mathbf{q}, u - \hat{u}, \varphi_{\mathbf{q}})\|^2 \leq 2\|\mathcal{F}(\zeta)\|_{\Omega_h}\|u\|_{\Omega_h} + 3\|\kappa^{1/2}l^{-1/2}\bar{\mathbf{g}}\|_{\Gamma_h}^2,$$

where

$$\|(\mathbf{v}, w, \mu)\| := \left(\|\kappa^{-1/2}\mathbf{v}\|_{\Omega_h}^2 + \|\tau^{1/2}w\|_{\partial\mathcal{T}_h}^2 + \|\kappa^{1/2}l^{-1/2}\mu\|_{\Gamma_h}^2 \right)^{1/2}. \quad (3.8)$$

Proof. Take $\zeta \in W_h$ and let $u = \mathcal{J}(\zeta) \in W_h$ be the corresponding solution satisfying (3.1). Integrating by parts the left hand side in (3.1b), testing (3.1) with $\mathbf{v} = \mathbf{q}$, $w = u$, and

$$\mu := \begin{cases} -\hat{\mathbf{q}} \cdot \mathbf{n} & \text{on } \Gamma_h, \\ -\hat{u} & \text{on } \partial\mathcal{T}_h \setminus \Gamma_h, \end{cases}$$

and adding the resulting equalities, we get

$$\|\kappa^{-1/2}\mathbf{q}\|_{\Omega_h}^2 + \|\tau^{1/2}(u - \hat{u})\|_{\partial\mathcal{T}_h}^2 = (\mathcal{F}(\zeta), u)_{\mathcal{T}_h} - \langle \varphi_{\mathbf{q}}, \hat{\mathbf{q}} \cdot \mathbf{n} \rangle_{\Gamma_h}.$$

Then, using the fact that $\hat{\mathbf{q}} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} + \tau(u - \hat{u})$ in combination with identity (3.7), we obtain

$$\|(\mathbf{q}, u - \hat{u}, \varphi_{\mathbf{q}})\|^2 \leq \|\mathcal{F}(\zeta)\|_{\Omega_h}\|u\|_{\Omega_h} + |\langle \varphi_{\mathbf{q}}, \delta_{\mathbf{q}} \rangle_{\Gamma_h}| + |\langle \varphi_{\mathbf{q}}, \tau(u - \hat{u}) \rangle_{\Gamma_h}| + |\langle \varphi_{\mathbf{q}}, \kappa l^{-1} \bar{\mathbf{g}} \rangle_{\Gamma_h}|.$$

By the Cauchy-Schwarz inequality and Lemma 3, we arrive at

$$\begin{aligned} \|(\mathbf{q}, u - \hat{u}, \varphi_{\mathbf{q}})\|^2 &\leq \|\mathcal{F}(\zeta)\|_{\Omega_h}\|u\|_{\Omega_h} + \frac{1}{2}\|\kappa^{1/2}l^{-1/2}\varphi_{\mathbf{q}}\|_{\Gamma_h}^2 + \frac{1}{2}\|\kappa^{-1/2}\mathbf{q}\|_{\Omega_h}^2 \\ &\quad + \frac{1}{2}\|\tau^{1/2}(u - \hat{u})\|_{\partial\mathcal{T}_h}^2 + \frac{3}{2}\|\kappa^{1/2}l^{-1/2}\bar{\mathbf{g}}\|_{\Gamma_h}^2. \end{aligned}$$

and the result follows. \square

Proof of Lemma 2. With all the previous technical results in place we are now in a position to prove the crucial result. In the arguments below, $\Pi_{\mathbf{V}}$ and Π_W are, respectively, the \mathbf{V} and W components of the HDG projector introduced in the Appendix A.

Proof. We will proceed using an auxiliary problem that generalizes the result of Lemma 3.3 in [10] to our semi-linear case. We will consider that, given $\Theta \in L^2(\Omega)$, the solution to the auxiliary problem

$$\kappa^{-1}\phi + \nabla\psi = 0 \quad \text{in } \Omega, \quad (3.9a)$$

$$\nabla \cdot \phi = \Theta \quad \text{in } \Omega, \quad (3.9b)$$

$$\psi = 0 \quad \text{on } \partial\Omega, \quad (3.9c)$$

satisfies the regularity estimate

$$\|\phi\|_{H^1(\Omega)} + \|\psi\|_{H^2(\Omega)} \leq C_{reg}\|\Theta\|_{\Omega}. \quad (3.10)$$

Then, following the argument of [9, Lemma 4.1] it is possible to show that if u satisfies (3.1) then

$$(u, \Theta)_{\mathcal{T}_h} = (\kappa^{-1} \mathbf{q}, \mathbf{\Pi}_V \phi - \phi)_{\mathcal{T}_h} + \langle \widehat{u}, \phi \cdot \mathbf{n} \rangle_{\Gamma_h} - \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, \psi \rangle_{\Gamma_h} + (\mathcal{F}(\zeta), \mathbf{\Pi}_W \psi)_{\mathcal{T}_h}.$$

We will now use this expression to bound the norm of u . In order to simplify the exposition, we will group some terms on the right hand side of this expression and treat them separately. Hence, we decompose the above expression as

$$(u, \Theta)_{\mathcal{T}_h} = \mathbb{T}_{\mathbf{q}} + \mathbb{T}_{\mathcal{F}} + \mathbb{T}_u, \quad (3.11)$$

by defining

$$\mathbb{T}_{\mathbf{q}} := (\kappa^{-1} \mathbf{q}, \mathbf{\Pi}_V \phi - \phi)_{\mathcal{T}_h}, \quad \mathbb{T}_u := \langle \widehat{u}, \phi \cdot \mathbf{n} \rangle_{\Gamma_h} - \langle \widehat{\mathbf{q}} \cdot \mathbf{n}, \psi \rangle_{\Gamma_h}, \quad \text{and} \quad \mathbb{T}_{\mathcal{F}} := (\mathcal{F}(\zeta), \mathbf{\Pi}_W \psi)_{\mathcal{T}_h}.$$

The terms $\mathbb{T}_{\mathbf{q}}$ and $\mathbb{T}_{\mathcal{F}}$ can be bounded by an application of the estimates (A.2a) in combination with the elliptic regularity (3.10), yielding

$$|\mathbb{T}_{\mathbf{q}}| \leq \underline{\kappa}^{-1/2} \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h} \|\mathbf{\Pi}_V \phi - \phi\|_{\Omega_h} \lesssim h \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h} \|\Theta\|_{\Omega}, \quad (3.12)$$

and

$$|\mathbb{T}_{\mathcal{F}}| \lesssim \|\mathcal{F}(\zeta)\|_{\Omega_h} \|\Theta\|_{\Omega}. \quad (3.13)$$

The treatment of the term \mathbb{T}_u requires more work. Denoting by Id_M the identity operator in M , considering (3.7) and equation (3.9a), \mathbb{T}_u can be written as $\mathbb{T}_u = \sum_{i=1}^5 \mathbb{T}_u^i$, where

$$\begin{aligned} \mathbb{T}_u^1 &:= -\langle \kappa l^{-1} \varphi_{\mathbf{q}}, \psi + l \partial_n \psi \rangle_{\Gamma_h}, & \mathbb{T}_u^2 &:= \langle \kappa \varphi_{\mathbf{q}}, (P_M - Id_M) \partial_n \psi \rangle_{\Gamma_h}, \\ \mathbb{T}_u^3 &:= \langle \delta_{\mathbf{q}}, \psi \rangle_{\Gamma_h}, & \mathbb{T}_u^4 &:= -\langle \tau(u - \widehat{u}), P_M \psi \rangle_{\Gamma_h}, \\ \mathbb{T}_u^5 &:= \langle \kappa l^{-1} \bar{\mathbf{g}}, \psi \rangle_{\Gamma_h}. \end{aligned}$$

We will now determine bounds for all the terms in the decomposition.

By Young's inequality and combining the fact that $l(\mathbf{x}) \lesssim Rh, \forall \mathbf{x} \in \Gamma_h$ with estimate (C.1c), we have

$$|\mathbb{T}_u^1| \leq \left| \langle \kappa^{1/2} l \kappa^{1/2} l^{-1/2} \varphi_{\mathbf{q}}, l^{-3/2} (\psi + l \partial_n \psi) \rangle_{\Gamma_h} \right| \lesssim Rh \|\kappa^{1/2} l^{-1/2} \varphi_{\mathbf{q}}\|_{\Gamma_h} \|\Theta\|_{\Omega}.$$

Analogously, we get

$$|\mathbb{T}_u^2| \lesssim R^{1/2} h \|\kappa^{1/2} l^{-1/2} \varphi_{\mathbf{q}}\|_{\Gamma_h} \|\Theta\|_{\Omega}.$$

To bound \mathbb{T}_u^3 , we employ (3.6a), (C.1d), and (3.2d) yielding

$$\begin{aligned} |\mathbb{T}_u^3| &\lesssim (Rh)^{1/2} \|l^{1/2} \delta_{\mathbf{q}}\|_{\Gamma_h} \|l^{-1} \psi\|_{\Gamma_h} \\ &\lesssim (Rh)^{1/2} \left(\sum_{e \in \mathcal{E}_h^{\partial}} \frac{1}{3} r_e^3 (C_{ext}^e C_{inv}^e)^2 \|\mathbf{q}\|_{T^e}^2 \right)^{1/2} \|\Theta\|_{\Omega} \\ &\lesssim R^2 h^{1/2} \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h} \|\Theta\|_{\Omega}. \end{aligned}$$

Similarly, using (C.1d) we can bound

$$|\mathbb{T}_u^4| \lesssim \bar{\tau}^{1/2} Rh \|\tau^{1/2} (u - \widehat{u})\|_{\partial \mathcal{T}_h} \|\Theta\|_{\Omega} \quad \text{and} \quad |\mathbb{T}_u^5| \lesssim (Rh)^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h} \|\Theta\|_{\Omega}.$$

Taking $\Theta = u$ in Ω_h and $\Theta = 0$ in Ω_h^c in (3.9) and considering the bounds for the terms \mathbb{T}_u^i , we can combine the decomposition (3.11) with the estimates (3.12) and (3.13), to obtain

$$\begin{aligned} \|u\|_{\Omega_h} &\lesssim h \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h} + h (R + R^{1/2}) \|\kappa^{1/2} l^{-1/2} \varphi_{\mathbf{q}}\|_{\Gamma_h} + R^2 h^{1/2} \|\kappa^{-1/2} \mathbf{q}\|_{\Omega_h} \\ &\quad + \bar{\tau}^{1/2} R h \|\tau^{1/2} (u - \hat{u})\|_{\partial\mathcal{T}_h} + \|\mathcal{F}(\zeta)\|_{\Omega_h} + (Rh)^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h}. \end{aligned}$$

Now, let

$$\tilde{c} := C \left\{ 1 + R^2 + R + R^{1/2} + \bar{\tau}^{1/2} R \right\}, \quad (3.14)$$

where $C > 0$ is the constant hidden in the symbol \lesssim . Then, since $h < 1$ by Lemma 4 and Young's inequality, we infer

$$\begin{aligned} \|u\|_{\Omega_h} &\leq \tilde{c} h^{1/2} \left(\sqrt{2} \|\mathcal{F}(\zeta)\|_{\Omega_h}^{1/2} \|u\|_{\Omega_h}^{1/2} + \sqrt{3} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h} \right) + \|\mathcal{F}(\zeta)\|_{\Omega_h} + \tilde{c} h^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h} \\ &\leq \tilde{c}^2 h \|\mathcal{F}(\zeta)\|_{\Omega_h} + \frac{1}{2} \|u\|_{\Omega_h} + \|\mathcal{F}(\zeta)\|_{\Omega_h} + \tilde{c} (\sqrt{3} + 1) h^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h} \end{aligned}$$

and thus

$$\|u\|_{\Omega_h} \leq 4 \max\{\tilde{c}^2 h, 1\} \|\mathcal{F}(\zeta)\|_{\Omega_h} + 2\tilde{c} (\sqrt{3} + 1) h^{1/2} \|\kappa^{1/2} l^{-1/2} \bar{\mathbf{g}}\|_{\Gamma_h}.$$

□

4 *A priori* error analysis

We now provide the *a priori* error bounds for the method. As we will see, some of the results presented in this section can be proven by using similar arguments to those of Section 3 and many details will be omitted. Given that the set of assumptions (3.2) is required to hold in order to ensure the well-posedness of the problem, in the present section they will be assumed as true and used for the error analysis without explicitly stating them in the results. Similarly, the regularity assumption (3.10) will be assumed to hold.

