A mixed-primal finite element method for the coupling of Brinkman-Darcy flow and nonlinear transport*

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Abstract

This paper is devoted to the mathematical and numerical analysis of a model describing the flow-transport interaction in a porous-fluidic domain. The medium consists of a highly permeable material, where the flow of an incompressible viscous fluid is governed by Brinkman equations (written in terms of vorticity, velocity and pressure), and a porous medium where Darcy's law describes fluid motion. Gravity and the local fluctuations of a scalar field (representing for instance, the solids volume fraction, or the concentration of a contaminant) are the main drivers of the fluid patterns on the whole domain, and the Brinkman-Darcy equations are coupled to a nonlinear transport equation accounting for mass balance of the scalar. We introduce a mixed-primal variational formulation of the problem and establish existence and uniqueness of solution using fixed-point arguments and small-data assumptions. A family of Galerkin discretisations that produce divergence-free discrete velocities is also presented and analysed using similar tools to those employed in the continuous problem. Convergence of the resulting mixed-primal finite element method is proven, and some numerical examples confirming the theoretical error bounds and illustrating the performance of the proposed discrete scheme are reported.

Key words: Nonlinear transport, Brinkman-Darcy coupling, vorticity-based formulation, fixed-point theory, mixed finite elements, error analysis.

Mathematics Subject Classifications (2000): 65N30, 76S05, 65N12, 65N15.

1 Introduction

The aim of this paper is to put together an extension of the results from [2, 3] and [4] dealing with augmented and fully mixed finite element approximations of coupled flow and transport problems, and coupled Brinkman and Darcy flow, respectively. The coupled system describes the interaction of flow and transport phenomena in two different domains separated by an interface. Such a formalism arises naturally, and has been systematically used, in hydrology and biological applications including

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for instance subsurface flow, hydraulic fractures, CO_2 sequestration, perfusion of soft living tissues, etc. In obtaining approximate solutions for the problem under consideration one faces marked difficulties. These are related to the coupling of mechanisms that act simultaneously, such as active transport and reaction of the solute and nonlinearities in the diffusion process and in the source term; as well as the heterogeneities, transmission conditions, and the need of preserving physical properties. Even if many numerical solutions are already available (see e.g. [11, 14, 15, 16] and the references therein), up to the authors' knowledge the only contributions addressing mathematical and numerical properties of somewhat similar couplings are the recent works [8], where existence and stability bounds of weak solutions is established also for the nonlinear Navier-Stokes-Darcy flow coupled with transport; [20], where a mixed finite element scheme approximates the Stokes-Darcy system and a local discontinuous Galerkin method is employed to discretise the transport equation; and [19] that analyses stabilised velocity-pressure-concentration formulations for a model where viscosity depends on the solute concentration.

The main difference of these works with respect to our contribution, is that we propose a formulation of the problem written in terms of Brinkman vorticity, and the transmission conditions we employ are slightly different. In addition, we introduce a mixed-primal finite element method for the Brinkman-Darcy-Transport coupling that produces divergence-free discrete velocities. Following our recent work [4], the coupling of subdomains is based on a vorticity based fully-mixed formulation for the Brinkman-Darcy problem, whereas a primal formulation for the transport problem is adapted from [3]. The solvability of such a coupling will be based on extending the fixed-point strategy introduced in [2] and [3] to the present context. In particular, we realise that the primal formulation for the transport problem requires further regularity for the global velocity, initially living in $\mathbf{H}(\mathrm{div}, \Omega)$. In turn, and in contrast with [2] and [3], we can not exploit augmentation techniques to recover $\mathbf{H}^1(\Omega)$ velocities. However, a different smoothness assumption is introduced at the level of the continuous analysis of the transport problem, and subsequently in the solvability of the Brinkman-Darcy-Transport coupling. More precisely, the derivation of existence of weak solutions relies on a strategy combining classical fixed-point arguments, suitable regularity assumptions on the decoupled problems, the Lax-Milgram Lemma, preliminary results from [4], and the Sobolev embedding and Rellich-Kondrachov compactness theorems. In addition, sufficiently small data allow us to establish uniqueness of weak solution. On the other hand, the well-posedness of the discrete problem is based on the Brouwer fixed-point theorem and analogous arguments to those employed in the continuous analysis. Finally, similar arguments as those utilised in [3, 4] allow us to derive the corresponding Céa estimates for both the Brinkman-Darcy and transport problems, and these lead to natural *a priori* error bounds for the Galerkin scheme.

Outline. This paper has been structured as follows. The remainder of this section presents some notation and preliminary definitions of spaces needed thereafter. The model problem along with boundary data are stated in Section 2. The weak formulation of the problem and its well-posedness analysis in the framework of the Schauder fixed-point theorem are collected in Section 3. The associated Galerkin scheme is then proposed in Section 4 and its solvability is established by the Brouwer fixed-point theorem. Next, we derive in Section 5 some *a priori* error estimates, and conclude in Section 6 with a few numerical examples in 2D and 3D, illustrating the good performance of the mixed-primal finite element method and confirming the theoretical rates of convergence.

Preliminaries. Standard notation will be adopted for Lebesgue and Sobolev spaces. In addition, by $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. For instance, if $\Theta \subseteq \mathrm{R}^n$, $n = 2, 3$ is a domain, $\Lambda \subseteq \mathrm{R}^n$

is a Lipschitz surface, and $r \in \mathrm{R}$, we define $\mathbf{H}^r(\Theta) := [\mathrm{H}^r(\Theta)]^n$ and $\mathbf{H}^r(\Lambda) := [\mathrm{H}^r(\Lambda)]^n$. We also recall the definition of the following Hilbert spaces

$$\mathbf{H}(\mathrm{div};\Theta) := \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Theta) : \quad \mathrm{div}\, \boldsymbol{v} \in \mathrm{L}^2(\Theta) \right\}, \quad \mathbf{H}(\mathbf{curl};\Theta) := \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Theta) : \quad \mathbf{curl}\, \boldsymbol{v} \in \mathbf{L}^2(\Theta) \right\},$$

normed, respectively, with

$$\|\boldsymbol{v}\|_{\mathrm{div};\Theta} := \left\{ \|\boldsymbol{v}\|^2_{0,\Theta} + \|\,\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Theta} \right\}^{1/2}, \quad \|\boldsymbol{v}\|_{\mathbf{curl};\Theta} := \left\{ \|\boldsymbol{v}\|^2_{0,\Theta} + \|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Theta} \right\}^{1/2},$$

where, for any vector field $\boldsymbol{v} := (v_1, \ldots, v_d)^{\mathtt{t}} \in \mathbf{L}^2(\Theta)$,

$$\mathrm{div}\,\boldsymbol{v} := \sum_{i=1}^{n} \partial_i v_i, \quad \mathbf{curl}\,\boldsymbol{v} := \nabla \times \boldsymbol{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} \text{ if } n = 3, \text{ and } \mathrm{curl}\,\boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1 \text{ if } n = 2.$$

In addition, we also recall the orthogonal decomposition

$$\mathrm{L}^2(\Theta) = \mathrm{L}^2_0(\Theta) \oplus P_0(\Theta),$$

where $P_0(\Theta)$ is the space of constant functions on Θ, and

$$\mathrm{L}^2_0(\Theta) = P_0(\Theta)^{\perp} := \left\{ q \in \mathrm{L}^2(\Theta) : \quad \int_\Theta q = 0 \right\}.$$

Equivalently, each $q \in \mathrm{L}^2(\Theta)$ can be uniquely decomposed as $q = q_0 + c$, with

$$q_0 := q - \frac{1}{|\Theta|} \int_\Theta q \in \mathrm{L}^2_0(\Theta) \quad \text{and} \quad c := \frac{1}{|\Theta|} \int_\Theta q \in \mathrm{R},$$

where $\mathrm{L}^2_0(\Theta)$ is endowed with the usual norm of $\mathrm{L}^2(\Theta)$, and it is easy to see that there holds

$$\|q\|^2_{0,\Theta} = \|q_0\|^2_{0,\Theta} + |\Theta|\, c^2.$$

By $\mathbf{0}$ we will denote the generic null vector (including the null functional and operator), and we will denote by C and c, with or without subscripts, bars, tildes or hats, generic constants independent of the discretisation parameters.

2 Governing equations

Let $\Omega \subset \mathrm{R}^n$, $n = 2, 3$, denote an heterogeneous porous domain composed of two regions: Ω_{B}, where the viscous flow patterns characterised by velocity $\boldsymbol{u}_{\mathrm{B}}$, vorticity $\boldsymbol{\omega}_{\mathrm{B}}$, and pressure p_{B} can be governed by the linear Brinkman equations; and Ω_{D}, where the flow of the immiscible fluid obeys to Darcy's law written in terms of velocity $\boldsymbol{u}_{\mathrm{D}}$ and pressure p_{D} in the porous domain. These subdomains are separated by an interface Σ, through which exchange of fluid velocities and pressures occurs. We also consider that a given scalar field ϕ (representing, for instance, the concentration of a chemical component, the fluid density, the temperature, or the volume fraction or saturation of a solid phase) is advected and diffused on the whole Ω according to the mass conservation principle (or energy conservation if the scalar field stands for e.g. temperature). The model problem can be summarised as follows:

$$\text{(Brinkman)} \qquad \left. \begin{aligned} \mu \mathbb{K}_{\mathrm{B}}^{-1} \boldsymbol{u}_{\mathrm{B}} + \mu\, \mathbf{curl}\, \boldsymbol{\omega}_{\mathrm{B}} + \nabla p_{\mathrm{B}} &= \phi \boldsymbol{f}_{\mathrm{B}} \\ \boldsymbol{\omega}_{\mathrm{B}} - \mathbf{curl}\, \boldsymbol{u}_{\mathrm{B}} &= \mathbf{0} \\ \mathrm{div}\, \boldsymbol{u}_{\mathrm{B}} &= 0 \end{aligned} \right\} \quad \text{in} \quad \Omega_{\mathrm{B}}, \tag{2.1}$$

coupled with

$$\text{(Darcy)} \qquad \left.\begin{aligned} \mu\mathbb{K}_{\mathrm{D}}^{-1}\boldsymbol{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} &= \phi\boldsymbol{f}_{\mathrm{D}} \\ \operatorname{div}\boldsymbol{u}_{\mathrm{D}} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\mathrm{D}}\,, \tag{2.2}$$

and

$$\text{(Transport)} \qquad \beta\phi - \operatorname{div}(\vartheta(\phi)\nabla\phi - \phi\boldsymbol{u} - f_{\mathrm{bk}}(\phi)\boldsymbol{g}) = 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

where $\mu > 0$ is the constant viscosity of the fluid in the entire domain Ω, the parameter β is the porosity of the medium (assumed constant inside each subdomain, but possibly discontinuous across Σ). Notice that $\boldsymbol{u}$ in (2.3) refers to the global velocity field defined in both Ω_{B} and Ω_{D}, that is $\boldsymbol{u} := \mathbf{1}_{\Omega_{\mathrm{B}}}\boldsymbol{u}_{\mathrm{B}} + \mathbf{1}_{\Omega_{\mathrm{D}}}\boldsymbol{u}_{\mathrm{D}}$, where $\mathbf{1}_{\Omega_\star}$ is the characteristic function, $\star \in \{\mathrm{B}, \mathrm{D}\}$. In addition, $\mathbb{K}_{\mathrm{B}}$ and $\mathbb{K}_{\mathrm{D}}$ are symmetric, bounded, and uniformly positive definite tensors $\mathbb{K}_{\mathrm{B}}, \mathbb{K}_{\mathrm{D}}$, which means that there exist $\alpha_{\mathbb{K}_{\mathrm{B}}} > 0$ and $\alpha_{\mathbb{K}_{\mathrm{D}}} > 0$ such that

$$\boldsymbol{v}^{\mathrm{t}}\mathbb{K}_{\mathrm{B}}^{-1}(\boldsymbol{x})\boldsymbol{v} \;\geq\; \alpha_{\mathbb{K}_{\mathrm{B}}}|\boldsymbol{v}|^2 \quad \forall\,\boldsymbol{v} \in \mathrm{R}^n, \forall \boldsymbol{x} \in \Omega_{\mathrm{B}}\,,$$

and

$$\boldsymbol{v}^{\mathrm{t}}\mathbb{K}_{\mathrm{D}}^{-1}(\boldsymbol{x})\boldsymbol{v} \;\geq\; \alpha_{\mathbb{K}_{\mathrm{D}}}|\boldsymbol{v}|^2 \quad \forall\,\boldsymbol{v} \in \mathrm{R}^n, \forall \boldsymbol{x} \in \Omega_{\mathrm{D}}.$$

In turn, the tensors $\mathbb{K}_{\mathrm{B}}$ and $\mathbb{K}_{\mathrm{D}}$ characterise the absolute permeability of the Brinkman and Darcy domains, respectively; the function ϑ is a nonlinear diffusivity, and f_{bk} is a nonlinear flux acting on the direction of the gravity acceleration $\boldsymbol{g}$, aligned with the negative x_n-axis. The specific forms of these variable coefficients will be made precise later. In addition, we assume that $\boldsymbol{f}_{\mathrm{B}} \in \mathbf{L}^\infty(\Omega_{\mathrm{B}})$ and $\boldsymbol{f}_{\mathrm{D}} \in \mathbf{L}^\infty(\Omega_{\mathrm{D}})$. We stress that the local fluctuations of ϕ drive the flow patterns only through the external load in the momentum equations. In this sense, the coupling mechanisms considered here are somehow weaker than those studied in [2, 3] for transport-flow in a single domain (where also viscosity was depending of ϕ).

We assume that Ω has a Lipschitz continuous boundary split into two disjoint sub-boundaries with positive measure, according to two criteria: firstly, $\partial\Omega = \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{D}}$, where $\Gamma_{\mathrm{B}} = \partial\Omega_{\mathrm{B}}\backslash\Sigma$ and $\Gamma_{\mathrm{D}} = \partial\Omega_{\mathrm{D}}\backslash\Sigma$ denote pure Brinkman and Darcy borders, respectively; and secondly $\partial\Omega = \Gamma_0 \cup \Gamma_{\mathrm{N}}$, where $\Gamma_0, \Gamma_{\mathrm{N}}$ denote the parts of the boundary where homogeneous Dirichlet or Neumann (zero flux) conditions are enforced for ϕ, respectively (see a rough diagram of domains and boundaries in Figure 2.1). The considered boundary and transmission conditions are:

$$\begin{gathered} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n} = \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} \quad \text{and} \quad p_{\mathrm{D}} = p_{\mathrm{B}} \quad \text{on} \quad \Sigma, \\ \boldsymbol{\omega}_{\mathrm{B}} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega_{\mathrm{B}} = \Sigma \cup \Gamma_{\mathrm{B}}\,,\; \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{B}}\,, \quad \text{and} \quad \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\,, \\ \phi = 0 \quad \text{on} \quad \Gamma_0, \quad \text{and} \quad (\vartheta(\phi)\nabla\phi - \phi\boldsymbol{u} - f_{\mathrm{bk}}(\phi)\boldsymbol{g}) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{N}}, \end{gathered} \tag{2.4}$$

where $\boldsymbol{n}$ denotes the outward normal at Ω_{B} and Ω_{D}. Note that interface conditions are not required in the transport equation, as the continuity of ϕ and of the corresponding fluxes is incorporated naturally in the formulation.

