A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy. *

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Abstract

In this work we present a new mixed finite element method for a class of natural convection models describing the behavior of non-isothermal incompressible fluids subject to a heat source. More precisely, we consider a system based on the coupling of the steady-state equations of momentum (Navier-Stokes) and thermal energy by means of the Boussinesq approximation. Our approach is based on the introduction of a modified pseudostress tensor depending on the pressure, and the diffusive and convective terms of the Navier–Stokes equations for the fluid and a vector unknown involving the temperature, its gradient and the velocity. The introduction of these further unknowns lead to a mixed formulation where the aforementioned pseudostress tensor and vector unknown, together with the velocity and the temperature, are the main unknowns of the system. Then the associated Galerkin scheme can be defined by employing Raviart–Thomas elements of degree k for the pseudostress tensor and the vector unknown, and discontinuous piece-wise polynomial elements of degree k for the velocity and temperature. With this choice of spaces, both, momentum and thermal energy, are conserved if the external forces belong to the velocity and temperature discrete spaces, respectively, which constitutes one of the main feature of our approach. We employ the Banach–Nečas–Babuška and Banach's fixed point theorems to prove unique solvability for both, the continuous and discrete problems. We provide the convergence analysis and particularly prove that the error decay with optimal rate of convergence. Further variables of interest, such as the fluid pressure, the fluid vorticity, the fluid velocity gradient, and the heat-flux can be easily approximated as a simple postprocess of the finite element solutions with the same rate of convergence. Finally, several numerical results illustrating the performance of the method are provided.

Key words: stationary Boussinesq equations, divergence-conforming elements, stress-velocity formulation, mixed finite element method, conservation of momentum, conservation of thermal energy

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

The motion of a liquid or gas, generated by some parts of the fluid being heavier than other parts, or in other words, produced by density differences as, for example, when a liquid in a vessel is heated from

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below, is a process known as natural convection. Different from what happens in forced convection, where the fluid flow is driven by a external source (like a suction device or a fan), the driving force is gravity and creates a circulating flow (convection). For several phenomena in nature and industry, in particular when the fluid behavior depend solely on the temperature of the fluid and density differences can be ignored except where they appear in terms multiplied by the acceleration due to gravity, such as in oceanic circulation, central heating and dense gas dispersion, the problem can be described by a system of equations commonly known as Boussinesq model, which consists in a coupling of the Navier-Stokes and heat equations, coupled by means of the so called Boussinesq approximation. More precisely, the stationary Boussinesq problem is a system of equations where the incompressible Navier–Stokes equation:

$$
\begin{gathered}
-\nu\,\Delta\mathbf{u} + (\nabla\mathbf{u})\mathbf{u} + \nabla p - \theta\,\mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\
\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad \int_\Omega p = 0,
\end{gathered} \tag{1.1}
$$

is coupled with the convection-diffusion equation:

$$
-\,\kappa\,\Delta\,\theta + \mathbf{u}\cdot\nabla\theta = 0 \quad \text{in} \quad \Omega, \quad \theta = \theta_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \quad \kappa\,\nabla\theta\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}, \tag{1.2}
$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \{2,3\}$, with polyhedral boundary Γ. Above, the unknowns are the velocity $\mathbf{u}$, the pressure p and the temperature θ of the fluid occupying the region Ω, and the given data are the fluid viscosity $\nu > 0$, the thermal conductivity $\kappa > 0$, the external force per unit mass $\mathbf{g} \in \mathbf{L}^2(\Omega)$, and the boundary temperature $\theta_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$.

Recently, in the literature it can observed an increasing interest in developing new numerical methods to approximate the solution of (1.1)-(1.2), motivated by the diverse applications of this coupled model (as those already mentioned above), and also by the increasing need of simpler, more accurate, and more efficient procedures to solve it. For instance, primal and mixed-type numerical formulations have been already considered in several works over the last decades (see, e.g. [6, 12, 21, 19, 16, 18, 24, 28, 27, 29, 30, 32, 33], respectively, and the references therein). The above list includes approaches with constant and temperature-dependent parameters as well as the steady-state and evolutive cases. In particular, in the context of dual-mixed formulations for (1.1)-(1.2), in [2] and [18] have been introduced augmented mixed formulations for the Boussinesq problem with temperature-dependent and constant viscosity, respectively. In both cases, the analysis is based on the introduction of a pseudostress tensor relating the diffusive and convective terms with the pressure and it is proved optimal convergence. In turn, in [19] and [15] the authors explore new numerical schemes for (1.1)-(1.2) considering constant (in [19]) and temperature-dependent viscosity (in [15]). There the authors introduce an alternative pseudostress tensor which allows them to derive a variational formulation with a skew-symmetric convective term. In this way, without augmenting the formulation as in [2] and [18], it is proved well-posedness and optimal convergence at the cost of not being able to utilize low order elements (Raviart-Thomas spaces of order $k \geq n-1$). Finally, the gradient of the velocity and the temperature are introduced in [24] to obtain a quasi-optimal mixed finite element method to approximate the solution of (1.1)-(1.2).

When the equations to be solved are conservation laws, specifically, conservation of mass, conservation of linear momentum, and conservation of energy as it is in this case, it is always desirable to employ numerical schemes respecting these laws. In this direction, in [1, 27] have been proposed two mass-conservative schemes to approximate the solution of the Boussinesq problem. In [27] the conservation of mass is numerically attained by utilizing the exactly divergence-free discontinuous Galerkin (DG) method proposed in [14] (see also [13]) for the discretization of fluid-flow problems. Later on, in [1] the authors consider a low order stabilized numerical scheme to discretize the fluid-flow equation and obtain the desired mass-conservative scheme. We emphasize that both works consider the temperature-dependent parameter case. We emphasize also that [27] has been replicated in [28] and [8] for the Boussinesq model with constant parameters and for double-diffusion equations in porous media, respectively.

Now, for flow problems in general, if the intention is to conserve momentum, probably one of the classical approaches to do that is the discretization by means of mixed finite element methods. In fact, since the equilibrium equation is discretized at the same time with the constitutive equation, by construction, they naturally conserve momentum. This is the case, for instance, of the pseudostress-based mixed method for the Navier-Stokes equation introduced in [9]. There, considering a non-standard mixed formulation posed in Banach spaces, it is proposed a new dual-mixed method for the Navier-Stokes problem where the pseudostress and the velocity are approximated using Raviart-Thomas elements of order k and discontinuous piecewise polynomials of degree k, respectively.

Going back to the Boussinesq equations, we observe that the mixed-type approaches [2] and [18] do not conserve momentum nor thermal energy because of the augmentation of the mixed formulation. The same lack of conservation of momentum and thermal energy can be observed in [19], [15] and [24] precisely because of the introduction of the aforementioned alternative pseudostress tensor (for [19], [15]) and the gradient of the velocity and the temperature (in [24]) as further unknowns.

Our main goal in this work is to extend the works [2, 18, 19, 15, 24] by introducing a new fully-mixed finite element method for the coupled system (1.1)-(1.2), allowing conservation of momentum and thermal energy. The latter is achieved by employing the pseudostress-based mixed formulation introduced in [9] for (1.1) and a similar approach for (1.2) based on the introduction of an additional vector unknown relating the gradient of the temperature with the convective term. In this way, the aforementioned pseudostress and vector unknowns, together with the velocity and the temperature, become the resulting unknowns of the coupled problem. As for the numerical scheme, the continuous problem is discretized by using a conforming scheme defined by Raviart-Thomas elements of order k for the pseudostress and vector unknowns and discontinuous piece-wise polynomials of degree k for the velocity and temperature. Since the resulting formulation is a nonlinear problem posed in nonstandard Banach spaces (due to the convective terms), for both, the continuous and discrete problems, we make use of the Banach–Nečas–Babuška and Banach's fixed point theorems to prove unique solvability. In addition, we show that the error decays with optimal rate of convergence. Further variables of interest, such as the fluid pressure, the fluid vorticity and the fluid velocity gradient, can be easily approximated as a simple postprocess of the finite element solution with the same rate of convergence.

The rest of this work is organized as follows. In Section 2, the fully-mixed formulation is proposed. Then, in Section 3 the well-posedness of the continuous problem is proved by means of the Banach–Nečas–Babuška and Banach's fixed point theorems. A similar argument is employed in Section 4, to prove the well-posedness of the Galerkin scheme. The corresponding a priori error estimates are derived in Section 5 and finally in Section 6 we present some numerical examples to validate the theoretical results and illustrate the good performance of our mixed finite element method.

We end this section by introducing some notation that will be used throughout the paper. Let us denote by $\Omega \subseteq \mathbb{R}^n$, $n \in \{2,3\}$, a given bounded domain with polyhedral boundary Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{r,p}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the spaces of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space S. In addition, we will denote by $\|(u,v)\| := \|(u,v)\|_{U\times V} := \|u\|_U + \|v\|_V$ the norm on the product space $U \times V$.

As usual $\mathbb{I}$ stands for the identity tensor in $\mathbb{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Also, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i\, w_j)_{i,j=1,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}.$$

We also recall the Hilbert space

$$\mathbf{H}(\operatorname{div};\Omega) := \big\{\mathbf{z}\in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{z}\in \mathrm{L}^2(\Omega)\big\},$$

with norm $\|\mathbf{z}\|^2_{\operatorname{div};\Omega} := \|\mathbf{z}\|^2_{0,\Omega} + \|\operatorname{div}\mathbf{z}\|^2_{0,\Omega}$, and introduce the tensor version of $\mathbf{H}(\operatorname{div};\Omega)$ given by

$$\mathbb{H}(\mathbf{div};\Omega) := \big\{\boldsymbol{\tau}\in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau}\in \mathbf{L}^2(\Omega)\big\},$$

whose norm will be denoted by $\|\cdot\|_{\mathbf{div};\Omega}$. Finally, given $p > \frac{2n}{n+2}$, in what follows we will also employ the non-standard Banach space $\mathbb{H}(\mathbf{div}_p,\Omega)$ defined by

$$\mathbb{H}(\mathbf{div}_p;\Omega) := \big\{\boldsymbol{\tau}\in \mathbb{L}^2(\Omega) \ : \ \mathbf{div}\,\boldsymbol{\tau}\in \mathbf{L}^p(\Omega)\big\},$$

endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} := \Big(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{\mathbf{L}^p(\Omega)}\Big)^{1/2}.$$

Finally, for any scalar function v, we define the sign function sgn, given by

$$\operatorname{sgn}(v) := \begin{cases} 1 & \text{if} \quad v \geq 0, \\ -1 & \text{if} \quad v < 0, \end{cases}$$

and observe that there holds $v\operatorname{sgn}(v) = |v|$.

2 The continuous weak formulation

In this section we derive the weak formulation for (1.1)-(1.2) which will allow us to propose later on our conforming scheme preserving linear momentum and thermal energy. To that end, and similarly to [9] and [18] (see also [11]) we introduce the tensor and vector variables

$$\boldsymbol{\sigma} := \nu\,\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega,$$

and

$$\boldsymbol{\rho} := \kappa\,\nabla\theta - \theta\,\mathbf{u} \quad \text{in} \quad \Omega,$$

and utilize the incompressibility condition div $\mathbf{u} = \mathrm{tr}\,(\nabla\,\mathbf{u}) = 0$ to rewrite the systems (1.1) and (1.2), respectively as the following equivalent first-order set of equations (see [9] and [18] for details):

$$\begin{gathered}\frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} + \frac{1}{\nu}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} = \nabla\mathbf{u} \quad \text{in} \quad \Omega, \quad \mathbf{div}\,\boldsymbol{\sigma} + \theta\,\mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega,\\ p = -\frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0,\end{gathered} \tag{2.1}$$

and

$$\begin{gathered}\kappa^{-1}\boldsymbol{\rho} + \kappa^{-1}\theta\,\mathbf{u} = \nabla\theta \quad \text{in} \quad \Omega, \quad \mathrm{div}\,\boldsymbol{\rho} = 0 \quad \text{in} \quad \Omega,\\ \theta = \theta_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \quad \boldsymbol{\rho}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}.\end{gathered} \tag{2.2}$$

Note that the third equation in (2.1) allows us to eliminate the pressure p from the system (which anyway can be approximated later on through a post-processing procedure), whereas the last equation takes care of the requirement that $\displaystyle\int_\Omega p = 0$.

Now, to derive our variational formulation, we begin by proceeding analogously to [9] for the first and second equations of (2.1), that is, we multiply the first equation of (2.1) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}\,;\Omega)$, integrate by parts, employ the identity $\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau} = \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}$ and the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on Γ, and test the second equation of (2.1) by $\mathbf{v} \in \mathbf{L}^4(\Omega)$, to obtain

$$\frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\boldsymbol{\tau} + \frac{1}{\nu}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau} = 0 \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}\,;\Omega), \tag{2.3}$$

and

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}\boldsymbol{\sigma} + \int_\Omega \theta\,\mathbf{g}\cdot\mathbf{v} = 0 \quad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega). \tag{2.4}$$

Next, for (2.2) we proceed similarly to (2.3)–(2.4). In fact, we define the Banach space

$$\mathbf{H} := \Big\{\boldsymbol{\eta} \in \mathbf{H}(\mathrm{div}_{4/3}\,;\Omega) : \quad \boldsymbol{\eta}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}\Big\},$$

and then, multiplying the first equation of (2.2) by $\boldsymbol{\eta} \in \mathbf{H}$ and integrating by parts, we get

$$\kappa^{-1}\int_\Omega \boldsymbol{\rho}\cdot\boldsymbol{\eta} + \int_\Omega \theta\,\mathrm{div}\,\boldsymbol{\eta} + \kappa^{-1}\int_\Omega \theta\,\mathbf{u}\cdot\boldsymbol{\eta} = \langle\boldsymbol{\eta}\cdot\mathbf{n}, \theta_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}} \quad \forall\,\boldsymbol{\eta} \in \mathbf{H}. \tag{2.5}$$

Observe that, similarly to [10, eq. (4.3)], it can be seen that for all $\boldsymbol{\eta} \in \mathbf{H}$, $\boldsymbol{\eta}\cdot\mathbf{n}|_{\Gamma_{\mathrm{D}}} \in \mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})$, thus the term $\langle\boldsymbol{\eta}\cdot\mathbf{n}, \theta_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}}$ is well defined.

In turn, the second equation of (2.2) is imposed weakly as

$$\int_\Omega \psi\,\mathrm{div}\,\boldsymbol{\rho} = 0 \quad \forall\,\psi \in \mathrm{L}^4(\Omega). \tag{2.6}$$

Notice that since $\mathbf{u} \in \mathbf{L}^4(\Omega)$ and since the term $\displaystyle\int_\Omega \boldsymbol{\rho}\cdot\boldsymbol{\eta}$ is well defined if $\boldsymbol{\rho}, \boldsymbol{\eta} \in \mathbf{L}^2(\Omega)$, the third term in the left-hand side of (2.5) forces θ, and consequently the test function ψ, to live in $\mathrm{L}^4(\Omega)$. This fact suggested the introduction of the space $\mathbf{H}$ for the unknown $\boldsymbol{\rho}$ and test $\boldsymbol{\eta}$.

According to the above, at first we are interested in finding $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_{4/3}\,;\Omega)$, $\mathbf{u} \in \mathbf{L}^4(\Omega)$, $\boldsymbol{\rho} \in \mathbf{H}$ and $\theta \in \mathrm{L}^4(\Omega)$, satisfying (2.3)–(2.6) and $\displaystyle\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$.