The total approximation error has a component due to the accuracy of the discretization, and a component due entirely to the approximation properties of the discrete subspace. This is made apparent using the HDG projection defined in (A.1) and defining *the projections of the errors*

$$\boldsymbol{\varepsilon}^{\mathbf{q}} := \mathbf{\Pi}_V \mathbf{q} - \mathbf{q}_h \quad \text{and} \quad \varepsilon^u := \mathbf{\Pi}_W u - u_h,$$

and the *error of the projections*

$$\mathbf{I}^{\mathbf{q}} := \mathbf{q} - \mathbf{\Pi}_V \mathbf{q} \quad \text{and} \quad I^u := u - \mathbf{\Pi}_W u.$$

Using these quantities we can decompose the error as follows

$$\mathbf{q} - \mathbf{q}_h = \boldsymbol{\varepsilon}^{\mathbf{q}} + \mathbf{I}^{\mathbf{q}} \quad \text{and} \quad u - u_h = \varepsilon^u + I^u.$$

In addition, we define $\widehat{\varepsilon}^u := P_M u - \widehat{u}_h$, where we recall that P_M is the L^2 projection into M_h .

It is not difficult to show that $(\boldsymbol{\varepsilon}^{\mathbf{q}}, \varepsilon^u, \widehat{\varepsilon}^u)$ belongs to $\mathbf{V}_h \times W_h \times M_h$ and satisfies

$$\langle \kappa^{-1} \boldsymbol{\varepsilon}^{\mathbf{q}}, \mathbf{v} \rangle_{\mathcal{T}_h} - \langle \varepsilon^u, \nabla \cdot \mathbf{v} \rangle_{\mathcal{T}_h} + \langle \widehat{\varepsilon}^u, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = -\langle \kappa^{-1} \mathbf{I}^{\mathbf{q}}, \mathbf{v} \rangle_{\mathcal{T}_h}, \quad (4.1a)$$

$$-\langle \boldsymbol{\varepsilon}^{\mathbf{q}}, \nabla w \rangle_{\mathcal{T}_h} + \langle \widehat{\varepsilon}^u, \mathbf{n}, w \rangle_{\partial\mathcal{T}_h} = \langle \mathcal{F}(u) - \mathcal{F}(u_h), w \rangle_{\mathcal{T}_h}, \quad (4.1b)$$

$$\langle \widehat{\varepsilon}^u, \mu \rangle_{\Gamma_h} = \langle \varphi - \varphi_h, \mu \rangle_{\Gamma_h}, \quad (4.1c)$$

$$\langle \widehat{\boldsymbol{\varepsilon}}^{\mathbf{q}} \cdot \mathbf{n}, \mu \rangle_{\partial\mathcal{T}_h \setminus \Gamma_h} = 0, \quad (4.1d)$$

for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h$, where $\boldsymbol{\varepsilon}^{\widehat{\mathbf{q}}} \cdot \mathbf{n} := \boldsymbol{\varepsilon}^{\mathbf{q}} \cdot \mathbf{n} + \tau(\varepsilon^u - \varepsilon^{\widehat{u}}) = P_M(\mathbf{q} \cdot \mathbf{n}) - \widehat{\mathbf{q}}_h \cdot \mathbf{n}$. This *error equations* will help us establishing two results that will eventually lead to the proof of the convergence of the method.

To abbreviate the notation in the following arguments it will be useful to define

$$\Lambda_{\mathbf{q}} := \left(\|\mathbf{I}^{\mathbf{q}}\|_{\Omega_h}^2 + \|h^\perp \partial_n(\mathbf{I}^{\mathbf{q}} \cdot \mathbf{n})\|_{\Omega_h^\varepsilon}^2 + \|(h^\perp)^{1/2} \mathbf{I}^{\mathbf{q}} \cdot \mathbf{n}\|_{\Gamma_h}^2 \right)^{1/2}, \quad (4.2a)$$

$$\Lambda_u := \left(\|(h^\perp)^{1/2} I^u\|_{\Gamma_h} + \|I^u\|_{\Omega_h} \right)^{1/2}. \quad (4.2b)$$

With respect to these quantities we point out that, if $\mathbf{q} \in \mathbf{H}^{k+1}(\Omega)$, $u \in H^{k+1}(\Omega)$ and $\tau = \mathcal{O}(1)$ then, by scaling arguments and the properties (A.2), both $\Lambda_{\mathbf{q}}$ and Λ_u are of order h^{k+1} .

The first of these auxiliary lemmas establishes the convergence of the discrete flux \mathbf{q}_h , the restriction to the mesh skeleton \widehat{u}_h , and the transferred boundary data φ_h as a consequence of the convergence of the primary scalar variable u_h and the errors of the projections I^u and $\mathbf{I}^{\mathbf{q}}$.

Lemma 5. *Let $\|\cdot\|$ be the norm defined in (3.8). There exists a positive constant $C > 0$, independent of h , such that*

$$\|(\boldsymbol{\varepsilon}^{\mathbf{q}}, \varepsilon^u - \varepsilon^{\widehat{u}}, \varphi - \varphi_h)\|^2 \leq 4L(\|\varepsilon^u\|_{\Omega_h} + \|I^u\|_{\Omega_h}) \|\varepsilon^u\|_{\Omega_h} + C \Lambda_{\mathbf{q}}^2. \quad (4.3)$$

Proof. Testing (4.1) with $\mathbf{v} := \boldsymbol{\varepsilon}^{\mathbf{q}}$, $w := \varepsilon^u$ and $\mu := \begin{cases} -\boldsymbol{\varepsilon}^{\widehat{\mathbf{q}}} \cdot \mathbf{n} & \text{on } \Gamma_h \\ -\varepsilon^u & \text{on } \partial\mathcal{T}_h \setminus \Gamma_h \end{cases}$, results in

$$\|\kappa^{-1/2} \boldsymbol{\varepsilon}^{\mathbf{q}}\|_{\Omega_h}^2 + \|\tau^{1/2} (\varepsilon^u - \varepsilon^{\widehat{u}})\|_{\partial\mathcal{T}_h} = -(\kappa^{-1} \mathbf{I}^{\mathbf{q}}, \boldsymbol{\varepsilon}^{\mathbf{q}})_{\mathcal{T}_h} + (\mathcal{F}(u) - \mathcal{F}(u_h), \varepsilon^u)_{\mathcal{T}_h} - \langle \varphi - \varphi_h, \boldsymbol{\varepsilon}^{\widehat{\mathbf{q}}} \cdot \mathbf{n} \rangle_{\Gamma_h},$$

then, owing to (3.7), we readily obtain $\boldsymbol{\varepsilon}^{\widehat{\mathbf{q}}} \cdot \mathbf{n} = \kappa l^{-1} (\varphi - \varphi_h) - \delta_{\boldsymbol{\varepsilon}^{\mathbf{q}}} - \delta_{\mathbf{I}^{\mathbf{q}}} - \mathbf{I}^{\mathbf{q}} \cdot \mathbf{n} + \tau(\varepsilon^u - \varepsilon^{\widehat{u}})$ on Γ_h . Substituting this above, we get

$$\begin{aligned} \|(\boldsymbol{\varepsilon}^{\mathbf{q}}, \varepsilon^u - \varepsilon^{\widehat{u}}, \varphi - \varphi_h)\|^2 &\leq |(\kappa^{-1} \mathbf{I}^{\mathbf{q}}, \boldsymbol{\varepsilon}^{\mathbf{q}})_{\mathcal{T}_h}| + L(\|\varepsilon^u\|_{\Omega_h} + \|I^u\|_{\Omega_h}) \|\varepsilon^u\|_{\Omega_h} \\ &\quad + |\langle \varphi - \varphi_h, \delta_{\boldsymbol{\varepsilon}^{\mathbf{q}}} + \delta_{\mathbf{I}^{\mathbf{q}}} + \mathbf{I}^{\mathbf{q}} \cdot \mathbf{n} - \tau(\varepsilon^u - \varepsilon^{\widehat{u}}) \rangle_{\Gamma_h}|. \end{aligned} \quad (4.4)$$

The estimates in Lemma 3, can be applied to the last term of (4.4) to arrive at

$$\begin{aligned} \langle \varphi - \varphi_h, \delta_{\mathbf{I}^{\mathbf{q}}} \rangle_{\Gamma_h} &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + \frac{1}{2} \frac{\kappa^{-1}}{\underline{\kappa}} \max_{e \in \mathcal{E}_h^\partial} \{r_e^2\} \|h^\perp \partial_n(\mathbf{I}^{\mathbf{q}} \cdot \mathbf{n})\|_{\Omega_h^\varepsilon}^2, \\ \langle \varphi - \varphi_h, \delta_{\boldsymbol{\varepsilon}^{\mathbf{q}}} \rangle_{\Gamma_h} &\leq \frac{1}{4} \|\kappa^{1/2} l^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + \frac{1}{3} \|\kappa^{-1/2} \boldsymbol{\varepsilon}^{\mathbf{q}}\|_{\Omega_h}^2, \\ \langle \varphi - \varphi_h, \mathbf{I}^{\mathbf{q}} \cdot \mathbf{n} \rangle_{\Gamma_h} &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + \frac{3}{2} \frac{\kappa^{-1}}{\underline{\kappa}} \max_{e \in \mathcal{E}_h^\partial} \{r_e\} \|(h^\perp)^{1/2} \mathbf{I}^{\mathbf{q}} \cdot \mathbf{n}\|_{\Gamma_h}^2, \\ |\langle \varphi - \varphi_h, \tau(\varepsilon^u - \varepsilon^{\widehat{u}}) \rangle_{\Gamma_h}| &\leq \frac{1}{6} \|\kappa^{1/2} l^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + \frac{1}{2} \|\tau^{1/2} (\varepsilon^u - \varepsilon^{\widehat{u}})\|_{\partial\mathcal{T}_h}^2. \end{aligned} \quad (4.5)$$

The estimate (4.3) is obtained with $C := 4\kappa^{-1} \max\{1, \frac{1}{2} R^2, \frac{3}{2} R\}$, applying Young's inequality to term $|(\kappa^{-1} \mathbf{I}^{\mathbf{q}}, \boldsymbol{\varepsilon}^{\mathbf{q}})_{\mathcal{T}_h}|$ and the estimates given in (4.5). \square

Due to the previous result, it is enough to show the convergence of ε^u to guarantee the convergence of the method. The next step then is to estimate $\|\varepsilon^u\|_{\Omega_h}$, which we will do through a duality argument very much in the spirit of the proof of Lemma 2. Indeed, given $\Theta \in L^2(\Omega)$, and considering the linear auxiliary problem (3.9), but now using equations (4.1) instead of (3.1), we can decompose

$$(\varepsilon^u, \Theta)_{\mathcal{T}_h} = \mathbb{T}^{\mathcal{F}} + \mathbb{T}^{\mathbf{q}} + \mathbb{T}^u \quad (4.6)$$

where

$$\begin{aligned}\mathbb{T}^{\mathcal{F}} &:= (\mathcal{F}(u) - \mathcal{F}(u_h), \Pi_W \psi)_{\mathcal{T}_h}, \\ \mathbb{T}^q &:= (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi}_V \phi)_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}^q, \nabla \psi)_{\mathcal{T}_h}, \\ \mathbb{T}^u &:= \langle \widehat{\boldsymbol{\varepsilon}}^u, \boldsymbol{\phi} \cdot \mathbf{n} \rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}}^q \cdot \mathbf{n}, \psi \rangle_{\Gamma_h}.\end{aligned}$$

In order to estimate the size of $\boldsymbol{\varepsilon}^u$ we will now treat each of these terms separately. The term $\mathbb{T}^{\mathcal{F}}$ is easy to bound, since

$$|\mathbb{T}^{\mathcal{F}}| \leq L \|u - u_h\|_{\Omega_h} \|\Pi_W \psi\|_{\Omega_h} \leq L (\|\boldsymbol{\varepsilon}^u\|_{\Omega_h} + \|I^u\|_{\Omega_h}) \|\Pi_W \psi\|_{\Omega_h} \lesssim L (\|\boldsymbol{\varepsilon}^u\|_{\Omega_h} + \|I^u\|_{\Omega_h}) \|\Theta\|_{\Omega}. \quad (4.7)$$

Now, by adding and subtracting $(\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\phi})_{\mathcal{T}_h}$ in the definition of the term \mathbb{T}^q , we obtain

$$\mathbb{T}^q = (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi}_V \boldsymbol{\phi} - \boldsymbol{\phi})_{\mathcal{T}_h} + (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\phi})_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}^q, \nabla \psi)_{\mathcal{T}_h}.$$

