NUMERICAL ANALYSIS OF A THREE-SPECIES CHEMOTAXIS MODEL

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Abstract. A reaction-diffusion system is formulated to describe three interacting species within the Hastings-Powell (HP) food chain structure with chemotaxis produced by three chemicals. The existence of a weak solution is proven and a finite volume (FV) scheme for this system is constructed. In combination with the non-negativity and the a priori estimate, the existence of a discrete solution of the FV scheme is proven, and it is shown that the scheme converges to the corresponding weak solution of the model. The convergence proof uses two ingredients of interest for various applications, namely the discrete Sobolev embedding inequalities with general boundary conditions and a space-time L^1 compactness argument. Finally, numerical tests illustrate the model and the behavior of the FV scheme.

1. Introduction

1.1. **Scope.** We consider a reaction-diffusion system describing three interacting species with respective density $u_i,\ i=1,2,3$ in the Hastings-Powell (HP) food chain structure [16, 20], where each species secretes a chemical substance of concentration y_i $i=1,2,3$, respectively. Each biological species is able to orient their movement towards a higher concentration of the chemical (chemotaxis) or away from it (chemorepulsion). The resulting model is presented as a strongly coupled nonlinear system of six PDEs with chemotactic terms, namely three parabolic equations describing the evolution of the densities u_i coupled with three elliptic equations for the concentrations y_i, $i=1,2,3$:

$$\begin{aligned}
\partial_t u_1 - D_1\Delta u_1 + \chi_1 \operatorname{div}(u_1\nabla y_2) &= F_1(\boldsymbol{u}),\\
\partial_t u_2 - D_2\Delta u_2 + \chi_2 \operatorname{div}(u_2\nabla (y_1-y_3)) &= F_2(\boldsymbol{u}),\\
\partial_t u_3 - D_3\Delta u_3 + \chi_3 \operatorname{div}(u_3\nabla y_2) &= F_3(\boldsymbol{u}),\\
-\mathcal{D}_i\Delta y_i + \theta_i y_i &= \delta_i u_i, \quad i=1,2,3, \quad (\boldsymbol{x},t)\in\Omega\times(0,T],
\end{aligned} \tag{1.1}$$

where $u_i(\boldsymbol{x},t),\ i=1,2,3$ are the population densities of the species at the lowest level of the food chain (prey; $i=1$), of the species that prey upon species 1 (predator, $i=2$), and of species 3 that preys upon species 2 (super-predator, $i=3$), and $\boldsymbol{u}(\boldsymbol{x},t) := (u_1(\boldsymbol{x},t),u_2(\boldsymbol{x},t),u_3(\boldsymbol{x},t))^{\mathrm{T}}$. Moreover, $y_i(\boldsymbol{x},t)$ denotes the concentration of the chemical substance secreted by species i at position $\boldsymbol{x}$ at time t, and $\boldsymbol{y}(\boldsymbol{x},t) = (y_1(\boldsymbol{x},t),y_2(\boldsymbol{x},t),y_3(\boldsymbol{x},t))^{\mathrm{T}}$. The chemotactic movement of the species is due to chemical substances secreted by the other species, which is determined by the sign of the chemotactic coefficients χ_i for $i=1,2,3$ [11]. In this work, we consider that the prey (species 1) moves toward low concentrations of the chemical secreted by species 2 trying to avoid it, which means that $\chi_1<0$, while the super-predator (species 3) moves toward higher concentrations of the chemical secreted by species 2, which means that $\chi_3>0$. On the other hand, the predator (species 2) moves towards the higher concentrations of the chemical secreted by species 1 and towards the low concentrations of the chemical secreted by species 2, such that $\chi_2>0$.

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The interaction due to the competition between the species is specified by the functional responses

$$\begin{aligned} F_1(\boldsymbol{u}) &:= \Big(1-\frac{u_1}{k}\Big)u_1 - \frac{L_2M_2u_1u_2}{R_0+u_1},\\ F_2(\boldsymbol{u}) &:= \frac{L_2M_2u_1u_2}{R_0+u_1} - L_2u_2 - \frac{L_3M_3u_2u_3}{C_0+u_2},\\ F_3(\boldsymbol{u}) &:= \frac{L_3M_3u_2u_3}{C_0+u_2} - L_3u_3 \end{aligned} \tag{1.2}$$

(see [16, 20]). Herein, the constant k is the carrying capacity of species 1, and R_0 and C_0 are the half-saturation densities of u_1 and u_2, respectively. Moreover, L_2 and L_3 are the mass-specific metabolic rates of species 2 and 3, respectively, M_2 is a measure of ingestion rate per unit metabolic rate of species 2, and M_3 denotes the ingestion rate for species 3 on prey. We impose, in addition, the boundary conditions

$$\begin{gathered} \big(\chi_j u_j \nabla y_2 - D_j \nabla u_j\big)\cdot \boldsymbol{n}|_{\partial\Omega} = \big(\chi_2 u_2 \nabla(y_1-y_3) - D_3\nabla u_3\big)\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \quad j=1,2,\\ \nabla y_i\cdot\boldsymbol{n}|_{\partial\Omega}=0, \quad i=1,2,3, \end{gathered} \tag{1.3}$$

where $\boldsymbol{n}$ stands for the outward unit normal vector to $\partial\Omega$, and the initial condition

$$u_i(\boldsymbol{x},0) = u_{i,0}(\boldsymbol{x}), \quad i=1,2,3. \tag{1.4}$$

It is the purpose of this work to prove the existence of weak solutions of the initial-boundary value problem (1.1)–(1.4), and to propose a convergent finite volume (FV) method for their numerical approximation. In addition, we will show numerically the chemotactic movement and the importance of the chemotactic coefficients in the movement of each species, either towards higher concentrations or towards low concentrations. Finally, with the specified numerical parameters, we show that this model exhibits spatial-temporal oscillatory behavior.

1.2. **Related work.** The classical Lotka–Volterra predator-prey model only (e.g [21, vol I]) reflects population changes due to predation in a situation where predator and prey densities are not spatially dependent. Variants of the model have been applied to medicine [25], biology [22], ecology [5, 15, 19, 24, 31], mathematics [21, 30], and other fields. This model does not take into account that population is usually not homogeneously distributed, or that predators and prey naturally develop strategies for survival. Both considerations involve spatial biological movement that is usually described by diffusion. The resulting models can become complicated as different concentration levels of prey and predators cause different population movements. Such movements can be determined by the concentration of the same species (diffusion) or that of other species (cross–diffusion). However, systems of two interacting species can account for only a small number of the phenomena that are commonly exhibited in nature. This limitation is particularly significant in community studies where the essence of the behavior of a complex system may only be understood when the interactions among a large number of species are incorporated; of course, the increasing number of differential equations and the increasing dimensionality raise additional problems.

The dynamics of interacting population with chemotaxis has been investigated by numerous researchers. Lin et al. [17] construct energy functionals, to investigate the asymptotic behavior of solutions under simple choices of parameter. Stability and asymptotic behavior of chemotactic systems with two biological species have been already studied in [28, 29], where the stability of homogeneous steady states is obtained for one chemical substance secreted, while in [11, 23] the authors established the asymptotic behavior and the global existence of solutions for two chemical substance secreted. In [2] a reaction-diffusion model for predator-prey interaction is analyzed, featuring both prey and predator taxis mediated by nonlocal sensing. The analysis is supported by some numerical experiments. On the other hand, Bürger et al. [8] propose and simulate a three-species spatio-temporal predator-prey system with infected prey where the biological movement is not directed by the gradient of a chemical, but rather by a non-local convolution of the density of infected prey that determines a convection term.

Mathematical developments also suggest that models which involve only two species may miss important ecological behavior. Results that are much more complicated than those seen in two-species models appeared in early theoretical studies of three species (e.g [26]) based on local stability analyses. Hastings and Powell

[16] studied the three-species food chain, and among other results they found that there is a "tea-cup" attractor in the system. In [9] the effects of size of forest remnants on trophically linked communities of plants, leaf-mining insects, and their parasitoids were evaluated. The time evolution and spatiotemporal pattern in the Lotka-Volterra model of three interacting species with noise and time delay were investigated by stochastic simulation in [33]. Anaya et al. [3] proposed a convergent semi-implicit FV scheme to describe three interacting species in the food chain structure with nonlocal and cross diffusion. The global existence and boundedness of solutions of the system in bounded domains of arbitrary spatial dimension and small prey-taxis sensitivity coefficients are proved in [32]. The model considered in that work is a reaction-diffusion system with prey taxis that models a two-predator-one-prey ecosystem in which the predators collaboratively take advantage of the prey's strategy.

1.3. **Outline.** The remainder of the paper is organized as follows. Section 2 is concerned with the proof of existence of a weak solution to the continuous problem. Before starting our results concerning the weak solutions, we collect in Section 2.1 some preliminary material, including relevant notation and assumptions on the data of the problem. Next, in Section 2.2 we define a weak solution to the continuous problem, while Section 2.3 is devoted to proving that any weak solution of (1.1)–(1.4) is non-negative. Then, in Section 2.4 we prove existence of a weak solution based on the Schauder fixed-point theorem. Next, in Section 3, we specify the FV method, starting with recalling in Section 3.1 the standard notation of an admissible mesh from [12]. Then, in Section 3.2 we specify the FV scheme to discretize (1.1)–(1.4). Since the scheme is implicit and requires the solution of nonlinear algebraic equations in each time step, we need to demonstrate that the scheme is well defined, that is, that it admits a unique solution in each time step. This is done in Section 4.3, where we first prove (in Section 4.1) that any (discrete) solution produced by the FV scheme is non-negative, and then establish (in Section 4.2) certain a priori L^2 estimates on the discrete solutions. These results allow us to prove in Section 4.3 the existence and uniqueness of a solution for the FV scheme. Section 5 is concerned with the proof of convergence of the FV scheme as the mesh is refined. To this end, we prove in Section 5.1 compactness for discrete solutions (in an appropiate sense) and prove in Section 5.2 that the limit of discrete solutions constitutes a weak solution of (1.1)–(1.4). In Section 6, we provide three numerical examples. Example 1 shows that species interact with each other via chemical substance, while, the Example 2 the prey do not interact by via chemical substance. Finally, the Example 3 compare the dynamics of the spatio-temporal model (1.1)–(1.4), with that of the non-spatial model.

2. Existence of weak solutions

2.1. **Preliminaries.** Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ be a bounded open domain with piecewise smooth boundary $\partial\Omega$. Namely we use standard Lebesgue and Sobolev spaces $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,p}(\Omega)$ and $L^p(\Omega)$ (with their usual norms [1]) for all $m \in \mathbb{N}$ and $p \in [1, \infty]$. We define for $p \in [1, \infty)$ the spaces

$$\mathcal{W}^p := \{u \in W^{2,p}(\Omega) : \nabla u \cdot \boldsymbol{n} = 0\}, \quad (L^p(\Omega))^+ := \left\{u : \Omega \longrightarrow \mathbb{R}_+ \text{ measurable and } \int_\Omega |u(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} < \infty\right\}.$$

Furthermore, for later use, we recall the Sobolev inequalities (see e.g. [7]) $W^{1,p}(\Omega) \hookrightarrow L^\theta$ with $\theta \in (2, +\infty)$ if $d = 2$, and $W^{1,p}(\Omega) \hookrightarrow L^\theta$ with $\theta = (2d)/(d-2) = 6$ if $d = 3$.

If X is a Banach space, $a < b$ and $p \in [1, \infty]$, then $L^p(a, b; X)$ denotes the space of all measurable functions $u : (a, b) \longrightarrow X$ such that $\|u(\cdot)\|_X \in L^p(a, b)$. Next T is a positive number and $\Omega_T := \Omega \times (0, T)$. We define

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} := \begin{pmatrix} y_2 \\ y_1 - y_3 \\ y_2 \end{pmatrix} = \boldsymbol{B}\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1^{\mathrm{T}} \\ \boldsymbol{b}_2^{\mathrm{T}} \\ \boldsymbol{b}_3^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The system (1.1) can then be written as

$$\partial_t u_i - D_i \Delta u_i + \chi_i \operatorname{div}\big(u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y})\big) = F_i(\boldsymbol{u}), \tag{2.1}$$

$$-\mathcal{D}_i \Delta y_i + \theta_i y_i = \delta_i u_i, \quad i = 1, 2, 3, \quad (\boldsymbol{x}, t) \in \Omega_T. \tag{2.2}$$

In matrix form, (2.1), (2.2) can be written as

$$\partial_t \boldsymbol{u} - \mathbf{div}\big(\boldsymbol{D}_1 \nabla \boldsymbol{u} - \mathcal{A}(\boldsymbol{u})\nabla(\boldsymbol{B}\boldsymbol{y}\boldsymbol{\chi}^{\mathrm{T}})\big) = \boldsymbol{F}(\boldsymbol{u}), \quad -\mathbf{div}(\boldsymbol{D}_2\nabla \boldsymbol{y}) + \boldsymbol{\Pi}_1 \boldsymbol{y} = \boldsymbol{\Pi}_2 \boldsymbol{u},$$

where $\boldsymbol{D}_1 := \operatorname{diag}(D_1, D_2, D_3)$, $\mathcal{A}(\boldsymbol{u}) := \operatorname{diag}(u_1, u_2, u_3)$, $\boldsymbol{\chi} := (\chi_1, \chi_2, \chi_3)^{\mathrm{T}}$, $\boldsymbol{D}_2 := \operatorname{diag}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, $\boldsymbol{\Pi}_1 := \operatorname{diag}(\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\Pi}_2 := \operatorname{diag}(\delta_1, \delta_2, \delta_3)$, and $\boldsymbol{F} := (F_1, F_2, F_3)^{\mathrm{T}}$. Furthermore, we assume that $D_i > 0$, $\mathcal{D}_i > 0$, $\theta_i \geqslant 0$, and $\delta_i \geqslant 0$ for $i = 1, 2, 3$.

For later use we note that the particular choice of the functions F_i allows us to write them as

$$F_i(\boldsymbol{u}) = \tilde{F}_i(\boldsymbol{u}) u_i, \tag{2.3}$$

with bounded functions $\tilde{F}_i$, $i = 1, 2, 3$.

