A five-field augmented fully-mixed finite element method for the Navier-Stokes/Darcy coupled problem*

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Abstract

In this work we introduce and analyze a new augmented fully-mixed formulation for the stationary Navier-Stokes/Darcy coupled problem. Our approach employs, on the free-fluid region, a technique previously applied to the stationary Navier-Stokes equations, which consists of the introduction of a modified pseudostress tensor involving the diffusive and convective terms, together with the pressure. In addition, by using the incompressibility condition, the pressure is eliminated, and since the convective term forces the free-fluid velocity to live in a smaller space than usual, we augment the resulting formulation with suitable Galerkin type terms arising from the constitutive and equilibrium equations. On the other hand, in the Darcy region we apply the usual dual-mixed formulation, which yields the introduction of the trace of the porous media pressure as an associated Lagrange multiplier. The latter is connected with the fact that one of the transmission conditions involving mass conservation becomes essential and must be imposed weakly. In this way, we obtain a five-field formulation where the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, and the aforementioned Lagrange multiplier, are the corresponding unknowns. The well-posedness analysis is carried out by combining the classical Babuška-Brezzi theory and the Banach fixed-point theorem. A proper adaptation of the arguments exploited in the continuous analysis allows us to state suitable hypotheses on the finite element subspaces ensuring that the associated Galerkin scheme is well-posed and convergent. In particular, Raviart-Thomas elements of lowest order for the pseudostress and the Darcy velocity, continuous piecewise linear polynomials for the free-fluid velocity, piecewise constants for the Darcy pressure, together with continuous piecewise linear elements for the Lagrange multiplier, constitute feasible choices. Finally, we provide several numerical results illustrating the performance of the Galerkin method and confirming the theoretical rates of convergence.

Key words: Navier-Stokes-Darcy, mixed finite element method, augmented formulation, Raviart-Thomas elements.

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1 Introduction

In this article we introduce a new finite element scheme to numerically solve the coupling of fluid flow, governed by the Navier-Stokes equations, with porous media flow, modeled by the Darcy law, coupled through interface conditions given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. More precisely, we employ a recent approach for the stationary Navier-Stokes equations based on the introduction of a pseudostress tensor relating the diffusive term with the convective term and the pressure, consider the standard dual-mixed formulation for the Darcy law, which yields the introduction of the trace of the porous media pressure as an associated Lagrange multiplier, and propose an augmented fully-mixed finite element method for the coupled problem, where the aforementioned pseudostress tensor and Lagrange multiplier, together with the fluid velocities in both domains and the Darcy pressure, are the main unknowns of the system. The pressure in the fluid region, as well as the fluid velocity gradient and the shear-stress tensor, can be easily recovered through simple post-processing procedures.

An important body of literature dealing with numerical techniques to solve this coupled system, or its linearized version where the Stokes equations are considered instead of the Navier-Stokes system, has been introduced in the last decades due to its applicability in different areas of interest, such as medicine, petroleum engineering, environmental science, etc. (see e.g. [2, 5, 6, 11, 13, 14, 15, 16, 19, 20, 21, 23, 25, 30, 32] and the references therein). The list above includes iterative subdomain and mortar methods, discontinuous Galerkin (DG) and hybridizable discontinuous Galerkin (HDG) schemes, as well as stabilized formulations. In general, most of the finite element formulations developed are based on velocity-pressure discretizations for the free-fluid part of the coupled system (see, for instance [2, 11, 13, 14, 16, 25, 30, 32]). However, in this work we give special attention to numerical schemes based on dual-mixed formulations for the fluid flow, which have gained considerable attention mainly due to the fact that, on the one hand, they allow to unify the analysis for Newtonian and non-Newtonian flows, and on the other hand, they permit to approximate diverse unknowns of physical interest, either directly through the formulation employed or using simple post-processing formulae.

Going back to the Stokes-Darcy model, new fully-mixed finite element methods have been introduced in [6, 21, 19] to approximate the solution of the coupled system, considering Newtonian (in [6, 21]) and Non-Newtonian flows (in [19]). There, the methods are based on the introduction of the pseudostress (in [19, 21]) or stress tensors (in [6]) as further unknowns, which permits, on one hand, to successfully unify the analysis, and on the other hand, to employ the same family of finite elements in both domains. In particular, in [6] two new fully-mixed formulations have been suggested for the linear coupled system. The first one extends [21] by introducing a new fully-mixed formulation where the stress tensor is considered in the fluid domain instead of the pseudostress, which yields the introduction of the vorticity as a further unknown. Next, the aforementioned stress-based formulation is partially augmented by introducing Galerkin least-squares type terms arising from the constitutive and equilibrium equations of the Stokes equation, and from the relation defining the vorticity in terms of the free fluid velocity, yielding, in this way, the second method. The main advantage of the latter is the flexibility of choosing discrete subspaces for the variables in the Stokes domain since no inf-sup conditions are needed to obtain the stability of the method.

More recently, the results obtained in [19] were extended in [10] to the coupled nonlinear Navier-Stokes and linear Darcy problems with constant density and variable viscosity in the fluid region. Due to the nonlinearity related with the viscosity, the velocity gradient is introduced as a further unknown, which together with the fluid stress, the fluid vorticity, the velocity in both domains, the porous media pressure and two Lagrange multipliers, namely the traces of the porous media pressure and the fluid velocity on the interface, constitute the main unknowns of the system. In addition, since the convective term of the Navier-Stokes model forces the velocity to live in a space smaller than L^2, we follow [7, 9, 8] and seek this unknown in H^1, so that the variational formulation is then augmented with residual terms arising from the constitutive and equilibrium equations for the fluid flow, and the formulae for the strain and vorticity tensors. As for the second method in [6], the latter yields more flexibility in the choice of discrete subspaces for the variables of the Navier-Stokes equations.

The purpose of the present work is to additionally contribute in the direction of mixed finite element schemes for the coupling of fluid flows with porous media flows by introducing a new augmented fully-mixed method for the steady state Navier-Stokes/Darcy coupled problem. Differently from [6] and [10], here we proceed analogously to [8] in the fluid region, by taking advantage of the fact that the fluid velocity is considered in H^1 and avoiding the introduction of the vorticity and the trace of the fluid velocity on the interface as further unknowns. In this way, we obtain a simpler method with only five unknowns. The rest of the work is organized as follows. In Section 2 we recall the model problem and rewrite it as a first-order system of equations. In Section 3 we derive the augmented mixed variational formulation, which, differently from [6, 10], does not include the vorticity nor the trace of the fluid velocity on the interface as auxiliary unknowns. Next, we proceed with the solvability analysis, mainly via the Babuška-Brezzi theory and the Banach fixed-point theorem, under a sufficiently small data assumption. In turn, in Section 4 we study the associated Galerkin scheme by using a discrete version of the fixed-point strategy developed in Section 3. Next, the a priori error estimate and the corresponding rates of convergence for a particular choice of discrete subspaces are derived in Section 4 under a similar assumption on the size of the data. Finally, a couple of numerical examples illustrating the performance of the method and confirming the theoretical rates of convergence, are reported in Section 5.

We end this section by recalling some definitions and fixing useful notations. Given the vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, with $n \in \{2,3\}$, we set the gradient, divergence, and tensor product operators, by

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div}\mathbf{v} := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor fields $\mathbf{S} := (S_{ij})_{i,j=1,n}$ and $\mathbf{R} := (R_{ij})_{i,j=1,n}$, we define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, by

$$\mathbf{S}^t := (S_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\mathbf{S}) := \sum_{i=1}^{n} S_{ii}, \quad \mathbf{S}:\mathbf{R} := \sum_{i,j=1}^{n} S_{ij}R_{ij}, \quad \text{and} \quad \mathbf{S}^{\mathbf{d}} := \mathbf{S} - \frac{1}{n}\operatorname{tr}(\mathbf{S})\mathrm{I},$$

where I is the identity matrix in $\mathbb{R}^{n\times n}$. When no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Additionally, we will utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, Γ is an open or closed Lipschitz curve (respectively surface in $\mathbb{R}^3$), and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [\mathrm{H}^r(\mathcal{O})]^n, \quad \mathbb{H}^r(\mathcal{O}) := [\mathrm{H}^r(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [\mathrm{H}^r(\Gamma)]^n,$$

and for $r = 0$ we write, as usual, $\mathrm{L}^2(\mathcal{O})$, $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathrm{H}^0(\mathcal{O})$, $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ for $\mathrm{H}^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$ and $\mathbb{H}^r(\mathcal{O})$, and $\|\cdot\|_{r,\Gamma}$ for $\mathrm{H}^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$. We also write $|\cdot|_{r,\mathcal{O}}$ for the $\mathbf{H}^r$-seminorm. In addition, we recall that

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \left\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \operatorname{div}\mathbf{w} \in \mathrm{L}^2(\mathcal{O})\right\},$$

is a standard Hilbert space (see, e.g. [4, 24]), and the space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div};\mathcal{O})$. The norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. In turn, the symbol for the $L^2(\Gamma)$– and $\mathbf{L}^2(\Gamma)$–inner products

$$\langle \xi,\lambda\rangle_\Gamma := \int_\Gamma \xi\lambda, \qquad \langle \boldsymbol{\xi},\boldsymbol{\lambda}\rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda}$$

will be also employed for their respective extensions as the duality products $\mathrm{H}^{-1/2}(\Sigma)\times\mathrm{H}^{1/2}(\Sigma)$ and $\mathbf{H}^{-1/2}(\Sigma)\times\mathbf{H}^{1/2}(\Sigma)$. On the other hand, given an integer $k\geq 0$ and a set $M\subseteq\mathbb{R}^n$, we let $P_k(M)$ be the space of polynomials on M of degree $\leq k$, and set $\mathbf{P}_k(M) := [P_k(M)]^n$ and $\mathbb{P}_k(M) := [P_k(M)]^{n\times n}$. Furthermore, we will use $\|\cdot\|$ with no subscripts, to denote the natural norm of either an element or an operator in any product functional space. In addition, C will stand for any positive constant independent of the meshsizes, but eventually depending on data and/or stabilization parameters, which may take different values at each occurrence. Finally, we employ $\mathbf{0}$ to mean a generic null vector, including the null functional and operator.

2 The model problem

We begin by describing the geometry of the problem. To that end we let Ω_{S} and Ω_{D} be two bounded and simply connected polygonal domains in $\mathbb{R}^n$, $n\in\{2,3\}$, such that $\partial\Omega_{\mathrm{S}}\cap\partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$ and $\Omega_{\mathrm{S}}\cap\Omega_{\mathrm{D}} = \emptyset$, and let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}}\setminus\bar{\Sigma}$ and $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}}\setminus\bar{\Sigma}$. On the boundaries we consider the normal unit vector field $\mathbf{n}$ which is chosen pointing outwards from $\Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D}, when seen on Σ). In addition, on Σ we consider a local orthonormal basis for its tangent hyperplane given by $\{\mathbf{t}_1,\cdots,\mathbf{t}_{n-1}\}$. See Fig. 2.1 below for a two–dimensional representation of the geometry of the problem, where we simply denote $\mathbf{t}=\mathbf{t}_1$.

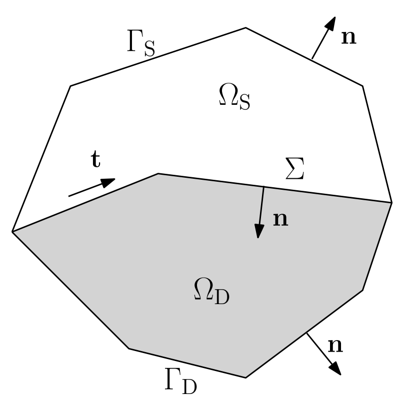

Figure 2.1: Geometric configuration for our Navier-Stokes/Darcy model

Then, our coupled problem consists of two set of equations describing the behaviour of the fluid in both domains, Ω_{S} and Ω_{D}, and a set of interface conditions on Σ. More precisely, in Ω_{S} the governing equations are those of the Navier–Stokes problem with constant viscosity $\nu>0$ and density $\rho>0$, that is

$$\begin{gathered}\boldsymbol{\sigma}_{\mathrm{S}} = 2\nu\,\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathrm{I} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \rho(\mathbf{u}_{\mathrm{S}}\cdot\nabla)\mathbf{u}_{\mathrm{S}} - \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \\ \mathrm{div}\,\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}\,, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\end{gathered} \tag{2.1}$$

where $\mathbf{u}_{\mathrm{S}}$ and p_{S} denote the velocity and the pressure of the fluid, respectively, whereas $\boldsymbol{\sigma}_{\mathrm{S}}$ is the Cauchy stress tensor, $\mathbf{f}_{\mathrm{S}}$ is a given external force living in a space to be specified later on, and $\mathbf{e}$ is the strain rate tensor given by

$$\mathbf{e}(\mathbf{u}_\mathrm{S}) := \frac{1}{2}(\nabla\mathbf{u}_\mathrm{S} + (\nabla\mathbf{u}_\mathrm{S})^t),$$

with the superscript ^{t} denoting transposition. While the standard strong Navier-Stokes equations are presented above to describe the behaviour of the fluid in Ω_S, in this work we make use of an equivalent version of (2.1) based on the introduction of a pseudostress tensor relating the stress tensor $\boldsymbol{\sigma}$ with the convective term. More precisely, analogously to [7] and [10], we introduce the nonlinear–pseudostress tensor

$$\mathbf{T}_\mathrm{S} \,:=\, \boldsymbol{\sigma}_\mathrm{S} \,-\, \rho(\mathbf{u}_\mathrm{S}\otimes\,\mathbf{u}_\mathrm{S}) \,=\, 2\nu\mathbf{e}(\mathbf{u}_\mathrm{S}) \,-p_\mathrm{S}\mathbf{I} \,-\, \rho(\mathbf{u}_\mathrm{S}\otimes\mathbf{u}_\mathrm{S}), \tag{2.2}$$

and owing to the incompressibility condition $\mathrm{tr}\,(\mathbf{e}(\mathbf{u}_\mathrm{S})) = \mathrm{div}\,\mathbf{u}_\mathrm{S} = 0$ in Ω_S, we deduce the following identities

$$p_\mathrm{S} \,=\, -\frac{1}{n}\Big\{\mathrm{tr}\,(\mathbf{T}_\mathrm{S}) \,+\, \rho\mathrm{tr}\,(\mathbf{u}_\mathrm{S}\otimes\mathbf{u}_\mathrm{S})\Big\} \quad \text{in} \quad \Omega_\mathrm{S} \quad \text{and} \quad -\mathbf{div}\,\mathbf{T}_\mathrm{S} = \mathbf{f}_\mathrm{S}. \tag{2.3}$$

Note that the first identity allows us to eliminate the unknown pressure in (2.1), obtaining

$$\mathbf{T}_\mathrm{S}^\mathtt{d} \,=\, 2\nu\mathbf{e}(\mathbf{u}_\mathrm{S}) \,-\, \rho(\mathbf{u}_\mathrm{S}\otimes\mathbf{u}_\mathrm{S})^\mathtt{d} \quad \text{in} \quad \Omega_\mathrm{S}\,.$$

Then, defining

$$\boldsymbol{\omega}(\mathbf{v}_\mathrm{S}) \,:=\, \frac{1}{2}(\nabla\mathbf{v}_\mathrm{S} \,-\, (\nabla\mathbf{v}_\mathrm{S})^t) \qquad \forall\,\mathbf{v}_\mathrm{S}\in\mathbf{H}^1(\Omega_\mathrm{S})\,, \tag{2.4}$$

the Navier-Stokes equations (2.1) can be rewritten equivalently as follows:

$$\begin{gathered}\mathbf{T}_\mathrm{S}^\mathtt{d} \,=\, 2\nu\nabla\mathbf{u}_\mathrm{S} \,-\, 2\nu\boldsymbol{\omega}(\mathbf{u}_\mathrm{S}) \,-\, \rho(\mathbf{u}_\mathrm{S}\otimes\mathbf{u}_\mathrm{S})^\mathtt{d} \quad \text{in} \quad \Omega_\mathrm{S}\,, \qquad -\mathbf{div}\,\mathbf{T}_\mathrm{S} \,=\, \mathbf{f}_\mathrm{S} \quad \text{in} \quad \Omega_\mathrm{S}\,,\\ \mathbf{T}_\mathrm{S} = \mathbf{T}_\mathrm{S}^t \quad \text{in} \quad \Omega_\mathrm{S}, \qquad \text{and} \quad \mathbf{u}_\mathrm{S} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{S}.\end{gathered} \tag{2.5}$$