Now, let us define the space

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}\,\boldsymbol{\tau} = 0 \right\},$$

and recall that there holds (see e.g. [9, Section 3])

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{P}_0(\Omega)\,\mathbb{I}, \tag{2.7}$$

where $\mathrm{P}_0(\Omega)$ is the space of constant polynomials on Ω. More precisely, each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ can be decomposed uniquely as:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathbb{I}, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c := \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}\ \boldsymbol{\tau} \in \mathbb{R}. \tag{2.8}$$

Then, if we define the tensor

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \left(\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})\right)\mathbb{I}, \tag{2.9}$$

it follows that $\boldsymbol{\sigma}$ satisfies $\displaystyle\int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$ if and only if $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Moreover, from (2.7) it can be readily seen that equations (2.3) and (2.4) can be rewritten in terms of $\boldsymbol{\sigma}_0$ as follows

$$\frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}_0^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\boldsymbol{\tau} + \frac{1}{\nu}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau} = 0 \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}\,;\Omega), \tag{2.10}$$

and

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}\boldsymbol{\sigma}_0 + \int_\Omega \theta\,\mathbf{g}\cdot\mathbf{v} = 0 \quad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega). \tag{2.11}$$

Consequently, for the sake of the subsequent analysis we reformulate the system (2.3)–(2.6) considering $\boldsymbol{\sigma}_0$ defined in (2.9) as the tensor unknown and the equations (2.10) and (2.11) instead of (2.3) and (2.4), respectively. More precisely, denoting by

$$\mathbb{X} := \mathbb{H}(\mathbf{div}_{4/3};\Omega), \quad \mathbb{X}_0 := \mathbb{X}\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \quad \mathbf{M} := \mathbf{L}^4(\Omega) \quad \text{and} \quad \mathrm{Q} := \mathrm{L}^4(\Omega)$$

and introducing the forms $a_{\mathtt{F}} : \mathbb{X}\times\mathbb{X} \to \mathbb{R}, b_{\mathtt{F}} : \mathbb{X}\times\mathbf{M} \to \mathbb{R}, c_{\mathtt{F}} : \mathbf{M}\times\mathbf{M}\times\mathbb{X} \to \mathbb{R}, d_{\mathtt{F}} : \mathrm{Q}\times\mathbf{M} \to \mathbb{R}, a_{\mathtt{T}} : \mathbf{H}\times\mathbf{H} \to \mathbb{R}, b_{\mathtt{T}} : \mathbf{H}\times\mathrm{Q} \to \mathbb{R}$, and $c_{\mathtt{T}} : \mathbf{M}\times\mathrm{Q}\times\mathbf{H} \to \mathbb{R}$:

$$\begin{gathered}
a_{\mathtt{F}}(\boldsymbol{\sigma},\boldsymbol{\tau}) := \frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}, \quad b_{\mathtt{F}}(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}\boldsymbol{\tau}, \\
c_{\mathtt{F}}(\mathbf{w};\mathbf{u},\boldsymbol{\tau}) := \frac{1}{\nu}\int_\Omega (\mathbf{w}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau}, \quad d_{\mathtt{F}}(\theta,\mathbf{v}) := \int_\Omega \theta\,\mathbf{g}\cdot\mathbf{v}, \\
a_{\mathtt{T}}(\boldsymbol{\rho},\boldsymbol{\eta}) := \kappa^{-1}\int_\Omega \boldsymbol{\rho}\cdot\boldsymbol{\eta}, \quad b_{\mathtt{T}}(\boldsymbol{\eta},\theta) := \int_\Omega \theta\,\mathrm{div}\,\boldsymbol{\eta}, \\
c_{\mathtt{T}}(\mathbf{w};\theta,\boldsymbol{\eta}) := \kappa^{-1}\int_\Omega \theta\,\mathbf{w}\cdot\boldsymbol{\eta},
\end{gathered} \tag{2.12}$$

and the functional $F_{\mathtt{T}} \in \mathbf{H}'$:

$$F_{\mathtt{T}}(\boldsymbol{\eta}) := \langle \boldsymbol{\eta}\cdot\mathbf{n}, \theta_{\mathrm{D}}\rangle_{\Gamma_{\mathrm{D}}}, \tag{2.13}$$

we arrive at our fully-mixed variational formulation: Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta) \in \mathbb{X}_0 \times \mathbf{M} \times \mathbf{H} \times \mathrm{Q}$, such that:

$$
\begin{aligned}
a_{\mathtt{F}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b_{\mathtt{F}}(\boldsymbol{\tau}, \mathbf{u}) + c_{\mathtt{F}}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}) &= 0 && \forall\, \boldsymbol{\tau} \in \mathbb{X}_0, \\
b_{\mathtt{F}}(\boldsymbol{\sigma}, \mathbf{v}) + d_{\mathtt{F}}(\theta, \mathbf{v}) &= 0 && \forall\, \mathbf{v} \in \mathbf{M}, \\
a_{\mathtt{T}}(\boldsymbol{\rho}, \boldsymbol{\eta}) + b_{\mathtt{T}}(\boldsymbol{\eta}, \theta) + c_{\mathtt{T}}(\mathbf{u}; \theta, \boldsymbol{\eta}) &= F_{\mathtt{T}}(\boldsymbol{\eta}) && \forall\, \boldsymbol{\eta} \in \mathbf{H}, \\
b_{\mathtt{T}}(\boldsymbol{\rho}, \psi) &= 0 && \forall\, \psi \in \mathrm{Q},
\end{aligned}
\tag{2.14}
$$

where, for the sake of simplicity, the subscript 0 from the new unknown $\boldsymbol{\sigma}_0$ has been dropped.

Remark 2.1 *We observe here that, according to the third equation of* (2.1) *and the identity* (2.9)*, the pressure can be recovered in terms of the pseudostress* $\boldsymbol{\sigma} \in \mathbb{X}_0$ *and the velocity* $\mathbf{u} \in \mathbf{M}$*, as follows*

$$
p \;=\; -\frac{1}{n}\left(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{|\Omega|}\int_\Omega \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})\right). \tag{2.15}
$$

Moreover, one can compute further variables of interest, such as the shear-stress tensor $\widetilde{\boldsymbol{\sigma}} := \nu\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{t}}) - p\,\mathbb{I}$*, the vorticity* $\boldsymbol{\omega} := \dfrac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}})$*, the velocity gradient* $\nabla\,\mathbf{u}$ *and the heat-flux* $\widetilde{\boldsymbol{\rho}} := -\kappa\,\nabla\theta$*, with the following post-processing formulas*

$$
\begin{aligned}
\widetilde{\boldsymbol{\sigma}} \;&=\; \boldsymbol{\sigma}^{\mathrm{d}} \,+\, (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} + \boldsymbol{\sigma}^{t} + \mathbf{u}\otimes\mathbf{u} - \left(\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})\right)\mathbb{I}, \\
\boldsymbol{\omega} \;&=\; \frac{1}{2\,\nu}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{t}}\right), \\
\nabla\,\mathbf{u} \;&=\; \frac{1}{\nu}\left(\boldsymbol{\sigma}^{\mathrm{d}} + (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\right), \\
\widetilde{\boldsymbol{\rho}} \;&=\; -(\boldsymbol{\rho} + \theta\,\mathbf{u}).
\end{aligned}
\tag{2.16}
$$

3 Analysis of the coupled problem

In this section we combine the classical Banach–Nečas–Babuška and Banach fixed-point theorems to prove the well-posedness of (2.14) under a suitable smallness assumption on the data. We begin by establishing the stability properties of the forms involved.

3.1 Stability properties

We start by recalling the well-known Hölder inequality

$$
\int_\Omega |fg| \le \|f\|_{\mathrm{L}^p(\Omega)}\|g\|_{\mathrm{L}^q(\Omega)}, \quad \forall\, f \in \mathrm{L}^p(\Omega),\ \forall\, g \in \mathrm{L}^q(\Omega), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{3.1}
$$

In turn, we recall that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^p(\Omega)$ for $p \ge 1$ if $n = 2$ or $p \in [1, 6]$ if $n = 3$. More precisely, we have the following inequality

$$
\|w\|_{\mathrm{L}^p(\Omega)} \le C_{\mathrm{S}}\,\|w\|_{1,\Omega} \quad \forall\, w \in \mathrm{H}^1(\Omega), \tag{3.2}
$$

with $C_{\mathrm{S}} > 0$ depending only on $|\Omega|$ and p (see [31, Theorem 1.3.4]). Then, owing to the Hölder inequality (3.1) and simple computations, we deduce that the forms $a_{\mathtt{F}}, b_{\mathtt{F}}, c_{\mathtt{F}}, d_{\mathtt{F}}, a_{\mathtt{T}}, b_{\mathtt{T}}$ and $c_{\mathtt{T}}$ (cf. (2.12)) are bounded:

$$
\big|a_{\mathtt{F}}(\boldsymbol{\sigma}, \boldsymbol{\tau})\big| \le \frac{1}{\nu}\,\|\boldsymbol{\sigma}\|_{\mathbb{X}}\|\boldsymbol{\tau}\|_{\mathbb{X}}, \quad \big|b_{\mathtt{F}}(\boldsymbol{\tau}, \mathbf{v})\big| \le \|\boldsymbol{\tau}\|_{\mathbb{X}}\|\mathbf{v}\|_{\mathbf{M}}, \tag{3.3}
$$

$$
\left|c_{\mathtt{F}}(\mathbf{w};\mathbf{v},\boldsymbol{\tau})\right| \leq \frac{1}{\nu}\,\|\mathbf{w}\|_{\mathbf{M}}\|\mathbf{v}\|_{\mathbf{M}}\|\boldsymbol{\tau}\|_{\mathbb{X}}, \quad \left|d_{\mathtt{F}}(\theta,\mathbf{v})\right| \leq \|\mathbf{g}\|_{0,\Omega}\|\theta\|_{\mathrm{Q}}\|\mathbf{v}\|_{\mathbf{M}}, \tag{3.4}
$$

$$
\left|a_{\mathtt{T}}(\boldsymbol{\rho},\boldsymbol{\eta})\right| \leq \frac{1}{\kappa}\,\|\boldsymbol{\rho}\|_{\mathbf{H}}\|\boldsymbol{\eta}\|_{\mathbf{H}}, \quad \left|b_{\mathtt{T}}(\boldsymbol{\eta},\psi)\right| \leq \|\boldsymbol{\eta}\|_{\mathbf{H}}\|\psi\|_{\mathrm{Q}}, \tag{3.5}
$$

$$
\left|c_{\mathtt{T}}(\mathbf{w};\psi,\boldsymbol{\eta})\right| \leq \frac{1}{\kappa}\,\|\mathbf{w}\|_{\mathbf{M}}\|\psi\|_{\mathrm{Q}}\|\boldsymbol{\eta}\|_{\mathbf{H}}. \tag{3.6}
$$

On the other hand, analogously to [9, Lemma 3.5], we observe that the functional $F_{\mathtt{T}}$ (cf. (2.13)) is bounded

$$
\left|F_{\mathtt{T}}(\boldsymbol{\eta})\right| \leq C_F\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{\eta}\|_{\mathbf{H}} \quad \forall\,\boldsymbol{\eta}\in\mathbf{H}, \tag{3.7}
$$

with C_F a positive constant depending on C_{S} (cf. (3.2)).

Now, we let $\mathbb{V}$ and $\mathbf{V}$ be the kernel of $b_{\mathtt{F}}$ and $b_{\mathtt{T}}$, respectively, that is

$$
\mathbb{V} := \Big\{\boldsymbol{\tau}\in\mathbb{X}_0 : \quad b_{\mathtt{F}}(\boldsymbol{\tau},\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in\mathbf{M}\Big\} = \Big\{\boldsymbol{\tau}\in\mathbb{X}_0 : \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega\Big\}, \tag{3.8}
$$

and

$$
\mathbf{V} := \Big\{\boldsymbol{\eta}\in\mathbf{H} : \quad b_{\mathtt{T}}(\boldsymbol{\eta},\psi) = 0 \quad \forall\,\psi\in\mathrm{Q}\Big\} = \Big\{\boldsymbol{\eta}\in\mathbf{H} : \quad \mathrm{div}\;\boldsymbol{\eta} = 0 \quad \text{in} \quad \Omega\Big\}, \tag{3.9}
$$

and recall from [9, Lemma 3.2] that there exists $C_{\mathrm{d}} > 0$, such that

$$
C_{\mathrm{d}}\,\|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathbf{L}^{4/3}(\Omega)}^2 \quad \forall\,\boldsymbol{\tau}\in\mathbb{X}_0. \tag{3.10}
$$

From (3.10) we easily realize that $a_{\mathtt{F}}$ satisfies

$$
a_{\mathtt{F}}(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \frac{C_{\mathrm{d}}}{\nu}\,\|\boldsymbol{\tau}\|_{\mathbb{X}}^2 \quad \forall\,\boldsymbol{\tau}\in\mathbb{V}, \tag{3.11}
$$

whereas for $a_{\mathtt{T}}$ we proceed similarly to [10, Lemma 2.2] to obtain

$$
a_{\mathtt{T}}(\boldsymbol{\eta},\boldsymbol{\eta}) \geq \frac{1}{\kappa}\|\boldsymbol{\eta}\|_{\mathbf{H}}^2 \quad \forall\,\boldsymbol{\eta}\in\mathbf{V}. \tag{3.12}
$$

Now, we recall from [9, Lemma 3.4] that $b_{\mathtt{F}}$ satisfies the inf-sup condition:

$$
\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{X}_0} \frac{b_{\mathtt{F}}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \beta_{\mathtt{F}}\,\|\mathbf{v}\|_{\mathbf{M}} \quad \forall\,\mathbf{v}\in\mathbf{M}. \tag{3.13}
$$

Similarly, we can obtain an analogous result for $b_{\mathtt{T}}$. This is established in the next lemma.

Lemma 3.1

$$
\sup_{\mathbf{0}\neq\boldsymbol{\eta}\in\mathbf{H}} \frac{b_{\mathtt{T}}(\boldsymbol{\eta},\psi)}{\|\boldsymbol{\eta}\|_{\mathbf{H}}} \geq \beta_{\mathtt{T}}\,\|\psi\|_{\mathrm{Q}} \quad \forall\,\psi\in\mathrm{Q}. \tag{3.14}
$$

Proof. Given $\psi\in\mathrm{L}^4(\Omega)$, we consider the variational problem

$$
-\Delta z = \mathrm{sgn}(\psi)|\psi|^3 \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}, \quad \nabla z\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}},
$$

and proceed analogously to the proof of [10, Lemma 2.1] to obtain the desired result. We omit further details. $\square$

Using the aforementioned stability properties, particularly (3.3), (3.11) and (3.13), and applying [23, Proposition 2.36] it is not difficult to see that the bilinear form $\mathcal{A}_{\mathtt{F}} : (\mathbb{X}\times\mathbf{M})\times(\mathbb{X}\times\mathbf{M})\to\mathbb{R}$ defined by

$$
\mathcal{A}_{\mathtt{F}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := a_{\mathtt{F}}(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_{\mathtt{F}}(\boldsymbol{\tau},\mathbf{u}) + b_{\mathtt{F}}(\boldsymbol{\sigma},\mathbf{v}), \tag{3.15}
$$

satisfies:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathcal{A}_{\mathtt{F}}((\boldsymbol{\zeta},\mathbf{z}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \geq \gamma_{\mathtt{F}}\,\|(\boldsymbol{\zeta},\mathbf{z})\| \quad \forall\,(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M}, \tag{3.16}$$

where $\gamma_{\mathtt{F}}$ is the positive constant defined by

$$\gamma_{\mathtt{F}} := C\frac{\min\{1,\nu\beta_{\mathtt{F}}\}}{\nu\beta_{\mathtt{F}}+1}, \tag{3.17}$$

with $C>0$ independent of ν.

