Identification of space distributed coefficients in an indirectly transmitted diseases model

Aníbal Coronel[†], **Fernando Huancas**[‡], **Mauricio Sepúlveda**[§]

[†] GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Chillán, Chile,

[‡] Departamento de Matemática, Facultad de Ciencias, Universidad de Tarapacá, Arica, Chile

[§] Centro de Investigación en Ingeniería Matemática (CI^{2}MA) y Departamento de Ingeniería Matemática (DIM), Universidad de Concepción, Concepción, Chile

E-mail: acoronel@ubiobio.cl,fihuanca@gmail.com,mauricio@ing-mat.udec.cl

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Abstract. We consider the coefficients identification problem in a mathematical model for indirect transmission of a disease between two independent host populations living in two non-coincident spatial domains. The direct problem is given by an initial boundary value problem for a set of seven differential equations: a single equation for the dynamics of propagation of the contaminant and six equations governing the dynamics of disease in each host population under the susceptible-infected-removed approach. The different rates of disease transmission are space dependent functions and are the coefficients in the reaction terms. The identification problem consists of the determination of the coefficients in the reaction terms from an observation of the state variables at the final time of the process. We apply a methodology based on optimization with partial differential equations as constraints. We reformulate the inverse problem as an optimization problem for an appropriate cost function. Our main results are: the proof of existence of solutions for the optimization problem, the introduction of a necessary optimality conditions, the stability of direct problem solution with respect to the unknown coefficients, the stability of the adjoint system solution with respect to the unknown coefficients and the observations, and the uniqueness up to an additive constant of the identification problem.

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1. Introduction

In the last decades there is a growing interest in inverse problems arising from mathematical models coming from several applications and where the governing equations are given in terms of partial differential equations, see for instance [4, 27–29].

In a broad sense there are at least four kinds of inverse problems: the determination of the domain or part of the domain, the determination of the initial condition, the determination of the boundary condition and the determination of the coefficients of the equations. In all cases we need some knowledge about the solution of the forward problem obtained from experimental measurements. Moreover, it is well known that the methods for analysis and properties of inverse problems are different from the standard theories used for direct problems, for instance a general common property for all kinds of inverse problems is the fact that they usually are ill-posed in uniqueness. In particular, the aim of the present study is to analyze the inverse problem arising from the coefficients determination problem for a reaction-diffusion system originated from the mathematical theory of epidemics.

In epidemiology, the consideration of differential equations as transmitted disease models go back to the work of Kermack and McKendrick [30]. They study the evolution of an epidemic in a closed host population of total size N which is divided into three classes of individuals: susceptibles (S) which capable of contracting the disease and became themselves infectives; infectives (I) which are capable of transmitting the disease to susceptibles; and removed (R) which have contracted the disease and being unable to transmitted the disease because have died or, if recovered, are permanently immune or have been isolated. Assuming that the transfer process from S to I and from I to R are given by a mass action and exponential decay laws, respectively; the basic model (called SIR model) is given by the following dynamical system

$$\frac{dS}{dt} = -kSI, \quad \frac{dI}{dt} = kSI - \lambda I, \quad \frac{dR}{dt} = \lambda I, \tag{1}$$

$$S(0) = S_0, \quad I(0) = I_0, \quad I(0) = R_0, \tag{2}$$

with k and λ some positive constants and S_0, I_0, and R_0 some nonegative constants such that $S_0 + I_0 + R_0 = N$. Later on, SIR models have been improved by several authors who have considered other factors of individuals (position, sex, age, diffusion, etc) and have developed a long and rich literature, see e.g. [6, 13–15, 25] and references therein. Particularly, by considering the position as a factor and assuming that the individuals disperse by means of Fickian diffusion, we obtain a a description of population densities by continuous in space and time functions satisfying a spatially-extended versions of (1) to the reaction-diffusion system

$$\partial_t S - \mathrm{div}(d_S(x)\nabla S) = k(x)SI, \qquad (x,t) \in \Omega \times \mathbb{R}_+, \tag{3}$$

$$\partial_t I - \mathrm{div}(d_I(x)\nabla I) = k(x)SI - \lambda(x)I, \qquad (x,t) \in \Omega \times \mathbb{R}_+, \tag{4}$$

$$\partial_t R - \mathrm{div}(d_R(x)\nabla R) = \lambda(x)I, \qquad (x,t) \in \Omega \times \mathbb{R}_+, \tag{5}$$

where d_S, d_S, d_S, k and λ are some positive functions defined on $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$. More complex and realistic models can be obtained for incorporating modifications in the diffusion and reaction terms, see for instance [3] for a model with nonlocal cross-diffusion and [2] for a reaction term obtained by application of Frequency-dependent transmission law for the transfer process from S to I.

In this work, the direct problem is the mathematical model introduced in [26] to describe the transmission through a contaminated environment of a microparasite between two independent host populations H_1 and H_2, living in two non-coincident spatial domains Ω_1 and Ω_2 of $\mathbb{R}^d$ $(d = 1, 2, 3)$, i.e. $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 \neq \Omega_i, i = 1, 2$; respectively. Assuming that the dynamic processes of transmitted disease in each population are governed by a SIR type reaction-diffusion system and the interaction of both populations is given by a contaminated environment, Fitzgibbon and collaborators [26] deduce the following set of differential equations

$$
\begin{aligned}
\partial_t \varphi - \operatorname{div}(d_{11}(x)\nabla\varphi) &= -\sigma_{11}(x)\frac{\varphi\psi}{H_1} - \sigma_{31}(x)c\varphi + (1-w_1)\lambda_1\psi \\
&\quad + b(x)H_1 - (m(x) + k(x)H_1)\varphi, \qquad \text{in } Q_{1,T} := \Omega_1 \times]0, T[, \quad (6)\\
\partial_t \psi - \operatorname{div}(d_{12}(x)\nabla\psi) &= \sigma_{11}(x)\frac{\varphi\psi}{H_1} + \sigma_{31}(x)c\varphi - \omega_1\lambda_1\psi \\
&\quad - (m(x) + k(x)H_1)\psi, \qquad \text{in } Q_{1,T}, \quad (7)\\
\partial_t \chi - \operatorname{div}(d_{13}(x)\nabla\chi) &= \omega_1\lambda_1\psi - (m(x) + k(x)H_1)\chi, \qquad \text{in } Q_{1,T}, \quad (8)\\
\partial_t u - \operatorname{div}(d_{21}(x)\nabla u) &= -\sigma_{32}(x)cu, \qquad \text{in } Q_{2,T} := \Omega_2 \times]0, T[, \quad (9)\\
\partial_t u - \operatorname{div}(d_{22}(x)\nabla u) &= \sigma_{32}(x)cu - \varepsilon\lambda_2 v, \qquad \text{in } Q_{2,T}, \quad (10)\\
\partial_t v - \operatorname{div}(d_{23}(x)\nabla w) &= 0, \qquad \text{in } Q_{2,T}, \quad (11)\\
\partial_t c &= \sigma_{13}(x)(1-c)\tilde{\varphi} + \sigma_{23}(x)(1-c)\tilde{v} - \delta(x)c, \qquad \text{in } Q_T := (\Omega_1 \cup \Omega_2) \times]0, T[, \quad (12)
\end{aligned}
$$

supplemented with the initial and boundary conditions

$$
\begin{aligned}
&\varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \chi(x,0) = \chi_0(x), && \text{in } \Omega_1, && (13)\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), && \text{in } \Omega_2, && (14)\\
&c(x,0) = c_0(x), && \text{in } \Omega_1 \cup \Omega_2, && (15)\\
&d_{11}(x)\nabla\varphi \cdot \boldsymbol{\eta}_1 = d_{12}(x)\nabla\psi \cdot \boldsymbol{\eta}_1 = d_{13}(x)\nabla\chi \cdot \boldsymbol{\eta}_1 = 0, && \text{in } \Gamma_{1,T} := \partial\Omega_1 \times]0, T[, && (16)\\
&d_{21}(x)\nabla u \cdot \boldsymbol{\eta}_2 = d_{22}(x)\nabla v \cdot \boldsymbol{\eta}_2 = d_{23}(x)\nabla w \cdot \boldsymbol{\eta}_2 = 0, && \text{in } \Gamma_{2,T} := \partial\Omega_2 \times]0, T[. && (17)
\end{aligned}
$$

The state variables φ, ψ and χ are defined on $Q_{1,T}$ and represent the population densities of the subclasses of susceptible, infected and recovered individuals from the total population $H_1 = \varphi + \psi + \chi$; the state variables u, v, and w are defined on $Q_{2,T}$ and are used to represent the population densities of the susceptible, infected and recovered subclasses of the total population $H_2 = u + v + w$; while the state variable c is defined on Q_T represents the proportion of the environment contaminant. The functions $\tilde{\varphi}$ and $\tilde{v}$ denote the prolongation by zero of φ and v, on $Q_{2,T}$ and $Q_{1,T}$, respectively. The functions d_{i1}, d_{i2}, d_{i3} are defined on Ω_i and denotes the diffusivity of susceptible, infected and recovered individuals from population H_i, $i = 1, 2$, respectively. The functions $\sigma_{11}, \sigma_{13}, \sigma_{23}, \sigma_{31}$ and σ_{32} are intrapopulation and contact with the contaminant transmission rates. The functions b, m and k are rates related with the vital dynamics (birth and mortality) on the population H_1. Further, δ is a rate modelling the unsustainable habitat contamination. Finally, $\lambda_1, \lambda_2, \omega_1$, and ε are positive constants; $\varphi_0, \psi_0, \chi_0, u_0, v_0, w_0$, and c_0 are the initial conditions; and $\boldsymbol{\eta}_i$ denotes the unit outward normal vector to the boundary of Ω_i given by $\partial\Omega_i$, $i = 1, 2$. We note that (16)-(17) are no-flux boundary conditions which means that there is no population flux across the boundaries of Ω_1 and Ω_2.

In this paper, we focus on the inverse problem of coefficients determination on the initial boundary value problem (6)-(17) from final time observation of the state variables. The inverse problem is motivated by the practical situation where the densities of the different classes (susceptible, infected, and recovered) of both populations and the concentration of the contaminant can be measured, on the other hand, however, the different rates (disease transmission, disease recovery, birth, mortality, etc) and other coefficients of the model, are very costly or even infeasible to measure. Moreover, we remark that form epidemiological viewpoint, the knowledge of the different coefficients is important at least for two reasons: permits the solution of (6)-(17) in a real application context and can be used to validate the model in order to simulate or improve some properties.

There is a huge list of articles where the problem of coefficients identification in reaction-diffusion equations is focused, see for instance [7–11, 16, 17, 19–23, 34–36]. The list is not exhaustive and there is a more extensive literature. The majority of the results are obtained for the case of scalar equations and were originally motivated by applications of heat transfer phenomena. The few results in the case of systems are reported in [17, 19, 21, 22, 35, 36]. In particular, we remark that in [35] the authors define the inverse problem for a susceptible-infective-susceptible (SIS) reaction-diffusion model in a multidimensional space but however some of their findings are reduced to one dimensional case. Recently, in [17] the authors of the present paper have extend the results of [35] for the multidimensional case.

In this article, we rewrite the inverse coefficient problem in an optimization problem for an appropriate cost function defined on an admissible set. Then, we introduce an adjoint state and prove the main contributions of the paper: (a) the existence of solutions for the inverse problem, (b) the introduction of first order optimality condition, (c) the stability of a direct problem solution with respect to the coefficients of the reaction term, (d) the stability of the adjoint problem solution with respect to the coefficients of the reaction term and the observations, (e) the uniqueness of the identification problem.

The organization of the paper is as follows. In section 2 we present the definition of direct and inverse problems, the notation and the enunciate of main results. In section 3, we present the proofs of the main results.