In turn, in the porous medium Ω_D we consider the Darcy model:

$$\begin{gathered}\mathbf{K}^{-1}\mathbf{u}_\mathrm{D} \,=\, -\nabla p_\mathrm{D} \,+\, \mathbf{f}_\mathrm{D} \quad \text{in} \quad \Omega_\mathrm{D}\,,\\ \mathrm{div}\,\mathbf{u}_\mathrm{D} \,=\, 0 \quad \text{in} \quad \Omega_\mathrm{D}\,, \qquad \mathbf{u}_\mathrm{D}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_\mathrm{D}\end{gathered} \tag{2.6}$$

where $\mathbf{u}_\mathrm{D}$ is the velocity and p_D is the pressure. The matrix–valued function $\mathbf{K}$, describing the permeability of Ω_D divided by the viscosity ν, satisfies $\mathbf{K}^t = \mathbf{K}$, has $L^\infty(\Omega_\mathrm{D})$ components and is uniformly elliptic, that is, there exists $C_\mathbf{K} > 0$, such that

$$\alpha\cdot\mathbf{K}(x)\alpha \geq C_\mathbf{K}\|\alpha\|^2, \tag{2.7}$$

for almost all $x \in \Omega_\mathrm{D}$ and for all $\alpha \in \mathbb{R}^n$. Finally $\mathbf{f}_\mathrm{D}$ is a given external force that accounts for gravity. We conclude the description of our coupled system by introducing the transmission conditions on the interface Σ:

$$\mathbf{u}_\mathrm{S}\cdot\mathbf{n} \,=\, \mathbf{u}_\mathrm{D}\cdot\mathbf{n} \qquad \text{on} \qquad \Sigma \tag{2.8}$$

and

$$\boldsymbol{\sigma}_\mathrm{S}\mathbf{n} \,+\, \sum_{l=1}^{n-1} w_l(\mathbf{u}_\mathrm{S}\cdot\mathbf{t}_l)\mathbf{t}_l \,=\, -p_\mathrm{D}\mathbf{n} \qquad \text{on} \qquad \Sigma\,, \tag{2.9}$$

where $\{w_1,\ldots,w_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. Condition (2.8) corresponds to mass conservation on Σ, whereas (2.9) can be decomposed into its normal and tangential components, as follows:

$$(\boldsymbol{\sigma}_\mathrm{S}\mathbf{n})\cdot\mathbf{n} \,=\, -p_\mathrm{D} \qquad \text{and} \qquad (\boldsymbol{\sigma}_\mathrm{S}\mathbf{n})\cdot\mathbf{t}_l \,=\, -w_l(\mathbf{u}_\mathrm{S}\cdot\mathbf{t}_l), \quad l = 1,\ldots,n-1. \tag{2.10}$$

The first condition in (2.10) corresponds to the balance of normal forces, whereas the second one, known as the Beavers-Joseph-Saffman law, establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidences, that $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{t}_l$, $l = 1, \ldots, n-1$, is negligible). We refer the reader to [3, 28, 31] for further details on this interface condition. Notice that equation (2.9) can be rewritten in terms of tensor $\mathbf{T}_{\mathrm{S}}$ as

$$\mathbf{T}_{\mathrm{S}}\mathbf{n} \;=\; -\rho(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})\mathbf{n} \;-\; \sum_{l=1}^{n-1} w_l(\mathbf{u}_{\mathrm{S}} \cdot \mathbf{t}_l)\mathbf{t}_l - p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.11}$$

which will be employed below in place of (2.9).

3 The continuous formulation

In this section we introduce our augmented fully-mixed variational formulation and address its solvability.

3.1 The augmented fully–mixed variational problem

In what follow we derive the variational formulation of our model problem based on equations (2.5),(2.6), (2.8) and (2.11). To this end, we first introduce the Hilbert spaces

$$\begin{aligned} \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \;&:=\; \Big\{ \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) : \quad \mathbf{v}_{\mathrm{S}} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}} \Big\}, \\ \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}) \;&:=\; \Big\{ \mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n} \;=\mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{D}} \Big\}, \end{aligned}$$

and recall the identities

$$\mathbf{T}^{\mathrm{d}} : \mathbf{R} \;=\; \mathbf{T}^{\mathrm{d}} : \mathbf{R}^{\mathrm{d}} \quad \text{and} \quad (\boldsymbol{\omega}(\mathbf{v}), \mathbf{R})_{\Omega_{\mathrm{S}}} \;=\; \frac{1}{2}(\mathrm{curl}\,(\mathbf{v}), \mathrm{as}(\mathbf{R}))_{\Omega_{\mathrm{S}}} \tag{3.1}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$, and for all $\mathbf{T}, \mathbf{R} \in \mathbb{L}^2(\Omega_{\mathrm{S}})$, with

$$\mathrm{curl}\,(\mathbf{v}) := \begin{cases} \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} & \text{in} \quad \mathbb{R}^2, \\ \nabla \times \mathbf{v} := \left(\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}, \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}, \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \right) & \text{in} \quad \mathbb{R}^3, \end{cases}$$

and

$$\mathrm{as}(\mathbf{R}) := \begin{cases} R_{21} - R_{12} & \text{in} \quad \mathbb{R}^2, \\ (R_{32} - R_{23}, R_{13} - R_{31}, R_{21} - R_{12}) & \text{in} \quad \mathbb{R}^3. \end{cases}$$

In addition, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, in what follows we denote:

$$(u, v)_\star \;:=\; \int_{\Omega_\star} uv, \qquad (\mathbf{u}, \mathbf{v})_{\Omega_\star} \;:=\; \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_\star} \;:=\; \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

First, for the set of equations (2.5) we proceed analogously to [9] (see also [1, Section 2.1] for a similar approach). More precisely, we multiply the first and second equations of (2.5) by test functions $\mathbf{R}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, respectively, and then perform integration by parts once for the equation multiplied by $\mathbf{R}_{\mathrm{S}}$ and twice for the one multiplied by $\mathbf{v}_{\mathrm{S}}$. In this way, utilizing the Dirichlet boundary condition $\mathbf{u}_{\mathrm{S}} = \mathbf{0}$ on Γ_{S} and the interface condition (2.11), and making use of the identity $\langle (\mathbf{u} \otimes \mathbf{w})\mathbf{n}, \mathbf{v} \rangle_\Sigma \,=\, \langle \mathbf{w} \cdot \mathbf{n}, \mathbf{u} \cdot \mathbf{v} \rangle_\Sigma$, we obtain

$$
\begin{aligned}
&(\mathbf{T}_{\mathrm{S}}^{\mathtt{d}}, \mathbf{R}_{\mathrm{S}}^{\mathtt{d}})_{\Omega_{\mathrm{S}}} + 2\,\nu(\mathbf{u}_{\mathrm{S}}, \mathbf{div}\,\mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} + \,\nu(\mathrm{curl}\,(\mathbf{u}_{\mathrm{S}}), \mathrm{as}(\mathbf{R}_{\mathrm{S}}))_{\Omega_{\mathrm{S}}} \\
&\quad + \rho\big((\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathtt{d}}, \mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} - 2\,\nu\,\langle \mathbf{R}_{\mathrm{S}}\mathbf{n}, \mathbf{u}_{\mathrm{S}} \rangle_\Sigma \,=\, 0 \qquad \forall\,\mathbf{R}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\,,
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&-2\nu(\mathbf{v}_{\mathrm{S}}, \mathbf{div}\,\mathbf{T}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} + 2\nu\langle \mathbf{T}_{\mathrm{S}}\mathbf{n}, \mathbf{v}_{\mathrm{S}} \rangle_\Sigma - \nu(\mathrm{curl}\,(\mathbf{v}_{\mathrm{S}}), \mathrm{as}(\mathbf{T}_{\mathrm{S}}))_{\Omega_{\mathrm{S}}} + 2\nu\rho\,\langle \mathbf{u}_{\mathrm{S}} \cdot \mathbf{n}, \mathbf{u}_{\mathrm{S}} \cdot \mathbf{v}_{\mathrm{S}} \rangle_\Sigma \\
&\quad + 2\nu \sum_{l=1}^{n-1} w_l \langle (\mathbf{u}_{\mathrm{S}} \cdot \mathbf{t}_l)\mathbf{t}_l, \mathbf{v}_{\mathrm{S}} \rangle_\Sigma + 2\nu\langle \mathbf{v}_{\mathrm{S}} \cdot \mathbf{n}, \lambda \rangle_\Sigma \,=\, 2\nu(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} \qquad \forall\,\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\,,
\end{aligned}
\tag{3.3}
$$

where $\lambda := p_{\mathrm{D}}|_\Sigma \,\in\, \mathrm{H}^{1/2}(\Sigma)$ is introduced as an additional unknown.

On the other hand, for the set of equations (2.6) we proceed analogously to [14, 21] to get

$$
\begin{aligned}
&2\nu(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} - 2\nu(p_{\mathrm{D}}, \mathrm{div}\,\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} - 2\nu\langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \lambda \rangle_\Sigma \\
&\quad = \; 2\nu(\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} \qquad \forall\,\mathbf{v}_{\mathrm{D}} \in \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})\,,
\end{aligned}
\tag{3.4}
$$

and

$$
2\nu(q_{\mathrm{D}}, \mathrm{div}\,\mathbf{u}_{\mathrm{D}}) \,=\, 0 \qquad \forall\, q_{\mathrm{D}} \quad \text{in} \quad \mathrm{L}^2(\Omega_{\mathrm{D}}), \tag{3.5}
$$

whereas (2.8) is imposed weakly through

$$
2\nu\langle \mathbf{u}_{\mathrm{S}} \cdot \mathbf{n}, \xi \rangle_\Sigma - 2\nu\langle \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}, \xi \rangle_\Sigma \,=\, 0 \qquad \forall \xi \,\in\, \mathrm{H}^{1/2}(\Sigma). \tag{3.6}
$$

Above, and for convenience of the forthcoming analysis, we have intentionally multiplied equations (3.3)–(3.6) by 2ν. Finally, we proceed analogously to [9] and add the following redundant terms arising from the constitutive and equilibrium equations

$$
\kappa_1\,(\mathbf{div}\,\mathbf{T}_{\mathrm{S}} \,+\, \mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} \,=\, 0 \qquad \forall\,\mathbf{R}_{\mathrm{S}} \,\in\, \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})\,, \tag{3.7}
$$

and

$$
\kappa_2\left(\mathbf{T}_{\mathrm{S}}^{\mathtt{d}} \,-\, 2\,\nu\mathbf{e}(\mathbf{u}_{\mathrm{S}}) \,+\, \rho\,(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathtt{d}}, \mathbf{e}(\mathbf{v}_{\mathrm{S}})\right)_\Omega = 0 \qquad \forall\,\mathbf{v}_{\mathrm{S}} \,\in\, \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\,, \tag{3.8}
$$

respectively, where κ_1 and κ_2 are positive parameters to be specified later. In this way, at first instance we arrive at the variational problem: Find

$$
\begin{aligned}
(\mathbf{T}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \quad &\text{in} \quad \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \\
(p_{\mathrm{D}}, \lambda) \quad &\text{in} \quad \mathrm{L}^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma),
\end{aligned}
$$

such that (3.2)–(3.8) hold.

Now, let us notice that if $\big((\mathbf{T}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (p_{\mathrm{D}}, \lambda)\big) \in \mathbf{X} \times \mathbf{Q}$ solves (3.2) - (3.8), then simple computations show that for all $c \,\in\, \mathbb{R}$, $((\mathbf{T}_{\mathrm{S}} - c\mathbf{I}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (p_{\mathrm{D}} + c, \lambda + c))$ is also a solution of (3.2)-(3.8), and consequently, uniqueness of solution of the coupled system fails. Then, in order to overcome this drawback, from now on we restrict the pressure space to $\mathrm{L}^2_0(\Omega_{\mathrm{D}})$, where

$$
\mathrm{L}^2_0(\Omega_{\mathrm{D}}) \,=\, \left\{ q \,\in\, \mathrm{L}^2(\Omega_{\mathrm{D}}) \,:\, \int_{\Omega_{\mathrm{D}}} q \,=\, 0 \right\}.
$$

Furthermore, recalling that there holds the decomposition

$$\mathbb{H}(\mathbf{div};\Omega_S) \,=\, \mathbb{H}_0(\mathbf{div};\Omega_S) \,\oplus\, P_0(\Omega_S)\,\mathrm{I}\,, \tag{3.9}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega_S) \,:=\, \left\{ \mathbf{R}_S \,\in\, \mathbb{H}(\mathbf{div};\Omega_S) : \quad \int_{\Omega_S} \operatorname{tr}\mathbf{R}_S = 0 \right\}, \tag{3.10}$$

we redefine the fluid pseudostress tensor $\mathbf{T}_S$ as

$$\mathbf{T}_S + \mu\mathbf{I} \quad \text{with the new unknowns} \quad \mathbf{T}_S \in \mathbb{H}_0(\mathbf{div};\Omega_S) \quad \text{and} \quad \mu \in \mathbb{R}\,,$$

whence equation (3.3) becomes

$$\begin{aligned}
-2\mu\,(\mathbf{v}_S, \mathbf{div}\,\mathbf{T}_S)_{\Omega_S} + 2\nu\langle \mathbf{T}_S, \mathbf{v}_S\rangle_\Sigma + 2\nu\mu\langle \mathbf{v}_S\cdot\mathbf{n}, 1\rangle_\Sigma - \mu\,(\operatorname{curl}(\mathbf{v}_S), \operatorname{as}(\mathbf{T}_S))_{\Omega_S} \\
+ 2\nu\rho\langle \mathbf{u}_S\cdot\mathbf{n}, \mathbf{u}_S\cdot\mathbf{v}_S\rangle_\Sigma + \sum_{l=1}^{n-1} 2\nu w_l\langle(\mathbf{u}_S\cdot\mathbf{t}_l)\mathbf{t}_l, \mathbf{v}_S\rangle_\Sigma \\
+ 2\nu\langle \mathbf{v}_S\cdot\mathbf{n}, \lambda\rangle_\Sigma \,=\, (\mathbf{f}_S, \mathbf{v}_S)_{\Omega_S} \qquad \forall\,\mathbf{v}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)\,.
\end{aligned} \tag{3.11}$$

In turn, decomposing the test function $\mathbf{R}_S$ of (3.2) according to (3.9), we obtain

$$\begin{aligned}
\big(\mathbf{T}_S^{\mathrm{d}}, \mathbf{R}_S^{\mathrm{d}}\big)_{\Omega_S} + 2\nu\,(\mathbf{u}_S, \mathbf{div}\,\mathbf{R}_S)_{\Omega_S} \,+\, \nu\,(\operatorname{curl}(\mathbf{u}_S), \operatorname{as}(\mathbf{R}_S))_{\Omega_S} \\
+\, \rho\,\big((\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathrm{d}}, \mathbf{R}_S\big)_{\Omega_S} \,-\, 2\nu\langle \mathbf{R}_S\mathbf{n}, \mathbf{u}_S\rangle_\Sigma \,=\, 0 \qquad \forall\,\mathbf{R}_S \in \mathbb{H}_0(\mathbf{div};\Omega_S)\,,
\end{aligned} \tag{3.12}$$

and

$$2\nu\eta\langle \mathbf{u}_S\cdot\mathbf{n}, 1\rangle_\Sigma \,=\, 0 \qquad \forall\,\eta \in \mathbb{R}\,. \tag{3.13}$$