Finally, and analogously to (3.16) we can obtain from [23, Proposition 2.36] that estimates (3.5), (3.12) and (3.14) imply that the bilinear form $\mathcal{A}_{\mathtt{T}} : (\mathbf{H}\times\mathrm{Q})\times(\mathbf{H}\times\mathrm{Q})\to\mathbb{R}$, defined by

$$\mathcal{A}_{\mathtt{T}}((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\psi)) := a_{\mathtt{T}}(\boldsymbol{\rho},\boldsymbol{\eta}) + b_{\mathtt{T}}(\boldsymbol{\eta},\theta) + b_{\mathtt{T}}(\boldsymbol{\rho},\psi), \quad \forall\,(\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}, \tag{3.18}$$

satisfies the inf-sup condition:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}} \frac{\mathcal{A}_{\mathtt{T}}((\boldsymbol{\varsigma},\varphi),(\boldsymbol{\eta},\psi))}{\|(\boldsymbol{\eta},\psi)\|} \geq \gamma_{\mathtt{T}}\,\|(\boldsymbol{\varsigma},\varphi)\| \quad \forall\,(\boldsymbol{\varsigma},\varphi)\in\mathbf{H}\times\mathrm{Q}, \tag{3.19}$$

where $\gamma_{\mathtt{T}}$ is the positive constant defined by

$$\gamma_{\mathtt{T}} := \frac{\kappa\,\beta_{\mathtt{T}}^2}{\kappa^2\,\beta_{\mathtt{T}}^2 + 4\,\kappa\,\beta_{\mathtt{T}} + 2}. \tag{3.20}$$

3.2 The fixed-point operator

Here, proceed similarly to [3] and [17] and describe the fixed-point strategy to be employed next to prove the well-posedness of (2.14). We start by introducing the associated fixed-point operator. To that end we define the auxiliary operators $\mathbf{R} : \mathbf{W}\times\mathrm{Q}\subseteq\mathbf{M}\times\mathrm{Q}\to\mathbb{X}_0\times\mathbf{M}$ and $\mathbf{S} : \mathbf{W}\subseteq\mathbf{M}\to\mathbf{H}\times\mathrm{Q}$ given by

$$\mathbf{R}(\mathbf{w},\phi) := (\mathbf{R}_1(\mathbf{w},\phi),\mathbf{R}_2(\mathbf{w},\phi)) = (\boldsymbol{\sigma},\mathbf{u}) \quad \forall\,(\mathbf{w},\phi)\in\mathbf{W}\times\mathrm{Q}, \tag{3.21}$$

with $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{X}_0\times\mathbf{M}$ satisfying

$$\begin{aligned} a_{\mathtt{F}}(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_{\mathtt{F}}(\boldsymbol{\tau},\mathbf{u}) + c_{\mathtt{F}}(\mathbf{w};\mathbf{u},\boldsymbol{\tau}) &= 0 && \forall\,\boldsymbol{\tau}\in\mathbb{X}_0,\\ b_{\mathtt{F}}(\boldsymbol{\sigma},\mathbf{v}) &= -d_{\mathtt{F}}(\phi,\mathbf{v}) && \forall\,\mathbf{v}\in\mathbf{M}. \end{aligned} \tag{3.22}$$

and

$$\mathbf{S}(\mathbf{w}) := (\mathbf{S}_1(\mathbf{w}),\mathbf{S}_2(\mathbf{w})) = (\boldsymbol{\rho},\theta) \quad \forall\,\mathbf{w}\in\mathbf{W}, \tag{3.23}$$

where $(\boldsymbol{\rho},\theta)\in\mathbf{H}\times\mathrm{Q}$ is such that

$$\begin{aligned} a_{\mathtt{T}}(\boldsymbol{\rho},\boldsymbol{\eta}) + b_{\mathtt{T}}(\boldsymbol{\eta},\theta) + c_{\mathtt{T}}(\mathbf{w};\theta,\boldsymbol{\eta}) &= F_{\mathtt{T}}(\boldsymbol{\eta}) && \forall\,\boldsymbol{\eta}\in\mathbf{H},\\ b_{\mathtt{T}}(\boldsymbol{\rho},\psi) &= 0 && \forall\,\psi\in\mathrm{Q}. \end{aligned} \tag{3.24}$$

Above, $\mathbf{W}$ is a bounded set (to be specified next) ensuring the well-definiteness of $\mathbf{R}$ and $\mathbf{S}$.

By virtue of the above, by defining the operator $\mathcal{J} : \mathbf{W}\subseteq\mathbf{M}\to\mathbf{M}$ as

$$\mathcal{J}(\mathbf{w}) := \mathbf{R}_2(\mathbf{w},\mathbf{S}_2(\mathbf{w})) \quad \forall\,\mathbf{w}\in\mathbf{W}, \tag{3.25}$$

it is clear that $(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\rho},\theta)$ is a solution to (2.14) if and only if $\mathbf{u}$ satisfies $\mathcal{J}(\mathbf{u})=\mathbf{u}$, and consequently, the well-posedness of (2.14) is equivalent to the unique solvability of the fixed-point problem: Find $\mathbf{u}\in\mathbf{M}$ such that

$$\mathcal{J}(\mathbf{u}) = \mathbf{u}. \tag{3.26}$$

In this way, in what follows we focus on proving the unique solvability of (3.26). Before doing that, we have to provide a suitable choice of $\mathbf{W}$ ensuring the well-definiteness of $\mathcal{J}$.

3.3 Well-definiteness of $\mathcal{J}$

Since operator $\mathcal{J}$ is defined in terms of $\mathbf{R}$ and $\mathbf{S}$, first we must study the well-definiteness of both operators, which evidently is equivalently to study the well-posedness of (3.22) and (3.24). We begin by analyzing the well-posedness of (3.22).

Lemma 3.2 *Let* $(\mathbf{w},\phi)\in\mathbf{M}\times\mathrm{Q}$ *and assume that*

$$\|\mathbf{w}\|_{\mathbf{M}} \le \frac{\nu\gamma_{\mathrm{F}}}{2}, \tag{3.27}$$

with γ_{F} *the positive constant in* (3.17). *Then, there exists a unique* $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{X}_0\times\mathbf{W}$ *solution to* (3.22). *In addition, there holds*

$$\|(\boldsymbol{\sigma},\mathbf{u})\| \le \frac{2}{\gamma_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\phi\|_{\mathrm{Q}}. \tag{3.28}$$

Proof. We proceed similarly as in the proof of [9, Theorem 3.6]. In fact, given $(\mathbf{w},\phi)\in\mathbf{M}\times\mathrm{Q}$, we begin by defining the bilinear form:

$$\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := \mathcal{A}_{\mathrm{F}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) \,+\, c_{\mathrm{F}}(\mathbf{w};\mathbf{u},\boldsymbol{\tau}), \tag{3.29}$$

where $\mathcal{A}_{\mathrm{F}}$ and c_{F} are the forms defined in (3.15) and (2.12), respectively, that is

$$\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := a_{\mathrm{F}}(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_{\mathrm{F}}(\boldsymbol{\tau},\mathbf{u}) + b_{\mathrm{F}}(\boldsymbol{\sigma},\mathbf{v}) + c_{\mathrm{F}}(\mathbf{w};\mathbf{u},\boldsymbol{\tau}).$$

Then, problem (3.22) can be rewritten equivalently as: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{X}_0\times\mathbf{M}$, such that

$$\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) = -d_{\mathrm{F}}(\phi,\mathbf{v}) \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}. \tag{3.30}$$

Therefore, to prove the well-definiteness of $\mathbf{R}$, in the sequel we equivalently prove that problem (3.30) is well-posed by means of the Banach–Nečas–Babuška theorem (see, for instance [23, Theorem 2.6]).

First, given $(\boldsymbol{\zeta},\mathbf{z}),(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})\in\mathbb{X}_0\times\mathbf{M}$ with $(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})\neq\mathbf{0}$, from (3.4) we observe that

$$\begin{aligned}\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\frac{\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\zeta},\mathbf{z}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} &\ge \frac{\big|\mathcal{A}_{\mathrm{F}}((\boldsymbol{\zeta},\mathbf{z}),(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}}))\big|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})\|} - \frac{\big|c_{\mathrm{F}}(\mathbf{w};\mathbf{z},\widehat{\boldsymbol{\tau}})\big|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})\|}\\ &\ge \frac{\big|\mathcal{A}_{\mathrm{F}}((\boldsymbol{\zeta},\mathbf{z}),(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}}))\big|}{\|(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})\|} - \frac{1}{\nu}\,\|\mathbf{w}\|_{\mathbf{M}}\|(\boldsymbol{\zeta},\mathbf{z})\|,\end{aligned}$$

which together with (3.16) and the fact that $(\widehat{\boldsymbol{\tau}},\widehat{\mathbf{v}})$ is arbitrary, implies

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\frac{\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\zeta},\mathbf{z}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \ge \left(\gamma_{\mathrm{F}} - \frac{1}{\nu}\,\|\mathbf{w}\|_{\mathbf{M}}\right)\|(\boldsymbol{\zeta},\mathbf{z})\|.$$

Hence, using the fact that $\mathbf{w}\in\mathbf{M}$ satisfies (3.27), we easily obtain

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\frac{\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\zeta},\mathbf{z}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \ge \frac{\gamma_{\mathrm{F}}}{2}\,\|(\boldsymbol{\zeta},\mathbf{z})\| \quad \forall\,(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M}. \tag{3.31}$$

On the other hand, for a given $(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M}$, we observe that

$$\begin{aligned}\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) &\ge \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\frac{\mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &= \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}}\frac{\mathcal{A}_{\mathrm{F}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) + c_{\mathrm{F}}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})}{\|(\boldsymbol{\tau},\mathbf{v})\|},\end{aligned}$$

which together with (3.4) implies

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \geq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{\mathcal{A}_{\mathrm{F}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} - \frac{1}{\nu}\,\|\mathbf{w}\|_{\mathbf{M}}\|(\boldsymbol{\zeta},\mathbf{z})\|. \tag{3.32}$$

Therefore, using the fact that $\mathcal{A}_{\mathrm{F}}(\cdot,\cdot)$ is symmetric, from (3.16) and (3.32) we obtain

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \geq \left(\gamma_{\mathrm{F}} - \frac{1}{\nu}\,\|\mathbf{w}\|_{\mathbf{M}}\right)\|(\boldsymbol{\zeta},\mathbf{z})\|,$$

which combined with (3.27), yields

$$\sup_{(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \mathcal{A}_{\mathrm{F},\mathbf{w}}((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\zeta},\mathbf{z})) \geq \frac{\gamma_{\mathrm{F}}}{2}\,\|(\boldsymbol{\zeta},\mathbf{z})\| > 0 \quad \forall\,(\boldsymbol{\zeta},\mathbf{z})\in\mathbb{X}_0\times\mathbf{M},\ (\boldsymbol{\zeta},\mathbf{z})\neq\mathbf{0}. \tag{3.33}$$

In this way, from (3.31) and (3.33) we obtain that $\mathcal{A}_{\mathrm{F},\mathbf{w}}(\cdot,\cdot)$ satisfies the hypotheses of the Banach–Nečas–Babuška theorem [23, Theorem 2.6], which allows us to conclude the existence of a unique $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{X}_0\times\mathbf{M}$ solution to (3.22), or equivalently, the existence of a unique $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{X}_0\times\mathbf{M}$ such that $\mathbf{R}(\mathbf{w},\phi) = (\boldsymbol{\sigma},\mathbf{u})$. Finally, from (3.30), using (3.31) with $(\boldsymbol{\zeta},\mathbf{z}) = (\boldsymbol{\sigma},\mathbf{u})$, the bound of d_{F} (cf. (3.4)), we readily obtain (3.28), which concludes the proof. □

Next, we provide the well-definiteness of $\mathbf{S}$, or equivalently, the well-posedness of (3.24).

Lemma 3.3 *Let $\mathbf{w}\in\mathbf{M}$ and assume that*

$$\|\mathbf{w}\|_{\mathbf{M}} \leq \frac{\kappa\gamma_{\mathrm{T}}}{2}. \tag{3.34}$$

Then, there exists a unique $(\boldsymbol{\rho},\theta)\in\mathbf{H}\times\mathrm{Q}$ solution to (3.24)*. Moreover, there holds*

$$\|(\boldsymbol{\rho},\theta)\| \leq \frac{2\,C_F}{\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}, \tag{3.35}$$

with C_F and γ_{T} the positive constants in (3.7) *and* (3.20)*, respectively.*

Proof. The proof follows analogously to the proof of Lemma 3.2 (see also [9, Theorem 3.6]). In fact, by defining the bilinear form:

$$\mathcal{A}_{\mathrm{T},\mathbf{w}}((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\psi)) := \mathcal{A}_{\mathrm{T}}((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\psi)) + c_{\mathrm{T}}(\mathbf{w};\theta,\boldsymbol{\eta}), \tag{3.36}$$

where $\mathcal{A}_{\mathrm{T}}$ and c_{T} are the forms defined in (3.18) and (2.12) respectively, we observe that problem (3.24) can be rewritten equivalently as: Find $(\boldsymbol{\rho},\theta)\in\mathbf{H}\times\mathrm{Q}$, such that

$$\mathcal{A}_{\mathrm{T},\mathbf{w}}((\boldsymbol{\rho},\theta),(\boldsymbol{\eta},\psi)) = F(\boldsymbol{\eta}) \quad \forall\,(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}. \tag{3.37}$$

In turn, using (3.6), (3.19) and (3.34), it can be easily deduced that $\mathcal{A}_{\mathrm{T},\mathbf{w}}$ satisfies

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}} \frac{\mathcal{A}_{\mathrm{T},\mathbf{w}}((\boldsymbol{\varsigma},\varphi),(\boldsymbol{\eta},\psi))}{\|(\boldsymbol{\eta},\psi)\|} \geq \frac{\gamma_{\mathrm{T}}}{2}\,\|(\boldsymbol{\varsigma},\varphi)\| \quad \forall\,(\boldsymbol{\varsigma},\varphi)\in\mathbf{H}\times\mathrm{Q}, \tag{3.38}$$

and

$$\sup_{(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}} \mathcal{A}_{\mathrm{T},\mathbf{w}}((\boldsymbol{\eta},\psi),(\boldsymbol{\varsigma},\varphi)) > 0 \quad \forall\,(\boldsymbol{\varsigma},\varphi)\in\mathbf{H}\times\mathrm{Q},\ (\boldsymbol{\varsigma},\varphi)\neq\mathbf{0},$$

which together with the Banach–Nečas–Babuška theorem imply the well–posedness of (3.24). Finally, from (3.37), applying (3.38) with $(\boldsymbol{\varsigma},\varphi) = (\boldsymbol{\rho},\theta)$ and the bound (3.7), we readily obtain (3.35). □

From Lemmas 3.2 and 3.3 we automatically deduce that if the set $\mathbf{W}$ defining $\mathbf{R}$ and $\mathbf{S}$ (cf. (3.21) and (3.23)) is such that

$$\mathbf{W} \subseteq \overline{B\left(\mathbf{0}, \frac{\nu\gamma_{\mathrm{F}}}{2}\right)} \cap \overline{B\left(\mathbf{0}, \frac{\kappa\gamma_{\mathrm{T}}}{2}\right)} = \overline{B\left(\mathbf{0}, \frac{\lambda}{2}\right)},$$

with $\lambda := \min\left\{\nu\,\gamma_{\mathrm{F}}, \kappa\,\gamma_{\mathrm{T}}\right\}$, then $\mathbf{R}$ and $\mathbf{S}$, thus $\mathscr{J}$ (cf. (3.25)), are well-defined. Moreover, from (3.28) and (3.35) we readily obtain that there hold, respectively

$$\|\mathbf{R}_2(\mathbf{w},\phi)\|_{\mathbf{M}} \leq \frac{2}{\gamma_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\phi\|_{\mathrm{Q}} \quad \forall\,(\mathbf{w},\phi) \in \mathbf{W}\times \mathrm{Q},$$

and

$$\|\mathbf{S}_2(\mathbf{w})\|_{\mathrm{Q}} \leq \frac{2\,C_F}{\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \quad \forall\,\mathbf{w}\in\mathbf{W}, \tag{3.39}$$

which combined imply

$$\|\mathscr{J}(\mathbf{w})\|_{\mathbf{M}} = \|\mathbf{R}_2(\mathbf{w},\mathbf{S}_2(\mathbf{w}))\|_{\mathbf{M}} \leq \frac{2}{\gamma_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\mathbf{S}_2(\mathbf{w})\|_{\mathrm{Q}} \leq \frac{4\,C_F}{\gamma_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}.$$

As a consequence of the above, if we define the bounded set $\mathbf{W}$ as follows

$$\mathbf{W} := \left\{\mathbf{w}\in\mathbf{M} : \quad \|\mathbf{w}\|_{\mathbf{M}} \leq \frac{4\,C_F}{\gamma_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\}, \tag{3.40}$$

and assume that the data satisfies,

$$\frac{8\,C_F}{\lambda\,\gamma_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \leq 1, \tag{3.41}$$

then we clearly deduce that the fixed-point operator $\mathscr{J}$ is well-defined and satistfies $\mathscr{J}(\mathbf{W}) \subseteq \mathbf{W}$. The above is summarize in the following result.

Theorem 3.1 *Let define the bounded set* $\mathbf{W}$ *as in* (3.40) *and assume that the data satisfies* (3.41)*. Then,* $\mathscr{J}$ *is well-defined and satisfies* $\mathscr{J}(\mathbf{W}) \subseteq \mathbf{W}$*.*

3.4 Solvability analysis of the fixed-point equation

Here we provide the main result of this section, namely, existence and uniqueness of solution of problem (2.14). We begin by establishing two lemmas that will allow us to derive conditions under which operator $\mathscr{J}$ is a contraction mapping.