2. Problem statement and main results

2.1. The direct problem

The direct problem is given by the mathematical model (6)-(17) formulated by Fitzgibbon, Langlais and Morgan [26]. Indeed, for completeness and in order to be more precise, we summarize the assumptions by the following list

(A0) There are two independent host populations H_1 and H_2 which are spatially distributed over non-coincident spatial domains Ω_1 and Ω_2 of $\subset \mathbb{R}^d$ $(d = 1, 2, 3)$; i.e. $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 \neq \Omega_i, i = 1, 2$; respectively. The region Ω_1 is a reservoir where lives a parasite which, in most of the cases of interest, is benign on the population H_1 and lethal on the population H_2.

(A1) There is a contaminated habitat or environment and the variable $c(x, t)$ represents the proportion of such contaminant at the position $x \in \Omega_1 \cup \Omega_2$ and time $t \in [0, T]$.

(A2) Each host population is subdivided into three subclasses: susceptible, infected, and removed individuals, with population densities φ, ψ and χ in the population H_1 and u, v, and w in the population H_2, respectively.

(A3) The dynamic in the host population H_1 is given as follows. The susceptible individuals in the host population H_1 can contract the disease from cross contacts with infected hosts from H_1 or with the environment. Moreover, we consider that the force of infection is modeled by a frequency-dependent transmission function of the form $\sigma_{11}(x)\varphi\psi/H_1$ with σ_{11} a transmission coefficient. Related with with infective individuals from H_1, we consider two facts: $1/\lambda_1$ represents the duration of the infective stage and a fixed proportion $\omega_1 \in [0, 1]$ of infective individuals become permanently immune and a proportion $1 - \omega_1 \in [0, 1]$ reentering in the susceptible class. Moreover, we consider that H_1 has vital dynamics with $b(x)$ and $m(x) + k(x)H_1$ identical in each subclass and representing the birth and mortality rates, respectively.

(A4) The dynamic in the host population H_2 is given as follows. The susceptible individuals in the host population H_2 are infected by contact with the environment, but there is neither cross infection from infected hosts from H_2 nor criss-cross infection with H_1. The demographic effects are ignored in H_2. Moreover, the disease can be lethal in the population H_2 with a fixed survival rate $\varepsilon_1 \in [0, 1]$.

(A5) The transmission via contact with the contaminant environment is modeled by mass action kinetics of the form $\sigma_{31}c\varphi$ and $\sigma_{32}cu$, which appears as loss terms for susceptible classes and gain terms for infective classes of both populations. Moreover the dynamic of the contaminant is modeled by the mass action and linear decreasing factor with terms of the form $\sigma_{13}(1-c)\psi$, $\sigma_{23}(1-c)v$ and δc, respectively. The functions σ_{13} and σ_{23} are identically to zero on $\overline{\Omega}_2$ and $\overline{\Omega}_1$, respectively.

(A6) The host population H_1 and H_2 are confined to Ω_1 and Ω_2 for all time, respectively.

From the assumptions (A0)-(A6) and the standard arguments for modelling the dynamic of populations, we deduce (6)-(17).

The mathematical model (6)-(17) can be written as a reaction-diffusion system of the following form

$$\partial_t \mathbf{h}_i - \operatorname{div}(\mathbb{D}_i(x)\nabla \mathbf{h}_i) = \mathbf{f}_i(x, \mathbf{h}_i, c; \boldsymbol{\theta}_i(x)), \qquad \text{in } Q_{i,T}, \quad i = 1, 2, \tag{18}$$

$$\partial_t c = g(x, \mathbf{h}_1, \mathbf{h}_2, c; \boldsymbol{\theta}_3(x)), \qquad \text{in } Q_T, \tag{19}$$

$$(\mathbb{D}_i(x)\nabla \mathbf{h}_i) \cdot \boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1, 2, \tag{20}$$

$$\mathbf{h}_1(x,0) = (\varphi_0, \psi_0, \chi_0)(x) \qquad \text{in } \Omega_1, \tag{21}$$

$$\mathbf{h}_2(x,0) = (u_0, v_0, w_0)(x) \qquad \text{in } \Omega_2, \tag{22}$$

$$c(x,0) = c_0(x) \qquad \text{in } \Omega_1 \cup \Omega_2, \tag{23}$$

where $\mathbf{h}_i$ are the state variables of the population H_i, $i = 1,2$, i.e. $\mathbf{h}_1 = (\phi, \psi, \chi)$ and $\mathbf{h}_2 = (u, v, w)$; $\boldsymbol{\theta}_1 = (\sigma_{11}, \sigma_{31}, b, m, k)$, $\boldsymbol{\theta}_2 = \sigma_{32}$ are the coefficients related with the equations governing the population H_i, and $\boldsymbol{\theta}_3 = (\sigma_{13}, \sigma_{23}, \delta)$ with the equation of the contaminant; $\mathbb{D}_i = \mathrm{diag}(d_{i1}, d_{i2}, d_{i3})$ are diffusion matrices of the population H_i; $\mathbf{f}_i = (f_{i1}, f_{i2}, f_{i3})$ are the reaction terms modelling the dynamics of the population H_i and g the dynamics of the contaminant which are defined by

$$\begin{aligned}
f_{11} &= -\sigma_{11}(x)\frac{\varphi\psi}{H_1} - \sigma_{31}(x)c\varphi + (1-w_1)\lambda_1\psi + b(x)H_1 - (m(x) + k(x)H_1)\varphi, \\
f_{12} &= \sigma_{11}(x)\frac{\varphi\psi}{H_1} + \sigma_{31}(x)c\varphi - \omega_1\lambda_1\psi - (m(x) + k(x)H_1)\psi, \\
f_{13} &= \omega_1\lambda_1\psi - (m(x) + k(x)H_1)\chi, \quad f_{21} = -\sigma_{32}(x)cu, \\
f_{22} &= \sigma_{32}(x)cu - \varepsilon\lambda_2 v, \quad f_{23} = 0, \\
g \ &= \sigma_{13}(x)(1-c)\tilde{\varphi} + \sigma_{23}(x)(1-c)\tilde{v} - \delta(x)c,
\end{aligned}$$

with the functions $\tilde{\varphi}$ and $\tilde{v}$ denoting the prolongation by zero of φ and v, on Ω_2 and Ω_1, respectively; $\boldsymbol{\eta}_i$ are the unit outward normal vectors to $\partial\Omega_i$; and $\varphi_0, \psi_0, \chi_0, u_0, v_0, w_0$, and c_0 are the initial conditions.

We consider the standard functional framework used in the analysis of parabolic equations, see for instance [31–33]. In particular, we use the notations $C^{k,\alpha}(\overline{\Omega})$ with $k \in \mathbb{N}$ and $\alpha \in]0,1]$, $L^p(\Omega)$ with $p \geq 1$, $W^{m,p}(\Omega)$ with $m \in \mathbb{N}$ and $p \geq 1$, for the Banach spaces of Hölder $k-$times continuously function whose k^{th}-partial derivatives are Hölder continuous with exponent α; the space of all functions from Ω to $\mathbb{R}$ which are p-integrable in the sense of Lebesgue; and the usual Sobolev spaces, respectively. In particular, we consider the notations $C^{\alpha}(\overline{\Omega})$ and $H^m(\overline{\Omega})$ instead of $C^{0,\alpha}(\overline{\Omega})$ and $W^{m,2}(\Omega)$, respectively. The vector valued spaces like $[C^{\infty}(\Omega)]^3$, $[L^p(\Omega)]^3$ and $[W^{m,p}(\Omega)]^3$ and others, are defined as usual, namely in the componentwise sense, and are denoted by bold symbols, for instance we denote by $\mathbf{C}^{\infty}(\Omega)$, $\mathbf{L}^p(\Omega)$ and $\mathbf{W}^{m,p}(\Omega)$, the spaces $[\mathbf{C}^{\infty}(\Omega)]^3$, $[L^p(\Omega)]^3$ and $[W^{m,p}(\Omega)]^3$, respectively. Moreover, in order to simplify the presentation of our results and proofs we consider the following notation

$$\mathscr{L}^p = \mathbf{L}^p(\Omega_1) \times \mathbf{L}^p(\Omega_2) \times L^p(\Omega_1 \cup \Omega_2), \tag{24}$$

$$\mathbb{L}^p \ = \Big[\mathbf{L}^p(\Omega_1)\Big]^5 \times \mathbf{L}^p(\Omega_2) \times \mathbf{L}^p(\Omega_1 \cup \Omega_2), \tag{25}$$

$$\mathscr{C}^{\alpha} \ = \Big[C^{0,\alpha}(\overline{\Omega}_1)\Big]^5 \times C^{0,\alpha}(\overline{\Omega}_2) \times \Big[C^{2,\alpha}\Big(\overline{\Omega_1 \cup \Omega_2 \backslash \mathscr{D}}\Big)\Big]^2 \times C^{2,\alpha}(\overline{\Omega_1 \cup \Omega_2}), \tag{26}$$

with $\mathscr{D} = (\partial\Omega_1 \cap \overline{\Omega}_2) \cup (\partial\Omega_2 \cap \overline{\Omega}_1)$; and analogously to $\mathscr{L}^p$ we consider the notation for the functional spaces $\mathscr{W}^{m,p}$ and $\mathscr{H}^m$.

The existence and uniqueness of the positive classical solution of the system (18)-(23) were developed in [26] by considering the assumptions:

(H0) The sets Ω_1 and Ω_2 are open bounded convex sets of $\mathbb{R}^d$ such that $\partial\Omega_i$ are of $C^{3,\alpha}$ regularity.

(H1) The functions modelling the initial conditions are non-negative and satisfying the following regularity conditions: φ_0, ψ_0, χ_0 are continuous on $\overline{\Omega_1}$; u_0, v_0, w_0 are continuous on $\overline{\Omega_2}$; and c_0 is continuous on $\overline{\Omega_1 \cup \Omega_2}\backslash\mathscr{D}$. Moreover, we assume that $c_0(x) \in [0,1]$ on $\Omega_1 \cup \Omega_2$.

(H2) The diffusion coefficients $d_{i,j}$ for $(i,j) \in \{1,2\} \times \{1,2,3\}$ are positive functions, bounded from below on Ω_i and belong $C^{2,\alpha}(\overline{\Omega_i}) \cap L^\infty(\Omega_i)$.

(H3) The coefficients are componentwise strictly positive on their domains of definition, i.e. $\sigma_{11}, \sigma_{31}, b, m$, and k are strictly positive on $\overline{\Omega}_1$; σ_{32} is strictly positive on $\overline{\Omega}_2$; δ is strictly positive on $\overline{\Omega_1 \cup \Omega_2}$; σ_{13} is strictly positive on $\overline{\Omega}_1$ and identically 0 outside of $\overline{\Omega}_1$; σ_{23} is strictly positive on $\overline{\Omega}_2$ and identically 0 outside of $\overline{\Omega}_2$. Moreover, $\boldsymbol{\theta} \in \mathscr{C}^\alpha$ and the birth and mortality rates are such that $b(x) - m(x)$ is strictly positive for all $x \in \Omega_1$.

Theorem 2.1. *If the requirements listed above in (H0)-(H3) are met, then the system (18)-(23) has a unique, classical, global nonnegative solution* $\varphi, \psi, \chi, u, v, w$, *and* c, *which is componentwise non-negative;* φ, ψ, *and* χ *are uniformly bounded on* $Q_1 = \Omega_1 \times]0,\infty[$, u, v, *and* w, *are uniformly bounded on* $Q_2 = \Omega_2 \times]0,\infty[$, *and* c *is uniformly bounded on* $Q = (\Omega_1 \cup \Omega_2) \times]0,\infty[$; *and* $c(x,t) \in [0,1]$ *on* Q.

The details of the proof for Theorem 2.1 are given in [26] and is divided in four big parts: the local existence is followed by Banach fixed point argument; the componentwise non-negativity is deduced by application of the weak maximum principle for scalar parabolic equations; the global well posedness is a consequence of L_∞ estimates of solution components; and the global existence is proved by using the results for discontinuous coefficients and uniform estimates using cut-off functions.