In this way, by replacing (3.2) and (3.3) by (3.12) - (3.13) and (3.11), respectively, the variational formulation of our coupled system can be stated as: Find $(\mathbf{T}_S, \mathbf{u}_S, \mathbf{u}_D) \in \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times \mathbf{H}_{\Gamma_D}(\operatorname{div};\Omega_D)$ and $(p_D, \lambda, \mu) \in \mathrm{L}^2_0(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbb{R}$ such that (3.4)–(3.8), (3.12) - (3.13), and (3.11) hold. Moreover, we show next that it can be rewritten in terms of suitable forms and functionals. In fact, we begin by grouping the unknowns, test functions, and spaces, as follows

$$\begin{aligned}
&\underline{\boldsymbol{\Phi}} := (\mathbf{T}_S, \mathbf{u}_S, \mathbf{u}_D)\,,\ \underline{\boldsymbol{\Psi}} = (\mathbf{R}_S, \mathbf{v}_S, \mathbf{v}_D) \,\in\, \mathbf{X} := \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times \mathbf{H}_{\Gamma_D}(\operatorname{div};\Omega_D)\,, \\
&\underline{\mathbf{p}} := (p_D, \lambda, \mu)\,,\ \underline{\mathbf{q}} = (q_D, \xi, \eta) \,\in\, \mathbf{Q} := \mathrm{L}^2_0(\Omega_D) \times \mathrm{H}^{1/2}(\Sigma) \times \mathbb{R}\,,
\end{aligned} \tag{3.14}$$

where $\mathbf{X}$ and $\mathbf{Q}$ are endowed with the norms $\|\cdot\|^2_{\mathbf{X}} := \|\cdot\|^2_{\mathbf{div};\Omega_S} + \|\cdot\|^2_{1,\Omega_S} + \|\cdot\|^2_{\operatorname{div};\Omega_D}$ and $\|\cdot\|^2_{\mathbf{Q}} := \|\cdot\|_{0,\Omega_D} + \|\cdot\|^2_{1/2,\Sigma} + |\cdot|$, respectively. In addition, given $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}(\Omega_S)$, we define the bilinear forms

$$\mathbf{A}_{\mathbf{w}_S}(\underline{\boldsymbol{\Phi}}, \underline{\boldsymbol{\Psi}}) \,:=\, \mathbf{A}_S\big((\mathbf{T}_S, \mathbf{v}_S), (\mathbf{R}_S, \mathbf{v}_S)\big) \,+\, \mathbf{C}_{\mathbf{w}_S}\big((\mathbf{T}_S, \mathbf{v}_S), (\mathbf{R}_S, \mathbf{v}_S)\big) \,+\, \mathbf{A}_D(\mathbf{u}_D, \mathbf{v}_D)\,, \tag{3.15}$$

and

$$\mathbf{B}(\underline{\boldsymbol{\Psi}}, \underline{\mathbf{q}}) \,:=\, -2\nu(q_D, \operatorname{div}\mathbf{v}_D)_{\Omega_D} \,+\, 2\nu\langle \mathbf{v}_S\cdot\mathbf{n} \,-\, \mathbf{v}_D\cdot\mathbf{n}, \xi\rangle_\Sigma \,+\, 2\nu\eta\langle \mathbf{v}_S\cdot\mathbf{n}, 1\rangle_\Sigma, \tag{3.16}$$

for all $\underline{\boldsymbol{\Phi}} = (\mathbf{T}_S, \mathbf{u}_S, \mathbf{u}_D)$, $\underline{\boldsymbol{\Psi}} = (\mathbf{R}_S, \mathbf{v}_S, \mathbf{v}_D) \in \mathbf{X}$, and for all $\underline{\mathbf{q}} = (q_D, \xi, \eta) \in \mathbf{Q}$, where $\mathbf{A}_S$, $\mathbf{C}_{\mathbf{w}_S}$, and $\mathbf{A}_D$ are in turn the bilinear forms given by

$$\begin{aligned}
\mathbf{A}_S\big((\mathbf{T}_S, \mathbf{u}_S), (\mathbf{R}_S, \mathbf{v}_S)\big) \,:=\, (\mathbf{T}_S^{\mathrm{d}}, \mathbf{R}_S^{\mathrm{d}})_{\Omega_S} \,+\, 2\nu(\mathbf{u}_S, \mathbf{div}\,\mathbf{R}_S)_{\Omega_S} \,-\, 2\nu(\mathbf{v}_S, \operatorname{div}\mathbf{T}_S)_{\Omega_S} \\
+\, \nu(\operatorname{curl}(\mathbf{u}_S), \operatorname{as}(\mathbf{R}_S))_{\Omega_S} \,-\, \nu(\operatorname{curl}(\mathbf{v}_S), \operatorname{as}(\mathbf{T}_S))_{\Omega_S} \,-\, 2\nu\langle \mathbf{R}_S\mathbf{n}, \mathbf{u}_S\rangle_\Sigma \,+\, 2\nu\langle \mathbf{T}_S\mathbf{n}, \mathbf{v}_S\rangle_\Sigma \\
+\, \kappa_1(\mathbf{div}\,\mathbf{T}_S, \mathbf{div}\,\mathbf{R}_S)_{\Omega_S} \,-\, \kappa_2(\mathbf{T}_S^{\mathrm{d}} \,-\, 2\nu\mathbf{e}(\mathbf{u}_S), \mathbf{e}(\mathbf{v}_S))_{\Omega_S} \,+\, 2\nu\sum_{l=1}^{n-1} w_l\langle \mathbf{u}_S\cdot\mathbf{t}_l, \mathbf{v}_S\cdot\mathbf{t}_l\rangle_\Sigma\,,
\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}\big((\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}}),(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})\big) \;:=\; & \rho((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathtt{d}},\mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} \,+\, 2\nu\rho\langle\mathbf{w}_{\mathrm{S}}\cdot\mathbf{n},\mathbf{u}_{\mathrm{S}}\cdot\mathbf{v}_{\mathrm{S}}\rangle_{\Sigma} \\
& -\,\kappa_2\rho((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathtt{d}},\mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\Omega_{\mathrm{S}}}\,,
\end{aligned}
\tag{3.18}
$$

and

$$
\mathbf{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}) \;:=\; (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}}\,. \tag{3.19}
$$

In addition, we define the functional $\mathbf{F}\in\mathbf{X}'$ as

$$
\mathbf{F}(\underline{\boldsymbol{\Psi}}) \;:=\; -2\nu(\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} \,+\, 2\nu(\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} \,-\, \kappa_1(\mathbf{f}_{\mathrm{S}},\operatorname{div}\mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}}\,, \tag{3.20}
$$

for all $\underline{\boldsymbol{\Psi}}=(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}})\in\mathbf{X}$. Consequently, we arrive to the coupled system: Find $(\underline{\boldsymbol{\Phi}},\underline{\mathbf{p}})\in\mathbf{X}\times\mathbf{Q}$ such that

$$
\begin{aligned}
\mathbf{A}_{\mathbf{u}_{\mathrm{S}}}(\underline{\boldsymbol{\Phi}},\underline{\boldsymbol{\Psi}}) + \mathbf{B}(\underline{\boldsymbol{\Psi}},\underline{\mathbf{p}}) &= \mathbf{F}(\underline{\boldsymbol{\Psi}}) \qquad \forall\,\underline{\boldsymbol{\Psi}}\in\mathbf{X},\\
\mathbf{B}(\underline{\boldsymbol{\Phi}},\underline{\mathbf{q}}) &= 0 \qquad\quad\;\; \forall\,\underline{\mathbf{q}}\in\mathbf{Q}.
\end{aligned}
\tag{3.21}
$$

3.2 Analysis of the continuous problem

Let us define the mapping

$$
\mathcal{J}:\mathbf{M}\subseteq\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\to\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}),\quad \mathbf{w}_{\mathrm{S}}\to\mathcal{J}(\mathbf{w}_{\mathrm{S}})=\mathbf{u}_{\mathrm{S}}, \tag{3.22}
$$

where $\mathbf{u}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ is the second component of $\underline{\boldsymbol{\Phi}}\in\mathbf{X}$, which, together with $\underline{\mathbf{p}}\in\mathbf{Q}$, constitutes the unique solution of the linearized version of problem (3.21): Find $(\underline{\boldsymbol{\Phi}},\underline{\mathbf{p}})\in\mathbf{X}\times\mathbf{Q}$ such that

$$
\begin{aligned}
\mathbf{A}_{\mathbf{w}_{\mathrm{S}}}(\underline{\boldsymbol{\Phi}},\underline{\boldsymbol{\Psi}}) + \mathbf{B}(\underline{\boldsymbol{\Psi}},\underline{\mathbf{p}}) &= \mathbf{F}(\underline{\boldsymbol{\Psi}}) \qquad \forall\,\underline{\boldsymbol{\Psi}}\in\mathbf{X},\\
\mathbf{B}(\underline{\boldsymbol{\Phi}},\underline{\mathbf{q}}) &= 0 \qquad\quad\;\; \forall\,\underline{\mathbf{q}}\in\mathbf{Q},
\end{aligned}
\tag{3.23}
$$

and $\mathbf{M}$ is a bounded set ensuring the well-definiteness of $\mathcal{J}$ (to be specified below). Then, noticing that $(\underline{\boldsymbol{\Phi}},\underline{\mathbf{p}})=((\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}),(p_{\mathrm{D}},\lambda,\mu))\in\mathbf{X}\times\mathbf{Q}$ is a solution of (3.21) if and only if $\mathbf{u}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ is a solution of the fixed-point problem: Find $\mathbf{u}_{\mathrm{S}}\in\mathbf{M}$, such that

$$
\mathcal{J}(\mathbf{u}_{\mathrm{S}})=\mathbf{u}_{\mathrm{S}}, \tag{3.24}
$$

it is clear that to prove the well-posedness of (3.21) it suffices to prove the unique solvability of problem (3.24). Before doing that we must study the well-definiteness of $\mathcal{J}$. The following section is devoted to this matter.

3.2.1 Well-definedness of the fixed-point operator

According to the mixed structure of the linearized problem (3.23), in what follows we apply the Babuška-Brezzi theory to prove its well-posedness, or equivalently the one of $\mathcal{J}$. We begin by establishing the continuity of the functional $\mathbf{F}$ and the bilinear forms $\mathbf{A}_{\mathrm{S}}$, $\mathbf{A}_{\mathrm{D}}$ and $\mathbf{B}$:

$$
|\mathbf{F}(\underline{\boldsymbol{\Psi}})| \;\le\; \Big\{\big(4\nu^2+\kappa_1^2\big)^{1/2}\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\,\|\underline{\boldsymbol{\Psi}}\|_{\mathbf{X}}, \tag{3.25}
$$

$$
|\mathbf{A}_{\mathrm{S}}((\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}}),(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}))| \le C_{\mathrm{S},1}\|(\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})\|\|(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})\|, \tag{3.26}
$$

$$
|\,\mathbf{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})\,| \le C_{\mathrm{D}}\|\mathbf{u}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}, \qquad \mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \tag{3.27}
$$

and

$$
|\mathbf{B}(\underline{\boldsymbol{\Phi}},\underline{\mathbf{q}})| \le C_{\mathbf{B}}\|\underline{\boldsymbol{\Phi}}\|_{\mathbf{X}}\|\underline{\mathbf{q}}\|_{\mathbf{Q}}, \tag{3.28}
$$

for all $\underline{\boldsymbol{\Phi}} \in \mathbf{X}$, $\mathbf{T}_{\mathrm{S}}, \mathbf{R}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, and $(\underline{\boldsymbol{\Phi}}, \underline{\mathbf{q}}) \in \mathbf{X}\times\mathbf{Q}$, with $C_{\mathrm{S},1}, C_{\mathrm{D}}, C_{\mathbf{B}} > 0$. The proofs of the previous estimates are straightforward.

Now, given $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, from the continuity of the embeddings $\mathbf{i}_\Sigma : \mathbf{H}^{1/2}(\Sigma) \to \mathbf{L}^4(\Sigma)$ and $\mathbf{i}_{\mathrm{S}} : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{L}^4(\Omega_{\mathrm{S}})$, and the continuity of the trace operator $\boldsymbol{\gamma}_{\mathrm{S}} : \mathbf{H}^1(\Omega_{\mathrm{S}}) \to \mathbf{L}^2(\partial\Omega_{\mathrm{S}})$, we obtain that there exists $C_{\mathrm{S},2} > 0$ such that

$$|\mathbf{C}_{\mathbf{w}_{\mathrm{S}}}((\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}}),(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}))| \le C_{\mathrm{S},2}\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|(\mathbf{T}_{\mathrm{S}},\mathbf{u}_{\mathrm{S}})\|\|(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})\|, \tag{3.29}$$

for all $\mathbf{T}_{\mathrm{S}}, \mathbf{R}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{u}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, with $C_{\mathrm{S},2} = \|\mathbf{i}_{\mathrm{S}}\|^2\left(\rho^2+\kappa_2^2\right)^{1/2} + 2\nu\|\mathbf{i}_\Sigma\|^2\|\boldsymbol{\gamma}_{\mathrm{S}}\|$. In particular, from (3.26) and (3.29) we easily deduce that, for a fixed $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, $\mathbf{A}_{\mathbf{w}_{\mathrm{S}}}$ (cf. (3.15)) is a continuous bilinear form, that is

$$|\mathbf{A}_{\mathbf{w}_{\mathrm{S}}}(\underline{\boldsymbol{\Phi}},\underline{\boldsymbol{\Psi}})| \le (C_{\mathbf{A}} + C_{\mathrm{S},2}\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}})\|\underline{\boldsymbol{\Phi}}\|_{\mathbf{X}}\|\underline{\boldsymbol{\Psi}}\|_{\mathbf{X}} \qquad \forall\,\underline{\boldsymbol{\Phi}},\underline{\boldsymbol{\Psi}} \in \mathbf{X}, \tag{3.30}$$

with $C_{\mathbf{A}} > 0$. Let us now define the subspace

$$\mathbf{V} := \Big\{\underline{\boldsymbol{\Psi}} \in \mathbf{X} : \mathbf{B}(\underline{\boldsymbol{\Psi}},\underline{\mathbf{q}}) := 0 \quad \forall\,\underline{\mathbf{q}} = (q_{\mathrm{D}},\xi,\eta) \in \mathbf{Q}\Big\}. \tag{3.31}$$

From the definition of $\mathbf{B}$ (cf. (3.16)), it follows that $\underline{\boldsymbol{\Psi}} = (\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{V}$, if and only if,

$$\begin{gathered}(q_{\mathrm{D}},\operatorname{div}\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} = 0 \quad \forall\, q_{\mathrm{D}} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})\,, \qquad \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n}, 1\rangle_\Sigma = 0,\\ \text{and} \quad \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_\Sigma = 0 \quad \forall\,\xi \in \mathrm{H}^{1/2}(\Sigma).\end{gathered} \tag{3.32}$$

Then, recalling that $\mathrm{L}^2(\Omega_{\mathrm{D}}) = \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \oplus \mathbb{R}$, and noticing from the second and third equations of (3.32) that $\int_{\Omega_{\mathrm{D}}} \operatorname{div}\mathbf{v}_{\mathrm{D}} = \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},1\rangle_\Sigma = \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n},1\rangle_\Sigma = 0$, we conclude together with the first equation of (3.32) that

$$(\operatorname{div}\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}}) = 0 \qquad \forall\, q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}).$$

Consequently, we can rewrite the subspace $\mathbf{V}$ as follows

$$\begin{aligned}\mathbf{V} := \big\{\underline{\boldsymbol{\Psi}} = (\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{X} : \operatorname{div}\mathbf{v}_{\mathrm{D}} = 0 \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma\\ \text{and} \quad \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n},1\rangle_\Sigma = 0\big\}.\end{aligned}$$

Now we address the ellipticity of $\mathbf{A}_{\mathbf{w}_{\mathrm{S}}}$ on $\mathbf{V}$ for suitable choices of $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$.

Lemma 3.1 *Assume that $\kappa_1 > 0$ and $0 < \kappa_2 < 4\nu$, and let $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that*

$$\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le \frac{\alpha}{C_{\mathrm{S},2}}, \tag{3.33}$$

where $\alpha = \frac{1}{2}\min\{\alpha_{\mathrm{S}}, C_{\mathbf{K}}\}$, α_{S} is the constant defined below in (3.38), and $C_{\mathbf{K}}$ and $C_{\mathrm{S},2}$ are the positive constants satisfying (2.7) and (3.29), respectively. Then, there holds

$$\mathbf{A}_{\mathbf{w}_{\mathrm{S}}}(\underline{\boldsymbol{\Psi}},\underline{\boldsymbol{\Psi}}) \ge \alpha\|\underline{\boldsymbol{\Phi}}\|_{\mathbf{X}} \qquad \forall\,\underline{\boldsymbol{\Psi}} \in \mathbf{V}\,. \tag{3.34}$$

Proof. Given $\underline{\boldsymbol{\Psi}} := \big(\mathbf{R}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}\big) \in \mathbf{V}$, we first recall from [17, Lemma 2.3] (see also [4, Chapter IV]) and [4, 24], respectively, that the following well known estimates hold

$$C_d\|\mathbf{R}\|^2_{0,\Omega_{\mathrm{S}}} \le \|\mathbf{R}^{\mathrm{d}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\mathbf{R}\|^2_{0,\Omega_{\mathrm{S}}} \tag{3.35}$$

and

$$C_k\|\mathbf{v}\|^2_{1,\Omega_\mathrm{S}} \le \|\mathbf{e}(\mathbf{v})\|^2_{0,\Omega_\mathrm{S}}\,, \tag{3.36}$$

with $C_d > 0$ and $C_k > 0$ depending only on Ω_S. Then, employing (2.7) we deduce that

$$\begin{aligned}
&\mathbf{A}_\mathrm{S}\big((\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}),(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S})\big) + \mathbf{A}_\mathrm{D}(\mathbf{v}_\mathrm{D},\mathbf{v}_\mathrm{D}) \\
&\quad\ge\ \frac{C_d}{2}\min\{\kappa_1,1\}\|\mathbf{R}_\mathrm{S}\|^2_{\mathbf{div};\Omega_\mathrm{S}} \,+\, C_\mathbf{K}\|\mathbf{v}_\mathrm{D}\|^2_{0,\Omega_\mathrm{D}} \,+\, \kappa_2\, C_k\left(2\nu - \frac{\kappa_2}{2}\right)\|\mathbf{v}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} \\
&\quad\ge\ \alpha_\mathrm{S}\|(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S})\|^2 \,+\, C_\mathbf{K}\|\mathbf{v}_\mathrm{D}\|^2_{\mathrm{div};\Omega_\mathrm{D}}\,,
\end{aligned} \tag{3.37}$$

with

$$\alpha_\mathrm{S} \,=\, \min\left\{\frac{C_d}{2}\min\{\kappa_1,1\}, C_k\kappa_2\left(2\nu - \frac{\kappa_2}{2}\right)\right\}. \tag{3.38}$$

Then, using that

$$\begin{aligned}
\mathbf{A}_{\mathbf{w}_\mathrm{S}}(\underline{\boldsymbol{\Psi}},\underline{\boldsymbol{\Psi}}) =\ & \mathbf{A}_\mathrm{S}((\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}),(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S})) + \mathbf{A}_\mathrm{D}(\mathbf{v}_\mathrm{D},\mathbf{v}_\mathrm{D}) + \mathbf{C}_{\mathbf{w}_\mathrm{S}}((\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}),(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S})),\\
\ge\ & \mathbf{A}_\mathrm{S}((\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}),(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S})) + \mathbf{A}_\mathrm{D}(\mathbf{v}_\mathrm{D},\mathbf{v}_\mathrm{D}) - |\mathbf{C}_{\mathbf{w}_\mathrm{S}}((\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}),(\mathbf{R}_\mathrm{S},\mathbf{v}_\mathrm{S}))|,
\end{aligned}$$

we conclude from (3.29), (3.37), and the hypotheses on the parameters κ_1 and κ_2, the required inequality (3.34). □

Now we turn to establish the inf-sup condition of $\mathbf{B}$. As we shall see next, this result can be derived from the following two estimates.