2.2. **Weak formulation.** In order to prove the existence and uniqueness of weak solutions, we introduce a weak formulation of (2.1). For each $i = 1, 2, 3$ and a given function $u_i \in L^2(\Omega)$, the elliptic equation

$$-\mathcal{D}_i \Delta y_i + \theta_i y_i = \delta_i u_i \quad \text{in } \Omega, \quad \nabla y_i \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{2.4}$$

admits a unique solution $y_i \in \mathcal{W}^2$. By elliptic regularity [14], $y_i \in \mathcal{W}^2$ and $\|y_i\|_{H^2(\Omega)} \leqslant C\|u_i\|_{L^2(\Omega)}$. Thus, for each $i = 1, 2, 3$ we may define an operator $G_i : L^2 \to \mathcal{W}^2$ by $G_i u_i = y_i$, where y_i denotes the unique solution of (2.4), and the vector $\boldsymbol{G}(\boldsymbol{u}) := (G_1 u_1, G_2 u_2, G_3 u_3)^{\mathrm{T}}$.

Definition 2.1. *A weak solution of* (1.1) *is set of non-negative functions* u_i, $i = 1, 2, 3$, *such that*

$$u_i \in L^2(0, T, H^1(\Omega)), \quad \partial_t u_i \in L^2(0, T, (H^1(\Omega))^*), \quad i = 1, 2, 3,$$

and for all test functions $\xi_i \in L^2(0, T, H^1(\Omega))$, $i = 1, 2, 3$, *the identities*

$$\begin{aligned} &\int_0^T \langle \partial_t u_i, \xi_i \rangle \,\mathrm{d}t + \iint_{\Omega_T} \Big(D_i \nabla u_i \cdot \nabla \xi_i - \chi_i u_i (\nabla \boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u})) \cdot \nabla \xi_i \Big) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t \\ &= \iint_{\Omega_T} F_i \cdot \xi_i \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t, \quad i = 1, 2, 3 \end{aligned} \tag{2.5}$$

are satisfied. Here, $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $H^1(\Omega)$ *and* $(H^1(\Omega))^*$.

2.3. **Non-negativity of weak solutions.**

Lemma 2.1. *Assume that* $u_{i,0} \in L^2(\Omega)$ *with* $u_{i,0} \geqslant 0$ *for* $i = 1, 2, 3$. *Then any solution* u_1, u_2, u_3 *of* (1.1) *is non-negative.*

Proof. For any $a \in \mathbb{R}$ we define $a^+ := \max\{a, 0\}$ and $a^- := -\min\{a, 0\}$, such that $a = a^+ - a^-$, and $\boldsymbol{u}^+ := (u_1^+, u_2^+, u_3^+)^{\mathrm{T}}$. The proof of Lemma 2.1 is then based on the penalized system

$$\partial_t u_i - D_i \Delta u_i + \chi_i \operatorname{div}\big(u_i \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\big) = F_i(\boldsymbol{u}^+) \quad \text{in } \Omega_T,\ i = 1, 2, 3. \tag{2.6}$$

Let us fix $i \in \{1, 2, 3\}$. Multiplying (2.6) by $-u_i^-$ and integrating the result over Ω_T we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i^-\|_{L^2(\Omega)}^2 + D_i \|\nabla u_i^-\|_{L^2(\Omega)}^2 \leqslant |\chi_i| \int_\Omega u_i^- \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u})) \nabla u_i^- \,\mathrm{d}\boldsymbol{x}. \tag{2.7}$$

Now we fix $\nu > N$, and use the Hölder, Gagliardo-Nirenberg, Young, and Sobolev inequalities to get

$$\begin{aligned} \int_\Omega u_i^- \nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u})) \nabla u_i^- \,\mathrm{d}\boldsymbol{x} &\leqslant \|\nabla u_i^-\|_{L^2(\Omega)} \big\|\nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\big\|_{L^\nu(\Omega)} \|u_i^-\|_{L^{(\nu-2)/(2\nu)}(\Omega)} \\ &\leqslant \tilde{C} \|\nabla u_i^-\|_{L^2(\Omega)} \big\|\nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\big\|_{L^\nu(\Omega)} \|\nabla u_i^-\|_{L^2(\Omega)}^{N/\nu} \|u_i^-\|_{L^2(\Omega)}^{(\nu-N)/\nu} \\ &= \tilde{C} \|\nabla u_i^-\|_{L^2(\Omega)}^{(\nu+N)/\nu} \big\|\nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\big\|_{L^\nu(\Omega)} \|u_i^-\|_{L^2(\Omega)}^{(\nu-N)/\nu} \\ &\leqslant \tilde{C} \big\|\nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\big\|_{H^1(\Omega)} \big(\varepsilon \|\nabla u_i^-\|_{L^2(\Omega)}^2 + \hat{C}(\varepsilon) \|u_i^-\|_{L^2(\Omega)}^2\big) \end{aligned} \tag{2.8}$$

for all $\varepsilon > 0$. Choosing $\varepsilon := D_i / (\tilde{C} |\chi_i| \|\nabla(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}))\|_{H^1(\Omega)})$ and inserting (2.8) into (2.7) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i^-\|_{L^2(\Omega)}^2 \leqslant \hat{C}(\varepsilon) \|u_i^-\|_{L^2(\Omega)}^2.$$

By Gronwall's inequality, this inquality implies that

$$\|u_i^-\|_{L^2(\Omega)}^2 \leqslant \hat{C}_1(\varepsilon)\|u_{i,0}^-\|_{L^2(\Omega)}^2. \tag{2.9}$$

The non-negativity of u_i follows from (2.9) and $u_{i,0} \geqslant 0$. This concludes the proof of Lemma 2.1. $\square$

2.4. **The fixed-point method.** We introduce the following closed and convex subset of $L^2(\Omega_T)$:

$$W \coloneqq \{\boldsymbol{u} = (u_1, u_2, u_3)^{\mathrm{T}} \in [L^2(\Omega_T)]^3 : \|u_i\|_X \leqslant C_i,\ i = 1,2,3\}, \tag{2.10}$$

where $X = L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $C_i > 0$, $i = 1,2,3$ are three constants that will be defined below. Now, we introduce the operator $\boldsymbol{S} : W \to [\mathcal{W}^2]^3$ defined by $\boldsymbol{S}(\phi) := \boldsymbol{y}$ for all $\phi \in W$, where $\boldsymbol{y}$ is the unique solution of (2.2). On the other hand, we let $\tilde{\boldsymbol{S}} : W \times [\mathcal{W}^2]^3 \to W$ be the map defined by

$$\tilde{\boldsymbol{S}}(\phi, \boldsymbol{y}) \coloneqq \tilde{\phi} \quad \text{for all } (\phi, \boldsymbol{y}) \in W \times [\mathcal{W}^2]^3,$$

where $\tilde{\phi}$ solves (2.1). Finally, we define a map $\boldsymbol{T} : W \to W$ by $\boldsymbol{T}(\phi) \coloneqq \tilde{\boldsymbol{S}}(\phi, \boldsymbol{S}(\phi))$ for all $\phi \in W$. Finding a solution of (2.5) is equivalent to seeking a fixed point of $\boldsymbol{T}$, that is to finding $\phi \in W$ such that $\boldsymbol{T}(\phi) = \phi$. We prove that $\boldsymbol{T}$ has a fixed point by appealing to the Schauder fixed-point theorem. For the proof we need the following lemma.

Lemma 2.2. *We define the sequence $\{\boldsymbol{u}_l\}_{l\in\mathbb{N}} \subset W$ by $\boldsymbol{u}_1 = \boldsymbol{T}(\boldsymbol{u}_0)$ and $\boldsymbol{u}_l = \boldsymbol{T}(\boldsymbol{u}_{l-1})$ for $l = 1,2,\dots$. Then for each $i = 1,2,3$ the solutions $u_{i,l}$ to system (2.1) satisfy*

- (a) *The sequences $\{u_{i,l}\}_{l\in\mathbb{N}}$ are bounded in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.*
- (b) *The sequences $\{u_{i,l}\}_{l\in\mathbb{N}}$ are relatively compact in $L^2(\Omega_t)$.*

Proof. We fix $i \in 1,2,3$. Testing in (2.1) by $u_{i,l}$ and integrating the resulting equations over Ω yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{i,l}\|_{L^2(\Omega)}^2 + D_1\|\nabla u_{i,l}\|_{L^2(\Omega)}^2 \leqslant |\chi_i| \int_\Omega u_{i,l}\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{u}))\nabla u_{i,l}\,\mathrm{d}\boldsymbol{x} + \int_\Omega F(\boldsymbol{u})u_{i,l}\,\mathrm{d}\boldsymbol{x}.$$

Reasoning in the same way as in (2.8), for all $\varepsilon_i > 0$ we have

$$\int_\Omega u_{i,l}\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{u}))\nabla u_{i,l}\,\mathrm{d}\boldsymbol{x} \leqslant C_{i,1}\|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{u}))\|_{H^1(\Omega)}\big(\varepsilon_i\|\nabla u_{i,l}\|_{L^2(\Omega)}^2 + C(\varepsilon_i)\|u_{i,l}\|_{L^2(\Omega)}^2\big) \tag{2.11}$$

for some constants $C_{i,1} > 0$. Using the assumption (2.3) on F and the non-negativity of $u_{i,l}$ (see Lemma 2.1) we deduce the estimate

$$\int_\Omega F_i(\boldsymbol{u})u_{i,l}\,\mathrm{d}\boldsymbol{x} \leqslant C_{i,2}\|u_{i,l}\|_{L^2(\Omega)}^2. \tag{2.12}$$

From (2.11) and (2.12) and taking ε_i sufficiently small we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{i,l}\|_{L^2(\Omega)}^2 + g_i\|\nabla u_{i,l}\|_{L^2(\Omega)}^2 \leqslant C_{i,3}\|u_{i,l}\|_{L^2(\Omega)}^2 \tag{2.13}$$

with $C_{i,3} > 0$ and $g_i = D_i - C_{i,1}|\chi_i|\|\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{u}))\|_{H^1(\Omega)}\varepsilon_i > 0$. By Gronwall's inequality, (2.13) implies that

$$\sup_{t\in(0,T)}\|u_{i,l}\|_{L^2(\Omega)}^2 \leqslant \exp(C_{i,3}T)\|u_{i,0}\|_{L^2(\Omega)}^2, \tag{2.14}$$

which proves the first part of (a).

From (2.13) and (2.14) one may also conclude that

$$\int_0^T \|\nabla u_{i,l}\|_{L^2(\Omega)}^2\,\mathrm{d}t \leqslant \frac{T}{g_i}\exp(C_{i,3}T)\|u_{i,0}\|_{L^2(\Omega)}^2, \tag{2.15}$$

which yields the second part of (a).

(b) Finally, testing in (2.5) by $\xi_i \in L^2(0,T,H^1(\Omega))$ and using the boundedness of F_i and (2.15) we find that there exists a constant $C_{i,4} > 0$ such that

$$\left|\int_0^T \langle \partial u_{i,l}, \xi_i\rangle\,\mathrm{d}t\right| \leqslant C_{i,4}\|\xi_i\|_{L^2(0,T,H^1(\Omega))}. \tag{2.16}$$

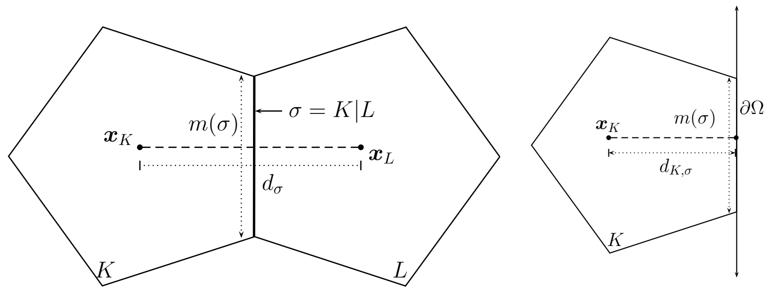

FIGURE 1. Admissible meshes.

Then, (b) is a consequence of (a), the uniform boundedness (2.16). □

Remark 2.1. *We may deduce from* (2.15) *that the constants* C_i *in* (2.10) *can be choosen as follows:*

$$C_i = \left(1 + \frac{T}{g_i}\right) \exp(C_{i,3}T) \|u_{i,0}\|^2_{L^2(\Omega)}, \quad i = 1,2,3.$$

Lemma 2.2 implies that there exist functions $u_i \in L^2(0,T,H^1(\Omega))$ for $i = 1,2,3$ such that up to extracting subsequences if necessary, $u_{i,l} \to u_i \in L^2(\Omega)$ strongly as $l \to \infty$, which in turn implies the continuity of $\boldsymbol{T}$ on W. Furthermore, by Lemma 2.2, $\boldsymbol{T}(W)$ is bounded on the set

$$\Theta \coloneqq \{\boldsymbol{u} \in L^2(0,T,H^1(\Omega)) : \partial_t \boldsymbol{u} \in L^2(0,T,(H^1(\Omega))^*)\} \tag{2.17}$$

In view of [18, Theorem 5.2] we conclude that $\Theta \hookrightarrow [L^2(\Omega_t)]^3$ is compact, thus $\boldsymbol{T}$ is compact. Then, by the Schauder fixed-point theorem, the operator $\boldsymbol{T}$ has a fixed point $\boldsymbol{u} \in W$. Then there exists a solution $\boldsymbol{u}$ of (2.5). Thus, we have proved the following theorem.