For the sake of our analysis, the variable coefficients need to satisfy the following requirements: there exist positive constants ϑ_1, ϑ_2, γ_1, γ_2, L_ϑ, and $L_{f_{\mathrm{bk}}}$, such that

$$\vartheta_1 \leq \vartheta(s) \leq \vartheta_2\,, \quad \text{and} \quad \gamma_1 \leq f_{\mathrm{bk}}(s) \leq \gamma_2 \qquad \forall\, s \in \mathrm{R}\,, \tag{2.5}$$

$$|\vartheta(s) - \vartheta(t)| \;\leq\; L_\vartheta\,|s - t| \qquad \forall\, s,\, t \in \mathrm{R}\,, \tag{2.6}$$

and

$$|f_{\mathrm{bk}}(s) - f_{\mathrm{bk}}(t)| \;\leq\; L_{f_{\mathrm{bk}}}\,|s - t| \qquad \forall\, s,\, t \in \mathrm{R}\,. \tag{2.7}$$

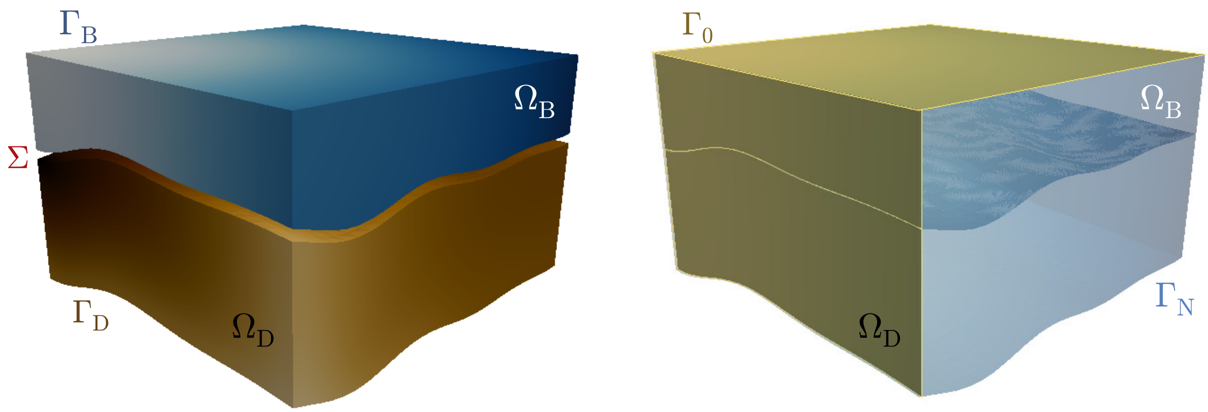

Figure 2.1: Sketch of the domains occupied by the incompressible fluid and by the porous medium (Ω_B and Ω_D, respectively), interface Σ, and corresponding boundaries.

In view of deriving a weak form of (2.1)-(2.3), and according to the boundary data (2.4), we introduce the following functional spaces

$$\begin{aligned}\mathbf{H}_B(\mathrm{div};\Omega_B) &:= \Big\{ \boldsymbol{v}_B \in \mathbf{H}(\mathrm{div};\Omega_B): \quad \boldsymbol{v}_B\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_B \Big\},\\ \mathbf{H}_0(\mathbf{curl};\Omega_B) &:= \Big\{ \boldsymbol{z}_B \in \mathbf{H}(\mathbf{curl};\Omega_B): \quad \boldsymbol{z}_B\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega_B \Big\},\\ \mathbf{H}_D(\mathrm{div};\Omega_D) &:= \Big\{ \boldsymbol{v}_D \in \mathbf{H}(\mathrm{div};\Omega_D): \quad \boldsymbol{v}_D\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D \Big\},\end{aligned}$$

and

$$\mathrm{H}^1_{\Gamma_0}(\Omega) := \{\psi \in \mathrm{H}^1(\Omega): \ \psi|_{\Gamma_0} = 0\},$$

for which, thanks to the generalised Poincaré inequality, there exists $c_p > 0$, depending only on Ω and Γ_0, such that

$$\|\psi\|_{1,\Omega} \le c_p\, |\psi|_{1,\Omega}, \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{2.8}$$

3 Weak formulation and its solvability analysis

In this section we proceed similarly as in [2] and [3] to derive a suitable variational formulation of (2.1)-(2.2)-(2.3)-(2.4) and analyse its corresponding solvability by using a fixed-point strategy.

3.1 A mixed-primal formulation

We first notice that the continuity of pressure across the interface Σ allows us to define its trace

$$\lambda := p_D|_\Sigma = p_B|_\Sigma \in \mathrm{H}^{1/2}(\Sigma)\,. \tag{3.1}$$

Then, after testing the momentum equation in (2.1) against $\boldsymbol{v}_B \in \mathbf{H}_B(\mathrm{div};\Omega_B)$, and integrating by parts, we get

$$\mu\int_{\Omega_B} \mathbb{K}_B^{-1}\boldsymbol{u}_B\cdot\boldsymbol{v}_B + \mu\int_{\Omega_B}\boldsymbol{v}_B\cdot\mathbf{curl}\,\boldsymbol{\omega}_B - \int_{\Omega_B} p_B\,\mathrm{div}\,\boldsymbol{v}_B + \langle \boldsymbol{v}_B\cdot\boldsymbol{n},\lambda\rangle_\Sigma = \int_{\Omega_B}\phi\boldsymbol{f}_B\cdot\boldsymbol{v}_B \quad \forall\,\boldsymbol{v}_B \in \mathbf{H}_B(\mathrm{div};\Omega_B).$$

Next, testing the constitutive equation in (2.1) against $\boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}})$, and integrating by parts, we obtain

$$\int_{\Omega_{\mathrm{B}}} \boldsymbol{\omega}_{\mathrm{B}} \cdot \boldsymbol{z}_{\mathrm{B}} - \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\,\boldsymbol{z}_{\mathrm{B}} = 0 \qquad \forall\, \boldsymbol{z}_{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}).$$

In turn, the incompressibility equation in (2.1) is tested as

$$\int_{\Omega_{\mathrm{B}}} q_{\mathrm{B}} \operatorname{div} \boldsymbol{u}_{\mathrm{B}} = 0 \qquad \forall\, q_{\mathrm{B}} \in \mathrm{L}^2(\Omega_{\mathrm{B}})\,.$$

On the other hand, testing the first equation of (2.2) with functions in $\mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})$, integrating by parts, using the corresponding boundary conditions, and employing (3.1), we get

$$\mu \int_{\Omega_{\mathrm{D}}} \mathbb{K}_{\mathrm{D}}^{-1} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} - \int_{\Omega_{\mathrm{D}}} p_{\mathrm{D}} \operatorname{div} \boldsymbol{v}_{\mathrm{D}} - \langle \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}, \lambda \rangle_{\Sigma} = \int_{\Omega_{\mathrm{D}}} \phi \boldsymbol{f}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \qquad \forall\, \boldsymbol{v}_{\mathrm{D}} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})\,.$$

In addition, similarly as for the incompressibility condition in Ω_{B}, the second equation in (2.2) is initially tested as

$$\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \operatorname{div} \boldsymbol{v}_{\mathrm{D}} = 0 \qquad \forall\, q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})\,.$$

Finally, the continuity of normal velocities across Σ (*cf.* first equation in (2.4)) is imposed weakly, that is

$$\langle \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{n} - \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{n}, \xi \rangle_{\Sigma} = 0 \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma)\,.$$

Therefore, given $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, we arrive at the following mixed formulation for the Brinkman-Darcy coupling: Find $\vec{\boldsymbol{u}} := (\boldsymbol{u}_{\mathrm{B}}, \boldsymbol{\omega}_{\mathrm{B}}, \boldsymbol{u}_{\mathrm{D}}) \in \mathbf{H}$ and $\vec{p} := (p_{\mathrm{B}}, p_{\mathrm{D}}, \lambda) \in \mathbf{Q}$, such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathcal{B}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathcal{F}_{\phi}(\vec{\boldsymbol{v}}) \quad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v}_{\mathrm{B}}, \boldsymbol{z}_{\mathrm{B}}, \boldsymbol{v}_{\mathrm{D}}) \in \mathbf{H}, \\ \mathcal{B}(\vec{\boldsymbol{u}}, \vec{q}) &= 0 \qquad\qquad \forall\, \vec{q} := (q_{\mathrm{B}}, q_{\mathrm{D}}, \lambda) \in \mathbf{Q}, \end{aligned} \tag{3.2}$$

where the product spaces are

$$\mathbf{H} := \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}}) \times \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}) \times \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}})\,, \quad \mathbf{Q} := \mathrm{L}^2(\Omega_{\mathrm{B}}) \times \mathrm{L}^2(\Omega_{\mathrm{D}}) \times \mathrm{H}^{1/2}(\Sigma)\,,$$

the bilinear forms $\mathcal{A} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $\mathcal{B} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$ are defined by

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) &:= \mu \int_{\Omega_{\mathrm{B}}} \mathbb{K}_{\mathrm{B}}^{-1} \boldsymbol{u}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{B}}} \boldsymbol{\omega}_{\mathrm{B}} \cdot \boldsymbol{z}_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{B}}} \boldsymbol{v}_{\mathrm{B}} \cdot \mathbf{curl}\,\boldsymbol{\omega}_{\mathrm{B}} \\ &\quad - \mu \int_{\Omega_{\mathrm{B}}} \boldsymbol{u}_{\mathrm{B}} \cdot \mathbf{curl}\,\boldsymbol{z}_{\mathrm{B}} + \mu \int_{\Omega_{\mathrm{D}}} \mathbb{K}_{\mathrm{D}}^{-1} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}}, \\ \mathcal{B}(\vec{\boldsymbol{v}}, \vec{q}) &:= - \int_{\Omega_{\mathrm{B}}} q_{\mathrm{B}} \operatorname{div} \boldsymbol{v}_{\mathrm{B}} - \int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}} \operatorname{div} \boldsymbol{v}_{\mathrm{D}} + \langle \boldsymbol{v}_{\mathrm{B}} \cdot \boldsymbol{n} - \boldsymbol{v}_{\mathrm{D}} \cdot \boldsymbol{n}, \xi \rangle_{\Sigma}, \end{aligned}$$

for all $\vec{\boldsymbol{v}} \in \mathbf{H}$, $\vec{q} \in \mathbf{Q}$, and $\mathcal{F} \in \mathbf{H}'$ is the functional defined by

$$\mathcal{F}_{\psi}(\vec{\boldsymbol{v}}) := \int_{\Omega_{\mathrm{B}}} \psi \boldsymbol{f}_{\mathrm{B}} \cdot \boldsymbol{v}_{\mathrm{B}} + \int_{\Omega_{\mathrm{D}}} \psi \boldsymbol{f}_{\mathrm{D}} \cdot \boldsymbol{v}_{\mathrm{D}} \quad \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}\,. \tag{3.3}$$

Next, we observe that the solution for (3.2) is not unique. Indeed, it suffices to consider $\vec{p} := (c, c, c)$, with $c \in \mathrm{R}$, and note that $(\mathbf{0}, \vec{p})$ is also solution of the associated homogeneous system (see [4, Theorem 3.1]). In order to guarantee the uniqueness of the solution to (3.2), and similarly to [4], we consider, instead of (3.2), the following mixed formulation for the Brinkman-Darcy coupling: Find $(\vec{\boldsymbol{u}}, \vec{p}) \in \mathbf{H} \times \mathbf{Q}_0$, such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathcal{B}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathcal{F}_\phi(\vec{\boldsymbol{v}}) \quad \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}, \\ \mathcal{B}(\vec{\boldsymbol{u}}, \vec{q}) &= 0 \qquad\quad\;\; \forall\, \vec{q} \in \mathbf{Q}_0, \end{aligned} \tag{3.4}$$

where $\mathbf{Q}_0 := \mathrm{L}^2_0(\Omega_\mathrm{B}) \times \mathrm{L}^2(\Omega_\mathrm{D}) \times \mathrm{H}^{1/2}(\Sigma)$.

On the other hand, given $\boldsymbol{u}$ in a suitable space (to be indicated later on in Lemma 3.2), testing with functions in $\mathrm{H}^1_{\Gamma_0}(\Omega)$, integrating by parts and using the boundary data, we deduce the following primal formulation for the transport problem: Find $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathcal{C}_{\boldsymbol{u}}(\phi, \psi) \,=\, \int_\Omega f_\mathrm{bk}(\phi)\, \boldsymbol{g} \cdot \nabla \psi \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \tag{3.5}$$

where, the form $\mathcal{C}_{\boldsymbol{u}}$ is defined by

$$\mathcal{C}_{\boldsymbol{u}}(\phi, \psi) \,:=\, \int_\Omega \vartheta(\phi)\, \nabla\phi \cdot \nabla\psi \,-\, \int_\Omega \phi\, \boldsymbol{u} \cdot \nabla\psi \,+\, \int_\Omega \beta\, \phi\, \psi \qquad \forall\, \phi,\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,.$$

In this way, the mixed-primal formulation of our original coupled problem (2.1)-(2.3);(2.4), reduces to (3.4)-(3.5), that is: Find $(\vec{\boldsymbol{u}}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathcal{B}(\vec{\boldsymbol{v}}, \vec{p}) &= \mathcal{F}_\phi(\vec{\boldsymbol{v}}) && \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}, \\ \mathcal{B}(\vec{\boldsymbol{u}}, \vec{q}) &= 0 && \forall\, \vec{q} \in \mathbf{Q}_0, \\ \mathcal{C}_{\boldsymbol{u}}(\phi, \psi) &= \int_\Omega f_\mathrm{bk}(\phi)\, \boldsymbol{g} \cdot \nabla\psi && \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,. \end{aligned} \tag{3.6}$$

The well-posedness of (3.6) is addressed below in Sections 3.3 and 3.4 employing a fixed-point strategy that is explained in the following section. We remark in advance that, in order to deal with the analysis of (3.5), and particularly to estimate the second term defining $\mathcal{C}_{\boldsymbol{u}}$, we will require further regularity for the global velocity. This assumption will be specified below in Section 3.3.

3.2 Fixed point strategy

We now describe our fixed-point framework for (3.6). According to the definition of the global velocity, we first introduce the operator $\mathbf{S} : \mathrm{H}^1_{\Gamma_0}(\Omega) \longrightarrow \mathbf{H}(\mathrm{div}; \Omega)$ defined as

$$\mathbf{S}(\phi) \,:=\, \boldsymbol{u} \qquad \forall\, \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,,$$

where $\boldsymbol{u}|_{\Omega_\mathrm{B}} = \boldsymbol{u}_\mathrm{B}$ and $\boldsymbol{u}|_{\Omega_\mathrm{D}} = \boldsymbol{u}_\mathrm{D}$ are the first and third components of $\vec{\boldsymbol{u}} \in \mathbf{H}$, which in turn is the first component of the unique solution (to be confirmed below) of the problem (3.4) with the given ϕ.

In turn, we also introduce the operator $\widetilde{\mathbf{S}} : \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega) \longrightarrow \mathrm{H}^1_{\Gamma_0}(\Omega)$ defined as

$$\widetilde{\mathbf{S}}(\phi, \boldsymbol{u}) \,:=\, \widetilde{\phi} \qquad \forall\, (\phi, \boldsymbol{u}) \,\in\, \mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div}; \Omega)\,,$$

where $\widetilde{\phi}$ is the unique solution (to be confirmed below) of the linear problem: Find $\widetilde{\phi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathcal{C}_{\phi, \boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) \,=\, \mathcal{H}_\phi(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \tag{3.7}$$

for fixed $(\phi, \boldsymbol{u})$, where the involved bilinear form is defined as

$$\mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) := \int_\Omega \vartheta(\phi)\, \nabla\widetilde{\phi} \cdot \nabla\widetilde{\psi} - \int_\Omega \widetilde{\phi}\, \boldsymbol{u} \cdot \nabla\widetilde{\psi} + \int_\Omega \beta\, \widetilde{\phi}\, \widetilde{\psi} \qquad \forall\, \widetilde{\phi},\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \tag{3.8}$$

and the linear functional is given by

$$\mathcal{H}_\phi(\widetilde{\psi}) := \int_\Omega f_{\mathrm{bk}}(\phi)\, \boldsymbol{g} \cdot \nabla\widetilde{\psi} \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,. \tag{3.9}$$

Here, we stress in advance that actually $\widetilde{\mathbf{S}}$ will be well-defined not in the whole space $\mathrm{H}^1_{\Gamma_0}(\Omega) \times \mathbf{H}(\mathrm{div};\Omega)$, but only in a subspace of it (see Lemma 3.2 below).