Lemma 3.2 *There exists $c_1 > 0$ such that*

$$S_1(q_\mathrm{D},\xi) \,:=\, \sup_{\substack{\mathbf{v}_\mathrm{D}\in\mathbf{H}_{\Gamma_\mathrm{D}}(\mathrm{div};\Omega_\mathrm{D})\\ \mathbf{v}_\mathrm{D}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_\mathrm{D},q_\mathrm{D}) \,+\, \langle\mathbf{v}_\mathrm{D}\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\mathbf{v}_\mathrm{D}\|_{\mathrm{div};\Omega_\mathrm{D}}} \,\ge\, c_1\Big\{\|q_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \,+\, \|\xi\|_{1/2,\Sigma}\Big\}. \tag{3.39}$$

for all $(q_\mathrm{D},\xi)\in L^2_0(\Omega_\mathrm{D})\times H^{1/2}(\Sigma)$.

Proof. See Lemma 3.3 in [22]. □

Lemma 3.3 *There exist $c_2, c_3 > 0$, such that*

$$S_2(\xi,\eta) \,:=\, \sup_{\substack{\mathbf{v}_\mathrm{S}\in\mathbf{H}_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})\\ \mathbf{v}_\mathrm{S}\neq\mathbf{0}}} \frac{\langle\mathbf{v}_\mathrm{S}\cdot\mathbf{n},\xi\rangle_\Sigma \,-\, \eta\langle\mathbf{v}_\mathrm{S}\cdot\mathbf{n},1\rangle_\Sigma}{\|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}} \,\ge\, c_2|\eta| \,-\, c_3\|\xi\|_{1/2,\Sigma}, \tag{3.40}$$

for all $(\xi,\eta)\in H^{1/2}(\Sigma)\times\mathbb{R}$.

Proof. We proceed similarly to the proofs of [22, Lemma 3.2] and [8, Lemma 3.2]. In fact, we let $\mathbf{v}_0$ be a fixed element in $\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ satisfying $\langle\mathbf{v}_0\cdot\mathbf{n},1\rangle_\Sigma \,\neq\, 0$ (see the proof of [8, Lemma 3.2] for the construction of such an element), and observe that for all $(\xi,\eta)\in H^{1/2}(\Sigma)\times\mathbb{R}$, there holds

$$\begin{aligned}
S_2(\xi,\eta) &\ge \frac{|\,\langle\mathbf{v}_0\cdot\mathbf{n},\xi\rangle_\Sigma \,-\, \eta\langle\mathbf{v}_0\cdot\mathbf{n},1\rangle_\Sigma\,|}{\|\mathbf{v}_0\|_{1,\Omega_\mathrm{S}}}\\
&\ge\ \frac{|\eta|\,|\langle\mathbf{v}_0\cdot\mathbf{n},1\rangle_\Sigma|}{\|\mathbf{v}_0\|_{1,\Omega_\mathrm{S}}} \,-\, \frac{|\langle\mathbf{v}_0\cdot\mathbf{n},\xi\rangle_\Sigma|}{\|\mathbf{v}_0\|_{1,\Omega_\mathrm{S}}} \,\ge\, c_2\,|\eta| \,-\, c_3\|\xi\|_{1/2,\Sigma}\,,
\end{aligned} \tag{3.41}$$

with $c_2 = \frac{|\langle\mathbf{v}_0\cdot\mathbf{n},1\rangle|}{\|\mathbf{v}_0\|_{1,\Omega_\mathrm{S}}}$ and $c_3 > 0$ the constant satisfying $|\langle\mathbf{v}_0\cdot\mathbf{n},\xi\rangle_\Sigma| \le c_3\|\mathbf{v}_0\|_{1,\Omega_\mathrm{S}}\|\xi\|_{1/2,\Gamma}$, which yields the desired result. □

Employing the previous two lemmas, we prove now the inf-sup condition of $\mathbf{B}$.

Lemma 3.4 *There exists $\beta > 0$, such that*

$$S(\underline{\mathbf{q}}) := \sup_{\substack{\underline{\mathbf{\Psi}} \in \mathbf{X} \\ \underline{\mathbf{\Psi}} \neq \mathbf{0}}} \frac{\mathbf{B}\big(\underline{\mathbf{\Psi}}, \underline{\mathbf{q}}\big)}{\|\underline{\mathbf{\Psi}}\|_{\mathbf{X}}} \geq \beta \|\underline{\mathbf{q}}\|_{\mathbf{Q}} \qquad \forall\, \underline{\mathbf{q}} \in \mathbf{Q}. \tag{3.42}$$

Proof. Given $\underline{\mathbf{q}} = (p_{\mathrm{D}}, \xi, \eta) \in \mathbf{Q}$, from the definition of the bilinear form $\mathbf{B}$ (cf. (3.16)), we observe that

$$S(\underline{\mathbf{q}}) \geq 2\nu S_1(q_{\mathrm{D}}, \xi) \quad \text{and} \quad S(\underline{\mathbf{q}}) \geq 2\nu S_2(\xi, \eta),$$

which together with Lemmas 3.2 and 3.3, yield

$$S(\underline{\mathbf{q}}) \geq 2\nu c_1 \left\{ \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|\xi\|_{1/2,\Sigma} \right\} \quad \text{and} \quad S(\underline{\mathbf{q}}) \geq 2\nu \left\{ c_2 |\eta| - c_3 \|\xi\|_{1/2,\Sigma} \right\}.$$

Then, from the latter estimates we easily obtain

$$\left(1 + \frac{c_2}{2c_3}\right) S(\underline{\mathbf{q}}) \geq \nu c_1 \min\left\{1, \frac{c_2}{c_3}\right\} \|\underline{\mathbf{q}}\|_{\mathbf{Q}}, \tag{3.43}$$

which implies the result with $\beta = \nu c_1 \min\{1, \frac{c_2}{c_3}\} \left(1 + \frac{c_2}{2c_3}\right)^{-1}$. $\square$

Now we are in position of providing the well-posedness of (3.23).

Lemma 3.5 *Assume that $\kappa_1 > 0$ and $0 < \kappa_2 < 4\nu$. Then, for each $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ satisfying (3.33) and each $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, there exists a unique $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{Q}$ solution to (3.23). In addition, the following estimates hold:*

$$\|\underline{\mathbf{\Phi}}\|_{\mathbf{X}} \leq \alpha^{-1}\Big\{ \left(4\nu^2 + \kappa_1^2\right)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \Big\}, \tag{3.44}$$

and

$$\|\underline{\mathbf{p}}\|_{\mathbf{Q}} \leq \beta^{-1}\big(1 + \alpha^{-1}(C_{\mathbf{A}} + C_{\mathrm{S},2}\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}})\big)\Big\{ \left(4\nu^2 + \kappa_1^2\right)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \Big\}. \tag{3.45}$$

Proof. The unique solvability of (3.23) is a direct consequence of Lemmas 3.1 and 3.4 and the classical Babuška-Brezzi theory. In turn, for the estimate (3.44) we use the fact that $\underline{\mathbf{\Phi}} \in \mathbf{V}$ and apply (3.34) and (3.25), whereas (3.45) follows from the inf-sup condition (3.42) and estimates (3.25), (3.30) and (3.44). We omit further details. $\square$

According to the previous lemma we conclude that if we choose the set $\mathbf{M}$ in (3.22) in such a way that $\mathbf{M} \subseteq \overline{B\left(\mathbf{0}, \frac{\alpha}{C_{\mathrm{S},2}}\right)} := \left\{ \mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega} \leq \frac{\alpha}{C_{\mathrm{S},2}} \right\}$, then $\mathscr{J}$ is clearly well-defined. If so, from (3.44) we obtain that for all $\mathbf{w}_{\mathrm{S}} \in \mathbf{M}$, there holds

$$\|\mathscr{J}(\mathbf{w}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} = \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \|\underline{\mathbf{\Phi}}\|_{\mathbf{X}} \leq \alpha^{-1}\left(\left(4\nu^2 + \kappa_1^2\right)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu \|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right). \tag{3.46}$$

In particular, if we consider the set

$$\mathbf{M} := \left\{ \mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \alpha^{-1}\Big\{(4\nu^2 + \kappa_1^2)^{1/2}\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\} \right\}, \tag{3.47}$$

and assume that

$$\frac{C_{\mathrm{S},2}}{\alpha^2}\Big\{(4\nu^2 + \kappa_1^2)^{1/2}\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\} < 1, \tag{3.48}$$

we readily obtain that $\mathbf{M} \subseteq \overline{B\left(\mathbf{0}, \frac{\alpha}{C_{\mathrm{S},2}}\right)}$, thus proving that $\mathcal{J}$ is well-defined. In addition from (3.46) we have that

$$\mathcal{J}(\mathbf{M}) \subseteq \mathbf{M}.$$

Therefore, in order to prove the well-posedness of (3.21), in what follows we consider $\mathbf{M}$ defined as in (3.47) and show, equivalently, that $\mathcal{J}$ has a unique fixed-point in $\mathbf{M}$ by means of the Banach fixed-point theorem and under the same small data assumption (3.48).

3.2.2 Unique solvability

The main result of this section is stated now.

Theorem 3.6 *Let* $\mathbf{f}_{\mathrm{S}} \in \mathrm{L}^2(\Omega_{\mathrm{S}})$ *and* $\mathbf{f}_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$ *be such that* (3.48) *holds. Then, the operator* $\mathcal{J}$ *(cf.* (3.22)*) has a unique fixed-point* $\mathbf{u}_{\mathrm{S}}$ *in* $\mathbf{M}$*. Equivalently, the coupled problem* (3.21) *has a unique solution* $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) = ((\mathbf{T}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (p_{\mathrm{D}}, \lambda, \mu)) \in \mathbf{X} \times \mathbf{Q}$*, with* $\mathbf{u}_{\mathrm{S}} \in \mathbf{M}$*. Moreover, there hold the following a priori estimates*

$$\begin{aligned}
\|\underline{\mathbf{\Phi}}\|_{\mathbf{X}} &\leq \alpha^{-1}\Big\{ \left(4\nu^2 + \kappa_1^2\right)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \Big\}, \\
\|\underline{\mathbf{p}}\|_{\mathbf{Q}} &\leq \beta^{-1}\left(2 + \alpha^{-1}C_{\mathbf{A}}\right)\Big\{ \left(4\nu^2 + \kappa_1^2\right)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \Big\}.
\end{aligned} \tag{3.49}$$

Proof. We begin by recalling from the previous analysis that assumption (3.48) ensures the well-definedness of $\mathcal{J}$. Now, let $\mathbf{z}_{\mathrm{S},1}$, $\mathbf{z}_{\mathrm{S},2}$, $\mathbf{u}_{\mathrm{S},1}$ and $\mathbf{u}_{\mathrm{S},2}$ in $\mathbf{M}$, such that $\mathbf{u}_{\mathrm{S},i} := \mathcal{J}(\mathbf{z}_{\mathrm{S},i})$, $i \in \{1,2\}$. According to the definition of $\mathcal{J}$ (cf. (3.22)) we have that for each $i \in \{1,2\}$ there exist $(\underline{\mathbf{\Phi}}_i, \underline{\mathbf{p}}_i) = ((\mathbf{T}_{\mathrm{S},i}, \mathbf{u}_{\mathrm{S},i}, \mathbf{u}_{\mathrm{D},i}), (p_{\mathrm{D},i}, \lambda_i, \mu_i)) \in \mathbf{X} \times \mathbf{Q}$, such that

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}_{\mathrm{S},i}}(\underline{\mathbf{\Phi}}_i, \underline{\mathbf{\Psi}}) + \mathbf{B}(\underline{\mathbf{\Psi}}, \underline{\mathbf{p}}_i) &= \mathbf{F}(\underline{\mathbf{\Psi}}) \quad \forall\, \underline{\mathbf{\Psi}} \in \mathbf{X}, \\
\mathbf{B}(\underline{\mathbf{\Phi}}_i, \underline{\mathbf{q}}) &= 0 \qquad\quad \forall\, \underline{\mathbf{q}} \in \mathbf{Q},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}_{\mathrm{S},1}}(\underline{\mathbf{\Phi}}_1, \underline{\mathbf{\Psi}}) - \mathbf{A}_{\mathbf{z}_{\mathrm{S},2}}(\underline{\mathbf{\Phi}}_2, \underline{\mathbf{\Psi}}) + \mathbf{B}(\underline{\mathbf{\Psi}}, \underline{\mathbf{p}}_1 - \underline{\mathbf{p}}_2) &= 0 \quad \forall\, \underline{\mathbf{\Psi}} \in \mathbf{X}, \\
\mathbf{B}(\underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2, \underline{\mathbf{q}}) &= 0 \quad \forall\, \underline{\mathbf{q}} \in \mathbf{Q}.
\end{aligned} \tag{3.50}$$

Then, observing that $\underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2 \in \mathbf{V}$, from the first equation of (3.50) with $\underline{\mathbf{\Psi}} = \underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2$, and simple computations, we deduce that

$$\begin{aligned}
\mathbf{A}_{\mathbf{z}_{\mathrm{S},1}}(\underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2, \underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2) &= -\mathbf{A}_{\mathbf{z}_{\mathrm{S},1}}(\underline{\mathbf{\Phi}}_2, \underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2) + \mathbf{A}_{\mathbf{z}_{\mathrm{S},2}}(\underline{\mathbf{\Phi}}_2, \underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2) \\
&= -\mathbf{C}_{\mathbf{z}_{\mathrm{S},1} - \mathbf{z}_{\mathrm{S},2}}((\mathbf{T}_{\mathrm{S},2}, \mathbf{u}_{\mathrm{S},2}), (\mathbf{T}_{\mathrm{S},1} - \mathbf{T}_{\mathrm{S},2}, \mathbf{u}_{\mathrm{S},1} - \mathbf{u}_{\mathrm{S},2})),
\end{aligned}$$

which, together with (3.34) and (3.30), imply

$$\begin{aligned}
\|\mathbf{u}_{\mathrm{S},1} - \mathbf{u}_{\mathrm{S},2}\|_{1,\Omega} \leq \|\underline{\mathbf{\Phi}}_1 - \underline{\mathbf{\Phi}}_2\|_{\mathbf{X}} &\leq \alpha^{-1} C_{\mathrm{S},2} \|\mathbf{z}_{\mathrm{S},1} - \mathbf{z}_{\mathrm{S},2}\|_{1,\Omega_{\mathrm{S}}} \|(\mathbf{T}_{\mathrm{S},2}, \mathbf{u}_{\mathrm{S},2})\| \\
&\leq \alpha^{-1} C_{\mathrm{S},2} \|\underline{\mathbf{\Phi}}_2\|_{\mathbf{X}} \|\mathbf{z}_{\mathrm{S},1} - \mathbf{z}_{\mathrm{S},2}\|_{1,\Omega_{\mathrm{S}}}.
\end{aligned}$$

Hence, recalling that $\underline{\mathbf{\Phi}}_2$ satisfies (3.44), we conclude from the foregoing inequality and (3.48) that $\mathcal{J}$ is a contraction mapping. Therefore, a straightforward application of the Banach fixed-point theorem implies the unique solvability of the fixed-point problem (3.24), or equivalently, the well-posedness of (3.21). Finally, letting $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) = ((\mathbf{T}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}), (p_{\mathrm{D}}, \lambda, \mu)) \in \mathbf{X} \times \mathbf{Q}$ be the unique solution of (3.21), with $\mathbf{u}_{\mathrm{S}} \in \mathbf{M}$ satisfying (3.24), it is clear that $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}})$ satisfies (3.23) with $\mathbf{w}_{\mathrm{S}} = \mathbf{u}_{\mathrm{S}}$. Therefore, noticing that estimates (3.46) and (3.48) imply $\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \frac{\alpha}{C_{\mathrm{S},2}}$, from (3.44) and (3.45) we clearly obtain (3.49), which concludes the proof. □

4 The Galerkin formulation

In this section we introduce the Galerkin scheme of problem (3.21) and provide sufficient conditions on the corresponding finite-dimensional spaces guaranteeing its unique solvability, stability, and Céa's estimate.