2.2. Definition and formulation of the inverse problem

We assume that a measurement of each subclass of both populations at final time T, and also of the contaminant are given and we need to determine the coefficients in the forward problem (18)-(23). More precisely, we have that the inverse coefficient problem is defined as follows:

> *Inverse problem.* Given the following data, $\varphi_0, \psi_0, \chi_0, u_0, v_0, w_0$ and c_0, namely the initial condition functions, the diffusion coefficients d_{ij} for $\{i,j\} \in \{1,2\} \times \{1,2,3\}$, and the observation functions $\varphi^{obs}, \psi^{obs}, \chi^{obs}, u^{obs}, v^{obs}, w^{obs}$ and c^{obs} defined at time $t = T$; find the functions $\sigma_{11}, \sigma_{13}, \sigma_{23}, \sigma_{31}, \sigma_{32}, b, m, k$ and δ such that the solution of the initial boundary value problem (18)-(23) is "as close as" possible to the observation functions at time $t = T$.

Note that the distinct functions are defined on Ω_1, Ω_2 or $\Omega_1 \cup \Omega_2$. In the context of inverse coefficient problems the term "as close as" is precised by considering an appropriate cost functional, see for instance [12] for the case of flux-diffusion determination in the mathematical modelling of sedimentation. In this paper, in order to analyze the inverse problem, we introduce a formulation of the inverse problem as an optimal control problem. The admissible set $U_{ad} := U_{ad}(\Omega_1, \Omega_2)$ and the cost function $J : U_{ad} \to \mathbb{R}$ are defined as follows

$$U_{ad} = \mathscr{A}(\Omega_1, \Omega_2) \cap \mathscr{H}^{[\![d/2]\!]+1}, \tag{27}$$

$$J(\boldsymbol{\theta}) := \frac{1}{2} \left\|(\mathbf{h}_1, \mathbf{h}_2, c)(\cdot, T) - (\mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, c^{obs})\right\|^2_{\mathscr{L}^2} + \frac{\Gamma}{2} \|\nabla\boldsymbol{\theta}\|^2_{\mathbb{L}^2}, \quad \Gamma > 0, \tag{28}$$

where $\mathbf{h}_1^{obs} = (\phi^{obs}, \psi^{obs}, \chi^{obs})$, $\mathbf{h}_2^{obs} = (u^{obs}, v^{obs}, w^{obs})$ and

$$\mathscr{A}(\Omega_1, \Omega_2) = \left\{\boldsymbol{\theta} := (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) \in \mathscr{C}^\alpha \ : \quad \mathrm{Ran}(\boldsymbol{\theta}) \subseteq \prod_{i=1}^{9} [\underline{r}_i, \overline{r}_i] \subset \mathbb{R}^9_+, \quad \nabla\boldsymbol{\theta} \in \mathbb{L}^2\right\}.$$

Here, $\mathrm{Ran}(\boldsymbol{\theta})$ and $\nabla\boldsymbol{\theta}$ denote the range and the gradient of a function $\boldsymbol{\theta}$ in the componentwise sense. Thus, the inverse problem is reformulated as the following optimization problem

$$\text{Find } \overline{\boldsymbol{\theta}} \in U_{ad} \ : \ J(\overline{\boldsymbol{\theta}}) = \inf_{\boldsymbol{\theta}\in U_{ad}} J(\boldsymbol{\theta}) \quad \text{subject to } (\mathbf{h}_1, \mathbf{h}_2, c) \text{ is solution of (18)-(23).} \tag{29}$$

We remark that the parameter Γ in (28) should be appropriately selected to get uniqueness of the inverse problem.

2.3. Main results

Let us consider that $\overline{\boldsymbol{\theta}} \in U_{ad}$ is a solution of the optimal control problem (29) and $(\overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c})$ is the corresponding solution of (18)-(23) with $\overline{\boldsymbol{\theta}}$ instead of $\boldsymbol{\theta}$. Then, we introduce the notation $\mathbf{p}_i : \Omega_i \to \mathbb{R}^3$ for $i = 1, 2$ and $s : \Omega_1 \cup \Omega_2 \to \mathbb{R}^3$, satisfying the following backward boundary value problem

$$\partial_t \mathbf{p}_i + \mathrm{div}(\mathbb{D}_i(x)\nabla\mathbf{p}_i) = \mathbf{q}_i(x, \mathbf{p}_i, s; \overline{\mathbf{h}}_i, \overline{c}, \overline{\boldsymbol{\theta}}_i(x)), \qquad \text{in } Q_{i,T}, \quad i = 1, 2, \tag{30}$$

$$\partial_t s = \varsigma(x, \mathbf{p}_1, \mathbf{p}_2, s; \overline{c}, \overline{\boldsymbol{\theta}}_3(x)), \qquad \text{in } Q_T, \tag{31}$$

$$(\mathbb{D}_i(x)\nabla\mathbf{h}_i) \cdot \boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1, 2, \tag{32}$$

$$\mathbf{p}_i(x, T) = \overline{\mathbf{h}}_i(x, T) - \mathbf{h}_i^{obs}(x), \qquad \text{in } \Omega_i, \quad i = 1, 2, \tag{33}$$

$$s(x, T) = \overline{c}(x, T) - c^{obs}(x), \qquad \text{in } \Omega_1 \cup \Omega_2, \tag{34}$$

where the functions $\mathbf{q}_i$ and ς are defined as follows

$$\begin{aligned}
q_{11} &= \left[\overline{\sigma}_{11}(x)\frac{\overline{\psi}(\overline{\varphi} + \overline{\psi})}{(H_1)^2} + \overline{\sigma}_{31}(x)\overline{c}\right](p_{12} - p_{11}) \\
&\quad + \left(\overline{b}(x) - \overline{m}(x)\right)p_{11} - \overline{k}(x)\left(2\overline{\varphi}p_{11} + \overline{\psi}p_{12} + \overline{\chi}p_{13}\right), \\
q_{12} &= \overline{\sigma}_{11}(x)\frac{\overline{\varphi}(\overline{\varphi} + \overline{\chi})}{(H_1)^2}(p_{12} - p_{11}) + (1 - \omega_1\lambda_1 + \overline{b}(x))p_{11} - \overline{m}(x)p_{12} + \omega_1\lambda_1(p_{13} - p_{12}) \\
&\quad - \overline{k}(x)\left(\overline{\varphi}p_{11} + 2\overline{\psi}p_{12} + \overline{\chi}p_{13}\right) + \overline{\sigma}_{13}(x)(1 - \overline{c})s, \\
q_{13} &= -\overline{\sigma}_{11}(x)\frac{\overline{\varphi}\overline{\psi}}{(H_1)^2}(p_{12} - p_{11}) - \overline{b}(x)p_{11} - \overline{k}(x)\left(\overline{\varphi}p_{11} + \overline{\psi}p_{12} + 2\overline{\chi}p_{13}\right),
\end{aligned}$$

$$q_{21}=\overline{\sigma}_{32}(x)\overline{c}(p_{22}-p_{21}),\qquad q_{22}=\varepsilon\lambda_2(p_{23}-p_{22})+\overline{\sigma}_{23}(x)(1-\overline{c})s,\qquad q_{23}=0,$$
$$\varsigma=\overline{\sigma}_{31}(x)\tilde{\overline{\psi}}(\tilde{p}_{12}-\tilde{p}_{11})+\overline{\sigma}_{32}(x)\tilde{\overline{v}}(\tilde{p}_{22}-\tilde{p}_{21})-(\overline{\sigma}_{13}(x)\tilde{\overline{\psi}}+\overline{\sigma}_{23}(x)\tilde{\overline{v}}+\overline{\delta}(x))s.$$

Here, the functions $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$ denote the prolongation by zero of $\mathbf{p}_1$ and $\mathbf{p}_2$ on Ω_2 and Ω_1, respectively. Similarly are defined $\tilde{\overline{\psi}}$ and $\tilde{\overline{v}}$. Moreover, we observe that the system (30)-(31) is a linear system and the analysis of existence can be developed by standard arguments for parabolic equations. Indeed, for a recent similar result, which can be straightforward extend to analyze the system (30)-(31), we refer to the work of Apreutesei [5].

Remark 2.1. *From Theorem 2.1, for $\overline{\boldsymbol{\theta}}\in U_{ad}$ the solution of the direct problem $(\mathbf{h}_1,\mathbf{h}_2,c)$ is classical and hence a weak solution. Thus, by convenience we work in the topology of $L^2(0,T,\mathscr{H}^1)$ without losing the viewpoint that we always are considering classical solutions for the direct problem.*

The main results of the paper are the existence, stability and uniqueness of the inverse problem, as is established in the following theorems:

Theorem 2.2. *Let us consider that (H0)-(H3) and the following hypothesis*

(H4) The observation function $(\mathbf{h}_1^{obs},\mathbf{h}_2^{obs},c^{obs})$ belongs $\mathscr{L}^2$,

are valid. Moreover consider the on $\mathscr{U}:=\mathscr{A}(\Omega_1,\Omega_2)\cap\mathscr{M}$ with $\mathscr{M}$ a bounded closed set of $\mathscr{H}^{[\![d/2]\!]+1}$ containing the constant functions. Then, there exists at least one solution of (29) on $\mathscr{U}$.

Theorem 2.3. *Assume that the hypothesis of Theorem 2.2 are satisfied, consider that $\overline{\boldsymbol{\theta}}$ is the solution of (29) and $(\overline{\mathbf{h}}_1,\overline{\mathbf{h}}_2,\overline{c})$ is the corresponding solution of (18)-(23) with $\overline{\boldsymbol{\theta}}$ instead of $\boldsymbol{\theta}$. Then, the adjoint system to (18)-(23) is given by the system (30)-(34). Moreover, the pair $(\mathbf{p}_1,\mathbf{p}_2)$ is bounded in $L^\infty(0,t;[H^2(\Omega_1)]^3\times[H^2(\Omega_2)]^3)$ for almost all time t in $]0,T]$ and the solution of (30)-(34) is bounded in $L^\infty(0,t;\mathscr{L}^\infty)$ for almost all time t in $]0,T]$.*

Theorem 2.4. *Assume that the hypothesis of Theorem 2.2 are satisfied and consider the notation $\overline{\boldsymbol{\theta}},(\overline{\mathbf{h}}_1,\overline{\mathbf{h}}_2,\overline{c})$ and $(\mathbf{p}_1,\mathbf{p}_2,s)$ as is given in Theorem 2.3. Then, the following inequality*

$$\begin{aligned}
&\int\int_{Q_{1,T}}\left\{\left[(\hat{\sigma}_{11}-\overline{\sigma}_{11})\frac{\overline{\varphi}\,\overline{\psi}}{\overline{H}_1}+(\hat{\sigma}_{31}-\overline{\sigma}_{31})\right](p_{11}-p_{12})+bH_1p_{11}-(m+kH_1)\overline{\mathbf{h}}_1\cdot\overline{\mathbf{p}}_1\right\}dxdt\\
&\quad+\int\int_{Q_{2,T}}(\hat{\sigma}_{32}-\overline{\sigma}_{32})(p_{21}-p_{22})dxdt+\int\int_{Q_T}\Big\{\Big(\hat{\sigma}_{13}-\overline{\sigma}_{13}\Big)(1-\overline{c})\tilde{\overline{\varphi}}s+\Big(\hat{\sigma_{23}}-\overline{\sigma_{23}}\Big)(1-\overline{c})\tilde{v}\\
&\qquad-(\hat{\delta}-\overline{\delta})\overline{c}\Big\}sdxdt+\Gamma\int_{\Omega_1}\nabla\overline{\boldsymbol{\theta}}_1\cdot\nabla(\hat{\boldsymbol{\theta}}_1-\overline{\boldsymbol{\theta}}_1)dx+\Gamma\int_{\Omega_2}\nabla\overline{\boldsymbol{\theta}}_2\cdot\nabla(\hat{\boldsymbol{\theta}}_2-\overline{\boldsymbol{\theta}}_2)dx\\
&\quad+\Gamma\int_{\Omega_1\cup\Omega_2}\nabla\overline{\boldsymbol{\theta}}_3\cdot\nabla(\hat{\boldsymbol{\theta}}_3-\overline{\boldsymbol{\theta}}_3)dx\Big]\geq 0,\quad\forall\hat{\boldsymbol{\theta}}\in U_{ad},
\end{aligned}\tag{35}$$

is satisfied.