Finally, we define the operator $\mathbf{T} : \mathrm{H}^1_{\Gamma_0}(\Omega) \longrightarrow \mathrm{H}^1_{\Gamma_0}(\Omega)$ as

$$\mathbf{T}(\phi) := \widetilde{\mathbf{S}}(\phi, \mathbf{S}(\phi)) \qquad \forall\, \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \tag{3.10}$$

and realise that solving (3.6) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ such that

$$\mathbf{T}(\phi) = \phi\,. \tag{3.11}$$

3.3 Well-posedness of the uncoupled problem

In this section, we show that the uncoupled problems (3.4) and (3.7) are in fact well-posed. We begin the solvability analysis with the following result, whose proof is a direct consequence of [4, Theorem 3.2]. Let us remark that similar vorticity-based formulations for Brinkman-Darcy equations can be analysed using a different approach, as done recently in [6].

Lemma 3.1 *For each $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, problem (3.4) has a unique solution $(\vec{\boldsymbol{u}}, \vec{p}) \in \mathbf{H} \times \mathbf{Q}_0$. Moreover, there exists $C_{\mathbf{S}} > 0$, independent of ϕ, such that*

$$\|\mathbf{S}(\phi)\|_{\mathrm{div},\Omega} \le \|(\vec{\boldsymbol{u}}, \vec{p})\|_{\mathbf{H}\times\mathbf{Q}_0} \le C_{\mathbf{S}}\, \|\phi\|_{0,\Omega} \Big\{ \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big\}, \qquad \forall\, \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega). \tag{3.12}$$

For the purpose of the next result, which provides the solvability of the uncoupled problem (3.7), we require that the global velocity $\boldsymbol{u}$ belong to $\mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega)$ for some $\delta \in (0,1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$). In turn, according to the aforementioned range for δ, we recall that the Sobolev embedding Theorem (*cf.* Ref. [1] [Theorem 4.12], Ref. [17] [Theorem 1.3.4]) establishes the continuous injection $\boldsymbol{i}_\delta : \mathbf{H}^\delta(\Omega) \longrightarrow \mathbf{L}^{\delta^*}(\Omega)$ with boundedness constant $\mathbf{C}^*_\delta$, where

$$\delta^* := \begin{cases} \frac{2}{1-\delta} & \text{if} \quad n = 2\,, \\ \frac{6}{3-2\delta} & \text{if} \quad n = 3\,. \end{cases} \tag{3.13}$$

and it also guarantees that the injection $\mathrm{i} : \mathrm{H}^1(\Omega) \to \mathrm{L}^{n/\delta}(\Omega)$ is compact, and hence continuous, with constant C_δ. In addition, for the subsequent writing we set $r_0 := \frac{\vartheta_1}{2c_p \mathrm{C}_\delta \mathbf{C}^*_\delta}$, where ϑ_1 and c_p are the constants given in (2.5) and (2.8), respectively.

Lemma 3.2 *Let $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, and $\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega)$ for some $\delta \in (0,1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$), such that $\|\boldsymbol{u}\|_{\delta,\Omega} < r_0$. Then, the problem (3.7) has a unique solution $\widetilde{\mathbf{S}}(\phi, \boldsymbol{u}) := \widetilde{\phi} \in \mathrm{H}^1_{\Gamma_0}(\Omega)$. Moreover, there exists $C_{\widetilde{\mathbf{S}}} > 0$, independent of $(\phi, \boldsymbol{u})$, such that*

$$\|\widetilde{\mathbf{S}}(\phi, \boldsymbol{u})\|_{1,\Omega} \le C_{\widetilde{\mathbf{S}}}\, \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|. \tag{3.14}$$

Proof. We first notice that $\mathcal{C}_{\phi,\boldsymbol{u}}$ (*cf.* (3.8)) is clearly a bilinear form. In turn, employing the upper bound of ϑ (*cf.* (2.5)), Cauchy-Schwarz's inequality, and Hölder's inequality, it readily follows from (3.8) that

$$|\mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi},\widetilde{\psi})| \,\le\, \vartheta_2\,|\widetilde{\phi}|_{1,\Omega}\,|\widetilde{\psi}|_{1,\Omega} \,+\, \|\widetilde{\phi}\|_{\mathrm{L}^{2q}(\Omega)}\,\|\boldsymbol{u}\|_{\mathbf{L}^{2p}(\Omega)}\,|\widetilde{\psi}|_{1,\Omega} \,+\, \beta\,\|\widetilde{\phi}\|_{0,\Omega}\,\|\widetilde{\psi}\|_{0,\Omega} \tag{3.15}$$

where $p,q\in[1,+\infty)$ are such that $1/p+1/q=1$. Next, choosing p such that $2p=\delta^*$ (*cf.* (3.13)), it readily follows that

$$2q \,:=\, \frac{2p}{p-1} \,=\, \frac{n}{\delta}. \tag{3.16}$$

In this way, applying the continuous injections $\boldsymbol{i}_\delta : \mathbf{H}^\delta(\Omega) \longrightarrow \mathbf{L}^{\delta^*}(\Omega)$, and $\mathrm{i} : \mathrm{H}^1(\Omega)\to \mathrm{L}^{n/\delta}(\Omega)$, the latter being consequence of the range of δ, and employing the bound for $\|\boldsymbol{u}\|_{\delta,\Omega}$ assumed here, we deduce from (3.15) the existence of a positive constant $\|\mathcal{C}\|$, depending on $\vartheta_1,\vartheta_2,\beta$, $\|\boldsymbol{i}\|$, $\|\mathrm{i}\|$, and c_p, such that

$$|\mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi},\widetilde{\psi})| \,\le\, \|\mathcal{C}\|\,\|\widetilde{\phi}\|_{1,\Omega}\,\|\widetilde{\psi}\|_{1,\Omega} \quad \forall\,\widetilde{\phi},\,\widetilde{\psi}\in\mathrm{H}^1_{\Gamma_0}(\Omega)\,,$$

which proves that $\mathcal{C}_{\phi,\boldsymbol{u}}$ is bounded independently of ϕ and $\boldsymbol{u}$. On the other hand, applying the same argument used for the derivation of second term on the right hand side of (3.15), and using (3.13), (3.16) and (2.8), we find that for each $\widetilde{\phi}\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ there holds

$$\begin{aligned}
\mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi},\widetilde{\phi}) \,&=\, \int_\Omega \vartheta(\phi)\,|\nabla\widetilde{\phi}|^2 \,-\, \int_\Omega \widetilde{\phi}\boldsymbol{u}\cdot\nabla\widetilde{\phi} \,+\, \beta\,\|\widetilde{\phi}\|^2_{0,\Omega}\\
&\ge\, \vartheta_1\,|\widetilde{\phi}|^2_{1,\Omega} \,-\, \mathbf{C}^*_\delta\,\|\widetilde{\phi}\|_{\mathrm{L}^{n/\delta}(\Omega)}\,\|\boldsymbol{u}\|_{\delta,\Omega}\,|\widetilde{\phi}|_{1,\Omega} \,+\, \beta\,\|\widetilde{\phi}\|^2_{0,\Omega}\\
&\ge\, (\vartheta_1 \,-\, c_p\mathrm{C}_\delta\mathbf{C}^*_\delta\,\|\boldsymbol{u}\|_{\delta,\Omega})\,|\widetilde{\phi}|^2_{1,\Omega}\\
&\ge\, \frac{\vartheta_1}{2}\,|\widetilde{\phi}|^2_{1,\Omega} \,\ge\, \frac{\vartheta_1}{2c_p^2}\,\|\widetilde{\phi}\|^2_{1,\Omega}\,,
\end{aligned} \tag{3.17}$$

which proves that $\mathcal{C}_{\phi,\boldsymbol{u}}$ is $\mathrm{H}^1_{\Gamma_0}(\Omega)$-elliptic with constant $\widetilde{\alpha} \,:=\, \frac{\vartheta_1}{2\,c_p^2}$, independently of both ϕ and $\boldsymbol{u}$. Next, applying Cauchy-Schwarz inequality and the upper bound for f_{bk} given in (2.5), we easily deduce that

$$|\mathcal{H}_\phi(\widetilde{\psi})| \,\le\, \gamma_2\,|\Omega|^{1/2}\,|\boldsymbol{g}|\,\|\widetilde{\psi}\|_{1,\Omega} \qquad \forall\,\widetilde{\psi}\,\in\,\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,,$$

which says that $\mathcal{H}_\phi\in\mathrm{H}^1_{\Gamma_0}(\Omega)'$ and $\|\mathcal{H}_\phi\| \,\le\, \gamma_2\,|\Omega|^{1/2}\,|\boldsymbol{g}|$. Consequently, a direct application of the Lax-Milgram Lemma implies the existence of a unique solution $\widetilde{\phi} \,:=\, \widetilde{\mathbf{S}}(\phi,\boldsymbol{u})\in\mathrm{H}^1_{\Gamma_0}(\Omega)$ of (3.7), and the corresponding continuous dependence result becomes (3.14) with $C_{\widetilde{\mathbf{S}}} = \dfrac{1}{\widetilde{\alpha}} = \dfrac{2c_p^2}{\vartheta_1}$. $\square$

At this point we remark that the restriction on $\|\boldsymbol{u}\|_{\delta,\Omega}$ in Lemma 3.2 could also have been taken as

$$\|\boldsymbol{u}\|_{\delta,\Omega} \,<\, \varepsilon\,\frac{\vartheta_1}{c_p\mathrm{C}_\delta\mathbf{C}^*_\delta}$$

with any $\varepsilon\,\in\,(0,1)$. However, we have chosen $\varepsilon \,=\, \frac{1}{2}$ for simplicity and because it yields a joint maximisation of the ellipticity constant of $\mathcal{C}_{\phi,\boldsymbol{u}}$. In addition, when dropping the term $\beta\,\|\widetilde{\phi}\|^2_{0,\Omega}$ in (3.17) we have first assumed that β is small and then utilised the Poincaré inequality (2.8). In turn, when β is sufficiently large, say $\beta\ge\vartheta_1$, then the aforementioned expression is kept along the whole derivation of (3.17), implying that the Poincaré inequality (2.8) is not required.

We end this section by introducing adequate regularity hypotheses on the operator $\mathbf{S}$ which will be employed to guarantee that the operator $\mathbf{T}$ is well defined. In addition, sufficient regularity of the operator $\widetilde{\mathbf{S}}$ is also assumed in order to establish its Lipschitz continuity, and then also that for $\mathbf{T}$. In fact, for the remainder of this paper we follow [3, Eq. (3.23) and Eq. (3.24)], and consider the following two hypotheses.

Regularity Hypothesis 3.1 *For $\boldsymbol{f}_{\mathrm{B}} \in \mathbf{L}^\infty(\Omega_{\mathrm{B}})$, $\boldsymbol{f}_{\mathrm{D}} \in \mathbf{L}^\infty(\Omega_{\mathrm{D}})$, and for each $\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ with $\|\phi\|_{1,\Omega} \le r$, $r > 0$ given, there holds $\mathbf{S}(\phi) \in \mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega)$, for some $\delta \in (0,1)$ (when $n = 2$) or $\delta \in (1/2, 1)$ (when $n = 3$), with*

$$\|\mathbf{S}(\phi)\|_{\delta,\Omega} \,\le\, \widehat{C}_{\mathbf{S}}(r)\, \|\phi\|_{0,\Omega} \Big\{ \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big\}, \tag{3.18}$$

where $\widehat{C}_{\mathbf{S}}(r)$ is a positive constant independent of ϕ, but depending on the upper bound r of its norm.

Regularity Hypothesis 3.2 *For each $(\varphi, \boldsymbol{w}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times (\mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^\delta(\Omega))$, with $\delta \in (0,1)$ (when $n = 2$) or $\delta \in (1/2,1)$ (when $n = 3$), and $\|\varphi\|_{1,\Omega} + \|\boldsymbol{w}\|_{\mathrm{div},\Omega} + \|\boldsymbol{w}\|_\delta \le r$, $r > 0$ given, there holds $\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w}) \in \mathrm{H}^{1+\delta}_{\Gamma_0}(\Omega)$, with*

$$\|\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{1+\delta,\Omega} \,\le\, \widehat{C}_{\widetilde{\mathbf{S}}}(r)\, C(\Omega)\, |\boldsymbol{g}|, \tag{3.19}$$

where $C(\Omega)$ is a positive constant depending only on the domain Ω, and $\widehat{C}_{\widetilde{\mathbf{S}}}(r)$ is a positive constant independent of $(\varphi, \boldsymbol{w})$, but depending on the upper bound r of its norm.

We remark that similar hypotheses have been employed in [3, Section 3.3]. We also point out, in advance, that Hypothesis 3.1 is needed in the proof of Lemma 3.3 to make use of Lemma 3.2, which is crucial to prove that the operator $\mathbf{T}$ is well-defined. Afterward, the estimate (3.18) is also employed in Lemma 3.5 to bound an expression of the form $\|\mathbf{S}(\phi - \varphi)\|_{\mathbf{L}^{2p}(\Omega)}$ in terms of $\|\mathbf{S}(\phi - \varphi)\|_{\delta,\Omega}$, and hence of the data at the right hand side of (3.18). In turn, the further regularity from Hypothesis 3.2 is used in the proof of Lemma 3.4 to bound an expression of the form $\|\nabla\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{\mathbb{L}^{2p}(\Omega)}$ in terms of $\|\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{1+\delta,\Omega}$, which subsequently, in the proof of Theorem 3.7, can be bounded by the data at the right hand side of (3.19).

3.4 Solvability of the fixed-point equation

The well-posedness of the uncoupled problems (3.4) and (3.7) confirms that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\mathbf{T}$ (*cf.* Section 3.2) are well defined, and hence now we can address the solvability of the fixed-point equation (3.11). To this end, we will proceed to verify the hypotheses of the Schauder fixed-point theorem (see, e.g. [10] [Theorem 9.12-1(b)]).

Lemma 3.3 *Given $r > 0$, we let $W^\phi := \big\{ \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega) : \quad \|\phi\|_{1,\Omega} \le r \big\}$ and assume that*

$$\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \,<\, \frac{r_0}{r\widehat{C}_{\mathbf{S}}(r)} \qquad \text{and} \qquad C_{\widetilde{\mathbf{S}}}\, \gamma_2\, |\Omega|^{1/2}\, |\boldsymbol{g}| \le r \tag{3.20}$$

where r_0 is the constant specified right before Lemma 3.2. Then $\mathbf{T}(\mathrm{W}^\phi) \subseteq \mathrm{W}^\phi$.

Proof. Given $\phi \in \mathrm{W}^\phi$, by virtue of Lemma 3.1, the estimate (3.18) together with the first condition in (3.20), and Lemma 3.2, it follows that $\widetilde{\mathbf{S}}(\phi, \mathbf{S}(\phi)) := \mathbf{T}(\phi)$ is well defined. Next, according to the definition of the operator $\mathbf{T}$ (*cf.* (3.10)), and the continuous dependence estimate (3.14), it readily follows that

$$\|\mathbf{T}(\phi)\|_{1,\Omega} \,=\, \|\widetilde{\mathbf{S}}(\phi, \mathbf{S}(\phi))\|_{1,\Omega} \,\le\, C_{\widetilde{\mathbf{S}}}\, \gamma_2\, |\Omega|^{1/2}\, |\boldsymbol{g}|,$$

which, due to the second inequality in (3.20), proves that $\mathbf{T}(\phi) \in \mathrm{W}^{\phi}$, thus finishing the proof. $\square$

Our next goal is to establish the continuity and compactness of $\mathbf{T}$, which is precisely the purpose of the following two lemmas.