Theorem 2.1. *Assume that* $u_{i,0} \in (L^2(\Omega))^+$ *for* $i = 1,2,3$. *Then the problem* (1.1) *possesses a weak solution.*

3. Finite Volume Scheme

3.1. **Admissible mesh.** Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ denote an open bounded polygonal with boundary $\partial\Omega$. An admissible FV mesh of Ω is given by a family $\mathscr{T}$ of control volumes (open and convex polygonal subsets of Ω), a family $\mathcal{E} \subset \bar{\Omega}$ of hyperplanes of $\mathbb{R}^d$ (edges in two-dimensional case or sides in three-dimensional) and a family of points $\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathscr{T}}$ that satisfy

$$\overline{\Omega} = \bigcup_{K\in\mathscr{T}} \overline{K}, \qquad \mathcal{E} = \bigcup_{K\in\mathscr{T}} \mathcal{E}_K, \qquad \partial K = \bigcup_{L\in\mathcal{N}(K)} \overline{\sigma}.$$

Let $K, L \in \mathscr{T}$ with $K \neq L$. If $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, then $\sigma = K|L$ (common edge). We introduce the set of interior (respectively boundary) edges denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), that is $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E} \ : \ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E} \ : \ \sigma \subset \partial\Omega\}$). The set of neighbours of K is given by $\mathcal{N}(K) = \{L \in \mathscr{T} \ : \ \exists\sigma \in \mathcal{E}, \overline{\sigma} = \overline{K} \cap \overline{L}\}$. The family $\mathcal{P}$ is such that for all $K \in \mathscr{T}$, $\boldsymbol{x}_K \in \overline{K}$, and, if $\sigma = K|L$, it is assumed that $\boldsymbol{x}_K \neq \boldsymbol{x}_L$, and that the segment $\overline{\boldsymbol{x}_K\boldsymbol{x}_L}$ is orthogonal to $\sigma = K|L$ (see Figure 1). Let $d_{_{K|L}}$ denote the Euclidean distance between $\boldsymbol{x}_k$ and $\boldsymbol{x}_L$ and by $d_{K,\sigma}$ the distance from $\boldsymbol{x}_K$ to σ. The transmissibility through $\sigma \in \mathcal{E}_{\text{int}}$ is defined by $\tau_{K|L} = m(K|L)/d_{_{K|L}} = m(\sigma)/d_\sigma$ and for $\sigma \in \mathcal{E}_{\text{ext}}$ by $\tau_{K,\sigma} = m(\sigma)/d_{K,\sigma}$. We require local regularity restrictions on the family of meshes $\mathscr{T}_h$; namely

$$\exists\gamma > 0 \ \forall h \ \forall K \in \mathscr{T}_h \ \forall L \in \mathcal{N}(K) : \quad \text{diam}(K) + \text{diam}(L) \leqslant \gamma d_{K,L} \tag{3.1}$$

$$\exists\gamma > 0 \ \forall h \ \forall K \in \mathscr{T}_h \ \forall L \in \mathcal{N}(K) : \quad m(K|L) d_{K,L} \leqslant \gamma m(K). \tag{3.2}$$

A discrete function on the mesh $\mathscr{T}_h$ is a set $(u_K)_{K\in\mathscr{T}_h}$. Whenever convenient, we identify it with the piecewise constant function $u_h \in \Omega$ such that $u_h|_K = u_K$. Finally, the discrete gradient $\nabla_h u_h$ of a constant per control volume function u_h is defined on $\overline{K}\cap\overline{L}$ by

$$\nabla_h u_h \cdot \boldsymbol{n}_{K|L} = \frac{u_L - u_K}{d_{K|L}}. \tag{3.3}$$

3.2. **Finite volume (FV) scheme.** Let $T > 0$. To discretize (1.1) we choose an admissible discretization of $\Omega_T = \Omega \times (0,T)$ consisting of an admissible mesh $\mathscr{T}_h$ of Ω and of a time step $\Delta t_h > 0$; both Δt_h and the size $\max_{K\in\mathscr{T}} \operatorname{diam}(K)$ tend to zero as $h \to 0$.We define $N_h > 0$ as the smallest integer such that $(N_h+1)\Delta t_h \geqslant T$, and set $t_n = n\Delta t_h$ for $n \in \{0,\dots,N_h\}$. Whenever Δt_h is fixed, we will drop the subscript h in the notation.

To formulate the resulting scheme, we introduce the terms

$$\mathcal{A}^{n+1}_{i,K,L} := \mathcal{A}_{i,i} \min\bigl\{(u^{n+1}_{i,K})^+, (u^{n+1}_{i,L})^+\bigr\}, \quad F^{n+1}_{i,K} := F_i\bigl((u^{n+1}_{1,K})^+, (u^{n+1}_{2,K})^+, (u^{n+1}_{3,K})^+\bigr), \quad i = 1,2,3. \tag{3.4}$$

The computation starts from the initial cell averages

$$u^{i,0}_K := \frac{1}{m(K)} \int_K u_{i,0}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad i = 1,2,3. \tag{3.5}$$

We state the FV scheme for (1.1) as follows: find $(u^{n+1}_{i,K})_{K\in\mathscr{T}_h}$, $i = 1,2,3$, such that

$$-\mathcal{D}_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\bigl(y^{n+1}_{i,L} - y^{n+1}_{i,K}\bigr) + \theta_i m(K) y^{n+1}_{i,K} = \delta_i m(K) u^n_{i,K}, \quad i = 1,2,3, \tag{3.6a}$$

$$\begin{aligned} m(K)\frac{u^{n+1}_{i,K} - u^n_{i,K}}{\Delta t} &- D_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\bigl(u^{n+1}_{i,L} - u^{n+1}_{i,K}\bigr) \\ &+ \chi_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\mathcal{A}^{n+1}_{i,K,L}\boldsymbol{b}_i^{\mathrm{T}}\bigl(\boldsymbol{y}^{n+1}_L - \boldsymbol{y}^{n+1}_K\bigr) = m(K)F^{n+1}_{i,K}, \quad i = 1,2,3 \end{aligned} \tag{3.6b}$$

for all $K \in \mathscr{T}_h$ and $n \in \{0,1,\dots,N_h\}$. As usual, homogeneous Neumann boundary conditions are taken into account implicitly. Indeed, the parts of ∂K that lie in $\partial\Omega$ do not contribute to the sums over $L \in \mathcal{N}(K)$ terms, which means that the flux is zero is imposed on the external edge of the mesh.

The sets of values $(u^{n+1}_{1,K}, u^{n+1}_{2,K}, u^{n+1}_{3,K})_{K\in\mathscr{T}_h, n\in\{0,1,\dots,N_h\}}$ satisfying (3.6) will be called a discrete solution. We associate a discrete solution of the scheme at $t = t_{n+1}$ with the triple $\boldsymbol{u}^{n+1}_h = (u^{n+1}_{1,h}, u^{n+1}_{2,h}, u^{n+1}_{3,h})^{\mathrm{T}}$ of the piecewise constant on Ω functions given by

$$u^{n+1}_{1,h}|_K = u^{n+1}_{1,K}, \quad u^{n+1}_{2,h}|_K = u^{n+1}_{2,K}, \quad u^{n+1}_{3,h}|_K = u^{n+1}_{3,K} \quad \text{for all } K \in \mathscr{T}_h \text{ and all } n \in \{0,1,\dots,N_h - 1\}.$$

Furthermore, we define the piecewise constant function

$$\boldsymbol{u}_h(\boldsymbol{x},t) = \bigl(u_{1,h}(\boldsymbol{x},t), u_{2,h}(\boldsymbol{x},t), u_{3,h}(\boldsymbol{x},t)\bigr)^{\mathrm{T}} := \sum_{\substack{K\in\mathscr{T}_h\\ n\in\{0,1,\dots,N_h\}}} \mathbb{1}_{(t_n,t_{n+1}]\times K}\boldsymbol{u}^{n+1}_K.$$

Finally, it is assumed that Δt satisfies the mild restriction

$$\Delta t < \max\left\{\frac{1}{2}, \frac{1}{2L_2M_2}, \frac{1}{2L_3M_3}\right\}, \tag{3.7}$$

which will be used to prove the existence of solution to the scheme.

4. Existence of a solution for the finite volume scheme

4.1. **Non-negativity.** The non-negativity of any (discrete) solution produced by the FV scheme is given in the following lemma.

Lemma 4.1. *Any solution $\boldsymbol{u}^{n+1} = (u^{n+1}_{1,K}, u^{n+1}_{2,K}, u^{n+1}_{3,K})^{\mathrm{T}}$, $K \in \mathscr{T}_h$, $n \in \{0,1,\dots,N_h\}$ of (3.6) is non-negative.*

Proof. We fix $i \in \{1,2,3\}$ and prove by induction that $\min\{u_{i,K}^{n+1} : K \in \mathscr{T}_h\} \geqslant 0$ for all $n \in \{0,1,\dots,N_h\}$. To this end, we recall that $u_{i,K}^0 \geqslant 0$ for all $K \in \mathscr{T}_h$. For $n \geqslant 0$, we fix $K \in \mathscr{T}_h$ such that $u_{i,K}^{n+1} \coloneqq \min\{u_L^{n+1} : L \in \mathscr{T}_h\}$. Multiplying (3.6b) by $\Delta t (u_{i,K}^{n+1})^-$, we deduce

$$m(K)\big(u_{i,K}^{n+1} - u_{i,K}^n\big)(u_{i,K}^{n+1})^- = S_1 + S_2 + S_3, \tag{4.1}$$

where we define

$$S_1 \coloneqq \Delta t D_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\big(u_{i,L}^{n+1} - u_{i,K}^{n+1}\big)(u_{i,K}^{n+1})^-, \quad S_2 \coloneqq \Delta t \chi_i \sum_{L\in\mathcal{N}(K)} \tau_{K|L}\mathcal{A}_{i,K,L}^{n+1}\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}_K^{n+1} - \boldsymbol{y}_L^{n+1}\big)(u_{i,K}^{n+1})^-,$$
$$S_3 \coloneqq \Delta t\, m(K) F_{i,K}^{n+1}(u_{i,K}^{n+1})^-.$$

By the choice of K, we have $S_1 \geqslant 0$, and the choice (3.4) of $\mathcal{A}_{i,K,L}^{n+1}$ implies that $S_2 = 0$. Similarly, by the definition of $F_{i,K}^{n+1}$, we obtain

$$S_3 = \Delta t\, m(K)\tilde{F}_i\big((\boldsymbol{u}_K^{n+1})^+\big)(u_{i,K}^{n+1})^+(u_{i,K}^{n+1})^- = 0.$$

Since $m(K) > 0$, (4.1) now means that

$$\big(u_{i,K}^{n+1} - u_{i,K}^n\big)(u_{i,K}^{n+1})^- \geqslant 0. \tag{4.2}$$

By definition, $(u_{i,K}^{n+1})^- \geqslant 0$. Since $u_{i,K}^n \geqslant 0$, (4.2) cannot be satisfied for $(u_{i,K}^{n+1})^- > 0$ (and therefore $u_{i,K}^{n+1} < 0$). We conclude that $(u_{i,K}^{n+1})^- = 0$, hence $u_{i,K}^{n+1} \geqslant 0$. By induction in n, we infer that $u_{i,L}^{n+1} \geqslant 0$ for all $n \in \{0,1,\dots,N_h\}$ and $L \in \mathscr{T}$. $\square$

4.2. **A priori estimates.**

Lemma 4.2. *Let* $(\boldsymbol{u}_K^{n+1})_{K\in\mathscr{T}_h, n\in\{0,1,\dots,N_h\}}$ *be a solution of* (3.6b). *Then there are constants* $C_i > 0$, $i = 1,2,3$ *depending on* Ω, T, $\|u_{i,0}\|_{L^2(\Omega)}$ *for* $i = 1,2,3$, L_j, M_j *for* $j = 2,3$, R_0, *and* C_0 *such that*

$$\sum_{i=1}^3 \left(\max_{n\in\{0,1,\dots,N_h\}} \sum_{K\in\mathscr{T}_h} m(K)\big|u_{i,K}^{n+1}\big|^2 \right) \leqslant C_1, \tag{4.3}$$

$$\frac{1}{2}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \tau_{K|L} \sum_{i=1}^3 D_i\big|u_{i,L}^{n+1} - u_{i,K}^{n+1}\big|^2 \leqslant C_2, \tag{4.4}$$

$$\frac{1}{2}\sum_{n=0}^{N_h} \Delta t \sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \tau_{K|L} \sum_{i=1}^3 |\chi_i|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_K^{n+1}\big|u_{i,L}^{n+1} - u_{i,K}^{n+1}\big|^2 \leqslant C_3. \tag{4.5}$$

Proof. We multiply (3.6b) by $\Delta t u_{i,K}^{n+1}$ and sum the result over $i = 1,2,3$, $K \in \mathscr{T}_h$, and n. This yields an identity $T_1 + T_2 + T_3 + T_4 = 0$, where

$$T_1 \coloneqq \sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^3\big(u_{i,K}^{n+1} - u_{i,K}^n\big)u_{i,K}^{n+1},$$

$$T_2 \coloneqq -\Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \tau_{K|L}\sum_{i=1}^3 D_i\big(u_{i,L}^{n+1} - u_{i,K}^{n+1}\big)u_{i,K}^{n+1},$$

$$T_3 \coloneqq \Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \tau_{K|L}\sum_{i=1}^3 \chi_i\mathcal{A}_{i,K,L}^{n+1}\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}_L^{n+1} - \boldsymbol{y}_K^{n+1}\big)u_{i,K}^{n+1},$$

$$T_4 = -\Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^3 F_{i,K}^{n+1}u_{i,K}^{n+1}.$$

Using the inequality $a(a-b) \geqslant \frac{1}{2}(a^2-b^2)$ for $a,b\in\mathbb{R}$, we obtain

$$T_1 \geqslant \frac{1}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}\left(\left|u_{i,K}^{n+1}\right|^2-\left|u_{i,K}^{n}\right|^2\right)=\frac{1}{2}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}\left(\left|u_{i,K}^{N_h+1}\right|^2-\left|u_{i,K}^{0}\right|^2\right).$$

Reordering over the set edges, we can write

$$T_2=\frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}D_i\left(u_{i,L}^{n+1}-u_{i,K}^{n+1}\right)^2.$$

Next, using summation by parts and the definition of $\mathcal{A}_{i,K,L}^{n+1}$, $i=1,2,3$, we get

$$\begin{aligned}
T_3 &\geqslant -\frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}|\chi_i|\mathcal{A}_{i,K,L}^{n+1}\left(u_{i,L}^{n+1}-u_{i,K}^{n+1}\right)\boldsymbol{b}_i^{\mathrm{T}}\left(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1}\right)\\
&= -\Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}|\chi_i|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_K^{n+1}\mathcal{A}_{i,K,L}^{n+1}\left(u_{i,K}^{n+1}-u_{i,L}^{n+1}\right)\\
&\geqslant \frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}|\chi_i|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_K^{n+1}\left(u_{i,L}^{n+1}-u_{i,K}^{n+1}\right)^2.
\end{aligned}$$