Theorem 2.5. *Assume that the hypothesis of Theorem 2.2 are valid. Then, considering the norm induced topologies of* $\mathbb{L}^2$, $L^\infty(0,t;\mathscr{L}^2)$, *and* $\mathbb{L}^2\times\mathscr{L}^2$ *we have that the assertions*

(i) The mapping $\boldsymbol{\theta} \mapsto (\mathbf{h}_1,\mathbf{h}_2,c)$ *is continuous from* $U_{ad}\subset\mathbb{L}^2$ *to* $L^\infty(0,t;\mathscr{L}^2)$ *for almost all time* t *in* $]0,T]$.

(ii) The mapping $(\boldsymbol{\theta},\mathbf{h}_1^{obs},\mathbf{h}_2^{obs},c^{obs}) \mapsto (\mathbf{p}_1,\mathbf{p}_2,s)$ *is continuous from* $U_{ad}\times\mathscr{L}^2\subset\mathbb{L}^2\times\mathscr{L}^2$ *to* $L^\infty(0,t;\mathscr{L}^2)$ *for almost all time* t *in* $]0,T]$.

are satisfied.

Theorem 2.6. *Let us define the set*

$$\mathscr{U}_{\mathbf{c}}=\left\{\boldsymbol{\theta}\in\mathscr{U}\ :\ \int_\Omega\boldsymbol{\theta}(x)dx=\mathbf{c},\quad \mathbf{c}=(c_1,\ldots,c_9)\in\mathbb{R}^9_+\right\}$$

with $\mathscr{U}$ *the set defined on Theorem 2.2. Then, for each* $\mathbf{c}$, *the solution of (29) is uniquely defined, up to an additive constant, on* $\mathscr{U}_{\mathbf{c}}$ *in the* $\mathbb{L}^2$ *sense for any large enough regularization parameter* Γ.

3. Proof of Main results

3.1. Proof of Theorem 2.2

We note that the admissible set $\mathscr{U}$ is a nonempty set and the cost function J is bounded on $\mathscr{U}$. To prove that $\mathscr{U}\neq\emptyset$ is enough to select the functions $\boldsymbol{\theta}(x)=(\underline{\mathbf{r}}+\overline{\mathbf{r}})/2$, which clearly is belong to $\mathscr{A}(\Omega_1,\Omega_2)$ and $\mathscr{M}$. The boundedness of J is deduced by the following three facts: the $\mathscr{L}^2$ norm of $(\overline{\mathbf{h}}_1,\overline{\mathbf{h}}_2,\overline{c})(\cdot,T)$ is bounded as consequence of Theorem 2.1 and hypothesis (H0), the hypothesis (H4) and the property that $\nabla\boldsymbol{\theta}\in\mathscr{L}^2$ by the definition of $\mathscr{A}(\Omega_1,\Omega_2)$. Then, we can consider that there exist $\{\boldsymbol{\theta}_n\}\subset\mathscr{U}$ a minimizing sequence of J.

On the other hand, we claim the compact embedding $\mathscr{H}^{[\![d/2]\!]+1}\subset\mathscr{C}^\alpha$ for $\alpha\in]0,1/2]$. Indeed, firstly we observe that $H^{[\![d/2]\!]+1}(\Omega)\subset C^\alpha(\Omega)$ for $\alpha\in]0,1/2]$, for any open convex subset $\Omega\subset\mathbb{R}^d$. This fact can be deduced using two results: the Theorem 6 [24, pp. 270] and the Theorem 1.3.1 [1, pp. 11], which enables the continuous embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^{1/2}(\Omega)$ and the compact embedding $C^{1/2}(\Omega)\subset C^\alpha(\Omega)$ for all $\alpha\in]0,1/2]$, respectively. Then, we can prove our claim by applying the the compact embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^\alpha(\Omega)$. Thus, the claim is proved by using the Cartesian product defining $\mathscr{H}^{[\![d/2]\!]+1}$ and C^α.

The compact embedding $\mathscr{H}^{[\![d/2]\!]+1}\subset\mathscr{C}^\alpha$ for $\alpha\in]0,1/2]$, implies that the minimizing sequence $\{\boldsymbol{\theta}_n\}$ is bounded in the strong topology of $\mathscr{C}^\alpha$ for all $\alpha\in]0,1/2]$, since there exists a positive constant C (independent of $\boldsymbol{\theta}_n$) such that: $\|\boldsymbol{\theta}_n\|_{\mathscr{C}^\alpha}\leq C\|\boldsymbol{\theta}_n\|_{\mathscr{H}^{[\![d/2]\!]+1}}$ for all $\alpha\in]0,1/2]$. Now, we note that $\{\boldsymbol{\theta}_n\}$ is bounded in $\mathscr{H}^{[\![d/2]\!]+1}$ by the definition of $\mathscr{U}$.

Let us denote by $(\mathbf{h}_1,\mathbf{h}_2,c)_n$ the solution of the initial boundary value problem (18)-(23) corresponding to $\boldsymbol{\theta}_n$. Then, by considering the fact that $\boldsymbol{\theta}_n\in\mathscr{C}^\alpha$ for all $\alpha\in]0,1/2]$, by Theorem 2.1, we have that $(\mathbf{h}_1,\mathbf{h}_2,c)_n$ are belong the Hölder space $\mathscr{C}^{2+\alpha,1+\frac{\alpha}{2}}$ and also $\{(\mathbf{h}_1, \mathbf{h}_2, c)_n\}$ is a bounded sequence in the strong topology of $\mathscr{C}^{2+\alpha,1+\frac{\alpha}{2}}$ for all $\alpha \in]0, 1/2]$.

The boundedness of the minimizing sequence and the corresponding sequence $\{(\mathbf{h}_1, \mathbf{h}_2, c)_n\}$, implies that there exist $\overline{\boldsymbol{\theta}} \in \mathscr{C}^{1/2} \cap \mathscr{U}$ and $(\overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c}) \in \mathscr{C}^{2+\frac{1}{2},1+\frac{1}{4}}$ and the subsequences again labeled by $\{\overline{\boldsymbol{\theta}}_n\}$ and $\{(\mathbf{h}_1, \mathbf{h}_2, c)_n\}$ such that $\overline{\boldsymbol{\theta}}_n \to \overline{\boldsymbol{\theta}}$ uniformly on $\mathscr{C}^\alpha$ and $(\mathbf{h}_1, \mathbf{h}_2, c)_n \to (\overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c})$ uniformly on $\mathscr{C}^{2+\alpha,1+\frac{\alpha}{2}}$. Moreover, we can deduce that $(\overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c})$ is the solution of the initial boundary value problem (18)-(23) corresponding to the coefficients $\overline{\boldsymbol{\theta}}$.

Hence, by Lebesgue's dominated convergence theorem, the weak lower-semicontinuity of L^2 norm, and the definition of the minimizing sequence, we have that $J(\overline{\boldsymbol{\theta}}) \le \lim_{n\to\infty} J(\boldsymbol{\theta}_n) = \inf_{\boldsymbol{\theta}\in U_{ad}(\Omega)} J(\boldsymbol{\theta})$. Then, $\overline{\boldsymbol{\theta}}$ is a solution of (29).

3.2. Proof of Theorem 2.3

The fact that $(\mathbf{p}_1, \mathbf{q}_2, s)$ satisfying the system (30)-(34) is the adjoint system for (18)-(23) can be proved by straightforward generalization to systems of the formal calculus presented in [12, 18] for the case of nonlinear scalar parabolic strongly degenerate equation. Now, to get the boundedness behavior of the solution, we observe that is enough to prove the following space estimates

$$\|(\mathbf{p}_1, \mathbf{q}_2, s)(\cdot, t)\|^2_{\mathscr{L}^2(\Omega)} \le P_1, \quad \|\nabla(\mathbf{p}_1, \mathbf{q}_2)(\cdot, t)\|^2_{\mathbf{L}^2(\Omega_1)\times\mathbf{L}^2(\Omega_2)} \le P_2, \tag{36}$$

$$\sum_{i=1}^{2} \|\mathrm{div}(\mathbb{D}_1(x)\nabla \mathbf{p}_i)(\cdot, t)\|^2_{\mathbf{L}^2(\Omega_i)} \le P_3, \quad \|(\mathbf{p}_1, \mathbf{q}_2, s)(\cdot, t)\|_{\mathscr{L}^\infty(\Omega)} \le P_4, \tag{37}$$

for any $t \in [0, T]$ and some positive constants $P_1, \dots, P_4$. Thus, the rest of the proof is focused on getting (36)-(37).

Let us introduce the change of variable $\tau = T - t$ for $t \in [0, T]$ and the notation

$$(\mathbf{v}_1, \mathbf{v}_2, \varrho)(x, \tau) = (\mathbf{p}_1, \mathbf{q}_2, s)(x, T - \tau), \quad (\mathbf{h}_1^*, \mathbf{h}_2^*, c^*)(x, \tau) = (\overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c})(x, T - \tau).$$

Then, the adjoint system (18)-(34) can be rewritten as follows

$$\partial_\tau \mathbf{v}_i - \mathrm{div}(\mathbb{D}_i(x)\nabla \mathbf{v}_i) = \mathbf{q}_i(x, \mathbf{v}_i, \rho; \mathbf{h}_i^*, c^*, \overline{\boldsymbol{\theta}}_i(x)), \qquad \text{in } Q_{i,T}, \quad i = 1, 2, \tag{38}$$

$$\partial_\tau \varrho = \varsigma(x, \mathbf{v}_1, \mathbf{v}_2, \varrho; c^*, \overline{\boldsymbol{\theta}}_3(x)), \qquad \text{in } Q_T, \tag{39}$$

$$(\mathbb{D}_i(x)\nabla \mathbf{v}_i) \cdot \boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1, 2, \tag{40}$$

$$\mathbf{v}_i(x, 0) = \overline{\mathbf{h}}_i(x, T) - \mathbf{h}_i^{obs}(x), \qquad \text{in } \Omega_i, \quad i = 1, 2, \tag{41}$$

$$\varrho(x, 0) = \overline{c}(x, T) - c^{obs}(x), \qquad \text{in } \Omega_1 \cup \Omega_2. \tag{42}$$

Now, in order to prove the estimates of the form (36) for $(\mathbf{p}_1, \mathbf{p}_2, s)$, we get the corresponding estimates for $(\mathbf{v}_1, \mathbf{v}_2, \varrho)$.