Lemma 3.4 *Assume that the data satisfies* (3.41)*. Then, there holds*

$$\|\mathbf{R}(\mathbf{w}_1,\phi_1) - \mathbf{R}(\mathbf{w}_2,\phi_2)\| \leq \frac{4}{\nu\,\gamma_{\mathrm{F}}^2}\,\|\mathbf{g}\|_{0,\Omega}\|\phi_2\|_{\mathrm{Q}}\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{M}} + \frac{2}{\gamma_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\phi_1-\phi_2\|_{\mathrm{Q}}, \tag{3.42}$$

for all $(\mathbf{w}_1,\phi_1),(\mathbf{w}_2,\phi_2)\in\mathbf{W}\times\mathrm{Q}$*, with* γ_{F} *the positive constant defined in* (3.17)*.*

Proof. Given $(\mathbf{w}_1,\phi_1),(\mathbf{w}_2,\phi_2)\in\mathbf{W}\times\mathrm{Q}$, we let $(\boldsymbol{\sigma}_1,\mathbf{u}_1)$, $(\boldsymbol{\sigma}_2,\mathbf{u}_2)\in\mathbb{X}_0\times\mathbf{M}$, such that $\mathbf{R}(\mathbf{w}_1,\phi_1) = (\boldsymbol{\sigma}_1,\mathbf{u}_1)$ and $\mathbf{R}(\mathbf{w}_2,\phi_2) = (\boldsymbol{\sigma}_2,\mathbf{u}_2)$. Then, from the definition of $\mathbf{R}$ and $\mathscr{A}_{\mathrm{F},\mathbf{w}}$ (cf. (3.21) and (3.29)), and after simple computations, we obtain

$$\mathscr{A}_{\mathrm{F},\mathbf{w}_1}((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2,\mathbf{u}_1-\mathbf{u}_2),(\boldsymbol{\tau},\mathbf{v})) = -c_{\mathrm{F}}(\mathbf{w}_1-\mathbf{w}_2;\mathbf{u}_2,\boldsymbol{\tau}) - d_{\mathrm{F}}(\phi_1-\phi_2,\mathbf{v}).$$

Hence, we employ (3.31) with $(\boldsymbol{\zeta},\mathbf{z}) = (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2)$, the upper bounds of c_{F} and d_{F} (cf. (3.4)), and the fact that $\|\mathbf{u}_2\|_{\mathbf{M}} \le \frac{2}{\gamma_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\phi_2\|_{\mathrm{Q}}$ (cf. (3.28)), to deduce

$$
\begin{aligned}
\frac{\gamma_{\mathrm{F}}}{2}\,\|(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2)\| &\le \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{X}_0\times\mathbf{M}} \frac{-c_{\mathrm{F}}(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{u}_2, \boldsymbol{\tau}) - d_{\mathrm{F}}(\phi_1 - \phi_2, \mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \\
&\le \frac{1}{\nu}\,\|\mathbf{u}_2\|_{\mathbf{M}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}} + \|\mathbf{g}\|_{0,\Omega}\|\phi_1 - \phi_2\|_{\mathrm{Q}} \\
&\le \frac{2}{\nu\,\gamma_{\mathrm{F}}}\,\|\mathbf{g}\|_{0,\Omega}\|\phi_2\|_{\mathrm{Q}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}} + \|\mathbf{g}\|_{0,\Omega}\|\phi_1 - \phi_2\|_{\mathrm{Q}},
\end{aligned}
$$

which implies (3.42). □

Lemma 3.5 *Assume that the data satisfies* (3.41)*. Then, there holds*

$$
\|\mathbf{S}(\mathbf{w}_1) - \mathbf{S}(\mathbf{w}_2)\|_{\mathrm{Q}} \le \frac{4\,C_F}{\kappa\,\gamma_{\mathrm{T}}^2}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}}, \tag{3.43}
$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$*, with* C_F *and* γ_{T} *the positive constants in* (3.7) *and* (3.20)*.*

Proof. Given $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$, we let $(\boldsymbol{\rho}_1,\theta_1),(\boldsymbol{\rho}_2,\theta_2) \in \mathbf{H}\times\mathrm{Q}$ be such that $\mathbf{S}(\mathbf{w}_1) = (\boldsymbol{\rho}_1,\theta_1)$ and $\mathbf{S}(\mathbf{w}_2) = (\boldsymbol{\rho}_2,\theta_2)$. Then, from the definitions of $\mathbf{S}$ and $\mathcal{A}_{\mathrm{T},\mathbf{w}}$ (cf. (3.23) and (3.36)), and after simple computations, we deduce that

$$
\mathcal{A}_{\mathrm{T},\mathbf{w}_1}((\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \theta_1 - \theta_2),(\boldsymbol{\eta},\psi)) = -c_{\mathrm{T}}(\mathbf{w}_1 - \mathbf{w}_2; \theta_2, \boldsymbol{\eta}).
$$

Thus, employing (3.38) with $(\boldsymbol{\varsigma},\varphi) = (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \theta_1 - \theta_2)$, the upper bound of c_{T} (cf. (3.6)), and the fact that $\|\theta_2\|_{\mathrm{Q}} \le \frac{2\,C_F}{\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}$ (cf. (3.35)), we get

$$
\begin{aligned}
\frac{\gamma_{\mathrm{T}}}{2}\,\|(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \theta_1 - \theta_2)\| &\le \sup_{\mathbf{0}\neq(\boldsymbol{\eta},\psi)\in\mathbf{H}\times\mathrm{Q}} \frac{-c_{\mathrm{T}}(\mathbf{w}_1 - \mathbf{w}_2; \theta_2, \boldsymbol{\eta})}{\|(\boldsymbol{\eta},\psi)\|} \\
&\le \frac{1}{\kappa}\,\|\theta_2\|_{\mathrm{Q}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}} \\
&\le \frac{2\,C_F}{\kappa\,\gamma_{\mathrm{T}}}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}},
\end{aligned}
$$

which implies (3.43). □

We are ready now to prove the main result of this section, that is, the existence and uniqueness of solution of problem (2.14).

Theorem 3.2 *Let define* $\lambda := \min\left\{\nu\,\gamma_{\mathrm{F}}, \kappa\,\gamma_{\mathrm{T}}\right\}$ *and assume that*

$$
\frac{16\,C_F}{\lambda\,\gamma_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} < 1. \tag{3.44}
$$

Then, the operator $\mathcal{J}$ *(cf.* (3.25)*) has a unique fixed-point* $\mathbf{u}$ *in* $\mathbf{W}$*. Equivalently, the coupled problem* (2.14) *has a unique solution* $(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\rho},\theta) \in \mathbb{X}_0\times\mathbf{M}\times\mathbf{H}\times\mathrm{Q}$ *with* $\mathbf{u}\in\mathbf{W}$*. Moreover, there hold*

$$
\|(\boldsymbol{\sigma},\mathbf{u})\| \le \frac{4\,C_F}{\gamma_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \quad \textit{and} \quad \|(\boldsymbol{\rho},\theta)\| \le \frac{2\,C_F}{\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}. \tag{3.45}
$$

Proof. We begin by recalling from the previous analysis that assumption (3.44) ensures the well-definedness of $\mathcal{J}$. Now, let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{W}$, be such that $\mathbf{u}_1 = \mathcal{J}(\mathbf{w}_1)$ and $\mathbf{u}_2 = \mathcal{J}(\mathbf{w}_2)$. According to the definition of $\mathcal{J}$ (cf. (3.25)), from estimates (3.39), (3.42) and (3.43), we deduce that

$$
\begin{aligned}
&\|\mathcal{J}(\mathbf{w}_1) - \mathcal{J}(\mathbf{w}_2)\|_{\mathbf{M}} = \|\mathbf{R}_2(\mathbf{w}_1, \mathbf{S}_2(\mathbf{w}_1)) - \mathbf{R}_2(\mathbf{w}_2, \mathbf{S}_2(\mathbf{w}_2))\|_{\mathbf{M}} \\
&\le \frac{4}{\nu\,\gamma_{\mathtt{F}}^2}\,\|\mathbf{g}\|_{0,\Omega}\|\mathbf{S}_2(\mathbf{w}_2)\|_{\mathrm{Q}}\,\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}} + \frac{2}{\gamma_{\mathtt{F}}}\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{S}_2(\mathbf{w}_1) - \mathbf{S}_2(\mathbf{w}_2)\|_{\mathrm{Q}} \\
&\le \frac{16\,C_F}{\lambda\gamma_{\mathtt{F}}\gamma_{\mathtt{T}}}\,\|\mathbf{g}\|_{0,\Omega}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}},
\end{aligned}
$$

which together with (3.44) and the Banach's fixed point theorem implies that $\mathcal{J}$ has a unique fixed-point in $\mathbf{W}$, which equivalently implies that there exists a unique $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta) \in \mathbb{X}_0 \times \mathbf{M} \times \mathbf{H} \times \mathrm{Q}$ solution to (2.14) with $\mathbf{u} \in \mathbf{W}$. Finally, since $(\boldsymbol{\sigma}, \mathbf{u})$ satisfies (3.22) with $\phi = \theta$ and $\mathbf{w} = \mathbf{u} \in \mathbf{W}$, and (ρ, θ) satisfies (3.24), with $\mathbf{w} = \mathbf{u} \in \mathbf{W}$, the estimates in (3.45) follow from (3.28) and (3.35). □

4 Galerkin scheme

In this section we introduce and analyze the Galerkin scheme of problem (2.14). We mention in advance that the well-posedness analysis follows straightforwardly by adapting the results derived for the continuous problem to the discrete case, reason why most of the details are omitted.

4.1 The discrete coupled system and its well-posedness

Let us begin by considering $\{\mathcal{T}_h\}_{h>0}$ a family of regular triangulation of $\overline{\Omega}$ made by triangles T when $n=2$ (or tetrahedra when $n=3$) of diameter h_T and define the meshsize $h := \max\{h_T \,:\, T \in \mathcal{T}_h\}$. Given an integer $l \ge 0$ and a subset S of $\mathbb{R}^n$, we denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S. Hence, for each integer $k \ge 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as (see, for instance [7]):

$$
\mathbf{RT}_k(T) := [P_k(T)]^n \oplus \widetilde{P}_k(T)\mathbf{x},
$$

where $\mathbf{x} := (x_1, \dots, x_n)^t$ is a generic vector of $\mathbb{R}^n$ and $\widetilde{P}_k(T)$ is the space of polynomials of total degree equal to k defined on T. In this way, we define the finite element subspaces:

$$
\begin{aligned}
\mathbb{X}_h &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{X} : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_T \in \mathbf{RT}_k(T) \quad \forall\,\mathbf{c} \in \mathbb{R}^n \quad \forall\,T \in \mathcal{T}_h\Big\}, \\
\mathbf{M}_h &:= \{\mathbf{v}_h \in \mathbf{M} : \quad \mathbf{v}_h|_T \in [P_k(T)]^n \quad \forall\,T \in \mathcal{T}_h\}, \\
\mathbf{H}_h &:= \Big\{\boldsymbol{\eta}_h \in \mathbf{H} : \quad \boldsymbol{\eta}_h|_T \in \mathbf{RT}_k(T) \quad \forall\,T \in \mathcal{T}_h\Big\}, \\
\mathrm{Q}_h &:= \{\phi_h \in \mathrm{Q} : \quad \phi_h|_T \in P_k(T) \quad \forall\,T \in \mathcal{T}_h\}.
\end{aligned}
$$

Then defining the subspace $\mathbb{X}_{h,0} := \mathbb{X}_h \cap \mathbb{X}_0$, the Galerkin scheme associated to problem (2.14) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ such that:

$$
\begin{aligned}
a_{\mathtt{F}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_{\mathtt{F}}(\boldsymbol{\tau}_h, \mathbf{u}_h) + c_{\mathtt{F}}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= 0 && \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_{h,0} \\
b_{\mathtt{F}}(\boldsymbol{\sigma}_h, \mathbf{v}_h) + d_{\mathtt{F}}(\theta_h, \mathbf{v}_h) &= 0 && \forall\,\mathbf{v}_h \in \mathbf{M}_h \\
a_{\mathtt{T}}(\boldsymbol{\rho}_h, \boldsymbol{\eta}_h) + b_{\mathtt{T}}(\boldsymbol{\eta}_h, \theta_h) + c_{\mathtt{T}}(\mathbf{u}_h; \theta_h, \boldsymbol{\eta}_h) &= F_{\mathtt{T}}(\boldsymbol{\eta}_h) && \forall\,\boldsymbol{\eta}_h \in \mathbf{H}_h \\
b_{\mathtt{T}}(\boldsymbol{\rho}_h, \boldsymbol{\psi}_h) &= 0 && \forall\,\psi_h \in \mathrm{Q}_h,
\end{aligned}
\tag{4.1}
$$

where the forms $a_{\mathtt{F}}, b_{\mathtt{F}}, c_{\mathtt{F}}, d_{\mathtt{F}}, a_{\mathtt{T}}, b_{\mathtt{T}}, c_{\mathtt{T}}$ and the functional $F_{\mathtt{T}}$ are defined in (2.12) and (2.13), respectively.

4.2 Analysis of the discrete problem

First we provide the stability properties of the associated forms on the discrete spaces defined above. We begin by observing that the boundedness of all the forms are inherited from the continuous case. In addition, since $\mathbf{div}\,\mathbb{X}_h \subseteq \mathbf{M}_h$ and $\mathrm{div}\,\mathbf{H}_h \subseteq \mathrm{Q}_h$, there hold that the discrete versions of $\mathbb{V}$ and $\mathbf{V}$ (cf. (3.8), (3.9)) become, respectively

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{X}_{h,0} : \quad b_{\mathtt{F}}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{M}_h \Big\} = \Big\{ \boldsymbol{\tau}_h \in \mathbb{X}_{h,0} : \quad \mathbf{div}\,\boldsymbol{\tau}_h = \mathbf{0} \quad \text{in} \quad \Omega \Big\},$$

and

$$\mathbf{V}_h := \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_h : \quad b_{\mathtt{T}}(\boldsymbol{\eta}_h, \psi_h) = 0 \quad \forall\, \psi_h \in \mathrm{Q}_h \Big\} = \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_h : \quad \mathrm{div}\ \boldsymbol{\eta}_h = 0 \quad \text{in} \quad \Omega \Big\},$$

thus, $\mathbb{V}_h \subseteq \mathbb{V}$ and $\mathbf{V}_h \subseteq \mathbf{V}$. As consequence, from (3.11) and (3.12), we obtain

$$a_{\mathtt{F}}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \frac{C_{\mathrm{d}}}{\nu}\, \|\boldsymbol{\tau}_h\|^2_{\mathbb{X}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h, \tag{4.2}$$

and

$$a_{\mathtt{T}}(\boldsymbol{\eta}_h, \boldsymbol{\eta}_h) \geq \frac{1}{\kappa} \|\boldsymbol{\eta}_h\|^2_{\mathbf{H}} \quad \forall\, \boldsymbol{\eta}_h \in \mathbf{V}_h. \tag{4.3}$$

We continue by recalling from [9, Lemma 4.4] that the bilinear form $b_{\mathtt{F}}$ satisfy the following discrete inf-sup condition:

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}} \frac{b_{\mathtt{F}}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \geq \widehat{\beta}_{\mathtt{F}}\, \|\mathbf{v}_h\|_{\mathbf{M}} \quad \forall\, \mathbf{v}_h \in \mathbf{M}_h, \tag{4.4}$$

with $\widehat{\beta}_{\mathtt{F}} > 0$ independent of h.