4.1 The discrete problem

Let us consider generic finite element subspaces

$$\begin{gathered}\mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{S}}), \qquad \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}}), \qquad \mathrm{H}^1_h(\Omega_{\mathrm{S}}) \subseteq \mathrm{H}^1(\Omega_{\mathrm{S}}),\\ \mathrm{L}_h(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}) \quad \text{and} \quad \Lambda_h(\Sigma) \subseteq \mathrm{H}^{1/2}(\Sigma),\end{gathered} \tag{4.1}$$

and let

$$\mathbb{H}_h(\Omega_{\mathrm{S}}) := \left\{ \mathbf{R}_h \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) \,:\, \mathbf{c}^t\mathbf{R}_h \in \mathbf{H}_h(\mathrm{div};\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{c} \in \mathbb{R}^n \right\}.$$

Then, defining the global finite element spaces as

$$\mathbf{X}_h := \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) \quad \text{and} \quad \mathbf{Q}_h := \mathrm{L}_{h,0}(\Omega_{\mathrm{D}}) \times \Lambda_h(\Sigma) \times \mathbb{R}, \tag{4.2}$$

with

$$\begin{aligned}&\mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) := \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}), \qquad &&\mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) := [\mathrm{H}^1_h(\Omega_{\mathrm{S}})]^n \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}),\\ &\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) := \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}}), \quad &&\mathrm{L}_{h,0}(\Omega_{\mathrm{S}}) := \mathrm{L}_h(\Omega_{\mathrm{D}}) \cap \mathrm{L}^2_0(\Omega_{\mathrm{D}}),\end{aligned} \tag{4.3}$$

the Galerkin scheme of (3.21) reads: Find $\big(\underline{\boldsymbol{\Phi}}_h, \underline{\mathbf{p}}_h\big) = \big((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}), (q_{h,\mathrm{D}}, \lambda_h, \mu_h)\big) \in \mathbf{X}_h \times \mathbf{Q}_h$, such that

$$\begin{aligned}\mathbf{A}_{\mathbf{u}_{h,\mathrm{S}}}(\underline{\boldsymbol{\Phi}}_h, \underline{\boldsymbol{\Psi}}_h) + \mathbf{B}\big(\underline{\boldsymbol{\Psi}}_h, \underline{\mathbf{p}}_h\big) &= \mathbf{F}(\underline{\boldsymbol{\Psi}}_h) \quad \forall\, \underline{\boldsymbol{\Psi}}_h \in \mathbf{X}_h,\\ \mathbf{B}\big(\underline{\boldsymbol{\Phi}}_h, \underline{\mathbf{q}}_h\big) &= 0 \qquad\quad \forall\, \underline{\mathbf{q}}_h \in \mathbf{Q}_h.\end{aligned} \tag{4.4}$$

In turn, in order to study the unique solvability of (4.4), and analogously to the continuous case, we realize that (4.4) can be rewritten equivalently as the fixed-point problem: Find $\mathbf{u}_{h,\mathrm{S}} \in \mathbf{M}_h$ such that

$$\mathcal{J}_h(\mathbf{u}_{h,\mathrm{S}}) = \mathbf{u}_{h,\mathrm{S}}, \tag{4.5}$$

where $\mathcal{J}_h$ is the discrete fixed-point operator defined as $\mathcal{J}_h : \mathbf{M}_h \subseteq \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \to \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, $\mathbf{w}_{h,\mathrm{S}} \to \mathcal{J}_h(\mathbf{w}_{h,\mathrm{S}}) = \mathbf{u}_{h,\mathrm{S}}$, where $\mathbf{u}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ is the second component of $\underline{\boldsymbol{\Phi}}_h$, which, together with $\underline{\mathbf{p}}_h \in \mathbf{Q}_h$, constitutes the unique solution of the linearized version of (4.4): Find $\mathbf{u}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that

$$\begin{aligned}\mathbf{A}_{\mathbf{w}_{h,\mathrm{S}}}(\underline{\boldsymbol{\Phi}}_h, \underline{\boldsymbol{\Psi}}_h) + \mathbf{B}\big(\underline{\boldsymbol{\Psi}}_h, \underline{\mathbf{p}}_h\big) &= \mathbf{F}(\underline{\boldsymbol{\Psi}}_h) \quad \forall\, \underline{\boldsymbol{\Psi}}_h \in \mathbf{X}_h,\\ \mathbf{B}\big(\underline{\boldsymbol{\Phi}}_h, \underline{\mathbf{q}}_h\big) &= 0 \qquad\quad \forall\, \underline{\mathbf{q}}_h \in \mathbf{Q}_h,\end{aligned} \tag{4.6}$$

and $\mathbf{M}_h \subseteq \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ is a subset ensuring the well-definedness of $\mathcal{J}_h$, or equivalently, the well-posedness of (4.6).

According to the above, now we focus on providing suitable hypotheses on the finite-dimensional spaces (4.1) that will allow us to prove the well-posedness of (4.6), and consequently, the unique solvability of problem (4.4). We begin by observing that, in order to properly define the spaces $\mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$ and $\mathrm{L}^2_0(\Omega_{\mathrm{D}})$, we need to be able to eliminate multiples of the identity matrix from $\mathbb{H}_h(\Omega_{\mathrm{S}})$ and constants polynomials from $\mathrm{L}^2(\Omega_{\mathrm{D}})$. These requests are certainly satisfied if we assume that:

(H.0) $\mathbf{P}_0(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}_h(\mathbf{div};\Omega_{\mathrm{S}})$ and $P_0(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}_h(\Omega_{\mathrm{D}})$.

Thus, it readily follows from ($\mathbf{H}.\mathbf{0}$) that the following decompositions hold

$$\mathbb{H}_h(\Omega_S) \,=\, \mathbb{H}_{h,0}(\Omega_S) \oplus P_0(\Omega_S)\,\mathrm{I} \quad \text{and} \quad \mathrm{L}_h(\Omega_D) \,=\, \mathrm{L}_{h,0}(\Omega_D) \oplus P_0(\Omega_D). \tag{4.7}$$

Now, we turn to establish sufficient conditions for the discrete inf-sup condition

$$S_h(\underline{\mathbf{q}}_h) := \sup_{\substack{\underline{\boldsymbol{\Psi}}_h \in \mathbf{X}_h \\ \underline{\boldsymbol{\Psi}}_h \neq \mathbf{0}}} \frac{\mathbf{B}(\underline{\boldsymbol{\Psi}}_h, \underline{\mathbf{q}}_h)}{\|\underline{\boldsymbol{\Psi}}_h\|_{\mathbf{X}}} \,\geq\, \widehat{\beta}\|\underline{\mathbf{q}}_h\|_{\mathbf{Q}} \qquad \forall\, \underline{\mathbf{q}}_h \,\in\, \mathbf{Q}_h, \tag{4.8}$$

where $\widehat{\beta} > 0$ is a constant required to be independent of the discretization parameter h. To that end, we apply the same arguments utilized in the proof of Lemma 3.4 and realize that the inf-sup condition (4.8) holds if we guarantee the following conditions:

($\mathbf{H}.\mathbf{1}$) there exists $\widehat{c}_1 > 0$, independent of h, such that

$$\begin{aligned} S_{1,h}(q_{h,D}, \xi_h) &:= \sup_{\substack{\mathbf{v}_{h,D} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D) \\ \mathbf{v}_{h,D} \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_{h,D}, q_{h,D})_D + \langle \mathbf{v}_{h,D} \cdot \mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{h,D}\|_{\operatorname{div};\Omega_D}} \\ &\geq\, \widehat{c}_1 \Big\{\|q_{h,D}\|_{0,\Omega_D} + \|\xi_h\|_{1/2,\Sigma}\Big\} \quad \forall\,(q_{h,D}, \xi_h) \in \mathrm{L}_{h,0}(\Omega_D) \times \Lambda_h(\Sigma)\,. \end{aligned} \tag{4.9}$$

($\mathbf{H}.\mathbf{2}$) there exists $\mathbf{v}_0 \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that $\mathbf{v}_0 \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S)$ for all h, and $\langle \mathbf{v}_0 \cdot \mathbf{n}, 1\rangle_\Sigma \neq 0$.

In particular, analogously to the proof of Lemma 3.3, the latter clearly implies

$$\begin{aligned} S_{2,h}(\xi_h, \eta_h) &:= \sup_{\substack{\mathbf{v}_{h,S} \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S) \\ \mathbf{v}_{h,S} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{h,S} \cdot \mathbf{n}, \xi_h\rangle_\Sigma - \eta_h\langle \mathbf{v}_{h,S} \cdot \mathbf{n}, 1\rangle_\Sigma}{\|\mathbf{v}_{h,S}\|_{1,\Omega_S}} \\ &\geq\, \widehat{c}_2|\eta_h| - \widehat{c}_3\|\xi_h\|_{1/2,\Sigma} \quad \forall\,(\xi_h, \eta_h) \in \Lambda_h(\Sigma) \times \mathbb{R}\,, \end{aligned} \tag{4.10}$$

with $\widehat{c}_2$, $\widehat{c}_3 > 0$ independent of h, which, together with (4.9), gives (4.8).

Finally, we look at the discrete kernel of $\mathbf{B}$, namely

$$\mathbf{V}_h := \Big\{\underline{\boldsymbol{\Psi}}_h \in \mathbf{X}_h \,:\, \mathbf{B}(\underline{\boldsymbol{\Psi}}_h, \underline{\mathbf{q}}_h) = 0 \quad \forall\, \underline{\mathbf{q}}_h \in \mathbf{Q}_h\Big\}\,. \tag{4.11}$$

In order to describe explicitly $\mathbf{V}_h$, we now introduce the following assumption:

($\mathbf{H}.\mathbf{3}$) $\operatorname{div} \mathbf{H}_h(\Omega_D) \subseteq \mathrm{L}_h(\Omega_D)$ and $P_0(\Sigma) \subseteq \Lambda_h(\Sigma)$

Using ($\mathbf{H}.\mathbf{3}$), and recalling the definition of $\mathbf{B}$ (cf. (3.16)), we have that $\underline{\boldsymbol{\Psi}}_h = (\mathbf{R}_{h,S}, \mathbf{v}_{h,S}, \mathbf{v}_{h,D}) \in \mathbf{V}_h$ if and only if

$$\operatorname{div} \mathbf{v}_{h,D} \in \mathbb{R}\,, \quad \langle \mathbf{v}_{h,S} \cdot \mathbf{n} - \mathbf{v}_{h,D} \cdot \mathbf{n}, \xi_h\rangle_\Sigma = 0 \quad \forall\, \xi_h \,\in\, \Lambda_h\,, \quad \text{and} \quad \langle \mathbf{v}_{h,S} \cdot \mathbf{n}, 1\rangle_\Sigma = 0\,. \tag{4.12}$$

In particular, starting from the third identity of (4.12), and then taking $\xi_h = 1$ in the second one, we obtain

$$0 = \langle \mathbf{v}_{h,S} \cdot \mathbf{n}, 1\rangle_\Sigma = \langle \mathbf{v}_{h,D} \cdot \mathbf{n}, 1\rangle_\Sigma = (\operatorname{div} \mathbf{v}_{h,D}, 1)_D\,,$$

which easily yields $\operatorname{div} \mathbf{v}_{h,D} = 0$ in Ω_D. In this way, we obtain the following characterization of $\mathbf{V}_h$:

$$\begin{aligned} \mathbf{V}_h = \Big\{\underline{\boldsymbol{\Psi}}_h \in \mathbf{X}_h : \quad & \operatorname{div} \mathbf{v}_{h,D} = 0 \quad \text{in} \quad \Omega_D\,, \quad \langle \mathbf{v}_{h,S} \cdot \mathbf{n}, 1\rangle_\Sigma = 0\,, \\ & \text{and} \quad \langle \mathbf{v}_{h,S} \cdot \mathbf{n} - \mathbf{v}_{h,D} \cdot \mathbf{n}, \xi_h\rangle_\Sigma = 0 \quad \forall\, \xi_h \,\in\, \Lambda_h(\Sigma)\Big\}\,. \end{aligned} \tag{4.13}$$

As a consequence of the above, we observe that the following discrete version of (3.37) holds:

$$\mathbf{A}_S\big((\mathbf{R}_{h,S}, \mathbf{v}_{h,S}), (\mathbf{R}_{h,S}, \mathbf{v}_{h,S})\big) + \mathbf{A}_D(\mathbf{v}_{h,D}, \mathbf{v}_{h,D}) \geq \alpha_S\|(\mathbf{R}_{h,S}, \mathbf{v}_{h,S})\|^2 \,+\, C_{\mathbf{K}}\|\mathbf{v}_{h,D}\|^2_{\operatorname{div};\Omega_D}\,, \tag{4.14}$$

for all $(\mathbf{R}_{h,S}, \mathbf{v}_{h,S}, \mathbf{v}_{h,D}) \in \mathbf{V}_h$, with α_S defined in (3.38) and $C_{\mathbf{K}}$ the positive constant satisfying (2.7).

4.2 Well-posedness of the discrete problem

We begin by establishing the well-definedness of $\mathcal{J}_h$, or equivalently, the well-posedness of (4.6). For this purpose, we first observe that the estimates (3.25), (3.26), (3.27), (3.28), and (3.29) certainly hold for the subspaces $\mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$, $\mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, $\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$, $\mathrm{L}_h(\Omega_{\mathrm{D}})$, and $\Lambda_h(\Sigma)$, of $\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, $\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, $\mathbf{H}^1_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$, $\mathrm{L}^2(\Omega_{\mathrm{D}})$, and $\mathrm{H}^{1/2}(\Sigma)$, respectively. In addition, (3.30) is also valid for $\mathbf{w}_{\mathrm{S}} = \mathbf{w}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$. In turn, we establish next the discrete analogue of Lemma 3.1.

Lemma 4.1 *Assume that $\kappa_1 > 0$ and $0 < \kappa_2 < 4\nu$, and let $\mathbf{w}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that*

$$\|\mathbf{w}_{h,\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le \frac{\alpha}{C_{\mathrm{S},2}}, \tag{4.15}$$

where $\alpha = \frac{1}{2}\min\{\alpha_{\mathrm{S}}, C_{\mathbf{K}}\}$, α_{S} is the constant defined in (3.38), and $C_{\mathbf{K}}$ and $C_{\mathrm{S},2}$ are the positive constants satisfying (2.7) and (3.29), respectively. Assume further that (**H.0**) *and* (**H.3**) *hold. Then, there holds*

$$\mathbf{A}_{\mathbf{w}_{h,\mathrm{S}}}(\underline{\mathbf{\Psi}}_h, \underline{\mathbf{\Psi}}_h) \,\ge\, \alpha\,\|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}} \qquad \forall\, \underline{\mathbf{\Psi}}_h \in \mathbf{V}_h\,. \tag{4.16}$$

Proof. Analogously to the proof of Lemma 3.1, (4.16) is a direct consequence of (3.29), (4.14) and assumption (4.15). We omit further details. □

Now we are in position of establishing the well-posedness of (4.6).