Finally, the non-negativity of $u_{i,K}^{n+1}$ implies that

$$\begin{aligned}
T_4 &= -\Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\Bigg\{\left(\left(1-\frac{u_{1,K}^{n+1}}{k}\right)u_{1,K}^{n+1}-\frac{L_2M_2u_{1,K}^{n+1}u_{2,K}^{n+1}}{R_0+u_{1,K}^{n+1}}\right)u_{1,K}^{n+1}\\
&\quad -\left(\frac{L_2M_2u_{1,K}^{n+1}u_{2,K}^{n+1}}{R_0+u_{1,K}^{n+1}}-L_2u_{2,K}^{n+1}-\frac{L_3M_3u_{2,K}^{n+1}u_{3,K}^{n+1}}{C_0+u_{2,K}^{n+1}}\right)u_{2,K}^{n+1}-\left(\frac{L_3M_3u_{2,K}^{n+1}u_{3,K}^{n+1}}{C_0+u_{2,K}^{n+1}}-L_3u_{3,K}^{n+1}\right)u_{3,K}^{n+1}\Bigg\}\\
&\geqslant -\Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\Big(\left(u_{1,K}^{n+1}\right)^2+L_2M_2(u_{2,K}^{n+1})^2+L_3M_3\left(u_{3,K}^{n+1}\right)^2\Big).
\end{aligned}$$

Collecting the previous inequalities we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}\left(\left|u_{i,K}^{N_h+1}\right|^2-\left|u_{i,K}^{0}\right|^2\right)+\frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}D_i\left(u_{i,L}^{n+1}-u_{i,K}^{n+1}\right)^2\\
&+\frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}|\chi_i|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_K^{n+1}\left(u_{i,L}^{n+1}-u_{i,K}^{n+1}\right)^2 \qquad (4.6)\\
&\leqslant \Delta t\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} m(K)\Big(\left(u_{1,K}^{n+1}\right)^2+L_2M_2(u_{2,K}^{n+1})^2+L_3M_3\left(u_{3,K}^{n+1}\right)^2\Big).
\end{aligned}$$

In view of the discrete Gronwall inequality, (4.3) follows from (4.6). Consequently, (4.6) entails the estimates (4.4) and (4.5). This concludes the proof. ☐

Lemma 4.3. *Let* $(\boldsymbol{y}_K^{n+1})_{K\in\mathscr{T}_h,n\in\{0,1,\dots,N_h\}}$ *be a solution of* (3.6a). *Then, there are constants* $C_4,C_5>0$ *depending on* Ω, T, $\|u_{i,0}\|_{L^2(\Omega)}$ *for* $i=1,2,3$, L_j, M_j *for* $j=2,3$, R_0, *and* C_0 *such that*

$$\sum_{i=1}^{3}\left(\max_{n\in\{0,1,\dots,N_h\}}\sum_{K\in\mathscr{T}_h} m(K)\left(y_{i,K}^{n+1}\right)^2\right)\leqslant C_4, \qquad (4.7)$$

$$\frac{\Delta t}{2}\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}\mathcal{D}_i\left(y_{i,L}^{n+1}-y_{i,K}^{n+1}\right)^2\leqslant C_5. \qquad (4.8)$$

Proof. We fix $i \in \{1,2,3\}$, multiply (3.6a) by $y_{i,K}^{n+1}$, and sum the result over $K \in \mathscr{T}_h$ to obtain

$$-\mathcal{D}_i \sum_{K\in\mathscr{T}_h} \sum_{L\in\mathcal{N}(K)} \tau_{K|L} \big(y_{i,L}^{n+1} - y_{i,K}^{n+1}\big) y_{i,K}^{n+1} + \theta_i \sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 = \delta_i \sum_{K\in\mathscr{T}_h} m(K) u_{i,K}^n y_{i,K}^{n+1}. \tag{4.9}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\frac{1}{2} \sum_{K\in\mathscr{T}_h} \sum_{L\in\mathcal{N}(K)} \tau_{K|L} \mathcal{D}_i \big(y_{i,L}^{n+1} - y_{i,K}^{n+1}\big)^2 + \theta_i \sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \\ &\leqslant \delta_i \left(\sum_{K\in\mathscr{T}_h} m(K) \big(u_{i,K}^n\big)^2 \right)^{1/2} \left(\sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \right)^{1/2}. \end{aligned} \tag{4.10}$$

From (4.10) we deduce

$$\delta_i \sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \leqslant \delta_i \left(\sum_{K\in\mathscr{T}_h} m(K) \big(u_{i,K}^n\big)^2 \right)^{1/2} \left(\sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \right)^{1/2}.$$

Considering the estimates for all $i = 1,2,3$, we may deduce (4.7) from (4.3). To get the discrete $L^2(\Omega)$ estimate for $\nabla_h y_{i,h}$ we use generalized Young's inequality and gathering by edges in (4.9) to obtain

$$\begin{aligned} &\frac{1}{2} \sum_{K\in\mathscr{T}_h} \sum_{L\in\mathcal{N}(K)} \tau_{K|L} \mathcal{D}_i \big(y_{i,L}^{n+1} - y_{i,K}^{n+1}\big)^2 + \theta_i \sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \\ &\leqslant \delta_i \left(C(\varepsilon) \sum_{K\in\mathscr{T}_h} m(K) \big(u_{i,K}^n\big)^2 + \varepsilon \sum_{K\in\mathscr{T}_h} m(K) \big(y_{i,K}^{n+1}\big)^2 \right) \end{aligned}$$

for all $\varepsilon > 0$. Taking $\varepsilon = \theta_i/\delta_i$ we have

$$\frac{1}{2} \sum_{K\in\mathscr{T}_h} \sum_{L\in\mathcal{N}(K)} \tau_{K|L} \mathcal{D}_i \big(y_{i,L}^{n+1} - y_{i,K}^{n+1}\big)^2 \leqslant \delta_i C(\varepsilon) \sum_{K\in\mathscr{T}_h} m(K) \big(u_{i,K}^n\big)^2.$$

Again, considering the estimates for all $i = 1,2,3$ we may deduce (4.8) from (4.3). □

4.3. **Existence and uniqueness of a solution for the finite volume scheme.** The following theorem shows that the scheme (3.6a) is well defined, and we prove the non-negativity of $y_{i,K}^{n+1}$ for $i = 1,2,3$.

Theorem 4.1. *Let $\mathcal{T}$ be an admissible discretization of Ω_T and the initial data (3.5). Then there exists a unique non-negative solution $\boldsymbol{y}_K^{n+1} = (y_{1,K}^{n+1}, y_{2,K}^{n+1}, y_{3,K}^{n+1})^{\mathrm{T}}$ for all $K \in \mathscr{T}_h$ and $n \in \{0,1,\dots,N_h\}$ to (3.6a).*

Proof. Utilizing an argument similar to that of [6, Section 3], we rewrite (3.6a) as the linear system

$$\boldsymbol{A}_i^{n+1} \boldsymbol{y}_i^{n+1} = \boldsymbol{R}_i \boldsymbol{u}_i^n, \quad \text{where } \boldsymbol{y}_i^n := (y_{i,K}^n)_{K\in\mathscr{T}} \text{ and } \boldsymbol{u}_i^n := (u_{i,K}^n)_{K\in\mathscr{T}},\ i = 1,2,3,$$

with the matrices

$$\boldsymbol{A}_i^{n+1} := (a_{iK,L}^{n+1})_{K,L\in\mathscr{T}} = \begin{cases} \theta_i m(K) + \sum_{L\in\mathcal{N}(K)} \mathcal{D}_i \tau_{K|L} & \text{for } K = L, \\ -\mathcal{D}_i \tau_{K|L} & \text{for } K \neq L, \end{cases},$$

$$\boldsymbol{R}_i := (r_{i,K,L})_{K,L\in\mathscr{T}} = \begin{cases} \delta_i m(K) & \text{for } K = L, \\ 0 & \text{for } K \neq L, \end{cases} \quad i = 1,2,3.$$

Since for all $L \in \mathscr{T}$ and $i = 1,2,3$,

$$\big|a_{i,L,L}^{n+1}\big| - \sum_{K\neq L} \big|a_{i,K,L}^{n+1}\big| = \theta_i m(L) + \sum_{L\in\mathcal{N}(K)} \mathcal{D}_i \tau_{K|L} - \sum_{L\in\mathcal{N}(K)} \mathcal{D}_i \tau_{K|L} = \theta_i m(L) > 0,$$

the matrix $\boldsymbol{A}_i^{n+1}$ is strictly dominant with respect to the columns and hence $(\boldsymbol{A}_i^{n+1})^{-1}$ is positive. Finally, under the assumption $u_i^0 > 0$ for $i = 1,2,3$ we obtain $y_{i,K}^n \geqslant 0$ for $i = 1,2,3$ and all $n \in \mathbb{N}$. □

Theorem 4.1 showed the existence and uniqueness of discrete solution of the FV scheme (3.6a). The following Theorem shows the existence for (3.6b).

Theorem 4.2. *Let $\mathcal{T}$ be admissible discretization of Ω. Then the discrete problem admits at least one solution $(u_{i,K}^{n+1})_{K\in\mathscr{T}_h, n\in\{0,1,\dots,N_h\}}$ for $i=1,2,3$.*

Proof. First we introduce the Hilbert space $H_h \coloneqq X_h\times X_h\times X_h$ of triples $\boldsymbol{u}_h^{n+1} = (u_{1,h}^{n+1}, u_{2,h}^{n+1}, u_{3,h}^{n+1})^{\mathrm{T}}$ of discrete functions on Ω. Here, we denote by $X_h\subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume K. We define the norm

$$\|\boldsymbol{u}_h^{n+1}\|_{H_h}^2 \coloneqq \sum_{i=1}^{3}\left(|u_{i,h}^{n+1}|_{X_h}^2 + \|u_{i,h}^{n+1}\|_{L^2(\Omega)}^2\right),$$

where the discrete seminorm $|\cdot|_{X_h}^2$ is given by

$$|w_h|_{X_h}^2 \coloneqq \frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)d_{K,L}\left|\frac{w_L-w_K}{d_{K,L}}\right|^2$$

We introduce the discrete operator $\boldsymbol{G}_h : H_h\to H_h$ defined by $\boldsymbol{G}_h(\boldsymbol{u}_h)\coloneqq \boldsymbol{y}_h$, and let $\boldsymbol{\Psi}_h = (\psi_{1,h},\psi_{2,h},\psi_{3,h})^{\mathrm{T}}$. Multiplying (3.6b) by $\psi_{i,h}$ and summing the result over $K\in\mathscr{T}_h$ we obtain

$$\begin{aligned}&\frac{1}{\Delta t}(\boldsymbol{B}_h(\boldsymbol{u}_h^{n+1},\boldsymbol{\Psi}_h^{n+1}) - \boldsymbol{B}_h(\boldsymbol{u}_h^{n},\boldsymbol{\Psi}_h^{n+1}))\\ &+\boldsymbol{a}_{1,h}(\boldsymbol{u}_h^{n+1},\boldsymbol{\Psi}_h^{n+1}) + \boldsymbol{a}_{2,h}(\boldsymbol{G}_h(\boldsymbol{u}_h^{n+1}),\boldsymbol{\Phi}_h^{n+1}) - \boldsymbol{B}_h(\boldsymbol{F}_h(\boldsymbol{u}_h^{n+1}),\boldsymbol{\Psi}_h^{n+1}) = 0,\end{aligned}$$

where the discrete bilinear forms are given by

$$\begin{aligned}\boldsymbol{B}_h(\boldsymbol{u}_h^{n+1},\boldsymbol{\Psi}_h^{n+1}) &\coloneqq \sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3} u_{i,K}^{n+1}\psi_{i,K}^{n+1}\\ \boldsymbol{a}_{1,h}(\boldsymbol{u}_h^{n+1},\boldsymbol{\Psi}_h^{n+1}) &\coloneqq \frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3} D_i(u_{i,L}^{n+1}-u_{i,K}^{n+1})(\psi_{i,L}^{n+1}-\psi_{i,K}^{n+1})\\ \boldsymbol{a}_{2,h}(\boldsymbol{G}_h(\boldsymbol{u}_h^{n+1}),\boldsymbol{\Psi}_h^{n+1}) &\coloneqq -\frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}\chi_i\mathcal{A}_{i,K,L}^{n+1}\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1}\big)(\psi_{i,L}^{n+1}-\psi_{i,K}^{n+1}).\end{aligned}$$

Now, we define, by duality, the mapping $\mathbb{P}$ from H_h into itself: for all $\boldsymbol{\Phi}_h\in H_h$

$$\begin{aligned}[\mathbb{P}(\boldsymbol{u}_h^{n+1}),\boldsymbol{\Phi}_h^{n+1}] = &\frac{1}{\Delta t}(\boldsymbol{B}_h(\boldsymbol{u}_h^{n+1},\boldsymbol{\Phi}_h^{n+1}) - \boldsymbol{B}_h(\boldsymbol{u}_h^{n},\boldsymbol{\Phi}_h^{n+1})) + \boldsymbol{a}_{1,h}(\boldsymbol{u}_h^{n+1},\boldsymbol{\Phi}_h^{n+1})\\ &+\boldsymbol{a}_{2,h}(\boldsymbol{G}_h(\boldsymbol{u}_h^{n+1}),\boldsymbol{\Psi}_h^{n+1}) - \boldsymbol{B}_h(\boldsymbol{F}_h^{n+1},\boldsymbol{\Phi}_h^{n+1}),\end{aligned}$$

where $\boldsymbol{F}_h^{n+1}\coloneqq (F_{i,h}^{n+1},F_{2,h}^{n+1},F_{3,h}^{n+1})^{\mathrm{T}}$. The continuity of $\mathbb{P}$ follows from that of the nonlinearities $\boldsymbol{F}_h^{n+1}$, $\mathcal{A}_h^{n+1}$ and of $a_{1,h}(\cdot,\cdot)$, $a_{2,h}(\cdot,\cdot)$ and $\boldsymbol{B}_h(\cdot,\cdot)$.

