

A Virtual Element Method for the von Kármán Equations

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Abstract

In this article we propose and analyze a Virtual Element Method (VEM) to approximate the isolated solutions of the von Kármán equations, which describes the deformation of very thin elastic plates. We consider a variational formulation in terms of two variables: the transverse displacement of the plate and the Airy stress function. The VEM scheme is conforming in H^2 for both variables and has the advantages of supporting general polygonal meshes and is simple in terms of coding aspects. We prove that the discrete problem is well posed for h small enough and optimal error estimates are obtained. Finally, numerical experiments are reported illustrating the behavior of the virtual scheme on different families of meshes.

Key words: Virtual element method, von Kármán equations, error estimates, polygonal meshes
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1. Introduction

The von Kármán equations is a fourth order system of nonlinear partial differential equations that model the deformation of very thin plates. This system consists of two unknowns, which describe the transverse displacement and the boundary stresses of the plate, respectively [30]. This model has attracted great interest in the scientific community since it is frequently encountered in several engineering applications such as the design of cars and aircrafts (see [35, 48]).

Results on existence of solutions of the von Kármán system have been stated in [30, 32]. In general the problem has not a unique solution. However, sufficient conditions that guarantee uniqueness of isolated solutions are established in [22]. Due to the importance of this problem, several finite element methods have been developed to approximate the isolated solutions of a von Kármán plate. For instance, a general technique based on any conforming discretization was introduced in [22], convergence and optimal error bounds in the energy norm are presented considering the standard formulation in H^2 . Then, in [39] a conforming finite element method was analyzed to approximate the isolated solutions of the von Kármán problem, using Bogner-Fox-Schmit elements, and they also obtained error estimates in H^1 and L^2 norms using a duality argument. On the other hand, to avoid C^1 finite elements, nonconforming discretizations based on Morley finite element methods were proposed in [38] and [40]. In these works, a priori error

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estimates for displacement and Airy stress functions have been established. Lately, a C^0 interior penalty method has been introduced in [19]. The method uses quadratic Lagrange elements to approximate both the transverse displacement and the Airy stress function. Optimal order error estimates are derived. More recently, a discontinuous Galerkin method has been developed in [27]. The authors prove a priori and a posteriori error estimates for the isolated solution of von Kármán equations.

It is well known that conforming finite element spaces of H^2 are of complex implementation and contain high order polynomials (see [31]), for instance, Argyris and Bell finite elements (21 and 18 degrees of freedom per triangle, respectively) or Bogner-Fox-Schmit finite elements (16 degrees of freedom in a rectangle), respectively. In this paper, we will propose a C^1 VEM to approximate the isolated solutions of the von Kármán problem which can be applied to general polygonal meshes (made by possibly non-convex elements). The method will make use of a very simple set of degrees of freedom. For instance, the total computational cost of the proposed VEM method will be $6N_v$, where N_v denotes the number of internal vertices of the polygonal mesh to approximate both the transverse displacement and the Airy stress function.

The VEM was introduced for the first time in [8], as a generalization of the finite element methods by considering polygonal or polyhedral meshes. One of its main characteristics is the possibility to construct and implement in an easy way discrete subspaces of C^α , $\alpha \in \mathbb{N}$. In recent years, the Virtual Elements Method has been a focus of great interest in the scientific community. Several virtual element methods based on conforming and non-conforming schemes have been developed to solve a wide variety of problems in Solid and Fluid Mechanics, for example [4, 5, 6, 9, 10, 11, 13, 18, 24, 26, 29, 41, 45, 46]. Moreover, the VEM for thin structures has been developed in [16, 23, 28, 29, 42, 44], whereas VEM for nonlinear problems have been introduced in [3, 15, 25, 33, 34, 49]

In this paper, we analyze a conforming C^1 Virtual Element Method to approximate the isolated solutions of the von Kármán equations. We consider a variational formulation in terms of the transverse displacement and the Airy stress function, which contains bilinear and trilinear forms. After introducing the local and global virtual space ([5, 23, 29]), we write the discrete problem by constructing discrete version of the bilinear and trilinear forms considering different projectors (polynomial functions) which are computable using only the information of the degrees of freedom of the discrete virtual space. For the analysis, we will adapt some ideas presented in [22] to deal with the variational crimes in the forms and in the right hand side. More precisely, in order to prove that the discrete scheme is well posed, an operator T is defined. Next, it is established that each solution of the discrete problem is a fixed point of this operator (and reciprocally). To prove existence and uniqueness the classical Banach fixed point theorem is employed and some assumptions on the mesh are considered. In particular, for h small enough, we establish that the operator T has a unique fixed point in a proper set which is the unique solution of the discrete problem. Optimal order of convergence in H^2 -norm is established in this work for both unknowns.

The outline of this work is organized as follows. In Section 2 the physical model problem is described. An auxiliary variable that depends of the horizontal load forces applied to the plate is introduced, which allows us to rewrite an equivalent nonlinear system of partial differential equations. Hence, a variational formulation is obtained from this system. In Section 3, we introduce a conforming virtual element discretization and some auxiliary local results are proved. Then, in Section 4, fixed-point arguments are employed to establish that our discrete scheme is well posed. In addition, optimal convergence rate is obtained in this section. Finally, in Section 5, we report some numerical tests that confirm the theoretical analysis developed.

In this article, we will employ standard notations for Sobolev spaces, norms and seminorms. In addition, we will denote by C a generic constant independent of the mesh parameter h , which may take different values in different occurrences. In addition, let X, Y be Hilbert spaces. If $\Pi : X \rightarrow Y$

is a linear and bounded operator, we will denote by $\mathbf{\Pi} : X \times X \rightarrow Y \times Y$ the operator defined by $\mathbf{\Pi}(x, \tilde{x}) := (\mathbf{\Pi}x, \mathbf{\Pi}\tilde{x}), \forall (x, \tilde{x}) \in X \times X$.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal bounded domain with boundary $\Gamma := \partial\Omega$. The von Kármán system can be reads as follows (see e.g [30, 32]): Given a load force $f \in L^2(\Omega)$ and lateral load forces $(\varphi_0, \varphi_1) \in H^{5/2}(\Gamma) \times H^{3/2}(\Gamma)$ (see Fig. 1), find u and ϕ such that

$$\begin{aligned} \Delta^2 u &= [\phi, u] + f && \text{in } \Omega, \\ \Delta^2 \phi &= -\frac{1}{2}[u, u] && \text{in } \Omega, \\ u = \partial_\nu u &= 0 && \text{on } \Gamma, \\ \phi &= \varphi_0 && \text{on } \Gamma, \\ \partial_\nu \phi &= \varphi_1 && \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $[\phi, u]$ is defined by

$$[\phi, u] := \partial_{11}\phi\partial_{22}u + \partial_{22}\phi\partial_{11}u - 2\partial_{12}\phi\partial_{12}u, \tag{2.2}$$

and $\nu := (\nu_1(x, y), \nu_2(x, y))$ denotes the outer unit normal vector to Γ , for all $(x, y) \in \Gamma$.

In the system (2.1), u and φ represent the transverse displacement and the boundary stresses of the plate, respectively. This model is also known as the canonical von Kármán equations. This is a non-linear system of fourth-order partial differential equations established by T. von Kármán in 1910 (see [47]). The existence of solutions of this problem has been proved in [30, 32]. Moreover, it can be seen in [32, Section 2.2] that introducing $\theta_0 \in H_0^2(\Omega)$ as the unique solution of the following problem: find $\theta_0 \in H_0^2(\Omega)$ such that

$$\Delta^2 \theta_0 = 0 \text{ in } \Omega, \quad \theta_0 = \varphi_0, \quad \partial_\nu \theta_0 = \varphi_1 \quad \text{on } \Gamma, \tag{2.3}$$

we have that the system (2.1) can be written equivalently as follows: find $(u, \psi) \in [H_0^2(\Omega)]^2$ such that

$$\begin{aligned} \Delta^2 u &= [\psi, u] + [\theta_0, u] + f && \text{in } \Omega, \\ \Delta^2 \psi &= -\frac{1}{2}[u, u] && \text{in } \Omega, \\ u = \partial_\nu u &= \psi = \partial_\nu \psi = 0 && \text{on } \Gamma, \end{aligned}$$

where $\psi = \phi - \theta_0$.

Moreover, if the lateral load forces on Γ are chosen in (2.3) such that

$$\varphi_0 := -\frac{\lambda}{2}(x^2 + y^2) \quad \text{on } \Gamma, \quad \varphi_1 := -\frac{\lambda}{2}\partial_\nu(x^2 + y^2) \quad \text{on } \Gamma,$$

where λ is a real number called bifurcation parameter that measures the intensity of the horizontal forces (see [32, Section 2.3] or [30, Section 5.9] for further details), hence we obtain that $\theta_0(x, y) = -\frac{\lambda}{2}(x^2 + y^2)$ in Ω solves problem (2.3). In addition, from the definition of $[\cdot, \cdot]$, we obtain that

$$[\theta_0, u] = -\lambda\Delta u \quad \text{in } \Omega.$$

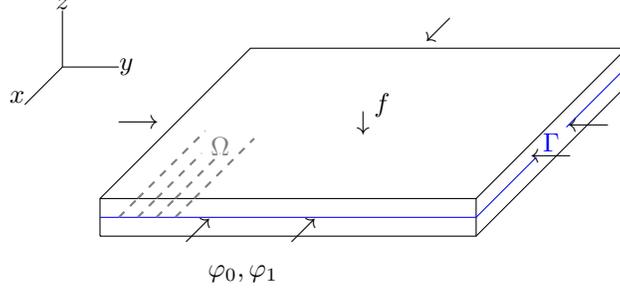


Figure 1: Plate subject to transversal and lateral forces $f \in L^2(\Omega)$ and $(\varphi_0, \varphi_1) \in H^{5/2}(\Gamma) \times H^{3/2}(\Gamma)$, respectively.

As a consequence of the previous definitions, from now on, our aim is to solve the following set of equations: given $(f, \lambda) \in L^2(\Omega) \times \mathbb{R}$, find $(u, \psi) \in [H_0^2(\Omega)]^2$ such that

$$\begin{aligned} \Delta^2 u &= [\psi, u] - \lambda \Delta u + f & \text{in } \Omega, \\ \Delta^2 \psi &= -\frac{1}{2}[u, u] & \text{in } \Omega, \\ u = \partial_\nu u &= \psi = \partial_\nu \psi = 0 & \text{on } \Gamma. \end{aligned} \quad (2.4)$$

Remark 2.1. *The von Kármán model (2.4) does not have a unique solution: For instance, when $f = 0$ and $\lambda > \lambda^*$, where λ^* is the lowest positive real number that satisfies*

$$\Delta^2 u = -\lambda \Delta u \quad \text{in } \Omega \quad \text{and} \quad u = \partial_\nu u = 0 \quad \text{on } \Gamma, \quad (2.5)$$

in this case, the problem (2.4) has at least three solutions: $u_0 = 0$, $u_1 \neq 0$, and $u_2 = -u_1$ (see [30, Theorem 5.9-2]). On the other hand, if f and λ are small enough then the system (2.4) has a unique solution.

Now, testing system (2.4) with functions in $H_0^2(\Omega)$, we arrive at the following weak formulation of the problem: given $(f, \lambda) \in L^2(\Omega) \times \mathbb{R}$, find $(u, \psi) \in [H_0^2(\Omega)]^2$ such that

$$a^\Delta(u, v) + \lambda a^\nabla(u, v) + b(u, \psi, v) + b(\psi, u, v) = F(v) \quad \forall v \in H_0^2(\Omega), \quad (2.6)$$

$$a^\Delta(\psi, \varphi) - b(u; u, \varphi) = 0 \quad \forall \varphi \in H_0^2(\Omega), \quad (2.7)$$

where $a^\Delta, a^\nabla : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}$ are bilinear forms, $b : H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}$ is a trilinear form and $F : H_0^2(\Omega) \rightarrow \mathbb{R}$ is a linear functional, all of them defined as follows:

$$a^\Delta(u, v) := \begin{cases} \int_\Omega \Delta u \Delta v & \forall u, v \in H_0^2(\Omega) \\ \text{or} \\ \int_\Omega D^2 u : D^2 v & \forall u, v \in H_0^2(\Omega), \end{cases} \quad (2.8)$$

$$a^\nabla(u, v) := - \int_\Omega \nabla u \cdot \nabla v \quad \forall u, v \in H_0^2(\Omega), \quad (2.9)$$

$$b(w; u, v) := -\frac{1}{2} \int_\Omega [w, u] v \quad \forall w, u, v \in H_0^2(\Omega), \quad (2.10)$$

$$F(v) := \int_\Omega f v \quad \forall v \in H_0^2(\Omega). \quad (2.11)$$

Remark 2.2. We have proposed two options to write the bilinear form associated to the bilaplacian operator. We will write the discrete method considering both options. In particular, we will employ the bilinear form $a^\Delta(u, v) := \int_\Omega D^2 u : D^2 v$ to construct the projector $\Pi_K^{2,D}$ (cf. (3.1a)-(3.1b)) which will be used to write the discrete schemes with the two options in (2.8).

On the other hand, we endow the space $\mathbb{H} := [H_0^2(\Omega)]^2$ with the corresponding product norm, which we will simply denote by $\|(\cdot, \cdot)\|$.

We rewrite (2.6)-(2.7), in the following equivalent form: given $(f, \lambda) \in L^2(\Omega) \times \mathbb{R}$, find $\mathbf{u} := (u, \psi) \in \mathbb{H}$ such that

$$A^\Delta(\mathbf{u}, \mathbf{v}) + \lambda A^\nabla(\mathbf{u}, \mathbf{v}) + B(\mathbf{u}; \mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} := (v, \varphi) \in \mathbb{H}, \quad (2.12)$$

where, A^Δ, A^∇, B, F are defined as follows:

$$A^\Delta(\mathbf{u}, \mathbf{v}) := a^\Delta(u, v) + a^\Delta(\psi, \varphi) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}, \quad (2.13)$$

$$A^\nabla(\mathbf{u}, \mathbf{v}) := a^\nabla(u, v) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}, \quad (2.14)$$

$$B(\mathbf{w}; \mathbf{u}, \mathbf{v}) := b(w; \psi, v) + b(\xi; u, v) - b(w; u, \varphi) \quad \forall \mathbf{w} := (w, \xi), \mathbf{u}, \mathbf{v} \in \mathbb{H}, \quad (2.15)$$

$$F(\mathbf{v}) := F(v) \quad \forall \mathbf{v} \in \mathbb{H}. \quad (2.16)$$

It is easy to see that the bilinear forms $a^\Delta(\cdot, \cdot)$ and $a^\nabla(\cdot, \cdot)$ are bounded and symmetric, the former is also positive definite on $H_0^2(\Omega) \times H_0^2(\Omega)$ and F is bounded. Moreover, from Lemma 2.2-2 in [32] we have that the trilinear form $b(\cdot; \cdot, \cdot)$ is bounded and symmetric independent of the arguments. Therefore, we have the following result.

Lemma 2.1. *The forms defined in (2.13)-(2.16) satisfy the following properties:*

$$\begin{aligned} |A^\Delta(\mathbf{u}, \mathbf{v})| &\leq \|\mathbf{u}\| \|\mathbf{v}\| \quad \text{and} \quad |A^\nabla(\mathbf{u}, \mathbf{v})| \leq |\lambda| \|\mathbf{u}\| \|\mathbf{v}\| && \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}, \\ |F(\mathbf{v})| &\leq \|f\|_{0,\Omega} \|\mathbf{v}\| && \forall \mathbf{v} \in \mathbb{H}, \\ |B(\mathbf{w}; \mathbf{u}, \mathbf{v})| &\leq C \|\mathbf{w}\| \|\mathbf{u}\| \|\mathbf{v}\| && \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbb{H}, \\ B(\mathbf{w}; \mathbf{u}, \mathbf{v}) &= B(\mathbf{u}; \mathbf{w}, \mathbf{v}) && \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbb{H}. \end{aligned}$$

Now, from [30, Theorem 5.8-3(b)] (see also [37]) we have that the variational formulation (2.12) has at least one solution. Moreover, we present the following additional regularity result for the solution of the von Kármán problem (2.12), which has been proved in [19, Theorem 2.4].