However, due to (3.9a), it holds that $(\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \boldsymbol{\phi})_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}^q, \nabla \psi)_{\mathcal{T}_h} = -(\mathbf{I}^q, \nabla \psi)_{\mathcal{T}_h}$. Let $\psi_h \in W_h$ be arbitrary. Then, by (A.1b), we have $(\mathbf{I}^q, \nabla \psi_h) = 0$. Combining these last two facts we obtain

$$\mathbb{T}^q = (\kappa^{-1}(\mathbf{q} - \mathbf{q}_h), \mathbf{\Pi}_V \boldsymbol{\phi} - \boldsymbol{\phi})_{\mathcal{T}_h} + (\mathbf{I}^q, \nabla(\psi - \psi_h))_{\mathcal{T}_h}.$$

Therefore, by choosing $\psi_h = \Pi_W \psi$, it follows that

$$\begin{aligned}|\mathbb{T}^q| &\leq \|\kappa^{-1/2}(\boldsymbol{\varepsilon}^q + \mathbf{I}^q)\|_{\Omega_h} \|\mathbf{\Pi}_V \boldsymbol{\phi} - \boldsymbol{\phi}\|_{\Omega_h} + \|\mathbf{I}^q\|_{\Omega_h} \|\nabla(\psi - \psi_h)\|_{\Omega_h} \\ &\lesssim h^{\min\{1, k\}} \|\kappa^{-1/2} \boldsymbol{\varepsilon}^q\|_{\Omega_h} \|\Theta\|_{\Omega} + h^{\min\{1, k\}} \|\mathbf{I}^q\|_{\Omega_h} \|\Theta\|_{\Omega}\end{aligned} \quad (4.8)$$

where we have used the elliptic regularity for the auxiliary problem, the approximation properties (A.1) and (A.2) of the HDG projector. Finally, we can further decompose $\mathbb{T}^u := \sum_{i=1}^7 \mathbb{T}_i^u$, where:

$$\begin{aligned}\mathbb{T}_1^u &:= -\langle \kappa l^{-1}(\varphi - \varphi_h), \psi + l \partial_n \psi \rangle_{\Gamma_h}, & \mathbb{T}_2^u &:= -\langle \kappa(\varphi - \varphi_h), (Id_M - P_M) \partial_n \psi \rangle_{\Gamma_h}, \\ \mathbb{T}_3^u &:= \langle \delta \mathbf{I}^q, \psi \rangle_{\Gamma_h}, & \mathbb{T}_4^u &:= \langle \mathbf{I}^q \cdot \mathbf{n}, (Id_M - P_M) \psi \rangle_{\Gamma_h}, \\ \mathbb{T}_5^u &:= -\langle \tau P_M I^u, \psi \rangle_{\Gamma_h}, & \mathbb{T}_6^u &:= \langle \delta \boldsymbol{\varepsilon}^q, \psi \rangle_{\Gamma_h}, \\ \mathbb{T}_7^u &:= -\langle \tau(\boldsymbol{\varepsilon}^u - \widehat{\boldsymbol{\varepsilon}}^u), P_M \psi \rangle_{\Gamma_h}.\end{aligned}$$

Bounding separately each of the terms above it is possible to estimate $\|\boldsymbol{\varepsilon}^u\|_{\mathcal{T}_h}$, as we show below.

Lemma 6. *Assume that the Lipschitz constant is such that L is small enough, and consider the discrete spaces to be of polynomial degree $k \geq 1$. Then,*

$$\|\boldsymbol{\varepsilon}^u\|_{\Omega_h} \lesssim ((Rh)^{1/2}(1 + \bar{\tau}^{1/2}) + h) \|(\boldsymbol{\varepsilon}^q, \boldsymbol{\varepsilon}^u - \widehat{\boldsymbol{\varepsilon}}^u, \varphi - \varphi_h)\| + (Rh^{1/2} + L)(\Lambda_q + \Lambda_u). \quad (4.9)$$

Proof. By applying Young's inequality to each term in the decomposition of \mathbb{T}^u , considering the estimates in Lemma C1, using the fact $l(\mathbf{x}) \lesssim Rh$, $\forall \mathbf{x} \in \Gamma_h$ and having in mind the estimates in (3.6), it is possible to deduce:

$$\begin{aligned}|\mathbb{T}_1^u| &\lesssim \bar{\kappa}^{1/2} R h \|\kappa^{1/2} l^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h} \|\Theta\|_{\Omega}, & |\mathbb{T}_5^u| &\lesssim \bar{\tau} R h^{1/2} \|(h^\perp)^{1/2} I^u\|_{\Gamma_h} \|\Theta\|_{\Omega}, \\ |\mathbb{T}_2^u| &\lesssim \bar{\kappa}^{1/2} R^{1/2} h \|\kappa^{1/2} l^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h} \|\Theta\|_{\Omega}, & |\mathbb{T}_6^u| &\lesssim \bar{\kappa}^{1/2} R^2 h^{1/2} \|\kappa^{-1/2} \boldsymbol{\varepsilon}^q\|_{\Omega_h} \|\Theta\|_{\Omega}, \\ |\mathbb{T}_3^u| &\lesssim R^{3/2} h^{1/2} \|h^\perp \partial_n \mathbf{I}^q \cdot \mathbf{n}\|_{\Gamma_h} \|\Theta\|_{\Omega}, & |\mathbb{T}_7^u| &\lesssim \bar{\tau}^{1/2} R h \|\tau^{1/2}(\boldsymbol{\varepsilon}^u - \widehat{\boldsymbol{\varepsilon}}^u)\|_{\Gamma_h} \|\Theta\|_{\Omega}, \\ |\mathbb{T}_4^u| &\lesssim h \|(h^\perp)^{1/2} \mathbf{I}^q \cdot \mathbf{n}\|_{\Gamma_h} \|\Theta\|_{\Omega},\end{aligned}$$

Then, recalling the definition of the norm $\|\cdot\|$ in (3.8), and of the terms Λ_q and Λ_u in (4.2), we get

$$\begin{aligned}|\mathbb{T}^u| &\lesssim \left(\bar{\kappa}^{1/2} R h + \bar{\kappa}^{1/2} R^{1/2} h + \bar{\kappa}^{1/2} R^2 h^{1/2} + \bar{\tau}^{1/2} R h \right) \|(\boldsymbol{\varepsilon}^q, \boldsymbol{\varepsilon}^u - \widehat{\boldsymbol{\varepsilon}}^u, \varphi - \varphi_h)\| \|\Theta\|_{\Omega} \\ &\quad + \max\{R^{1/2}, \bar{\tau}\} R h^{1/2} (\Lambda_u + \Lambda_q) \|\Theta\|_{\Omega} + h \Lambda_q \|\Theta\|_{\Omega}.\end{aligned} \quad (4.10)$$

Finally, taking $\Theta = \boldsymbol{\varepsilon}^u$ in Ω_h and $\Theta = 0$ in Ω_h^c in (4.6) and using the estimates (4.7), (4.8) and (4.10), and considering assumption (3.2c), the estimate (4.9) is obtained. \square

Combining Lemmas 5 and 6, we can bound the error in terms of the error of the projection I^u and I^q as we do below.

Theorem 2. *Assume that $6L((Rh)^{1/2}(1+\bar{\tau}^{1/2})+h) < 1$, τ is of order one, and the discrete spaces are of polynomial degree $k \geq 1$, then*

$$\|(\boldsymbol{\varepsilon}^q, \varepsilon^u - \widehat{\varepsilon}^u, \varphi - \varphi_h)\| \lesssim \Lambda_q + \Lambda_u \quad (4.11a)$$

and

$$\|\varepsilon^u\|_{\Omega_h} \lesssim ((Rh)^{1/2} + L + h)(\Lambda_u + \Lambda_q). \quad (4.11b)$$

Proof. It follows from Lemma 5 and the estimate in (4.9), that

$$\begin{aligned} \|(\boldsymbol{\varepsilon}^q, \varepsilon^u - \widehat{\varepsilon}^u, \varphi - \varphi_h)\|^2 &\leq 6L\|\varepsilon^u\|_{\Omega_h}^2 + 2L\Lambda_u^2 + C\Lambda_q^2 \\ &\leq 6L((Rh)^{1/2}(1+\bar{\tau}^{1/2})+h)\|(\boldsymbol{\varepsilon}^q, \varepsilon^u - \widehat{\varepsilon}^u, \varphi - \varphi_h)\|^2 + \max\{6L(Rh^{1/2}+L, C, 2L)\}(\Lambda_u^2 + \Lambda_q^2), \end{aligned}$$

where C is the constant defined in Lemma 5. Then, due to $6L((Rh)^{1/2}(1+\bar{\tau}^{1/2})+h) < 1$, the estimate (4.11a) is fulfilled. Finally, (4.9) and (4.11a) imply (4.11b). \square

As a byproduct of the previous result, we are now in the position to establish the asymptotic convergence rate of the discretization.

Corollary 1. *Suppose that assumptions of Theorem 2 hold. If $u \in H^{k+1}(\Omega)$ and $q \in \mathbf{H}^{k+1}(\Omega)$, then*

$$\|q - q_h\|_{\Omega} + \|u - u_h\|_{\Omega} \lesssim h^{k+1}(|u|_{k+1, \Omega} + |q|_{k+1, \Omega}). \quad (4.12)$$

Proof. It follows from Theorem 2, Lemma 4.9, and the approximation properties (A.2), combined with Lemmas 3.7 and 3.8 in [10]. \square

5 A posteriori error analysis

A residual-based error estimator. In order to prevent the proliferation of high order (with respect to h) oscillatory terms that would only make the analysis more cumbersome, we will suppose for the remainder of this sections that φ is the trace of a function in $W_h^* \cap H^1(\Omega_h)$. Let $(\mathcal{T}_h, \mathcal{E}_h^{\circ}, \mathcal{E}_h^{\partial})$ refer to the elements, interior faces and boundary faces of the computational mesh respectively. In each element $T \in \mathcal{T}_h$, we define the following residual-type local error estimator:

$$\begin{aligned} \eta_T(\mathbf{q}_h, u_h^*, \varphi_h) &:= \left(h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 + \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 \right. \\ &\quad \left. + \sum_{e \in \mathcal{E}^{\circ} \cap T} \left(h_e \|[\![\mathbf{q}_h]\!] \|_e^2 + h_e^{-1} \|[\![u_h^*]\!] \|_e^2 \right) + \sum_{e \in \mathcal{E}^{\partial} \cap T} h_e^{-1} \|(\varphi_h - u_h^*)\|_e^2 \right)^{1/2}, \end{aligned} \quad (5.1)$$

and introduce the data oscillation term

$$\text{osc}^2(\mathcal{F}, \mathcal{T}_h) := \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*)\|_T^2. \quad (5.2)$$

We will show that the global error estimator, given by

$$\eta := \left(\sum_{T \in \mathcal{T}_h} \eta_T^2(\mathbf{q}_h, u_h^*, \varphi_h) \right)^{1/2}, \quad (5.3)$$

constitutes a reliable and efficient local *a posteriori* estimator for the error

$$e_h^2 := \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|u - u_h^*\|_{\Omega_h}^2 + \|h_e^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h}^2. \quad (5.4)$$

The remaining part of this section is devoted to proving one of the main contributions of this work, which is the efficiency and reliability of the local error estimator (5.1). We state the result here, and will proceed to develop the tools required for its proof. This will follow readily from Theorem 3 and Theorem 4, the proof of which is lengthy and requires a few technical lemmas.

Reliability and local efficiency. *If the Lipschitz constant L associated to the source term \mathcal{F} and the distance between Γ_h and Γ is small enough (in a sense that will be made clear in the hypothesis of Theorem 3), then the error estimator η is reliable, i.e.,*

$$e_h^2 \lesssim \eta^2 + \text{osc}^2(\mathcal{F}, \mathcal{T}_h).$$

Moreover, η is locally efficient, meaning that

$$\eta_T^2(\mathbf{q}_h, u_h^*, \varphi_h) \lesssim \sum_{T \in \mathcal{U}_h(e)} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|u - u_h^*\|_T^2 \right) + h_e^{-1} \|\varphi - \varphi_h\|_e^2 + \text{osc}^2(\mathcal{F}, \mathcal{U}_h(e))$$

where $\mathcal{U}_h(e)$ is the set of elements that have e as an face. Namely, $\mathcal{U}_h(e) := \{T \in \mathcal{T}_h : T \cap \mathcal{E}_h = e\}$.

Before setting out to show the validity of this result, we would like to make a few remarks regarding the steps required for the proof. The efficiency of the estimator can be established by adapting some of the arguments in [11] to account for the semi-linearity of the problem and for the approximation of the boundary data due to the curved boundaries. This will be addressed at the end in Theorem 4. Reliability, however, requires a much lengthier argument and the proof will be divided in several steps. Lemma 7 establishes the connection between the residual of the HDG equation (2.2a) and the post-processed solution u_h^* that appears in the local error estimator. To show that each of the terms in the estimator are indeed upper bounds for the error in the flux, the term $\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2$ will be decomposed in four components which will be treated separately in Lemma 8. This shows that the error on the flux can be successfully bounded by the estimator, plus some additional terms involving the error the scalar variable u and the data in the boundary φ . Next, Lemma 9 shows an estimation for the error in the variable u in terms of that of the transferred boundary condition and the flux. All these results are then consolidated in Theorem 3, which establishes that the error can be controlled by a combination of the estimator η and the data oscillation, that is, the reliability is finally proven.