Now, using the definition of $F_{i,K}^{n+1}$, for $i=1,2,3$ and the Young's inequality we deduce

$$\begin{aligned}&[\mathbb{P}(\boldsymbol{u}_h^{n+1}),\boldsymbol{u}_h^{n+1}]\\ &\geqslant \frac{1}{\Delta t}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}(u_{i,K}^{n+1})^2 - \frac{1}{\Delta t}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}u_{i,K}^{n}u_{i,K}^{n+1} + \frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}D_i(u_{i,L}^{n+1}-u_{i,K}^{n+1})^2\\ &\quad+\frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}|\chi_i|\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_K^{n+1}(u_{i,L}^{n+1}-u_{i,K}^{n+1})^2\\ &\quad-\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}\big((u_{1,K}^{n+1})^2 + L_2M_2(u_{2,K}^{n+1})^2 + L_3M_3(u_{3,K}^{n+1})^2\big)\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{1}{2\Delta t}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}(u_{i,K}^{n+1})^2 - \frac{1}{2\Delta t}\sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}(u_{i,K}^{n})^2 + \frac{1}{2}\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\sum_{i=1}^{3}D_i(u_{i,L}^{n+1}-u_{i,K}^{n+1})^2\\
&\quad - \sum_{K\in\mathscr{T}_h} m(K)\sum_{i=1}^{3}\big((u_{1,K}^{n+1})^2 + L_2M_2(u_{2,K}^{n+1})^2 + L_3M_3(u_{3,K}^{n+1})^2\big)\\
&= \left(\frac{1}{2\Delta t}-1\right)\|u_{1,h}^{n+1}\|_{L^2(\Omega)}^2 + D_1|u_{1,h}^{n+1}|_{X_h} + \left(\frac{1}{2\Delta t}-L_2M_2\right)\|u_{2,h}^{n+1}\|_{L^2(\Omega)}^2 + D_2|u_{2,h}^{n+1}|_{X_h}\\
&\quad + \left(\frac{1}{2\Delta t}-L_3M_3\right)\|u_{3,h}^{n+1}\|_{L^2(\Omega)}^2 + D_3|u_{3,h}^{n+1}|_{X_h} - C_1(\Delta t,\boldsymbol{u}_h^n)\\
&\geqslant C_2\left(\|\boldsymbol{u}_h^{n+1}\|_{H_h}^2\right) - C_1(\Delta t,\boldsymbol{u}_h^n).
\end{aligned}
$$

The constant C_2 depends on D_i, L_j, M_j, and Δt, for $i=1,2,3$ and $j=2,3$, Moreover, due to (3.7), $C_2>0$. This implies that $[\mathbb{P}(\boldsymbol{u}_h^{n+1}),\boldsymbol{u}_h^{n+1}]>0$ whenever $\|\boldsymbol{u}_h^{n+1}\|_{H_h}=r$, where $r>C_1/C_2$. By induction on n, we deduce (see for e.g [10, 18]) the existence of at least one solution to the discrete problem. □

5. Convergence

In this section we prove that the solution approximated by the finite volume scheme constitute a weak solution of (1.1). We start by proving that the family of the discrete solutions u_i, are relatively compact in $L^1(\Omega)$ for $i=1,2,3$.

5.1. **Compactness argument.** Using the following lemma proven in [4, Appendix A], we prove that the family of discrete solutions $u_{i,h}$ are relatively compact in L^1.

Lemma 5.1. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $T>0$, and $\Omega_T\coloneqq\Omega\times(0,T)$. Let $(\mathscr{T}^h)_h$ be an admissible family of meshes on Ω satisfying (3.1); let $(\Delta t_h)_h$ be the associated time steps. For all $h>0$, assume that the discrete functions (u_h^{n+1}), (f_h^{n+1}) and and discrete fields $(\boldsymbol{\mathcal{F}}_h^{n+1})$ for $n\in\{0,1,\dots,N_h\}$ satisfy the discrete evolution equations*

$$
\frac{u_h^{n+1}-u_h^n}{\Delta t} = \operatorname{div}[\boldsymbol{\mathcal{F}}_h^{n+1}] + f_h^{n+1} \quad \text{for } n\in\{0,1,\dots,N_h\} \tag{5.1}
$$

with a family $(u_h^0)_h$ of initial data. Assume that for all $\Omega'\subseteq\Omega$, there exists a constant $M(\Omega')$ such that

$$
\sum_{n=0}^{N_h}\Delta t\|u_h^{n+1}\|_{L^1(\Omega')} + \sum_{n=0}^{N_h}\Delta t\|f_h^{n+1}\|_{L^1(\Omega')} + \sum_{n=0}^{N_h}\Delta t\|\boldsymbol{\mathcal{F}}_h^{n+1}\|_{L^1(\Omega')} \leqslant M(\Omega'), \tag{5.2}
$$

and, moreover,

$$
\sum_{n=0}^{N_h}\Delta t\|\nabla_h u_h^{n+1}\|_{L^1(\Omega')} \leqslant M(\Omega'). \tag{5.3}
$$

Assume that the family $(u_0^h)_h$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$. Then there exists a measurable function u on Ω_T such that, along a subsequence,

$$
\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h} u_k^{n+1}\mathbb{1}_{(t_n,t_{n+1}]\times K} \to u \quad \text{in } L^1_{\mathrm{loc}}(\Omega\times[0,T]) \text{ as } h\to 0.
$$

We have the following convergence results along a subsequence.

Lemma 5.2. *There exists $\boldsymbol{u}\in[L^r(\Omega_T)]^3\cap[L^2(0,T;H^1(\Omega))]^3$ with $r\in(0,4)$ if d=2, and $r\in(0,10/3)$ if $d=3$, and subsequences of $\boldsymbol{u}_h=(u_{1,h},u_{2,h},u_{3,h})^{\mathrm{T}}$ not labeled, such that for $i=1,2,3$ and as $h\to 0$,*

(i) $u_{i,h}\to u_i$ *in* $L^1(\Omega_T)$, *a.e in* Ω_T,
(ii) $\nabla_h u_{i,h}\to\nabla u_i$ *weakly in* $[L^2(\Omega_T)]^{2d}$
(iii) $\mathcal{A}_{i,h}\nabla_h(\boldsymbol{b}_i^T\boldsymbol{y})\to\boldsymbol{\mathcal{A}}_{i,i}\nabla(\boldsymbol{b}_i^T\boldsymbol{y})$ *weakly in* $[L^1(\Omega_T)]^{2d}$
(iv) $F_i(\boldsymbol{u}_h)\to F_i(\boldsymbol{u})$ *weakly in* $L^1(\Omega_T)$,

Proof. We fix $i \in \{1,2,3\}$. The evolution of the first component $(u_{i,h}^{n+1})$ of the solution is governed by the system of discrete equations

$$\frac{u_{i,K}^{n+1} - u_{i,K}^{n}}{\Delta t} = \frac{1}{m(K)} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \boldsymbol{\mathcal{F}}_{K,L}^{n+1} \cdot \boldsymbol{n}_{K,L} + F_{i,K}^{n+1}, \tag{5.4}$$

where $F_{i,K}^{n+1} \coloneqq F_i(u_{1,K}^{n+1}, u_{2,K}^{n+1}, u_{3,K}^{n+1})$ and

$$\begin{aligned} \boldsymbol{\mathcal{F}}_{i,K,L}^{n+1} \cdot \boldsymbol{n}_{K,L} &\coloneqq D_i \frac{u_{i,L}^{n+1} - u_{i,K}^{n+1}}{d_{K,L}} \boldsymbol{n}_{K,L} - \chi_i \mathcal{A}_{i,K,L}^{n+1} \frac{\boldsymbol{b}_i^{\mathrm{T}} \big(\boldsymbol{y}_L^{n+1} - \boldsymbol{y}_K^{n+1}\big)}{d_{K,L}} \cdot \boldsymbol{n}_{K,L} \\ &= \nabla_{K,L} u_{i,h}^{n+1} - \chi_i \mathcal{A}_{i,K,L}^{n+1} \nabla_{K,L} \big(\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y}_K^{n+1}\big). \end{aligned}$$

Equations (5.4) have the form (5.1) required in Lemma 5.1.

It remains to check that the local L^1 bounds (5.2), (5.3) are satisfied. We actually have the global $L^1(\Omega_T)$ uniform estimates on the families

$$u_{i,h} \coloneqq \sum_{\substack{K \in \mathscr{T}_h \\ n \in \{0,1,\dots,N_h\}}} u_{i,K}^{n+1} \mathbb{1}_{(t_n,t_{n+1}] \times K}, \quad \boldsymbol{\mathcal{F}}_{i,h} \coloneqq \frac{1}{2} \sum_{n=0}^{N_h} \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \boldsymbol{\mathcal{F}}_{i,K,L}^{n+1} \mathbb{1}_{(t_n,t_{n+1}] \times K},$$

$$F_{i,h} \coloneqq \sum_{\substack{K \in \mathscr{T}_h \\ n \in \{0,1,\dots,N_h\}}} F_{i,K}^{n+1} \mathbb{1}_{(t_n,t_{n+1}] \times K}, \quad \nabla_h u_{i,h} \coloneqq \frac{1}{2} \sum_{n=0}^{N_h} \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \nabla_{K,L} u_{i,h}^{n+1} \mathbb{1}_{(t_n,t_{n+1}] \times K}$$

Indeed, the non-negativity of the discrete solutions, the assumption (2.3) and the Cauchy-Schwarz inequality ensure, for $i = 1,2,3$, the existence of $M_1(\Omega_T), M_2(\Omega_T) > 0$ such that $\|F_{i,h}\|_{L^1(\Omega_T)} \leqslant M_1(\Omega_T)$ and

$$\|u_{i,h}\|_{L^1(\Omega_T)} \leqslant \left(\sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} m(K) |u_{i,K}^{n+1}|^2 \right)^{1/2} \left(\sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} m(K) \right)^{1/2} \leqslant M_2(\Omega_T).$$

The estimate (4.4) and the restriction (3.2) guarantee the existence of $M_3(\Omega_T) > 0$ satisfying

$$\begin{aligned} &\|\nabla_h u_{i,h}\|_{L^1(\Omega_T)} \\ &= \frac{1}{2} \sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} m(K|L) |u_{i,L}^{n+1} - u_{i,K}^{n+1}| \\ &= \frac{1}{2} \sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \sqrt{m(K|L) d_{K|L}} \sqrt{m(K|L)} \frac{|u_{i,L}^{n+1} - u_{i,K}^{n+1}|}{\sqrt{d_{K|L}}} \\ &\leqslant \left(\frac{1}{2} \sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \tau_{K|L} (u_{i,L}^{n+1} - u_{i,K}^{n+1})^2 \right)^{1/2} \left(\frac{1}{2} \sum_{n=0}^{N_h} \Delta t \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \gamma m(K) \right)^{1/2} \leqslant M_3(\Omega_T). \end{aligned} \tag{5.5}$$

Using the critical discrete Sobolev embedding (see [4, Appendix B, Prop. B.1]) and the interpolation between $L^{p_t}(0,T;L^{p_x}(\Omega))$ spaces, from the $L^\infty(0,T;L^2(\Omega))$ estimate (4.3) and the discrete $L^2(0,T;H^1(\Omega))$ estimate (4.4) we get a uniform $L^r(\Omega_T)$ bound on $u_{i,h}$, and uniform $L^1(\Omega_T)$ bound on $\mathcal{A}_{i,h}$ (moreover, they are uniformly integrable). The quantity

$$\mathcal{A}_{i,h} \coloneqq \frac{1}{2} \sum_{n=0}^{N_h} \sum_{K \in \mathscr{T}_h} \sum_{L \in \mathcal{N}(K)} \sum_{i=1}^{3} \mathcal{A}_{i,K,L}^{n+1} \mathbb{1}_{(t_n,t_{n+1}] \times K}$$

satisfies the estimate

$$
\begin{aligned}
\|\mathcal{A}_{i,h}\nabla_h(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_h)\|_{L^1(\Omega_T)} &= \frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)\mathcal{A}_{i,K,L}^{n+1}|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1})| \\
&\leqslant \frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)u_{i,K}^{n+1}|\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1})| \\
&\leqslant \left(\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \tau_{K|L}\big(\boldsymbol{b}_i^{\mathrm{T}}(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1})\big)^2\right)^{1/2} \\
&\quad\times\left(\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} \gamma m(K)(u_K^{n+1})^2\right)^{1/2} \leqslant M_4(\Omega_T).
\end{aligned}
\tag{5.6}
$$

Since we can write

$$
\boldsymbol{\mathcal{F}}_{i,h} = \nabla_h u_{i,h} - \chi_i\mathcal{A}_{i,h}^{n+1}\nabla_h(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_h),
$$

we deduce an $L^1(\Omega_T)$ bound on $\boldsymbol{\mathcal{F}}_h$ from (5.5) and (5.6). Thus (5.2) and (5.3) are satisfied; the uniform $L^1(\Omega)$ bound on the initial data $u_{0,h}$ is also clear from (3.5), and Lemma 5.1 can be applied to derive the $L^1(\Omega_T)$ compactness of $(\boldsymbol{u}_h)_h$. Thus we can define the limits $\boldsymbol{u}=(u_1,u_2,u_3)$ of $\boldsymbol{u}_h$ and from this obtain the claim (i). Furthermore, to deduce the claim (ii), we use (4.4) to bound $\nabla_h u_{i,h}$ in $L^2(\Omega_T)$. Upon extraction of a further subsequence, we have e.g. $u_{i,h}\to u_i\in L^2(\Omega_T)$ and $\nabla_h u_{i,h}\to\zeta_i$ in $[L^2(\Omega_T)]^d$. Let us show (like as in [13]) that $u_i\in L^2(0,T;H^1(\Omega))$ and $\zeta_i=\nabla u_i$ for $i=1,2,3$. Let $\Psi_i\in C_{\mathrm{c}}^{\infty}([0,T)\times\overline{\Omega}))$ be given and

$$
\begin{aligned}
T_{i,1} &\coloneqq \int_0^T\int_\Omega \nabla_h u_{i,h}(\boldsymbol{x},t)\Psi_i(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}(u_{i,L}^{n+1}-u_{i,K}^{n+1})\cdot\Psi_{i,K}^{n+1}, \\
T_{i,2} &\coloneqq \frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)(u_{i,L}^{n+1}-u_{i,K}^{n+1})\boldsymbol{n}_{K|L}\cdot\Psi_{i,K|L}^{n+1},
\end{aligned}
$$

where $\boldsymbol{n}_{K|L}$ denotes the unit normal vector to $K|L$ outward to K and we define

$$
\Psi_{i,K}^{n+1} \coloneqq \frac{1}{m(K)}\int_K \Psi_i(x,t_{n+1})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \quad\text{and}\quad \Psi_{i,K|L}^{n+1} \coloneqq \frac{1}{m(K|L)}\int_{\sigma_{K|L}}\Psi_i(x,t_{n+1})\,\mathrm{d}\gamma(\boldsymbol{x}).
$$