Theorem 2.1. *Let \mathbf{u} be a solution of von Kármán problem (2.12). Then, there exist $s \in (1/2, 1]$ and $C > 0$ such that $\mathbf{u} \in [H^{2+s}(\Omega)]^2$ and*

$$\|\mathbf{u}\|_{2+s,\Omega} \leq C \|f\|_{0,\Omega},$$

where, the constant $s \in (1/2, 1]$ is the Sobolev regularity for the biharmonic equation with the right-hand side in $H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions (see e.g. [36]).

Now, we will introduce some definitions that will be used to establish an existence result of the isolated solutions of the problem (2.12). Given $\mathbf{u} \in \mathbb{H}$ we introduce the following global form.

$$A_{\mathbf{u}}(\mathbf{w}, \mathbf{v}) := A^\Delta(\mathbf{w}, \mathbf{v}) + \lambda A^\nabla(\mathbf{w}, \mathbf{v}) + 2B(\mathbf{u}; \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{w}, \mathbf{v} \in \mathbb{H}. \quad (2.17)$$

Definition 2.1. (see [22]) A solution \mathbf{u} of the system (2.12) is said to be isolated if and only if the linearized problem: given $\mathbf{g} \in [L^2(\Omega)]^2$, find $\mathbf{w} \in \mathbb{H}$ such that

$$A_{\mathbf{u}}(\mathbf{w}, \mathbf{v}) = \int_{\Omega} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{H},$$

has a unique solution and satisfies the following a priori estimates

$$\|\mathbf{w}\| \leq C\|\mathbf{g}\|_{0,\Omega} \quad \text{and} \quad \|\mathbf{w}\|_{2+s,\Omega} \leq C\|\mathbf{g}\|_{0,\Omega},$$

where, the constant $s \in (1/2, 1]$ is the Sobolev regularity for the biharmonic problem with the right-hand side in $H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions (see e.g. [36]).

Now, the following result gives a sufficient condition to obtain an isolated solution of the system (2.12) (see for instance [19, Remark 3.1], or [39, Remark 2.1]–[39, Theorem 2.3]).

Theorem 2.2. If $(f, \lambda) \in L^2(\Omega) \times \mathbb{R}$ are small enough, then the von Kármán system (2.12) has a unique solution and it is isolated.

We finish this section with the following result which will be used to approach the isolated solution of the von Kármán problem.

Theorem 2.3. Assume that the bilinear form $A_{\mathbf{u}}(\cdot, \cdot)$ (cf. (2.17)) is non singular on $\mathbb{H} \times \mathbb{H}$, this means (see e.g [22, Lemma 1]) that there exist two positive constants c_1 and c_2 such that

$$\sup_{\substack{\mathbf{v} \in \mathbb{H} \\ \|\mathbf{v}\|=1}} A_{\mathbf{u}}(\mathbf{w}, \mathbf{v}) \geq c_1 \|\mathbf{w}\| \quad \forall \mathbf{w} \in \mathbb{H}, \quad \text{and} \quad \sup_{\substack{\mathbf{w} \in \mathbb{H} \\ \|\mathbf{w}\|=1}} A_{\mathbf{u}}(\mathbf{w}, \mathbf{v}) \geq c_2 \|\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbb{H}. \quad (2.18)$$

Then, there exists a positive constant δ such that $A_{\tilde{\mathbf{u}}}(\cdot, \cdot)$ is non singular on $\mathbb{H} \times \mathbb{H}$, for all $\tilde{\mathbf{u}}$ that satisfies

$$\|\mathbf{u} - \tilde{\mathbf{u}}\| \leq \delta.$$

Proof. The proof can be obtained repeating the arguments in [22, Lemma 1]. □

3. Discrete problem

In this section, we will introduce a C^1 -VEM discretization to approximate the isolated solutions of a von Kármán plate (cf. Theorem 2.2).

We begin with the mesh construction and the assumptions considered to introduce the discrete virtual element spaces (see e.g [2, 5, 8]). Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements K . We will denote by h_K the diameter of the element K and by h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K \in \mathcal{T}_h} h_K$. In addition, we denote by N_K the number of vertices of K , by e a generic edge of $\{\mathcal{T}_h\}_h$ and for all $e \in \partial K$, we define a unit normal vector ν_K^e that points outside of K .

Moreover, we will make the following assumptions: there exists a positive real number $C_{\mathcal{T}}$ such that, for every h and every $K \in \mathcal{T}_h$:

A1: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}}h_K$;

A2: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$.

The hypotheses **A1** and **A2** though not too restrictive in several practical cases, can be further relaxed, as established in [14].

In order to write the method, we first define the following finite dimensional space (see [5]).

$$\begin{aligned} \tilde{H}_h^K := \{v_h \in H^2(K) : \Delta^2 v_h \in \mathbb{P}_2(K), v_h|_{\partial K} \in C^0(\partial K), v_h|_e \in \mathbb{P}_3(e) \forall e \in \partial K, \\ \nabla v_h|_{\partial K} \in C^0(\partial K)^2, \partial_{\nu_K^e} v_h|_e \in \mathbb{P}_1(e) \forall e \in \partial K\}. \end{aligned}$$

It is easy to see that any $v_h \in \tilde{H}_h^K$ satisfies the following conditions:

- the trace (and the trace of the gradient) on the boundary of K is continuous;
- $\mathbb{P}_2(K) \subseteq \tilde{H}_h^K$.

On the other hand, we introduce two sets of linear operators from \tilde{H}_h^K into \mathbb{R} . For all $v_h \in \tilde{H}_h^K$, they are defined as follows:

D₁: evaluation of v_h at the N_K vertices of K ;

D₂: evaluation of ∇v_h at the N_K vertices of K .

Now, we will introduce some preliminary definitions in order to construct the discrete scheme. Let $a_K^D : H^2(K) \times H^2(K) \rightarrow \mathbb{R}$ be defined as follows:

$$a_K^D(u, v) := \int_K D^2 u : D^2 v, \quad u, v \in H^2(K).$$

We build the projection operator $\Pi_K^{2,D} : H^2(K) \rightarrow \mathbb{P}_2(K) \subseteq \tilde{H}_h^K$, defined by the unique solution of the following local problem:

$$a_K^D(\Pi_K^{2,D} v, q) = a_K^D(v, q) \quad \forall q \in \mathbb{P}_2(K), \quad (3.1a)$$

$$\widehat{\Pi_K^{2,D} v} = \widehat{v}, \quad \widehat{\nabla \Pi_K^{2,D} v} = \widehat{\nabla v}, \quad (3.1b)$$

where \widehat{v} is defined as follows:

$$\widehat{v} := \frac{1}{N_K} \sum_{i=1}^{N_K} v(\mathbf{v}_i) \quad \forall v \in C^0(\partial K)$$

and $\mathbf{v}_i, 1 \leq i \leq N_K$, are the vertices of K .

We note that the bilinear form $a_K^D(\cdot, \cdot)$ has a non-trivial kernel, given by $\mathbb{P}_1(K)$. Thus, the role of condition (3.1b) is to choose an element of the kernel of the operator. Hence, we have that the operator $\Pi_K^{2,D}$ is well defined on \tilde{H}_h^K and is computable from the output values of the sets **D₁** and **D₂** (see [5, 23]).

Next, we introduce our local virtual space:

$$H_h^K := \left\{ v_h \in \tilde{H}_h^K : \int_K (v_h - \Pi_K^{2,D} v_h) q = 0, \quad \forall q \in \mathbb{P}_2(K) \right\}.$$

Note that $H_h^K \subseteq \tilde{H}_h^K$. This allows us to obtain the well definition of $\Pi_K^{2,D}$ on H_h^K , and therefore to prove that $\Pi_K^{2,D}$ is computable from the output values of operators **D₁** and **D₂**. Moreover, it is

easy to check that $\mathbb{P}_2(K) \subseteq H_h^K$. This will guarantee the good approximation properties for the virtual space.

On the other hand, the following result which has been proved in [5] guarantees that any function $v_h \in H_h^K$ is uniquely determined by the output values of the sets \mathbf{D}_1 and \mathbf{D}_2 .

Lemma 3.1. *The set of operators \mathbf{D}_1 and \mathbf{D}_2 constitutes a set of degrees of freedom for the space H_h^K .*

Now, we are in a position to introduce our global virtual space for the transverse displacement and the Airy stress function:

$$H_h := \{v_h \in H_0^2(\Omega) : v_h|_K \in H_h^K\}. \quad (3.2)$$

3.1. Construction of bilinear and trilinear forms and the loading term.

In this subsection we will propose discrete version of the local forms to construct the discrete virtual scheme.

We begin by introducing new projectors: For $\ell = 0, 1, 2$, we consider $\Pi_K^\ell : L^2(K) \rightarrow \mathbb{P}_\ell(K)$ the standard L^2 -orthogonal projector defined as follows:

$$\int_K \Pi_K^\ell v q = \int_K v q \quad \forall q \in \mathbb{P}_\ell(K) \quad \forall v \in L^2(K). \quad (3.3)$$

Now, due to the particular property appearing in definition of the space H_h^K , it can be seen that the right hand side in (3.3) is computable using $\Pi_K^{2,D} v$. Thus, $\Pi_K^\ell v$, $\ell = 0, 1, 2$ depends only on the values of the degrees of freedom given by the sets \mathbf{D}_1 and \mathbf{D}_2 . Furthermore, it is easy to check that on the space H_h^K the projectors Π_K^2 and $\Pi_K^{2,D}$ are the same operator. In fact:

$$\int_K (\Pi_K^2 v_h) q = \int_K v_h q = \int_K (\Pi_K^{2,D} v_h) q \quad \forall q \in \mathbb{P}_2(K), \quad \forall v_h \in H_h^K.$$

We introduce the projector $\Pi_K^{2,\nabla} : H^1(K) \rightarrow \mathbb{P}_2(K)$, for each $v \in H^1(K)$ as the solution of

$$\begin{aligned} a_K^\nabla(\Pi_K^{2,\nabla} v, q) &= a_K^\nabla(v, q) \quad \forall q \in \mathbb{P}_2(K), \\ \int_K \Pi_K^{2,\nabla} v &= \int_K v. \end{aligned}$$

The following lemma proved in [5] establishes that operator $\Pi_K^{2,\nabla}$ is fully computable on the local virtual space H_h^K .

Lemma 3.2. *The operator $\Pi_K^{2,\nabla} : H_h^K \rightarrow \mathbb{P}_2(K) \subseteq H_h^K$ is well defined and depends only on the values of the degrees of freedom given by the sets \mathbf{D}_1 and \mathbf{D}_2 .*

Following the standard procedure in VEM literature (see for instance [2, 8, 23, 29]), we propose the following (computable) discrete local bilinear forms:

$$a_{h,K}^\Delta : H_h^K \times H_h^K \rightarrow \mathbb{R}; \quad \text{and} \quad a_{h,K}^\nabla : H_h^K \times H_h^K \rightarrow \mathbb{R}; \quad (3.5)$$

defined by

$$a_{h,K}^\Delta(u_h, v_h) := \begin{cases} \int_K \Delta \Pi_K^{2,D} u_h \Delta \Pi_K^{2,D} v_h + \bar{\alpha} s_K^D (u_h - \Pi_K^{2,D} u_h, v_h - \Pi_K^{2,D} v_h), \\ \text{or} \\ \int_K D^2 \Pi_K^{2,D} u_h : D^2 \Pi_K^{2,D} v_h + \bar{\alpha} s_K^D (u_h - \Pi_K^{2,D} u_h, v_h - \Pi_K^{2,D} v_h), \end{cases} \quad (3.6)$$

$$a_{h,K}^{\nabla}(u_h, v_h) := -\left\{ \int_K \mathbf{\Pi}_K^1 \nabla u_h \cdot \mathbf{\Pi}_K^1 \nabla v_h + \widehat{\alpha} s_K^{\nabla}(u_h - \Pi_K^{2,\nabla} u_h, v_h - \Pi_K^{2,\nabla} v_h) \right\}, \quad (3.7)$$

respectively, where $\mathbf{\Pi}_K^1 : L^2(K)^2 \rightarrow \mathbb{P}_1(K)^2$ is the standard L^2 -orthogonal projector and $s_K^{\Delta}(\cdot, \cdot)$ and $s_K^{\nabla}(\cdot, \cdot)$ are two symmetric positive definite bilinear forms satisfying the following conditions:

$$c_0 a_K^D(v_h, v_h) \leq s_K^D(v_h, v_h) \leq c_1 a_K^D(v_h, v_h) \quad \forall v_h \in H_h^K \quad \text{with} \quad \Pi_K^{2,D} v_h = 0, \quad (3.8)$$

$$c_2 a_K^{\nabla}(v_h, v_h) \leq s_K^{\nabla}(v_h, v_h) \leq c_3 a_K^{\nabla}(v_h, v_h) \quad \forall v_h \in H_h^K \quad \text{with} \quad \Pi_K^{2,\nabla} v_h = 0, \quad (3.9)$$

respectively. In addition, in (3.6)-(3.7), $\bar{\alpha}$ and $\widehat{\alpha}$ are constants which depend on the physical parameters.

Remark 3.1. In (3.7) the vector function $\mathbf{\Pi}_K^1 \nabla v_h$ is fully computable from the degrees of freedom given by the sets \mathbf{D}_1 and \mathbf{D}_2 (see for instance [44, Section 3] for further details).

Now, we define the following global discrete bilinear forms on H_h .

$$a_h^{\Delta}(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_{h,K}^{\Delta}(u_h, v_h), \quad u_h, v_h \in H_h,$$

$$a_h^{\nabla}(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_{h,K}^{\nabla}(u_h, v_h), \quad u_h, v_h \in H_h.$$

The following result establishes the usual properties of consistency and stability for the local virtual forms.