Lemma 3.4 *There exists a positive constant $\widetilde{C} > 0$, depending on $L_{f_{\mathrm{bk}}}, L_{\vartheta}, \mathbf{C}^*_{\delta}$, $\widetilde{\alpha}$ (*cf. (2.7), (2.6), *Lemma* 3.2*), and the boundedness constant C^*_{δ} of the injection $\mathrm{i}_{\delta} : \mathrm{H}^{\delta}(\Omega) \to \mathrm{L}^{\delta^*}(\Omega)$, such that for all $(\phi, \boldsymbol{u}), (\varphi, \boldsymbol{w}) \in \mathrm{H}^1_{\Gamma_0}(\Omega) \times (\mathbf{H}(\mathrm{div};\Omega) \cap \mathbf{H}^{\delta}(\Omega))$, with $\|\boldsymbol{u}\|_{\delta,\Omega}, \|\boldsymbol{w}\|_{\delta,\Omega} < r_0$ (*cf. *Lemma* 3.2*), there holds*

$$
\begin{aligned}
\|\widetilde{\mathbf{S}}(\phi, \boldsymbol{u}) - \widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{1,\Omega} \,\le\, \widetilde{C} \Big\{ &|\boldsymbol{g}| \, \|\phi - \varphi\|_{0,\Omega} \,+\, \|\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{\mathrm{L}^{n/\delta}(\Omega)} \|\boldsymbol{u} - \boldsymbol{w}\|_{\delta,\Omega} \\
&+ \|\widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})\|_{1+\delta,\Omega} \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)} \Big\}.
\end{aligned}
\tag{3.21}
$$

Proof. Given $(\phi, \boldsymbol{u}), (\varphi, \boldsymbol{w})$ as stated, we let $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi, \boldsymbol{u})$ and $\widetilde{\varphi} := \widetilde{\mathbf{S}}(\varphi, \boldsymbol{w})$, that is (*cf.* (3.7))

$$
\mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) \,=\, \mathcal{H}_{\phi}(\widetilde{\psi}) \quad \text{and} \quad \mathcal{C}_{\varphi,\boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\psi}) \,=\, \mathcal{H}_{\varphi}(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_0}(\Omega) \,.
$$

Then, according to the ellipticity of $\mathcal{C}_{\phi,\boldsymbol{u}}$ with constant $\widetilde{\alpha}$, subtracting and adding $\mathcal{H}_{\varphi}(\widetilde{\phi} - \widetilde{\varphi}) = \mathcal{C}_{\varphi,\boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})$, it follows that

$$
\begin{aligned}
\widetilde{\alpha} \, \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1,\Omega} \,&\le\, \mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\phi} - \widetilde{\varphi}) \,-\, \mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) \\
&= \quad \mathcal{H}_{\phi}(\widetilde{\phi} - \widetilde{\varphi}) - \mathcal{H}_{\varphi}(\widetilde{\phi} - \widetilde{\varphi}) \,+\, \mathcal{C}_{\varphi,\boldsymbol{w}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) - \mathcal{C}_{\phi,\boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) \\
&= \quad \int_{\Omega} \big(f_{\mathrm{bk}}(\phi) - f_{\mathrm{bk}}(\varphi)\big) \, \boldsymbol{g} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}) \,+\, \int_{\Omega} \widetilde{\varphi} \, (\boldsymbol{u} - \boldsymbol{w}) \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}) \\
&+ \quad \int_{\Omega} \Big(\vartheta(\varphi) - \vartheta(\phi)\Big) \nabla\widetilde{\varphi} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}) \,,
\end{aligned}
\tag{3.22}
$$

where for the last equality we have employed definitions (3.8) and (3.9). Then applying Cauchy-Schwarz's inequality, Hölder's inequality, the further regularity in Hypothesis 3.2, the Lipschitz-continuity (2.6)-(2.7), and proceeding similarly as in (3.17) (see also [3, Eq. (3.29)]) on the last two terms in (3.22), we obtain

$$
\begin{aligned}
\widetilde{\alpha} \, \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1,\Omega} \,\le\, &\Big\{ L_{f_{\mathrm{bk}}} \, |\boldsymbol{g}| \, \|\phi \,-\, \varphi\|_{0,\Omega} \,+\, \mathbf{C}^*_{\delta} \, \|\widetilde{\varphi}\|_{\mathrm{L}^{n/\delta}(\Omega)} \, \|\boldsymbol{u} - \boldsymbol{w}\|_{\delta,\Omega} \Big\} \, |\widetilde{\phi} - \widetilde{\varphi}|_{1,\Omega} \\
&+ \quad L_{\vartheta} \, \mathrm{C}^*_{\delta} \, \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)} \, \|\nabla\widetilde{\varphi}\|_{\delta,\Omega} \, |\widetilde{\phi} - \widetilde{\varphi}|_{1,\Omega} \,,
\end{aligned}
\tag{3.23}
$$

In this way, inequalities (3.22) and (3.23) imply (3.21), which finishes the proof. $\square$

The following result is a straightforward consequence of Lemma 3.4

Lemma 3.5 *Given $r > 0$, we let $\mathrm{W}^{\phi} := \{\phi \in \mathrm{H}^1_{\Gamma_0}(\Omega) : \quad \|\phi\|_{1,\Omega} \le r\}$ and assume* (3.20)*. Then, for all $\phi, \varphi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, there holds*

$$
\begin{aligned}
\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega} \,\le\, &\Big\{ \widetilde{C} \, |\boldsymbol{g}| \,+\, r \, \widetilde{C} \, \widehat{C}_{\mathbf{S}}(r) (\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}) \, \|\mathbf{T}(\varphi)\|_{\mathrm{L}^{n/\delta}(\Omega)} \Big\} \|\phi - \varphi\|_{0,\Omega} \\
&+ \widetilde{C} \, \|\mathbf{T}(\varphi)\|_{1+\delta,\Omega} \|\phi - \varphi\|_{\mathrm{L}^{n/\delta}(\Omega)},
\end{aligned}
\tag{3.24}
$$

*where $\widetilde{C}$ and $\widehat{C}_{\mathbf{S}}(r)$ are the constants given in *Lemma* 3.4 *and estimate* (3.18)*, respectively.*

Proof. It suffices to recall from Section 3.2 that $\mathbf{T}(\phi) = \widetilde{\mathbf{S}}(\phi, \mathbf{S}(\phi)) \quad \forall \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega)$, and then apply Lemmas 3.3, 3.4, the linearity of $\mathbf{S}$, and the estimate (3.18). $\square$

The announced properties of $\mathbf{T}$ are proved now.

Lemma 3.6 *Given $r > 0$, we let $\mathrm{W}^\phi := \left\{ \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega) : \quad \|\phi\|_{1,\Omega} \le r \right\}$, and assume* (3.20) *(*cf. *Lemma* 3.3*). Then, $\mathbf{T} : \mathrm{W}^\phi \to \mathrm{W}^\phi$ is continuous and $\overline{\mathbf{T}(\mathrm{W}^\phi)}$ is compact.*

Proof. It follows almost verbatim as the proof of [2, Lemma 3.12]. Indeed, it is basically a consequence of the Rellich-Kondrachov compactness Theorem (*cf.* [1, Theorem 6.3], [17, Theorem 1.3.5]), the specified range of the constant δ involved in the further regularity Hypotheses 3.1 and 3.2, and the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence. We omit further details. $\square$

Finally, the main result of this section is stated as follows.

Theorem 3.7 *Assume that the hypotheses of the Lemmas* 3.3-3.6 *are met. Then, the mixed-primal problem* (3.6) *has at least one solution $(\vec{\boldsymbol{u}}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ with $\phi \in \mathrm{W}^\phi$, satisfying*

$$\|(\vec{\boldsymbol{u}}, \vec{p})\|_{\mathbf{H}\times\mathbf{Q}_0} \le r\, C_{\mathbf{S}} \Big\{ \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big\}, \tag{3.25}$$

and

$$\|\phi\|_{1,\Omega} \le C_{\widetilde{\mathbf{S}}}\, \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|, \tag{3.26}$$

where $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$ are the constants specified in Lemmas 3.1 *and* 3.2*, respectively. Moreover, if the data $\boldsymbol{f}_{\mathrm{B}}, \boldsymbol{f}_{\mathrm{D}}$ and $\boldsymbol{g}$ are sufficiently small so that, with the constants $\widetilde{C}$, $\widehat{C}_{\mathbf{S}}(r)$, $\widehat{C}_{\widetilde{\mathbf{S}}}(r)$ and $C(\Omega)$ from Lemma* 3.4*, and estimates* (3.18) *and* (3.19)*, and denoting by C_δ the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\delta}(\Omega)$, there holds*

$$\widetilde{C}(1 + \widehat{C}_{\widetilde{\mathbf{S}}}(r)\, \mathrm{C}_\delta C(\Omega))\, |\boldsymbol{g}| + r^2\, \widetilde{C}\, \mathrm{C}_\delta\, \widehat{C}_{\mathbf{S}}(r) \Big(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big) < 1, \tag{3.27}$$

then the solution ϕ is unique in W^ϕ.

Proof. According to the equivalence between (3.6) and the fixed-point equation (3.11), and thanks to Lemmas 3.3 and 3.6, the existence of solution is a direct consequence of the Schauder fixed-point theorem (*cf.* [10, Theorem 9.12-1(b)]). In turn, the estimates (3.25) and (3.26) follow from (3.12) (*cf.* Lemma 3.1) and (3.14) (*cf.* Lemma 3.2), respectively. Finally, given another solution $\varphi \in \mathrm{W}^\phi$ of (3.11), the estimates (3.24),

$$\|\mathbf{T}(\varphi)\|_{1,\Omega} = \|\varphi\|_{1,\Omega} \le r, \qquad \|\widetilde{\varphi}\|_{1+\delta,\Omega} \le \widehat{C}_{\widetilde{\mathbf{S}}}(r)\, C(\Omega)\, |\boldsymbol{g}| \quad (\textit{cf. } (3.19)),$$

and

$$\|\psi\|_{\mathrm{L}^{n/\delta}(\Omega)} \le \mathrm{C}_\delta\, \|\psi\|_{1,\Omega} \quad \forall\, \psi \in \mathrm{H}^1(\Omega)\,,$$

confirm (3.27) as a sufficient condition for concluding that $\phi = \varphi$. $\square$

4 Galerkin scheme

Let $\mathcal{T}_h$ be a regular family of triangulations of $\bar{\Omega}_{\mathrm{B}} \cup \bar{\Omega}_{\mathrm{D}}$ by tetrahedra K of diameter h_K with meshsize $h := \max\{h_K : \ K \in \mathcal{T}_h\}$, such that $\mathcal{T}_h(\Omega_\star) := \Big\{ K \in \mathcal{T}_h : \ K \subseteq \bar{\Omega}_\star \Big\}$ is a triangulation of $\Omega_\star$ for each $\star \in \{\mathrm{B}, \mathrm{D}\}$. We denote by $\mathcal{T}_h(\Sigma)$ the triangulation on Σ induced by $\mathcal{T}_h$ (either from Ω_{B} or Ω_{D}). Also, we introduce an independent triangulation $\mathcal{T}_{\tilde{h}}(\Sigma)$ of Σ by triangles $\widetilde{T}$ of diameter $h_{\widetilde{T}}$, and define $\tilde{h} := \max\{h_{\widetilde{T}} : \widetilde{T} \in \mathcal{T}_{\tilde{h}}(\Sigma)\}$. We now introduce the following finite dimensional subspaces of the test and trial spaces appearing in Section 3:

$$
\begin{gathered}
\mathbf{H}_h^{\mathrm{B}} \subseteq \mathbf{H}_{\mathrm{B}}(\mathrm{div};\Omega_{\mathrm{B}}), \qquad \mathbf{H}_{0,h}^{\mathrm{B}} \subseteq \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}), \qquad \mathbf{H}_h^{\mathrm{D}} \subseteq \mathbf{H}_{\mathrm{D}}(\mathrm{div};\Omega_{\mathrm{D}}), \qquad \mathbf{X}_h \subset \mathbf{H}(\mathrm{div};\Omega), \\
\mathrm{Q}_h^{\mathrm{B}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{B}}), \qquad \mathrm{Q}_h^{\mathrm{D}} \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}), \qquad \mathrm{Q}_h^{\Sigma} \subseteq \mathrm{H}^{1/2}(\Sigma), \qquad \mathrm{H}_h^{\phi} \subseteq \mathrm{H}^1_{\Gamma_0}(\Omega).
\end{gathered} \tag{4.1}
$$

Hence, setting the global spaces

$$
\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}} \times \mathbf{H}_{0,h}^{\mathrm{B}} \times \mathbf{H}_h^{\mathrm{D}} \quad \text{and} \quad \mathbf{Q}_{0,h} := \mathrm{Q}_{h,0}^{\mathrm{B}} \times \mathrm{Q}_h^{\mathrm{D}} \times \mathrm{Q}_h^{\Sigma},
$$

the Galerkin scheme for (3.6) becomes: Find $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^{\phi}$ such that

$$
\begin{array}{rcll}
\mathcal{A}(\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h) + \mathcal{B}(\vec{\boldsymbol{v}}_h, \vec{p}_h) & = & \mathcal{F}_{\phi_h}(\vec{\boldsymbol{v}}_h) & \forall\, \vec{\boldsymbol{v}}_h \in \mathbf{H}_h, \\
\mathcal{B}(\vec{\boldsymbol{u}}_h, \vec{q}_h) & = & 0 & \forall\, \vec{q}_h \in \mathbf{Q}_{0,h}, \\
\mathcal{C}_{\boldsymbol{u}_h}(\phi_h, \psi_h) & = & \mathcal{H}_{\phi_h}(\psi_h) & \forall\, \psi_h \in \mathrm{H}_h^{\phi}.
\end{array} \tag{4.2}
$$

In order to guarantee the well-posedness of the discrete scheme associated to (3.4), and hence of the Galerkin scheme (4.2), the subspaces introduced in (4.1) can be chosen as follows (see [4, Section 4.1])

$$
\begin{aligned}
\mathbf{H}_h^{\star} &:= \Big\{ \boldsymbol{v}_h^{\star} \in \mathbf{H}_{\star}(\mathrm{div};\Omega_{\star}) : \quad \boldsymbol{v}_h^{\star}|_K \in \mathbb{RT}_0(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_{\star}) \Big\}, \\
\mathrm{Q}_h^{\star} &:= \Big\{ q_h \in \mathrm{L}^2(\Omega_{\star}) : \quad q_h|_K \in P_0(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_{\star}) \Big\}, \\
\mathrm{Q}_{h,0}^{\star} &:= \mathrm{Q}_h^{\star} \cap \mathrm{L}^2_0(\Omega_{\star}), \\
\mathrm{H}_h^{\phi} &:= \Big\{ \psi_h \in C(\Omega) \cap \mathrm{H}^1_{\Gamma_0}(\Omega) : \quad \psi_h|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \mathcal{T}_h \Big\},
\end{aligned}
$$

with $\star \in \{\mathrm{B}, \mathrm{D}\}$, and for any $K \in \mathcal{T}_h(\Omega_{\star})$

$$
\mathbb{RT}_0(K) := \mathbf{P}_0(K) \oplus P_0(K)\, \boldsymbol{x}
$$

is the local Raviart-Thomas space of lowest order. In addition, we set

$$
\mathbf{H}_{0,h}^{\mathrm{B}} := \Big\{ \boldsymbol{z}_h^{\mathrm{B}} \in \mathbf{H}_0(\mathbf{curl};\Omega_{\mathrm{B}}) : \quad \boldsymbol{z}_h^{\mathrm{B}}|_K \in \mathbb{ND}_1(K) \quad \forall\, K \in \mathcal{T}_h(\Omega_{\mathrm{B}}) \Big\},
$$

where for any $K \in \mathcal{T}_h(\Omega_{\mathrm{B}})$

$$
\mathbb{ND}_1(K) := \mathbf{P}_0(K) \oplus \mathbf{P}_0(K) \times \boldsymbol{x}
$$

is the local edge space of Nédélec, that is

$$
\mathbb{ND}_1(K) := \Big\{ w : K \to \mathbb{C}^3 : \quad w(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} \quad \forall\, \boldsymbol{x} \in K,\, \boldsymbol{a},\, \boldsymbol{b} \in \mathbb{C}^3 \Big\}.
$$