Lemma 4.2 *Assume that* (**H.0**), (**H.1**), (**H.2**), *and* (**H.3**) *hold, and that $\kappa_1 > 0$ and $0 < \kappa_2 < 4\nu$. Then, for each $\mathbf{w}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ satisfying (4.15) and each $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ and $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^2(\Omega_{\mathrm{D}})$, there exists a unique $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ solution to (4.6). In addition, there hold the following a priori estimates*

$$\begin{aligned}
\|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}} &\le \alpha^{-1}\Big\{ \big(4\nu^2 \,+\, \kappa_1^2\big)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\,,\\
\|\underline{\mathbf{p}}_h\|_{\mathbf{Q}} &\le \widehat{\beta}^{-1}\big(1 + \alpha^{-1}(C_{\mathbf{A}} + C_{\mathrm{S},2}\|\mathbf{w}_{h,\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}})\big)\Big\{ \big(4\nu^2 \,+\, \kappa_1^2\big)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\,.
\end{aligned} \tag{4.17}$$

Proof. The unique solvability of (4.6) follows straightforwardly from (4.8), (4.16), and the classical Babuška-Brezzi theory. In turn, by applying the same steps employed in the proof of Lemma 3.5, one can obtain the estimates (4.17). □

According to the previous lemma, and analogously to the continuous case (cf. (3.47)), we now introduce the bounded set

$$\mathbf{M}_h := \Big\{ \mathbf{v}_{h,\mathrm{S}} \in \mathbf{H}^1_{h,\mathrm{S}}(\Omega_{\mathrm{S}}) \,:\, \|\mathbf{v}_{h,\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \,\le\, \alpha^{-1}\Big\{(4\nu^2 \,+\, \kappa_1^2)^{1/2}\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\Big\}\,.$$

Then, assuming that (3.48) holds, it follows that the fixed operator $\mathcal{J}_h$ defined through (4.6) is well-defined and satisfies $\mathcal{J}_h(\mathbf{M}_h) \subseteq \mathbf{M}_h$. Moreover, analogously to the continuous case, we can prove the well-posedness of problem (4.4). This result is established now.

Theorem 4.3 *Assume that* (**H.0**), (**H.1**), (**H.2**), *and* (**H.3**) *hold, and that $\kappa_1 > 0$ and $0 < \kappa_2 < 4\nu$. Assume further that the external forces $\mathbf{f}_{\mathrm{S}}$ and $\mathbf{f}_{\mathrm{D}}$ satisfy (3.48). Then, there exists a unique $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ solution to (4.4). In addition, there hold the following a priori estimates*

$$\begin{aligned}
\|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}} &\le \alpha^{-1}\Big\{ \big(4\nu^2 \,+\, \kappa_1^2\big)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\,,\\
\|\underline{\mathbf{p}}_h\|_{\mathbf{Q}} &\le \beta^{-1}\big(2 + \alpha^{-1}C_{\mathbf{A}}\big)\Big\{ \big(4\nu^2 \,+\, \kappa_1^2\big)^{1/2} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \,+\, 2\nu\|\mathbf{f}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\Big\}\,.
\end{aligned} \tag{4.18}$$

Proof. By applying the same tools employed in the proof Theorem 3.6, that is, estimates (4.16), (3.29), and assumption (3.48), it can be easily deduced that $\mathcal{J}_h$ is a contraction mapping on $\mathbf{M}_h$, which, together with the Banach fixed-point theorem, implies the unique solvability of the fixed-point problem (4.5), or equivalently, the well-posedness of (4.4). Moreover, analogously to the proof of Theorem 3.6, estimates (4.18) follow from (3.48) and the fact that the solution $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) = \big((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}), (p_{h,\mathrm{D}}, \lambda_h, \mu_h)\big) \in \mathbf{X}_h \times \mathbf{Q}_h$ satisfies the estimates (4.17), the second of them with $\mathbf{w}_{h,\mathrm{S}} = \mathbf{u}_{h,\mathrm{S}} \in \mathbf{M}_h$. $\square$

4.3 The Cea estimate

Our next goal is to provide the Cea estimate for our Galerkin scheme (4.4). For this purpose, we let $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) = \big((\mathbf{T}_\mathrm{S}, \mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}), (p_\mathrm{D}, \lambda, \mu)\big) \in \mathbf{X} \times \mathbf{Q}$ and $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) = \big((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}), (p_{h,\mathrm{D}}, \lambda_h, \mu_h)\big) \in \mathbf{X}_h \times \mathbf{Q}_h$ be the unique solutions of (3.21) and (4.4), respectively, and observe that the following orthogonality-type relation holds

$$
\begin{aligned}
\mathbf{A}_{\mathbf{u}_\mathrm{S}}(\underline{\mathbf{\Phi}}, \underline{\mathbf{\Psi}}_h) - \mathbf{A}_{\mathbf{u}_{h,\mathrm{S}}}(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{\Psi}}_h) + \mathbf{B}(\underline{\mathbf{\Psi}}_h, \underline{\mathbf{p}} - \underline{\mathbf{p}}_h) &= 0 \quad \forall\, \underline{\mathbf{\Psi}}_h \in \mathbf{X}_h, \\
\mathbf{B}(\underline{\mathbf{\Phi}} - \underline{\mathbf{\Phi}}_h, \underline{\mathbf{q}}_h) &= 0 \quad \forall\, \underline{\mathbf{q}}_h \in \mathbf{Q}_h.
\end{aligned}
\tag{4.19}
$$

In turn, for the sake of simplicity we denote the corresponding errors as

$$
\mathbf{e}_{\underline{\mathbf{\Phi}}} = \underline{\mathbf{\Phi}} - \underline{\mathbf{\Phi}}_h, \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{p}}} = \underline{\mathbf{p}} - \underline{\mathbf{p}}_h,
\tag{4.20}
$$

and for given $\underline{\boldsymbol{\varphi}}_h = \big(\mathbf{S}_{h,\mathrm{S}}, \mathbf{z}_{h,\mathrm{S}}, \mathbf{z}_{h,\mathrm{D}}\big) \in \mathbf{V}_h$ and $\underline{\mathbf{r}}_h = (r_{h,\mathrm{D}}, \vartheta_h, \zeta_h) \in \mathbf{Q}_h$, we write

$$
\mathbf{e}_{\underline{\mathbf{\Phi}}} = \boldsymbol{\delta}_{\underline{\mathbf{\Phi}}} + \boldsymbol{\eta}_{\underline{\mathbf{\Phi}}} := (\underline{\mathbf{\Phi}} - \underline{\boldsymbol{\varphi}}_h) + (\underline{\boldsymbol{\varphi}}_h - \underline{\mathbf{\Phi}}_h) \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{p}}} = \boldsymbol{\delta}_{\underline{\mathbf{p}}} + \boldsymbol{\eta}_{\underline{\mathbf{p}}} := (\underline{\mathbf{p}} - \underline{\mathbf{r}}_h) + (\underline{\mathbf{r}}_h - \underline{\mathbf{p}}_h).
\tag{4.21}
$$

Then, we have the following main result.

Theorem 4.4 *Assume that* $(\mathbf{H.0})$, $(\mathbf{H.1})$, $(\mathbf{H.2})$, *and* $(\mathbf{H.3})$ *hold, and that* $\kappa_1 > 0$ *and* $0 < \kappa_2 < 4\nu$. *Assume further that*

$$
\frac{C_{\mathrm{S},2}}{\alpha^2}\Big\{(4\nu^2 + \kappa_1^2)^{1/2}\|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + 2\nu\|\mathbf{f}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}\Big\} \leq \frac{1}{2}\,.
\tag{4.22}
$$

Then, there exists $C_{cea} > 0$, *independent of* h, *such that*

$$
\|(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) - (\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{Q}} \leq C_{cea} \inf_{(\underline{\mathbf{\Psi}}_h, \underline{\mathbf{q}}_h)\in \mathbf{X}_h\times\mathbf{Q}_h} \|(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) - (\underline{\mathbf{\Psi}}_h, \underline{\mathbf{q}}_h)\|_{\mathbf{X}\times\mathbf{Q}}\,.
\tag{4.23}
$$

Proof. From the first equation of (4.19), adding and subtracting suitable terms, and recalling the definition of $\mathbf{A}_{\mathbf{u}_{h,\mathrm{S}}}$ (cf. (3.15)), we arrive at

$$
\begin{aligned}
\mathbf{A}_{\mathbf{u}_\mathrm{S}}(\mathbf{e}_{\underline{\mathbf{\Phi}}}, \underline{\mathbf{\Psi}}_h) &= -\mathbf{A}_{\mathbf{u}_{h,\mathrm{S}}}(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{\Psi}}_h) + \mathbf{A}_{\mathbf{u}_\mathrm{S}}(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{\Psi}}_h) - \mathbf{B}(\underline{\mathbf{\Psi}}_h, \mathbf{e}_{\underline{\mathbf{p}}}) \\
&= -\mathbf{C}_{\mathbf{u}_\mathrm{S} - \mathbf{u}_{h,\mathrm{S}}}((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}), (\mathbf{R}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{S}})) - \mathbf{B}(\underline{\mathbf{\Psi}}_h, \mathbf{e}_{\underline{\mathbf{p}}})\,,
\end{aligned}
\tag{4.24}
$$

which, using that $\mathbf{e}_{\underline{\mathbf{\Phi}}} = \boldsymbol{\delta}_{\underline{\mathbf{\Phi}}} + \boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}$ and $\mathbf{e}_{\underline{\mathbf{p}}} = \boldsymbol{\delta}_{\underline{\mathbf{p}}} + \boldsymbol{\eta}_{\underline{\mathbf{p}}}$, can be rewritten as

$$
\begin{aligned}
\mathbf{A}_{\mathbf{u}_\mathrm{S}}(\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}, \underline{\mathbf{\Psi}}_h) = -\mathbf{A}_{\mathbf{u}_\mathrm{S}}(\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}, \underline{\mathbf{\Psi}}_h) &- \mathbf{C}_{\mathbf{u}_\mathrm{S} - \mathbf{u}_{h,\mathrm{S}}}((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}), (\mathbf{R}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{S}})) \\
&- \mathbf{B}(\underline{\mathbf{\Psi}}_h, \boldsymbol{\delta}_{\underline{\mathbf{p}}}) - \mathbf{B}(\underline{\mathbf{\Psi}}_h, \boldsymbol{\eta}_{\underline{\mathbf{p}}})\,,
\end{aligned}
$$

for all $\underline{\mathbf{\Psi}}_h = (\mathbf{R}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{D}}) \in \mathbf{X}_h$. Next, we notice from the second equation of (4.4) that $\underline{\mathbf{\Phi}}_h \in \mathbf{V}_h$, whence $\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}} := \underline{\boldsymbol{\varphi}}_h - \underline{\mathbf{\Phi}}_h$ belongs to $\mathbf{V}_h$ as well. Then, taking $\underline{\mathbf{\Psi}}_h = \boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}$ in (4.16), employing the estimates (3.29) and (3.28), and denoting $\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}} = \mathbf{u}_\mathrm{S} - \mathbf{z}_{h,\mathrm{S}}$ and $\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}} = \mathbf{z}_{h,\mathrm{S}} - \mathbf{u}_{h,\mathrm{S}}$, we readily obtain

$$
\begin{aligned}
\alpha \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|^2_{\mathbf{X}} \,\leq\, & \big(C_{\mathbf{A}} + C_{\mathrm{S},2}\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\big)\, \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \\
& + C_{\mathrm{S},2}\,\big(\|\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega} + \|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega}\big)\, \|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}} \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_{\mathbf{B}} \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,,
\end{aligned}
$$

which, together with the fact that $\|\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega} \leq \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}}$ and $\|\boldsymbol{\eta}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega} \leq \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}}$, implies

$$
\big(\alpha - C_{\mathrm{S},2}\|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}}\big)\, \|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,\leq\, \big(C_{\mathbf{A}} + C_{\mathrm{S},2}\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + C_{\mathrm{S},2}\|\underline{\mathbf{\Phi}}_h\|_{\mathbf{X}}\big)\, \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_{\mathbf{B}}\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,.
$$

Then, recalling that $\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq \|\underline{\mathbf{\Phi}}\|_{\mathbf{X}}$, from (3.44), (4.17), and assumption (4.22), we deduce that

$$
\|\boldsymbol{\eta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,\leq\, C_1\, \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_2\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,,
$$

with constants C_1, $C_2 > 0$, independent of h, which yields

$$
\|\mathbf{e}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,\leq\, (1 + C_1)\, \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_2\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,. \tag{4.25}
$$

On the other hand, noticing from (4.24) that there holds

$$
\mathbf{B}(\underline{\mathbf{\Psi}}_h, \boldsymbol{\eta}_{\underline{\mathbf{p}}}) = -\mathbf{A}_{\mathbf{u}_\mathrm{S}}(\mathbf{e}_{\underline{\mathbf{\Phi}}}, \underline{\mathbf{\Psi}}_h) - \mathbf{C}_{\mathbf{u}_\mathrm{S} - \mathbf{u}_{h,\mathrm{S}}}((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}), (\mathbf{R}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{S}})) \,-\, \mathbf{B}(\underline{\mathbf{\Psi}}_h, \boldsymbol{\delta}_{\underline{\mathbf{p}}}),
$$

for all $\underline{\mathbf{\Psi}}_h = (\mathbf{R}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{S}}, \mathbf{v}_{h,\mathrm{D}}) \in \mathbf{X}_h$, and using the inf-sup condition (4.8), the estimates (3.28), (3.29), (3.30), (4.17), and (3.46) (with $\mathbf{w}_\mathrm{S} = \mathbf{u}_\mathrm{S} \in \mathbf{M}$), and the fact that $\|\mathbf{u}_\mathrm{S} - \mathbf{u}_{h,\mathrm{S}}\|_{1,\Omega_\mathrm{S}} \leq \|\mathbf{e}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}}$, we conclude that

$$
\widehat{\beta} \|\boldsymbol{\eta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}} \,\leq\, \sup_{\substack{\underline{\mathbf{\Psi}}_h \in \mathbf{X}_h \\ \underline{\mathbf{\Psi}}_h \neq \mathbf{0}}} \frac{\mathbf{B}\big(\underline{\mathbf{\Psi}}_h, \boldsymbol{\eta}_{\underline{\mathbf{p}}}\big)}{\|\underline{\mathbf{\Psi}}_h\|_{\mathbf{X}}} \,\leq\, C_3\, \|\mathbf{e}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_4\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,,
$$

which, combined with (4.25), gives

$$
\|\mathbf{e}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}} \,\leq\, \|\boldsymbol{\eta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}} \,+\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}} \,\leq\, C_5\, \|\boldsymbol{\delta}_{\underline{\mathbf{\Phi}}}\|_{\mathbf{X}} \,+\, C_6\, \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbf{Q}}\,, \tag{4.26}
$$

with constants C_3, C_4, C_5, $C_6 > 0$, independent of h. Finally, recalling that the inf-sup condition (4.8) implies the estimate (see [17, estimate (2.89)])

$$
\inf_{\underline{\mathbf{\Psi}}_h \in \mathbf{V}_h} \|\underline{\mathbf{\Phi}} - \underline{\mathbf{\Psi}}_h\|_{\mathbf{X}} \,\leq\, c \inf_{\underline{\mathbf{\Psi}}_h \in \mathbf{X}_h} \|\underline{\mathbf{\Phi}} - \underline{\mathbf{\Psi}}_h\|_{\mathbf{X}}, \tag{4.27}
$$

with $c > 0$, independent of h, from (4.25) and (4.26) and the fact that $\underline{\boldsymbol{\varphi}}_h = \big(\mathbf{S}_{h,\mathrm{S}}, \mathbf{z}_{h,\mathrm{S}}, \mathbf{z}_{h,\mathrm{D}}\big) \in \mathbf{V}_h$ and $\underline{\mathbf{r}}_h = (r_{h,\mathrm{D}}, \vartheta_h, \zeta_h) \in \mathbf{Q}_h$ are arbitrary, we obtain the desired result. $\square$

4.4 Computing further variables of interest

In this section we introduce suitable approximations for further variables of interest, such as the pressure p, the vorticity $\boldsymbol{\omega} := \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^t)$, the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma} := \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) - p\mathrm{I}$, all them written in terms of $\big(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h\big) = \big((\mathbf{T}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}}), (q_{h,\mathrm{D}}, \lambda_h, \mu_h)\big) \in \mathbf{X}_h \times \mathbf{Q}_h$, solution of the discrete problem (4.4). In fact, observing that at the continuous level there hold

$$
\begin{aligned}
& p \,=\, -\frac{1}{n}\left(\mathrm{tr}\,(\mathbf{T}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{|\Omega|}(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega\right), \qquad \mathbf{G} \,=\, \frac{1}{\nu}(\mathbf{T}^d \,+\, (\mathbf{u}\otimes\mathbf{u})^d), \\
& \boldsymbol{\sigma} \,=\, \mathbf{T}^d + \mathbf{T}^t - \frac{1}{n|\Omega|}(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega \mathrm{I} + \mathbf{u}\otimes\mathbf{u} \,+\, (\mathbf{u}\otimes\mathbf{u})^d \quad \text{and} \quad \boldsymbol{\omega} \,=\, \frac{1}{2\nu}(\mathbf{T} - \mathbf{T}^t)\,,
\end{aligned}
$$

we propose the following approximations for the aforementioned variables

$$p_h \;=\; -\frac{1}{n}\left(\operatorname{tr}(\mathbf{T}_h)+\operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)-\frac{1}{|\Omega|}(\operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h),1)_\Omega\right), \qquad \mathbf{G}_h \;=\; \frac{1}{\nu}(\mathbf{T}_h^d \,+\, (\mathbf{u}_h\otimes\mathbf{u}_h)^d),$$
$$\boldsymbol{\sigma}_h = \mathbf{T}_h^d+\mathbf{T}_h^t-\frac{1}{n|\Omega|}(\operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h),1)_\Omega\mathrm{I}+\mathbf{u}_h\otimes\mathbf{u}_h+(\mathbf{u}_h\otimes\mathbf{u}_h)^d \quad\text{and}\quad \boldsymbol{\omega}_h \;=\; \frac{1}{2\nu}(\mathbf{T}_h-\mathbf{T}_h^t)\,.$$