As mentioned in Section 2.3, the inclusion of the nonlinear source term \mathcal{F} in the definition of u_h^* helps obtaining the following result, which is important for the estimate in Lemma 8, that will link the post processed solution with Equation (2.2b).

Lemma 7. *Let (u_h, \mathbf{q}_h) be the solutions to (2.2) and u_h^* be the post-processing of u_h given by (2.3). It holds*

$$(P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, w)_{\mathcal{T}_h} = - \langle \mathbf{q}_h \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} - (\kappa \nabla u_h^* + \mathbf{q}_h, \nabla w)_{\mathcal{T}_h}, \quad \forall w \in W_{1,h}^c,$$

where $W_{1,h}^c := \{w \in H_0^1(\Omega) : w|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}$, and $\mathbb{P}_1(T)$ is the space of piecewise linear polynomials on T .

Proof. Considering $w \in W_{1,h}^c$ and integrating by parts in the equation (2.2b) we obtain, for all $T \in \mathcal{T}_h$

$$(\nabla \cdot \mathbf{q}_h, w)_T - \langle \mathbf{q}_h \cdot \mathbf{n}, w \rangle_{\partial T} + \langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial T} = (\mathcal{F}(u_h), w)_T.$$

Then, due to $\langle \widehat{\mathbf{q}}_h \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = 0$ and using (2.3), we have

$$(\nabla \cdot \mathbf{q}_h, w)_{\mathcal{T}_h} - \langle \mathbf{q}_h \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (\kappa \nabla u_h^*, \nabla w)_{\mathcal{T}_h} + (\mathcal{F}(u_h^*), w)_{\mathcal{T}_h} + (\mathbf{q}_h, \nabla w)_{\mathcal{T}_h},$$

which concludes the proof. \square

In what follows, $\tilde{u}_h^* \in W_h^* \cap H^1(\Omega_h)$ such that $\tilde{u}_h^* = \varphi$ on Γ_h , will be used to denote the so-called Oswald interpolation of u_h^* defined in D5. Now, we apply the Lemma D5, with $|\alpha| = 1$, to get

$$\begin{aligned} \|\nabla(\tilde{u}_h^* - u_h^*)\|_{\Omega_h}^2 &\leq C_O \left(\sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \|[[u_h^*]]\|_e^2 + \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|\varphi - u_h^*\|_e^2 \right) \\ &\leq C_O \left(\sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \|[[u_h^*]]\|_e^2 + 2 \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|\varphi - \varphi_h\|_e^2 + 2 \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|\varphi_h - u_h^*\|_e^2 \right). \end{aligned}$$

where $C_O > 0$ is a constant independent of h arising from the approximation properties of the Oswald's interpolant. Similarly, for $|\alpha| = 0$, we have

$$\|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2 \leq C_O \left(\sum_{e \in \mathcal{E}_h^\circ} h_e \|[[u_h^*]]\|_e^2 + 2 \sum_{e \in \mathcal{E}_h^\partial} h_e \|\varphi - \varphi_h\|_e^2 + 2 \sum_{e \in \mathcal{E}_h^\partial} h_e \|\varphi_h - u_h^*\|_e^2 \right).$$

Since for a fine enough mesh $h_e \leq h_e^{-1}$, the two inequalities above can be combined into

$$\|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2 + \|\nabla(\tilde{u}_h^* - u_h^*)\|_{\Omega_h}^2 \leq 2C_O \left(\sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \|[[u_h^*]]\|_e^2 + \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|\varphi_h - u_h^*\|_e^2 + \|h_e^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h}^2 \right). \quad (5.5)$$

The following three results allow us to find a preliminary estimate for each term of our error defined in (5.4). We begin rewrite $\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\mathcal{T}_h}^2$ in a suitable manner. Note first that using (1.3a), and adding and subtracting \tilde{u}_h^* , it follows

$$\begin{aligned} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 &= (\mathbf{q} - \mathbf{q}_h, \kappa^{-1}(\mathbf{q} - \mathbf{q}_h))_T \\ &= -(\mathbf{q} - \mathbf{q}_h, \nabla(u - \tilde{u}_h^*))_T - (\mathbf{q} - \mathbf{q}_h, \nabla \tilde{u}_h^* - \kappa^{-1} \mathbf{q}_h)_T \\ &= (\nabla \cdot (\mathbf{q} - \mathbf{q}_h), u - \tilde{u}_h^*)_T - \langle (\mathbf{q} - \mathbf{q}_h) \cdot \mathbf{n}, u - \tilde{u}_h^* \rangle_{\partial T} - (\mathbf{q} - \mathbf{q}_h, \nabla \tilde{u}_h^* - \kappa^{-1} \mathbf{q}_h)_T. \end{aligned}$$

Adding and subtracting $\mathcal{F}(u_h^*)$ and $P_W \mathcal{F}(u_h^*)$ in the first term above, and using (1.3b) to replace $\nabla \cdot \mathbf{q}$ by $\mathcal{F}(u)$ yields

$$\begin{aligned} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 &= (\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*) + P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h + \mathcal{F}(u) - \mathcal{F}(u_h^*), u - \tilde{u}_h^*)_T \\ &\quad - \langle (\mathbf{q} - \mathbf{q}_h) \cdot \mathbf{n}, u - \tilde{u}_h^* \rangle_{\partial T} - (\mathbf{q} - \mathbf{q}_h, \nabla \tilde{u}_h^* - \kappa^{-1} \mathbf{q}_h)_T. \end{aligned}$$

Thus, since $\mathbf{q} \in H(\text{div}; \Omega_h)$ and $u - \tilde{u}_h^* \in H_0^1(\Omega_h)$, we can add over the entire grid to obtain

$$\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 := \sum_{i=1}^4 \mathbb{T}_i, \quad (5.6)$$

where

$$\begin{aligned}\mathbb{T}_1 &:= (\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*), u - \tilde{u}_h^*)_{\mathcal{T}_h}, & \mathbb{T}_3 &:= -(\mathbf{q} - \mathbf{q}_h, \nabla \tilde{u}_h^* - \kappa^{-1} \mathbf{q}_h)_{\mathcal{T}_h} \\ \mathbb{T}_2 &:= (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, u - \tilde{u}_h^*)_{\mathcal{T}_h} + \langle \mathbf{q}_h \cdot \mathbf{n}, u - \tilde{u}_h^* \rangle_{\partial \mathcal{T}_h}, & \mathbb{T}_4 &:= (\mathcal{F}(u) - \mathcal{F}(u_h^*), u - \tilde{u}_h^*)_{\mathcal{T}_h}.\end{aligned}$$

In the following estimates, for a given function v , let $Q_k(v)$ be the averaged Taylor polynomial of degree k associated to v . For smooth functions this polynomial coincides with the ‘‘usual’’ Taylor polynomial, whereas for functions with Sobolev regularity it is defined by mollification of the weakly defined Taylor polynomial [12, Section 4.1].

Lemma 8. *There exists $\bar{C}_1 > 0$, independent of h such that*

$$\begin{aligned}\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 &\leq \bar{C}_1 \left(\text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \sum_{T \in \mathcal{T}_h} h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 \right. \\ &\quad \left. + \sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \sum_{e \in \mathcal{E}_h^\circ} \left(h_e \|\llbracket \mathbf{q}_h \rrbracket\|_e^2 + h_e^{-1} \|\llbracket u_h^* \rrbracket\|_e^2 \right) + \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|\varphi_h - u_h^*\|_e^2 \right) \\ &\quad + \bar{C}_1 \|h_e^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + \bar{C}_1 L \|u - u_h^*\|_{\Omega_h}^2.\end{aligned}$$

Proof. To prove the result, we will bound each of the terms \mathbb{T}_i in the decomposition separately. The final result will come as a consequence of the individual estimates. In some cases we will make use of a free parameter $\epsilon_j > 0$.

Bound for \mathbb{T}_1 . Consider $Q_0(u - \tilde{u}_h^*)$, the zeroth order averaged Taylor polynomial associated to $u - \tilde{u}_h^*$. Since $(\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*), Q_0(u - \tilde{u}_h^*))_T = 0$, then by Young’s inequality and the Bramble-Hilbert lemma with constant $c > 0$, independent of h , we obtain

$$(h_T(\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*)), h_T^{-1}(u - \tilde{u}_h^* - Q_0(u - \tilde{u}_h^*)))_T \leq \frac{h_T^2}{4\epsilon_1} \|\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*)\|_T^2 + c\epsilon_1 \|\nabla(u - \tilde{u}_h^*)\|_T^2.$$

Using (1.4a) in the last term of the above expression to replace ∇u by $\kappa^{-1} \mathbf{q}$ along with adding and subtracting ∇u_h^* and $\kappa^{-1} \mathbf{q}_h$, we obtain

$$\|\nabla(u - \tilde{u}_h^*)\|_T^2 \leq \frac{3}{\underline{\kappa}} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \|\kappa^{1/2} \nabla u_h^* - \kappa^{1/2} \nabla \tilde{u}_h^*\|_T^2 \right). \quad (5.7)$$

Thus

$$\begin{aligned}|\mathbb{T}_1| &\leq \hat{C}_1 \epsilon_1 \sum_{T \in \mathcal{T}_h} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 \|\kappa^{1/2} \nabla u_h^* - \kappa^{1/2} \nabla \tilde{u}_h^*\|_T^2 \right) \\ &\quad + \frac{\hat{C}_1}{4\epsilon_1} \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*)\|_T^2,\end{aligned} \quad (5.8)$$

where $\hat{C}_1 := \max\{1, 3c\underline{\kappa}^{-1}\}$.

Bound for \mathbb{T}_2 . We begin by rewriting \mathbb{T}_2 as

$$\begin{aligned}\mathbb{T}_2 &= \langle \mathbf{q}_h \cdot \mathbf{n}, u - \tilde{u}_h^* \rangle_{\partial \mathcal{T}_h} + (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, (u - \tilde{u}_h^*) - \mathcal{C}_h(u - \tilde{u}_h^*))_{\mathcal{T}_h} \\ &\quad + (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, \mathcal{C}_h(u - \tilde{u}_h^*))_{\mathcal{T}_h},\end{aligned}$$

where \mathcal{C}_h is the Clément interpolation operator defined in Appendix D. Rearranging terms above, using $u = \tilde{u}_h^* = \varphi$ on Γ_h , and applying Lemma 7, we have

$$\begin{aligned} \mathbb{T}_2 &= \langle \mathbf{q}_h \cdot \mathbf{n}, u - \tilde{u}_h^* \rangle_{\partial\mathcal{T}_h} + (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, (Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*))_{\mathcal{T}_h} - \langle \mathbf{q}_h \cdot \mathbf{n}, \mathcal{C}_h(u - \tilde{u}_h^*) \rangle_{\partial\mathcal{T}_h} \\ &\quad - (\kappa \nabla u_h^* + \mathbf{q}_h, \nabla \mathcal{C}_h(u - \tilde{u}_h^*))_{\mathcal{T}_h} \\ &= \langle \mathbf{q}_h \cdot \mathbf{n}, (Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*) \rangle_{\partial\mathcal{T}_h \setminus \Gamma_h} + (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, (Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*))_{\mathcal{T}_h} \\ &\quad - (\kappa \nabla u_h^* + \mathbf{q}_h, \nabla \mathcal{C}_h(u - \tilde{u}_h^*))_{\mathcal{T}_h} \end{aligned}$$

Then, by Young's inequality,

$$\begin{aligned} |\mathbb{T}_2| &\leq \frac{1}{4\epsilon_2} \sum_{T \in \mathcal{T}_h} h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 + \epsilon_2 \sum_{T \in \mathcal{T}_h} h_T^{-2} \|(Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*)\|_T^2 \\ &\quad + \frac{1}{4\epsilon_2} \sum_{e \in \mathcal{E}_h^\circ} h_e \|\llbracket \mathbf{q}_h \rrbracket\|_e^2 + \epsilon_2 \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \|(Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*)\|_e^2 \\ &\quad + \frac{\bar{\kappa}}{4\epsilon_2} \sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \epsilon_2 \sum_{T \in \mathcal{T}_h} |\mathcal{C}_h(u - \tilde{u}_h^*)|_{1,T}^2. \end{aligned}$$