Applying summation by parts and Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|T_{i,1}-T_{i,2}| \\
&= \left|\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)\big(u_{i,L}^{n+1}-u_{i,K}^{n+1}\big)\big(\boldsymbol{n}_{K|L}\cdot\big(\Psi_{i,K}^{n+1}-\Psi_{i,K|L}^{n+1}\big)\big)\right| \\
&\leqslant \left(\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}|u_{i,L}^{n+1}-u_{i,K}^{n+1}|^2\right)^{1/2}\left(\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)d_{K|L}(R_{i,K|L})^2\right)^{1/2},
\end{aligned}
$$

where we define $R_{i,K|L}\coloneqq\Psi_{i,K}^{n+1}-\Psi_{i,K|L}^{n+1}$. Regularity properties of the function Ψ_i give the existence of $C_{i,\Psi_i}>0$ only depending on Ψ_i, such that $|R_{i,K|L}|\leqslant C_{i,\Psi_i}h$. Therefore,

$$
\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)} m(K|L)d_{K|L}\leqslant dm(\Omega_T)
$$

and the estimate (4.4) imply that $T_{i,1}-T_{i,2}\to 0$ as $h\to 0$. Applying summation by parts yields

$$T_{i,2}=-\frac{1}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}m(K|L)u_{i,K}^{n+1}\boldsymbol{n}_{K|L}\cdot\Psi_{i,K|L}^{n+1}=-\int_0^T\int_\Omega u_{i,h}(\boldsymbol{x},t)\operatorname{div}\big(\Psi_i(\boldsymbol{x},t)\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

such that

$$T_{i,1}\longrightarrow-\int_0^T\int_\Omega u_i(\boldsymbol{x},t)\operatorname{div}\big(\Psi_i(\boldsymbol{x},t)\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\quad\text{as } h\to 0.$$

This proves that $u_i\in L^2(0,T;H^1(\Omega_T))$ and the function $\zeta_i\in[L^2(\Omega_T)]^d$ is almost everywhere equal to ∇u_i for $i=1,2,3$ in Ω_T, and the uniqueness of the limit implies that the whole family $\nabla_h u_{i,h}$ weakly convergence in $[L^2(\Omega_T)]^d$ to ∇u_i as $h\to 0$. Now, from the a.e convergence of $u_{i,h}$ to u_i and the Vitali theorem one has $\mathcal{A}_{i,h}$ weakly converge to $\boldsymbol{\mathcal{A}}_{i,i}$. Then, we pass to the limit to obtain (iii).

To prove (iv), we use the uniform $L^2(\Omega_T)$ estimation of $u_{i,h}$ and the assumption (2.3) of F_i to prove that the family $(F_i(\boldsymbol{u}_h))_h$ is uniformly integrable. Finally, using the a.e. convergence of $\boldsymbol{u}_h$ to $\boldsymbol{u}$ and by the Vitali theorem we get the a.e. convergence of $F_i(\boldsymbol{u}_h)$ to $F_i(\boldsymbol{u})$ in $L^1(\Omega_T)$ and from this we get (iv). □

5.2. **Convergence Analysis.** Our final goal is to show that the limit functions $\boldsymbol{u}$ constructed in Lemma 5.2 constitute a weak solution of system (1.1). We start by passing the limit (keep in mind that $i=1,2,3$) in (3.6b) to get the equality in (2.5).

Let $\psi_i\in C_c^\infty([0,T)\times\overline{\Omega})$. Set $\psi_K^n:=\psi_i(\boldsymbol{x}_K,t_n)$ for all $K\in\mathscr{T}_h$ and $n\in[0,N_h+1]$. Then multiply the discrete equation (3.6b) by $\Delta t\psi_{i,K}^{n+1}$ and summing the result over $K\in\mathscr{T}_h$ and $n\in\{0,1,\dots,N_h\}$.

$$T_{i,h}^1+T_{i,h}^2+T_{i,h}^3=T_{i,h}^4,$$

where

$$T_{i,h}^1=\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}m(K)(u_{i,K}^{n+1}-u_{i,K}^n)\psi_{i,K}^{n+1},$$

$$T_{i,h}^2=-D_i\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}(u_{i,L}^{n+1}-u_{i,K}^{n+1})\psi_{i,K}^{n+1},$$

$$T_{i,h}^3=\chi_i\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\mathcal{A}_{i,K,L}^{n+1}\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}_L^{n+1}-\boldsymbol{y}_K^{n+1}\big)\psi_{i,K}^{n+1},$$

$$T_{i,h}^4=\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}m(K)F_{i,K}^{n+1}\psi_{i,K}^{n+1}.$$

Item (iv) of Lemma 5.2 implies that

$$T_{i,h}^4\longrightarrow\int_0^T\int_\Omega F_i(\boldsymbol{u})\psi_i\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\qquad\text{as}\qquad h\to 0.$$

By a summation by parts in time and keeping in mind that $\psi^{N_h+1}=0$ we get

$$\begin{aligned}T_{i,h}^1&=-\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}m(K)u_{i,K}^{n+1}(\psi_{i,K}^{n+1}-\psi_{i,K}^n)-\sum_{K\in\mathscr{T}_h}m(K)u_{i,K}^0\psi_{i,K}^0\\&=-\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\int_{t_n}^{t_{n+1}}\int_K u_{i,K}^{n+1}\partial_t\psi_i(\boldsymbol{x}_K,t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\sum_{K\in\mathscr{T}_h}\int_K u_{i,K}^0\psi_{i,K}^0\,\mathrm{d}\boldsymbol{x}=:-T_{i,h}^{1,1}-T_{i,h}^{1,2}.\end{aligned}$$

We compare $T_{i,h}^1$ with

$$\tilde{T}_{i,h}^1:=-\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\int_{t_n}^{t_{n+1}}\int_K u_{i,h}(\boldsymbol{x},t)\partial_t(\psi_i(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_\Omega u_{i,0}(\boldsymbol{x})\psi_i(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}=-\tilde{T}_{i,h}^{1,1}-\tilde{T}_{i,h}^{1,2}$$

to obtain

$$\begin{aligned}
&\left|T^{1,2}_{i,h}-\tilde T^{1,2}_{i,h}\right| \\
&=\left|\int_\Omega u_{i,0}(\boldsymbol{x})\psi_i(x,0)\,\mathrm{d}\boldsymbol{x}-\sum_{K\in\mathscr{T}_h} m(K)u^0_{i,K}\psi^0_{i,K}-\right|=\left|\int_\Omega u_{i,0}(\boldsymbol{x})\left(\sum_{K\in\mathscr{T}_h}(\psi_i(\boldsymbol{x},0)-\psi_i(\boldsymbol{x}_K,0))\right)\mathrm{d}\boldsymbol{x}\right| \\
&\leqslant\left(\int_\Omega |u_{i,0}|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\sum_{K\in\mathscr{T}_h}\int_K|\psi_i(\boldsymbol{x},0)-\psi_i(\boldsymbol{x}_K,0)|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\leqslant C_{i,1}h
\end{aligned}$$

due to the Lipschitz continuity of ψ_i. Using the regularity properties of $\partial_t\psi_i$ and the estimates (4.3) we get

$$\begin{aligned}
|T^{1,1}_{i,h}-\tilde T^{1,1}_{i,h}| &= \left|\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\int_{t_n}^{t_{n+1}}\int_K u_{i,h}(\boldsymbol{x},t)\partial_t\psi_i(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\sum_{n=0}^{N_h}\sum_{K\in\mathscr{T}_h}\int_{t_n}^{t_{n+1}}\int_K u^{n+1}_{i,K}\partial_t\psi_i(\boldsymbol{x}_K,t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\
&\leqslant\left|\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}m(K)u^{n+1}_{i,K}\int_{t_n}^{t_{n+1}}\int_K(\partial_t\psi_i(\boldsymbol{x},t)-\partial_t\psi_i(\boldsymbol{x}_K,t))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\
&\leqslant C_2h\left(\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}m(K)|u^{n+1}_{i,K}|^2\right)^{1/2}\leqslant C_3h.
\end{aligned}$$

Thus $T^{1,1}_{i,h}\to\tilde T^{1,1}_{i,h}$ and $T^{1,2}_{i,h}\to\tilde T^{1,2}_{i,h}$ as $h\to0$, which proves that

$$T^1_{i,h}\longrightarrow-\int_0^T\int_\Omega u(\boldsymbol{x},t)\partial_t\psi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\int_\Omega u_0(\boldsymbol{x})\partial_t\psi(x,0)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\quad\text{as } h\to0.$$

Next, we deal with $T_{2,h}$ and $T_{3,h}$. Gathering by edges and using the definition (3.3) of $\nabla_h u_h$, we get

$$\begin{aligned}
T^2_{i,h}&=\frac{D_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}(u^{n+1}_{i,L}-u^{n+1}_{i,K})(\psi^{n+1}_{i,L}-\psi^{n+1}_{i,K}) \\
&=\frac{D_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}m(K|L)d_{K|L}\frac{u^{n+1}_{i,L}-u^{n+1}_{i,K}}{d_{K|L}}\frac{\psi^{n+1}_{i,L}-\psi^{n+1}_{i,K}}{d_{K|L}} \\
&=\frac{D_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}m(K|L)d_{K|L}(\nabla_{K|L}u^{n+1}_{i,h}\cdot\boldsymbol{n}_{K,L})(\nabla_{K|L}\psi_i(\overline{\boldsymbol{x}_{K,L}},t_{n+1}\cdot\boldsymbol{n}_{K,L}),
\end{aligned}$$

and

$$\begin{aligned}
T^3_{i,h}&=-\frac{\chi_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}\tau_{K|L}\mathcal{A}^{n+1}_{i,K,L}\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}^{n+1}_L-\boldsymbol{y}^{n+1}_K\big)(\psi^{n+1}_{i,L}-\psi^{n+1}_{i,K}) \\
&=-\frac{\chi_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}m(K|L)d_{K|L}\mathcal{A}^{n+1}_{i,K,L}\frac{\boldsymbol{b}_i^{\mathrm{T}}\big(\boldsymbol{y}^{n+1}_L-\boldsymbol{y}^{n+1}_K\big)}{d_{K|L}}\frac{(\psi^{n+1}_{i,L}-\psi^{n+1}_{i,K})}{d_{K|L}} \\
&=-\frac{\chi_i}{2}\sum_{n=0}^{N_h}\Delta t\sum_{K\in\mathscr{T}_h}\sum_{L\in\mathcal{N}(K)}m(K|L)d_{K|L}\mathcal{A}^{n+1}_{i,K,L}(\nabla_{K|L}(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}^{n+1}_h)\cdot\boldsymbol{n}_{K,L})(\nabla_{K|L}\psi_i(\overline{\boldsymbol{x}_{K,L}},t_{n+1}\cdot\boldsymbol{n}_{K,L}),
\end{aligned}$$

where $\overline{\boldsymbol{x}_{K,L}}$ is some point on the segment with the endpoints $\boldsymbol{x}_K,\ \boldsymbol{x}_L$.Since the values $\nabla_{K,L}$ are directed by $\boldsymbol{n}_{K,L}$, we have

$$\begin{aligned}
\big(\nabla_{K|L}u^{n+1}_h\cdot\boldsymbol{n}_{K,L}\big)\big(\nabla_{K|L}\psi(\overline{\boldsymbol{x}_{K,L}},t_{n+1})\cdot\boldsymbol{n}_{K,L}\big)&\equiv\big(\nabla_{K|L}u^{n+1}_h\big)\cdot\big(\nabla_{K|L}\psi(\overline{\boldsymbol{x}_{K,L}},t_{n+1})\big) \\
\big(\nabla_{K|L}\big(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}^{n+1}_h\big)\cdot\boldsymbol{n}_{K,L}\big)\big(\nabla_{K|L}\psi(\overline{\boldsymbol{x}_{K,L}},t_{n+1})\cdot\boldsymbol{n}_{K,L}\big)&\equiv\big(\nabla_{K|L}\big(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}^{n+1}_h\big)\big)\cdot\big(\nabla_{K|L}\psi(\overline{\boldsymbol{x}_{K,L}},t_{n+1})\big),
\end{aligned}$$

Then

$$T^2_{i,h} = D_i \int_0^T \int_\Omega \nabla_h u_h (\nabla\psi)_h \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \quad T^3_{i,h} = -\chi_i \int_0^T \int_\Omega \mathcal{A}_{i,h} \nabla_{K|L} (\boldsymbol{b}_i^{\mathrm{T}} \boldsymbol{y}_h^{n+1}) (\nabla\psi)_h \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

where $(\nabla\psi_i)_h|_{(t_n,t_{n+1})\times K} := \nabla\psi_i(\overline{\boldsymbol{x}_K \boldsymbol{x}_L}, t_{n+1})$. Here the construction of the diamond $T_{K,L}$ from the neighboring centers $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ and the interface $\sigma = K|L$ (see e.g [3, 4]) has been used.