In turn, we set $\mathbf{X}_h := \mathbf{1}_{\Omega_{\mathrm{B}}} \mathbf{H}_h^{\mathrm{B}} + \mathbf{1}_{\Omega_{\mathrm{D}}} \mathbf{H}_h^{\mathrm{D}}$, whereas for the interface Σ, we consider the following finite element subspace

$$
\mathrm{Q}_{\tilde{h}}^{\Sigma} := \Big\{ \lambda_{\tilde{h}} \in \mathcal{C}^0(\Sigma) : \quad \lambda_{\tilde{h}}|_{\tilde{T}} \in P_1(\tilde{T}) \quad \forall\, \tilde{T} \in \mathcal{T}_{\tilde{h}}(\Sigma) \Big\}.
$$

4.1 Fixed point strategy

We begin by noticing that the further regularity hypotheses employed in the proof of Lemma 3.2 and Lemma 3.4, respectively, neither are needed nor could be applied in the discrete case. It is therefore not possible to extend the fixed-point strategy introduced in Section 3.2 to the present context. Instead, and in order to guarantee the solvability of (4.2), we introduce a new approach where the operator associated with the discrete version of Brinkman-Darcy problem (3.4) must satisfy a uniform boundedness (see below Hypothesis 4.1 in Section 4.2). In what follows, for simplicity of the presentation, we will restrict the fixed-point scheme and its analysis to the 2D case. Given $r > 0$, we first define

$$W_h^{\boldsymbol{u}} := \Big\{ \boldsymbol{u}_h \in \mathbf{X}_h : \quad ||\boldsymbol{u}_h||_\star \leq r \Big\}, \tag{4.3}$$

where $||\cdot||_\star := \|\cdot\|_{\mathrm{div},\Omega} + \|\cdot\|_{\mathbf{L}^s(\Omega)}$, with $s > 2$. At this point, we anticipate that the stipulated range for s will allow us to employ suitable Sobolev embeddings which will be required for the analysis in the forthcoming Sections (see below proof of Lemma 4.2 and Theorem 4.7).

We now set $Y_h := \mathrm{H}_h^\phi \times W_h^{\boldsymbol{u}}$ and introduce the operator $\widetilde{\mathbf{S}}_h : Y_h \to \mathrm{H}_h^\phi$ defined by

$$\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h) := \varphi_h \qquad \forall\, (\phi_h, \boldsymbol{u}_h) \in Y_h,$$

where φ_h is the unique solution (to be confirmed below) of the linear problem: Find $\varphi_h \in \mathrm{H}_h^\phi$ such that

$$\mathcal{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \psi_h) = \mathcal{H}_{\phi_h}(\psi_h) \qquad \forall\, \psi_h \in \mathrm{H}_h^\phi, \tag{4.4}$$

for given $(\phi_h, \boldsymbol{u}_h)$, where

$$\mathcal{C}_{\phi_h, \boldsymbol{u}_h}(\varphi_h, \psi_h) := \int_\Omega \vartheta(\phi_h)\, \nabla\varphi_h \cdot \nabla\psi_h - \int_\Omega \varphi_h\, \boldsymbol{u}_h \cdot \nabla\psi_h + \int_\Omega \beta\, \varphi_h\, \psi_h \qquad \forall\, \varphi_h,\, \psi_h \in \mathrm{H}_h^\phi,$$

and

$$\mathcal{H}_{\phi_h}(\psi_h) := \int_\Omega f_{\mathrm{bk}}(\phi_h)\, \boldsymbol{g} \cdot \nabla\psi_h \qquad \forall\, \psi_h \in \mathrm{H}_h^\phi.$$

In turn, we define the operator $\mathbf{S}_h : \mathrm{H}_h^\phi \to \mathbf{X}_h$ as

$$\mathbf{S}_h(\varphi_h) := \boldsymbol{w}_h \qquad \forall \varphi_h \in \mathrm{H}_h^\phi, \tag{4.5}$$

where $\boldsymbol{w}_h|_{\Omega_\mathrm{B}} = \boldsymbol{w}_h^\mathrm{B}$ and $\boldsymbol{w}_h|_{\Omega_\mathrm{D}} = \boldsymbol{w}_h^\mathrm{D}$ are the first and third components of $\vec{\boldsymbol{w}}_h \in \mathbf{H}_h$, which in turn is the first component of the unique solution (to be confirmed below) of the discrete problem associated to (3.4): Find $(\vec{\boldsymbol{w}}_h, \vec{r}_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$, such that

$$\begin{aligned} \mathcal{A}(\vec{\boldsymbol{w}}_h, \vec{\boldsymbol{v}}_h) + \mathcal{B}(\vec{\boldsymbol{v}}_h, \vec{r}_h) &= \mathcal{F}_{\varphi_h}(\vec{\boldsymbol{v}}_h) && \forall\, \vec{\boldsymbol{v}}_h \in \mathbf{H}_h, \\ \mathcal{B}(\vec{\boldsymbol{w}}_h, \vec{q}_h) &= 0 && \forall\, \vec{q}_h \in \mathbf{Q}_{0,h}, \end{aligned} \tag{4.6}$$

with φ_h given. Therefore, by introducing the operator $\mathbf{T}_h : Y_h \to \mathrm{H}_h^\phi \times \mathbf{X}_h$ as

$$\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) := (\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h), \mathbf{S}_h(\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h))) \qquad \forall\, (\phi_h, \boldsymbol{u}_h) \in Y_h, \tag{4.7}$$

we see that solving (4.2) is equivalent to finding a fixed point of $\mathbf{T}_h$, that is: Find $(\phi_h, \boldsymbol{u}_h) \in Y_h$ such that

$$\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) = (\phi_h, \boldsymbol{u}_h). \tag{4.8}$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.4) and (4.6) are well-posed. This is precisely the purpose of the next section.

4.2 Well-posedness of the uncoupled problem

In this section, we establish the well-posedness of both (4.6) and (4.4), thus confirming that $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, and hence $\mathbf{T}_h$, are well-defined.

Lemma 4.1 *For each $\phi_h \in \mathrm{H}_h^{\phi}$, the problem (4.6) has a unique solution $(\vec{\boldsymbol{u}}_h, \vec{p}_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h}$. Moreover, there exists a positive constant C_2, independent of h, such that*

$$\|\mathbf{S}_h(\phi_h)\|_{\mathrm{div},\Omega} \,\le\, \|(\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \,\le\, C_2\,\|\phi_h\|_{0,\Omega}\left\{\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}\right\} \qquad \forall \phi_h \in \mathrm{H}_h^{\phi}. \tag{4.9}$$

Proof. It follows directly from [4, Theorem 4.1]. □

Lemma 4.2 *Assume that $r \in (0, \frac{\vartheta_1}{2\mathrm{C}_s c_p})$, where C_s is the boundedness constant of the injection $\mathrm{i}_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{2s}{s-2}}(\Omega)$, with $s > 2$. Then, for each $(\phi_h, \boldsymbol{u}_h) \in Y_h$, the problem (4.4) has a unique solution $\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h) := \varphi_h \in \mathrm{H}_h^{\phi}$. Moreover, by denoting $C_1 := \frac{1}{\widetilde{\alpha}}$, with $\widetilde{\alpha}$ as in the proof of Lemma 3.2, there holds*

$$\|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{1,\Omega} \,\le\, C_1\,\gamma_2|\Omega|^{1/2}\,|\boldsymbol{g}|, \qquad \forall(\phi_h, \boldsymbol{u}_h) \in Y_h. \tag{4.10}$$

Proof. It procceds by similar ideas used in the proof of Lemma 3.2. Indeed, employing the same arguments used for the derivation of (3.15), to the present context, we deduce that

$$|\mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h)| \,\le\, \vartheta_2\,|\widetilde{\phi}_h|_{1,\Omega}\,|\widetilde{\psi}_h|_{1,\Omega} + \|\widetilde{\phi}_h\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)}\,\|\boldsymbol{u}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)}\,|\widetilde{\psi}_h|_{1,\Omega} + \beta\,\|\widetilde{\phi}_h\|_{0,\Omega}\,\|\widetilde{\psi}_h\|_{0,\Omega} \tag{4.11}$$

where $\widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/\widetilde{p} + 1/\widetilde{q} = 1$. Thus, choosing $\widetilde{p}$ such that $2\widetilde{p} = s$, with $s > 2$, it readily follows that

$$2\widetilde{q} := \frac{2s}{s-2} > 1. \tag{4.12}$$

In this way, having in mind that $\|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)} \le r$ (*cf.* (4.3)), and the fact that, for the 2D case, the injection $\mathrm{i}_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{2s}{s-2}}(\Omega)$ is compact, and hence continuous with constant C_s, we deduce the existence of a positive constant $\|\mathcal{C}\|$, depending on ϑ_2, β, r and C_s, such that

$$|\mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h)| \,\le\, \|\mathcal{C}\|\,\|\widetilde{\phi}_h\|_{1,\Omega}\,\|\widetilde{\psi}_h\|_{1,\Omega} \quad \forall\, \widetilde{\phi}_h,\, \widetilde{\psi}_h \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \tag{4.13}$$

which proves that $\mathcal{C}_{\phi_h,\boldsymbol{u}_h}$ is bounded independently of ϕ_h and $\boldsymbol{u}_h$. On the other hand, applying the same argument to handle the trilinear term on the right hand side of (4.11) in the derivation of (4.13), and employing the estimates (2.5) and (2.8), we find that for each $\widetilde{\phi}_h \in \mathrm{H}^1_{\Gamma_0}(\Omega)$ there holds

$$\begin{aligned}
\mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\phi}_h) \,&\ge\, \vartheta_1|\widetilde{\phi}_h|^2_{1,\Omega} - \|\widetilde{\phi}_h\|_{\mathrm{L}^{2s/s-2}(\Omega)}\|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)}|\widetilde{\phi}_h|_{1,\Omega} + \beta\|\widetilde{\phi}_h\|^2_{0,\Omega}\\
&\ge\, (\vartheta_1 - \mathrm{C}_s\,c_p r)|\widetilde{\phi}_h|^2_{1,\Omega}\\
&\ge\, \frac{\vartheta_1}{2}|\widetilde{\phi}_h|^2_{1,\Omega} \,\ge\, \frac{\vartheta_1}{2c_p^2}\|\widetilde{\phi}_h\|^2_{1,\Omega},
\end{aligned}$$

which proves that $\mathcal{C}_{\phi_h,\boldsymbol{u}_h}$ is elliptic on $\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\phi}$, with the same constant $\widetilde{\alpha}$ from Lemma 3.2. In addition, the fact that $\|\mathcal{H}_{\phi}\|$ is bounded independently of ϕ (*cf.* Proof of Lemma 3.2), confirms the same upper bound for $\|\mathcal{H}_{\phi_h}\|_{(\mathrm{H}_h^{\phi})'}$. The rest of the proof is a direct application of Lax-Milgram's Lemma. □

We point out that the manipulation of the term $\beta\|\widetilde{\phi}_h\|^2_{0,\Omega}$ in the derivation of the ellipticity of $\mathcal{C}_{\phi_h,\boldsymbol{u}_h}$ is the same that was described at the end of the proof of Lemma 3.2 (see Section 3.3).

We end this section with an hypothesis of uniform boundedness on the operator $\mathbf{S}_h$ (*cf.* (4.5)), which will be required in the forthcoming Section to guarantee that the operator $\mathbf{T}_h$, given in (4.7) is well defined and continuous in a certain ball.

Hypothesis 4.1 *There exists $s > 2$ such that the operator $\mathbf{S}_h : (\mathrm{H}_h^\phi, \|\cdot\|_{1,\Omega}) \to (\mathbf{X}_h, \|\cdot\|_{s,\Omega})$ (*cf.* (4.5)) is uniformly bounded, that is*

$$\|\mathbf{S}_h(\phi_h)\|_{\mathbf{L}^s(\Omega)} \le \widetilde{C}_s \|\phi_h\|_{1,\Omega} \quad \forall\, \phi_h \in \mathrm{H}_h^\phi, \tag{4.14}$$

where $\widetilde{C}_s$ is a positive constant independent of h.

We remark in advance that the estimate (4.14) is needed in the proof of Lemma 4.3 (see Section 4.3 below) to bound an expression of the form $\|\mathbf{S}_h(\widetilde{\mathbf{S}}_h(\,\cdot\,))\|_{\mathbf{L}^s(\Omega)}$ in terms of $\|\widetilde{\mathbf{S}}_h(\,\cdot\,)\|_{1,\Omega}$, which in turn is bounded by data (*cf.* (4.10)). Afterward, the estimate (4.14) will be required to properly handle the expression $\|\mathbf{S}_h(\widetilde{\mathbf{S}}_h(\phi_h)) - \mathbf{S}_h(\widetilde{\mathbf{S}}_h(\varphi_h))\|_{\mathbf{L}^s(\Omega)}$ in order to derive a Lipschitz continuity property for $\mathbf{T}_h$ (see below Lemma 4.6).

4.3 Solvability of the fixed-point equation

We now aim to show the solvability of (4.2) by analyzing the equivalent fixed-point equation (4.8). To this end, we will proceed to verify the hypotheses of the Brouwer fixed-point theorem (*cf.* [10, Theorem 9.9-2]).

We start by defining the following set

$$\mathbf{W}_h := \{(\phi_h, \boldsymbol{u}_h) \in \mathrm{H}_h^\phi \times \mathbf{X}_h : |||(\phi_h, \boldsymbol{u}_h)||| \le r\}, \tag{4.15}$$

where r was previously fixed in (4.3), and

$$|||(\phi_h, \boldsymbol{u}_h)||| := \|\phi_h\|_{1,\Omega} + \|\boldsymbol{u}_h\|_\star = \|\phi_h\|_{1,\Omega} + \|\boldsymbol{u}_h\|_{\mathrm{div},\Omega} + \|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)}.$$

Lemma 4.3 *Let $\mathbf{W}_h$ be as in (4.15), and assume that the data $\boldsymbol{g}$, $\boldsymbol{f}_\mathrm{B}$ and $\boldsymbol{f}_\mathrm{D}$ are sufficiently small so that*

$$(\widetilde{C}_s + C_1)\gamma_2|\Omega|^{1/2}|\boldsymbol{g}| + C_1\, C_2\, \gamma_2|\Omega|^{1/2}|\boldsymbol{g}| \left\{\|\boldsymbol{f}_\mathrm{B}\|_{\infty,\Omega_\mathrm{B}} + \|\boldsymbol{f}_\mathrm{D}\|_{\infty,\Omega_\mathrm{D}}\right\} \le r. \tag{4.16}$$

Then $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$.

Proof. Given $(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$, we get from (4.7), and the estimates (4.14), (4.9), and (4.10), that

$$\begin{aligned} |||\mathbf{T}_h(\phi_h, \boldsymbol{u}_h)||| &= |||(\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h), \mathbf{S}_h(\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)))||| \\ &= \|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{1,\Omega} + \|\mathbf{S}_h(\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h))\|_{\mathrm{div},\Omega} + \|\mathbf{S}_h(\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h))\|_{\mathbf{L}^s(\Omega)} \\ &\le (\widetilde{C}_s + C_1)\gamma_2|\Omega|^{1/2}|\boldsymbol{g}| + C_1\, C_2\, \gamma_2|\Omega|^{1/2}|\boldsymbol{g}| \left\{\|\boldsymbol{f}_\mathrm{B}\|_{\infty,\Omega_\mathrm{B}} + \|\boldsymbol{f}_\mathrm{D}\|_{\infty,\Omega_\mathrm{D}}\right\}, \end{aligned}$$

and hence, employing the condition (4.16), we conclude that $\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$. □

In order to prove the continuity of $\mathbf{T}_h$, in the following two lemmas, we derive Lipschitz type-estimates for $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$.