The following result establishes the discrete version of Lemma 3.1

Lemma 4.1 *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_{\mathrm{N}} \subseteq \partial B$. Then there exists $\widehat{\beta}_{\mathtt{T}} > 0$ independent of h, such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\eta}_h \in \mathbf{H}_h} \frac{b_{\mathtt{T}}(\boldsymbol{\eta}_h, \psi_h)}{\|\boldsymbol{\eta}_h\|_{\mathbf{H}}} \geq \widehat{\beta}_{\mathtt{T}}\, \|\psi_h\|_{\mathrm{Q}} \quad \forall\, \psi_h \in \mathrm{Q}_h. \tag{4.5}$$

Proof. We proceed similarly to the proof of [10, Lemma 3.3]. In fact, given $\psi_h \in \mathrm{Q}_h$, and similarly to [4, Lemma 3.9] we let $z \in \mathrm{W}^{1,4/3}(B)$ be the unique weak solution of the boundary value problem:

$$\Delta z = \widetilde{\psi}_h := \begin{cases} \mathrm{sgn}(\psi_h)|\psi_h|^3 & \text{in } \ \Omega \\ \dfrac{-1}{|B \setminus \overline{\Omega}|} \displaystyle\int_{\Omega} \mathrm{sgn}(\psi_h)|\psi_h|^3 & \text{in } \ B \setminus \overline{\Omega} \end{cases}, \qquad \nabla z \cdot \mathbf{n} = 0 \ \text{ on } \ \partial B, \qquad \int_{\Omega} z = 0.$$

Since, B is a convex domain, it is well known that $z \in \mathrm{W}^{2,4/3}(B)$ (see [26, Theorem 1.1]) and

$$\|z\|_{\mathrm{W}^{2,4/3}(B)} \leq c\|\widetilde{\psi}_h\|_{\mathrm{L}^{4/3}(B)} \leq C\||\psi_h|^3\|_{\mathrm{L}^{4/3}(\Omega)} = C\|\psi_h\|^3_{\mathrm{L}^4(\Omega)},$$

Then we let $\widehat{\boldsymbol{\eta}} = \nabla z|_{\Omega} \in \mathbf{W}^{1,4/3}(\Omega)$, and observe that $\operatorname{div}\widehat{\boldsymbol{\eta}} = sgn(\psi_h)|\psi_h|^3$ in Ω, $\widehat{\boldsymbol{\eta}}\cdot\mathbf{n} = 0$ on Γ_{N} (since $\Gamma_{\mathrm{N}} \subseteq \partial B$) and

$$\|\widehat{\boldsymbol{\eta}}\|_{\mathbf{W}^{1,4/3}(\Omega)} \leq C\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}. \tag{4.6}$$

Moreover, from the latter, and the fact that $W^{1,4/3}(\Omega)$ is continuously embedded into $\mathrm{L}^2(\Omega)$, we obtain

$$\|\widehat{\boldsymbol{\eta}}\|_{0,\Omega} \leq C\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}. \tag{4.7}$$

Now, we let $\widehat{\boldsymbol{\eta}}_h \in \mathbf{H}_h$ be the Raviart-Thomas interpolation of $\boldsymbol{\eta}$ (see [25, Section 3.4] and [9, Section 4.2.1]). From [15, Lemma 5.4] we have that there exists $C>0$, independent of h, such that

$$\|\widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}_h\|_{0,\Omega} \leq Ch^{1-n/4}\|\widehat{\boldsymbol{\eta}}\|_{\mathbf{W}^{1,4/3}(\Omega)},$$

which together with (4.6) and (4.7), implies

$$\|\widehat{\boldsymbol{\eta}}_h\|_{0,\Omega} \leq \|\widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}_h\|_{0,\Omega} + \|\widehat{\boldsymbol{\eta}}\|_{0,\Omega} \leq Ch^{1-n/4}\|\widehat{\boldsymbol{\eta}}\|_{\mathbf{W}^{1,4/3}(\Omega)} + C\|\psi_h\|^3_{\mathrm{L}^4(\Omega)} \leq \hat{C}\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}. \tag{4.8}$$

In turn, it is well known that the following identity holds

$$\operatorname{div}\widehat{\boldsymbol{\eta}}_h = \mathcal{P}_h(\operatorname{div}\widehat{\boldsymbol{\eta}}) = \mathcal{P}_h(\operatorname{sgn}(\psi_h)|\psi_h|^3), \tag{4.9}$$

with $\mathcal{P}_h : \mathrm{L}^4(\Omega) \to \mathrm{Q}_h$ being the usual orthogonal projection with respect to the $L^2(\Omega)$-inner product. Hence, using the fact that $\mathcal{P}_h$ is a continuous operator, from (4.8) and (4.9), we easily obtain

$$\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{H}} \leq \widehat{C}\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}, \tag{4.10}$$

with $\widehat{C}>0$ independent of h. In this way, from (4.9) and (4.10), we find that

$$\sup_{\mathbf{0}\neq\boldsymbol{\eta}_h\in\mathbf{H}_h} \frac{b_{\mathtt{T}}(\boldsymbol{\eta}_h,\psi_h)}{\|\boldsymbol{\eta}_h\|_{\mathbf{H}}} \geq \frac{b_{\mathtt{T}}(\widehat{\boldsymbol{\eta}}_h,\psi_h)}{\|\widehat{\boldsymbol{\eta}}_h\|_{\mathbf{H}}} \geq \frac{\displaystyle\int_{\Omega}\psi_h \operatorname{sgn}(\psi_h)|\psi_h|^3}{\widehat{C}\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}} = \widehat{C}^{-1}\frac{\|\psi_h\|^4_{\mathrm{L}^4(\Omega)}}{\|\psi_h\|^3_{\mathrm{L}^4(\Omega)}} = \widehat{C}^{-1}\,\|\psi_h\|_{\mathrm{L}^4(\Omega)},$$

which concludes the proof. □

Analogously to the continuous case, owing to (3.3), (3.5), (4.2), (4.3), (4.4), (4.5) and [23, Proposition 2.36], it can be deduced that the bilinear forms $\mathcal{A}_{\mathtt{F}}$ and $\mathcal{A}_{\mathtt{T}}$ defined in (3.15) and (3.18), satisfy:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h} \frac{\mathcal{A}_{\mathtt{F}}((\boldsymbol{\zeta}_h,\mathbf{z}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h))}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|} \geq \widehat{\gamma}_{\mathtt{F}}\,\|(\boldsymbol{\zeta}_h,\mathbf{z}_h)\| \quad \forall\,(\boldsymbol{\zeta}_h,\mathbf{z}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h, \tag{4.11}$$

and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\psi_h)\in\mathbf{H}_h\times\mathrm{Q}_h} \frac{\mathcal{A}_{\mathtt{T}}((\boldsymbol{\varsigma}_h,\varphi_h),(\boldsymbol{\eta}_h,\psi_h))}{\|(\boldsymbol{\eta}_h,\psi_h)\|} \geq \widehat{\gamma}_{\mathtt{T}}\,\|(\boldsymbol{\varsigma}_h,\varphi_h)\| \quad \forall\,(\boldsymbol{\varsigma}_h,\varphi_h)\in\mathbf{H}_h\times\mathrm{Q}_h, \tag{4.12}$$

with

$$\widehat{\gamma}_{\mathtt{F}} := C\frac{\min\{1,\nu\widehat{\beta}_{\mathtt{F}}\}}{\nu\widehat{\beta}_{\mathtt{F}}+1},$$

and

$$\widehat{\gamma}_{\mathtt{T}} := \frac{\kappa\,\widehat{\beta}_{\mathtt{T}}^2}{\kappa^2\,\widehat{\beta}_{\mathtt{T}}^2 + 4\,\kappa\,\widehat{\beta}_{\mathtt{T}} + 2}.$$

Employing (4.11) and (4.12) it can be proved the following result.

Lemma 4.2 *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_{\mathrm{N}} \subseteq \partial B$. Let $\widehat{\lambda} := \min\left\{\nu\,\widehat{\gamma}_{\mathtt{F}}, \kappa\,\widehat{\gamma}_{\mathtt{T}}\right\}$ and given $\mathbf{w}_h \in \mathbf{M}_h$, let $\mathcal{A}_{\mathtt{F},\mathbf{w}_h}$ and $\mathcal{A}_{\mathtt{T},\mathbf{w}_h}$ be the bilinear forms defined in (3.29) and (3.36), respectively. Then, for all $\mathbf{w}_h \in \mathbf{M}_h$ such that $\|\mathbf{w}_h\|_{\mathbf{M}} \leq \widehat{\lambda}$, there hold*

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h} \frac{\mathcal{A}_{\mathtt{F},\mathbf{w}_h}((\boldsymbol{\zeta}_h,\mathbf{z}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h))}{\|(\boldsymbol{\tau}_h,\mathbf{v}_h)\|} \geq \frac{\widehat{\gamma}_{\mathtt{F}}}{2}\,\|(\boldsymbol{\zeta}_h,\mathbf{z}_h)\| \quad \forall\,(\boldsymbol{\zeta}_h,\mathbf{z}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h, \tag{4.13}$$

and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_h,\psi_h)\in\mathbf{H}_h\times\mathrm{Q}_h} \frac{\mathcal{A}_{\mathtt{T},\mathbf{w}_h}((\boldsymbol{\varsigma}_h,\varphi_h),(\boldsymbol{\eta}_h,\psi_h))}{\|(\boldsymbol{\eta}_h,\psi_h)\|} \geq \frac{\widehat{\gamma}_{\mathtt{T}}}{2}\,\|(\boldsymbol{\varsigma}_h,\varphi_h)\| \quad \forall\,(\boldsymbol{\varsigma}_h,\varphi_h)\in\mathbf{H}_h\times\mathrm{Q}_h. \tag{4.14}$$

Proof. The proofs of (4.13) and (4.14) follow using the same steps employed to obtain (3.31) in Lemma 3.2. We omit further details. □

Now, let us define the bounded set

$$\mathbf{W}_h := \left\{\mathbf{w}_h \in \mathbf{M}_h : \quad \|\mathbf{w}_h\|_{\mathbf{M}} \leq \frac{4\,C_F}{\widehat{\gamma}_{\mathtt{F}}\,\widehat{\gamma}_{\mathtt{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right\},$$

and the discrete operators $\mathbf{R}_h : \mathbf{W}_h \times \mathrm{Q}_h \to \mathbb{X}_{h,0}\times\mathbf{M}_h$ and $\mathbf{S}_h : \mathbf{W}_h \to \mathbf{H}_h\times\mathrm{Q}_h$, defined respectively by

$$\mathbf{R}_h(\mathbf{w}_h,\phi_h) := (\mathbf{R}_{1,h}(\mathbf{w}_h,\phi_h),\mathbf{R}_{2,h}(\mathbf{w}_h,\phi_h)) = (\boldsymbol{\sigma}_h,\mathbf{u}_h) \quad \forall\,(\mathbf{w}_h,\phi_h)\in\mathbf{W}_h\times\mathrm{Q}_h,$$

where $(\boldsymbol{\sigma}_h,\mathbf{u}_h)$ is the unique solution of problem: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{X}_{h,0}\times\mathbf{M}_h$ such that

$$\begin{aligned} a_{\mathtt{F}}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + b_{\mathtt{F}}(\boldsymbol{\tau}_h,\mathbf{u}_h) + c_{\mathtt{F}}(\mathbf{w}_h;\mathbf{u}_h,\boldsymbol{\tau}_h) &= 0 && \forall\,\boldsymbol{\tau}_h\in\mathbb{X}_{h,0},\\ b_{\mathtt{F}}(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= -d_{\mathtt{F}}(\phi_h,\mathbf{v}_h) && \forall\,\mathbf{v}_h\in\mathbf{M}_h, \end{aligned}$$

and

$$\mathbf{S}_h(\mathbf{w}_h) := (\mathbf{S}_{1,h}(\mathbf{w}_h),\mathbf{S}_{2,h}(\mathbf{w}_h)) = (\boldsymbol{\rho}_h,\theta_h) \quad \forall\,\mathbf{w}_h\in\mathbf{W}_h,$$

where $(\boldsymbol{\rho}_h,\theta_h)$ is the unique solution of problem: Find $(\boldsymbol{\rho}_h,\theta_h)\in\mathbf{H}_h\times\mathrm{Q}_h$ such that

$$\begin{aligned} a_{\mathtt{T}}(\boldsymbol{\rho}_h,\boldsymbol{\eta}_h) + b_{\mathtt{T}}(\boldsymbol{\eta}_h,\theta_h) + c_{\mathtt{T}}(\mathbf{w}_h;\theta_h,\boldsymbol{\eta}_h) &= F_{\mathtt{T}}(\boldsymbol{\eta}_h) && \forall\,\boldsymbol{\eta}_h\in\mathbf{H}_h,\\ b_{\mathtt{T}}(\boldsymbol{\rho}_h,\psi_h) &= 0 && \forall\,\psi_h\in\mathrm{Q}_h. \end{aligned}$$

Utilizing Lemma 4.2 and proceeding exactly as for the continuous case, it can be easily deduced that both operators are well-defined if there holds

$$\frac{8\,C_F}{\widehat{\lambda}\,\widehat{\gamma}_{\mathtt{F}}\,\widehat{\gamma}_{\mathtt{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \leq 1. \tag{4.15}$$

Then, analogously to the continuous case we define the operator $\mathcal{J}_h : \mathbf{W}_h \subseteq \mathbf{M}_h \to \mathbf{M}_h$ as

$$\mathcal{J}_h(\mathbf{w}_h) = \mathbf{R}_{2,h}(\mathbf{w}_h,\mathbf{S}_{2,h}(\mathbf{w}_h)) \quad \forall\,\mathbf{w}_h\in\mathbf{W}_h, \tag{4.16}$$

which is clearly well-defined and satisfies $\mathcal{J}_h(\mathbf{W}_h)\subseteq\mathbf{W}_h$ provided (4.15), and realize that (4.1) is equivalent to the fixed-point problem: Find $\mathbf{u}_h\in\mathbf{W}_h$ such that

$$\mathcal{J}_h(\mathbf{u}_h) = \mathbf{u}_h. \tag{4.17}$$

The following theorem provides the main result of this section, namely, existence and uniqueness of solution of the fixed-point problem (4.17), or equivalently, the well-posedness of problem (4.1).

Theorem 4.1 *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_{\mathrm{N}} \subseteq \partial B$. Let define $\widehat{\lambda} := \min\left\{\nu\,\widehat{\gamma}_{\mathrm{F}}, \kappa\,\widehat{\gamma}_{\mathrm{T}}\right\}$ and assume that*

$$\frac{16\,C_F}{\widehat{\lambda}\,\widehat{\gamma}_{\mathrm{F}}\,\widehat{\gamma}_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} < 1. \tag{4.18}$$

Then, the operator $\mathcal{J}_h$ (cf. (4.16)) has a unique fixed-point $\mathbf{u}_h$ in $\mathbf{W}_h$. Equivalently, the coupled problem (4.1) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ with $\mathbf{u}_h \in \mathbf{W}_h$. Moreover, there hold

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \le \frac{4\,C_F}{\widehat{\gamma}_{\mathrm{F}}\,\widehat{\gamma}_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \quad \text{and} \quad \|(\boldsymbol{\rho}_h, \theta_h)\| \le \frac{2\,C_F}{\widehat{\gamma}_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}. \tag{4.19}$$

Proof. First we observe that, as for the continuous case (see the proof of Theorem 3.2), assumption (4.18) ensures the well-definiteness of operators $\mathbf{S}_h$ and $\mathbf{R}_h$, and consequently the well-definiteness of $\mathcal{J}_h$. Now, adapting the arguments utilized in Section 3.4 (see Lemmas 3.4 and 3.5) one can obtain the following estimates

$$\|\mathbf{R}_h(\mathbf{w}_1, \phi_1) - \mathbf{R}_h(\mathbf{w}_2, \phi_2)\| \le \frac{4}{\nu\,\widehat{\gamma}_{\mathrm{F}}^2}\,\|\mathbf{g}\|_{0,\Omega}\|\phi_2\|_{\mathrm{Q}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}} + \frac{2}{\widehat{\gamma}_{\mathrm{F}}}\|\mathbf{g}\|_{0,\Omega}\|\phi_1 - \phi_2\|_{\mathrm{Q}},$$

and

$$\|\mathbf{S}_h(\mathbf{w}_1) - \mathbf{S}_h(\mathbf{w}_2)\|_{\mathrm{Q}} \le \frac{4\,C_F}{\kappa\,\widehat{\gamma}_{\mathrm{T}}^2}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}},$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}_h$ and $\phi_1, \phi_2 \in \mathrm{Q}_h$, which together with the definition of $\mathcal{J}_h$ (cf. (4.16)), yield

$$\|\mathcal{J}_h(\mathbf{w}_1) - \mathcal{J}_h(\mathbf{w}_2)\|_{\mathbf{M}} \le \frac{16\,C_F}{\widehat{\lambda}\widehat{\gamma}_{\mathrm{F}}\widehat{\gamma}_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{M}},$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}_h$. In this way, using estimate (4.18) we obtain that $\mathcal{J}_h$ is a contraction mapping on $\mathbf{W}_h$, thus problem (4.17), or equivalently (4.1) is well-posed. Finally, analogously to the proof of Theorem 3.2 we can obtain (4.19), which concludes the proof. $\square$

5 A priori error analysis

In this section we aim to provide the convergence of the Galerkin scheme (4.1) and derive the corresponding rate of convergence. We begin by deriving the corresponding Cea's estimate.

5.1 Cea's estimate

From now on we assume that the hypotheses of Theorems 3.2 and 4.1 hold and let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta) \in \mathbb{X}_0 \times \mathbf{M} \times \mathbf{H} \times \mathrm{Q}$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ be the unique solutions of (2.14) and (4.1), respectively.