From the continuity of $\nabla\psi$ we get $(\nabla\psi_i)_h \to \nabla\psi_i$ in $L^\infty(\Omega_T)$. Hence using the weak L^2 convergence of $\nabla_h u_{i,h}$ to ∇u_i, and the weak L^1 convergence of $\mathcal{A}_{i,h}\nabla_h(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_h)$ to $\boldsymbol{\mathcal{A}}_{i,i}\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y}_h)$, we obtain

$$T^2_{i,h} \to D_i \int_0^T \int_\Omega \nabla u \nabla\psi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \quad T^3_{i,h} \to -\chi_i \int_0^T \int_\Omega \mathcal{A}_i(u_i)\nabla(\boldsymbol{b}_i^{\mathrm{T}}\boldsymbol{y})\nabla\psi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \text{as } h \to 0.$$

Gathering the results obtained, we can justify the equality in (2.5). This concludes the proof of the following theorem.

Theorem 5.1. *Assume that $u_{i,0} \in (L^2(\Omega))^+$ for $i = 1,2,3$. Let $\boldsymbol{u}_h = (u_{1,h}, u_{2,h}, u_{3,h})^{\mathrm{T}}$ be the discrete solution generated by the finite volume scheme* (3.6) *on a family of meshes satisfying* (3.1) *and* (3.2)*. Then, as $h \to 0$, $\boldsymbol{u}_h$ converges (along a subsequence) a.e on Ω_T to a limit $\boldsymbol{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ that is a weak solution of the system* (1.1), (1.2).

6. Numerical examples

We present in this section some numerical results obtained with the finite volume scheme (3.6a)–(3.6b) . To obtain the numerical test, we will reduce the number of the parameters in the model (1.1)-(1.2). For this reason we non-dimensionalize the system following [16]. We choose $U_i = u_i/k$ for $i = 1,2,3$. Making the substitution and simplifying, we obtain

$$\begin{aligned}
F_1(\boldsymbol{U}) &:= (1-U_1)U_1 - \frac{a_1 U_1}{1+b_1 U_1} U_2, \\
F_2(\boldsymbol{U}) &:= \frac{a_1 U_1}{1+b_1 U_1} U_2 - \frac{a_2 U_2}{1+b_2 U_2} U_3 - e_1 U_2, \\
F_3(\boldsymbol{U}) &:= \frac{a_2 U_2}{1+b_2 U_2} U_3 - e_2 U_3.
\end{aligned}$$

On the domain $\Omega := (-2,2)\times(-2,2)$ we define a uniform Cartesian grid

$$\mathscr{T}_h = \{K_{ij} \subseteq \Omega \;:\; K_{ij} = ((i-1)N_x, iN_x) \times ((j-1)N_x, jN_x)\}$$

with $N_x \times N_y$ control volumes. For the simulations, we choose $N_x = N_y = 256$ and we take the parameters

$$a_1 = 5.0, \qquad a_2 = 0.1, \qquad b_1 = 2.0, \qquad b_2 = 2.0, \qquad e_1 = 0.4, \qquad e_2 = 0.01$$

used in [16]. The corresponding diffusion coefficients are given by $D_i = \mathcal{D}_i = \theta_i = 1$, for $i = 1,2,3$. The sensitivity chemotactic parameters are chosen by

$$\delta_1 = 100, \qquad \delta_2 = 20, \qquad \delta_3 = 10.$$

The initial distribution for u_1, u_2 and u_3 species correspond to a constant $u_{1,0} = u_{2,0} = 0.8$ and $u_{3,0} = 1$.

In order to illustrate the convergence of the numerical scheme and due to the lack of exact solutions for each example, we compute approximate errors in different norms using a numerical solution on an extremely fine mesh as reference. To measure errors between such a reference solution u_{ref} and an approximate solution u_h at time t^n, we will use the L^2-error

$$e_2^n(u) = \|u^n_{\mathrm{ref}} - u^n_h\|_2 = \left(\sum_{K\in\mathscr{T}_h} \frac{1}{m(K)} |u^n_{\mathrm{ref},K} - u^n_{h,K}|^2 \right)^{1/2}.$$

Here, $u^n_{\mathrm{ref},K}$ stands for the projection of the reference solution onto control volume K. For solving the corresponding nonlinear system arising from the implicit finite FV, we use the Newton method, where at

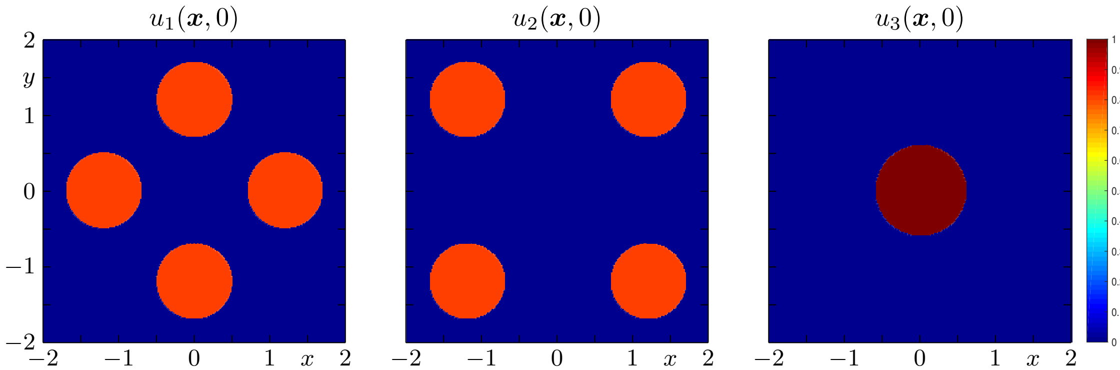

FIGURE 2. Example 1: initial condition for the u, v and s species.

$N_x \times N_y$	h	$e_2^n(u_1)$	$e_2^n(u_2)$	$e_2^n(u_3)$
32×32	1.25e-1	1.33e-03	3.09e-03	4.97e-04
64×64	6.25e-2	2.82e-04	6.61e-04	2.09e-04
128×128	3.12e-2	4.41e-05	1.16e-04	3.39e-05
256×256	1.56e-2	1.07e-05	3.20e-05	8.00e-06

TABLE 1. Example 1: approximate L^2-errors for each species at simulated time $t = 0.02$.

each time step, only a few iterations are required to achieve convergence. In addition, the linear systems involved in Newton method are solved by the GMRES method.

6.1. **Example 1 (species the interacting via chemical substance).** For this numerical test, the chemotactic coefficients are $\chi_1 = -0.8$, $\chi_2 = 0.8$ and $\chi_3 = 2$, where $\chi_1 < 0$ means that movement of the prey is against the presence of the predator. For the initial condition, the super-predators are concentrated in small pockets at a one spatial point while de predators and preys are concentrated in small pockets at four spatial points (see Figure 2).

In Figure 3, we display the numerical solution for each species at three different simulated times. Initially, at simulated time $t = 0.02$ (Figure 3, top), we can observe the effect of the chemotaxis for the super-predators (u_3) predators (u_2) feeling their respective preys, and the preys feeling the presence of the predator. At simulation time $t = 0.04$ (Figure 3, middle). We notice the rapid movement of the super-predators towards the regions occupied by the predators and at the same time predators spread out to the areas where the preys (u_1) are located, but it does not move towards the area occupied by the predator. The prey moves to the regions where the predator is not located. At $t = 0.06$ (Figure 3, bottom), we can see that the super-predators continue moving towards the area occupied by the predators, the predators occupy almost the entire area, except the region where the super-predators is located while the prey move toward (running away) the area where the predators are not located. In Table 1 we show the L^2-error for each species at simulated time $t = 0.02$, we observe convergence of the numerical scheme.

6.2. **Example 2 (prey do not interact via chemical substances).** In Example 2, we choose $\chi_1 = 0$, $\chi_2 = 0.8$ and $\chi_3 = 2$. In this case we do not consider chemotactic movement of the prey. The initial distribution is as in Example 1, but the super-predators, predators, and prey are concentrated in small pockets at a one spatial point (see Figure 4). We display in Figure 5 the numerical solution for each species at three different simulations time. We notice the rapid movement of the super-predators towards the regions occupied by the predators and at the same time predators spread out to the areas where the preys are located, while the prey present isotropic and homogeneous diffusion (due to the choice of the chemotactic coefficients

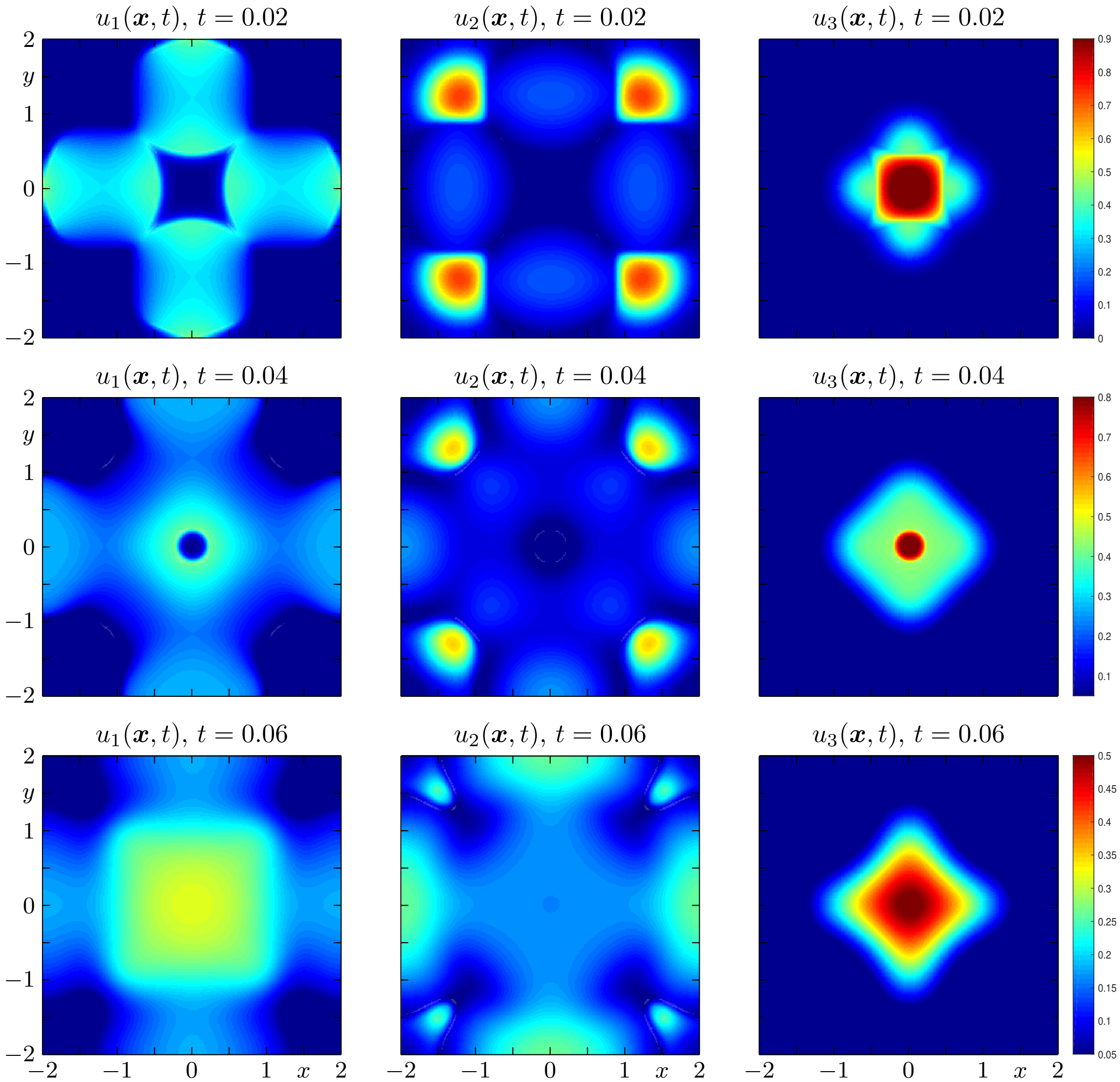

FIGURE 3. Example 1: interaction of the three species at different times $t = 0.02,\ 0.04,\ 0.06$.

$N_x \times N_y$	h	$e_2^n(u_1)$	$e_2^n(u_2)$	$e_2^n(u_3)$
32×32	1.25e-1	1.13e-3	1.60e-3	1.56e-3
64×64	6.25e-2	5.47e-4	8.09e-4	7.56e-4
128×128	3.12e-2	2.74e-4	4.09e-4	3.71e-4
256×256	1.56e-2	1.36e-4	2.08e-4	1.84e-4

TABLE 2. Example 2: approximate L^2-errors for each species at simulated time $t = 0.04$.

$\chi_1 = 0$). In Table 2 we show the L^2-error for each species at simulated time $t = 0.04$, we observe the convergence of the numerical scheme.

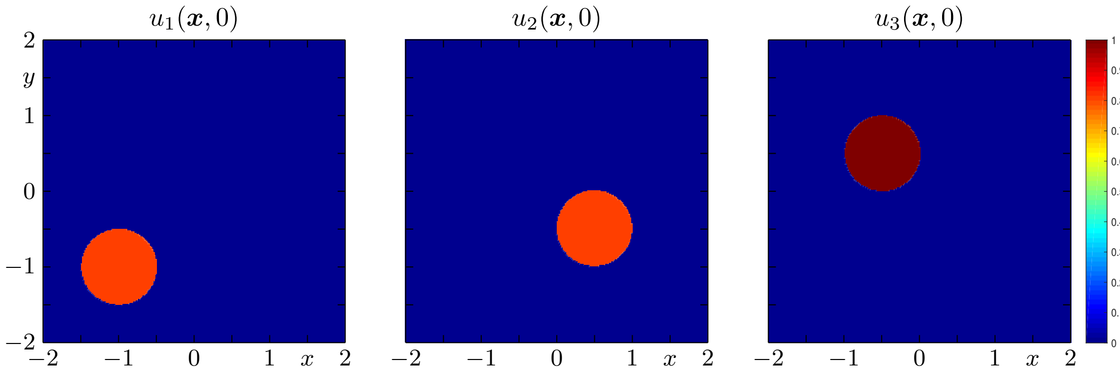

FIGURE 4. Example 2: initial condition for the u, v and s species.