On the other hand, the properties of Clément's interpolant—Lemma D4—and the Poincaré inequality with constant c_p imply that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T^{-2} \|(Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*)\|_T^2 &\lesssim \sum_{T \in \mathcal{T}_h} \|u - \tilde{u}_h^*\|_{\Delta_T}^2 \leq \hat{c}_1 c_p \sum_{T \in \mathcal{T}_h} |u - \tilde{u}_h^*|_{1,T}^2, \\ \sum_{e \in \mathcal{E}_h^\circ} h_e^{-1} \|(Id_M - \mathcal{C}_h)(u - \tilde{u}_h^*)\|_e^2 &\lesssim \sum_{e \in \mathcal{E}_h^\circ} \|u - \tilde{u}_h^*\|_{\Delta_e}^2 \leq \hat{c}_2 c_p \sum_{T \in \mathcal{T}_h} |u - \tilde{u}_h^*|_{1,T}^2, \\ \sum_{T \in \mathcal{T}_h} |\mathcal{C}_h(u - \tilde{u}_h^*)|_{1,T}^2 &\lesssim \sum_{T \in \mathcal{T}_h} \|u - \tilde{u}_h^*\|_T^2 \leq \hat{c}_3 c_p \sum_{T \in \mathcal{T}_h} |u - \tilde{u}_h^*|_{1,T}^2. \end{aligned}$$

Above, the sets Δ_T and Δ_e correspond to the macro element surrounding the element T and face e respectively, i.e.

$$\Delta_T := \{T' \in \mathcal{T}_h : \bar{T} \cap \bar{T}' \neq \emptyset\} \quad \text{and} \quad \Delta_e = \{T' \in \mathcal{T}_h : \bar{T}' \cap \bar{e} \neq \emptyset\}.$$

Then, applying (5.7) to the right side terms of the last three inequalities, one arrives at

$$\begin{aligned} |\mathbb{T}_2| &\leq \frac{\hat{C}_2}{4\epsilon_2} \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 + \sum_{e \in \mathcal{E}_h^\circ} h_e \|\llbracket \mathbf{q}_h \rrbracket\|_e^2 \right) \\ &\quad + \epsilon_2 \hat{C}_2 \sum_{T \in \mathcal{T}_h} \left(\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 + \left(\frac{1}{4\epsilon_2} + 1 \right) \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \|\kappa^{1/2} \nabla (u - \tilde{u}_h^*)\|_T \right) \end{aligned} \quad (5.9)$$

with $\hat{C}_2 = \max\{1, \bar{\kappa}, 3c_p \underline{\kappa}^{-1}(\hat{c}_1 + \hat{c}_2 + \hat{c}_3)\}$.

Bound for \mathbb{T}_3 . From Young's inequality, it follows that

$$|\mathbb{T}_3| \leq \sum_{T \in \mathcal{T}_h} \left(\frac{1}{2\epsilon_3} \left(\|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \|\kappa^{1/2} \nabla (\tilde{u}_h^* - u_h^*)\|_T^2 \right) + \epsilon_3 \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 \right). \quad (5.10)$$

Bound for \mathbb{T}_4 . Adding and subtracting u_h^* , and using the Lipschitz continuity of \mathcal{F} , we have

$$|\mathbb{T}_4| \leq L \sum_{T \in \mathcal{T}_h} \left(\|u - u_h^*\|_T^2 + \|u - u_h^*\|_T \|u_h^* - \tilde{u}_h^*\|_T \right) \leq \frac{L}{2} \sum_{T \in \mathcal{T}_h} \left(3\|u - u_h^*\|_T^2 + \|u_h^* - \tilde{u}_h^*\|_T^2 \right), \quad (5.11)$$

where the second inequality follows from Young's inequality.

Wrap-up. By the decomposition (5.6) and the bounds (5.8) - (5.11) obtained for the terms \mathbb{T}_i , we deduce that

$$\begin{aligned} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 &\leq \frac{\widehat{C}_1}{4\epsilon_1} \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathcal{F}(u_h^*) - P_W \mathcal{F}(u_h^*)\|_{0,T}^2 + \frac{\widehat{C}_2}{4\epsilon_2} \sum_{T \in \mathcal{T}_h} h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 \\ &+ \left(\widehat{C}_1 \epsilon_1 + \widehat{C}_2 \epsilon_2 + \frac{\widehat{C}_2}{4\epsilon_2} + \frac{1}{2\epsilon_3} \right) \sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla u_h^* + \kappa^{-1/2} \mathbf{q}_h\|_T^2 + \frac{\widehat{C}_2}{4\epsilon_2} \sum_{e \in \mathcal{E}_h^o} h_e \|[[\mathbf{q}_h]]\|_e^2 \\ &+ \left(\widehat{C}_1 \epsilon_1 + \widehat{C}_2 \epsilon_2 + \frac{1}{2\epsilon_3} \right) \bar{\kappa} \sum_{T \in \mathcal{T}_h} \|\nabla(u_h^* - \tilde{u}_h^*)\|_T^2 + \frac{L}{2} \sum_{T \in \mathcal{T}_h} \|u_h^* - \tilde{u}_h^*\|_T^2 + \frac{3L}{2} \sum_{T \in \mathcal{T}_h} \|u - u_h^*\|_T^2 \\ &+ (\widehat{C}_1 \epsilon_1 + \widehat{C}_2 \epsilon_2 + \epsilon_3) \sum_{T \in \mathcal{T}_h} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2. \end{aligned}$$

Finally, considering values of ϵ_1, ϵ_2 , and ϵ_3 such that $\widehat{C}_1 \epsilon_1 + \widehat{C}_2 \epsilon_2 + \epsilon_3 < 1/2$, and the estimate for the terms that involve \tilde{u}_h^* , given in (5.5), the proof is concluded with \bar{C}_1 dependent only of \widehat{C}_1 and \widehat{C}_2 . \square

Now, we bound the second term of the error e_h^2 (see (5.4)).

Lemma 9. *Under all the previous assumptions, the following bound for the error in the post processed solution holds*

$$\begin{aligned} \|u - u_h^*\|_{\Omega_h}^2 &\leq \bar{C}_2 \left(\sum_{T \in \mathcal{T}_h} \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2 + \sum_{e \in \mathcal{E}_h^o} h_e^{-1} \|[[u_h^*]]\|_e^2 + \sum_{e \in \mathcal{E}_h^o} h_e^{-1} \|\varphi_h - u_h^*\|_e^2 \right. \\ &\quad \left. + \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|h_e^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h}^2 \right) \end{aligned}$$

where $\bar{C}_2 > 0$ is a positive constant independent of h .

Proof. First, note that, since $u - \tilde{u}_h^* \in H_0^1(\Omega_h)$, then thanks to the triangle and Poincaré inequalities with constant c_p , it follows that

$$\|u - u_h^*\|_{\Omega_h}^2 \leq 2 \|u - \tilde{u}_h^*\|_{\Omega_h}^2 + 2 \|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2 \leq 2 c_p^2 \|\nabla u - \nabla \tilde{u}_h^*\|_{\Omega_h}^2 + 2 \|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2.$$

then, since $\mathbf{q} = -\kappa^{-1} \nabla u$ (see (1.3a)) adding $\pm \kappa^{-1} \mathbf{q}_h$, we get

$$\|u - u_h^*\|_{\Omega_h}^2 \leq 4 c_p^2 \underline{\kappa}^{-1} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla \tilde{u}_h^*\|_{\Omega_h}^2 \right) + 2 \|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2.$$

Now, adding $\pm \kappa^{-1/2} \nabla u_h^*$, results in

$$\begin{aligned} \|u - u_h^*\|_{\Omega_h}^2 &\leq 4 c_p^2 \underline{\kappa}^{-1} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + 8 c_p^2 \underline{\kappa}^{-1} \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_{\Omega_h}^2 \\ &\quad + 2 \max\{4 c_p^2 \underline{\kappa}^{-1} \bar{\kappa}, 1\} \left(\|\nabla(u_h^* - \tilde{u}_h^*)\|_{\Omega_h}^2 + \|\tilde{u}_h^* - u_h^*\|_{\Omega_h}^2 \right). \end{aligned} \quad (5.12)$$

Finally, the proof is concluded by substituting (5.5) into (5.12). \square

We conclude this part with an estimate for the last term of our error,

Lemma 10. *Assume that all previous assumptions are satisfied. Then, there exists a positive constant \bar{C}_3 , independent of h such that*

$$\begin{aligned} \|h_e^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h}^2 &\leq \bar{C}_3 \max_{e \in \mathcal{E}_h^\partial} \{r_e^2, r_e (C_{ext}^e)^2\} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + h^2 L^2 \|u - u_h^*\|_{\Omega_h}^2 \right. \\ &\quad \left. + \sum_{T \in \mathcal{T}_h} h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 + \text{osc}^2(\mathcal{F}, \mathcal{T}_h) \right). \end{aligned}$$

Proof. We first notice that this term depends on what happens in the domain Ω_h^c , that is

$$\|h_e^{-1/2}(\varphi - \varphi_h)\|_{\Gamma_h}^2 \lesssim \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h^c}^2. \quad (5.13)$$

Then, for each $T \in \mathcal{T}_h$, we have

$$\begin{aligned} h_T \|\nabla \cdot (\mathbf{q} - \mathbf{q}_h)\|_T &= h_T \|\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h\|_T \\ &\leq h_T \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T + h_T \|\mathcal{F}(u) - \mathcal{F}(u_h^*)\|_T + h_T \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T. \end{aligned} \quad (5.14)$$

Now we will need to consider the approximation error measured in a function space with additional regularity. For $T \in \mathcal{T}_h$ let $E_T : \mathbf{H}(\mathbf{div}; T) \rightarrow \mathbf{H}(\mathbf{div}; \mathbb{R}^d)$ be any local extension operator, and $Q_k(E_T(\mathbf{q})) \in \mathbb{P}_k(\mathbb{R}^d)$ the averaged averaged Taylor polynomial of degree k introduced in the proof of Lemma 7. Let also $E : \mathbf{H}(\mathbf{div}; \mathcal{T}_h) \rightarrow \mathbf{H}(\mathbf{div}; \mathbb{R}^d)$ be a global extension such that $E(\mathbf{v})|_T := E_T(\mathbf{v})$ for all $T \in \mathcal{T}_h$ and $\mathbf{v} \in \mathbf{H}(\mathbf{div}; \mathcal{T}_h)$. Note that

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_h\|_{\Omega_h^c} &\leq \|Q_k(E(\mathbf{q})) - \mathbf{q}_h\|_{\Omega_h^c} + \|\mathbf{q} - Q_k(E(\mathbf{q}))\|_{\Omega_h^c} \\ &= \|\mathbf{q} - Q_k(E(\mathbf{q}))\|_{\Omega_h^c} + \left(\sum_{e \in \mathcal{E}_h^\partial} \|Q_k(E(\mathbf{q})) - \mathbf{q}_h\|_{T_{ext}^e}^2 \right)^{1/2} \\ &\leq \|E(\mathbf{q}) - Q_k(E(\mathbf{q}))\|_{\Omega_h^c} + \left(\sum_{e \in \mathcal{E}_h^\partial} r_e (C_{ext}^e)^2 \|Q_k(E(\mathbf{q})) - \mathbf{q}_h\|_{T^e}^2 \right)^{1/2}. \end{aligned}$$

Since $\|Q_k(E(\mathbf{q})) - \mathbf{q}_h\|_{T^e}^2 = \|Q_k(E(\mathbf{q}) - \mathbf{q}_h)\|_{T^e}^2 \lesssim \|E(\mathbf{q}) - \mathbf{q}_h\|_{T^e}^2 = \|\mathbf{q} - \mathbf{q}_h\|_{T^e}^2$, we obtain

$$\|\mathbf{q} - \mathbf{q}_h\|_{\Omega_h^c} \lesssim \|E(\mathbf{q}) - Q_k(E(\mathbf{q}))\|_{\Omega_h^c} + \max_{e \in \mathcal{E}_h^\partial} \{r_e^{1/2} C_{ext}^e\} \|\mathbf{q} - \mathbf{q}_h\|_{\Omega_h}.$$

Moreover, to bound the first term on the right hand side, we observe that

$$\begin{aligned} \|E_{T^e}(\mathbf{q}) - Q_k(E_{T^e}(\mathbf{q}))\|_{T_{ext}^e}^2 &= \|E_{T^e}(\mathbf{q} - \mathbf{q}_h) - Q_k(E_{T^e}(\mathbf{q}) - E_{T^e}(\mathbf{q}_h))\|_{T_{ext}^e}^2 \\ &\lesssim \|E_{T^e}(\mathbf{q} - \mathbf{q}_h)\|_{T_{ext}^e}^2 \lesssim r_e^2 \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{T^e}^2 + r_e^2 h_T^2 \|\nabla \cdot (\mathbf{q} - \mathbf{q}_h)\|_{T^e}^2, \end{aligned}$$

where we have used the estimate in (C.2a). Thus,

$$\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h^c} \lesssim \max_{e \in \mathcal{E}_h^\partial} \{r_e^2, r_e (C_{ext}^e)^2\} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + h_T^2 \|\nabla \cdot (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 \right).$$

The result follows combining the last inequality with (5.13). \square

With all the pieces in place, we can now show that the error in the flux can be successfully estimated if one considers the data oscillation.