Proposition 3.1. The local bilinear forms $a_{h,K}^{\Delta}(\cdot, \cdot)$ and $a_{h,K}^{\nabla}(\cdot, \cdot)$ on each element K satisfy

- *Consistency:* for all $h > 0$ and for all $K \in \mathcal{T}_h$, we have that

$$a_{h,K}^{\Delta}(q, v_h) = a_K^{\Delta}(q, v_h) \quad \forall q \in \mathbb{P}_2(K), \quad \forall v_h \in H_h^K, \quad (3.10)$$

$$a_{h,K}^{\nabla}(q, v_h) = a_K^{\nabla}(q, v_h) \quad \forall q \in \mathbb{P}_2(K), \quad \forall v_h \in H_h^K. \quad (3.11)$$

- *Stability and boundedness:* There exist positive constants $\alpha_i, i = 1, \dots, 4$ independent of K , such that:

$$\alpha_1 a_K^D(v_h, v_h) \leq a_{h,K}^{\Delta}(v_h, v_h) \leq \alpha_2 a_K^D(v_h, v_h) \quad \forall v_h \in H_h^K, \quad (3.12)$$

$$\alpha_3 a_K^{\nabla}(v_h, v_h) \leq a_{h,K}^{\nabla}(v_h, v_h) \leq \alpha_4 a_K^{\nabla}(v_h, v_h) \quad \forall v_h \in H_h^K. \quad (3.13)$$

Proof. Since the proof can be followed from standard arguments in the Virtual Element literature (see [2, 18]), it is omitted. \square

On the other hand, we will propose on each element K the following local (and computable) approximation for the trilinear form $b(\cdot; \cdot, \cdot)$ (cf. (2.10)):

$$b_{h,K}(w_h; u_h, v_h) := -\frac{1}{2} \int_K [\Pi_K^{2,D} w_h, \Pi_K^{2,D} u_h] \Pi_K^2 v_h. \quad (3.14)$$

Now, we are going to introduce the discrete version of (2.13)-(2.16). First, let us define $\mathbb{H}_h := H_h \times H_h$, $\mathbb{H}_h^K := H_h^K \times H_h^K$ and let $A_h^\Delta, A_h^\nabla, F_h, B_h$ be the discrete forms given by:

$$\begin{aligned} A_h^\Delta : \mathbb{H}_h \times \mathbb{H}_h &\rightarrow \mathbb{R}; & A_h^\Delta(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{K \in \mathcal{T}_h} A_{h,K}^\Delta(\mathbf{u}_h, \mathbf{v}_h); \\ A_h^\nabla : \mathbb{H}_h \times \mathbb{H}_h &\rightarrow \mathbb{R}; & A_h^\nabla(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{K \in \mathcal{T}_h} A_{h,K}^\nabla(\mathbf{u}_h, \mathbf{v}_h); \\ F_h : \mathbb{H}_h &\rightarrow \mathbb{R}; & F_h(\mathbf{v}_h) &:= \sum_{K \in \mathcal{T}_h} F_{h,K}(\mathbf{v}_h); \\ B_h : \mathbb{H}_h \times \mathbb{H}_h \times \mathbb{H}_h &\rightarrow \mathbb{R}; & B_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) &:= \sum_{K \in \mathcal{T}_h} B_{h,K}(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h); \end{aligned}$$

where

$$\begin{aligned} A_{h,K}^\Delta(\mathbf{u}_h, \mathbf{v}_h) &:= a_{h,K}^\Delta(u_h, v_h) + a_{h,K}^\Delta(\psi_h, \varphi_h); \\ A_{h,K}^\nabla(\mathbf{u}_h, \mathbf{v}_h) &:= a_{h,K}^\nabla(u_h, v_h); \\ F_{h,K}(\mathbf{v}_h) &:= \int_K \Pi_K^2 f v_h \equiv \int_K \Pi_K^2 f \Pi_K^{2,D} v_h \equiv \int_K f \Pi_K^{2,D} v_h; \\ B_{h,K}(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) &:= b_{h,K}(w_h; \psi_h, v_h) + b_{h,K}(\xi_h; u_h, v_h) - b_{h,K}(w_h; u_h, \varphi_h) \\ &= -\frac{1}{2} \int_K \left\{ [\Pi_K^{2,D} w_h, \Pi_K^{2,D} \psi_h] \Pi_K^2 v_h + [\Pi_K^{2,D} \xi_h, \Pi_K^{2,D} u_h] \Pi_K^2 v_h - [\Pi_K^{2,D} w_h, \Pi_K^{2,D} u_h] \Pi_K^2 \varphi_h \right\}, \end{aligned}$$

with $\mathbf{u}_h := (u_h, \psi_h)$, $\mathbf{v}_h := (v_h, \varphi_h)$, $\mathbf{w}_h := (w_h, \xi_h) \in \mathbb{H}_h$.

Now, we are ready to propose the virtual element scheme to approximate the isolated solutions of the von Kármán problem: given $(f, \lambda) \in L^2(\Omega) \times \mathbb{R}$, find $\mathbf{u}_h := (u_h, \psi_h) \in \mathbb{H}_h$ such that

$$A_h^\Delta(\mathbf{u}_h, \mathbf{v}_h) + \lambda A_h^\nabla(\mathbf{u}_h, \mathbf{v}_h) + B_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = F_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h := (v_h, \varphi_h) \in \mathbb{H}_h. \quad (3.15)$$

In the next section we are going to prove the well posedness of the discrete problem (3.15), provided that (f, λ) is small enough. To do that we will use a fix point argument. With this aim, we present that following discrete version of Lemma 2.1.

Lemma 3.3. *There exist positive constants α, C and C_B independent of h such that*

$$|A_h^\Delta(\mathbf{u}_h, \mathbf{v}_h)| \leq \|\mathbf{u}_h\| \|\mathbf{v}_h\| \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h; \quad (3.16a)$$

$$|A_h^\nabla(\mathbf{u}_h, \mathbf{v}_h)| \leq |\lambda| \|\mathbf{u}_h\| \|\mathbf{v}_h\| \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h; \quad (3.16b)$$

$$A_h^\Delta(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha \|\mathbf{v}_h\|^2 \quad \forall \mathbf{v}_h \in \mathbb{H}_h; \quad (3.16c)$$

$$|F_h(\mathbf{v}_h)| \leq \|f\|_{0,\Omega} \|\mathbf{v}_h\| \quad \forall \mathbf{v}_h \in \mathbb{H}_h; \quad (3.16d)$$

$$|B_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h)| \leq C \|\mathbf{w}_h\| \|\mathbf{u}_h\| \|\mathbf{v}_h\| \quad \forall \mathbf{w}_h, \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h; \quad (3.16e)$$

$$B_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) = B_h(\mathbf{u}_h; \mathbf{w}_h, \mathbf{v}_h) \quad \forall \mathbf{w}_h, \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h. \quad (3.16f)$$

Proof. The proof follows from (3.8)-(3.9) and Proposition 3.1. \square

We end this section with some definitions and results which will be used in Section 4 to prove the solvability of the discrete problem (3.15).

First, we introduce the following broken seminorm and projectors:

$$|v|_{\ell,h}^2 := \sum_{K \in \mathcal{T}_h} |v|_{\ell,K}^2 \quad \forall v \in L^2(\Omega) : v|_K \in H^\ell(K) \quad \ell = 1, 2.$$

Now, for all $v \in L^2(\Omega)$ such that $v|_K \in H^2(K)$ for all $K \in \mathcal{T}_h$, we define $\Pi_h^{2,D}$ in $L^2(\Omega)$ as follows

$$(\Pi_h^{2,D} v)|_K := \Pi_K^{2,D}(v|_K) \quad \forall K \in \mathcal{T}_h. \quad (3.17)$$

Next, we present the following standard approximation results. Proposition 3.2 is derived by interpolation between Sobolev spaces from the analogous result for integer values of s . In fact, this result for integer values is stated in [8, Proposition 4.2] and follows from the classical Scott-Dupont theory (see [20] and [5, Proposition 3.1]). Proposition 3.3 has been proved in [16, Proposition 4.2]. Proposition 3.4 and Lemma 3.4 can be seen for instance in [44, Section 3] and [42, Lemma 3.5], respectively.

Proposition 3.2. *If the assumption **A1** is satisfied, then there exists a constant $C > 0$, such that for every $v \in H^\delta(K)$ there exists $v_\pi \in \mathbb{P}_k(K)$, $k \geq 0$ such that*

$$|v - v_\pi|_{\ell, K} \leq Ch_K^{\delta - \ell} |v|_{\delta, K} \quad 0 \leq \delta \leq k + 1, \ell = 0, \dots, [\delta],$$

with $[\delta]$ denoting the largest integer equal or smaller than $\delta \in \mathbb{R}$.

Proposition 3.3. *Assume **A1**–**A2** are satisfied, let $v \in H^{2+s}(\Omega)$ with $s \in (1/2, 1]$. Then, there exist $v_I \in H_h$ and $C > 0$ independent of h such that*

$$|v - v_I|_{2, \Omega} \leq Ch_K^s |v|_{2+s, \Omega}.$$

Proposition 3.4. *There exists $C > 0$ independent of h such that for all $\mathbf{v} \in H^\delta(K)^2$*

$$\|\mathbf{v} - \mathbf{\Pi}_K^1 \mathbf{v}\|_{0, K} \leq Ch_K^\delta |\mathbf{v}|_{\delta, K} \quad 0 \leq \delta \leq 2,$$

where $\mathbf{\Pi}_K^1 : L^2(K)^2 \rightarrow \mathbb{P}_1(K)^2$ is the standard L^2 -orthogonal projector (cf. Remark 3.1).

Lemma 3.4. *Let $v_h \in H_h$. Then, there exists $C > 0$ such that*

$$\|v_h - \mathbf{\Pi}_h^{2,D} v_h\|_{0, \Omega} \leq Ch^2 \|v_h\|_{2, \Omega}.$$

4. Analysis of the discrete problem

The purpose of this section is to prove that problem (3.15) admits a unique solution. With this end, from now on, we assume that \mathbf{u} is an isolated solution of the von Kármán system (2.12).

Now, from Theorem 2.1, we know that $\mathbf{u} = (u, \psi) \in [H^{2+s}(\Omega)]^2$ with $s \in (1/2, 1]$. Then, from Proposition 3.3 we have that there exists $\mathbf{u}_I := (u_I, \psi_I) \in \mathbb{H}_h$ (from now on \mathbf{u}_I denotes the interpolated of \mathbf{u}) such that

$$\|\mathbf{u} - \mathbf{u}_I\| \leq Ch^s \|\mathbf{u}\|_{2+s, \Omega}, \quad (4.18)$$

where, u_I and ψ_I are the interpolants of u and ψ , respectively.

In order to establish the well posedness of the discrete problem (3.15), we need to introduce some definitions. Let $A_{h, \mathbf{u}_I} : \mathbb{H}_h \times \mathbb{H}_h \rightarrow \mathbb{R}$ be the discrete form defined by

$$A_{h, \mathbf{u}_I}(\mathbf{w}_h, \mathbf{v}_h) := A_h^\Delta(\mathbf{w}_h, \mathbf{v}_h) + \lambda A_h^\nabla(\mathbf{w}_h, \mathbf{v}_h) + 2B_h(\mathbf{u}_I; \mathbf{w}_h, \mathbf{v}_h) \quad \forall \mathbf{w}_h, \mathbf{v}_h \in \mathbb{H}_h. \quad (4.19)$$

We also define the operator

$$\begin{aligned} T : \mathbb{H}_h &\longrightarrow \mathbb{H}_h \\ \mathbf{w}_h &\longmapsto T \mathbf{w}_h, \end{aligned}$$

where, $T \mathbf{w}_h \in \mathbb{H}_h$ is the unique solution (to be proved below) of the following problem. Find $T \mathbf{w}_h \in \mathbb{H}_h$ such that

$$A_{h, \mathbf{u}_I}(T \mathbf{w}_h, \mathbf{v}_h) = 2B_h(\mathbf{u}_I; \mathbf{w}_h, \mathbf{v}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \mathbf{v}_h) + F_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{H}_h. \quad (4.20)$$

It is easy to check that any solution \mathbf{u}_h of the discrete problem (3.15) is a fixed point of T and reciprocally.

Now, we focus on proving that T is well defined and then we will use contraction and fixed point arguments to establish that T has a unique fixed point. To do that, we first need to prove an auxiliary lemma, which follows the argument presented in the proof of [22, Lemma 2].

Lemma 4.1. *Let $\tilde{\mathbf{v}} \in \mathbb{H}$ such that $\|\tilde{\mathbf{v}}\| = 1$. Then, the following problem: find $\tilde{\mathbf{v}}_h \in \mathbb{H}_h$ such that*

$$A_h^\Delta(\tilde{\mathbf{v}}_h, \mathbf{z}_h) = A^\Delta(\tilde{\mathbf{v}}, \mathbf{z}_h) \quad \forall \mathbf{z}_h \in \mathbb{H}_h, \quad (4.21)$$

has a unique solution and satisfies the following a priori estimates

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega} \leq Ch^t \quad \text{and} \quad \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{\infty,\Omega} \leq Ch^{t/4},$$

where, the constant $t \in (1/2, 1]$ is the Sobolev regularity for the biharmonic equation with the right-hand side in $H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions (see e.g. [36]), and C is a positive constant independent of h .

Proof. We know from Lemma 2.1 that $A^\Delta(\tilde{\mathbf{v}}, \mathbf{z}_h) \leq C\|\mathbf{z}_h\|$ for all $\mathbf{z}_h \in \mathbb{H}_h$. Then, from (3.16a), (3.16c), and Lax-Milgram Lemma, we obtain that the problem (4.21) has a unique solution and satisfies the following estimate (see [20])

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\| \leq C\|\tilde{\mathbf{v}}\| = C \quad (4.22)$$

for some positive constant C independent of h .

On the other hand, we will use duality arguments to obtain an error bound for $\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}$. In fact, we consider the following problem: find $\mathbf{r} \in \mathbb{H}$ such that

$$\begin{aligned} \Delta^2 \mathbf{r} &= \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h & \text{in } \Omega, \\ \mathbf{r} = \partial_\nu \mathbf{r} &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (4.23)$$

It is well known that problem (4.23) has a unique solution (see for instance [36]) and that there exists a positive constant $t \in (1/2, 1]$ such that

$$\|\mathbf{r}\|_{2+t,\Omega} \leq C\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}. \quad (4.24)$$

Now, from (4.24) and Proposition 3.3 we have that there exist $\mathbf{r}_I \in \mathbb{H}_h$ and $C > 0$ independent of h such that

$$\|\mathbf{r} - \mathbf{r}_I\|_{2,\Omega} \leq Ch^t \|\mathbf{r}\|_{2+t,\Omega}. \quad (4.25)$$

Next, by multiplying the system (4.23) by $\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h \in \mathbb{H}$, integrating by parts twice, adding and subtracting \mathbf{r}_I , using the symmetry of $A_h^\Delta(\cdot, \cdot)$ and (4.21) we obtain

$$\begin{aligned} \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}^2 &= A^\Delta(\mathbf{r}, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) = A^\Delta(\mathbf{r} - \mathbf{r}_I, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) + A^\Delta(\mathbf{r}_I, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) \\ &= A^\Delta(\mathbf{r} - \mathbf{r}_I, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) + A^\Delta(\tilde{\mathbf{v}}, \mathbf{r}_I) - A^\Delta(\tilde{\mathbf{v}}_h, \mathbf{r}_I) \\ &= A^\Delta(\mathbf{r} - \mathbf{r}_I, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) + A_h^\Delta(\tilde{\mathbf{v}}_h, \mathbf{r}_I) - A^\Delta(\tilde{\mathbf{v}}_h, \mathbf{r}_I). \end{aligned} \quad (4.26)$$

Now, from (4.24) and Proposition 3.2 there exists $\mathbf{r}_\pi \in [\mathbb{P}_2(K)]^2$ such that

$$\|\mathbf{r} - \mathbf{r}_\pi\|_{2,K} \leq Ch^t \|\mathbf{r}\|_{2+t,K} \quad \forall K \in \mathcal{T}_h. \quad (4.27)$$

Thus, adding and subtracting \mathbf{r}_π , and using the consistency property (3.10) on the right hand side of (4.26), we obtain that

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}^2 = A^\Delta(\mathbf{r} - \mathbf{r}_I, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) + \sum_{K \in \mathcal{T}_h} \{A_{h,K}^\Delta(\tilde{\mathbf{v}}_h, \mathbf{r}_I - \mathbf{r}_\pi) + A_K^\Delta(\tilde{\mathbf{v}}_h, \mathbf{r}_\pi - \mathbf{r}_I)\}.$$