Testing the equation (38) by $\mathbf{v}_i$, we have that

$$\int_{\Omega_i} (\mathbf{v}_i)_\tau \cdot \mathbf{v}_i \, dx + \int_{\Omega_i} (\mathbb{D}(x)\mathbf{v}_i) \cdot \mathbf{v}_i^t \, dx = \int_{\Omega_i} \mathbf{q}_i(x, \mathbf{v}_i, s; \mathbf{h}_i^*, c^*, \overline{\boldsymbol{\theta}}_i(x)) \cdot \mathbf{v}_i \, dx, \tag{43}$$

where $\mathbf{v}_i^t$ denotes the transpose of $\mathbf{v}_i$. By the strict positivity of $d_{i,j}$, we have that

$$\begin{aligned}
&\min\left\{\frac{1}{2}, \inf_{\Omega_i} d_{i1}, \inf_{\Omega_i} d_{i2}, \inf_{\Omega_i} d_{i3}\right\}\left[\frac{d}{d\tau}\|\mathbf{v}_i(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)} + \|\nabla\mathbf{v}_i(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)}\right] \\
&\qquad \le \int_{\Omega_1} (\mathbf{v}_i)_\tau \cdot \mathbf{v}_i\, dx + \int_{\Omega_i} (\mathbb{D}(x)\mathbf{v}_i)\cdot \mathbf{v}_i^t\, dx. \qquad (44)
\end{aligned}$$

We observe that the integrands in the right hand side of (43) are homogeneous polynomials of degree two in the components of $\mathbf{v}_i$ with the coefficients are depending on $(\mathbf{h}_1^*, \mathbf{h}_2^*, c^*)$ and $\overline{\boldsymbol{\theta}}_i$. Indeed, for $i=1$ we can rewrite the integrand as follows

$$\begin{aligned}
&\mathbf{q}_1(x,\mathbf{v}_1,s;\mathbf{h}_1^*,c^*,\overline{\boldsymbol{\theta}}_1(x))\cdot\mathbf{v}_1 \\
&= \left\{\left[\overline{\sigma}_{11}\frac{\psi^*(\varphi^*+\psi^*)}{(H_1)^2} + \overline{\sigma}_{31}c^*\right](v_{12}-v_{11}) + \Big(\overline{b}-\overline{m}\Big)v_{11} - \overline{k}\Big(2\varphi^* v_{11} + \psi^* v_{12} + \chi^* v_{13}\Big)\right\} v_{11} \\
&\quad + \left\{\overline{\sigma}_{11}\frac{\varphi^*(\varphi^*+\chi^*)}{(H_1)^2}(v_{12}-v_{11}) + (1-\omega_1\lambda_1+\overline{b})v_{11} - \overline{m}v_{12} + \omega_1\lambda_1(v_{13}-v_{12}) \right. \\
&\qquad\qquad \left. -\overline{k}\Big(\varphi^* v_{11} + 2\psi^* v_{12} + \chi^* v_{13}\Big) + \overline{\sigma}_{13}(1-c^*)\varrho\right\} v_{12} \\
&\quad + \left\{-\overline{\sigma}_{11}\frac{\varphi^*\psi^*}{(H_1)^2}(v_{12}-v_{11}) - \overline{b}v_{11} - \overline{k}\Big(\varphi^* v_{11} + \psi^* v_{12} + 2\chi^* v_{13}\Big)\right\} v_{13} \\
&= \Upsilon_1 v_{11}^2 + \Upsilon_2 v_{12}^2 + \Upsilon_3 v_{13}^2 + \Upsilon_4 v_{11}v_{12} + \Upsilon_5 v_{11}v_{13} + \Upsilon_6 v_{12}v_{13} + \Upsilon_7 v_{12}\varrho, \qquad (45)
\end{aligned}$$

with Υ_i the coefficients defined by

$$\begin{aligned}
&\Upsilon_1 = -\left[\overline{\sigma}_{11}\frac{\psi^*(\varphi^*+\psi^*)}{(H_1)^2} + \overline{\sigma}_{31}c^*\right] + \overline{b} - \overline{m} - 2\overline{k}\varphi^*, \\
&\Upsilon_2 = \overline{\sigma}_{11}\frac{\psi^*(\varphi^*+\chi^*)}{(H_1)^2} - \overline{m} - \omega_1\lambda_1 - 2\overline{k}\psi^*, \qquad \Upsilon_3 = -2\overline{k}\chi^*, \\
&\Upsilon_4 = \left[\overline{\sigma}_{11}\frac{\psi^*(\varphi^*+\psi^*)}{(H_1)^2} + \overline{\sigma}_{31}c^*\right] - \overline{k}\psi^* v_{12} - \overline{\sigma}_{11}\frac{\varphi^*(\varphi^*+\chi^*)}{(H_1)^2} + (1-\omega_1\lambda_1+\overline{b}) - \overline{k}\varphi^*, \\
&\Upsilon_5 = -\overline{k}\chi^* + \overline{\sigma}_{11}\frac{\varphi^*\psi^*}{(H_1)^2} - \overline{b} - \overline{k}\varphi^*, \quad \Upsilon_6 = \omega_1\lambda_1 - \overline{k}\chi^* - \overline{\sigma}_{11}\frac{\varphi^*\psi^*}{(H_1)^2} - \overline{k}\psi^*, \\
&\Upsilon_7 = \overline{\sigma}_{13}(1-c^*).
\end{aligned}$$

Similarly, for $i=2$, we have that

$$\begin{aligned}
&\mathbf{q}_2(x,\mathbf{v}_2,s;\mathbf{h}_2^*,c^*,\overline{\boldsymbol{\theta}}_2(x))\cdot\mathbf{v}_2 \\
&\qquad = -\overline{\sigma}_{32}c^* v_{21}^2 - \varepsilon\lambda_2 v_{22}^2 + \overline{\sigma}_{32}c^* v_{21}v_{22} + \varepsilon\lambda_2 v_{21}v_{23} + \overline{\sigma}_{32}(1-c^*)\varrho v_{22}. \qquad (46)
\end{aligned}$$

Now, from (45) and (46), the Cauchy-Schwarz inequality gives the bound

$$\begin{aligned}
&\int_{\Omega_1} \mathbf{q}_1(x,\mathbf{p}_1,s;\mathbf{h}_1^*,c^*,\overline{\boldsymbol{\theta}}_1(x))\cdot\mathbf{v}_1 dx \\
&\quad \le \left(\|\Upsilon_1\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_4\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_5\|_{L^\infty(\Omega_1)}\right)\|v_{11}\|^2_{L^2(\Omega_1)} \\
&\qquad + \left(\|\Upsilon_2\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_4\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_6\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_7\|_{L^\infty(\Omega_1)}\right)\|v_{12}\|^2_{L^2(\Omega_1)} \\
&\qquad + \left(\|\Upsilon_3\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_5\|_{L^\infty(\Omega_1)} + \frac{1}{2}\|\Upsilon_6\|_{L^\infty(\Omega_1)}\right)\|v_{13}\|^2_{L^2(\Omega_1)}
\end{aligned}$$

$$+\frac{1}{2}\|\Upsilon_7\|_{L^\infty(\Omega_1)}\|\varrho\|^2_{L^2(\Omega_1)}, \tag{47}$$

$$\begin{aligned}
&\int_{\Omega_2}\mathbf{q}_2(x,\mathbf{p}_2,s;\mathbf{h}_2^*,c^*,\overline{\boldsymbol{\theta}}_2(x))\cdot\mathbf{v}_2dx\\
&\quad\le\left(\frac{3}{2}\|\overline{\sigma}_{32}c^*\|_{L^\infty(\Omega_2)}+\frac{\varepsilon\lambda_2}{2}\right)\|\upsilon_{21}\|^2_{L^2(\Omega_2)}+\varepsilon\lambda_2\|\upsilon_{23}\|^2_{L^2(\Omega_2)}\\
&\qquad+\|\overline{\sigma}_{32}(1-c^*)\|_{L^\infty(\Omega_2)}\|\varrho\|^2_{L^2(\Omega_2)}.
\end{aligned}\tag{48}$$

Then adding the equations (43) for $i=1,2$; applying the bounds (44), (47), and (48); and using the facts that $\overline{\boldsymbol{\theta}}_i$ is bounded by definition of U_{ad} and $(\mathbf{h}_1^*,\mathbf{h}_2^*,c^*)$ is bounded by Theorem 2.1, we get that there exists a positive constant Ξ_1 such that

$$\begin{aligned}
&\frac{d}{d\tau}\Big[\|\mathbf{v}_1(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)}+\|\mathbf{v}_2(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_2)}\Big]+\|\nabla\mathbf{v}_1(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)}+\|\nabla\mathbf{v}_2(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_2)}\\
&\qquad\le\Xi_1\Big[\|\mathbf{v}_1(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)}+\|\mathbf{v}_1(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)}+\|\varrho\|^2_{L^2(\Omega_1\cup\Omega_2)}\Big],
\end{aligned}\tag{49}$$

for any $\tau\in[0,T]$. Similarly, we can test the equation (39) by ϱ and obtain the following estimate

$$\frac{d}{d\tau}\|\varrho(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_2)}\le\Xi_2\left[\|\mathbf{v}_1(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)}\|\mathbf{v}_2(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_2)}+\|\varrho(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)}\right]. \tag{50}$$

for any $\tau\in[0,T]$ and some positive constant Ξ_2. Now, summing the inequalities (49) and (50), we deduce that

$$\frac{d}{d\tau}\|(\mathbf{v}_1,\mathbf{v}_2,s)(\cdot,\tau)\|^2_{\mathscr{L}^2}+\|\nabla(\mathbf{v}_1,\mathbf{v}_2)(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_1)\times\mathbf{L}^2(\Omega_2)}\le\underline{C}\|(\mathbf{v}_1,\mathbf{v}_2,s)(\cdot,\tau)\|^2_{\mathscr{L}^2}, \tag{51}$$

for any $\tau\in[0,T]$ and with $\underline{C}=\max\{\Xi_1,\Xi_2\}$. By the Gronwall inequality we get $\|(\mathbf{v}_1,\mathbf{v}_2,s)(\cdot,\tau)\|^2_{\mathscr{L}^2}\le e^{\underline{C}T}\|(\mathbf{v}_1,\mathbf{v}_2,s)(\cdot,0)\|^2_{\mathscr{L}^2}$, which implies the first estimate in (36) with $P_1=e^{\underline{C}T}\|(\mathbf{w},\mathbf{v},s)(\cdot,0)\|^2_{\mathscr{L}^2}$. Moreover, the $\mathscr{L}^2$-estimate used to bound the right hand side of (51), proves the second inequality in (36) with $P_2=\underline{C}P_1$.

On the other hand, using the fact that

$$\begin{aligned}
&\int_{\Omega_i}(\mathbf{v}_i)_\tau\cdot\mathrm{div}(\mathbb{D}_i(x)\nabla\mathbf{v}_i)\,dx\\
&\qquad=-\int_{\Omega_i}\Big(\mathbb{D}_i(x)\nabla[(\mathbf{v}_i)_\tau]\Big)\cdot\nabla\mathbf{v}_i\,d\mathbf{x}+\int_{\partial\Omega_i}(\mathbf{v}_i)_\tau\cdot\Big(\Big[\mathbb{D}_i(x)\nabla(\mathbf{v}_i)\Big]\boldsymbol{\eta}_i^t\Big)\,dS\\
&\qquad\le-\frac{1}{2}\inf_{x\in\Omega_i}\Big\{d_{i1}(x),d_{i2}(x),d_{i3}(x)\Big\}\frac{d}{d\tau}\|\nabla\mathbf{v}_i(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)},
\end{aligned}$$

multiplying (38) by $\mathrm{div}(\mathbb{D}_i(x)\nabla\mathbf{v}_i)$; adding the results for $i=1,2$; and applying the Cauchy-Schwarz inequality with ϵ, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{2}\frac{d}{d\tau}\Big\{\|\nabla\mathbf{v}_i(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)}+\|\mathrm{div}(\mathbb{D}_i(x)\nabla\mathbf{v}_i)(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)}\Big\}\\
&\quad\le C^*\left[\epsilon\sum_{i=1}^{2}\|\mathbf{v}_i(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)}+\epsilon\|\varrho(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)}+\frac{1}{4\epsilon}\sum_{i=1}^{2}\|\Delta\mathbf{w}(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)}\right],
\end{aligned}\tag{52}$$

with C^* a positive constant and any $\epsilon > 0$. Now, selecting $\epsilon > C^*/4$ and using the $\mathscr{L}^2$-estimate, we get

$$\sum_{i=1}^{2} \|\mathrm{div}(\mathbb{D}_i(x)\nabla \mathbf{v}_i)(\cdot,\tau)\|^2_{\mathbf{L}^2(\Omega_i)} \leq \frac{4\epsilon^2 \max\{C_1, C_2\}}{4\epsilon - \max\{C_1, C_2\}} P_1.$$

Thus the first inequality in (37) is proved.

From (36) and the first estimate in (37), we have that the norm of $(\mathbf{p}_1, \mathbf{p}_2)$ is bounded in the norm of $[H^2(\Omega_1)]^3 \times [H^2(\Omega_2)]^3$. Then, by the standard embedding theorem of $H^2 \subset L^\infty$, we easily deduce that $(\mathbf{p}_1, \mathbf{p}_2)$ is bounded in $[L^\infty(\Omega_1)]^3 \times [L^\infty(\Omega_2)]^3$. Moreover, using the equation (39) and the fact that $(\mathbf{p}_1, \mathbf{p}_2)$ are bounded we deduce that ϱ is bounded in $L^\infty(\Omega_1 \cup \Omega_2)$. Thus, we have that the second inequality in (37) is also satisfied and we conclude the proof of theorem.