Theorem 3. *Assume that the hypotheses of Lemmas 8–10 hold. In addition, if*

$$\overline{C}_1 \overline{C}_3 \max_{e \in \mathcal{E}_h^{\partial}} \{r_e^2, r_e (C_{ext}^e)^2\} < 1/2, \quad (5.15a)$$

$$\overline{C}_2 L (L h^2 + 2 \overline{C}_1) < 1/2, \quad (5.15b)$$

$$(2 \overline{C}_2 + 1) \overline{C}_3 \max_{e \in \mathcal{E}_h^{\partial}} \{r_e^2, r_e (C_{ext}^e)^2\} < 1/2, \quad (5.15c)$$

where \overline{C}_i , $i \in \{1, 2, 3\}$ are defined in the Lemmas 8, 9 and 10. Then, there exists a positive constant C_{rel} , such that

$$e_h^2 \leq C_{rel} \left(\eta^2 + \text{osc}^2(\mathcal{F}, \mathcal{T}_h) \right).$$

Proof. We first replace the estimation of the Lemma 10 into Lemma 8 and, together with assumption (5.15a), obtain

$$\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 \leq L (L h^2 + 2 \overline{C}_1) \|u - u_h^*\|_{\Omega_h}^2 + (2 \overline{C}_1 + 1) \left(\text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2 \right). \quad (5.16)$$

Combining the assumption (5.15b) with (5.16) into the Lemma 9, we obtain

$$\|u - u_h^*\|_{\Omega_h}^2 \leq 2 \overline{C}_2 \|h_e^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + 2 (\overline{C}_1 + 1) \overline{C}_2 \left(\text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2 \right). \quad (5.17)$$

Note that, thanks to (5.17) and assumption (5.15b), the estimation (5.16) can be rewritten as

$$\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 \leq \|h_e^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 + (3 \overline{C}_1 + 2) \left(\text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2 \right). \quad (5.18)$$

Combining (5.17) with (5.18) and using the Lemma 10 again, we arrives at

$$\begin{aligned} & \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|u - u_h^*\|_{\Omega_h}^2 \\ & \leq (2 \overline{C}_2 + 1) \overline{C}_3 \max_{e \in \mathcal{E}_h^{\partial}} \{r_e^2, r_e (C_{ext}^e)^2\} \left(\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|u - u_h^*\|_{\Omega_h}^2 \right) + \widehat{c} \left(\text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2 \right). \end{aligned}$$

Then, by assumption (5.15c), we deduce

$$\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_{\Omega_h}^2 + \|u - u_h^*\|_{\Omega_h}^2 \lesssim \text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2.$$

Finally, observe that the above estimation allows us rewritten the Lemma 10 as

$$\|h_e^{-1/2} (\varphi - \varphi_h)\|_{\Gamma_h}^2 \lesssim \text{osc}^2(\mathcal{F}, \mathcal{T}_h) + \eta^2.$$

which concludes the proof. \square

Having established the reliability of the estimator we can now adapt arguments from the linear case to show that the estimator is locally efficient as well. This will follow readily from the following estimantes.

Theorem 4. *Suppose that $Lh \leq 1$. Then we can assert the following local estimates*

$$\begin{aligned}
\|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T &\lesssim \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T, \\
h_e^{-1} \|\llbracket u_h^* \rrbracket\|_e^2 &\lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 \quad \forall e \in \mathcal{E}_h^\circ, \\
h_e^{-1} \|\varphi_h - u_h^*\|_e &\lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|h_e^{-1/2} (\varphi - \varphi_h)\|_e^2 \quad \forall e \in \mathcal{E}_h^\partial, \\
h_e \|\llbracket \mathbf{q}_h \rrbracket\|_e^2 &\lesssim \sum_{T \in \mathcal{U}_h(e)} \left(\|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 + h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 \right. \\
&\quad \left. + \|u - u_h^*\|_T^2 \right) + \text{osc}^2(\mathcal{F}, \mathcal{U}_h(e)) \quad \forall e \in \mathcal{E}_h^\circ \\
h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T &\lesssim \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 + h_T^2 \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T^2 + \|u - u_h^*\|_T^2.
\end{aligned}$$

Proof. Note that due to the presence of the non-linear source term, the post-processing defining u_h^* is also non linear, and a direct application of the results in [11, Lemmas 4.4–4.5] and [13, Lemmas 3.4–3.7] is not possible. We then proceed to adapt those arguments to the current semi-linear case and treat each of the estimates above separately in what follows. Local efficiency will follow by combining these estimates.

Bound for $\|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T$. This term can be bounded by an application of [13, Lemma 3.7], that is

$$\|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T \lesssim \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T \quad (5.19)$$

Bound for $h_e^{-1} \|\llbracket u_h^* \rrbracket\|_e^2$. We begin by splitting $\llbracket u_h^* \rrbracket$ into its component in the space $M_0 := \{\mu \in L^2(\partial \mathcal{T}_h) : \mu|_e \in \mathbb{P}_0(e), \forall e \in \mathcal{E}_h\}$ and its orthogonal complement. Considering P_{M_0} , the $L^2(\Omega)$ –orthogonal projection into M_0 , and Id_M the identity operator on the same space we have

$$h_e^{-1} \|\llbracket u_h^* \rrbracket\|_e^2 \lesssim h_e^{-1} \|P_{M_0} \llbracket u_h^* \rrbracket\|_e^2 + h_e^{-1} \|(Id_M - P_{M_0}) \llbracket u_h^* \rrbracket\|_e^2. \quad (5.20)$$

Each of these terms can be bounded by an application of [13, Lemma 3.4. and Lemma 3.5.] to all the interior faces of the triangulations. That is, for each $e \in \mathcal{E}_h^\circ$,

$$h_e^{-1} \|P_{M_0} \llbracket u_h^* \rrbracket\|_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2, \quad (5.21a)$$

$$h_e^{-1} \|(Id_M - P_{M_0}) \llbracket u_h^* \rrbracket\|_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \|\nabla(u - u_h^*)\|_T^2. \quad (5.21b)$$

Now, adding and subtracting $\kappa^{-1/2} \mathbf{q}_h$ to $\nabla(u - u_h^*)$ and using the definition of the flux it follows that

$$\|\nabla(u - u_h^*)\|_T^2 \lesssim \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2. \quad (5.22)$$

Moreover, using the fact that $\|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2 \lesssim \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2$ (see (5.19)), we can bound the second term above. The same argument can be applied to (5.21a), and combining these two results we arrive at

$$h_e^{-1} \|\llbracket u_h^* \rrbracket\|_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2} (\mathbf{q} - \mathbf{q}_h)\|_T^2 \quad \forall e \in \mathcal{E}_h^\circ. \quad (5.23)$$

Bound for $h_e^{-1}\|\varphi_h - u_h^*\|_e^2$. First, we define for each $T \in \mathcal{T}_h$, the local Raviart-Thomas [14] space of order k as

$$\mathbb{RT}_k(T) := [\mathbb{P}_k(T)]^d \oplus \mathbb{P}_k(T) \mathbf{x},$$

where $\mathbb{P}_k(T)$ denotes the space of polynomials of degree at most k defined in $T \in \mathcal{T}_h$ (see Section 2.2).

Taking as test in (2.2a) $\mathbf{v} \in \mathbb{RT}_0(T)$, it is possible to use the second equation defining the postprocessing u_h^* , (2.3b), for $\nabla \cdot \mathbf{v}$ belongs to the space of piece-wise constant functions $\mathbb{P}_0(T)$, to obtain

$$(\kappa^{-1} \mathbf{q}_h, \mathbf{v})_T - (u_h^*, \nabla \cdot \mathbf{v})_T + \langle \hat{u}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} = (\kappa^{-1} \mathbf{q}_h + \nabla u_h^*, \mathbf{v})_T + \langle \hat{u}_h - u_h^*, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} = 0.$$

On the other hand, if we consider $\mathbf{v} \in \mathbf{H}(\text{div}, \mathcal{U}_h(e))$ for each $e \in \mathcal{E}_h^\partial$, then by summing over all $T \in \mathcal{U}_h(e)$, we arrive at

$$\sum_{T \in \mathcal{U}_h(e)} (\kappa^{-1} \mathbf{q}_h + \nabla u_h^*, \mathbf{v})_T + \sum_{T \in \mathcal{U}_h(e)} \sum_{F \in \partial T \setminus e} \langle \hat{u}_h - u_h^*, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} - \langle \varphi_h - u_h^*, \mathbf{v} \cdot \mathbf{n} \rangle_e.$$

Since $\mathbf{v} \in \mathbf{H}(\text{div}, \mathcal{U}_h(e))$ is arbitrary, we can choose it such that, on each $T \in \mathcal{U}_h(e)$, belongs to $\mathbb{RT}_0(T)$ and satisfies

$$\begin{aligned} \int_e \mathbf{v} \cdot \mathbf{n} &= \int_e P_{M_0}(\varphi_h - u_h^*) \cdot \mathbf{n} && \text{for the face } e, \\ \int_F \mathbf{v} \cdot \mathbf{n} &= 0 && \forall F \in \partial T \setminus e, \end{aligned}$$

we obtain

$$\|P_{M_0}(\varphi_h - u_h^*)\|_e^2 = \sum_{T \in \mathcal{U}_h(e)} (\kappa^{-1} \mathbf{q}_h + \nabla u_h^*, \mathbf{v})_T.$$

Then, from the Cauchy–Schwarz inequality and a standard scaling argument $\|\mathbf{v}\|_T \lesssim h_e^{1/2} \|\mathbf{v} \cdot \mathbf{n}\|_e$, we get

$$h_e^{-1} \|P_{M_0}(\varphi_h - u_h^*)\|_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2 \quad (5.24)$$

Now, analogously to [13, Lema 3.5], it follows that for each boundary face $e \in \Gamma_h$,

$$\begin{aligned} h_e^{-1} \|(Id_M - P_{M_0})(\varphi_h - u_h^*)\|_e^2 &= h_e^{-1} \|(Id_M - P_{M_0})(\varphi_h - u_h^*)\|_e^2 \\ &\lesssim h_e^{-1} \|(Id_M - P_{M_0})(u - u_h^*)\|_e^2 + h_e^{-1} \|(Id_M - P_{M_0})(\varphi - \varphi_h)\|_e^2 \\ &\lesssim \sum_{T \in \mathcal{U}_h(e)} \|\nabla(u - u_h^*)\|_T^2 + h_e^{-1} \|\varphi - \varphi_h\|_e^2. \end{aligned}$$

Therefore, by (5.22) it follows that

$$\begin{aligned} h_e^{-1} \|(Id_M - P_{M_0})(\varphi_h - u_h^*)\|_e^2 &\lesssim \sum_{T \in \mathcal{U}_h(e)} (\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2) \\ &\quad + h_e^{-1} \|\varphi - \varphi_h\|_e^2. \end{aligned} \quad (5.25)$$

Finally, we decompose

$$\varphi_h - u_h^* = P_{M_0}(\varphi_h - u_h^*) + (Id_M - P_{M_0})(\varphi_h - u_h^*)$$

and apply (5.24) and (5.25), to arrive at

$$\begin{aligned} h_e^{-1} \|\varphi_h - u_h^*\|_e &\lesssim \sum_{T \in \mathcal{U}_h(e)} (\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|\kappa^{-1/2} \mathbf{q}_h + \kappa^{1/2} \nabla u_h^*\|_T^2) + h_e^{-1} \|\varphi - \varphi_h\|_e^2 \\ &\lesssim \sum_{T \in \mathcal{U}_h(e)} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + \|h_e^{-1/2}(\varphi - \varphi_h)\|_e^2, \end{aligned} \quad (5.26)$$

where we have applied (5.19) in the second line.