Next, using Lemma 2.1, Cauchy-Schwarz inequality, Poincaré inequality, (3.12) and finally adding and subtracting \mathbf{r} on the right hand side of the above term, we obtain

$$\begin{aligned} & \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}^2 \\ & \leq C\|\mathbf{r} - \mathbf{r}_I\| \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\| + \sum_{K \in \mathcal{T}_h} \{A_{h,K}^\Delta(\tilde{\mathbf{v}}_h, \tilde{\mathbf{v}}_h)^{1/2} A_{h,K}^\Delta(\mathbf{r}_I - \mathbf{r}_\pi, \mathbf{r}_I - \mathbf{r}_\pi)^{1/2} \\ & \quad + A_K^\Delta(\tilde{\mathbf{v}}_h, \tilde{\mathbf{v}}_h)^{1/2} A_K^\Delta(\mathbf{r}_I - \mathbf{r}_\pi, \mathbf{r}_I - \mathbf{r}_\pi)^{1/2}\} \\ & \leq C\|\mathbf{r} - \mathbf{r}_I\| \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\| + (\alpha_2 + 1) \sum_{K \in \mathcal{T}_h} |\tilde{\mathbf{v}}_h|_{2,K} |\mathbf{r}_I - \mathbf{r}_\pi|_{2,K} \\ & \leq C\|\mathbf{r} - \mathbf{r}_I\| \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\| + (\alpha_2 + 1) \sum_{K \in \mathcal{T}_h} |\tilde{\mathbf{v}}_h|_{2,K} \left(|\mathbf{r} - \mathbf{r}_\pi|_{2,K} + |\mathbf{r} - \mathbf{r}_I|_{2,K} \right). \end{aligned}$$

Then, using (4.25), (4.22), (4.27), and (4.24) in the above term, we infer

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega}^2 \leq Ch^t \|\mathbf{r}\|_{2+t,\Omega} \|\tilde{\mathbf{v}}\| = Ch^t \|\mathbf{r}\|_{2+t,\Omega} \leq Ch^t \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega},$$

and therefore, we conclude that

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega} \leq Ch^t. \quad (4.28)$$

Now, using interpolation theory [20, Chapter 14] on the estimates (4.22) and (4.28), we obtain that for all $\delta \in [0, 2]$ the following estimate holds true

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{\delta,\Omega} \leq Ch^{t(2-\delta)/2}.$$

Then, using the Sobolev injection of $H^{1+\sigma}(\Omega)$ (for all $\sigma > 0$) in $L^\infty(\Omega)$ (see e.g [1, Theorem 4.12]) we have in particular that

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{\infty,\Omega} \leq \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{\tilde{\delta},\Omega} \leq Ch^{t(2-\tilde{\delta})/2}$$

for some $\tilde{\delta} \in (1, 2)$. In particular, taking $\tilde{\delta} = 3/2$, we conclude the proof. \square

Now, we will use Lemma 4.1 to prove that operator T is well defined. More precisely, we will show in the following result that $A_{h,\mathbf{u}_I}(\cdot, \cdot)$ is non singular (cf. (2.18)) on $\mathbb{H}_h \times \mathbb{H}_h$.

Lemma 4.2. *Let $\mathbf{u} = (u, \psi) \in \mathbb{H}$ be an isolated solution of problem (2.12). Then, for h small enough we have that the bilinear form $A_{h,\mathbf{u}_I}(\cdot, \cdot)$ (cf. (4.19)) is non singular on $\mathbb{H}_h \times \mathbb{H}_h$.*

Proof. Since the discrete space $\mathbb{H}_h \subseteq \mathbb{H}$ we will proceed as in [22, Lemma 2]. However, we have to deal with the approximation of the bilinear and trilinear forms in our case. We recall that at the discrete level it is enough to verify one of the two inequality in (2.18) for bilinear form $A_{h,\mathbf{u}_I}(\cdot, \cdot)$. We will prove that there exists a constant $C > 0$ independent of h such that

$$\sup_{\substack{\mathbf{v}_h \in \mathbb{H}_h \\ \|\mathbf{v}_h\|=1}} A_{h,\mathbf{u}_I}(\mathbf{w}_h, \mathbf{v}_h) \geq C\|\mathbf{w}_h\| \quad \forall \mathbf{w}_h \in \mathbb{H}_h. \quad (4.29)$$

Indeed, because of \mathbf{u} is isolated, we have from Definition 2.1 that $A_{\mathbf{u}}(\cdot, \cdot)$ (cf. (2.17)) is non singular on $\mathbb{H} \times \mathbb{H}$. Then, the following result is a consequence of the fact that $A_{\mathbf{u}}(\cdot, \cdot)$ is non singular, $\mathbf{u}_I, \mathbf{w}_h \in \mathbb{H}_h \subset \mathbb{H}$, (4.18) and Theorem 2.3.

$$\sup_{\substack{\mathbf{v} \in \mathbb{H} \\ \|\mathbf{v}\|=1}} A_{\mathbf{u}_I}(\mathbf{w}_h, \mathbf{v}) \geq \bar{c}_1 \|\mathbf{w}_h\| \quad \forall \mathbf{w}_h := (w_h, \xi_h) \in \mathbb{H}_h.$$

Next, we can choose $\tilde{\mathbf{v}} := (\tilde{v}, \tilde{\varphi}) \in \mathbb{H}$ with $\|\tilde{\mathbf{v}}\| = 1$ such that

$$A_{\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}}) \geq \bar{c}_1 \|\mathbf{w}_h\|. \quad (4.30)$$

Moreover, from Lemma 4.1 we have that: given $\tilde{\mathbf{v}} = (\tilde{v}, \tilde{\varphi})$, there exists $\tilde{\mathbf{v}}_h := (\tilde{v}_h, \tilde{\varphi}_h)$ and $t \in (1/2, 1]$ such that

$$\|\tilde{v} - \tilde{v}_h\|_{0,\Omega} \leq \|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{0,\Omega} \leq Ch^t, \quad (4.31)$$

and

$$\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h\|_{\infty,\Omega} \leq Ch^{t/4}. \quad (4.32)$$

Now, from the left hand side of (4.29), normalizing $\tilde{\mathbf{v}}_h$, using the definition of $A_{h,\mathbf{u}_I}(\cdot, \cdot)$ in (4.20) and $A_{\mathbf{u}_I}(\cdot, \cdot)$ in (2.17), we obtain

$$\begin{aligned} \sup_{\substack{\mathbf{v}_h \in \mathbb{H}_h \\ \|\mathbf{v}_h\|=1}} A_{h,\mathbf{u}_I}(\mathbf{w}_h, \mathbf{v}_h) &\geq A_{h,\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}}_h) = A_{\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}}) + \{A_{h,\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}}_h) - A_{\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}})\} \\ &= A_{\mathbf{u}_I}(\mathbf{w}_h, \tilde{\mathbf{v}}) + \lambda \left\{ A_h^\nabla(\mathbf{w}_h, \tilde{\mathbf{v}}_h) - A^\nabla(\mathbf{w}_h, \tilde{\mathbf{v}}) \right\} + 2 \left\{ B_h(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) - B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}) \right\} \\ &\geq \bar{c}_1 \|\mathbf{w}_h\| + \lambda \underbrace{\left\{ A_h^\nabla(\mathbf{w}_h, \tilde{\mathbf{v}}_h) - A^\nabla(\mathbf{w}_h, \tilde{\mathbf{v}}) \right\}}_{E_1} + 2 \underbrace{\left\{ B_h(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) - B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}) \right\}}_{E_2}, \end{aligned} \quad (4.33)$$

where, in the last step we have used (4.30). We note that the terms E_1 and E_2 have appeared because of the approximation of the bilinear and trilinear forms.

In what follows, we will prove that the terms E_1, E_2 are bounded. Indeed, for E_1 we use the definition of $\mathbf{\Pi}_K^1$, add and subtract the term $\nabla w_h \cdot \nabla \tilde{v}_h$ and integrate by parts to obtain

$$\begin{aligned} E_1 &= \sum_{K \in \mathcal{T}_h} \left\{ \int_K \{ \nabla w_h \cdot \nabla \tilde{v} - \mathbf{\Pi}_K^1 \nabla w_h \cdot \mathbf{\Pi}_K^1 \nabla \tilde{v}_h \} + s_K^\nabla (w_h - \mathbf{\Pi}_K^{2,\nabla} w_h, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h) \right\} \\ &= \sum_{K \in \mathcal{T}_h} \left\{ \int_K \nabla w_h \cdot \{ \nabla \tilde{v} - \mathbf{\Pi}_K^1 \nabla \tilde{v}_h \} + s_K^\nabla (w_h - \mathbf{\Pi}_K^{2,\nabla} w_h, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h) \right\} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \left\{ \nabla w_h \cdot \{ \nabla \tilde{v}_h - \mathbf{\Pi}_K^1 \nabla \tilde{v}_h \} + \nabla w_h \cdot \nabla (\tilde{v} - \tilde{v}_h) + s_K^\nabla (w_h - \mathbf{\Pi}_K^{2,\nabla} w_h, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h) \right\} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \left\{ \nabla w_h \cdot \{ \nabla \tilde{v}_h - \mathbf{\Pi}_K^1 \nabla \tilde{v}_h \} + s_K^\nabla (w_h - \mathbf{\Pi}_K^{2,\nabla} w_h, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h) \right\} + \int_\Omega \Delta w_h (\tilde{v}_h - \tilde{v}). \end{aligned}$$

Next, applying Cauchy-Schwarz inequality three times, using (3.9) and the definition of $\mathbf{\Pi}_K^{2,\nabla}$ on the right hand side above, we get

$$\begin{aligned}
E_1 &\leq \sum_{K \in \mathcal{T}_h} \left\{ |w_h|_{1,K} \|\Pi_K^1 \nabla \tilde{v}_h - \nabla \tilde{v}_h\|_{0,K} + s_K^\nabla (w_h - \Pi_K^{2,\nabla} w_h, w_h - \Pi_K^{2,\nabla} w_h)^{1/2} \right. \\
&\quad \left. \times s_K^\nabla (\tilde{v}_h - \Pi_K^{2,\nabla} \tilde{v}_h, \tilde{v}_h - \Pi_K^{2,\nabla} \tilde{v}_h)^{1/2} \right\} + \|\Delta w_h\|_{0,\Omega} \|\tilde{v} - \tilde{v}_h\|_{0,\Omega} \\
&\leq \sum_{K \in \mathcal{T}_h} \left\{ |w_h|_{1,K} \|\Pi_K^1 \nabla \tilde{v}_h - \nabla \tilde{v}_h\|_{0,K} + c_3 |w_h - \Pi_K^{2,\nabla} w_h|_{1,K} |\tilde{v}_h - \Pi_K^{2,\nabla} \tilde{v}_h|_{1,K} \right\} \\
&\quad + \|\Delta w_h\|_{0,\Omega} \|\tilde{v} - \tilde{v}_h\|_{0,\Omega} \\
&\leq \sum_{K \in \mathcal{T}_h} \left\{ |w_h|_{1,K} \|\Pi_K^1 \nabla \tilde{v}_h - \nabla \tilde{v}_h\|_{0,K} + c_3 |w_h - w_h^\pi|_{1,K} |\tilde{v}_h - \tilde{v}_h^\pi|_{1,K} \right\} \\
&\quad + \|\Delta w_h\|_{0,\Omega} \|\tilde{v} - \tilde{v}_h\|_{0,\Omega},
\end{aligned}$$

for $w_h^\pi, \tilde{v}_h^\pi \in \mathbb{P}_2(K)$ such that Proposition 3.2 holds true with respect to w_h and \tilde{v}_h . Therefore, from Propositions 3.4 and 3.2, and (4.31), we conclude that

$$E_1 \leq C \sum_{K \in \mathcal{T}_h} \left\{ h_K |w_h|_{1,K} |\nabla \tilde{v}_h|_{1,K} + c_3 C h_K^2 |w_h|_{2,K} |\tilde{v}_h|_{2,K} \right\} + C h^t \|\Delta w_h\|_{0,\Omega} \leq C h^t \|w_h\|, \quad (4.34)$$

with $t \in (1/2, 1]$ and where we have used in the last inequality the fact that $\|\tilde{v}_h\|_{2,\Omega} \leq \|\tilde{\mathbf{v}}_h\| = 1$.

Now, let us find a bound for the term E_2 in (4.33). Indeed, in E_2 , we add and subtract $B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h)$ to obtain

$$\begin{aligned}
E_2 &= \{B_h(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) - B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h)\} + B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h - \tilde{\mathbf{v}}) \\
&= B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h - \tilde{\mathbf{v}}) + \{B_h(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) - B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h)\}. \quad (4.35)
\end{aligned}$$

Now, for first term on the right hand side above we have the following estimate,

$$B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h - \tilde{\mathbf{v}}) = \int_{\Omega} [\mathbf{u}_I, \mathbf{w}_h] (\tilde{\mathbf{v}}_h - \tilde{\mathbf{v}}) \leq C \|\mathbf{u}_I\| \|\mathbf{w}_h\| \|\tilde{\mathbf{v}}_h - \tilde{\mathbf{v}}\|_{\infty,\Omega} \leq C h^{t/4} \|\mathbf{u}_I\| \|\mathbf{w}_h\|, \quad (4.36)$$

with, $t \in (1/2, 1]$ and where we have used the fact that $(\tilde{\mathbf{v}} - \tilde{\mathbf{v}}_h) \in L^\infty(\Omega)$ and estimate (4.32).

On the other hand, for the second term on right hand side of (4.35), we use the definitions of $B_h(\cdot; \cdot, \cdot)$ and $B(\cdot; \cdot, \cdot)$ to get

$$\begin{aligned}
&B_h(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) - B(\mathbf{u}_I; \mathbf{w}_h, \tilde{\mathbf{v}}_h) \\
&= -\frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K \left\{ \left([\Pi_K^{2,D} u_I, \Pi_K^{2,D} \xi_h] \Pi_K^2 \tilde{v}_h - [u_I, \xi_h] \tilde{v}_h \right) + \left([\Pi_K^{2,D} \psi_I, \Pi_K^{2,D} w_h] \Pi_K^2 \tilde{v}_h - [\psi_I, w_h] \tilde{v}_h \right) \right. \\
&\quad \left. - \left([\Pi_K^{2,D} u_I, \Pi_K^{2,D} w_h] \Pi_K^2 \tilde{\varphi}_h - [u_I, w_h] \tilde{\varphi}_h \right) \right\} =: -\frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_K \left\{ \mathbf{B}^{1,K} + \mathbf{B}^{2,K} - \mathbf{B}^{3,K} \right\}. \quad (4.37)
\end{aligned}$$

Now, we will prove that the terms $\int_K \mathbf{B}^{1,K}$, $\int_K \mathbf{B}^{2,K}$ and $\int_K (-\mathbf{B}^{3,K})$ are bounded. Indeed, for

$\int_K \mathbf{B}^{1,K}$, we use the definitions of Π_K^2 and $[\cdot, \cdot]$ (cf. (3.3) and (2.2), respectively) to obtain

$$\begin{aligned}
\int_K \mathbf{B}^{1,K} &= \int_K \left([\Pi_K^{2,D} u_I, \Pi_K^{2,D} \xi_h] - [u_I, \xi_h] \right) \tilde{v}_h \\
&= \int_K \left\{ \left\{ \left(\partial_{xx} \Pi_K^{2,D} u_I \right) \left(\partial_{yy} \Pi_K^{2,D} \xi_h \right) \tilde{v}_h - \left(\partial_{xx} u_I \right) \left(\partial_{yy} \xi_h \right) \tilde{v}_h \right\} \right. \\
&\quad + \left\{ \left(\partial_{yy} \Pi_K^{2,D} u_I \right) \left(\partial_{xx} \Pi_K^{2,D} \xi_h \right) \tilde{v}_h - \left(\partial_{yy} u_I \right) \left(\partial_{xx} \xi_h \right) \tilde{v}_h \right\} \\
&\quad \left. - 2 \left\{ \left(\partial_{xy} \Pi_K^{2,D} u_I \right) \left(\partial_{xy} \Pi_K^{2,D} \xi_h \right) \tilde{v}_h - \left(\partial_{xy} u_I \right) \left(\partial_{xy} \xi_h \right) \tilde{v}_h \right\} \right\} =: \int_K \left\{ \boldsymbol{\alpha} + \boldsymbol{\beta} - 2\boldsymbol{\gamma} \right\}. \quad (4.38)
\end{aligned}$$