3.3. Proof of Theorem 2.4

Let us select arbitrarily $\widetilde{\boldsymbol{\theta}} \in U_{ad}$ and introduce the notation

$$\boldsymbol{\theta}^\epsilon = (1-\epsilon)\overline{\boldsymbol{\theta}} + \epsilon\widetilde{\boldsymbol{\theta}} \in U_{ad},$$
$$J_\epsilon = J(\boldsymbol{\theta}^\epsilon) = \frac{1}{2}\|(\mathbf{h}_1^\epsilon, \mathbf{h}_2^\epsilon, c^\epsilon)(\cdot, T) - (\mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, c^{obs})\|^2_{\mathscr{L}^2} + \frac{\Gamma}{2}\|\nabla\boldsymbol{\theta}^\epsilon\|^2_{\mathbb{L}^2}$$

where $(\mathbf{h}_1^\varepsilon, \mathbf{h}_2^\varepsilon, c^\varepsilon)$ is the solution of (18)-(23) with $\boldsymbol{\theta}^\epsilon$ instead of $\boldsymbol{\theta}$. Now, using the hypothesis that $\bar{\boldsymbol{\theta}}$ is an optimal solution of (29) and taking the Fréchet derivative of J_ε, we have that

$$\frac{dJ_\epsilon}{d\epsilon}\Big|_{\epsilon=0} = \sum_{i=1}^{2}\int_{\Omega_i} \Big(\mathbf{h}_i^\epsilon(x,T) - \mathbf{h}_i^{obs}(x)\Big) \cdot \frac{\partial \mathbf{h}_i^\epsilon}{\partial\epsilon}(x,T)\Big|_{\epsilon=0} dx$$
$$+ \int_{\Omega_1\cup\Omega_2} \Big(c^\epsilon(x,T) - c^{obs}(x)\Big) \cdot \frac{\partial c^\epsilon}{\partial \epsilon}(x,T)\Big|_{\epsilon=0} dx + \Gamma\int_{\Omega_1\cup\Omega_2} \nabla\bar{\boldsymbol{\theta}} \cdot \nabla\left(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right) dx \geq 0, \quad (53)$$

where $\partial_\epsilon \mathbf{h}_1^\epsilon$ and $\partial_\epsilon \mathbf{h}_2^\epsilon$ for $\epsilon = 0$ are calculated by analyzing the sensitivities of solutions for (18)-(23) with respect to perturbations of $\boldsymbol{\theta}$.

From the definition of $(\mathbf{h}_1^\epsilon, \mathbf{h}_2^\epsilon, c^\epsilon)$ and $(\bar{\mathbf{h}}_1, \bar{\mathbf{h}}_2, \bar{c})$, we have that

$$\partial_t \mathbf{h}_i^\epsilon - \mathrm{div}(\mathbb{D}_i(x)\nabla\mathbf{h}_i^\epsilon) = \mathbf{f}_i(x, \mathbf{h}_i^\epsilon, c^\epsilon; \boldsymbol{\theta}_i^\epsilon(x)), \qquad \text{in } Q_{i,T}, \quad i = 1,2, \quad (54)$$
$$\partial_t c^\epsilon = g(x, \mathbf{h}_1^\epsilon, \mathbf{h}_2^\epsilon, c^\epsilon; \boldsymbol{\theta}_3^\epsilon(x)), \qquad \text{in } Q_T, \quad (55)$$
$$(\mathbb{D}_i(x)\nabla\mathbf{h}_i^\epsilon)\cdot\boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1,2, \quad (56)$$
$$\mathbf{h}_1^\epsilon(x,0) = (\varphi_0, \psi_0, \chi_0)(x) \qquad \text{in } \Omega_1, \quad (57)$$
$$\mathbf{h}_2^\epsilon(x,0) = (u_0, v_0, w_0)(x) \qquad \text{in } \Omega_2, \quad (58)$$
$$c^\epsilon(x,0) = c_0(x) \qquad \text{in } \Omega_1\cup\Omega_2, \quad (59)$$

and

$$\partial_t \overline{\mathbf{h}}_i - \mathrm{div}(\mathbb{D}_i(x)\nabla\overline{\mathbf{h}}_i) = \mathbf{f}_i(x, \overline{\mathbf{h}}_i, \overline{c}; \boldsymbol{\theta}_i^\epsilon(x)), \qquad \text{in } Q_{i,T}, \quad i = 1,2, \quad (60)$$
$$\partial_t \overline{c} = g(x, \overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c}; \overline{\boldsymbol{\theta}}_3(x)), \qquad \text{in } Q_T, \quad (61)$$
$$(\mathbb{D}_i(x)\nabla\overline{\mathbf{h}}_i)\cdot\boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1,2, \quad (62)$$

$$\overline{\mathbf{h}}_1(x,0) = (\varphi_0, \psi_0, \chi_0)(x) \qquad \text{in } \Omega_1, \tag{63}$$
$$\overline{\mathbf{h}}_2(x,0) = (u_0, v_0, w_0)(x) \qquad \text{in } \Omega_2, \tag{64}$$
$$\overline{c}(x,0) = c_0(x) \qquad \text{in } \Omega_1 \cup \Omega_2. \tag{65}$$

Subtracting the system (60)-(65) from (54)-(59), dividing by ϵ and using the notation $(\mathbf{z}_1, \mathbf{z}_2, \xi) = \epsilon^{-1}\left(\mathbf{h}_1^\varepsilon - \bar{\mathbf{h}}_1, \mathbf{h}_2^\varepsilon - \bar{\mathbf{h}}_2, c^\varepsilon - \bar{c}\right)$, we deduce the system

$$\partial_t \mathbf{z}_i - \operatorname{div}(\mathbb{D}_i(x)\nabla \mathbf{z}_i) = \mathbf{F}_i(x, \mathbf{z}_i, \xi; \overline{\mathbf{h}}_i, \overline{c}, \overline{\boldsymbol{\theta}}_i(x)), \qquad \text{in } Q_{i,T}, \quad i = 1,2, \tag{66}$$
$$\partial_t \xi = \kappa(x, \mathbf{z}_1, \mathbf{z}_2, \xi; \overline{\mathbf{h}}_1, \overline{\mathbf{h}}_2, \overline{c}, \overline{\boldsymbol{\theta}}_3(x)), \qquad \text{in } Q_T, \tag{67}$$
$$(\mathbb{D}_i(x)\nabla \mathbf{z}_i) \cdot \boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1,2, \tag{68}$$
$$\mathbf{z}_1(x,0) = 0, \qquad \text{in } \Omega_1, \tag{69}$$
$$\mathbf{z}_2(x,0) = 0, \qquad \text{in } \Omega_2, \tag{70}$$
$$\xi(x,0) = 0, \qquad \text{in } \Omega_1 \cup \Omega_2, \tag{71}$$

where

$$\begin{aligned}
F_{11} = &-\Big(\hat{\sigma}_{11}(x) - \overline{\sigma}_{11}(x)\Big)\frac{\overline{\varphi}\overline{\psi}}{\overline{H}_1} - \overline{\sigma}_{11}(x)\left\{\frac{\overline{\psi}(\overline{\psi}+\overline{\chi})}{(\overline{H}_1)^2}z_{11} + \frac{\overline{\varphi}(\overline{\varphi}+\overline{\chi})}{(\overline{H}_1)^2}z_{12} - \frac{\overline{\varphi}\overline{\psi}}{(\overline{H}_1)^2}z_{13}\right\}\\
&- \Big(\hat{\sigma}_{31}(x) - \overline{\sigma}_{31}(x)\Big)\overline{c\varphi} - \overline{\sigma}_{31}(x)\Big(\overline{c}z_{12} + \overline{\varphi}\xi\Big) + (1-w_1)\lambda_1 z_{12} + \Big(\hat{b}(x) - \overline{b}(x)\Big)\overline{H}_1\\
&+ \hat{b}(x)(z_{11}+z_{12}+z_{13}) - \Big(\hat{m}(x) - \overline{m}(x)\Big)\overline{\varphi} - \overline{m}(x)z_{11} - \Big(\hat{k}(x) - \overline{k}(x)\Big)\overline{H}_1\overline{\varphi}\\
&- \overline{k}(x)\overline{\varphi}\Big((2\overline{\varphi}+\overline{\psi})z_{11} + \overline{\varphi}z_{12} + \overline{\varphi}z_{13}\Big),\\
F_{12} = &\Big(\hat{\sigma}_{11}(x) - \overline{\sigma}_{11}(x)\Big)\frac{\overline{\varphi}\overline{\psi}}{\overline{H}_1} + \overline{\sigma}_{11}(x)\left\{\frac{\overline{\psi}(\overline{\psi}+\overline{\chi})}{(\overline{H}_1)^2}z_{11} + \frac{\overline{\varphi}(\overline{\varphi}+\overline{\chi})}{(\overline{H}_1)^2}z_{12} - \frac{\overline{\varphi}\overline{\psi}}{(\overline{H}_1)^2}z_{13}\right\}\\
&+ \Big(\hat{\sigma}_{31}(x) - \overline{\sigma}_{31}(x)\Big)\overline{c\varphi} + \overline{\sigma}_{31}(x)\Big(\overline{c}z_{11} + \overline{\varphi}\xi\Big) - \Big(\hat{m}(x) - \overline{m}(x)\Big)\overline{\psi}\\
&- \overline{m}(x)z_2 - \Big(\hat{k}(x) - \overline{k}(x)\Big)\overline{H}_1\overline{\psi} - \overline{k}(x)\overline{\psi}\Big(z_{11} + 2z_{12} + z_{13}\Big),\\
F_{13} = &\, w_1\lambda_1 z_{12} - \Big(\hat{m}(x) - \overline{m}(x)\Big)\overline{\chi} - \overline{m}(x)z_{13} - \Big(\hat{k}(x) - \overline{k}(x)\Big)\overline{H}_1\overline{\chi}\\
&- \overline{k}(x)\overline{\chi}\Big(z_{12} + z_{12} + 2z_{13}\Big),\\
F_{21} = &-\Big(\hat{\sigma}_{32}(x) - \overline{\sigma}_{32}(x)\Big)\overline{cu} - \overline{\sigma}_{32}(x)\Big(\overline{c}z_{21} + \overline{u}\xi\Big),\\
F_{22} = &\Big(\hat{\sigma}_{32}(x) - \overline{\sigma}_{32}(x)\Big)\overline{cu} + \overline{\sigma}_{32}(x)\Big(\overline{c}z_{21} + \overline{u}\xi\Big) - \varepsilon\lambda_2 z_{23},\\
F_{23} = &\, \varepsilon\lambda_2 z_{23},\\
\kappa \;\; = &\left\{\Big(\hat{\sigma}_{13}(x) - \overline{\sigma}_{13}(x)\Big)\hat{\overline{\varphi}} + \Big(\hat{\sigma}_{23}(x) - \overline{\sigma}_{23}(x)\Big)\hat{\overline{v}}\right\}(1-\overline{c}) - \Big(\overline{\sigma}_{13}(x)\hat{\overline{\varphi}} + \overline{\sigma}_{23}(x)\hat{\overline{v}}\Big)\xi\\
&+ \Big(\overline{\sigma}_{13}(x) + \hat{\overline{v}}\Big)(1-\overline{c})z_{22} - \Big[\Big(\hat{\delta}(x) - \overline{\delta}(x)\Big)\overline{c} + \overline{\delta}(x)\xi\Big].
\end{aligned}$$