4.5 A particular choice of finite elements

In this section we proceed similarly to [6] and [22] and specify concrete examples of finite element subspaces in 2D and 3D satisfying the hypotheses $(\mathbf{H}.\mathbf{0})$ – $(\mathbf{H}.\mathbf{3})$. To this end, we let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles (in $\mathbb{R}^2$) or tetrahedra (in $\mathbb{R}^3$) of diameter h_T, assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}}\cup\mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$, and denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). We let $h_\star := \max\{h_T : T\in\mathcal{T}_h^\star\}$ $(\star\in\{\mathrm{S},\mathrm{D}\})$ and $h:=\max\{h_{\mathrm{S}},h_{\mathrm{D}}\}$. In addition, we denote by $\mathbf{x}:=(x_1,...,x_n)^{\mathrm{t}}$ a generic vector of $\mathbb{R}^n$ and for each $T\in\mathcal{T}_h^{\mathrm{S}}\cup\mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of order 0, given by

$$\mathrm{RT}_0(T) \;:=\; \mathbf{P}_0(T)+P_0(T)\mathbf{x}.$$

4.5.1 Finite element subspaces in 2D

Here we propose to choose the finite element subspaces $\mathbf{H}_h^1(\Omega_{\mathrm{S}})$, $\mathbf{H}_h(\Omega_\star)$ $(\star\in\{\mathrm{S},\mathrm{D}\})$, and $\mathrm{L}_h(\Omega_{\mathrm{D}})$ in (4.1) as follows

$$\begin{aligned}
\mathbf{H}_h^1(\Omega_{\mathrm{S}}) \;&:=\; \Big\{\mathbf{v}_h\,\in\,[\mathcal{C}(\bar\Omega_{\mathrm{S}})]^2:\quad \mathbf{v}_h|_T\,\in\,\mathbf{P}_1(T)\quad\forall\,T\,\in\,\mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbf{H}_h(\Omega_\star) \;&:=\; \Big\{\tau_h\,\in\,\mathbf{H}(\mathrm{div};\Omega_\star):\quad \tau_h|_T\,\in\,\mathrm{RT}_0(T)\quad\forall\,T\,\in\,\mathcal{T}_h^\star\Big\},\quad \star\in\{\mathrm{S},\mathrm{D}\},\\
\mathrm{L}_h(\Omega_{\mathrm{D}}) \;&:=\; \Big\{q_h\in L^2(\Omega_{\mathrm{D}}):\quad q_h|_T\,\in\,P_0(T)\quad\forall\,T\,\in\,\mathcal{T}_h^{\mathrm{D}}\Big\}.
\end{aligned} \tag{4.28}$$

Observe that $\mathbf{H}_h(\Omega_{\mathrm{S}})$ and $\mathrm{L}_h(\Omega_{\mathrm{D}})$ clearly satisfy $(\mathbf{H}.\mathbf{0})$. In addition, $(\mathbf{H}.\mathbf{2})$ is easy to verify if the sequence of subspaces is nested or if we are able to find a coarser space where $(\mathbf{H}.\mathbf{2})$ holds. For further details on the construction of $\mathbf{v}_0\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ satisfying $(\mathbf{H}.\mathbf{2})$, we refer to [22, Section 3.2] (see, also [21, Section 3.2] or [6, Lemma 3.2]).

Now, we turn to define the finite dimensional subspace $\Lambda_h(\Sigma)$. For this purpose, let us assume that the number of edges of Σ_h is even and let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h (if the number of edges of Σ_h is odd, we simply reduce to the even case by joining any pair of two adjacent elements and then construct Σ_{2h} from this reduced partition). Then, we set

$$\Lambda_h(\Sigma) := \Big\{\xi_h\,\in\,\mathcal{C}(\Sigma):\quad \xi_h|_e\,\in\,P_1(e)\quad\forall\,e\,\in\,\Sigma_{2h}\Big\}, \tag{4.29}$$

and denote $h_\Sigma := \max\big\{h_e:\quad e\in\Sigma_{2h}\big\}$. Observe that $P_0(\Sigma)\subseteq\Lambda_h(\Sigma)$. Also, it is easy to see that $\operatorname{div}\mathbf{H}_h(\Omega_{\mathrm{D}})\subseteq\mathrm{L}_h(\Omega_{\mathrm{D}})$, whence hypothesis $(\mathbf{H}.\mathbf{3})$ holds.

It remains to prove that $(\mathbf{H}.\mathbf{1})$ is satisfied as well. To this end, we recall from [21] that the set of normal traces of $\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})=\mathbf{H}_h(\Omega_{\mathrm{D}})\cap\mathbf{H}_{\Gamma_{\mathrm{D}}}(\mathrm{div};\Omega_{\mathrm{D}})$ on Σ is defined by the subspace of $L^2(\Sigma)$ given by

$$\Theta_h(\Sigma) := \Big\{\phi_h:\Sigma\to\mathbb{R}:\quad \phi_h|_e\,\in\,P_0(e)\quad\forall\,e\,\in\,\Sigma_h\Big\}. \tag{4.30}$$

Then, analogously to [29, Lemma A.1], we can deduce that there exists a discrete lifting operator $\mathcal{L}^h : \Theta_h(\Sigma) \to \mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})$, satisfying

$$\|\mathcal{L}^h(\phi_h)\|_{\mathrm{div};\Omega_\mathrm{D}} \leq c\|\phi_h\|_{-1/2,\Sigma} \quad \text{and} \quad \mathcal{L}^h(\phi_h)\cdot\mathbf{n} = \phi_h \quad \text{on} \quad \Sigma, \tag{4.31}$$

for all $\phi_h \in \Theta_h(\Sigma)$. Additionally, we recall from [21, Lemma 5.1] that there exists $\hat{\beta}_\Sigma > 0$ such that the pair of subspaces $(\Theta_h(\Sigma), \Lambda_h(\Sigma))$ satisfies the discrete inf-sup condition

$$\sup_{\substack{\phi_h\in\Theta_h(\Sigma)\\ \phi\neq 0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geq \hat{\beta}_\Sigma \,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h(\Sigma)\,. \tag{4.32}$$

Then, owing to the existence of $\mathcal{L}_h$ and estimate (4.32), it is easy to see (see [21, Lemma 4.2]) that there exists $C > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_{h,\mathrm{D}}\in\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})\\ \mathbf{v}_{h,\mathrm{D}}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{h,\mathrm{D}}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{h,\mathrm{D}}\|_{\mathrm{div};\Omega_\mathrm{D}}} \geq C\|\xi_h\|_{1/2,\Sigma}. \tag{4.33}$$

According to the above discussion, we are in position of proving next the inf-sup condition (4.9).

Lemma 4.5 *There exists $\widehat{c}_1 > 0$, independent of h, such that*

$$\begin{aligned} S_{1,h}(q_{h,\mathrm{D}}, \xi_h) := & \sup_{\substack{\mathbf{v}_{h,\mathrm{D}}\in\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})\\ \mathbf{v}_{h,\mathrm{D}}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{h,\mathrm{D}}, q_{h,\mathrm{D}})_{\Omega_\mathrm{D}} + \langle \mathbf{v}_{h,\mathrm{D}}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{h,\mathrm{D}}\|_{\mathrm{div};\Omega_\mathrm{D}}} \\ \geq & \quad \widehat{c}_1 \left\{ \|q_{h,\mathrm{D}}\|_{0,\Omega_\mathrm{D}} + \|\xi_h\|_{1/2,\Sigma} \right\}, \end{aligned} \tag{4.34}$$

for all $(q_{h,\mathrm{D}}, \xi_h) \in \mathrm{L}_{h,0}(\Omega_\mathrm{D}) \times \Lambda_h(\Sigma)$.

Proof. Given $(q_{h,\mathrm{D}}, \xi_h) \in \mathrm{L}_{h,0}(\Omega_\mathrm{D}) \times \Lambda_h(\Sigma)$, we first observe that there holds

$$S_{1,h}(q_{h,\mathrm{D}}, \xi_h) \geq \sup_{\substack{\mathbf{v}_{h,\mathrm{D}}\in\widetilde{\mathbf{H}}_h(\Omega_\mathrm{D})\\ \mathbf{v}_{h,\mathrm{D}}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{h,\mathrm{D}}, q_{h,\mathrm{D}})_{\Omega_\mathrm{D}}}{\|\mathbf{v}_{h,\mathrm{D}}\|_{\mathrm{div};\Omega_\mathrm{D}}},$$

where $\widetilde{\mathbf{H}}_h(\Omega_\mathrm{D}) := \Big\{ \mathbf{v}_{h,\mathrm{D}} \in \mathbf{H}_h(\Omega_\mathrm{D}) : \quad \mathbf{v}_{h,\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega_\mathrm{D} \Big\}$. Then, employing the analysis from [17, Section 4.3], we get

$$S_{1,h}(q_{h,\mathrm{D}}, \xi_h) \geq \hat{C}\|q_{h,\mathrm{D}}\|_{0,\Omega_\mathrm{D}}. \tag{4.35}$$

On the other hand, it is clear that

$$S_{1,h}(q_{h,\mathrm{D}}, \xi_h) \geq \sup_{\substack{\mathbf{v}_{h,\mathrm{D}}\in\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D})\\ \mathbf{v}_{h,\mathrm{D}}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{h,\mathrm{D}}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{h,\mathrm{D}}\|_{\mathrm{div};\Omega_\mathrm{D}}} - \|q_{h,\mathrm{D}}\|_{0,\Omega}, \tag{4.36}$$

which, combined with (4.33), yields

$$S_{1,h}(q_{h,\mathrm{D}}, \xi_h) \;\geq\; C\,\|\xi_h\|_{1/2,\Sigma} \;-\; \|q_{h,\mathrm{D}}\|_{0,\Omega_\mathrm{D}}\,.$$

Finally, from the latter estimate and (4.35) we readily obtain (4.34), which concludes the proof. □

Having verified hypotheses (**H.0**) – (**H.3**), a straightforward application of Theorem 4.3 yields the well-posedness of (4.4) and the corresponding Céa estimate.

Theorem 4.6 *Let $\mathbf{X}_h$ and $\mathbf{Q}_h$ be the finite element subspaces defined by (4.2) in terms of the specific discrete spaces given by (4.28) and (4.29), and assume that the hypotheses of Theorem 4.4 hold. Then the Galerkin scheme (4.4) has a unique solution $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$, which satisfies the estimates (4.17) and (4.23).*

Proof. Since assumption (4.22) implies (3.48) and hypotheses (**H.0**) – (**H.3**) hold, the result follows from a straightforward application of Theorems 4.3 and 4.4. □

Finally, by employing the approximations properties of the finite element subspaces involved (see, e.g. [4, 17, 24, 27]), and the a priori estimate (4.23), we can easily obtain the following result.

Theorem 4.7 *Assume that the hypotheses of Theorem 4.4 hold. Let $(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) \in \mathbf{X} \times \mathbf{Q}$ and $(\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ be the unique solutions of the continuous and discrete problem (3.21) and (4.4), respectively. Assume further that there exists $\delta > 0$ such that $\mathbf{T}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{div}\,\mathbf{T}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, $\operatorname{div}\mathbf{u}_{\mathrm{D}} \in \mathrm{H}^\delta(\Omega_{\mathrm{D}})$, $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{\delta+1}(\Omega_{\mathrm{S}})$, and $\mathbf{f}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$. Then, $p_{\mathrm{D}} \in \mathrm{H}^{\delta+1}(\Omega_{\mathrm{D}})$, $\lambda \in \mathrm{H}^{\delta+1/2}(\Sigma)$, and there exists $C > 0$, independent of h and the continuous and discrete solutions, such that*

$$\begin{aligned} \|(\underline{\mathbf{\Phi}}, \underline{\mathbf{p}}) - (\underline{\mathbf{\Phi}}_h, \underline{\mathbf{p}}_h)\|_{\mathbf{X}\times\mathbf{Q}} \,\le\, C\, h^\delta \Big\{ &\|\mathbf{T}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\,\mathbf{T}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} \\ &+ \|\mathbf{u}_{\mathrm{S}}\|_{\delta+1,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\operatorname{div}\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|p_{\mathrm{D}}\|_{\delta+1,\Omega_{\mathrm{D}}} \Big\}. \end{aligned} \tag{4.37}$$

Proof. From the first equation of (3.21) (cf. (3.4)) we find that $\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}} = -\nabla p_{\mathrm{D}} + \mathbf{f}_{\mathrm{D}}$ in Ω_{D}, which implies that $p_{\mathrm{D}} \in \mathrm{H}^{1+\delta}(\Sigma)$, whence $\lambda = p_{\mathrm{D}}\,|_\Sigma \in \mathrm{H}^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the a priori estimate (4.23), the approximation properties of the discrete spaces involved and the fact that, owing to the trace theorem in Ω_{D}, there holds $\|\lambda\|_{\delta+1/2,\Sigma} \le c\|p_{\mathrm{D}}\|_{\delta+1,\Omega_{\mathrm{D}}}$. □

4.5.2 Finite element subspaces in 3D

Let us now consider the discrete spaces

$$\begin{aligned} \mathbf{H}_h^1(\Omega_{\mathrm{S}}) &:= \Big\{ \mathbf{v}_h \in [\mathcal{C}(\bar{\Omega}_{\mathrm{S}})]^3 : \quad \mathbf{v}_h|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\ \mathbf{H}_h(\Omega_\star) &:= \Big\{ \tau_h \in \mathbf{H}(\mathrm{div};\Omega_\star) : \quad \tau_h|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^\star \Big\}, \quad \star \in \{\mathrm{S},\mathrm{D}\}, \\ \mathrm{L}_h(\Omega_{\mathrm{D}}) &:= \Big\{ q_h \in L^2(\Omega_{\mathrm{D}}) : \quad q_h|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \Big\}. \end{aligned} \tag{4.38}$$

Next, in order to define the subspace approximating the unknown λ, we introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}_K$, and define $\widetilde{h}_\Sigma := \max\big\{ h_K : \quad K \in \Sigma_h \big\}$ and $\widehat{h}_\Sigma := \max\big\{ \widehat{h}_K : \quad K \in \Sigma_{\widehat{h}} \big\}$. Then, we define

$$\Lambda_h(\Sigma) := \Big\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_K \in P_1(K) \quad \forall K \in \Sigma_{\widehat{h}} \Big\}. \tag{4.39}$$

In this way, we define the global spaces $\mathbf{X}_h$ and $\mathbf{Q}_h$ by combining (4.1), (4.2), (4.38), and (4.39).

Now, for the verification of the required hypotheses for the corresponding discrete analysis, we first observe that the same arguments from the 2D case imply the verification of **(H.0)**, **(H.2)** and **(H.3)** in 3D. However, for the inf-sup conditions in **(H.1)**, we need to proceed slightly different to the 2D case and apply [18, Lemma 7.5]. More precisely, utilizing [18, Lemma 7.5], we conclude that there exists $C_0 \in (0,1)$ such that for each pair $(\widetilde{h}_\Sigma, \widehat{h}_\Sigma)$ verifying $\widetilde{h}_\Sigma \le C_0 \widehat{h}_\Sigma$, the inf-sup condition (4.33) holds. According to this, we can proceed analogously to the proof of Lemma 4.5 to verify **(H.1)**.

Having verified hypotheses **(H.0)**–**(H.3)**, we conclude that the Galerkin scheme (4.4) defined with the spaces in (4.38) is well posed. In addition, owing again to the approximations properties of the finite element subspaces involved (see, e.g. [4, 17, 24, 27]), and the a priori estimate (4.23), we obtain exactly the same Theorem 4.7 for the 3D case as well.

5 Numerical results

In this section we present two numerical examples in 2D illustrating the performance of our augmented mixed finite element scheme (4.4) on a set of uniform triangulations of the corresponding domains, and considering the finite element spaces introduced in Section 4.5.1. Our implementation is based on a *FreeFem++* code (see [26]), in conjunction with the direct linear solver UMFPACK (see [12]). Regarding the implementation of the iterative strategy generated by the Newton method applied to (4.4), we remark that the corresponding iterations are terminated once the relative error of the entire coefficient vectors between two consecutive ones is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{X}_h$ and $\mathbf{Q}_h$, and tol is a fixed tolerance to be specified in each case. For the examples shown below we simply take $(\mathbf{0},\mathbf{0})$ as initial guess.