In order to simplify the subsequent analysis, we write $\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, $\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h$, $\mathbf{e}_{\boldsymbol{\rho}} = \boldsymbol{\rho} - \boldsymbol{\rho}_h$, and $e_\theta = \theta - \theta_h$. As usual, for a given $(\widehat{\boldsymbol{\tau}}_h, \widehat{\mathbf{v}}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$ and $(\widehat{\boldsymbol{\eta}}_h, \widehat{\psi}_h) \in \mathbf{H}_h \times \mathrm{Q}_h$, we shall then decompose these errors into

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}}, \quad \mathbf{e}_{\mathbf{u}} = \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, \quad \mathbf{e}_{\boldsymbol{\rho}} = \boldsymbol{\xi}_{\boldsymbol{\rho}} + \boldsymbol{\chi}_{\boldsymbol{\rho}}, \quad e_\theta = \xi_\theta + \chi_\theta, \tag{5.1}$$

with

$$\boldsymbol{\xi_\sigma} = \boldsymbol{\sigma} - \widehat{\boldsymbol{\tau}}_h, \quad \boldsymbol{\chi_\sigma} = \widehat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h, \quad \boldsymbol{\xi}_{\mathbf{u}} = \mathbf{u} - \widehat{\mathbf{v}}_h, \quad \boldsymbol{\chi}_{\mathbf{u}} = \widehat{\mathbf{v}}_h - \mathbf{u}_h,$$

$$\boldsymbol{\xi_\rho} = \boldsymbol{\rho} - \widehat{\boldsymbol{\eta}}_h, \quad \boldsymbol{\chi_\rho} = \widehat{\boldsymbol{\eta}}_h - \boldsymbol{\rho}_h, \quad \xi_\theta = \theta - \widehat{\psi}_h, \quad \chi_\theta = \widehat{\psi}_h - \theta_h.$$

Consequently, subtracting (2.14) and (4.1), and utilizing the definition of $\mathcal{A}_{\mathtt{F}}$ and $\mathcal{A}_{\mathtt{T}}$ (cf. (3.15) and (3.18), respectively), we obtain the following identities:

$$\mathcal{A}_{\mathtt{F}}((\mathbf{e}_{\boldsymbol{\sigma}}, \mathbf{e}_{\mathbf{u}}), (\boldsymbol{\tau}_h, \mathbf{v}_h)) + c_{\mathtt{F}}(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}_h) - c_{\mathtt{F}}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) = -d_{\mathtt{F}}(e_\theta, \mathbf{v}_h) \tag{5.2}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$, and

$$\mathcal{A}_{\mathtt{T}}((\mathbf{e}_{\boldsymbol{\rho}}, e_\theta), (\boldsymbol{\eta}_h, \psi_h)) + c_{\mathtt{T}}(\mathbf{u}; \theta, \boldsymbol{\eta}_h) - c_{\mathtt{T}}(\mathbf{u}_h; \theta_h, \boldsymbol{\eta}_h) = 0 \tag{5.3}$$

for all $(\boldsymbol{\eta}_h, \psi_h) \in \mathbf{H}_h \times \mathrm{Q}_h$.

We start providing the following auxiliary results.

Lemma 5.1 *Assume that*

$$\frac{8\, C_F}{\nu \gamma_{\mathtt{F}} \widehat{\gamma}_{\mathtt{F}}\, \gamma_{\mathtt{T}}} \|\mathbf{g}\|_{0,\Omega} \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \leq \frac{1}{2} \tag{5.4}$$

Then there exist $C_1, C_2 > 0$, independent of h, such that

$$\|(\boldsymbol{\chi_\sigma}, \boldsymbol{\chi}_{\mathbf{u}})\| \leq C_1 \,\|(\boldsymbol{\xi_\sigma}, \boldsymbol{\xi}_{\mathbf{u}})\| + C_2 \,\|\xi_\theta\|_{\mathrm{Q}} + \frac{4}{\widehat{\gamma}_{\mathtt{F}}} \|\mathbf{g}\|_{0,\Omega} \,\|\chi_\theta\|_{\mathrm{Q}} \tag{5.5}$$

Proof. First, from (5.1), (5.2), the definition of the bilinear form $\mathcal{A}_{\mathtt{F},\mathbf{w}}$ (cf. (3.29)), and simple computations it can be obtained the identity

$$\begin{aligned} \mathcal{A}_{\mathtt{F},\mathbf{u}_h}((\boldsymbol{\chi_\sigma}, \boldsymbol{\chi}_{\mathbf{u}}), (\boldsymbol{\tau}_h, \mathbf{v}_h)) &= -a_{\mathtt{F}}(\boldsymbol{\xi_\sigma}, \boldsymbol{\tau}_h) - b_{\mathtt{F}}(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}) - b_{\mathtt{F}}(\boldsymbol{\xi_\sigma}, \mathbf{v}_h) \\ &- c_{\mathtt{F}}(\mathbf{u}_h; \boldsymbol{\xi}_{\mathbf{u}}, \boldsymbol{\tau}_h) - c_{\mathtt{F}}(\boldsymbol{\xi}_{\mathbf{u}}; \mathbf{u}, \boldsymbol{\tau}_h) - c_{\mathtt{F}}(\boldsymbol{\chi}_{\mathbf{u}}; \mathbf{u}, \boldsymbol{\tau}_h) - d_{\mathtt{F}}(e_\theta, \mathbf{v}_h). \end{aligned}$$

Then, utilizing the discrete inf-sup condition (4.13) with $(\boldsymbol{\zeta}_h, \mathbf{z}_h) = (\boldsymbol{\chi_\sigma}, \boldsymbol{\chi}_{\mathbf{u}}) \in \mathbb{X}_{h,0} \times \mathbf{M}_h$, and the continuity properties of $a_{\mathtt{F}}$, $b_{\mathtt{F}}$, $c_{\mathtt{F}}$ and $d_{\mathtt{F}}$ (cf. (3.3) and (3.4)), we obtain

$$\begin{aligned} \frac{\widehat{\gamma}_{\mathtt{F}}}{2} \|(\boldsymbol{\chi_\sigma}, \boldsymbol{\chi}_{\mathbf{u}})\| \leq &\left(1 + \frac{1}{\nu}\right) \|\boldsymbol{\xi_\sigma}\|_{\mathbb{X}} + \left(1 + \frac{1}{\nu} \|\mathbf{u}_h\|_{\mathbf{M}} + \frac{1}{\nu} \|\mathbf{u}\|_{\mathbf{M}}\right) \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}} \\ &+ \frac{1}{\nu} \|\mathbf{u}\|_{\mathbf{M}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{g}\|_{0,\Omega} \|e_\theta\|_{\mathrm{Q}}. \end{aligned} \tag{5.6}$$

In this way, using the fact that $\mathbf{u} \in \mathbf{W}$ and $\mathbf{u}_h \in \mathbf{W}_h$, from (5.6) we deduce that there exists $C > 0$, independent of h, such that

$$\frac{\widehat{\gamma}_{\mathtt{F}}}{2} \|(\boldsymbol{\chi_\sigma}, \boldsymbol{\chi}_{\mathbf{u}})\| \leq C \,\|(\boldsymbol{\xi_\sigma}, \boldsymbol{\xi}_{\mathbf{u}})\| + \frac{4\, C_F}{\nu \gamma_{\mathtt{F}}\, \gamma_{\mathtt{T}}} \|\mathbf{g}\|_{0,\Omega} \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{g}\|_{0,\Omega} \|e_\theta\|_{\mathrm{Q}},$$

which together with (5.4) implies (5.5) and concludes the proof. □

Lemma 5.2 *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_{\mathrm{N}} \subseteq \partial B$. Then there exist $C_3, C_4 > 0$, independent of h, such that*

$$\|(\boldsymbol{\chi_\rho}, \chi_\theta)\| \leq C_3 \|(\boldsymbol{\xi_\rho}, \xi_\theta)\| + C_4 \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}} + \frac{4\, C_F}{\kappa \widehat{\gamma}_{\mathtt{T}} \gamma_{\mathtt{T}}} \|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}}. \tag{5.7}$$

Proof. We proceed similarly to the proof of Lemma 5.1. In fact, from (5.3), the definition of the bilinear form $\mathcal{A}_{\mathrm{T},\mathbf{w}}$ (cf. (3.36)), the decomposition (5.1), and simple algebraic manipulations, it can be obtained the identity

$$\begin{aligned}\mathcal{A}_{\mathrm{T},\mathbf{u}_h}((\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta),(\boldsymbol{\eta}_h,\psi_h)) &= -a_{\mathrm{T}}(\boldsymbol{\xi}_{\boldsymbol{\rho}},\boldsymbol{\eta}_h) - b_{\mathrm{T}}(\boldsymbol{\eta}_h,\xi_\theta) - b_{\mathrm{T}}(\boldsymbol{\xi}_{\boldsymbol{\rho}},\psi_h)\\ &\quad - c_{\mathrm{T}}(\mathbf{u}_h;\xi_\theta,\boldsymbol{\eta}_h) - c_{\mathrm{T}}(\boldsymbol{\xi}_{\mathbf{u}};\theta,\boldsymbol{\eta}_h) - c_{\mathrm{T}}(\boldsymbol{\chi}_{\mathbf{u}};\theta,\boldsymbol{\eta}_h).\end{aligned}$$

Then, applying the discrete inf-sup condition (4.14) with $(\boldsymbol{\varsigma}_h,\varphi_h) = (\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta) \in \mathbf{H}_h \times \mathrm{Q}_h$, and the continuity properties of a_{T}, b_{T} and c_{T} (cf. (3.5) and (3.6)), we obtain

$$\frac{\widehat{\gamma}_{\mathrm{T}}}{2}\,\|(\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta)\| \le \left(1+\frac{1}{\kappa}\right)\,\|\boldsymbol{\xi}_{\boldsymbol{\rho}}\|_{\mathbf{H}} + \left(1+\frac{1}{\kappa}\|\mathbf{u}_h\|_{\mathbf{M}}\right)\,\|\xi_\theta\|_{\mathrm{Q}} + \frac{1}{\kappa}\,\|\theta\|_{\mathrm{Q}}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}} + \frac{1}{\kappa}\,\|\theta\|_{\mathrm{Q}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}},$$

which together with the fact that $\mathbf{u}_h \in \mathbf{W}_h$ and that θ satisfies $\|\theta\|_{\mathrm{Q}} \le \|(\boldsymbol{\rho},\theta)\| \le \frac{2\,C_F}{\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}$ (see (3.45)), imply that there exists a positive constant C, independent of h, such that

$$\frac{\widehat{\gamma}_{\mathrm{T}}}{2}\,\|(\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta)\| \le C\|(\boldsymbol{\xi}_{\boldsymbol{\rho}},\xi_\theta)\| + \frac{2\,C_F}{\kappa\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{M}} + \frac{2\,C_F}{\kappa\gamma_{\mathrm{T}}}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}},$$

from which we deduce (5.7). □

Now we are in position of establishing the aforementioned Cea's estimate.

Theorem 5.1 *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_{\mathrm{N}} \subseteq \partial B$. Let define $\widetilde{\lambda} := \min\left\{\nu\,\gamma_{\mathrm{F}},\kappa\,\widehat{\gamma}_{\mathrm{T}}\right\}$ and assume further that*

$$\frac{16\,C_F}{\widetilde{\lambda}\,\widehat{\gamma}_{\mathrm{F}}\,\gamma_{\mathrm{T}}}\,\|\mathbf{g}\|_{0,\Omega}\,\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \le \frac{1}{2}. \tag{5.8}$$

Then, there exists $C > 0$, independent of h, such that

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{e}_{\boldsymbol{\rho}}\|_{\mathbf{H}} + \|e_\theta\|_{\mathrm{Q}} \le C\left\{\mathrm{dist}\,((\boldsymbol{\sigma},\mathbf{u}),\mathbb{X}_{h,0}\times\mathbf{M}_h) + \mathrm{dist}\,((\boldsymbol{\rho},\theta),\mathbf{H}_h\times\mathrm{Q}_h)\right\}. \tag{5.9}$$

Proof. We begin by observing that estimate (5.8) implies (5.4), thus estimate (5.5) holds. Now, since $\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{M}} \le \|(\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}})\|$, combining (5.5) and (5.7), it is not difficult to see that there exist positive constants c_1, c_2, independent of h, such that

$$\begin{aligned}\|(\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta)\| \le\ & c_1\|(\boldsymbol{\xi}_{\boldsymbol{\rho}},\xi_\theta)\| + c_2\|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\| + \frac{16\,C_F}{\kappa\widehat{\gamma}_{\mathrm{T}}\widehat{\gamma}_{\mathrm{F}}\gamma_{\mathrm{T}}}\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\chi_\theta\|_{\mathrm{Q}}\\ \le\ & c_1\|(\boldsymbol{\xi}_{\boldsymbol{\rho}},\xi_\theta)\| + c_2\|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\| + \frac{16\,C_F}{\widetilde{\lambda}\widehat{\gamma}_{\mathrm{F}}\gamma_{\mathrm{T}}}\|\mathbf{g}\|_{0,\Omega}\|\theta_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\,\|\chi_\theta\|_{\mathrm{Q}}\end{aligned}$$

which combined with (5.8) implies

$$\|(\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta)\| \le \hat{c}_1\|(\boldsymbol{\xi}_{\boldsymbol{\rho}},\xi_\theta)\| + \hat{c}_2\|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\|, \tag{5.10}$$

with $\hat{c}_1, \hat{c}_2 > 0$, independent of h. In turn, from (5.5), (5.10) and estimate $\|\chi_\theta\|_{\mathrm{Q}} \le \|(\boldsymbol{\chi}_{\boldsymbol{\rho}},\chi_\theta)\|$ we easily deduce that

$$\|(\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}})\| \le c_3\|(\boldsymbol{\xi}_{\boldsymbol{\rho}},\xi_\theta)\| + c_4\|(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\xi}_{\mathbf{u}})\|, \tag{5.11}$$

with $c_3, c_4 > 0$, independent of h. In this way, estimate (5.9) follows from (5.1), (5.10), (5.11), the triangle inequality and the fact that $(\widehat{\boldsymbol{\tau}}_h,\widehat{\mathbf{v}}_h) \in \mathbb{X}_{h,0}\times\mathbf{M}_h$ and $(\widehat{\boldsymbol{\eta}}_h,\widehat{\psi}_h) \in \mathbf{H}_h\times\mathrm{Q}_h$ are arbitrary. □

5.2 Rate of convergence

In order to establish the rate of convergence of our Galerkin scheme (4.1), we first recall the approximation properties of the discrete spaces involved:

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) For each $0 \le l \le k$ and for each $\boldsymbol{\tau} \in \mathbb{H}^{l+1}(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}\boldsymbol{\tau} \in \mathbf{W}^{l+1,4/3}(\Omega)$, there holds

$$\mathrm{dist}\left(\boldsymbol{\tau}, \mathbb{X}_{h,0}\right) := \inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{h,0}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \le C\, h^{l+1} \Big\{ \|\boldsymbol{\tau}\|_{l+1,\Omega} + \|\mathbf{div}\boldsymbol{\tau}\|_{\mathbf{W}^{l+1,4/3}(\Omega)} \Big\}, \tag{5.12}$$

($\mathbf{AP}_h^{\mathbf{u}}$) For each $0 \le l \le k$ and for each $\mathbf{v} \in \mathbf{W}^{l+1,4}(\Omega)$, there holds

$$\mathrm{dist}\left(\mathbf{v}, \mathbf{M}_h\right) := \inf_{\mathbf{v}_h \in \mathbf{M}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{L}^4(\Omega)} \le C\, h^{l+1}\, \|\mathbf{v}\|_{\mathbf{W}^{l+1,4}(\Omega)}, \tag{5.13}$$

($\mathbf{AP}_h^{\boldsymbol{\rho}}$) For each $0 \le l \le k$ and for each $\boldsymbol{\eta} \in \mathbf{H}^{l+1}(\Omega)$ with $\mathrm{div}\,\boldsymbol{\eta} \in \mathrm{W}^{l+1,4/3}(\Omega)$, there holds

$$\mathrm{dist}\left(\boldsymbol{\eta}, \mathbf{H}_h\right) := \inf_{\boldsymbol{\eta}_h \in \mathbf{H}_h} \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{\mathrm{div}\,{4/3};\Omega} \le C\, h^{l+1} \Big\{ \|\boldsymbol{\eta}\|_{l+1,\Omega} + \|\mathrm{div}\,\boldsymbol{\eta}\|_{\mathrm{W}^{l+1,4/3}(\Omega)} \Big\}, \tag{5.14}$$

($\mathbf{AP}_h^{\theta}$) For each $0 \le l \le k$ and for each $\psi \in \mathrm{W}^{l+1,4}(\Omega)$, there holds

$$\mathrm{dist}\left(\psi, \mathrm{Q}_h\right) := \inf_{\psi_h \in \mathrm{Q}_h} \|\psi - \psi_h\|_{\mathrm{L}^4(\Omega)} \le C\, h^{l+1}\, \|\psi\|_{\mathrm{W}^{l+1,4}(\Omega)}. \tag{5.15}$$

For (5.12) and (5.14) we refer to [9, eq. (4.8)] and [10, eq. (3.8)], which are consequences of [23, Lemma B.67, Lemma 1.101] and [25, Section 3.4.4], whereas for (5.13) and (5.15) we refer to [23, Proposition 1.134, Section 1.6.3].