On the other hand, from (30)-(34) and (66)-(71), we deduce the following identities

$$\begin{aligned}
&\frac{\partial}{\partial t}(\mathbf{p}_1 \cdot \mathbf{z}_1) = \Big(\mathbf{p}_1 \cdot \operatorname{div}(\mathbb{D}_1(x)\nabla \mathbf{z}_1) - \mathbf{z}_1 \cdot \Delta\operatorname{div}(\mathbb{D}_1(x)\nabla \mathbf{p}_1)\Big)\\
&+ \left[(\hat{\sigma}_{11} - \overline{\sigma}_{11})\frac{\overline{\varphi}\,\overline{\psi}}{\overline{H}_1} + (\hat{\sigma}_{31} - \overline{\sigma}_{31})\right](p_{11} - p_{12}) + bH_1 p_{11} - (m + kH_1)\overline{\mathbf{h}}_1 \cdot \overline{\mathbf{p}}_1,
\end{aligned}$$

$$\frac{\partial}{\partial t}\left(\mathbf{p}_2\cdot\mathbf{z}_2\right)=\Big(\mathbf{p}_2\cdot\mathrm{div}(\mathbb{D}_2(x)\nabla\mathbf{z}_1)-\mathbf{z}_2\cdot\Delta\mathrm{div}(\mathbb{D}_2(x)\nabla\mathbf{p}_2)\Big)+(\hat{\sigma}_{32}-\overline{\sigma}_{32})(p_{21}-p_{22}),$$
$$\frac{\partial}{\partial t}\left(s\xi\right)=\Big(\hat{\sigma}_{13}-\overline{\sigma}_{13}\Big)(1-\overline{c})\tilde{\overline{\varphi}}s+\Big(\hat{\sigma_{23}}-\overline{\sigma_{23}}\Big)(1-\overline{c})\tilde{v}-(\hat{\delta}-\overline{\delta})\overline{c}$$

which implies that

$$\begin{aligned}
&\int\int_{Q_{1,T}}\frac{\partial}{\partial t}\left(\mathbf{p}_1\cdot\mathbf{z}_1\right)dxdt+\int\int_{Q_{2,T}}\frac{\partial}{\partial t}\left(\mathbf{p}_2\cdot\mathbf{z}_2\right)dxdt+\int\int_{Q_T}\frac{\partial}{\partial t}\left(s\xi\right)dxdt\\
&=\int\int_{Q_{1,T}}\left\{\left[(\hat{\sigma}_{11}-\overline{\sigma}_{11})\frac{\overline{\varphi}\,\overline{\psi}}{\overline{H}_1}+(\hat{\sigma}_{31}-\overline{\sigma}_{31})\right](p_{11}-p_{12})+bH_1p_{11}-(m+kH_1)\overline{\mathbf{h}}_1\cdot\overline{\mathbf{p}}_1\right\}dxdt\\
&\quad+\int\int_{Q_{2,T}}(\hat{\sigma}_{32}-\overline{\sigma}_{32})(p_{21}-p_{22})dxdt\\
&\quad+\int\int_{Q_T}\left\{\Big(\hat{\sigma}_{13}-\overline{\sigma}_{13}\Big)(1-\overline{c})\tilde{\overline{\varphi}}s+\Big(\hat{\sigma_{23}}-\overline{\sigma_{23}}\Big)(1-\overline{c})\tilde{v}-(\hat{\delta}-\overline{\delta})\overline{c}\right\}dxdt,
\end{aligned}\tag{72}$$

by integration on $Q_{1,T},Q_{2,T}$ and Q_T, respectively. Moreover, we notice that

$$\begin{aligned}
&\int\int_{Q_{1,T}}\frac{\partial}{\partial t}\left(\mathbf{p}_1\cdot\mathbf{z}_1\right)dxdt+\int\int_{Q_{2,T}}\frac{\partial}{\partial t}\left(\mathbf{p}_2\cdot\mathbf{z}_2\right)dxdt+\int\int_{Q_T}\frac{\partial}{\partial t}\left(s\xi\right)dxdt\\
&\quad=\sum_{i=1}^{2}\int_{\Omega_i}(\mathbf{h}_i(x,T)-\mathbf{h}_i^{obs}(x))\cdot\mathbf{z}_i(x,T)dx+\int_{\Omega_1\cup\Omega_2}(c(x,T)-c^{obs}(x))\xi(x,T)dx.
\end{aligned}\tag{73}$$

Then, from (72) and (73) we deduce that

$$\begin{aligned}
&\int\int_{Q_{1,T}}\left\{\left[(\hat{\sigma}_{11}-\overline{\sigma}_{11})\frac{\overline{\varphi}\,\overline{\psi}}{\overline{H}_1}+(\hat{\sigma}_{31}-\overline{\sigma}_{31})\right](p_{11}-p_{12})+bH_1p_{11}-(m+kH_1)\overline{\mathbf{h}}_1\cdot\overline{\mathbf{p}}_1\right\}dxdt\\
&+\int\int_{Q_{2,T}}(\hat{\sigma}_{32}-\overline{\sigma}_{32})(p_{21}-p_{22})dxdt\\
&+\int\int_{Q_T}\left\{\Big(\hat{\sigma}_{13}-\overline{\sigma}_{13}\Big)(1-\overline{c})\tilde{\overline{\varphi}}s+\Big(\hat{\sigma}_{23}-\overline{\sigma}_{23}\Big)(1-\overline{c})\tilde{\overline{v}}-(\hat{\delta}-\overline{\delta})\overline{c}\right\}dxdt\\
&\quad=\sum_{i=1}^{2}\int_{\Omega_i}(\mathbf{h}_i(x,T)-\mathbf{h}_i^{obs}(x))\cdot\mathbf{z}_i(x,T)dx+\int_{\Omega_1\cup\Omega_2}(c(x,T)-c^{obs}(x))\xi(x,T)dx.
\end{aligned}\tag{74}$$

We can conclude the proof of (35) by replacing (74) in (53).

3.4. Proof of Theorem 2.5

3.4.1. Proof of part (i). Let us consider that $(\mathbf{h}_1,\mathbf{h}_2,s)$ and $(\hat{\mathbf{h}}_1,\hat{\mathbf{h}}_2,\hat{s})$ are solutions to the system (18)-(23) with coefficients $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$, respectively. Then, to prove the part *(i)* of the theorem is enough to get that there exist the positive constant Ψ_1 such that the estimates

$$\left\|\Big((\hat{\mathbf{h}}_1,\hat{\mathbf{h}}_2,\hat{s})-(\hat{\mathbf{h}}_1,\hat{\mathbf{h}}_2,\hat{s})\Big)(\cdot,t)\right\|^2_{\mathscr{L}^2}\le\Psi_1\left\|\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right\|^2_{\mathbb{L}^2},\tag{75}$$

holds for any $t\in[0,T]$. Indeed, let us consider the notation $\delta\mathbf{h}_i=\hat{\mathbf{h}}_i-\mathbf{h}_i$, $\delta s=\hat{s}-s$, and $\delta\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}$. Then, by the definition of $(\hat{\mathbf{h}}_1,\hat{\mathbf{h}}_2,\hat{s})$ and $(\mathbf{h}_1,\mathbf{h}_2,s)$, we have that

$(\delta\mathbf{h}_1, \delta\mathbf{h}_2, \delta s)$ is the solution of the system

$$\partial_t(\delta\mathbf{h}_i) - \operatorname{div}(\mathbb{D}_i(x)\nabla(\delta\mathbf{h}_i)) = \delta\mathbf{f}_i(x, \mathbf{h}_i, c; \boldsymbol{\theta}_i(x)), \qquad \text{in } Q_{i,T}, \quad i = 1, 2, \tag{76}$$

$$\partial_t(\delta c) = \delta g(x, \mathbf{h}_1, \mathbf{h}_2, c; \boldsymbol{\theta}_3(x)), \qquad \text{in } Q_T, \tag{77}$$

$$(\mathbb{D}_i(x)\nabla\mathbf{h}_i) \cdot \boldsymbol{\eta}_i = 0, \qquad \text{on } \Gamma_{i,T}, \quad i = 1, 2, \tag{78}$$

$$\delta\mathbf{h}_1(x, 0) = 0 \qquad \text{in } \Omega_1, \tag{79}$$

$$\delta\mathbf{h}_2(x, 0) = 0 \qquad \text{in } \Omega_2, \tag{80}$$

$$\delta c(x, 0) = 0 \qquad \text{in } \Omega_1 \cup \Omega_2, \tag{81}$$

where

$$\begin{aligned}
\delta f_{11} &= -\hat{\sigma}_{11}\left[\frac{\hat{\varphi}\hat{\psi}}{\hat{H}_1} - \frac{\varphi\psi}{H_1}\right] - \delta\sigma_{11}\frac{\varphi\psi}{H_1} - \hat{\sigma}_{31}(\hat{c}\hat{\varphi} - c\varphi) - \delta\sigma_{31}c\varphi + (1-\omega_1)\lambda_1\Big(\hat{\psi} - \psi\Big) \\
&\quad + \hat{b}(\hat{H}_1 - H_1) + \delta b H_1 - \Big((\hat{m} + \hat{k}\hat{H}_1)\hat{\varphi} - (m + kH_1)\varphi\Big) - (\delta m + \delta k H_1)\varphi, \\
\delta f_{12} &= \hat{\sigma}_{11}\left[\frac{\hat{\varphi}\hat{\psi}}{\hat{H}_1} - \frac{\varphi\psi}{H_1}\right] + \delta\sigma_{11}\frac{\varphi\psi}{H_1} + \hat{\sigma}_{31}(\hat{c}\hat{\varphi} - c\varphi) + \delta\sigma_{31}c\varphi - \omega_1\lambda_1\Big(\hat{\psi} - \psi\Big) \\
&\quad - \Big((\hat{m} + \hat{k}\hat{H}_1)\hat{\psi} - (m + kH_1)\psi\Big) - (\delta m + \delta k H_1)\psi, \\
\delta f_{13} &= \omega_1\lambda_1(\hat{\psi} - \psi) - \Big((\hat{m} + \hat{k}\hat{H}_1)\hat{\chi} - (m + kH_1)\chi\Big) - (\delta m + \delta k H_1)\chi, \\
\delta f_{21} &= -\hat{\sigma}_{32}\Big(\hat{c}\hat{u} - cu\Big) - \delta\sigma_{32}cu, \\
\delta f_{22} &= -\hat{\sigma}_{32}\Big(\hat{c}\hat{u} - cu\Big) + \delta\sigma_{32}cu - \varepsilon\lambda_2(\hat{v} - v), \quad \delta f_{23} = 0, \\
\delta g \ &= \hat{\sigma}_{13}\Big((1-\hat{c})\hat{\tilde{\varphi}} -)(1-c)\tilde{\varphi}\Big) + \delta\sigma_{13}(1-c)\tilde{\varphi} + \hat{\sigma}_{23}\Big((1-\hat{c})\hat{\tilde{v}} - (1-c)\tilde{v}\Big) \\
&\quad + \delta\sigma_{23}(1-c)\tilde{v} - \hat{\delta}(\hat{c} - c) - \delta c.
\end{aligned}$$

Moreover, we observe that the following identities

$$\begin{aligned}
&\frac{\hat{\varphi}\hat{\psi}}{\hat{H}_1} - \frac{\varphi\psi}{H_1} = \frac{(\hat{\varphi} + \hat{\psi})\hat{\psi}}{\hat{H}_1 H_1}(\hat{\varphi} - \varphi) + \frac{(\varphi + \hat{\psi})\hat{\varphi}}{\hat{H}_1 H_1}(\hat{\psi} - \psi) - \frac{\varphi\hat{\psi}}{\hat{H}_1 H_1}(\hat{\chi} - \chi), \\
&(\hat{c}\hat{\varphi} - c\varphi) = \hat{\varphi}(\hat{c} - c) + c(\hat{\varphi} - \varphi), \\
&\hat{H}_1 - H_1 = (\hat{\varphi} - \varphi) + (\hat{\psi} - \psi) + (\hat{\chi} - \chi), \\
&(\hat{m} + \hat{k}\hat{H}_1)\hat{\varphi} - (m + kH_1)\varphi = \Big[\hat{m} + \hat{k}(\hat{H}_1 + \varphi)\Big](\hat{\varphi} - \varphi) \\
&\qquad\qquad\qquad + (\hat{k}\varphi)(\hat{\psi} - \psi) + (\hat{k}\varphi)(\hat{\chi} - \chi) + H_1\varphi(\hat{k} - k), \\
&\hat{c}\hat{u} - cu = \hat{c}(\hat{u} - u) + u(\hat{c} - c),
\end{aligned}$$

are satisfied. Now, in order to prove (75), we test the equations (76) and (77) by $\delta\mathbf{h}_i$ and δc, respectively. Then, adding the results and applying the Cauchy inequality, we deduce the following inequality