Lemma 4.4 *Let C_2 be the constant given in Lemma 4.1. Then, there holds*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_{\mathrm{div},\Omega} \le C_2\, \|\phi_h - \varphi_h\|_{0,\Omega} \left\{\|\boldsymbol{f}_\mathrm{B}\|_{\infty,\Omega_\mathrm{B}} + \|\boldsymbol{f}_\mathrm{D}\|_{\infty,\Omega_\mathrm{D}}\right\} \qquad \forall \phi_h, \varphi_h \in \mathrm{H}_h^\phi. \tag{4.17}$$

Proof. Given $\phi_h, \varphi_h \in \mathrm{H}_h^\phi$, we let $\vec{u}_h := (\boldsymbol{u}_h^{\mathrm{B}}, \boldsymbol{\omega}_h^{\mathrm{B}}, \boldsymbol{u}_h^{\mathrm{D}}) \in \mathbf{H}_h$, $\vec{p}_h := (p_h^{\mathrm{B}}, p_h^{\mathrm{D}}, \lambda_h) \in \mathbf{Q}_{0,h}$ and $\vec{w}_h = (\boldsymbol{w}_h^{\mathrm{B}}, \boldsymbol{x}_h^{\mathrm{B}}, \boldsymbol{w}_h^{\mathrm{D}}) \in \mathbf{H}_h, \vec{r}_h := (r_h^{\mathrm{B}}, r_h^{\mathrm{D}}, \chi_h) \in \mathbf{Q}_{0,h}$ be the corresponding solutions of (4.6), so that $\boldsymbol{u}_h = \boldsymbol{u}_h^{\mathrm{B}} + \boldsymbol{u}_h^{\mathrm{D}} =: \mathbf{S}_h(\phi_h)$ and $\boldsymbol{w}_h = \boldsymbol{w}_h^{\mathrm{B}} + \boldsymbol{w}_h^{\mathrm{D}} =: \mathbf{S}_h(\varphi_h)$. Then, employing the linearity of the forms $\mathcal{A}$ and $\mathcal{B}$, we deduce from (4.6) that

$$
\begin{aligned}
\mathcal{A}(\vec{\boldsymbol{u}}_h - \vec{\boldsymbol{w}}_h, \vec{\boldsymbol{v}}_h) + \mathcal{B}(\vec{\boldsymbol{v}}_h, \vec{p}_h - \vec{r}_h) &= \mathcal{F}_{\phi_h - \varphi_h}(\vec{\boldsymbol{v}}_h) \quad \forall\, \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_h^{\mathrm{B}}, \boldsymbol{z}_h^{\mathrm{B}}, \boldsymbol{v}_h^{\mathrm{D}}) \in \mathbf{H}, \\
\mathcal{B}(\vec{\boldsymbol{u}}_h - \vec{\boldsymbol{w}}_h, \vec{q}_h) &= 0 \qquad\qquad\quad \forall\, \vec{q}_h := (q_h^{\mathrm{B}}, q_h^{\mathrm{D}}, \xi_h) \in \mathbf{Q}_{0,h}.
\end{aligned}
$$

In this way, due to the fact that $\mathbf{S}_h(\phi_h - \varphi_h) = \mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)$, the bound (4.17) follows directly from estimate (4.9). $\square$

Lemma 4.5 *Let $L_{f_{\mathrm{bk}}}, L_\vartheta$, and $\widetilde{\alpha}$ be the constants given in* (2.7), (2.6), *and Lemma* 3.2, *respectively. Then, there holds*

$$
\begin{aligned}
\|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h) - \widetilde{\mathbf{S}}_h(\varphi_h, \boldsymbol{w}_h)\|_{1,\Omega} \le \frac{1}{\widetilde{\alpha}} \Big\{ & L_{f_{\mathrm{bk}}} |\boldsymbol{g}| \|\phi_h - \varphi_h\|_{0,\Omega} + L_\vartheta \|\nabla \widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^4(\Omega)} \|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \\
& + \|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)} \Big\}
\end{aligned} \tag{4.18}
$$

$\forall\, (\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h) \in \mathrm{H}_h^\phi \times W_h^{\boldsymbol{u}}$, *where $\widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/\widetilde{p} + 1/\widetilde{q} = 1$.*

Proof. Given $(\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h)$ as stated, we let $\widetilde{\phi}_h := \widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)$ and $\widetilde{\varphi}_h := \widetilde{\mathbf{S}}_h(\varphi_h, \boldsymbol{w}_h)$, that is (*cf.* (4.4))

$$
\mathcal{C}_{\phi_h, \boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h) = \mathcal{H}_{\phi_h}(\widetilde{\psi}_h) \quad \text{and} \quad \mathcal{C}_{\varphi_h, \boldsymbol{w}_h}(\widetilde{\varphi}_h, \widetilde{\psi}_h) = \mathcal{H}_{\varphi_h}(\widetilde{\psi}_h) \qquad \forall\, \widetilde{\psi}_h \in \mathrm{H}_h^\phi .
$$

Next, we proceed analogously as in the proof of Lemma 3.4. In fact, applying Cauchy-Schwarz's inequality, the Lipschitz-continuity estimates (2.6)-(2.7), Hölder's inequality to the second term on the right hand side for the discrete version of (3.22), and a $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument for the corresponding last term, we deduce that

$$
\begin{aligned}
\widetilde{\alpha}\, \|\widetilde{\phi}_h - \widetilde{\varphi}_h\|_{1,\Omega}^2 \le\ & \Big\{ L_{f_{\mathrm{bk}}} |\boldsymbol{g}| \, \|\phi_h - \varphi_h\|_{0,\Omega} + \|\widetilde{\varphi}_h\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)} \Big\} |\widetilde{\phi}_h - \widetilde{\varphi}_h|_{1,\Omega} \\
& + \ L_\vartheta \|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \|\nabla \widetilde{\varphi}_h\|_{\mathrm{L}^4(\Omega)} |\widetilde{\phi}_h - \widetilde{\varphi}_h|_{1,\Omega} .
\end{aligned}
$$

Then, since the elements of H_h^ϕ are piecewise polynomials, it follows that $\|\nabla \widetilde{\varphi}_h\|_{\mathrm{L}^4(\Omega)} < +\infty$, and hence the foregoing equation yields (4.18). Further details are omitted. $\square$

We now can establish the following result providing a Lipschitz continuity type-estimate for the operator $\mathbf{T}_h$.

Lemma 4.6 *Given $r > 0$, we let $\mathbf{W}_h := \{(\phi_h, \boldsymbol{u}_h) \in \mathrm{H}_h^\phi \times \mathbf{X}_h : |||(\phi_h, \boldsymbol{u}_h)||| \le r\}$, and assume* (4.16) *(*cf. *Lemma* 4.3*). Then, there exist constants $C_3, C_4, C_5 > 0$, depending only on $L_{f_{\mathrm{bk}}}, L_\vartheta, \widetilde{\alpha}$, C_2, $\widetilde{C}_s$ (*cf. (2.7), (2.6), *Lemma* 3.2, *Lemma* 4.4, (4.14)*) and the data, such that, for all $(\phi_h, \boldsymbol{u}_h), (\varphi_h, \boldsymbol{w}_h) \in \mathbf{W}_h$, there holds*

$$
\begin{aligned}
|||\mathbf{T}_h(\phi_h, \boldsymbol{u}_h) - \mathbf{T}_h(\varphi_h, \boldsymbol{w}_h)||| \le C_3 \|\phi_h - \varphi_h\|_{0,\Omega} &+ C_4 \|\nabla \widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^4(\Omega)} \|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \\
&+ C_5 \|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)} \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)} ,
\end{aligned}
$$

where $\widetilde{p}, \widetilde{q} \in [1, +\infty)$ are such that $1/\widetilde{p} + 1/\widetilde{q} = 1$.

Proof. It suffices to recall from Section 4.1 that

$$\mathbf{T}_h(\psi_h, \boldsymbol{v}_h) := (\widetilde{\mathbf{S}}_h(\psi_h, \boldsymbol{v}_h), \mathbf{S}_h(\widetilde{\mathbf{S}}_h(\psi_h, \boldsymbol{v}_h))) \qquad \forall\,(\psi_h, \boldsymbol{v}_h) \in Y_h,$$

and then apply the estimate (4.14), and Lemmas 4.4 and 4.5. □

Consequently, from the foregoing Lemma, choosing $2\widetilde{p}$ and $2\widetilde{q}$ as in the proof of Lemma 4.2, that is $2\widetilde{p} = s$ and hence $2\widetilde{q} := \frac{2s}{s-2} > 1$, and employing the continuous injection $\mathrm{i} : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$, the continuity of $\mathbf{T}_h$ is derived. In this way, we conclude, thanks to the Brouwer fixed-point theorem (*cf.* [10, Theorem 9.9-2]) and Lemmas 4.3 and 4.6, the main result of this section.

Theorem 4.7 *Under the assumptions of Lemma* 4.3*, the Galerkin scheme* (4.2) *has at least one solution* $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^\phi$ *with* $(\phi_h, \boldsymbol{u}_h) \in \mathbf{W}_h$*, and there holds*

$$\|\phi_h\|_{1,\Omega} \,\leq\, C_1 \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}|,$$

and

$$\|(\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \,\leq\, C_1\, C_2\, \gamma_2 |\Omega|^{1/2} |\boldsymbol{g}| \left\{ \|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} \,+\, \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \right\},$$

where C_1*,* C_2*, and* γ_2*, are the constants provided by Lemmas* 4.1 *and* 4.2*, and* (2.5) *, respectively.*

We end this section pointing out that the extension to 3D case of our discrete analysis of fixed-point to solve (4.2), is basically based on a new range for the parameter s (*cf.* (4.3), Lemma 4.2, and Hypotheses 4.14). More precisely, for the 3D case, we need to take $s > 3$ in (4.3) to then guarantee the compactness, and hence the continuity, of the injection $\mathrm{i}_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^{\frac{3s}{s-3}}(\Omega)$, which is crucial in the proof of the corresponding Lemma 4.2.

5 A priori error estimate

Given $(\vec{\boldsymbol{u}}, \vec{p}, \phi) \in \mathbf{H} \times \mathbf{Q}_0 \times \mathrm{H}^1_{\Gamma_0}(\Omega)$ with $\phi \in \mathrm{W}$, and $(\vec{\boldsymbol{u}}_h, \vec{p}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_{0,h} \times \mathrm{H}_h^\phi$ with $(\phi_h, \boldsymbol{u}_h) \in W_h$ (*cf.* (4.15)), solutions of (3.6) and (4.2), respectively, we now aim to derive a corresponding *a priori* error estimate. To this end, we first observe from (3.6) and (4.2), that the above problems can be rewritten as follows:

$$(BD) \left\{ \begin{array}{rcll} \mathcal{A}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) + \mathcal{B}(\vec{\boldsymbol{v}}, \vec{p}) &=& \mathcal{F}_\phi(\vec{\boldsymbol{v}}) & \forall\, \vec{\boldsymbol{v}} \in \mathbf{H}, \\ \mathcal{B}(\vec{\boldsymbol{u}}, \vec{q}) &=& 0 & \forall\, \vec{q} \in \mathbf{Q}_0, \end{array} \right.$$

$$(BD_h) \left\{ \begin{array}{rcll} \mathcal{A}(\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h) + \mathcal{B}(\vec{\boldsymbol{v}}_h, \vec{p}_h) &=& \mathcal{F}_{\phi_h}(\vec{\boldsymbol{v}}_h) & \forall\, \vec{\boldsymbol{v}}_h \in \mathbf{H}_h, \\ \mathcal{B}(\vec{\boldsymbol{u}}_h, \vec{q}_h) &=& 0 & \forall\, \vec{q}_h \in \mathbf{Q}_{0,h}, \end{array} \right.$$

and

$$\begin{array}{lrcll} (T) & \mathcal{C}_{\boldsymbol{u}}(\phi, \psi) &=& \mathcal{H}_\phi(\psi) & \forall\, \psi \in \mathrm{H}^1_{\Gamma_0}(\Omega)\,, \\ (T_h) & \mathcal{C}_{\boldsymbol{u}_h}(\phi_h, \psi_h) &=& \mathcal{H}_{\phi_h}(\psi_h) & \forall\, \psi_h \in \mathrm{H}_h^\phi\,. \end{array}$$

We begin our analysis by establishing the following result concerning $\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0}$.

Lemma 5.1 *There exists* $C_{\mathrm{B,D}} > 0$*, independent of* h*, such that*

$$\begin{aligned} &\|(\vec{\boldsymbol{u}}, \vec{p}) - (\vec{\boldsymbol{u}}_h, \vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \\ &\quad \leq\, C_{\mathrm{B,D}} \left\{ \mathrm{dist}(\vec{\boldsymbol{u}}, \mathbf{H}_h) \,+\, \mathrm{dist}(\vec{p}, \mathbf{Q}_{0,h}) \,+\, \|\phi - \phi_h\|_{1,\Omega} \Big(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \Big) \right\}. \end{aligned} \tag{5.1}$$

Proof. Basically, the proof follows from the corresponding Strang-type error estimate for (BD) and (BD_h). Indeed, procceding anlogously as in [13, Section 4] (also see [18]), we deduce the existence of a positive constant $C_{\mathrm{B,D}}$, independent of h, such that

$$\|(\vec{\boldsymbol{u}},\vec{p}) - (\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \,\leq\, C_{\mathrm{B,D}}\left\{\mathrm{dist}(\vec{\boldsymbol{u}},\mathbf{H}_h) \,+\, \mathrm{dist}(\vec{p},\mathbf{Q}_{0,h}) \,+\, \|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{\mathbf{H}_h'}\right\}. \tag{5.2}$$

Next, according to the definition of $\mathcal{F}_\phi$ and $\mathcal{F}_{\phi_h}$ (*cf.* (3.3)), and applying Cauchy-Schwarz's inequality, we easily deduce that

$$\|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{\mathbf{H}_h'} \,\leq\, \|\phi-\phi_h\|_{0,\Omega}\Big(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + \|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}\Big). \tag{5.3}$$

In this way, by replacing (5.3) into (5.2), we arrive at (5.1), which ends the proof. □

We now derive a Céa estimate for the error $\|\phi-\phi_h\|_{1,\Omega}$ under the 2D-dimensional context. To this end, and in order to simplify the subsequent writing, we introduce the following constants, independent of the data $\boldsymbol{g}$, $\boldsymbol{f}_{\mathrm{B}}$, and $\boldsymbol{f}_{\mathrm{D}}$,

$$K_1 \,:=\, C_{\widetilde{\mathbf{S}}}\left\{L_{f_{\mathrm{bk}}} \,+\, L_\vartheta\,\mathrm{C}_\delta\,\mathrm{C}_\delta^*\,\widehat{C}_{\widetilde{\mathbf{S}}}(r)\,\gamma_2\,|\Omega|^{1/2}\right\}, \quad K_2 \,:=\, C_{\widetilde{\mathbf{S}}}\,(r\mathrm{C}_s \,+\, \beta \,+\, \vartheta_2) + 1, \quad \text{and} \quad K_3 \,=\, C_{\widetilde{\mathbf{S}}}.$$

where $\widehat{C}_{\widetilde{\mathbf{S}}}(r)$ and $C_{\widetilde{\mathbf{S}}}$ are the constants given in (3.19) and (3.14), and C_s, C_δ, C_δ^*, are the boundedness constants of the continuous injections

$$\mathrm{i}_s : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{L}^{\frac{2s}{s-2}}(\Omega), \quad \mathrm{i} : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{L}^{2/\delta}(\Omega), \quad \mathrm{i}_\delta : \mathrm{H}^\delta(\Omega) \longrightarrow \mathrm{L}^{\delta^*}(\Omega), \tag{5.4}$$

respectively, where $s>2$, $\delta\in(0,1)$, and $\delta^* := 2/(1-\delta)$. In addition, in order to suitably handle one of the terms in the derivation of the Céa estimate for $\|\phi-\phi_h\|_{1,\Omega}$, we will additionally assume that $\phi\in\mathrm{L}^\infty(\Omega)$.