Next, we will estimate the term $\int_K \boldsymbol{\alpha}$. The terms $\int_K \boldsymbol{\beta}$ and $\int_K (-2)\boldsymbol{\gamma}$ can be treated with the same arguments. Thus, applying the identity $\partial_{ij} \Pi_K^{2,D} v_h = \Pi_K^0(\partial_{ij} v_h)$ for all $v_h \in H_h^K$ and for all $i, j = x, y$, (which is a consequence of the definitions of the projectors $\Pi_K^{2,D}$ and Π_K^0 (cf. (3.1a) and (3.3), respectively)) in the expression $\int_K \boldsymbol{\alpha}$, we obtain

$$\int_K \boldsymbol{\alpha} = \int_K \left\{ \Pi_K^0(\partial_{xx} u_I) \Pi_K^0(\partial_{yy} \xi_h) \tilde{v}_h - (\partial_{xx} u_I) (\partial_{yy} \xi_h) \tilde{v}_h \right\}. \quad (4.39)$$

Now, adding and subtracting the terms $\Pi_K^0(\partial_{xx} u) \Pi_K^0(\partial_{yy} \xi_h) \tilde{v}_h$, $\Pi_K^0(\partial_{xx} u) (\partial_{yy} \xi_h) \tilde{v}_h$ and $(\partial_{yy} \xi_h) (\partial_{xx} u) \tilde{v}_h$, and using the definitions of Π_K^0 on the right hand side of (4.39), we get

$$\begin{aligned}
\int_K \boldsymbol{\alpha} &= \int_K \left\{ \left\{ \Pi_K^0(\partial_{xx}(u_I - u)) \right\} \Pi_K^0(\partial_{yy} \xi_h) \tilde{v}_h + \Pi_K^0(\partial_{xx} u) \left\{ (\Pi_K^0 - I)(\partial_{yy} \xi_h) \right\} \tilde{v}_h \right. \\
&\quad \left. + (\partial_{yy} \xi_h) \left\{ (\Pi_K^0 - I)(\partial_{xx} u) \right\} \tilde{v}_h + (\partial_{yy} \xi_h) (\partial_{xx}(u - u_I)) \tilde{v}_h \right\} \\
&= \int_K \left\{ \left\{ \Pi_K^0(\partial_{xx}(u_I - u)) \right\} \Pi_K^0(\partial_{yy} \xi_h) \tilde{v}_h + \Pi_K^0(\partial_{xx} u) \left\{ (\Pi_K^0 - I)(\partial_{yy} \xi_h) \right\} \left\{ (I - \Pi_K^0) \tilde{v}_h \right\} \right. \\
&\quad \left. + (\partial_{yy} \xi_h) \left\{ (\Pi_K^0 - I)(\partial_{xx} u) \right\} \tilde{v}_h + (\partial_{yy} \xi_h) (\partial_{xx}(u - u_I)) \tilde{v}_h \right\}
\end{aligned}$$

Then, applying Cauchy-Schwarz and Hölder inequalities, using the fact that $\tilde{v}_h \in L^\infty(K)$, and Π_K^0 is bounded in the L^2 -norm, on the right hand side of the last equality, we obtain

$$\begin{aligned}
\int_K \boldsymbol{\alpha} &\leq \|\Pi_K^0(\partial_{xx}(u_I - u))\|_{0,K} \|\Pi_K^0(\partial_{yy} \xi_h)\|_{0,K} \|\tilde{v}_h\|_{\infty,K} \\
&\quad + \|\Pi_K^0(\partial_{xx} u)\|_{L^4(K)} \|(\Pi_K^0 - I)(\partial_{yy} \xi_h)\|_{0,K} \|(I - \Pi_K^0) \tilde{v}_h\|_{L^4(K)} \\
&\quad + \|\partial_{yy} \xi_h\|_{0,K} \|(\Pi_K^0 - I)(\partial_{xx} u)\|_{0,K} \|\tilde{v}_h\|_{\infty,K} + \|\partial_{yy} \xi_h\|_{0,K} \|\partial_{xx}(u - u_I)\|_{0,K} \|\tilde{v}_h\|_{\infty,K} \\
&\leq \|\partial_{xx}(u_I - u)\|_{0,K} \|\partial_{yy} \xi_h\|_{0,K} \|\tilde{v}_h\|_{\infty,K} \\
&\quad + \|\Pi_K^0(\partial_{xx} u)\|_{L^4(K)} \|\partial_{yy} \xi_h\|_{0,K} \|(I - \Pi_K^0) \tilde{v}_h\|_{L^4(K)} \\
&\quad + \|\partial_{yy} \xi_h\|_{0,K} \|(\Pi_K^0 - I)(\partial_{xx} u)\|_{0,K} \|\tilde{v}_h\|_{\infty,K} + \|\partial_{yy} \xi_h\|_{0,K} \|\partial_{xx}(u - u_I)\|_{0,K} \|\tilde{v}_h\|_{\infty,K} \\
&\leq Ch_K^s |u|_{2+s,K} \|\xi_h\|_{2,K} + \|\Pi_K^0(\partial_{xx} u)\|_{L^4(K)} \|\xi_h\|_{2,K} \|(I - \Pi_K^0) \tilde{v}_h\|_{L^4(K)}, \quad (4.40)
\end{aligned}$$

where in the last step, we have used Theorem 2.1, Propositions 3.3 and 3.2, and the fact that $\|\tilde{v}_h\|_{\infty,K} \leq \|\tilde{v}_h\|_{2,K} \leq \|\tilde{\mathbf{v}}_h\| \leq C\|\tilde{\mathbf{v}}\| = C$.

Now, to bound the second term on the right hand side of (4.40), it is easy to check that $\|\Pi_K^0(\partial_{xx}u)\|_{L^4(K)} \leq C\|\partial_{xx}u\|_{L^4(K)}$ (see Proposition 3.3 in [15]). Moreover, using the fact that $H^s(K) \hookrightarrow L^4(K)$ for $s \in (1/2, 1]$ (see [7, Theorem 7.3.7(a)]), we obtain

$$\begin{aligned} \int_K \boldsymbol{\alpha} &\leq Ch_K^s |u|_{2+s,K} |\xi_h|_{2,K} + |\partial_{xx}u|_{s,K} |\xi_h|_{2,K} \|(I - \Pi_K^0)\tilde{v}_h\|_{L^4(K)} \\ &\leq Ch_K^s |u|_{2+s,K} |\xi_h|_{2,K} + Ch_K |\partial_{xx}u|_{s,K} |\xi_h|_{2,K} |\tilde{v}_h|_{W^{1,4}(K)} \\ &\leq Ch_K^s |u|_{2+s,K} |\xi_h|_{2,K} + Ch_K |\partial_{xx}u|_{s,K} |\xi_h|_{2,K} \|\tilde{v}_h\|_{2,K} \\ &\leq Ch_K^s |u|_{2+s,K} |\xi_h|_{2,K}, \end{aligned} \quad (4.41)$$

where, we have used that $\|(I - \Pi_K^0)\tilde{v}_h\|_{L^4(K)} \leq Ch_K |\tilde{v}_h|_{W^{1,4}(K)}$ (see [20, Section 4.4]) and the Sobolev imbedding $H^2(K) \hookrightarrow W^{1,4}(K)$ (see [7, Theorem 7.3.7(b)]).

Moreover, repeating the same steps used in (4.39)-(4.41), we can prove the following estimates

$$\int_K \boldsymbol{\beta} \leq Ch_K^s |\xi_h|_{2,K} |u|_{2+s,K} \quad \text{and} \quad \int_K (-2)\boldsymbol{\gamma} \leq Ch_K^s |\xi_h|_{2,K} |u|_{2+s,K}. \quad (4.42)$$

Hence, from (4.38) and (4.40),(4.42), we obtain

$$\int_K \mathbf{B}^{1,K} \leq Ch_K^s |u|_{2+s,K} |\xi_h|_{2,K}. \quad (4.43)$$

Now, we observe that the terms in (4.37) can be bounded repeating the same arguments used to bound $\mathbf{B}^{1,K}$. Thus,

$$\int_K \mathbf{B}^{2,K} \leq Ch_K^s |\psi|_{2+s,K} |w_h|_{2,K} \quad \text{and} \quad \int_K (-1)\mathbf{B}^{3,K} \leq Ch_K^s |u|_{2+s,K} |w_h|_{2,K}.$$

As a consequence, we have that

$$\int_K \mathbf{B}^{2,K} - \int_K \mathbf{B}^{3,K} \leq Ch_K^s \{|\psi|_{2+s,K} |w_h|_{2,K} + |u|_{2+s,K} |w_h|_{2,K}\}. \quad (4.44)$$

Therefore, using (4.36) and taking sum over K in (4.43) and (4.44), we have the following bound for the term E_2 in (4.35)

$$E_2 \leq Ch^{\tilde{s}} \|\mathbf{u}\|_{2+s,\Omega} \|\mathbf{w}_h\|, \quad (4.45)$$

with $\tilde{s} := \min\{s, t/4\}$, and where $s, t \in (1/2, 1]$.

Hence, from (4.34), (4.45) and Theorem 2.1 we get the following estimate

$$\lambda E_1 + 2E_2 \leq Ch^{\tilde{s}} \|\mathbf{u}\|_{2+\tilde{s},\Omega} \|\mathbf{w}_h\| \leq Ch^{\tilde{s}} \|f\|_{0,\Omega} \|\mathbf{w}_h\|. \quad (4.46)$$

Finally, from (4.46), there exists $h_0 := \left(\frac{\bar{c}_1}{2C\|f\|_{0,\Omega}}\right)^{1/\tilde{s}} > 0$ such that for all $h \leq h_0$, the following result is deduced from (4.33)

$$\sup_{\substack{\mathbf{v}_h \in \mathbb{H}_h \\ \|\mathbf{v}_h\|=1}} A_{h,\mathbf{u}_I}(\mathbf{w}_h, \mathbf{v}_h) \geq \bar{c}_1 \|\mathbf{w}_h\| - Ch^{\tilde{s}} \|f\|_{0,\Omega} \|\mathbf{w}_h\| \geq \frac{\bar{c}_1}{2} \|\mathbf{w}_h\|.$$

The proof is complete. \square

Now, in what follows, we will focus on proving that the operator T satisfies the hypotheses of Banach fixed-point theorem [21, Theorem 5.7]. With this aim, first from Lemma 4.2 we will prove that the operator T maps the ball

$$\mathcal{B}(\mathbf{u}_I, R) := \{\mathbf{w}_h \in \mathbb{H}_h : \|\mathbf{w}_h - \mathbf{u}_I\| \leq R\}$$

to itself, where $R := R(h) > 0$ is a positive real number depending on h which will be specified later in Lemma 4.4 and we recall that \mathbf{u}_I is the interpolant of the isolated solution \mathbf{u} . We first need to prove a technical lemma.

Lemma 4.3. *Let $\mathbf{w}_h \in \mathbb{H}_h$. For h small enough, there exists a positive constant C , independent of h , such that*

$$\|T \mathbf{w}_h - \mathbf{u}_I\| \leq C(h^s + \|\mathbf{u}_I - \mathbf{w}_h\|^2),$$

where $s \in (1/2, 1]$ is the Sobolev exponent for the solution of the von Kármán problem (see Theorem 2.1).

Proof. Let $\mathbf{w}_h \in \mathbb{H}_h$. Since $T \mathbf{w}_h - \mathbf{u}_I \in \mathbb{H}_h$, we have from Lemma 4.2 (cf. (4.29)) that there exists $\tilde{c}_1 > 0$ such that

$$\tilde{c}_1 \|T \mathbf{w}_h - \mathbf{u}_I\| \leq \sup_{\substack{\mathbf{v}_h \in \mathbb{H}_h \\ \|\mathbf{v}_h\|=1}} A_{h, \mathbf{u}_I}(T \mathbf{w}_h - \mathbf{u}_I, \mathbf{v}_h).$$

Now, we can choose $\bar{\mathbf{v}}_h \in \mathbb{H}_h \subseteq \mathbb{H}$ with $\|\bar{\mathbf{v}}_h\| = 1$ such that

$$\tilde{c}_1 \|T \mathbf{w}_h - \mathbf{u}_I\| \leq A_{h, \mathbf{u}_I}(T \mathbf{w}_h - \mathbf{u}_I, \bar{\mathbf{v}}_h). \quad (4.47)$$

Next, using the definitions of T and $A_{h, \mathbf{u}_I}(\cdot, \cdot)$ (cf. (4.20) and (4.19)), and adding the continuous variational formulation (2.12) tested with $\bar{\mathbf{v}}_h$ on the right hand side of (4.47), we obtain

$$\begin{aligned} \tilde{c}_1 \|T \mathbf{w}_h - \mathbf{u}_I\| &\leq A_{h, \mathbf{u}_I}(T \mathbf{w}_h - \mathbf{u}_I, \bar{\mathbf{v}}_h) = A_{h, \mathbf{u}_I}(T \mathbf{w}_h, \bar{\mathbf{v}}_h) - A_{h, \mathbf{u}_I}(\mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &= 2B_h(\mathbf{u}_I; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) + F_h(\bar{\mathbf{v}}_h) - A_h^\Delta(\mathbf{u}_I, \bar{\mathbf{v}}_h) - \lambda A_h^\nabla(\mathbf{u}_I, \bar{\mathbf{v}}_h) - 2B_h(\mathbf{u}_I; \mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &= 2B_h(\mathbf{u}_I; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) + F_h(\bar{\mathbf{v}}_h) - A_h^\Delta(\mathbf{u}_I, \bar{\mathbf{v}}_h) - \lambda A_h^\nabla(\mathbf{u}_I, \bar{\mathbf{v}}_h) - 2B_h(\mathbf{u}_I; \mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &\quad + A^\Delta(\mathbf{u}, \bar{\mathbf{v}}_h) + \lambda A^\nabla(\mathbf{u}, \bar{\mathbf{v}}_h) + B(\mathbf{u}; \mathbf{u}, \bar{\mathbf{v}}_h) - F(\bar{\mathbf{v}}_h) \\ &= \underbrace{\left\{ 2B_h(\mathbf{u}_I; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{u}_I; \mathbf{u}_I, \bar{\mathbf{v}}_h) \right\}}_{G_1} + \underbrace{\left\{ F_h(\bar{\mathbf{v}}_h) - F(\bar{\mathbf{v}}_h) \right\}}_{G_2} \\ &\quad + \underbrace{\left\{ A^\Delta(\mathbf{u}, \bar{\mathbf{v}}_h) - A_h^\Delta(\mathbf{u}_I, \bar{\mathbf{v}}_h) \right\}}_{G_3} + \underbrace{\lambda \left\{ A^\nabla(\mathbf{u}, \bar{\mathbf{v}}_h) - A_h^\nabla(\mathbf{u}_I, \bar{\mathbf{v}}_h) \right\}}_{G_4} + \underbrace{\left\{ B(\mathbf{u}; \mathbf{u}, \bar{\mathbf{v}}_h) - B_h(\mathbf{u}_I; \mathbf{u}_I, \bar{\mathbf{v}}_h) \right\}}_{G_5}. \end{aligned} \quad (4.48)$$

In what follows, we want to bound all the terms $G_i, i = 1, \dots, 5$ defined in (4.48). Indeed, using the properties of the trilinear form, we rewrite G_1 , then applying the identity (3.16f) and (3.16e) and the fact that $\|\bar{\mathbf{v}}_h\| = 1$, we get

$$\begin{aligned} G_1 &= B_h(\mathbf{u}_I; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{u}_I; \mathbf{u}_I, \bar{\mathbf{v}}_h) + B_h(\mathbf{u}_I; \mathbf{w}_h, \bar{\mathbf{v}}_h) \\ &= B_h(\mathbf{u}_I - \mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) + B_h(\mathbf{u}_I; \mathbf{w}_h - \mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &= B_h(\mathbf{u}_I - \mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) + B_h(\mathbf{u}_I - \mathbf{w}_h; -\mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &= B_h(\mathbf{u}_I - \mathbf{w}_h; \mathbf{w}_h - \mathbf{u}_I, \bar{\mathbf{v}}_h) \\ &\leq C \|\mathbf{u}_I - \mathbf{w}_h\| \|\mathbf{w}_h - \mathbf{u}_I\| \|\bar{\mathbf{v}}_h\| = C \|\mathbf{u}_I - \mathbf{w}_h\|^2. \end{aligned} \quad (4.49)$$

For G_2 , we use Cauchy-Schwarz inequality and Lemma 3.4 to obtain

$$G_2 = \sum_{K \in \mathcal{T}_h} \int_K f(\Pi_K^{2,D} \tilde{v}_h - \tilde{v}_h) \leq \|f\|_{0, \Omega} \|\Pi_K^{2,D} \tilde{v}_h - \tilde{v}_h\|_{0, \Omega} \leq Ch^2 \|f\|_{0, \Omega} \|\bar{\mathbf{v}}_h\| = Ch^2 \|f\|_{0, \Omega}. \quad (4.50)$$

Next, to bound G_3 , we use Theorem 2.1 and Proposition 3.2 to find $\mathbf{u}_\pi \in [\mathbb{P}_k(K)]^2$ such that

$$\|\mathbf{u} - \mathbf{u}_\pi\|_{2,K} \leq Ch^s \|\mathbf{u}\|_{2+s,K} \leq Ch^s \|f\|_{0,K} \quad \forall K \in \mathcal{T}_h, \quad (4.51)$$

with $s \in (1/2, 1]$.