Bound for $h_e \llbracket \mathbf{q}_h \rrbracket_e^2$. For the interior faces, we have that for any $w \in H_0^1(\mathcal{U}_h(e))$, then

$$\begin{aligned} \langle \llbracket \mathbf{q}_h \rrbracket, w \rangle_e &= \sum_{T \in \mathcal{U}_h(e)} \langle (\mathbf{q} - \mathbf{q}_h) \cdot \mathbf{n}, w \rangle_{\partial T} = \sum_{T \in \mathcal{U}_h(e)} \left((\mathbf{q} - \mathbf{q}_h), \nabla w \right)_T + (\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h, w)_T \\ &\leq \sum_{T \in \mathcal{U}_h(e)} \left(\bar{\kappa}^{1/2} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T \|\nabla w\|_T + h_T \|\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h\|_T h_T^{-1} \|w\|_T \right) \\ &\leq \sum_{T \in \mathcal{U}_h(e)} \left(\bar{\kappa}^{1/2} \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T + h_T \|\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h\|_T \right) \left(\|\nabla w\|_T + h_T^{-1} \|w\|_T \right). \end{aligned}$$

By choosing a test function of the form $w = B_e \llbracket \mathbf{q}_h \rrbracket \in \mathbb{P}_{k+d}(T)$, which being B_e is a face bubble function defined in (C.3), it follows that

$$\int_e B_e \llbracket \mathbf{q}_h \rrbracket^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T + h_T \|\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h\|_T \right) h_T^{-1} h_e^{1/2} \|B_e \llbracket \mathbf{q}_h \rrbracket\|_e,$$

then, due to $h_T^{-1} h_e^{1/2} \lesssim h_e^{-1/2}$ and the fact that

$$\int_e B_e^2 \llbracket \mathbf{q}_h \rrbracket^2 \lesssim \int_e \llbracket \mathbf{q}_h \rrbracket^2 \lesssim \int_e B_e \llbracket \mathbf{q}_h \rrbracket^2$$

one arrives at

$$h_e \llbracket \llbracket \mathbf{q}_h \rrbracket \rrbracket_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + h_T^2 \|\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h\|_T^2 \right).$$

Now, using (5.14) and the Lipschitz continuity of \mathcal{F} , due to $Lh < 1$ we get

$$h_e \llbracket \llbracket \mathbf{q}_h \rrbracket \rrbracket_e^2 \lesssim \sum_{T \in \mathcal{U}_h(e)} \left(\|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2 + \|u - u_h^*\|_T^2 \right) + \text{osc}^2(\mathcal{F}, \mathcal{U}_h(e)). \quad (5.27)$$

Bound for $h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T^2$. For each element $T \in \mathcal{T}_h$ and any function $w \in H_0^1(T)$, we have that

$$\begin{aligned} (P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h, w)_T &= (P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*), w)_T + (\mathcal{F}(u_h^*) - \mathcal{F}(u), w)_T + (\mathcal{F}(u) - \nabla \cdot \mathbf{q}_h, w)_T \\ &= (P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*), w)_T + (\mathcal{F}(u_h^*) - \mathcal{F}(u), w)_T - (\mathbf{q} - \mathbf{q}_h, \nabla w)_T. \end{aligned}$$

We now consider the element bubble function B_T defined in Lemma C3 and take $w := B_T v$, with $v := P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h$. Then, the equation above yields

$$(v, B_T v)_T \lesssim \left(h_T^{-1} \|\kappa^{-1}(\mathbf{q} - \mathbf{q}_h)\|_T + \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T + L \|u - u_h^*\|_T \right) \left(h_T \|\nabla(B_T v)\|_T + \|B_T v\|_T \right).$$

Then, due to (C.3) and the inverse inequality $h_T \|\nabla w\|_T + \|w\|_T \lesssim \|w\|_T$, we obtain

$$\|v\|_T^2 \lesssim \left(h_T^{-1} \|\kappa^{-1}(\mathbf{q} - \mathbf{q}_h)\|_T + \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T + L \|u - u_h^*\|_T \right) \|B_T v\|_T.$$

Since $\|B_T v\|_T \lesssim \|v\|_T$ also by (C.3), we have

$$\|v\|_T \lesssim h_T^{-1} \|\kappa^{-1}(\mathbf{q} - \mathbf{q}_h)\|_T + \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T + L \|u - u_h^*\|_T.$$

Equivalently, since $Lh < 1$, the estimate above can be rewritten as

$$h_T^2 \|P_W \mathcal{F}(u_h^*) - \nabla \cdot \mathbf{q}_h\|_T \lesssim \|\kappa^{-1/2}(\mathbf{q} - \mathbf{q}_h)\|_T^2 + h_T^2 \|P_W \mathcal{F}(u_h^*) - \mathcal{F}(u_h^*)\|_T^2 + \|u - u_h^*\|_T^2. \quad (5.28)$$

□

Putting together the bounds from the previous theorem and the definition of the local error estimator, the local efficiency follows as an easy corollary.

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A HDG projection

In order to make this manuscript self-contained, in this section we provide previous results that will help us to analyze our discrete scheme. First of all we recall the HDG projection operators introduced by [9]. Given constants $l_u, l_q \in [0, k]$ and a pair of functions $(\mathbf{q}, u) \in H^{1+l_q}(T) \times H^{1+l_u}(T)$, we denote by $\mathbf{\Pi}(\mathbf{q}, u) := (\mathbf{\Pi}_v \mathbf{q}, \mathbf{\Pi}_w u)$ the projection over $\mathbf{V}_h \times W_h$ defined as the unique element-wise solutions of

$$(\mathbf{\Pi}_v \mathbf{q}, \mathbf{v})_T = (\mathbf{q}, \mathbf{v})_T \quad \forall \mathbf{v} \in [\mathbb{P}_{k-1}(T)]^d, \quad (\text{A.1a})$$

$$(\mathbf{\Pi}_w u, w)_T = (u, w)_T \quad \forall w \in \mathbb{P}_{k-1}(T), \quad (\text{A.1b})$$

$$\langle \mathbf{\Pi}_v \mathbf{q} \cdot \mathbf{n} + \tau \mathbf{\Pi}_w u, \mu \rangle_F = \langle \mathbf{q} \cdot \mathbf{n} + \tau u, \mu \rangle_F \quad \forall \mu \in \mathbb{P}_k(F), \quad (\text{A.1c})$$

for every element $T \in \mathcal{T}_h$, and $F \in \partial T$. The L^2 projection into M_h will be denoted as P_M . If the stabilization function is chosen so that $\tau_T^{\max} := \max \tau|_{\partial T} > 0$, then by [9] there is a constant $C > 0$ independent of T and τ such that

$$\|\mathbf{\Pi}_v \mathbf{q} - \mathbf{q}\|_T \leq Ch_T^{l_q+1} |\mathbf{q}|_{\mathbf{H}^{l_q+1}(T)} + Ch_T^{l_u+1} \tau_T^* |u|_{H^{l_u+1}(T)}, \quad (\text{A.2a})$$

$$\|\mathbf{\Pi}_w u - u\|_T \leq Ch_T^{l_u+1} |u|_{H^{l_u+1}(T)} + C \frac{h_T^{l_q+1}}{\tau_T^{\max}} |\nabla \cdot \mathbf{q}|_{H^{l_q}(T)}. \quad (\text{A.2b})$$

Here $\tau_T^* := \max \tau|_{\partial T \setminus F^*}$ and F^* is a face of T at which $\tau|_{\partial T}$ is maximum. As is customary, the symbol $|\cdot|_{H^s}$ is to be understood as the Sobolev semi norm of order $s \in \mathbb{R}$.

B Proof of Lemma 1

In this section we present the proof of Lemma 1, relating to the well posedness of the auxiliary non linear local problem that leads to the post processed approximation u_h^* . We re state the Lemma here for convenience.

Lemma 1. *The local post processing u_h^* is well defined for L small enough. Moreover, if $Lh^2 < 1$ and $k \geq 1$, then*

$$\|u - u_h^*\|_{0, \mathcal{T}_h} \lesssim (Rh)^{1/2} (h^{l_u+1} |u|_{l_u+2, \mathcal{T}_h} + h^{l_u+1} |\mathbf{q}|_{l_q+2, \mathcal{T}_h}) + h^{l_u+2} |u|_{l_u+2, \mathcal{T}_h} + Lh^{l_u+1} |u|_{l_u+2, \mathcal{T}_h}, \quad (\text{B.1a})$$

$$|u - u_h^*|_{1, T} \lesssim h_T^{l_u+1} |u|_{l_u+1, T} + Lh_T \|\varepsilon^u\|_{0, T} + \|\mathbf{q} - \mathbf{q}_h\|_{0, T} + Lh_T \|u - u_h\|_{0, T}, \quad (\text{B.1b})$$

and

$$\sum_{e \in \mathcal{E}_h^\partial} h_e^{1/2} \|[[u_h^*]]\|_e \lesssim \|u - u_h^*\|_{0, \mathcal{T}_h}^{1/2} \left(\|u - u_h^*\|_{0, \mathcal{T}_h}^2 + h^2 |u - u_h^*|_{1, \mathcal{T}_h}^2 \right)^{1/4}. \quad (\text{B.1c})$$

Proof. We will prove first that the problem (2.3) is well posed. For this, we will use a fixed point argument. Let $T \in \mathcal{T}_h$. We define the operator $S : \mathbb{P}_{k+1}(T) \rightarrow \mathbb{P}_{k+1}(T)$ as $S(\zeta) = z$, where z is the only solution of

$$(\kappa \nabla z, \nabla w)_T = -(q_h, \nabla w)_T + (\mathcal{F}(u_h), w)_T - (\mathcal{F}(\zeta), w)_T, \quad \forall w \in \mathbb{P}_{k+1}(T), \quad (\text{B.2a})$$

$$(z, w)_T = (u_h, w)_T, \quad \forall w \in \mathbb{P}_0(T). \quad (\text{B.2b})$$

Note that S is surjective because (B.2) is well-posed. We will show now that S has a unique fixed point and in that case it is the solution of (2.3). Let $\zeta_1, \zeta_2 \in \mathbb{P}_{k+1}(T)$ such that $S(\zeta_1) = z_1$ and $S(\zeta_2) = z_2$, with z_1 and z_2 satisfying (B.2). We observe that $\zeta_1 - \zeta_2 \in \mathbb{P}_{k+1}(T)$ and

$$(\kappa \nabla(z_1 - z_2), \nabla w)_T = -(\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2), w)_T, \quad \forall w \in \mathbb{P}_{k+1}(T), \quad (\text{B.3a})$$

$$(z_1 - z_2, w)_T = 0, \quad \forall w \in \mathbb{P}_0(T). \quad (\text{B.3b})$$

Then, for $i = 1$ and 2 , we set $\bar{z}_i := \frac{1}{|T|} \int_T z_i$ and noticing that $\bar{z}_1 = \bar{z}_2$ by equation (B.3b), we have

$$\|z_1 - z_2\|_T^2 = \|(z_1 - \bar{z}_1) - (z_2 - \bar{z}_2)\|_T^2 \leq C_F^2 \|\kappa^{1/2} \nabla(z_1 - z_2)\|_T^2,$$

where we have used the Friedrichs inequality with constant $C_F > 0$. Taking $w = z_1 - z_2$ in (B.3a), and recalling that \mathcal{F} is Lipschitz continuous with constant L , we obtain

$$\|z_1 - z_2\|_T^2 \leq C_F^2 (\mathcal{F}(\zeta_2) - \mathcal{F}(\zeta_1), z_1 - z_2)_T \leq C_F^2 L \|\zeta_2 - \zeta_1\|_T \|z_1 - z_2\|_T.$$

Thus, the operator S is a contraction as long as $C_F^2 L < 1$. If that is indeed the case, it has a unique fixed point.

For the inequality (B.1a), let P_0 and P_{W^*} be the L^2 -projectors into the space of constants and into W_h^* respectively and decompose

$$u - u_h^* = (I - P_{W^*})u + P_0(P_{W^*}u - u_h^*) + (I - P_0)(P_{W^*}u - u_h^*), \quad (\text{B.4})$$

We will now proceed to bound each of the terms on the right hand side of this expression separately in order to estimate the difference $u - u_h^*$. For the first term it is easy to see that

$$\|(I - P_{W^*})u\|_{0,T} \lesssim h_T^{l_u+2} |u|_{l_u+2,T}. \quad (\text{B.5})$$

For the second term we first notice that, since W^* is a space of piecewise polynomials, the definitions of P_{W^*} and Π_W , since $k \geq 1$, imply $P_0 P_{W^*} u = P_0 u = P_0 \Pi_W u$

$$\|P_0(P_{W^*}u - u_h^*)\|_{0,T} = \|P_0(\Pi_W u - u_h)\|_{0,T} \leq \|\Pi_W u - u_h\|_{0,T} = \|\varepsilon^u\|_{0,T}. \quad (\text{B.6})$$

In the first equality we have made use of the fact that, due the definition of u_h^* in equation (2.3b), we have $P_0 u_h^* = P_0 u_h$.