Lemma 5.2 *Assume that $\phi\in\mathrm{H}^1(\Omega)\cap\mathrm{L}^\infty(\Omega)$, and that the data $\boldsymbol{g}$ satisfy*

$$K_1\,|\boldsymbol{g}| \,\leq\, \frac{1}{2}. \tag{5.5}$$

Then, there holds

$$\|\phi-\phi_h\|_{1,\Omega} \,\leq\, 2\,K_2\,\mathrm{dist}(\phi,\mathrm{H}_h^\phi) \,+\, 2\,K_3\,\|\phi\|_{\infty,\Omega}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}. \tag{5.6}$$

Proof. It proceeds almost verbatim as in the proof of [3, Lemma 5.3]. In fact, we first observe by triangle inequality that

$$\|\phi-\phi_h\|_{1,\Omega} \,\leq\, \|\phi-\varphi_h\|_{1,\Omega} \,+\, \|\phi_h-\varphi_h\|_{1,\Omega} \qquad \forall\,\varphi_h\in\mathrm{H}_h^\phi. \tag{5.7}$$

Then employing the ellipticity of the bilinear form $\mathcal{C}_{\phi_h,\boldsymbol{u}_h}$ with constant $\widetilde{\alpha}$, and then adding and subtracting suitable expressions, we deduce that

$$\begin{aligned}\widetilde{\alpha}\,\|\phi_h-\varphi_h\|_{1,\Omega}^2 \,&\leq\, \mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\phi_h-\varphi_h,\phi_h-\varphi_h)\\ &\leq\, |\mathcal{H}_{\phi_h}(\phi_h-\varphi_h) \,-\, \mathcal{H}_\phi(\phi_h-\varphi_h)|\\ &\quad +\, |\mathcal{C}_{\phi,\boldsymbol{u}}(\phi,\phi_h-\varphi_h) - \mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\varphi_h,\phi_h-\varphi_h)|\,.\end{aligned} \tag{5.8}$$

Next, according to the definition of $\mathcal{H}_\phi$ and $\mathcal{H}_{\phi_h}$ (*cf.* (3.9)), and applying Cauchy-Schwarz's inequality, we get

$$|\mathcal{H}_{\phi_h}(\phi_h-\varphi_h) \,-\, \mathcal{H}_\phi(\phi_h-\varphi_h)| \,\leq\, L_{f_{\mathrm{bk}}}\,|\boldsymbol{g}|\,\|\phi-\phi_h\|_{0,\Omega}\,|\phi_h-\varphi_h|_{1,\Omega}\,. \tag{5.9}$$

In turn, adding and subtracting suitable expressions, and then applying Hölder's inequality, the upper bound of ϑ (*cf.* (2.5)) and its Lipschitz continuity (*cf.* (2.6)), and the assumption that $\phi \in \mathrm{L}^\infty(\Omega)$, we find that

$$\begin{aligned}
&|\mathcal{C}_{\phi,\boldsymbol{u}}(\phi,\phi_h-\varphi_h)-\mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\varphi_h,\phi_h-\varphi_h)| \\
&\quad\leq\quad L_\vartheta\,\|\phi-\phi_h\|_{\mathrm{L}^{2q}(\Omega)}\,\|\nabla\phi\|_{\mathrm{L}^{2p}(\Omega)}\,|\phi_h-\varphi_h|_{1,\Omega}\,+\,\vartheta_2\,|\phi-\varphi_h|_{1,\Omega}\,|\phi_h-\varphi_h|_{1,\Omega} \\
&\quad+\quad \|\phi\|_{\infty,\Omega}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,|\phi_h-\varphi_h|_{1,\Omega}\,+\,\|\phi-\varphi_h\|_{\mathrm{L}^{2\widetilde{q}}(\Omega)}\,\|\boldsymbol{u}_h\|_{\mathbf{L}^{2\widetilde{p}}(\Omega)}\,|\phi_h-\varphi_h|_{1,\Omega} \\
&\quad+\quad \beta\,\|\phi-\varphi_h\|_{0,\Omega}\,\|\phi_h-\varphi_h\|_{0,\Omega}\,,
\end{aligned}\tag{5.10}$$

where $p,q,\widetilde{p},\widetilde{q}\in[1,+\infty)$ are such that $1/p+1/q=1$ and $1/\widetilde{p}+1/\widetilde{q}=1$. In this way, choosing $2p$ and $2q$ as in the proof of Lemma 3.2 (*cf.* (3.16)), $2\widetilde{p}$ and $2\widetilde{q}$ as in the proof of Lemma 4.2 (*cf.* (4.12)), and applying the continuous embeddings i, i_δ, i_s (*cf.* (5.4)), the estimate (3.19), and the fact that $\|\boldsymbol{u}_h\|_{\mathbf{L}^s(\Omega)}\leq r$, it follows from (5.10) that

$$\begin{aligned}
&|\mathcal{C}_{\phi,\boldsymbol{u}}(\phi,\phi_h-\varphi_h)-\mathcal{C}_{\phi_h,\boldsymbol{u}_h}(\varphi_h,\phi_h-\varphi_h)| \\
&\quad\leq\quad L_\vartheta\,\mathrm{C}_\delta\,\mathrm{C}_\delta^*\,\widehat{C}_{\widetilde{\mathbf{S}}}(r)\,\gamma_2|\Omega|^{1/2}|\boldsymbol{g}|\,\|\phi-\phi_h\|_{1,\Omega}\,\|\phi_h-\varphi_h\|_{1,\Omega} \\
&\quad+\quad \vartheta_2\,\|\phi-\varphi_h\|_{1,\Omega}\,\|\phi_h-\varphi_h\|_{1,\Omega}\,+\,r\,\mathrm{C}_s\,\|\phi-\varphi_h\|_{1,\Omega}\,\|\phi_h-\varphi_h\|_{1,\Omega} \\
&\quad+\quad \|\phi\|_{\infty,\Omega}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,\|\phi_h-\varphi_h\|_{1,\Omega}\,+\,\beta\,\|\phi-\varphi_h\|_{1,\Omega}\,\|\phi_h-\varphi_h\|_{1,\Omega}\,.
\end{aligned}\tag{5.11}$$

Thus, by replacing (5.9) and (5.11) into (5.8), and then the resulting estimate into (5.7), employing the constants defined previously to the statement of the present lemma, and recalling from the proof of Lemma 3.2 that $\widetilde{\alpha}=C_{\widetilde{\mathbf{S}}}^{-1}$, we find, after several algebraic manipulations, that

$$\|\phi-\phi_h\|_{1,\Omega}\,\leq\,K_1\,|\boldsymbol{g}|\,\|\phi-\phi_h\|_{1,\Omega}\,+\,K_2\,\|\phi-\varphi_h\|_{1,\Omega}\,+\,\|\phi\|_{\infty,\Omega}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\qquad\forall\,\varphi_h\in\mathrm{H}_h^\phi\,,$$

which, according to the assumption (5.5), and taking the infimum on $\varphi_h\in\mathrm{H}_h^\phi$, yields (5.2) and completes the proof. □

At this point we mention that for the proof of the 3D case of Lemma 5.2, it is required to choose the parameters δ and δ^*, and hence $2p$ and $2q$, as in proof of Lemma (3.2) (*cf.* (3.16)) for this case. In turn, and according to the remark at the end of Section 4.3, for the present case we need to take $s>3$ and then to choose $2\widetilde{p}$ and $2\widetilde{q}$ analogously as in the proof of Lemma 4.2 (*cf.* (4.12)), in order to make use of the continuous injection $\mathrm{i}_s:\mathrm{H}^1(\Omega)\to\mathrm{L}^{\frac{3s}{s-3}}(\Omega)$.

We now combine the inequalities provided by Lemmas 5.1 and 5.2 to derive the Céa estimate for the total error $\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0}+\|\phi-\phi_h\|_{1,\Omega}$. Indeed, by replacing the estimate for $\|\phi-\phi_h\|_{1,\Omega}$ given by (5.6) into the second term on the right hand side of (5.1), we find that

$$\begin{aligned}
\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0}&\leq C_{\mathrm{B,D}}\Big\{\mathrm{dist}(\vec{\boldsymbol{u}},\mathbf{H}_h)+\mathrm{dist}(\vec{p},\mathbf{Q}_{0,h})\Big\}\,+\,\widetilde{K}\,\mathrm{dist}(\phi,\mathrm{H}_h^\phi)\\
&+\,2C_{\mathrm{B,D}}\|\phi\|_{\infty,\Omega}\Big(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}}+\|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}\Big)\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,,
\end{aligned}$$

where

$$\widetilde{K}\,:=\,2K_2C_{\mathrm{B,D}}\left(\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}}+\|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}}\right).$$

In this way, assuming now that the data $\boldsymbol{f}_{\mathrm{B}}$ and $\boldsymbol{f}_{\mathrm{D}}$ satisfy

$$C_{\mathrm{B,D}}\|\phi\|_{\infty,\Omega}\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + C_{\mathrm{B,D}}\|\phi\|_{\infty,\Omega}\|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \leq \frac{1}{4}$$

we conclude from the foregoing equations that

$$\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} \leq 2\,C_{\mathrm{B,D}}\Big\{\mathrm{dist}(\vec{\boldsymbol{u}},\mathbf{H}_h)+\mathrm{dist}(\vec{p},\mathbf{Q}_{0,h})\Big\} + 2\,\widetilde{K}\,\mathrm{dist}(\phi,\mathrm{H}_h^{\phi}). \tag{5.12}$$

Consequently, we can establish the following result providing the Céa estimate for the total error $\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0}+\|\phi-\phi_h\|_{1,\Omega}$.

Theorem 5.3 *Assume that $\phi\in\mathrm{H}^1(\Omega)\cap\mathrm{L}^\infty(\Omega)$, and that the data $\boldsymbol{f}_{\mathrm{B}},\boldsymbol{f}_{\mathrm{D}}$ and $\boldsymbol{g}$ are sufficiently small so that*

$$K_1\,|\boldsymbol{g}| \leq \frac{1}{2} \quad \textit{and} \quad C_{\mathrm{B,D}}\|\phi\|_{\infty,\Omega}\|\boldsymbol{f}_{\mathrm{B}}\|_{\infty,\Omega_{\mathrm{B}}} + C_{\mathrm{B,D}}\|\phi\|_{\infty,\Omega}\|\boldsymbol{f}_{\mathrm{D}}\|_{\infty,\Omega_{\mathrm{D}}} \leq \frac{1}{4}.$$

Then, there exists a positive constant C depending only on data, parameters, $\|\phi\|_{\infty,\Omega}$, and other constants, all them independent of h, such that

$$\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} + \|\phi-\phi_h\|_{1,\Omega} \leq C\left\{\mathrm{dist}(\vec{\boldsymbol{u}},\mathbf{H}_h) + \mathrm{dist}(\vec{p},\mathbf{Q}_{0,h}) + \mathrm{dist}(\phi,\mathrm{H}_h^{\phi})\right\}. \tag{5.13}$$

Proof. It follows straightforward from (5.12) and (5.6). □

The following theorem provides the rate of convergence of our Galerkin scheme (4.2).

Theorem 5.4 *Let $\mathbf{H}_h := \mathbf{H}_h^{\mathrm{B}}\times\mathbf{H}_{0,h}^{\mathrm{B}}\times\mathbf{H}_h^{\mathrm{D}}$, $\mathbf{Q}_{h,0} := \mathrm{Q}_{h,0}^{\mathrm{B}}\times\mathrm{Q}_h^{\mathrm{D}}\times\mathrm{Q}_{\tilde{h}}^{\Sigma}$ and H_h^{ϕ}, be the subspaces specified in the Section 4. Let $(\vec{\boldsymbol{u}},\vec{p},\phi) := ((\boldsymbol{u}_{\mathrm{B}},\boldsymbol{\omega}_{\mathrm{B}},\boldsymbol{u}_{\mathrm{D}}),(p_{\mathrm{B}},p_{\mathrm{D}},\lambda),\phi)\in\mathbf{H}\times\mathbf{Q}_0\times\mathrm{H}^1_{\Gamma_0}(\Omega)$ and $(\vec{\boldsymbol{u}}_h,\vec{p}_h,\phi_h) := ((\boldsymbol{u}_h^{\mathrm{B}},\boldsymbol{\omega}_h^{\mathrm{B}},\boldsymbol{u}_h^{\mathrm{D}}),(p_h^{\mathrm{B}},p_h^{\mathrm{D}},\lambda_{\tilde{h}}),\phi_h)\in\mathbf{H}_h\times\mathbf{Q}_{0,h}\times\mathrm{H}_h^{\phi}$ be the solutions of the continuous and discrete problems (3.6) and (4.2), respectively. Assume that $\boldsymbol{u}_\star\in\mathbf{H}^l(\Omega_\star)$, $\mathrm{div}\,\boldsymbol{u}_\star\in\mathrm{H}^l(\Omega_\star)$, $p_\star\in\mathrm{H}^l(\Omega_\star)$ where $\star\in\{\mathrm{B},\mathrm{D}\}$, $\boldsymbol{\omega}_{\mathrm{B}}\in\mathbf{H}^l(\mathbf{curl};\Omega_{\mathrm{B}})$, $\lambda\in\mathrm{H}^{1/2+l}(\Sigma)$ and $\phi\in\mathrm{H}^{1+l}_{\Gamma_0}(\Omega)$, for some $l\in(0,1)$ (when $n=2$) or $l\in(1/2,1)$ (when $n=3$). Then, there exists $C>0$ and $\widetilde{C}>0$ independent of h and $\tilde{h}$ such that*

$$\begin{aligned}
&\|(\vec{\boldsymbol{u}},\vec{p})-(\vec{\boldsymbol{u}}_h,\vec{p}_h)\|_{\mathbf{H}\times\mathbf{Q}_0} + \|\phi-\phi_h\|_{1,\Omega}\\
&\quad\leq Ch^{\min\{l,k+1\}}\Big\{\|\boldsymbol{u}_{\mathrm{B}}\|_{l,\Omega_{\mathrm{B}}}+\|\mathrm{div}\,(\boldsymbol{u}_{\mathrm{B}})\|_{l,\Omega_{\mathrm{B}}} + \|\boldsymbol{\omega}_{\mathrm{B}}\|_{\mathbf{H}^l(\mathbf{curl};\Omega_{\mathrm{B}})} + \|\boldsymbol{u}_{\mathrm{D}}\|_{l,\Omega_{\mathrm{D}}}\\
&\quad + \|\mathrm{div}\,(\boldsymbol{u}_{\mathrm{D}})\|_{l,\Omega_{\mathrm{D}}} + \|p_{\mathrm{B}}\|_{l,\Omega_{\mathrm{B}}}+\|p_{\mathrm{D}}\|_{l,\Omega_{\mathrm{D}}} + \|\phi\|_{1+l,\Omega}\Big\} + \widetilde{C}\tilde{h}^l\|\lambda\|_{l+1/2,\Sigma}\,.
\end{aligned}$$

Proof. It follows directly from the Céa estimate (5.13) and the approximation properties of the discrete subspace specified in the Section 4 (*cf.* Ref. [4, Section 4.2.2] and [9]). □