Now, in the definition of G_3 , we add and subtract the term \mathbf{u}_π , use the definition of $A_{h,K}^\Delta(\cdot, \cdot)$, property (3.10), Cauchy-Schwarz inequality, and apply the estimate (3.12) in the definition of $A_{h,K}(\cdot, \cdot)$ to obtain

$$\begin{aligned} G_3 &= \sum_{K \in \mathcal{T}_h} \left\{ A_K^\Delta(\mathbf{u} - \mathbf{u}_\pi, \bar{\mathbf{v}}_h) + A_{h,K}^\Delta(\mathbf{u}_\pi - \mathbf{u}_I, \bar{\mathbf{v}}_h) \right\} \\ &\leq \sum_{K \in \mathcal{T}_h} \left\{ \|\mathbf{u} - \mathbf{u}_\pi\|_{2,K} \|\bar{\mathbf{v}}_h\|_{2,K} + A_{h,K}^\Delta(\mathbf{u}_\pi - \mathbf{u}_I, \mathbf{u}_\pi - \mathbf{u}_I)^{1/2} A_{h,K}^\Delta(\bar{\mathbf{v}}_h, \bar{\mathbf{v}}_h)^{1/2} \right\} \\ &\leq \sum_{K \in \mathcal{T}_h} \left\{ \|\mathbf{u} - \mathbf{u}_\pi\|_{2,K} \|\bar{\mathbf{v}}_h\|_{2,K} + \alpha_2 \|\mathbf{u}_\pi - \mathbf{u}_I\|_{2,K} \|\bar{\mathbf{v}}_h\|_{2,K} \right\}. \end{aligned}$$

Next, adding and subtracting the term \mathbf{u} , using the estimates (4.51) and (4.18) on the right hand side of the above term, we get

$$\begin{aligned} G_3 &\leq \sum_{K \in \mathcal{T}_h} \left\{ (1 + \alpha_2) \|\mathbf{u} - \mathbf{u}_\pi\|_{2,K} \|\bar{\mathbf{v}}_h\|_{2,K} + \alpha_2 \|\mathbf{u} - \mathbf{u}_I\|_{2,K} \|\bar{\mathbf{v}}_h\|_{2,K} \right\} \\ &\leq Ch^s \|\mathbf{u}\|_{2+s,\Omega} \|\bar{\mathbf{v}}_h\| = Ch^s \|\mathbf{u}\|_{2+s,\Omega} \leq Ch^s \|f\|_{0,\Omega}. \end{aligned} \quad (4.52)$$

For the term G_4 , we use the definition of $\mathbf{\Pi}_K^1$, Cauchy Schwarz inequality, and (3.9) to obtain

$$\begin{aligned} G_4 &= \sum_{K \in \mathcal{T}_h} \left\{ \int_K \{ \mathbf{\Pi}_K^1 \nabla u_I - \nabla u \} \cdot \nabla \tilde{v}_h + s_K^\nabla (\mathbf{\Pi}_K^{2,\nabla} u_I - u_I, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h) \right\} \\ &\leq \sum_{K \in \mathcal{T}_h} \left\{ \|\mathbf{\Pi}_K^1 \nabla u_I - \nabla u\|_{0,K} \|\tilde{v}_h\|_{1,K} + s_K^\nabla (\mathbf{\Pi}_K^{2,\nabla} u_I - u_I, \mathbf{\Pi}_K^{2,\nabla} u_I - u_I)^{1/2} \right. \\ &\quad \left. + s_K^\nabla (\tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h, \tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h)^{1/2} \right\}. \\ &\leq \sum_{K \in \mathcal{T}_h} \left\{ \|\mathbf{\Pi}_K^1 \nabla u_I - \nabla u\|_{0,K} \|\tilde{v}_h\|_{1,K} + c_3 \|u_I - \mathbf{\Pi}_K^{2,\nabla} u_I\|_{1,K} \|\tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h\|_{1,K} \right\} \\ &\leq \sum_{K \in \mathcal{T}_h} \left\{ \|\mathbf{\Pi}_K^1 \nabla u_I - \nabla u_I\|_{0,K} \|\tilde{v}_h\|_{1,K} + \|\nabla(u - u_I)\|_{0,K} \|\tilde{v}_h\|_{1,K} \right. \\ &\quad \left. + c_3 \|u_I - \mathbf{\Pi}_K^{2,\nabla} u_I\|_{1,K} \|\tilde{v}_h - \mathbf{\Pi}_K^{2,\nabla} \tilde{v}_h\|_{1,K} \right\}, \end{aligned}$$

where in the last step we have added and subtracted ∇u_I .

Now, on the right hand side above, we apply Propositions 3.4 and 3.3, the definition of $\mathbf{\Pi}_K^{2,\nabla}$, Proposition 3.2 and finally Theorem 2.1, to deduce that

$$\begin{aligned} G_4 &\leq C \sum_{K \in \mathcal{T}_h} \left\{ h_K \|u_I\|_{2,K} + h_K^{1+s} \|u\|_{2,K} \right\} \|\tilde{v}_h\|_{1,K} + h_K^2 \|u_I\|_{2,K} \|\tilde{v}_h\|_{2,K} \\ &\leq Ch \|\mathbf{u}\|_{2+s,\Omega} \|\bar{\mathbf{v}}_h\| = Ch \|\mathbf{u}\|_{2+s,\Omega} \leq Ch \|f\|_{0,\Omega}. \end{aligned} \quad (4.53)$$

It is easy to check that G_5 can be bounded using the same arguments as those applied to estimate (4.37). Hence, we obtain the following result

$$G_5 \leq Ch^s \|\mathbf{u}\|_{2+s,\Omega} \|\bar{\mathbf{v}}_h\| = Ch^s \|\mathbf{u}\|_{2+s,\Omega} \leq Ch^s \|f\|_{0,\Omega}. \quad (4.54)$$

Therefore, by combining (4.48) and the estimates (4.49)-(4.50), (4.52)-(4.54), we obtain that

$$\|T \mathbf{w}_h - \mathbf{u}_I\| \leq C(h^s + \|\mathbf{u}_I - \mathbf{w}_h\|^2),$$

where C is a positive constant depending on $\|f\|_{0,\Omega}$, and $s \in (1/2, 1]$ is the Sobolev exponent for the solution of the von Kármán problem (see Theorem 2.1). \square

Now, we are in position to prove that T maps the ball $\mathcal{B}(\mathbf{u}_I, R)$ to itself.

Lemma 4.4. *For h small enough, there exists a positive constant $R(h)$, depending on h , such that*

$$T(\mathcal{B}(\mathbf{u}_I, R(h))) \subseteq \mathcal{B}(\mathbf{u}_I, R(h)).$$

Proof. Let $\mathbf{z}_h \in T(\mathcal{B}(\mathbf{u}_I, R))$, then there exists $\tilde{\mathbf{w}}_h \in \mathbb{H}_h$ such that $\mathbf{z}_h = T\tilde{\mathbf{w}}_h$ and $\|\tilde{\mathbf{w}}_h - \mathbf{u}_I\| \leq R$. Then, applying Lemma 4.3, we have

$$\|\mathbf{z}_h - \mathbf{u}_I\| = \|T\tilde{\mathbf{w}}_h - \mathbf{u}_I\| \leq \tilde{C}(h^s + \|\mathbf{u}_I - \tilde{\mathbf{w}}_h\|^2).$$

Now, for all $h \leq h_1 := (2\tilde{C})^{-2/s}$, we choose $2\tilde{C}h^s =: R(h) = R$ and obtain

$$\|\mathbf{z}_h - \mathbf{u}_I\| \leq \tilde{C}h^s + \tilde{C}\|\tilde{\mathbf{w}}_h - \mathbf{u}_I\|^2 \leq \tilde{C}h^s + \tilde{C}R(h)^2 = \frac{R(h)}{2} + \frac{R(h)}{2}4\tilde{C}^2h^s \leq R(h).$$

Therefore $\mathbf{z}_h \in \mathcal{B}(\mathbf{u}_I, R(h))$. \square

In the following result, we will prove that the operator T is a contraction in $\mathcal{B}(\mathbf{u}_I, R(h))$.

Lemma 4.5. *For h small enough, the operator T is a contraction in $\mathcal{B}(\mathbf{u}_I, R(h))$.*

Proof. Let $\mathbf{w}_h, \tilde{\mathbf{w}}_h \in \mathcal{B}(\mathbf{u}_I, R(h))$, hence

$$\|\mathbf{w}_h - \mathbf{u}_I\|, \|\tilde{\mathbf{w}}_h - \mathbf{u}_I\| \leq R(h) = 2\tilde{C}h^s. \quad (4.55)$$

Now, from the definition of operator T (cf. (4.20)), we have

$$\begin{aligned} A_{h,\mathbf{u}_I}(T \mathbf{w}_h, \mathbf{v}_h) &= 2B_h(\mathbf{u}_I; \mathbf{w}_h, \mathbf{v}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \mathbf{v}_h) + F_h(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbb{H}_h; \\ A_{h,\mathbf{u}_I}(T \tilde{\mathbf{w}}_h, \mathbf{v}_h) &= 2B_h(\mathbf{u}_I; \tilde{\mathbf{w}}_h, \mathbf{v}_h) - B_h(\tilde{\mathbf{w}}_h; \tilde{\mathbf{w}}_h, \mathbf{v}_h) + F_h(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbb{H}_h, \end{aligned}$$

which implies

$$A_{h,\mathbf{u}_I}(T(\mathbf{w}_h - \tilde{\mathbf{w}}_h), \mathbf{v}_h) = 2B_h(\mathbf{u}_I; \mathbf{w}_h - \tilde{\mathbf{w}}_h, \mathbf{v}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \mathbf{v}_h) + B_h(\tilde{\mathbf{w}}_h; \tilde{\mathbf{w}}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{H}_h. \quad (4.56)$$

Using Lemma 4.2, we can choose $\bar{\mathbf{v}}_h \in \mathbb{H}_h$ with $\|\bar{\mathbf{v}}_h\| = 1$ such that

$$C\|T \mathbf{w}_h - T \tilde{\mathbf{w}}_h\| = C\|T(\mathbf{w}_h - \tilde{\mathbf{w}}_h)\| \leq A_{h,\mathbf{u}_I}(T(\mathbf{w}_h - \tilde{\mathbf{w}}_h), \bar{\mathbf{v}}_h). \quad (4.57)$$

Therefore, by combining (4.56) and (4.57), using (3.16e) and (3.16f), we get

$$\begin{aligned} \|T \mathbf{w}_h - T \tilde{\mathbf{w}}_h\| &\leq \frac{1}{C} \left\{ 2B_h(\mathbf{u}_I; \mathbf{w}_h - \tilde{\mathbf{w}}_h, \bar{\mathbf{v}}_h) - B_h(\mathbf{w}_h; \mathbf{w}_h, \bar{\mathbf{v}}_h) + B_h(\tilde{\mathbf{w}}_h; \tilde{\mathbf{w}}_h, \bar{\mathbf{v}}_h) \right\} \\ &= \frac{1}{C} \left\{ B_h(\tilde{\mathbf{w}}_h - \mathbf{w}_h; \mathbf{w}_h - \mathbf{u}_I, \bar{\mathbf{v}}_h) + B_h(\tilde{\mathbf{w}}_h - \mathbf{u}_I; \tilde{\mathbf{w}}_h - \mathbf{w}_h, \bar{\mathbf{v}}_h) \right\} \\ &\leq \frac{1}{C} \left\{ \|\tilde{\mathbf{w}}_h - \mathbf{w}_h\| \|\mathbf{w}_h - \mathbf{u}_I\| \|\bar{\mathbf{v}}_h\| + \|\tilde{\mathbf{w}}_h - \mathbf{w}_h\| \|\tilde{\mathbf{w}}_h - \mathbf{u}_I\| \|\bar{\mathbf{v}}_h\| \right\} \\ &= \frac{1}{C} \|\tilde{\mathbf{w}}_h - \mathbf{w}_h\| \left\{ \|\mathbf{w}_h - \mathbf{u}_I\| + \|\tilde{\mathbf{w}}_h - \mathbf{u}_I\| \right\}. \end{aligned} \quad (4.58)$$

Finally, for all $h \leq h_2 := (\frac{1}{8})^{1/s}$, we apply (4.55) on the right hand side of (4.58) to obtain

$$\|T \mathbf{w}_h - T \tilde{\mathbf{w}}_h\| \leq \frac{2R(h)}{C} \|\mathbf{w}_h - \tilde{\mathbf{w}}_h\| = 4h^s \|\mathbf{w}_h - \tilde{\mathbf{w}}_h\| \leq \frac{1}{2} \|\mathbf{w}_h - \tilde{\mathbf{w}}_h\|.$$

Therefore, we have finished the proof. \square

Finally, we are ready to prove that the discrete problem (3.15) admits a unique solution.

Theorem 4.1. *Let \mathbf{u} be an isolated solution of (2.12). Then, for h small enough, the discrete problem (3.15) has a unique solution $\mathbf{u}_h \in \mathbb{H}_h$. Moreover, we have that*

$$\|\mathbf{u}_h - \mathbf{u}_I\| = \|T \mathbf{u}_h - \mathbf{u}_I\| \leq Ch^s.$$

Proof. We know that the solution of (3.15) is a fixed point of operator T . Thus, the proof follows from Lemmas 4.4 and 4.5, and Banach fixed-point theorem. \square

We finish this section presenting the following result, which provides the rate of convergence of our virtual element scheme.