6.3. **Example 3: Spatio-temporal model versus non-spatial ODE model.** In this numerical example, we wish to compare the dynamics of the spatio-temporal model (1.1)–(1.4), with that of the non-spatial model

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = F_i\bigl(u_1(t), u_2(t), u_3(t)\bigr), \quad i = 1, 2, 3, \tag{6.1}$$

where the diffusion and chemotaxis movement are not present. To this end we determine for each species i at simulated time t_n the quantities

$$\mathcal{I}(u_i, t^n) \coloneqq \sum_{K \in \mathscr{T}_h} m(K) u_K^n \approx \int_\Omega u(\boldsymbol{x}, t^n)\, \mathrm{d}\boldsymbol{x},$$

which represents the approximate total number in Ω of individuals of compartment u at time t_n and

$$u_{i,\max}^n \coloneqq \max_{K \in \mathscr{T}_h} u_K^n, \quad u_{i,\min}^n \coloneqq \min_{K \in \mathscr{T}_h} u_K^n.$$

. We consider the diffusion coefficients $D_1 = 0.02$, $D_2 = 0.5$ and $D_3 = 5$, the sensitivity chemotactic parameters are chosen by $\delta_1 = 6$, $\delta_2 = 4$ and $\delta_3 = 2$ and the chemotactic coefficients $\chi_1 = -2$, $\chi_2 = 4$ and $\chi_3 = 6$. The other parameters are the same as in Examples 1 and 2. The initial condition for $i = 1, 2, 3$ is a spatially distributed random perturbation of the respective values $u_1 = 0.9$, $u_2 = 0.1$ and $u_3 = 12.75$, which is displayed in Figure 6. The "random" initial datum has been chosen to test whether small perturbations would give rise to large-scale regular structures akin to the well-known mechanism of pattern formation, or rather, the small fluctuations in the initial datum would simply be smoothed out. In Figure 7 we display the numerical solution at four different times. It turns out that each species aggregates in a kind of groups structure which forming zones of different densities. This structure varies with time (not show here), moreover in Figure 8 we can observe that the quantities $\mathcal{I}(u_i, t)$ and the solution u_i of ODE problem (6.1) have the same behavior even when the total variation of each species $u_{i,\max} - u_{i,\min}$ have a oscillatory behavior and remains bounded along the time, which lends further support to the conjecture that this model (at least with the parameters chosen) exhibits spatial-temporal oscillatory behavior.

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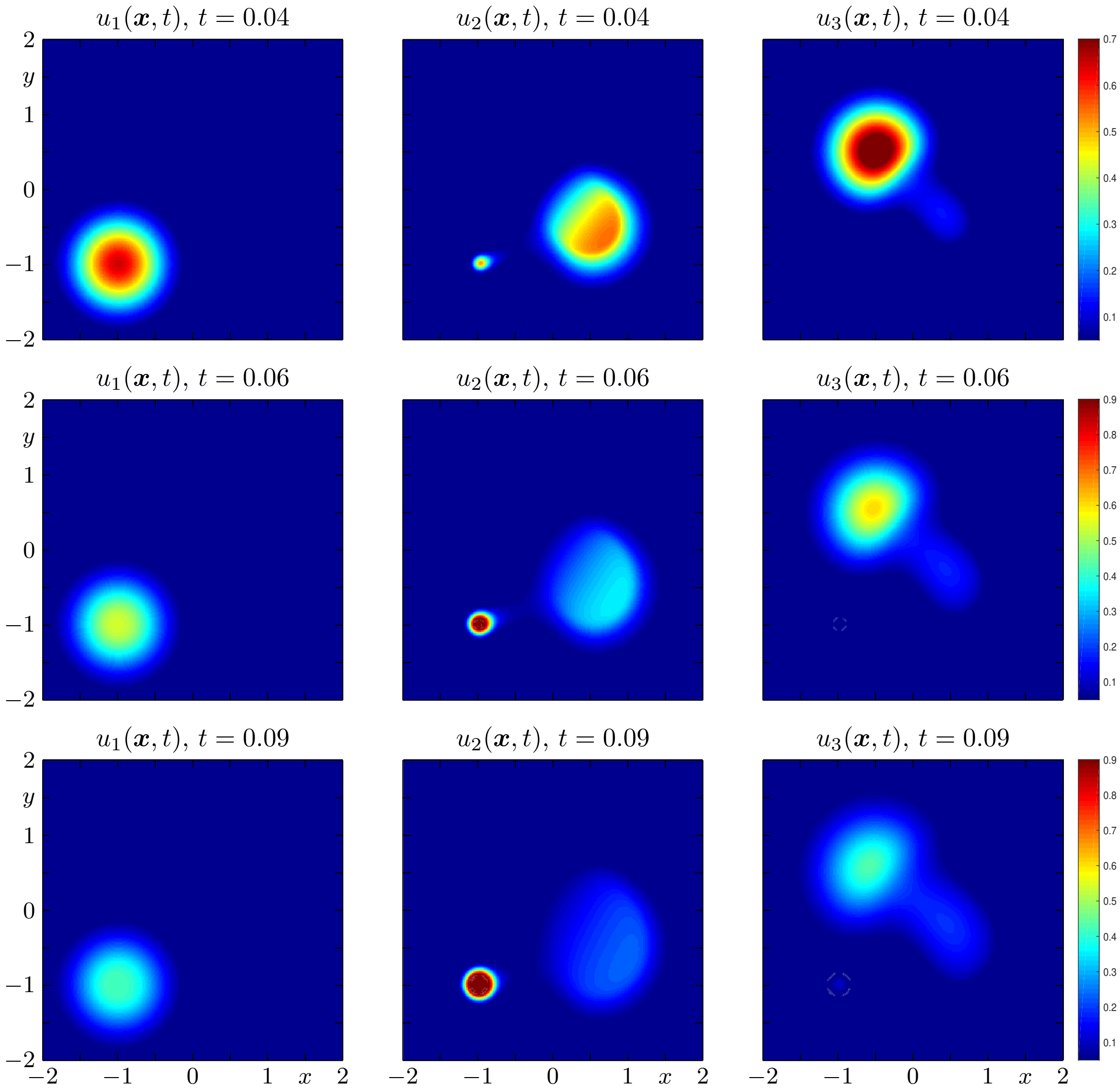

FIGURE 5. Example 2: interaction of the three species at different times $t = 0.04,\ 0.06,\ 0.09$.

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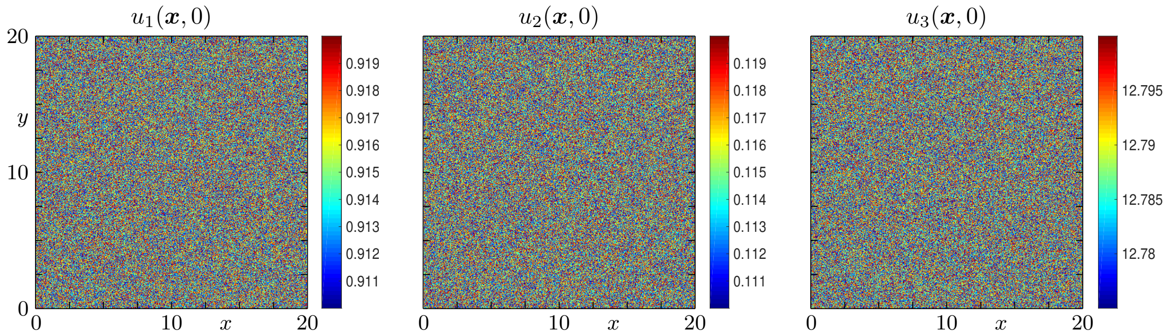

FIGURE 6. Example 3: initial conditions for the u_1, u_2 and u_3 species.

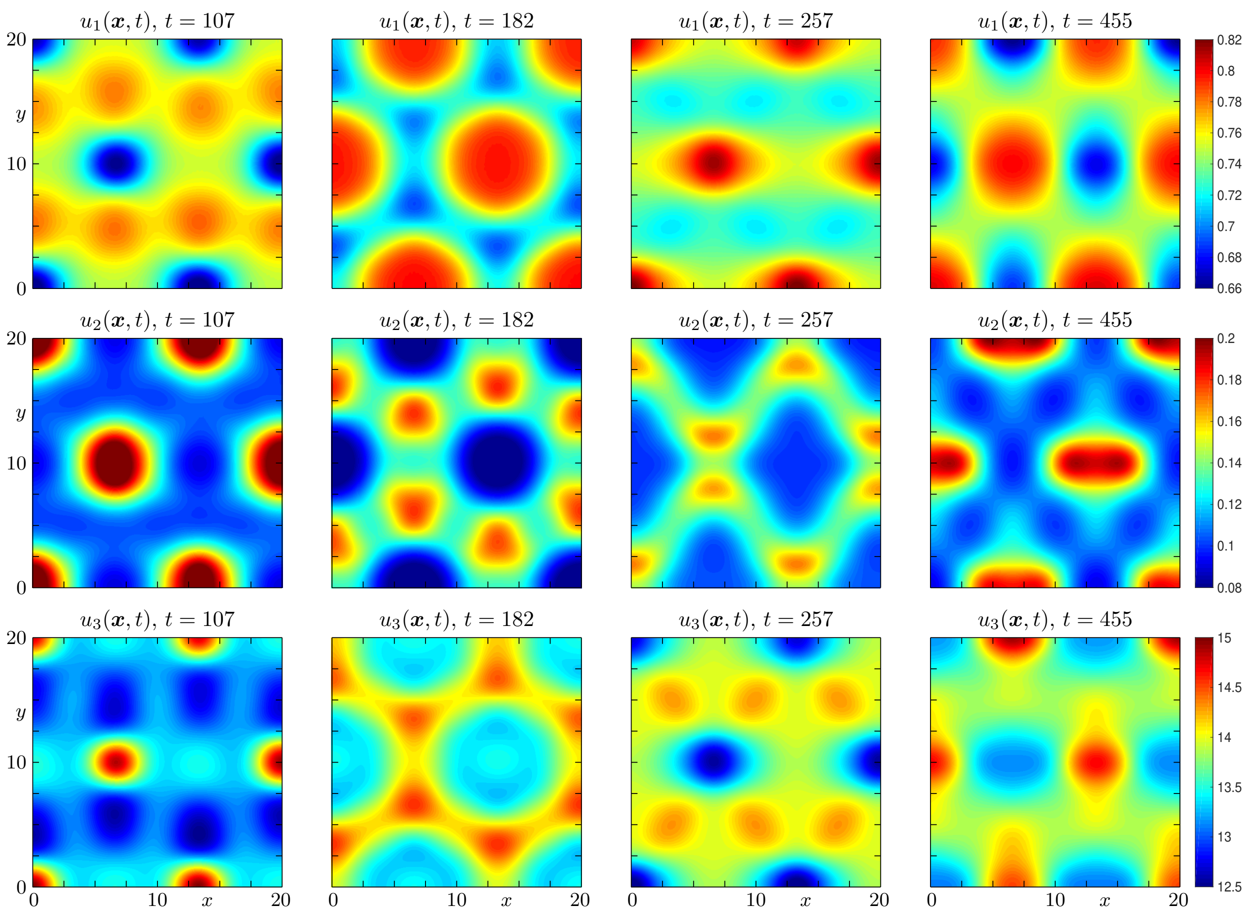

FIGURE 7. Example 3: numerical solution at four different times.

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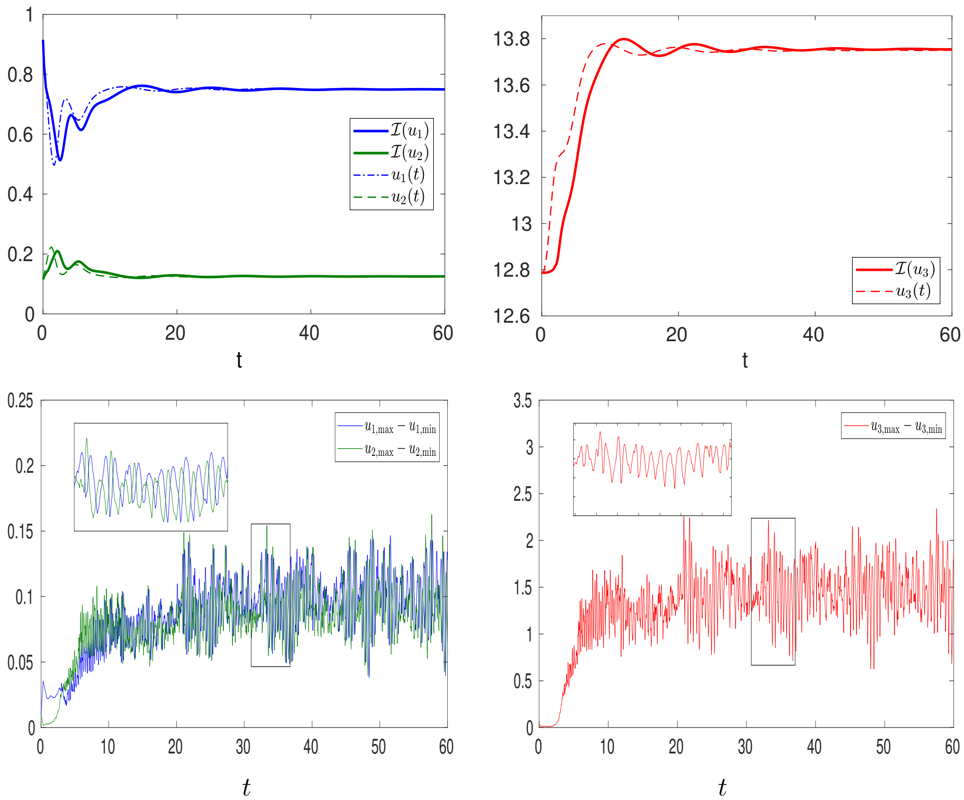

FIGURE 8. Example 3: spatial-temporal model versus non-spatial ODE model and time evolution of the total variation for each species.

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