Now we are in position of establishing the rates of convergence associated to the Galerkin scheme (4.1).

Theorem 5.2 *Assume that the hypotheses of Theorem 5.1 hold and let* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta) \in \mathbb{X}_0 \times \mathbf{M} \times \mathbf{H} \times \mathrm{Q}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.14) *and* (4.1)*, respectively. Assume further that* $\boldsymbol{\sigma} \in \mathbb{H}^{l+1}(\Omega)$*,* $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{W}^{l+1,4/3}(\Omega)$*,* $\mathbf{u} \in \mathbf{W}^{l+1,4}(\Omega)$*,* $\boldsymbol{\rho} \in \mathbf{H}^{l+1}(\Omega)$*,* $\mathrm{div}\,\boldsymbol{\rho} \in \mathrm{W}^{l+1,4/3}(\Omega)$ *and* $\theta \in \mathrm{W}^{l+1,4}(\Omega)$ *, for* $0 \le l \le k$*. Then there exists* $C_{rate} > 0$*, independent of* h*, such that*

$$\begin{aligned} \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{e}_{\boldsymbol{\rho}}\|_{\mathbf{H}} + \|e_\theta\|_{\mathrm{Q}} \le \quad & C_{rate}\, h^{l+1} \Big\{ \|\boldsymbol{\sigma}\|_{l+1,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{\mathbf{W}^{l+1,4/3}(\Omega)} + \|\mathbf{u}\|_{\mathbf{W}^{l+1,4}(\Omega)} \\ & + \|\boldsymbol{\rho}\|_{l+1,\Omega} + \|\mathrm{div}\,\boldsymbol{\rho}\|_{\mathrm{W}^{l+1,4/3}(\Omega)} + \|\theta\|_{\mathrm{W}^{l+1,4}(\Omega)} \Big\}. \end{aligned}$$

Proof. The result is a straightforward application of Theorem 5.1 and the approximation properties ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{\boldsymbol{\rho}}$), and ($\mathbf{AP}_h^{\theta}$). $\square$

5.3 Computing further variables of interest

In this section we introduce suitable approximations for further variables of interest, such as the pressure p, the stress tensor $\widetilde{\boldsymbol{\sigma}}$, the vorticity $\boldsymbol{\omega}$, the velocity gradient $\nabla\mathbf{u}$ and the heat-flux vector $\widetilde{\boldsymbol{\rho}}$, all of them written in terms of the solution of the discrete problem (4.1). To that end we let $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_h \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ be the discrete solution of problem (4.1). Then, inspired by the formulas in (2.15) and (2.16), we propose the following approximations for the aforementioned variables:

$$
\begin{aligned}
p_h &= -\frac{1}{n}\left(\operatorname{tr}(\boldsymbol{\sigma}_h) + \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h) - \frac{1}{|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h)\right), \\
\widetilde{\boldsymbol{\sigma}}_h &= \boldsymbol{\sigma}_h^d + (\mathbf{u}_h \otimes \mathbf{u}_h)^d + \boldsymbol{\sigma}_h^t + \mathbf{u}_h \otimes \mathbf{u}_h - \left(\frac{1}{n|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h)\right)\mathbb{I}
\end{aligned} \tag{5.16}
$$

$$
\boldsymbol{\omega}_h = \frac{1}{2\,\nu}\left(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathrm{t}}\right), \quad \mathbf{G}_h = \frac{1}{\nu}\left(\boldsymbol{\sigma}_h^{\mathrm{d}} + (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}}\right), \quad \widetilde{\boldsymbol{\rho}}_h = -(\boldsymbol{\rho}_h + \theta_h\,\mathbf{u}_h).
$$

The following corollary establishes the convergence result for this post-processing procedure.

Corollary 5.3 *Assume that the hypotheses of Theorem 5.1 hold and let* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \theta) \in \mathbb{X}_0 \times \mathbf{M} \times \mathbf{H} \times \mathrm{Q}$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\rho}_h, \theta_h) \in \mathbb{X}_{h,0} \times \mathbf{M}_h \times \mathbf{H}_h \times \mathrm{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.14) *and* (4.1)*, respectively. Let* p_h*,* $\widetilde{\boldsymbol{\sigma}}_h$*,* $\boldsymbol{\omega}_h$*,* $\mathbf{G}_h$ *and* $\widetilde{\boldsymbol{\rho}}_h$ *given by* (5.16)*. Assume further that* $\boldsymbol{\sigma} \in \mathbb{H}^{l+1}(\Omega)$*,* $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{W}^{l+1,4/3}(\Omega)$*,* $\mathbf{u} \in \mathbf{W}^{l+1,4}(\Omega)$*,* $\boldsymbol{\rho} \in \mathbf{H}^{l+1}(\Omega)$*,* $\operatorname{div}\boldsymbol{\rho} \in \mathrm{W}^{l+1,4/3}(\Omega)$ *and* $\theta \in \mathrm{W}^{l+1,4}(\Omega)$ *, for* $0 \le l \le k$*. Then there exists* $\widehat{C}_{rate} > 0$*, independent of* h*, such that*

$$
\begin{aligned}
& \|p - p_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\nabla\mathbf{u} - \mathbf{G}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\rho}} - \widetilde{\boldsymbol{\rho}}_h\|_{0,\Omega} \\
& \le \widehat{C}_{rate}\, h^{l+1}\Big\{\|\boldsymbol{\sigma}\|_{l+1,\Omega} + \|\mathbf{div}\boldsymbol{\sigma}\|_{\mathbf{W}^{l+1,4/3}(\Omega)} + \|\mathbf{u}\|_{\mathbf{W}^{l+1,4}(\Omega)} \\
& + \|\boldsymbol{\rho}\|_{l+1,\Omega} + \|\operatorname{div}\boldsymbol{\rho}\|_{\mathrm{W}^{l+1,4/3}(\Omega)} + \|\theta\|_{\mathrm{W}^{l+1,4}(\Omega)}\Big\}.
\end{aligned}
$$

Proof. Recalling the formulas given in (2.16) and (5.16), and employing suitable algebraic manipulations it is not difficult to show that there exist $\widehat{C}_1, \widehat{C}_2 > 0$, independents of h, such that the following estimates hold:

$$
\|p - p_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} + \|\nabla\mathbf{u} - \mathbf{G}_h\|_{0,\Omega} \le \widehat{C}_1\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbb{X}} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{M}}\right\},
$$

and

$$
\|\widetilde{\boldsymbol{\rho}} - \widetilde{\boldsymbol{\rho}}_h\|_{0,\Omega} \le \widehat{C}_2\Big\{\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{\mathbf{H}} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{M}} + \|\theta - \theta_h\|_{\mathrm{Q}}\Big\}.
$$

Then, the result follows straightforwardly from Theorem 5.2. We omit further details. □

6 Numerical results

In this section we present three numerical examples to illustrate the performance of our mixed finite element scheme (4.1) on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a *FreeFem++* code, in conjunction with the direct linear solver *UMFPACK*. Regarding the resolution of the non-linear problem, we utilize the algorithm utilized to define the fixed-point operator $\mathcal{J}_h$. More precisely, starting with $(\mathbf{u}_h^0, \theta_h^0) \in \mathbf{M}_h \times \mathrm{Q}_h$, we propose the following iterative process: for each $i = 1, 2, \ldots,$ solve

$$
\begin{aligned}
a_{\mathtt{T}}(\boldsymbol{\rho}_h^i, \boldsymbol{\eta}_h) + b_{\mathtt{T}}(\boldsymbol{\eta}_h, \theta_h^i) + c_{\mathtt{T}}(\mathbf{u}_h^{(i-1)}; \theta_h^i, \boldsymbol{\eta}_h) &= F_{\mathtt{T}}(\boldsymbol{\eta}_h) \quad \forall\, \boldsymbol{\eta}_h \in \mathbf{H}_h, \\
b_{\mathtt{T}}(\boldsymbol{\rho}_h^i, \psi_h) &= 0 \qquad\quad \forall\, \psi_h \in \mathrm{Q}_h,
\end{aligned}
$$

and

$$
\begin{aligned}
a_{\mathtt{F}}(\boldsymbol{\sigma}_h^i, \boldsymbol{\tau}_h) + b_{\mathtt{F}}(\boldsymbol{\tau}_h, \mathbf{u}_h^i) + c_{\mathtt{F}}(\mathbf{u}_h^{(i-1)}; \mathbf{u}_h^i, \boldsymbol{\tau}_h) &= 0 \qquad\qquad\quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{h,0}, \\
b_{\mathtt{F}}(\boldsymbol{\sigma}_h^i, \mathbf{v}_h) &= -d_{\mathtt{F}}(\theta_h^i, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{M}_h.
\end{aligned}
$$

The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|}{\|\mathbf{coeff}^{m+1}\|} \leq \ tol,$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathbb{R}^{dof}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{X}_h$, $\mathbf{M}_h$, $\mathbf{H}_h$ and Q_h, and tol is a specified tolerance.

Now, we introduce some additional notations. The individual errors are denoted by $e(\boldsymbol{\sigma})$, $e(\boldsymbol{\rho})$, $e(\mathbf{u})$, $e(\theta)$, $e(p)$, $e(\widetilde{\boldsymbol{\sigma}})$, $e(\boldsymbol{\omega})$, $e(\nabla\mathbf{u})$ and $e(\widetilde{\boldsymbol{\rho}}_h)$. Also, we let $r(\boldsymbol{\sigma})$, $r(\boldsymbol{\rho})$, $r(\mathbf{u})$, $r(\theta)$, $r(p)$, $r(\widetilde{\boldsymbol{\sigma}})$, $r(\boldsymbol{\omega})$, $r(\nabla\,\mathbf{u})$ and $r(\widetilde{\boldsymbol{\rho}}_h)$ be the experimental rates of convergence given by

$$\begin{aligned}
r(\boldsymbol{\sigma}) &:= \frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad & r(\boldsymbol{\rho}) &:= \frac{\log(e(\boldsymbol{\rho})/e'(\boldsymbol{\rho}))}{\log(h/h')}, \quad & r(\mathbf{u}) &:= \frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')},\\
r(\theta) &:= \frac{\log(e(\theta)/e'(\theta))}{\log(h/h')}, \quad & r(p) &:= \frac{\log(e(p)/e'(p))}{\log(h/h')}, \quad & r(\widetilde{\boldsymbol{\sigma}}) &:= \frac{\log(e(\widetilde{\boldsymbol{\sigma}})/e'(\widetilde{\boldsymbol{\sigma}}))}{\log(h/h')},\\
r(\boldsymbol{\omega}) &:= \frac{\log(e(\boldsymbol{\omega})/e'(\boldsymbol{\omega}))}{\log(h/h')}, \quad & r(\nabla\mathbf{u}) &:= \frac{\log(e(\nabla\mathbf{u})/e'(\nabla\mathbf{u}))}{\log(h/h')}, \quad & r(\widetilde{\boldsymbol{\rho}}) &:= \frac{\log(e(\widetilde{\boldsymbol{\rho}})/e'(\widetilde{\boldsymbol{\rho}}))}{\log(h/h')},
\end{aligned}$$

where h and h' denote two consecutive mesh sizes with their respective errors e and e'.

Example 1. In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega = (0,1)\times(0,1)$ considering the partition of the boundary $\Gamma_{\mathrm{N}} = [0,1]\times\{1\}$ and $\Gamma_{\mathrm{D}} = \partial\Omega\backslash\Gamma_{\mathrm{N}}$. We consider the thermal conductivity $\kappa = 1$, the viscosity of the fluid $\nu = 1$, the external force $g = (0,-1)^t$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\begin{aligned}
\mathbf{u}(x,y) :=& \begin{pmatrix} 2x^2y(x-1)^2(y-1)(2y-1)\\ -2y^2x(x-1)(y-1)^2(2x-1)\end{pmatrix},\\
p(x,y) :=& \ 3x^2+y^2-\frac{4}{3},\\
\theta(x,y) :=& \ \frac{1}{2}\sin(\pi x)\cos^2(\frac{\pi}{2}(y+1)).
\end{aligned}$$

We show in Tables 6.1 and 6.2 the convergence history for a sequence of quasi-uniform mesh refinements when the finite element spaces described in Section 4.1 are used with $k=0$ and $k=1$, respectively. It can be observed there that the rates of convergence are the ones expected from Theorem 5.2 and Corollary 5.3, that is $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively.

Example 2. In our second example we assess the capability of a 3D implementation of the Galerkin scheme (4.1), considering a manufactured exact solution defined on $\Omega = (0,1)^3$ with $\Gamma_{\mathrm{D}} = [0,1]\times[0,1]\times\{0\}$ and $\Gamma_{\mathrm{N}} = \partial\Omega\backslash\Gamma_{\mathrm{D}}$. We consider the thermal conductivity $\kappa = 1$, the viscosity of the fluid $\nu = 1$, the external force $g = (0,0,-1)^t$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\begin{aligned}
\mathbf{u}(x,y,z) :=& \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z)\\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z)\\ \cos(\pi x)\cos(\pi y)\sin(\pi z)\end{pmatrix},\\
p(x,y,z) :=& \ (x-1/2)^3\sin(y+z),\\
\theta(x,y,z) :=& \ \sin^2(\pi x)\sin^2(\pi y)(z-1)^2.
\end{aligned}$$

Errors and rates of convergence for the $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation

h	DOF	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\theta)$	$r(\theta)$	Iter
0.373	294	4.79e-01	–	2.00e-02	–	6.57e-01	–	6.68e-02	–	4
0.196	1188	2.29e-01	1.149	5.51e-03	2.016	2.86e-01	1.302	3.23e-02	1.135	3
0.097	4626	1.13e-01	0.999	1.53e-03	1.819	1.43e-01	0.983	1.66e-02	0.946	3
0.048	18312	5.75e-02	0.960	5.87e-04	1.350	6.96e-02	1.015	7.87e-03	1.053	3
0.025	72939	2.88e-02	1.033	2.63e-04	1.200	3.49e-02	1.034	3.97e-03	1.025	3
0.013	294363	1.42e-02	1.084	1.26e-04	1.135	1.73e-02	1.075	1.96e-03	1.085	3

Postprocessed variables

$e(p)$	$r(p)$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$e(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$e(\widetilde{\boldsymbol{\rho}})$	$r(\widetilde{\boldsymbol{\rho}})$
1.72e-01	–	4.82e-01	–	7.41e-02	–	2.33e-01	–	1.78e-01	–
7.87e-02	1.221	2.45e-01	1.058	3.20e-02	1.310	1.18e-01	1.062	8.33e-02	1.185
3.75e-02	1.052	1.21e-01	1.001	1.51e-02	1.066	5.84e-02	0.999	4.13e-02	0.995
1.88e-02	0.972	6.22e-02	0.939	7.40e-03	1.007	3.00e-02	0.941	2.05e-02	0.989
9.34e-03	1.049	3.12e-02	1.033	3.68e-03	1.044	1.50e-02	1.031	1.04e-02	1.020
4.56e-03	1.099	1.53e-02	1.090	1.84e-03	1.062	7.42e-03	1.085	5.13e-03	1.079

Table 6.1: EXAMPLE 1: Meshsizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximations of the Boussinesq equations.

Errors and rates of convergence for the $\mathbf{RT}_1 - P_1^{\mathrm{dc}} - \mathbf{RT}_1 - P_1^{\mathrm{dc}}$ approximation

h	DOF	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\theta)$	$r(\theta)$	Iter
0.373	912	3.22e-02	–	1.04e-03	–	7.09e-02	–	7.44e-03	–	3
0.196	3744	7.43e-03	2.291	2.62e-04	2.154	1.78e-02	2.161	1.58e-03	2.417	3
0.097	14688	1.92e-03	1.917	6.17e-05	2.050	4.37e-03	1.987	3.75e-04	2.042	3
0.048	58368	4.81e-04	1.956	1.53e-05	1.968	1.15e-03	1.893	1.08e-04	1.761	3
0.025	232944	1.22e-04	2.048	3.97e-06	2.023	2.86e-04	2.076	2.64e-05	2.108	3
0.013	941040	3.02e-05	2.147	9.79e-07	2.145	6.94e-05	2.172	6.41e-06	2.167	3

Postprocessed variables

$e(p)$	$r(p)$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$e(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$e(\widetilde{\boldsymbol{\rho}})$	$r(\widetilde{\boldsymbol{\rho}})$
8.86e-03	–	2.62e-02	–	3.12e-03	–	1.23e-02	–	1.75e-02	–
1.95e-03	2.364	6.18e-03	2.259	6.63e-04	2.417	2.92e-03	2.251	3.76e-03	2.399
4.67e-04	2.027	1.51e-03	1.999	1.54e-04	2.066	7.12e-04	2.000	9.45e-04	1.958
1.17e-04	1.955	3.82e-04	1.939	3.86e-05	1.959	1.81e-04	1.937	2.29e-04	2.000
2.98e-05	2.045	9.81e-05	2.036	9.90e-06	2.036	4.64e-05	2.035	5.96e-05	2.018
7.25e-06	2.169	2.39e-05	2.165	2.43e-06	2.153	1.13e-05	2.163	1.46e-05	2.159

Table 6.2: EXAMPLE 1: Meshsizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbf{RT}_1 - P_1^{\mathrm{dc}} - \mathbf{RT}_1 - P_1^{\mathrm{dc}}$ approximations of the Boussinesq equations.