Theorem 4.2. *Let \mathbf{u} and \mathbf{u}_h the isolated solution of (2.12) and the unique solution of the discrete problem (3.15), respectively. Then, there exists a positive constant C , independent of h , such that for all $h \leq \min\{h_1, h_2\}$ we have that*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^s,$$

where $s \in (1/2, 1]$ is the Sobolev exponent for the solution of von Kármán problem (see Theorem 2.1).

Proof. For all $h \leq \min\{h_1, h_2\}$, we have from Theorem 4.1 the following estimate

$$\|\mathbf{u}_h - \mathbf{u}_I\| \leq Ch^s. \tag{4.59}$$

Hence, applying triangle inequality in the term $\|\mathbf{u} - \mathbf{u}_h\|$, and using the estimates (4.18) and (4.59), we deduce

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{u} - \mathbf{u}_I\| + \|\mathbf{u}_I - \mathbf{u}_h\| \leq Ch^s.$$

The proof is complete. \square

5. Numerical results

We report in this section a series of numerical experiments to approximate the isolated solutions of the von Kármán problem (2.12), using the Virtual Element Method proposed and analyzed in this paper. We have implemented in a MATLAB code the proposed VEM on arbitrary polygonal meshes (see [12]).

We will test the method by using different families of meshes (see Figure 2):

- \mathcal{T}_h^1 : trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertices $(0, 0)$, $(1/2, 0)$, $(1/2, 2/3)$ and $(0, 1/3)$;

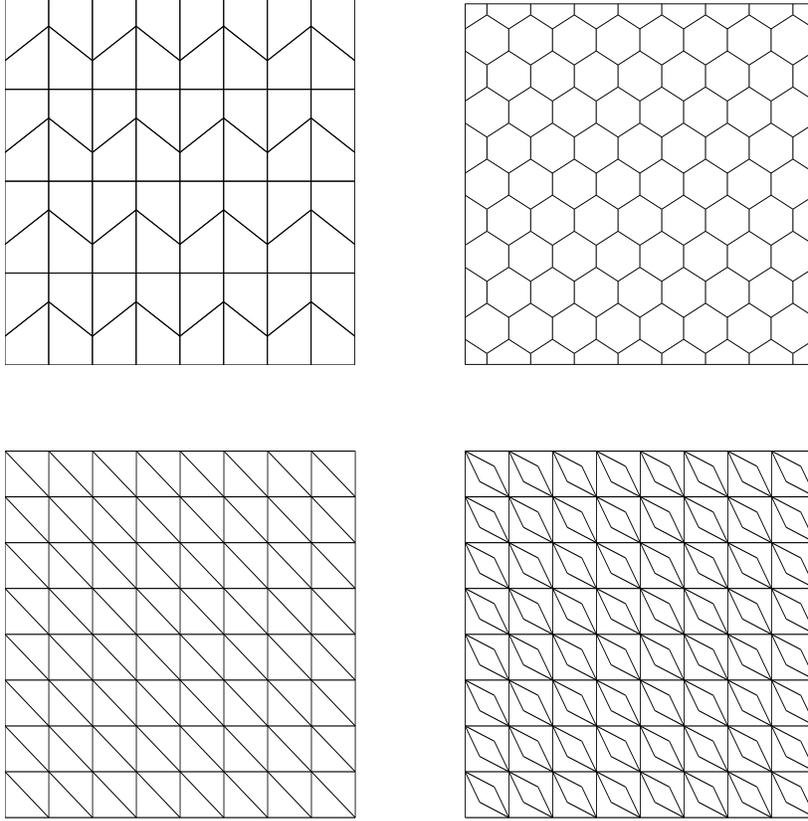


Figure 2: Sample meshes: \mathcal{T}_h^1 (top left), \mathcal{T}_h^2 (top right), \mathcal{T}_h^3 (bottom left) and \mathcal{T}_h^4 (bottom right).

- \mathcal{T}_h^2 : hexagonal meshes;
- \mathcal{T}_h^3 : triangular meshes;
- \mathcal{T}_h^4 : distorted concave rhombic quadrilaterals.

Now, in order to complete the choice of the VEM, we have to fix the bilinear forms $S_K^D(\cdot, \cdot)$ and $S_K^\nabla(\cdot, \cdot)$ satisfying (3.8) and (3.9), respectively. First, we consider the following symmetric bilinear forms (see for instance [8, 42]):

$$s_K^D(u_h, v_h) := \sigma_K \sum_{i=1}^{N_K} [u_h(\mathbf{v}_i)v_h(\mathbf{v}_i) + h_{\mathbf{v}_i}^2 \nabla u_h(\mathbf{v}_i) \cdot \nabla v_h(\mathbf{v}_i)] \quad \forall u_h, v_h \in H_h^K,$$

$$s_K^\nabla(u_h, v_h) := \bar{\sigma}_K \sum_{i=1}^{N_K} [u_h(\mathbf{v}_i)v_h(\mathbf{v}_i) + \nabla u_h(\mathbf{v}_i) \cdot \nabla v_h(\mathbf{v}_i)] \quad \forall u_h, v_h \in H_h^K,$$

where, $\mathbf{v}_1, \dots, \mathbf{v}_{N_K}$ denote the vertices of K , $h_{\mathbf{v}_i}$ is chosen as the maximum diameter of the elements K with \mathbf{v}_i as a vertex. Moreover, σ_K and $\bar{\sigma}_K$ are multiplicative factors to consider the

h -scaling and the physical constant of the problem. For instance, in the numerical tests we have considered $\sigma_K, \bar{\sigma}_K > 0$ as the mean value of the eigenvalues of the local matrices $a_K^\Delta(\Pi_K^{2,D}\phi_i, \Pi_K^{2,D}\phi_j)$ and $a_K^\nabla(\Pi_K^{2,\nabla}\phi_i, \Pi_K^{2,\nabla}\phi_j)$, respectively, where $\{\phi_i\}_{i=1}^{\dim H_h^K}$ corresponds to a basis of H_h^K .

Then, we set $S_K^D(\cdot, \cdot)$ and $S_K^\nabla(\cdot, \cdot)$ as follows

$$\begin{aligned} S_K^D(\mathbf{u}_h, \mathbf{v}_h) &:= s_K^D(u_h, v_h) + s_K^D(\psi_h, \varphi_h) & \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h^K, \\ S_K^\nabla(\mathbf{u}_h, \mathbf{v}_h) &:= s_K^\nabla(u_h, v_h) & \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbb{H}_h^K. \end{aligned}$$

In order to present the numerical tests, we have taken as computational domain $\Omega := (0, 1)^2$. The discrete solution associated to problem (3.15) was obtained by a classical Newton method. In particular, we have considered the usual incremental loading procedure (see for instance [13, Section 3.2]) to approximate the discrete solution of the nonlinear von Kármán problem: given a positive integer \hat{N} , let $F_h^\ell(\mathbf{v}_h) = (\ell/\hat{N})F_h(\mathbf{v}_h) \forall \ell = 1, 2, \dots, \hat{N}$ be the partial loadings. Therefore, given the initial guest \mathbf{u}_h^0 (for instance the zero function), one applies for $\ell = 1, 2, \dots, \hat{N}$ the following iterative procedure

Given \mathbf{u}_h^0 **do**

for $\ell = 1 : \hat{N}$ **do**

$$\left| \begin{array}{l} F_h^\ell(\mathbf{v}_h) = (\ell/\hat{N})F_h(\mathbf{v}_h) \\ \text{Solve Newton iterates} \\ A_h^\Delta(\mathbf{u}_h^\ell, \mathbf{v}_h) + \lambda A_h^\nabla(\mathbf{u}_h^\ell, \mathbf{v}_h) + 2B_h(\mathbf{u}_h^{\ell-1}; \mathbf{v}_h, \mathbf{v}_h) = B_h(\mathbf{u}_h^{\ell-1}; \mathbf{u}_h^{\ell-1}, \mathbf{v}_h) + F_h^\ell(\mathbf{v}_h) \end{array} \right.$$

end

Thus, the final solution is $\mathbf{u}_h = \mathbf{u}_h^{\hat{N}}$.

Moreover, we define the individual errors by:

$$e_0(w_h) := \|w - \Pi_h^{2,D}w_h\|_{0,\Omega}, \quad e_1(w_h) := |w - \Pi_h^{2,D}w_h|_{1,h}, \quad e_2(w_h) := |w - \Pi_h^{2,D}w_h|_{2,h},$$

where, $\Pi_h^{2,D}$ has been defined in (3.17).

We have computed the experimental rates of convergence for each individual errors as follows:

$$\text{rc}(\cdot) := \frac{\log(e_\star(\cdot)/e'_\star(\cdot))}{\log(N_{dof}/N'_{dof})} \quad \text{for all subscripts } \star \in \{0, 1, 2\},$$

with N_{dof} and N'_{dof} denote the degrees of freedom of two consecutive polygonal decomposition with respectively errors e_\star and e'_\star . For each mesh \mathcal{T}_h , the degrees of freedom are $N_{dof} = 6N_v$, where N_v denotes the number of interior vertices of the polygonal mesh.

In what follows, we present three numerical tests illustrating the performance of our virtual element scheme. For reasons of brevity, we do not report the results obtained with all meshes for all test problems. However, all non reported results are in accordance with the ones shown.

5.1. Test 1

In this test, we have taken $\lambda = 5$. In addition, we modify the right hand side of the second equation in (2.4) to compare the discrete solution with the continuous one, i.e.

$$\left\{ \begin{array}{lll} \Delta^2 u + \lambda \Delta u - [\psi, u] & = f & \text{in } \Omega, \\ \Delta^2 \psi + \frac{1}{2}[u, u] & = g & \text{in } \Omega, \\ u = \partial_\nu u & = 0 & \text{on } \Gamma, \\ \psi = \partial_\nu \psi & = 0 & \text{on } \Gamma. \end{array} \right. \quad (5.60)$$

Thus, we consider f and g so that the exact solution of (5.60) are

$$u(x, y) = x^2 y^2 (x - 1)^2 (y - 1)^2 \quad \text{in } \Omega, \quad \text{and} \quad \psi(x, y) = \sin^2(\pi x) \sin^2(\pi y) \quad \text{in } \Omega.$$

We report in Table 1 the convergence history of our virtual element scheme (3.15). For each ℓ the Newton's method used up to 5 iterations with a tolerance $\text{Tol} = 10^{-9}$. In particular, Table 1 summarizes the convergence history for the transverse displacements u_h and for the Airy stress function ψ_h . As predicted by Theorem 4.2, an $\mathcal{O}(h)$ of convergence is clearly seen for $e_2(u_h)$ and $e_2(\psi_h)$. We also report $e_0(u_h)$, $e_1(u_h)$, $e_0(\psi_h)$ and $e_1(\psi_h)$, where an $\mathcal{O}(h^2)$ is observed. The discrete solutions obtained with the virtual element method proposed in this work, are depicted in Figure 3.

Table 1: Test 1: Errors and experimental convergence rates $e_0(u_h)$, $e_1(u_h)$, $e_2(u_h)$, $e_0(\psi_h)$, $e_1(\psi_h)$ and $e_2(\psi_h)$ of the discrete solution to the von Kármán problem.

\mathcal{T}_h	N_{dof}	$e_0(u_h)$	$\text{rc}(u_h)$	$e_1(u_h)$	$\text{rc}(u_h)$	$e_2(u_h)$	$\text{rc}(u_h)$
\mathcal{T}_h^3	54	0.000431704768438	-	0.003262724422857	-	0.031728922182966	-
	294	0.000118318604331	1.87	0.000935470949060	1.80	0.016255461918515	0.96
	1350	0.000029867487367	1.99	0.000239063199567	1.97	0.008150175184542	1.00
	5766	0.000007461004475	2.00	0.000059929912000	2.00	0.004079547948378	1.00
	23814	0.000001863069256	2.00	0.000014979870677	2.00	0.002040881649304	1.00
\mathcal{T}_h^4	246	0.000480588388535	-	0.003317993901837	-	0.031341253980477	-
	1062	0.000136460315064	1.72	0.000988414232088	1.66	0.015733875049155	0.94
	4422	0.000034563080364	1.93	0.000256091744078	1.89	0.007800044026566	0.98
	18054	0.000008594114090	1.98	0.000064363787275	1.96	0.003890338441267	0.99
	72966	0.000002136931492	1.99	0.000016086825337	1.99	0.001944197375647	0.99
\mathcal{T}_h	N_{dof}	$e_0(\psi_h)$	$\text{rc}(\psi_h)$	$e_1(\psi_h)$	$\text{rc}(\psi_h)$	$e_2(\psi_h)$	$\text{rc}(\psi_h)$
\mathcal{T}_h^1	54	0.073050506944122	-	0.846040099702635	-	8.159441640704209	-
	294	0.012011400077328	2.60	0.292568564494940	1.53	4.341739275488559	0.91
	1350	0.002294277174721	2.39	0.083090515350743	1.81	2.174289254445207	1.00
	5766	0.000499349745906	2.20	0.021622526054148	1.94	1.083299492719920	1.01
	23814	0.000119140487292	2.08	0.005465045349110	1.98	0.540966031106735	1.00
\mathcal{T}_h^2	192	0.061884511675239	-	0.679874909970360	-	7.512913582695741	-
	768	0.026172828966453	1.46	0.249463389030945	1.71	4.077763689322426	1.04
	3072	0.008664638440498	1.74	0.071485859700407	1.97	2.055397431297525	1.08
	12288	0.002491920671679	1.88	0.018172983843658	2.06	1.024040359048393	1.05
	49152	0.000669515058352	1.94	0.004531873249613	2.05	0.510867678265516	1.03

5.2. Test 2

In this test we consider the canonical von Kármán equations (cf. (2.1)) with non-homogeneous boundary conditions. In addition, we modify the right side of the second equation in (2.1) to compare the discrete solution with the continuous one, i.e.

$$\begin{cases} \Delta^2 u - [\psi, u] = f & \text{in } \Omega, \\ \Delta^2 \psi + \frac{1}{2}[u, u] = g & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \Gamma, \\ \psi = \varphi_0 & \text{on } \Gamma, \\ \partial_\nu \psi = \varphi_1 & \text{on } \Gamma. \end{cases} \quad (5.61)$$

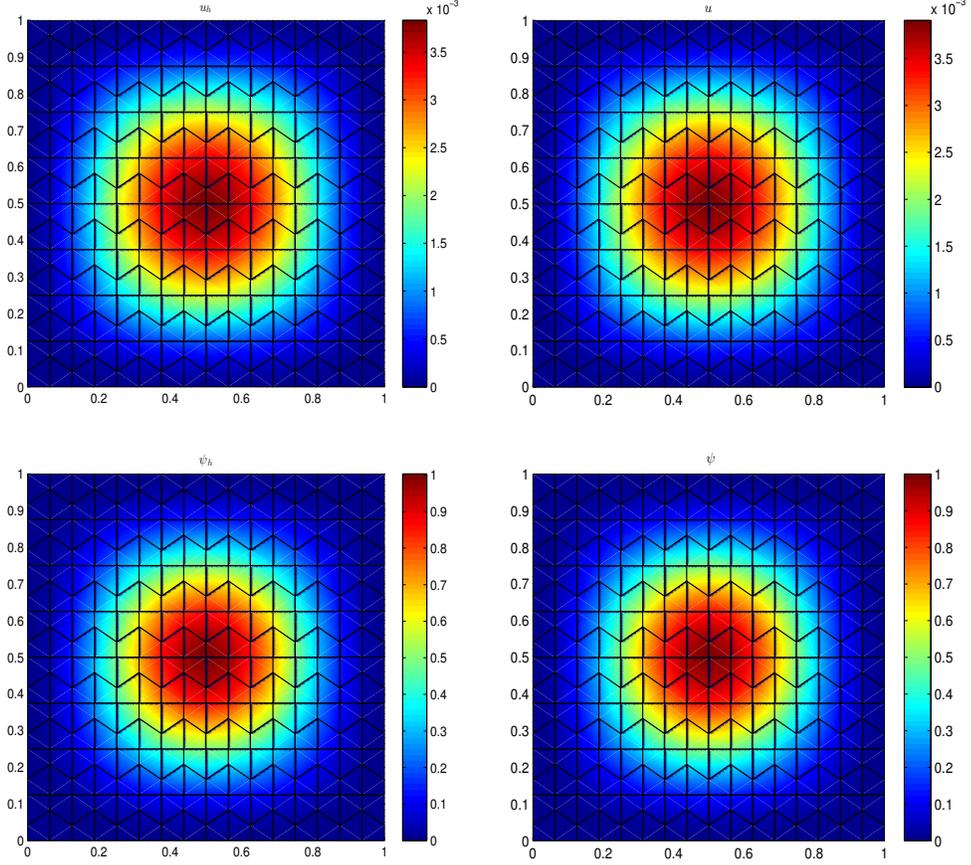


Figure 3: Test 1. u_h (top left), u (top right), ψ_h (bottom left) and ψ (bottom right).