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\delta\mathbf{h}_1, \delta\mathbf{h}_2, \delta c)(\cdot, t)\|^2_{\mathscr{L}^2} + \sum_{i=1}^{2}\|\sqrt{\mathbb{D}}_i\nabla(\delta\mathbf{h}_i)(\cdot, t)\|^2_{\mathbf{L}^2(\Omega_i)} \\
\leq \overline{C}\Big(\|(\delta\mathbf{h}_1, \delta\mathbf{h}_2, \delta c)(\cdot, t)\|^2_{\mathscr{L}^2} + \|\delta\Theta(\cdot, t)\|^2_{\mathbb{L}^2}\Big),
\end{aligned}$$

with $\sqrt{\mathbb{D}}_i = \mathrm{diag}(\sqrt{d_{i1}}, \sqrt{d_{i2}}, \sqrt{d_{i3}})$ and $\overline{C}$ a positive constant. Then, applying the Gronwall inequality, we deduce that

$$\|(\delta\mathbf{h}_1, \delta\mathbf{h}_2, \delta c)(\cdot,t)\|^2_{\mathscr{L}^2} \leq \exp(\overline{C}T)\|(\delta\mathbf{h}_1, \delta\mathbf{h}_2, \delta c)(\cdot,0)\|^2_{\mathscr{L}^2} + \overline{C}T\|\delta\Theta(\cdot,t)\|^2_{\mathbb{L}^2},$$

which implies (75) by using (79)-(81).

3.4.2. Proof of part (ii). Let us consider the notation $(\mathbf{h}_1, \mathbf{h}_2, s)$, $(\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{s})$, $\delta\mathbf{h}_i$, δs, and $\delta\boldsymbol{\theta}$ introduced in the proof of part *(i)*. Moreover, we consider the notation $(\mathbf{p}_1, \mathbf{p}_2, s)$ and $(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{s})$ to the solution of systems of the form (30)-(34) with coefficients and observations $\boldsymbol{\theta}, \mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, s^{obs}$ and $\hat{\boldsymbol{\theta}}, \hat{\mathbf{h}}_1^{obs}, \hat{\mathbf{h}}_2^{obs}, \hat{s}^{obs}$, respectively. Then, the proof of the part *(ii)* of Theorem 2.5 is reduced to get that there exist two positive constant Ψ_2, Ψ_3 such that the estimate

$$\begin{aligned}
&\|\Big((\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{s}) - (\mathbf{p}_1, \mathbf{p}_2, s)\Big)(\cdot,t)\|^2_{\mathscr{L}^2} \\
&\qquad\qquad \leq \Psi_2\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{\mathbb{L}^2} + \Psi_3\|(\hat{\mathbf{p}}_1^{obs}, \hat{\mathbf{p}}_2^{obs}, \hat{s}^{obs}) - (\mathbf{p}_1^{obs}, \mathbf{p}_2^{obs}, s^{obs})\Big)\|^2_{\mathscr{L}^2}, \qquad (82)
\end{aligned}$$

holds for any $t \in [0,T]$. Indeed, let us consider the notation $\delta\mathbf{p}_i = \hat{\mathbf{p}}_i - \mathbf{p}_i$ and $\delta s = \hat{s} - s$. Then, to prove the inequality (82) we proceed in a similar way to the proof of the part *(i)*, by performing the following three steps, which algebraic details are omitted: (a) we consider the definition of $(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{s})$ and $(\mathbf{p}_1, \mathbf{p}_2, s)$ and deduce a system for $(\delta\mathbf{p}_1, \delta\mathbf{p}_2, \delta s)$. In this new system we rewritten the reactive terms as linear combination of $\delta\mathbf{p}_i, \delta s$ and $\delta\boldsymbol{\theta}$; (b) we test the the new system by $\delta\mathbf{p}_1, \delta\mathbf{p}_2$ and δs to get a estimate of the following type

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|\Big(\delta\mathbf{p}_1, \delta\mathbf{p}_2, \delta s\Big)(\cdot,t)\|^2_{\mathscr{L}^2} &+ \sum_{i=1}^{2}\|\sqrt{\mathbb{D}_i}\nabla\delta\mathbf{p}_i(\cdot,t)\|^2_{\mathbf{L}^2(\Omega_i)} \\
&\leq \tilde{E}_1\|\Big(\delta\mathbf{p}_1, \delta\mathbf{p}_2, \delta s\Big)(\cdot,t)\|^2_{\mathscr{L}^2} + \tilde{E}_2\|\delta\boldsymbol{\theta}\|^2_{\mathbb{L}^2},
\end{aligned}$$

for some positive constants $\tilde{E}_1$ and $\tilde{E}_2$; (c) applying the estimate (75), rearranging some terms; integrating on $[t,T]$; and using the end conditions; we can deduce (82).

3.5. Proof of Theorem 2.6

We prove the uniqueness by using adequately the stability result of Theorem 2.5 and the necessary optimality condition of Theorem 2.4. To be more precise, let us consider that the sets of functions $\{\mathbf{h}_1, \mathbf{h}_2, c, \mathbf{p}_1, \mathbf{p}_2, s\}$ and $\{\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{c}, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{s}\}$ are solutions to the systems (18)-(23) and (30)-(34) with the data $\{\boldsymbol{\theta}, \mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, c^{obs}\}$ and $\{\hat{\boldsymbol{\theta}}, \hat{\mathbf{h}}_1^{obs}, \hat{\mathbf{h}}_2^{obs}, \hat{c}^{obs}\}$, respectively. From Theorem 2.4 and the hypothesis that $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are solutions of (29) we have that the inequalities

$$\begin{aligned}
&\int\int_{Q_{1,T}} \left\{\left[(\overline{\overline{\sigma}}_{11} - \sigma_{11})\frac{\varphi\,\psi}{H_1} + (\overline{\overline{\sigma}}_{31} - \sigma_{31})\right](p_{11} - p_{12}) + bH_1 p_{11} - (m + kH_1)\mathbf{h}_1\cdot\mathbf{p}_1\right\} dxdt \\
&\quad + \int\int_{Q_{2,T}} (\overline{\overline{\sigma}}_{32} - \sigma_{32})(p_{21} - p_{22})dxdt + \int\int_{Q_T} \left\{\Big(\overline{\overline{\sigma}}_{13} - \sigma_{13}\Big)(1-c)\tilde{\varphi}s + \Big(\overline{\overline{\sigma}}_{23} - \sigma_{23}\Big)(1-c)\tilde{v}\right.
\end{aligned}$$

$$- (\overline{\overline{\delta}} - \delta)c\Big\}sdxdt + \delta \int_{\Omega_1} \nabla\boldsymbol{\theta}_1 \cdot \nabla(\overline{\overline{\boldsymbol{\theta}}}_1 - \boldsymbol{\theta}_1)dx + \delta \int_{\Omega_2} \nabla\boldsymbol{\theta}_2 \cdot \nabla(\overline{\overline{\boldsymbol{\theta}}}_2 - \boldsymbol{\theta}_2)dx$$

$$+ \delta \int_{\Omega_1 \cup \Omega_2} \nabla\boldsymbol{\theta}_3 \cdot \nabla(\overline{\overline{\boldsymbol{\theta}}}_3 - \boldsymbol{\theta}_3)dx\Big] \geq 0, \quad \forall \overline{\overline{\boldsymbol{\theta}}} \in U_{ad}, \tag{83}$$

and

$$\int\int_{Q_{1,T}} \left\{ \left[(\underline{\underline{\sigma}}_{11} - \hat{\sigma}_{11}) \frac{\hat{\varphi}\,\hat{\psi}}{\hat{H}_1} + (\underline{\underline{\sigma}}_{31} - \hat{\sigma}_{31}) \right] (\hat{p}_{11} - \hat{p}_{12}) + \hat{b}\hat{H}_1\hat{p}_{11} - (\hat{m} + \hat{k}\hat{H}_1)\hat{\mathbf{h}}_1 \cdot \hat{\mathbf{p}}_1 \right\} dxdt$$

$$+ \int\int_{Q_{2,T}} (\underline{\underline{\sigma}}_{32} - \hat{\sigma}_{32})(\hat{p}_{21} - \hat{p}_{22})dxdt + \int\int_{Q_T} \Big\{ \Big(\underline{\underline{\sigma}}_{13} - \hat{\sigma}_{13}\Big)(1-\hat{c})\tilde{\varphi}s + \Big(\underline{\underline{\sigma}}_{23} - \hat{\sigma}_{23}\Big)(1-\hat{c})\tilde{\hat{v}}$$

$$- (\underline{\underline{\delta}} - \delta)\hat{c}\Big\}sdxdt + \delta \int_{\Omega_1} \nabla\boldsymbol{\theta}_1 \cdot \nabla(\underline{\underline{\boldsymbol{\theta}}}_1 - \hat{\boldsymbol{\theta}}_1)dx + \delta \int_{\Omega_2} \nabla\boldsymbol{\theta}_2 \cdot \nabla(\underline{\underline{\boldsymbol{\theta}}}_2 - \hat{\boldsymbol{\theta}}_2)dx$$

$$+ \delta \int_{\Omega_1 \cup \Omega_2} \nabla\boldsymbol{\theta}_3 \cdot \nabla(\underline{\underline{\boldsymbol{\theta}}}_3 - \hat{\boldsymbol{\theta}}_3)dx\Big] \geq 0, \quad \forall \underline{\underline{\boldsymbol{\theta}}} \in U_{ad}, \tag{84}$$

are satisfied, respectively. In particular, selecting $\overline{\overline{\boldsymbol{\theta}}} = \hat{\boldsymbol{\theta}}$ in (83) and $\underline{\underline{\boldsymbol{\theta}}} = \boldsymbol{\theta}$ in (84), adding both inequalities, rearranging some terms, applying the Cauchy-Schwarz inequality, and Theorem 2.5 we get

$$\begin{aligned}
&\Gamma\|\nabla(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\|^2_{\mathbb{L}^2} \\
&\quad \leq C_1\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{\mathbb{L}^2} + C_2\|(\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{c}) - (\mathbf{h}_1, \mathbf{h}_2, c)\|^2_{\mathscr{L}^2} + C_3\|(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{s}) - (\mathbf{p}_1, \mathbf{p}_2, s)\|^2_{\mathscr{L}^2} \\
&\quad \leq (C_1 + C_2 + C_3)\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2_{\mathbb{L}^2} + C_3\|(\hat{\mathbf{h}}_1^{obs}, \hat{\mathbf{h}}_2^{obs}, \hat{c}^{obs}) - (\mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, c^{obs})\|^2_{\mathscr{L}^2},
\end{aligned} \tag{85}$$

for some positive constants C_1, C_2, C_3. Now, considering that $\hat{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathscr{U}_{\mathbf{c}}$, by the generalized Poincaré inequality, we have that

$$\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\mathscr{L}^2} \leq C_{poi}\Big(\|\nabla(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\|_{\mathscr{L}^2} + \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\mathscr{L}^1}\Big) = C_{poi}\|\nabla(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\|_{\mathscr{L}^2}.$$

Then, in (85) we have that

$$\Big(\Gamma - (C_1 + C_2 + C_3)C_{poi}\Big)\|\nabla(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\|^2_{\mathbb{L}^2} \leq C_3\|(\hat{\mathbf{h}}_1^{obs}, \hat{\mathbf{h}}_2^{obs}, \hat{c}^{obs}) - (\mathbf{h}_1^{obs}, \mathbf{h}_2^{obs}, c^{obs})\|^2_{\mathscr{L}^2}.$$

Thus, selecting $\Gamma^* = (C_1 + C_2 + C_3)C_{poi}$ we deduce the uniqueness up an additive constant.

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