Errors and rates of convergence for the $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation.

h	DOF	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\theta)$	$r(\theta)$	Iter
0.141	74400	2.62e+01	–	1.24e-01	–	7.03e-01	–	3.82e-02	–	4
0.118	127872	2.18e+01	0,995	1.04e-01	0.990	5.87e-01	0.988	3.19e-02	0.986	4
0.101	202272	1.87e+01	0.997	8.90e-02	0.993	5.04e-01	0.992	2.74e-02	0.990	4
0.088	301056	1.64e+01	0.998	7.79e-02	0.995	4.41e-01	0.993	2.40e-02	0.993	4
0.079	427680	1.46e+01	0.998	6.93e-02	0.996	3.92e-01	0.995	2.13e-02	0.994	4

Postprocessed variables

$e(p)$	$r(p)$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$e(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$e(\widetilde{\boldsymbol{\rho}})$	$r(\widetilde{\boldsymbol{\rho}})$
1.33e-01	–	7.46e-01	–	6.32e-01	–	4.75e-01	–	1.70e-01	–
1.10e-01	1.047	6.23e-01	0.994	5.27e-01	0.997	3.97e-01	0.992	1.42e-01	0.981
9.34e-02	1.063	5.34e-01	0.998	4.52e-01	0.997	3.40e-01	0.994	1.22e-01	0.986
8.10e-01	1.069	4.67e-01	1.000	3.96e-01	0.998	2.98e-01	0.995	1.07e-01	0.990
7.14e-02	1.070	4.15e-01	1.001	3.52e-01	0.998	2.65e-01	0.996	9.51e-02	0.992

Table 6.3: EXAMPLE 2: Meshsizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximations of the three–dimensional Boussinesq equations.

In Table 6.3, we summarize the convergence history for Example 2 considering a sequence of quasi-uniform triangulations. We observe there that the rates of convergence $O(h)$ predicted by Theorem 5.2 and Corollary 5.3 are attained all for the unknowns and for all the post–processed variables. Moreover, in Figures 6.1, 6.2 and 6.3 we compare the exact heat flux vector field, heat velocity vector field and temperature with their approximate counterparts, respectively. There we can observe that the approximate solution captures satisfactorily the behavior of the exact solution.

Example 3. In our third example we study the behavior of a fluid in a square cavity $\Omega = (0,1)^2$ with differentially heated walls. Here the boundary $\partial\Omega$ has been partitioned considering $\Gamma_{\mathrm{N}} = [0,1] \times \{1\}$ and $\Gamma_{\mathrm{D}} = \partial\Omega\backslash\Gamma_{\mathrm{N}}$. This phenomenon has been widely studied with different types of boundary conditions (see, e.g. [5, 20, 22]). In particular, we are interested in the problem with dimensionless numbers: Find $(\mathbf{u}, p, \theta)$ such that

$$\begin{aligned} -\mathrm{Ra}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathrm{Pr}\ \mathrm{Ra}\,\mathbf{g}\,\theta &= 0 \quad \text{in} \quad \Omega, \\ \mathrm{div}\,\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \Gamma, \\ -\kappa\,\Delta\theta + \mathbf{u}\cdot\nabla\theta &= 0 \quad \text{in} \quad \Omega, \\ \theta &= \theta_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}}, \\ \kappa\nabla\theta\cdot\mathbf{n} &= 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}, \end{aligned}$$

where Pr and Ra are the Prandtl and Rayleigh numbers. Here we fix the Prandtl and Rayleigh numbers as $\mathrm{Pr} = 0.5$ and $\mathrm{Ra} = 2000$, the thermal conductivity $\kappa = 1$, and similarly to [20] we choose the boundary condition $\theta_{\mathrm{D}}(x,y) = 0.5(1-\cos(2\pi x))(1-y)$ on Γ_{D}. Here, since the analytical solution is unknown, we construct the convergence history by considering a solution calculated with 1,161,246 DOF as the exact solution, and employing tolerance $tol = 1e-6$ and a $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation on a sequence of uniform triangulations.

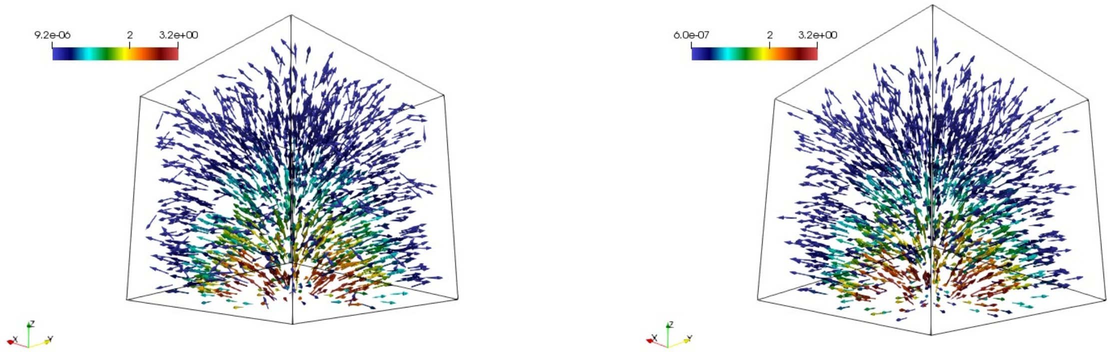

Figure 6.1: EXAMPLE 2: Approximate (left) and exact (right) heat flux vector fields, with $h = 0.079$.

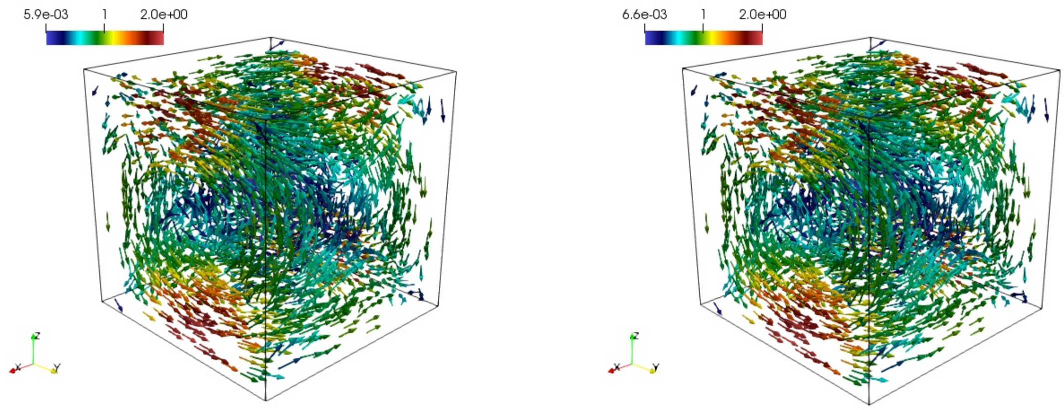

Figure 6.2: EXAMPLE 2: Approximate (left) and exact (right) velocity vector fields, with $h = 0.079$.

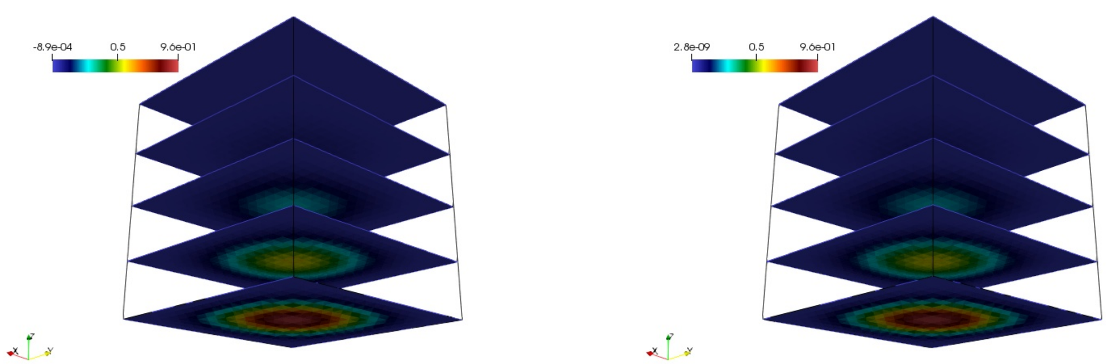

Figure 6.3: EXAMPLE 2: Transversal cuts of the approximate (left) and exact (right) temperatures, with $h = 0.079$.

Errors and rates of convergence for the $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation

h	DOF	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\theta)$	$r(\theta)$	Iter
0.373	294	5.38e+01	–	7.65e-04	–	4.56e-01	–	1.05e-01	–	3
0.196	1188	2.25e+01	1.165	2.58e-04	1.696	2.55e-01	0.909	5.47e-02	1.016	3
0.097	4626	1.30e+01	0.959	8.39e-05	1.594	1.32e-01	0.935	2.86e-02	0.919	3
0.048	18312	6.21e+00	1.042	3.08e-05	1.417	6.67e-02	0.963	1.34e-02	1.074	3
0.025	72939	3.19e+00	0.996	1.40e-05	1.179	3.37e-02	1.023	6.89e-03	0.993	3
0.013	294363	1.64e+00	1.020	6.73e-06	1.122	1.72e-02	1.027	3.51e-03	1.033	3

Postprocessed variables

$e(p)$	$r(p)$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(\boldsymbol{\omega})$	$r(\boldsymbol{\omega})$	$e(\nabla\mathbf{u})$	$r(\nabla\mathbf{u})$	$e(\widetilde{\boldsymbol{\rho}})$	$r(\widetilde{\boldsymbol{\rho}})$
1.35e+01	–	3.40e+01	–	9.65e+03	–	1.56e+01	–	4.56e-01	–
5.96e+00	1.281	1.81e+01	0.980	4.71e+03	1.120	8.69e+00	0.914	2.55e-01	0.909
2.94e+00	0.999	9.54e+00	0.911	2.31e+03	1.011	4.59e+00	0.905	1.32e-01	0.935
1.35e+00	1.101	4.54e+00	1.051	1.17e+03	0.965	2.22e+00	1.030	6.67e-02	0.963
6.93e-01	0.998	2.34e+00	0.990	5.91e+02	1.016	1.14e+00	0.992	3.37e-02	1.023
3.55e-01	1.024	1.20e+00	1.020	3.03e+02	1.027	5.87e-01	1.020	1.72e-02	1.027

Table 6.4: Example 3: Meshsizes, degrees of freedom, errors, rates of convergence, and number of iterations for the mixed $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximations of the Boussinesq equations.

In Figure 6.4 we show the approximated pressure and temperature (top left and bottom left, respectively), along with the approximated velocity and heat-flux vector fields (top right and bottom right, respectively). There, it is possible to see the expected physical behaviour from [20], that is, convection currents form inside the cavity in a symmetric configuration and, due to the relatively low Rayleigh number, the heat transfer throughout the fluid is mainly due to conduction. On the other hand, since the solution is smooth, it makes sense to expect convergence of $O(h)$ when our method is applied with $k = 0$; a fact that can be verified from the results in Table 6.4. Finally, in order to illustrate the conservativity property of our method, in Table 6.5 we display the l^∞-norm of $\mathbf{div}\boldsymbol{\sigma}_h + \mathbf{g}\theta_h$ and $\operatorname{div}\boldsymbol{\rho}_h$ for the mixed $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation of the Boussinesq equations. Since $\mathbf{div}\boldsymbol{\sigma}_h$ and $\mathbf{g}\theta_h$ belong to $\mathbf{M}_h$, it should be expected to obtain values close to zero for $\|\mathbf{div}\boldsymbol{\sigma}_h + \mathbf{g}\theta_h\|_{l^\infty}$ and similarly for $\|\operatorname{div}\boldsymbol{\rho}_h\|_{l^\infty}$. The latter is confirmed in Table 6.5.

h	$\|\mathbf{div}\boldsymbol{\sigma}_h + \mathbf{g}\theta_h\|_{l^\infty}$	$\|\operatorname{div}\boldsymbol{\rho}_h\|_{l^\infty}$
0.373	7.105e-14	3.553e-15
0.196	2.274e-13	7.105e-15
0.097	9.095e-13	1.421e-14
0.048	2.274e-12	5.684e-14
0.025	7.276e-12	1.137e-13
0.013	1.455e-11	3.411e-13

Table 6.5: Example 3: Meshsizes and l^∞-norms of $\mathbf{div}\boldsymbol{\sigma}_h + \mathbf{g}\theta_h$ and $\operatorname{div}\boldsymbol{\rho}_h$ for the mixed $\mathbf{RT}_0 - P_0 - \mathbf{RT}_0 - P_0$ approximation of the Boussinesq equations.

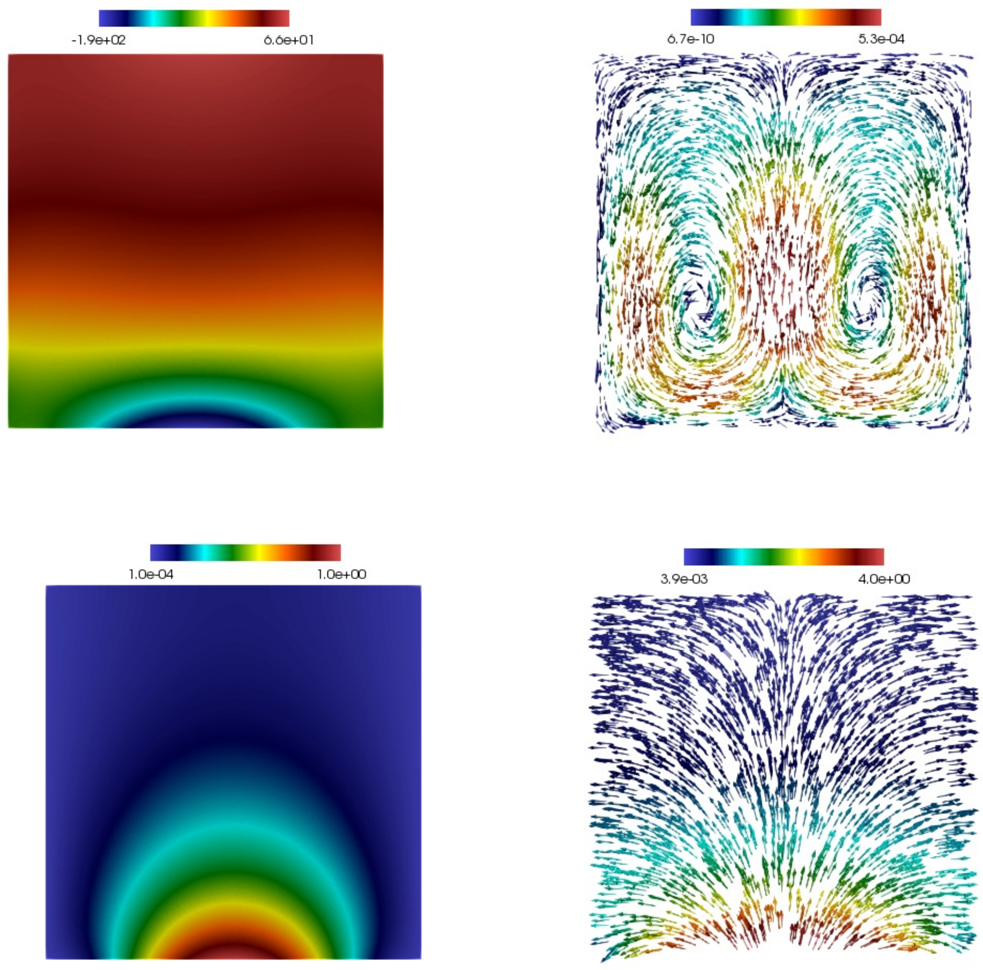

Figure 6.4: EXAMPLE 3: Pressure, velocity vector field (from the left to the right, at the top), temperature and heat flux vector field (from the left to the right, at the bottom).

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