We now introduce some additional notations. The individual errors are denoted by $\mathbf{e}_{\mathbf{T}} := \mathbf{T}_{\mathrm{S}} - \mathbf{T}_h$, $\mathbf{e}_{\mathbf{u}_{\mathrm{S}}} = \mathbf{u}_{\mathrm{S}} - \mathbf{u}_h$, $\mathbf{e}_{\mathbf{u}_{\mathrm{D}}} = \mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}$, $\mathbf{e}_{p_{\mathrm{D}}} = p_{\mathrm{D}} - p_{h,\mathrm{D}}$ and $\mathbf{e}_{\lambda} = \lambda - \lambda_h$. Also, we let $\mathbf{r}_{\mathbf{T}_{\mathrm{S}}}$, $\mathbf{r}_{\mathbf{u}_{\mathrm{S}}}$, $\mathbf{r}_{\mathbf{u}_{\mathrm{D}}}$, $\mathbf{r}_{p_{\mathrm{D}}}$ and $\mathbf{r}_{\lambda}$ be the experimental rates of convergence given by

$$\mathbf{r}_{\mathbf{T}_{\mathrm{S}}} := \frac{\log(\mathbf{e}_{\mathbf{T}_{\mathrm{S}}}/\mathbf{e}'_{\mathbf{T}_{\mathrm{S}}})}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})}, \quad \mathbf{r}_{\mathbf{u}_{\mathrm{S}}} := \frac{\log(\mathbf{e}_{\mathbf{u}_{\mathrm{S}}}/\mathbf{e}'_{\mathbf{u}_{\mathrm{S}}})}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})}, \quad \mathbf{r}_{\mathbf{u}_{\mathrm{D}}} := \frac{\log(\mathbf{e}_{\mathbf{u}_{\mathrm{D}}}/\mathbf{e}'_{\mathbf{u}_{\mathrm{D}}})}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})},$$
$$\mathbf{r}_{p_{\mathrm{D}}} := \frac{\log(e_{p_{\mathrm{D}}}/\mathbf{e}'_{p_{\mathrm{D}}})}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})}, \quad \mathbf{r}_{\lambda} := \frac{\log(e_{\lambda}/\mathbf{e}'_{\lambda})}{\log(h_{\Sigma}/h'_{\Sigma})},$$

where $h_\star$ and $h'_\star$ ($\star \in \{\mathrm{S},\mathrm{D},\Sigma\}$) denote two consecutive mesh sizes with their respective errors $\mathbf{e},\mathbf{e}'$ (or e,e'). For each example below we assume $\alpha_{\mathrm{D}} = 1$, $\rho = 1$, and $\mathbf{K} = \mathbf{I}$.

In Example 1 we take the porous domain $\Omega_{\mathrm{D}} := (-1/2, 1/2) \times (0, -1/2)$ coupled with a semi-disk-shaped fluid domain $\Omega_{\mathrm{S}} := \Big\{(x_1,x_2) : \quad x_1^2 + x_2^2 \leq (1/2)^2 \quad \text{and} \quad x_2 > 0\Big\}$. In addition, we consider the viscosity $\nu = 1$, the parameters $\kappa_1 = 1$ and $\kappa_2 = 2\nu$, and the data $\mathbf{f}_{\mathrm{S}}$ and $\mathbf{f}_{\mathrm{D}}$ are chosen so that the exact solution in the tombstone-shaped domain $\Omega = \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) := \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2) \\ -\cos(\pi x_2)\sin(\pi x_1) \end{pmatrix} \quad \forall\, \mathbf{x} = (x_1,x_2) \in \Omega_{\mathrm{S}},$$

$$\mathbf{u}_{\mathrm{D}}(\mathbf{x}) := \begin{pmatrix} -\dfrac{64}{\pi} x_2(x_2^2 - 0.25)\cos(\pi x_1) \\ -16\sin(\pi x_1)(x_2^2 - 0.25)^2 \end{pmatrix} \quad \forall\, \mathbf{x} = (x_1,x_2) \in \Omega_{\mathrm{D}}\,,$$

and

$$p_{\mathrm{D}}(\mathbf{x}) := \cos(x_1) + a x_2 \quad \forall\, \mathbf{x} = (x_1,x_2) \in \Omega_{\mathrm{D}}\,,$$

where the constant a is chosen in such a way that $\int_{\Omega_{\mathrm{D}}} p_{\mathrm{D}} = 0$. Notice that the foregoing solution satisfies $\mathbf{u}_{\mathrm{S}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}$ on Σ and the boundary condition $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} = 0$ on Γ_{D}. However, the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_{S} is non-homogeneous. Then, we need to modify accordingly the functional

$$\mathbf{F}(\underline{\mathbf{\Psi}}) := 2\nu(\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} + 2\nu(\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} - \kappa_1(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\,\mathbf{R}_{\mathrm{S}})_{\Omega_{\mathrm{S}}} + 2\mu\langle \mathbf{R}_{\mathrm{S}}\mathbf{n}, \mathbf{u}_{\mathrm{S}}\rangle_{\Gamma_{\mathrm{S}}} \qquad \forall \underline{\mathbf{\Psi}} \in \mathbf{X}.$$

In Table 5.1 we summarize the convergence history of the first example for a sequence of uniform triangulations, considering the finite element subspaces described in Section 4.5.1, and solving the nonlinear problem with a tolerance $tol = 10^{-6}$. Note that the rate of convergence $\mathcal{O}(h)$ predicted by Theorem 4.7 is attained in all the cases. Next, in Figures 5.1, 5.2 and 5.3 we display the first and second component of the discrete velocity $\mathbf{u}_h = (\mathbf{u}_{h,\mathrm{S}}, \mathbf{u}_{h,\mathrm{D}})$, the discrete velocity vector field and the approximate pressure $p_{h,\mathrm{D}}$ together with the $\{1,1\}$-component of tensor $\mathbf{T}_h$, respectively, with $N = 411915$. Observe there that the second components of $\mathbf{u}_{h,\mathrm{S}}$ and $\mathbf{u}_{h,\mathrm{D}}$ coincide on Σ, whence $\mathbf{u}_{h,\mathrm{S}} \cdot \mathbf{n} = \mathbf{u}_{h,\mathrm{D}} \cdot \mathbf{n}$ on Σ as expected, which is confirmed in Figure 5.3 where we clearly observe that the flux on Σ is continuous.

N	h_{S}	$\mathsf{e}_{\mathbf{T}_{\mathrm{S}}}$	$\mathsf{r}_{\mathbf{T}_{\mathrm{S}}}$	$\mathsf{e}_{\mathbf{u}_{\mathrm{S}}}$	$\mathsf{r}_{\mathbf{u}_{\mathrm{S}}}$
458	0.1901	1.1196	-	0.3459	-
1707	0.0911	0.5638	0.9302	0.1679	0.9801
6588	0.0486	0.2825	1.0972	0.0833	1.1122
26399	0.0242	0.1374	1.0353	0.0414	1.0064
103855	0.0134	0.0696	1.0822	0.0208	1.0935
411915	0.0077	0.0352	1.1468	0.0104	1.1630
N	h_{D}	$\mathsf{e}_{\mathbf{u}_{\mathrm{D}}}$	$\mathsf{r}_{\mathbf{u}_{\mathrm{D}}}$	$e_{p_{\mathrm{D}}}$	$r_{p_{\mathrm{D}}}$
458	0.2001	0.0056	-	0.0872	-
1707	0.0937	0.0027	0.9807	0.0442	0.8979
6588	0.0470	0.0012	1.1002	0.0219	1.0138
26399	0.0250	0.0006	1.1380	0.0107	1.1257
103855	0.0129	0.0003	1.0944	0.0054	1.0338
411915	0.0068	0.0001	1.0811	0.0027	1.0838

N	h_{Σ}	e_{λ}	r_{λ}
458	0.1250	0.0169	-
1707	0.0625	0.0069	1.3016
6588	0.0313	0.0036	0.9391
26399	0.0156	0.0019	0.9338
103855	0.0078	0.0009	1.0351
411915	0.0039	0.0005	0.9732

Table 5.1: Degrees of freedom N, mesh sizes $h_\star$ ($\star \in \{\mathrm{S}, \mathrm{D}, \Sigma\}$), errors and rates of convergence for the augmented-mixed approximation of the coupled Navier-Stokes/Darcy (Example 1).

In Example 2 we focus on the performance of the iterative method with respect to the viscosity ν. To this end, we take the domain $\Omega = \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$, with $\Omega_{\mathrm{S}} := (-1/2, 3/2) \times (0, 1/2)$ and $\Omega_{\mathrm{D}} := (-1/2, 3/2) \times (0, -1/2)$. In addition, we consider the parameters $\kappa_1 = \nu$ and $\kappa_2 = 2\nu$, so that the ellipticity constant α (see Lemma 3.1) becomes $\alpha = C\nu$ for small values of ν, with C independent of ν. In turn, the terms on the right-hand side are adjusted so that the exact solution is given by the functions

$$\mathbf{u}_{\mathrm{S}}(\mathbf{x}) := \begin{pmatrix} 1 - e^{\gamma x_1}\cos(2\pi x_2) \\ \dfrac{\gamma}{2\pi} e^{\gamma x_1}\sin(2\pi x_2) \end{pmatrix} \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \Omega_{\mathrm{S}}\,,$$

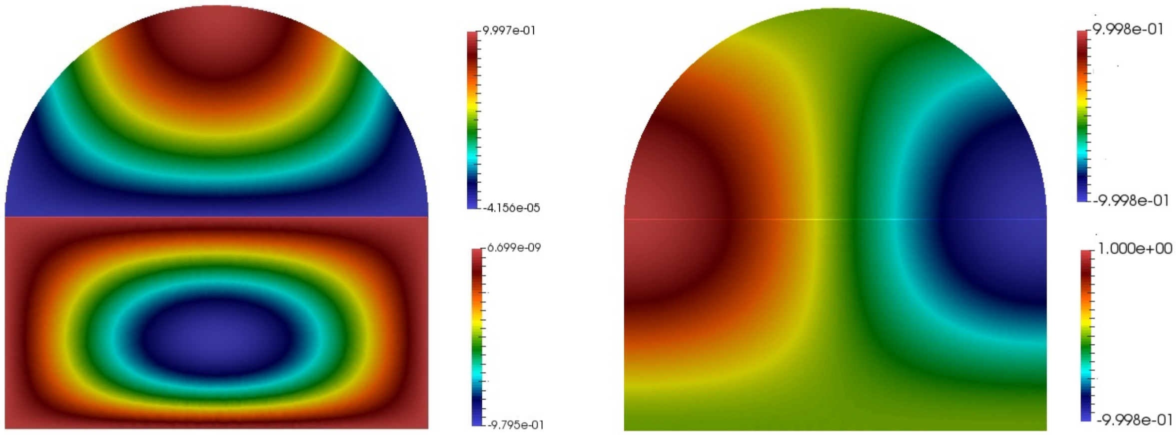

Figure 5.1: first components of $\mathbf{u}_{S,h}$ and $\mathbf{u}_{D,h}$ (left) and second components of $\mathbf{u}_{S,h}$ and $\mathbf{u}_{D,h}$ (right) (Example 1)

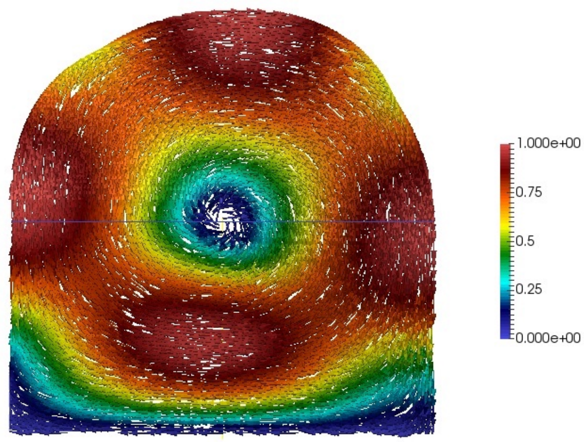

Figure 5.2: velocity vector field $\mathbf{u}_h$ (Example 1).

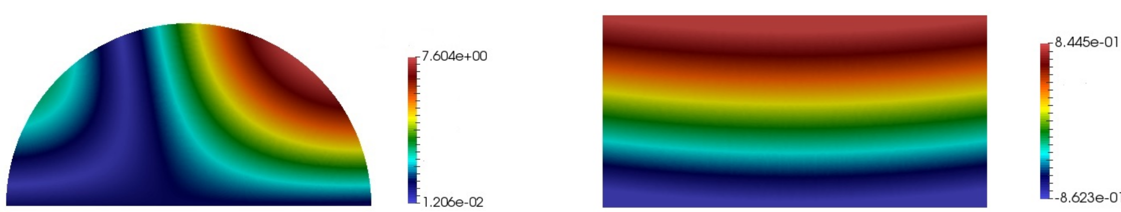

Figure 5.3: component $\mathbf{T}_h^{1,1}$ of the approximate tensor $\mathbf{T}_h$ (left) and approximate pressure $p_{h,D}$ (right) (Example 1).

$$\mathbf{u}_D(\mathbf{x}) := \begin{pmatrix} (x_1+0.5)(x_1-1.5) \\ -(x_2+2)(2x_1-1) \end{pmatrix} \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \Omega_D\,,$$

and

$$p_D(\mathbf{x}) := (x_1-0.5)^3(x_2+1) \qquad \forall\, \mathbf{x} = (x_1, x_2) \in \Omega_D\,,$$

with

$$\gamma := \frac{-8\pi}{\sqrt{\mu^{-2} + 16\pi^2} + \mu^{-1}}\,.$$

Notice that $\mathbf{u}_\mathrm{S}$ is the well known analytical solution for the Navier-Stokes problem obtained by Kovasznay in [26], which presents a boundary layer at $\{-1/2\} \times (0,2)$. In Table 5.2 we show the behavior of the iterative method as a function of the viscosity ν, considering different mesh sizes h, and a tolerance $tol = 10^{-6}$. We observe there that the smaller the viscosity, the larger the number of iterations. Numerical experiments for smaller values of ν are not reported since, in that case, the maximum number of iterations established in the code (100) is attained for all the meshes. Next, in Table 5.3 we show the convergence history considering the viscosity $\nu = 0.1$. We see there that the rate of convergence $\mathcal{O}(h)$ predicted by Theorem 4.7 is attained by all the unknowns.

μ	$h = 0.37499$	$h = 0.20009$	$h = 0.09576$	$h = 0.04915$	$h = 0.02698$	$h = 0.01392$
1	5	4	4	4	4	4
0.1	10	8	8	8	9	9
0.01	-	-	-	53	65	68

Table 5.2: Number of iterations of the iterative method with respect to ν (Example 2).

N	h_S	$\mathbf{e}_{\mathbf{T}_\mathrm{S}}$	$\mathbf{r}_{\mathbf{T}_\mathrm{S}}$	$\mathbf{e}_{\mathbf{u}_\mathrm{S}}$	$\mathbf{r}_{\mathbf{u}_\mathrm{S}}$
284	0.3536	2.2340	-	1.8824	-
1034	0.2001	0.9226	1.5537	0.8833	1.3293
4125	0.0958	0.3279	1.4037	0.4215	1.0040
14886	0.0492	0.1353	1.3278	0.2136	1.0189
60055	0.0270	0.0618	1.3013	0.1061	1.1660
231080	0.0139	0.0301	1.0904	0.0531	1.0477
N	h_D	$\mathbf{e}_{\mathbf{u}_\mathrm{D}}$	$\mathbf{r}_{\mathbf{u}_\mathrm{D}}$	e_{p_D}	r_{p_D}
284	0.3750	0.2980	-	0.0665	-
1034	0.2001	0.1474	1.1205	0.0317	1.1786
4125	0.0950	0.0679	1.0436	0.0145	1.0509
14886	0.0485	0.0347	0.9939	0.0072	1.0360
60055	0.0254	0.0172	1.0864	0.0037	1.0426
231080	0.0160	0.0086	1.4990	0.0018	1.5167

N	h_Σ	e_λ	r_λ
284	0.1250	0.1674	-
1034	0.0625	0.0651	1.3622
4125	0.0313	0.0232	1.4877
14886	0.0156	0.0084	1.4611
60055	0.0078	0.0031	1.4434
231080	0.0039	0.0012	1.4117

Table 5.3: Degrees of Freedom N, mesh sizes $h_\star$ ($\star \in \{\mathrm{S}, \mathrm{D}, \Sigma\}$), errors and rates of convergence for the augmented-mixed approximation of the Navier-Stokes/Darcy problem with $\nu = 0.1$, $\kappa_1 = \nu^2/3$ and $\kappa_2 = 3\nu$ (Example 2).

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