Now we move on to the third term in (B.4) and note that for every \mathbf{v} in the space of vector valued functions with components belonging to W_h^* and $T \in \mathcal{T}$ it holds that

$$(\kappa \nabla(u - u_h^*), \mathbf{v})_T = (\kappa \nabla(P_{W^*}u - u_h^*), \mathbf{v})_T = (\kappa \nabla(I - P_0)(P_{W^*}u - u_h^*), \mathbf{v})_T. \quad (\text{B.7})$$

Moreover, for the exact solutions (u, \mathbf{q}) , we have $\kappa \nabla u = -\mathbf{q}$ so that the difference $u - u_h^*$ satisfies

$$(\kappa \nabla(u - u_h^*), \nabla w)_T = -(\mathbf{q} - \mathbf{q}_h, \nabla w)_T + (\mathcal{F}(u_h^*) - \mathcal{F}(u), w)_T - (\mathcal{F}(u_h) - \mathcal{F}(u), w)_T$$

for every $w \in W_h^*$ and $T \in \mathcal{T}$. Letting $w := (I - P_0)(P_{W^*}u - u_h^*) \in W^*$ and ∇w be the test functions above, and using conditions (B.7) leads to

$$(\kappa \nabla w, \nabla w)_T = -(\mathbf{q} - \mathbf{q}_h, \nabla w)_T + (\mathcal{F}(u_h^*) - \mathcal{F}(u), w)_T + (\mathcal{F}(u) - \mathcal{F}(u_h), w)_T.$$

From this equation, using the the scaling argument $\|w\|_{0,T} \lesssim h_T |w|_{1,T}$ and the inverse inequality $|w|_{1,T} \lesssim h_T^{-1} \|w\|_{0,T}$ we arrive at

$$h_T^{-2} \|w\|_{0,T}^2 \lesssim \bar{\kappa} |w|_{1,T}^2 \leq \|\mathbf{q} - \mathbf{q}_h\|_{0,T} |w|_{1,T} + L (\|u - u_h^*\|_{0,T} + \|u - u_h\|_{0,T}) \|w\|_{0,T}$$

from which we conclude that

$$\|w\|_{0,T} \lesssim h_T \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + L h_T^2 (\|u - u_h^*\|_{0,T} + \|u - u_h\|_{0,T}).$$

Recalling the decomposition (B.4), and the estimates (B.5), (B.6) we can bound the term $\|u - u_h^*\|_{0,T}$ on the right hand side yielding

$$(1 - Lh_T^2)\|w\|_{0,T} \lesssim h_T \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + L h_T^2 \left(h_T^{l_u+2} |u|_{l_u+2,T} + \|\varepsilon^u\|_{0,T} + \|u - u_h\|_{0,T} \right) \quad (\text{B.8})$$

Combining (B.8) above with (B.5) and (B.6) once more we arrive at

$$\begin{aligned} (1 - Lh_T^2) \|u - u_h^*\|_{0,T} &\lesssim (1 - Lh_T^2) h_T^{l_u+2} |u|_{l_u+2,T} + (1 - Lh_T^2) \|\varepsilon^u\|_{0,T} + h_T \|\mathbf{q} - \mathbf{q}_h\|_{0,T} \\ &\quad + L h_T^2 \left(h_T^{l_u+2} |u|_{l_u+2,T} + \|\varepsilon^u\|_{0,T} + \|u - u_h\|_{0,T} \right) \\ &\lesssim h_T^{l_u+2} |u|_{l_u+2,T} + \|\varepsilon^u\|_{0,T} + h_T \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + L h_T^2 \|u - u_h\|_{0,T}. \end{aligned}$$

So, assuming $Lh_T^2 < 1$ for each $T \in \mathcal{T}_h$, results

$$\|u - u_h^*\|_{0,T} \lesssim h_T^{l_u+2} |u|_{l_u+2,T} + \|\varepsilon^u\|_{0,T} + h_T \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + Lh_T^2 \|u - u_h\|_{0,T}.$$

By adding on each $T \in \mathcal{T}_h$, the estimate (B.1a) is concluded after considering the results in Theorem 2. Now, if we apply the inverse inequality to the estimate above, we arrive at

$$(1 - Lh_T^2) |w|_{1,T} \lesssim \|\mathbf{q} - \mathbf{q}_h\|_{0,T} + Lh_T \left(h_T^{l_u+2} |u|_{l_u+2,T} + \|\varepsilon^u\|_{0,T} + \|u - u_h\|_{0,T} \right).$$

Assuming again $Lh_T^2 < 1$ for each $T \in \mathcal{T}_h$, (B.1b) follows.

Finally, using the trace inequality, the fact that $h_e \|v\|_{0,e}^2 \lesssim \|v\|_{0,T} \left(\|v\|_{0,T}^2 + h_T^2 |v|_{1,T}^2 \right)^{1/2}$ for any $v \in [H^1(K)]^d$, and the estimates (B.1a) and (B.1b), we have

$$\begin{aligned} \sum_{e \in \mathcal{E}_h} h_e \|[u_h^*]\|_{0,e}^2 &\lesssim \sum_{e \in \mathcal{E}_h} \sum_{T' \in \omega_e} h_e \|u - u_h^*\|_{T'}^2 \\ &\lesssim \sum_{e \in \mathcal{E}_h} \sum_{T' \in \omega_e} \|u - u_h^*\|_{0,T'} \left(\|u - u_h^*\|_{0,T'}^2 + h_{T'}^2 |u - u_h^*|_{1,T'}^2 \right)^{1/2}, \end{aligned}$$

which implies (B.1c). □

C Auxiliary estimates

The following results were used throughout the text. We include them here for completeness.

The first lemma was needed to bound the terms in the decomposition of \mathbb{T}^u carried out in Lemma 6.

Lemma C1. [10, Lemma 5.5] *Suppose Assumption (3.2d) and the elliptic regularity inequality (3.10) hold. Then,*

$$\|(h^\perp)^{-1/2} (Id_M - P_M) \psi\|_{\Gamma_h} \lesssim h \|\Theta\|_{\Omega}, \quad (\text{C.1a})$$

$$\|l^{1/2} (Id_M - P_M) \partial_n \psi\|_{\Gamma_h} \lesssim R^{1/2} h \|\Theta\|_{\Omega}, \quad (\text{C.1b})$$

$$\|l^{-3/2} (\psi + l \partial_n \psi)\|_{\Gamma_h} \lesssim \|\Theta\|_{\Omega}, \quad (\text{C.1c})$$

$$\|l^{-1} \psi\|_{\Gamma_h} \lesssim \|\Theta\|_{\Omega}. \quad (\text{C.1d})$$

The result below is used when deducing the bound for the term of the estimator involving the jump in the flux.

Lemma C2. *Let $e \in \mathcal{E}_h^\partial$ and $\mathbf{v} \in \mathbf{H}(\mathbf{div}; T^e)$. It holds*

$$\|E_{T^e}(\mathbf{v})\|_{T_{ext}^e}^2 \lesssim r_e^2 \|\mathbf{v}\|_{T^e}^2 + r_e^2 h_T^2 \|\nabla \cdot \mathbf{v}\|_{T^e}^2. \quad (\text{C.2a})$$

Proof. We employ a scaling argument. Let $\Phi : T^e \rightarrow \widehat{T}$ be the affine mapping from T^e to the reference element \widehat{T} and set $\widehat{T}_{ext}^e := \Phi^{-1}(T_{ext}^e)$. We have

$$\begin{aligned} \|E_{T^e}(\mathbf{v})\|_{T_{ext}^e}^2 &= 2|T_{ext}^e| \|\widehat{E}(\widehat{\mathbf{v}})\|_{\widehat{T}_{ext}^e}^2 \lesssim |T_{ext}^e| \|\widehat{\mathbf{v}}\|_{H(\mathbf{div}; \widehat{T})}^2 = |T_{ext}^e| \left(\|\widehat{\mathbf{v}}\|_{\widehat{T}}^2 + \|\widehat{\nabla} \cdot \widehat{\mathbf{v}}\|_{\widehat{T}}^2 \right) \\ &\lesssim |T_{ext}^e| \left(\frac{1}{|T^e|} \|\mathbf{v}\|_{T^e}^2 + \|\nabla \cdot \mathbf{v}\|_{T^e}^2 \right). \end{aligned}$$

Thus, considering that $|T_{ext}^e| \lesssim (H_e^\perp)^2 = R_e^2 (h_e^\perp)^2 \leq r_e^2 h_T^2$, and $|T^e| \lesssim h_T^2$, the inequality (C.2a) can be deduced. \square

The following result pertaining bubble functions is useful when addressing the local efficiency of the error estimator.

Lemma C3. *[15, Lemma 3.3.] Let $B_T := \prod_{i=1}^{d+1} \lambda_i$ be the element-bubble function associated to $T \in \mathcal{T}_h$, where $\{\lambda_i\}_{i=1}^{d+1}$ are the barycentric coordinates of T , and $B_e := \prod_{\substack{i=1 \\ i \neq j}}^{d+1} \lambda_i$ be the face-bubble function associated to $e \subset \partial T$, where λ_j vanishes on e . Then, the following estimates hold*

$$\begin{aligned} \|\mathbf{v}\|_T^2 &\lesssim (\mathbf{v}, B_T \mathbf{v})_T, & \|B_T \mathbf{v}\|_T &\lesssim \|\mathbf{v}\|_T, & \|B_T \mathbf{v}\|_{1,T} &\lesssim h_T^{-1} \|\mathbf{v}\|_T, \\ \|\boldsymbol{\mu}\|_e^2 &\lesssim (\boldsymbol{\mu}, B_e \boldsymbol{\mu})_e, & \|B_e \boldsymbol{\mu}\|_{\Delta_e} &\lesssim h_e^{1/2} \|\boldsymbol{\mu}\|_e, & \|B_e \boldsymbol{\mu}\|_{1,\Delta_e} &\lesssim h_e^{-1/2} \|\boldsymbol{\mu}\|_e, \end{aligned} \quad (\text{C.3})$$

for all $\mathbf{v} \in [\mathbb{P}_k(T)]^d$, $T \in \mathcal{T}_h$ and for each $\boldsymbol{\mu} \in [\mathbb{P}_k(e)]^d$, $e \in \mathcal{E}_h$.

D Clément and Oswald interpolants

The following two interpolants are useful in the arguments leading to the reliability of the estimator. They allow to control the behavior of functions with piecewise H^1 regularity by representatives belonging to the global $H_0^1(\Omega)$ space.

First, in the next lemma, we state the approximation properties of the Clément interpolation operator $\mathcal{C}_h : L^2(\Omega_h) \rightarrow W_h^{1,c} \cap H_0^1(\Omega)$, introduced in [16] as

$$\mathcal{C}_h w := \sum_{z \in \mathcal{N}_h} \left(\frac{1}{|\Omega_z|} \int_{\Omega_z} w \, dx \right) \phi_z$$

where ϕ_z is the \mathbb{P}_1 nodal basic functions associated to the interior vertex z , $\Omega_z = \text{supp } \phi_z$, and $W_h^{1,c} := \{w \in C(\Omega) : w|_T \in \mathcal{P}_1(T), T \in \mathcal{T}_h\}$.

Lemma D4. *[15, Lemma 3.2] For any $T \in \mathcal{T}_h$, $e \in \mathcal{E}_h^i$ and $0 \leq m \leq 1$, the following estimates hold, for all $w \in H_0^1(\Omega)$*

$$\|\mathcal{C}_h w\|_{m,\Omega} \lesssim \|w\|_{m,\Omega}, \quad \|w - \mathcal{C}_h w\|_{0,T} \lesssim h_T \|w\|_{1,\Delta_T}, \quad \|w - \mathcal{C}_h w\|_{0,e} \lesssim h_e^{1/2} \|w\|_{1,\Delta_e},$$

where $\Delta_T := \{T' \in \mathcal{T}_h : \overline{T} \cap \overline{T'} \neq \emptyset\}$ and $\Delta_e := \{T' \in \mathcal{T}_h : \overline{T'} \cap \overline{e} \neq \emptyset\}$.

The next results shows that an element w of W_h^* can be approximated by a continuous function $\tilde{w} \in W_h^*$, sometimes referred to as *Oswald interpolant*, and that the approximation error can be controlled by the size of the inter-element jumps of w .

Lemma D5. [17, Theorem 2.2.] *For any $w_h \in W_h^*$ and any multi-index with $|\alpha| = 0, 1$, the following approximation results holds: Let u_D be the restriction to Γ_h of a function in $W_h^* \cap H^1(\Omega_h)$. then there exists a function $\tilde{w}_h \in W_h^* \cap H^1(\Omega_h)$ satisfying $\tilde{w}_h|_\Gamma = u_D$, and*

$$\sum_{T \in \mathcal{T}_h} \|D^\alpha(w_h - \tilde{w}_h)\|_T^2 \leq C_O \left(\sum_{e \in \mathcal{E}_h^\circ} h_e^{1-2|\alpha|} \|[w_h]\|_e^2 + \sum_{e \in \mathcal{E}_h^\partial} h_e^{1-2|\alpha|} \|u_D - w_h\|_e^2 \right),$$

above, C_O is a positive constant independent of the mesh size.