Next, we consider the following lateral forces φ_0 and φ_1

$$\begin{aligned} \varphi_0(x, y) &= \sin^2(\pi x) && \text{on } \Gamma, \\ \varphi_1(x, y) &= 2\pi\nu_1(x, y) \cos(\pi x) \sin(\pi x) && \text{on } \Gamma, \end{aligned}$$

and transverse loads f and g so that the exact solution of (5.61) is

$$u(x, y) = x^2 \sin(\pi y)^2 \log(2 - x)^2 \quad \text{in } \Omega, \quad \text{and} \quad \psi(x, y) = \sin^2(\pi x) \quad \text{in } \Omega.$$

Table 2 reports the convergence history of our virtual element scheme (3.15) applied to solve (5.61). Once again, for each ℓ , the Newton's method used up to 5 iterations with a tolerance $\text{Tol} = 10^{-9}$. In particular, Table 1 summarizes the convergence history for the transverse displacements u_h (resp. to Airy stress function ψ_h). Once again an $\mathcal{O}(h)$ of convergence is clearly seen for $e_2(u_h)$ (resp. $e_2(\psi_h)$). We again report the errors $e_0(u_h)$ and $e_1(u_h)$ (resp. $e_0(\psi_h)$ and $e_1(\psi_h)$), where an $\mathcal{O}(h^2)$ is observed. The discrete solutions associated to the Test 2, and obtained with the virtual element method proposed in this work, are depicted in Figure 4.

5.3. Test 3

In this test we present a numerical example, illustrating the performance of our virtual element scheme applied to the von Kármán system (2.4), with $f = 0$ and different values of the parameter

Table 2: Test 2: Errors and experimental convergence rates $e_0(u_h)$, $e_1(u_h)$, $e_2(u_h)$, $e_0(\psi_h)$, $e_1(\psi_h)$ and $e_2(\psi_h)$ of the discrete solution to the von Kármán problem.

\mathcal{T}_h	N_{dof}	$e_0(u_h)$	$rc(u_h)$	$e_1(u_h)$	$rc(u_h)$	$e_2(u_h)$	$rc(u_h)$
\mathcal{T}_h^1	54	0.003331813500703	-	0.040541588728628	-	0.392326399686893	-
	294	0.000666755182746	1.90	0.013573656090400	1.29	0.211617144042376	0.73
	1350	0.000139272569159	2.05	0.003744148938304	1.69	0.107073487968784	0.89
	5766	0.000032059141547	2.02	0.000964443541553	1.87	0.053613559930229	0.95
	23814	0.000007817172339	1.99	0.000243151609540	1.94	0.026821738028166	0.98
\mathcal{T}_h^2	192	0.003338044048468	-	0.029905766623389	-	0.331352206962454	-
	768	0.001214026245280	1.46	0.011071959441869	1.43	0.188066935651525	0.82
	3072	0.000374019547265	1.70	0.003255655646633	1.77	0.098467208145884	0.93
	12288	0.000107769516445	1.80	0.000862676994761	1.92	0.050139745754829	0.97
	49152	0.000029635952218	1.86	0.000221290022671	1.96	0.025284764141485	0.99
\mathcal{T}_h	N_{dof}	$e_0(\psi_h)$	$rc(\psi_h)$	$e_1(\psi_h)$	$rc(\psi_h)$	$e_2(\psi_h)$	$rc(\psi_h)$
\mathcal{T}_h^3	54	0.029494648554614	-	0.412210903728138	-	6.003264759584535	-
	294	0.004591865781598	2.20	0.101028788029534	1.66	3.069362969416609	0.79
	1350	0.000940086309599	2.08	0.025008571679680	1.83	1.540413925828407	0.90
	5766	0.000222681115380	1.98	0.006215447146812	1.92	0.770497464565608	0.95
	23814	0.000055123897439	1.97	0.001548657099701	1.96	0.385190062299381	0.98
\mathcal{T}_h^4	246	0.027814402349946	-	0.372966458416872	-	5.590747627283569	-
	1062	0.004522053574370	2.62	0.094464604014345	1.98	2.885848302050357	0.95
	4422	0.000957131357883	2.24	0.023604212432399	2.00	1.453857116146261	0.99
	18054	0.000229063918352	2.06	0.005877183318359	2.01	0.728469592800096	1.00
	72966	0.000056858787426	2.01	0.001464852130236	2.00	0.364490348651261	1.00

λ (cf. Remark 2.1). Let $\lambda^* = 52.34469\dots$ (see for instance [17, 43, 44]) be the smallest eigenvalue λ of the following buckling spectral problem (cf. (2.5)):

$$a^\Delta(u, v) = -\lambda a^\nabla(u, v) \quad \forall v \in H_0^2(\Omega),$$

where, the bilinear forms $a^\Delta(\cdot, \cdot)$ and $a^\nabla(\cdot, \cdot)$ were defined in (2.8) and (2.9), respectively. In this particular case as predicted by the theory in [30, Theorem 5.9-2], there exist at least 3 solutions of the problem (2.4) for $\lambda > \lambda^*$ (see Remark 2.1).

Once again, we have solved our discrete problem (3.15), using three values for λ , namely: $\lambda = 53$, $\lambda = 55$ and $\lambda = 60$. In addition, for each value of the parameter λ , we used two different initial guess and trapezoidal meshes \mathcal{T}_h^1 . On the left hand side of Figures 5-7, we illustrate the approximation of $u_h =: u_1$ obtained with the initial guess $\mathbf{u}_h^0(x, y) = (\frac{1}{4}(yx^2 + 1), \frac{1}{4}(yx^2 + 1))$, and $\lambda = 53, 55$ and 60 , respectively. While, on the right hand side of Figures 5-7, we illustrate the approximation of $u_h =: u_2$ obtained with the initial guess $\mathbf{u}_h^0(x, y) = -(\frac{1}{4}(yx^2 + 1), \frac{1}{4}(yx^2 + 1))$ and $\lambda = 53, 55$ and 60 , respectively. We can appreciate that $u_1 \neq 0$ and $u_2 = -u_1$, which confirm the existence of non zero solutions with opposite displacements, as established in [30, Theorem 5.9-2]. On the other hand, for $\mathbf{u}_h^0 \equiv 0$, we obtained $\mathbf{u}_h = 0$ for any value of λ , as established in [30, Theorem 5.9-2].

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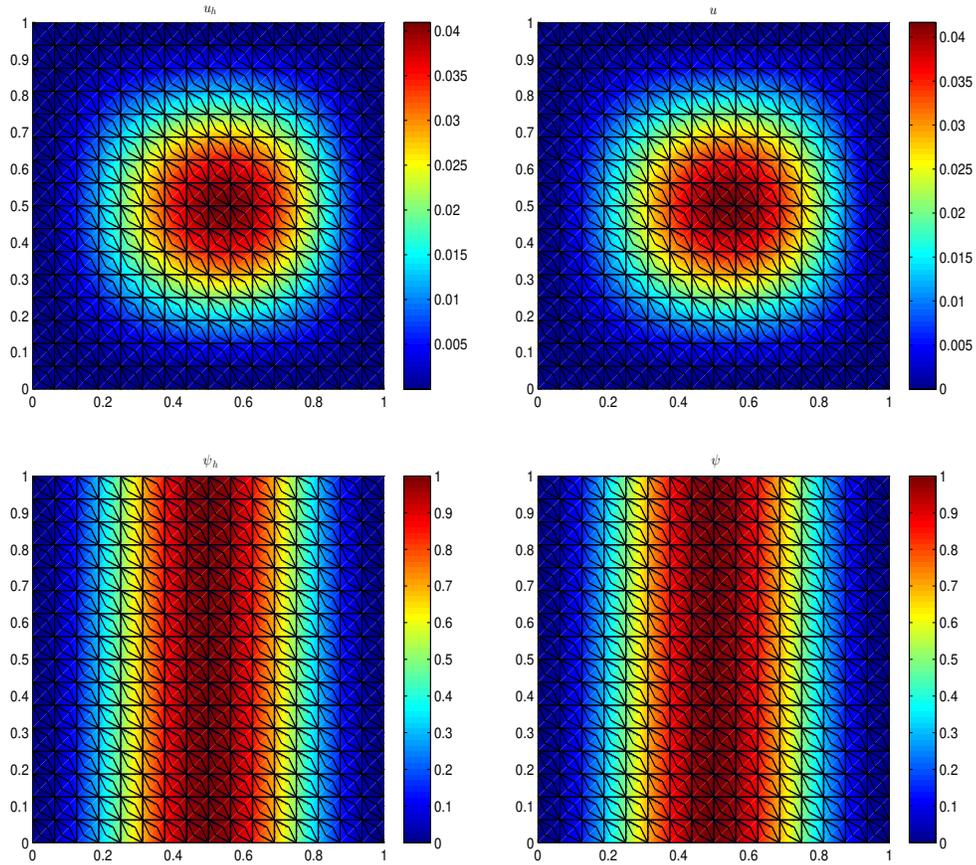


Figure 4: Test 2. u_h (top left), u (top right), ψ_h (bottom left) and ψ (bottom right).

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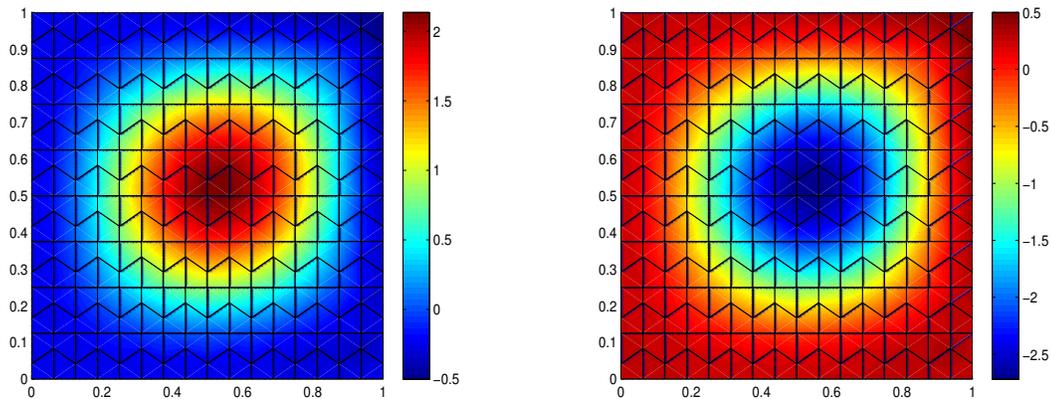


Figure 5: Test 3. $u_1 := u_h$ obtained from Newton iterates, with $\lambda = 53$, $f = 0$ and $\mathbf{u}^0(x, y) = (\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (left). $u_2 := u_h$ obtained from Newton iterates, with $\lambda = 53$, $f = 0$ and $\mathbf{u}^0(x, y) = -(\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (right).

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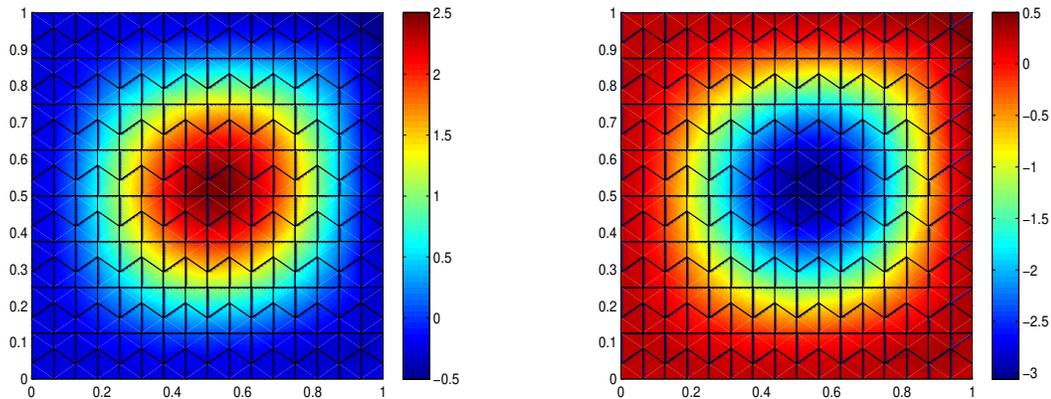


Figure 6: Test 3. $u_1 := u_h$ obtained from Newton iterates, with $\lambda = 55$, $f = 0$ and $\mathbf{u}^0(x, y) = (\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (left). $u_2 := u_h$ obtained from Newton iterates, with $\lambda = 55$, $f = 0$ and $\mathbf{u}^0(x, y) = -(\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (right).

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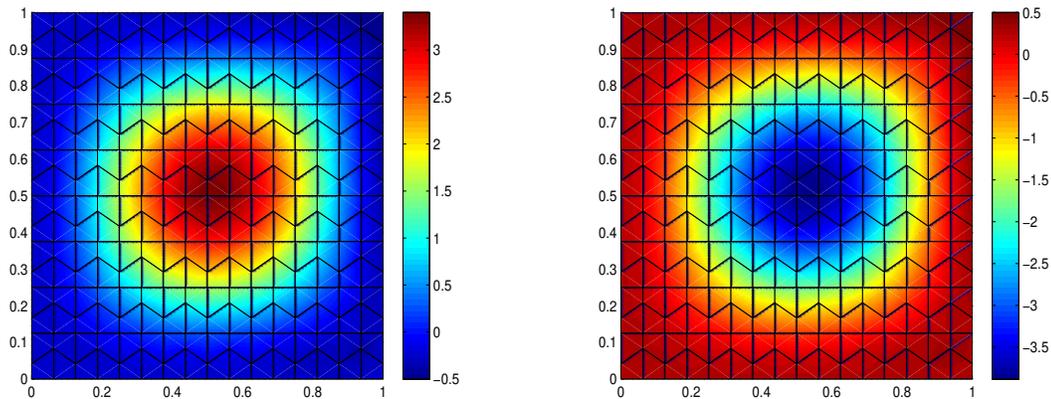


Figure 7: Test 3. $u_1 := u_h$ obtained from Newton iterates, with $\lambda = 60$, $f = 0$ and $\mathbf{u}^0(x, y) = (\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (left). $u_2 := u_h$ obtained from Newton iterates, with $\lambda = 60$, $f = 0$ and $\mathbf{u}^0(x, y) = -(\frac{1}{4}(yx^2+1), \frac{1}{4}(yx^2+1))$ (right).

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