6 Numerical examples

Test 1. We begin this section with an accuracy test, where we construct smooth solutions satisfying (2.1)-(2.3) on $\Omega=(0,2)\times(0,1)$. The Brinkman and Darcy domains are on the left and right parts of Ω, respectively, and are separated by the interface Σ defined by the parameterisation

$$(0,1)\ni t\mapsto(x_1,x_2)=(1+0.15[1/2-|t-1/2|]\cos(6\pi t-3\pi),t),$$

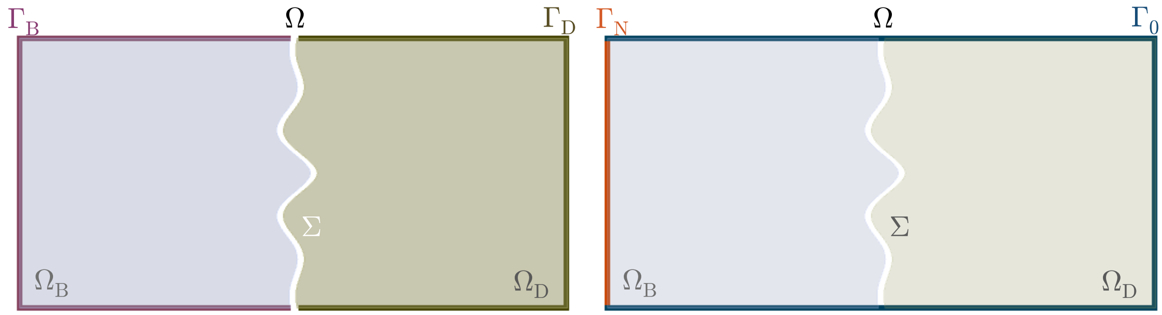

Figure 6.1: Test 1. Domain configuration for the Brinkman-Darcy problem, where $\partial\Omega = \Gamma_{\rm B} \cup \Gamma_{\rm D}$ (left panel), and for the transport equation where $\partial\Omega = \Gamma_{\rm N} \cup \Gamma_0$ (right). The unit normal vector on the interface points towards $\Omega_{\rm B}$.

on which the normal vector is considered pointing to $\Omega_{\rm B}$. The proposed exact solutions are given by

$$\boldsymbol{u} = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad \boldsymbol{u}_{\rm B} = \boldsymbol{u}|_{\Omega_{\rm B}}, \quad \boldsymbol{u}_{\rm D} = \boldsymbol{u}|_{\Omega_{\rm D}}, \quad \boldsymbol{\omega}_{\rm B} = 2\pi \sin(\pi x_1)\sin(\pi x_2),$$
$$p = (x_1 - 1/2)(x_2 - 1/2), \quad p_{\rm B} = p|_{\Omega_{\rm B}}, \quad p_{\rm D} = p|_{\Omega_{\rm D}}, \quad \phi = \frac{5}{2} x_1^2 (2 - x_1) x_2 (1 - x_2),$$

from which the necessary forcing, boundary, and source terms are generated. The concentration-dependent functions are $\vartheta(\phi) = \phi + (1 - c\phi)^2$, and $f_{\rm bk}(\phi) = c\phi(1 - c\phi)^2$, and the remaining physical parameters assume the values $c = 1/2$, $\boldsymbol{f}_{\rm B} = (1,0)^2$, $\boldsymbol{f}_{\rm D} = (0.1,0)^{\tt t}$, $\mathbb{K}_{\rm B} = 0.05$, $\mathbb{K}_{\rm D} = 0.01$, $\beta = 0.4\chi|_{\Omega_{\rm B}} + 0.1\chi|_{\Omega_{\rm D}}$, $\boldsymbol{k} = (0,-1)^{\tt t}$. We recall that two different splittings of the domain boundary $\partial\Omega$ are assumed. First, the distribution of the Brinkman and Darcy boundaries follows the sketch presented in the left panel of Figure 6.1. According to (2.4), on $\Gamma_{\rm B}$ we set slip velocities $\boldsymbol{u}_{\rm B} \cdot \boldsymbol{n} = 0$ and zero tangential vorticity (in this 2D case, it translates to fix the scalar vorticity to zero), but on Σ we prescribe the vorticity by its exact solution. Normal Darcy velocities are fixed on $\Gamma_{\rm D}$: $\boldsymbol{u}_{\rm D} \cdot \boldsymbol{n} = 0$. Secondly, by construction, the concentration normal flux is zero on the left side of $\Gamma_{\rm B}$, which constitutes the Neumann boundary $\Gamma_{\rm N}$. The remainder of $\partial\Omega$ conforms the Dirichlet boundary Γ_0, where we impose $\phi = 0$ (see Figure 6.1, right). Both domains are rendered with a gap on the interface, for visualisation purposes.

As usual, to determine the convergence of the method we generate a sequence of successively refined triangulations of Ω (and conforming partitions for $\Omega_{\rm B}$, $\Omega_{\rm D}$ and Σ) and proceed to compute errors and decay rates according to

$$e(\boldsymbol{u}_{\rm B}) = \|\boldsymbol{u}_{\rm B} - \boldsymbol{u}_{{\rm B}h}\|_{{\rm div},\Omega_{\rm B}},\ e(\boldsymbol{\omega}_{\rm B}) = \|\boldsymbol{\omega}_{\rm B} - \boldsymbol{\omega}_{{\rm B}h}\|_{\mathbf{curl},\Omega_{\rm B}},\ e(\boldsymbol{u}_{\rm D}) = \|\boldsymbol{u}_{\rm D} - \boldsymbol{u}_{{\rm D}h}\|_{{\rm div},\Omega_{\rm D}},$$
$$e(p_{\rm B}) = \|p_{\rm B} - p_{{\rm B}h}\|_{0,\Omega_{\rm B}},\ e(p_{\rm D}) = \|p_{\rm D} - p_{{\rm D}h}\|_{0,\Omega_{\rm D}},\ e(\lambda) = \|\lambda - \lambda_h\|_{0,\Sigma}\|\lambda - \lambda_h\|_{1,\Sigma},$$
$$e(\phi) = \|\phi - \phi_h\|_{1,\Omega},\ r(\cdot) = -2\log(e(\cdot)/\hat{e}(\cdot))[\log(N/\hat{N})]^{-1},$$

where e and $\hat{e}$ denote errors produced on two consecutive meshes associated to schemes with N and $\hat{N}$ D.o.f. (degrees of freedom), respectively. The results are collected in Figure 6.2, where we plot the decaying of individual errors with the meshsize, for a lowest-order scheme. All panels indicate an $O(h)$ convergence, as anticipated by Theorem 5.4. We point out that an average of 7 Picard steps (accounting for the coupling between the Brinkman-Darcy and transport problems) are required to reach the stopping tolerance of $1e-6$, whereas an average of 3 Newton steps are sufficient to achieve

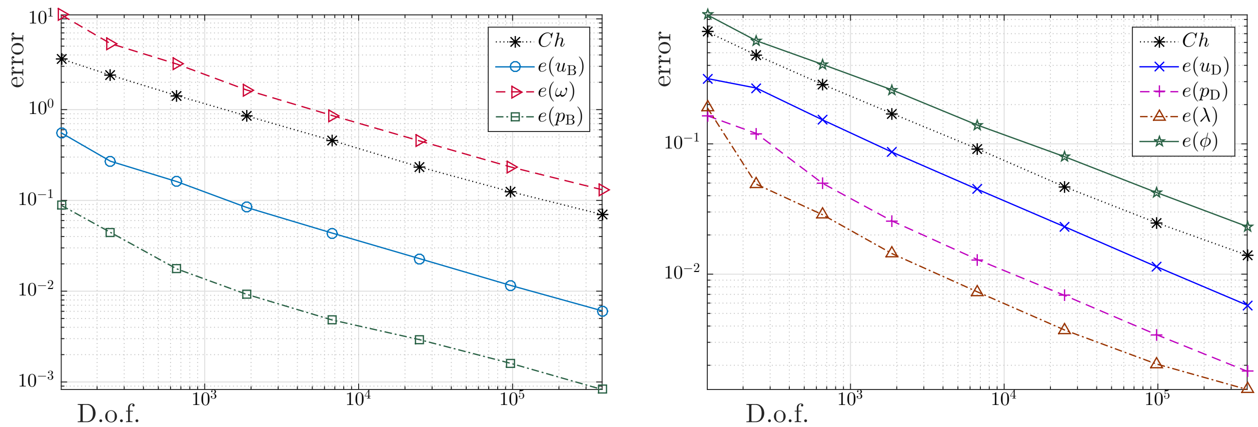

Figure 6.2: Test 1. Convergence history for the lowest-order approximation of the coupled Brinkman-Darcy-Transport problem.

convergence of the inner linearisation step (with a tolerance of $1e-7$) for the nonlinear advection-diffusion problem. We also portray the approximate solutions obtained with the proposed method on a fine mesh (see Figure 6.3).

An important assumption in the solvability analysis was an additional regularity for the discrete velocity, as stated in Hypothesis 4.1. Even if proving this assumption can be very difficult, we can at least provide numerical evidence of its validity for the finite element spaces we are employing here. For instance, we can observe that taking the regularity index as $s = 2.5$, and obtaining approximate solutions in the same refinement levels as mentioned above (whose error history is depicted in Figure 6.2), the values reported in Table 6.1 are produced. The ratios between the $\mathbf{L}^s-$norm of the discrete velocity and the H^1-norm of the concentration are tabulated in the last column and they suggest that the constant $\widetilde{C}_s$ is indeed uniform.

D.o.f.	h	$e(\boldsymbol{u}_\mathrm{B})$	$r(\boldsymbol{u}_\mathrm{B})$	$e(\boldsymbol{u}_\mathrm{D})$	$r(\boldsymbol{u}_\mathrm{D})$	$e(\phi)$	$r(\phi)$	$\\|\mathbf{S}(\phi_h)\\|_{\mathbf{L}^s(\Omega)}$	$\\|\phi_h\\|_{1,\Omega}$	$\widetilde{C}_s$
111	0.723	0.5621	–	0.371	–	0.823	–	0.644	1.791	0.521
237	0.479	0.269	1.786	0.267	0.791	0.622	0.678	0.920	1.825	0.570
664	0.285	0.160	0.997	0.155	1.045	0.407	0.819	1.022	1.890	0.569
1920	0.170	0.082	1.308	0.086	1.149	0.264	0.837	0.903	1.894	0.568
6625	0.099	0.043	1.160	0.044	1.117	0.161	0.975	1.082	1.907	0.567
24907	0.048	0.022	0.931	0.022	0.950	0.083	1.011	1.080	1.911	0.565
97907	0.023	0.011	0.956	0.011	0.962	0.022	1.041	1.079	1.911	0.564
382115	0.014	0.006	1.066	0.005	1.161	0.012	1.008	1.078	1.911	0.564

Table 6.1: Test 1. Illustration of Hypothesis 4.1 for the lowest-order scheme, and using $s = 2.5$.

Test 2. Our second example addresses the applicability of the formulation and the associated numerical scheme in the simulation of groundwater flow, where we have followed the setup adopted in Test 4.1 of [7]. The computational domain now corresponds to the rectangle $\Omega = (0, 12) \times (0, 6)$ (in square meters), where the Brinkman domain (with a maximum height of 4 m) is on the top and the Darcy subdomain (with a maximum height of 2.25 m) on the bottom. The subdomains are separated

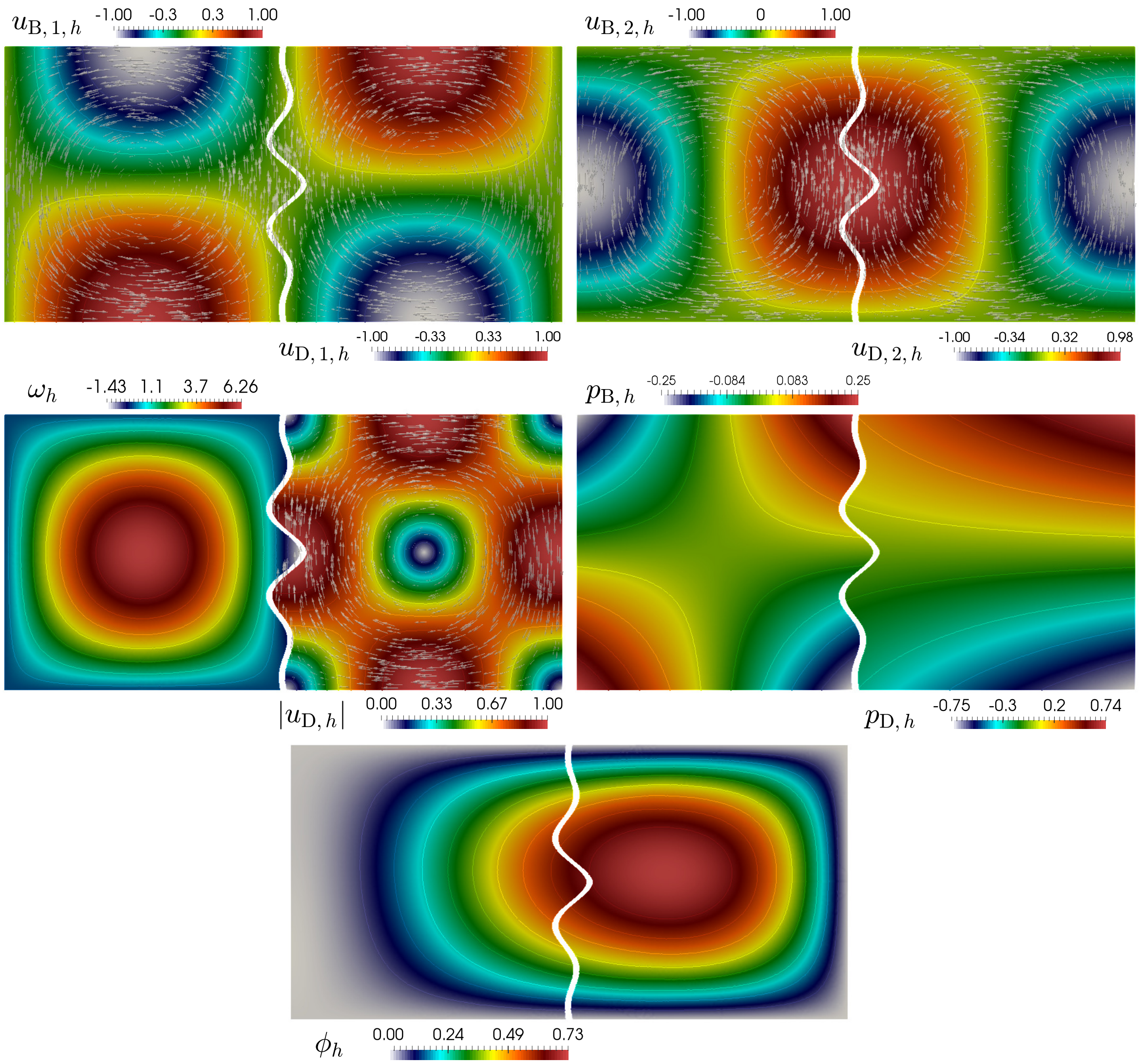

Figure 6.3: Test 1. Approximate velocity components, vorticity, pressure, and concentration, rendered on both Brinkman and Darcy domains.

by a step-shaped polygonal interface (see a sketch in the top-left panel of Figure 6.4, where we also depict sample triangular meshes). We consider $\mu = 1\,\mathrm{Pa{\cdot}s}$, $\beta = 0.75$, and the permeabilities are again isotropic and assume the values $\mathbb{K}_\mathrm{B} = \mathbb{K}_\mathrm{D} = 1e-6\,\mathrm{m}^2$. Normal velocities are imposed everywhere on $\partial\Omega$. On the top segment of Γ_B and in all Γ_D these are simply zero, whereas on the left and right sides of the Brinkman domain we prescribe the parabolic profiles

$$\boldsymbol{u}_\mathrm{B}\cdot\boldsymbol{n} = \frac{1}{4}(y-4)(y-8), \quad \text{and} \quad \boldsymbol{u}_\mathrm{B}\cdot\boldsymbol{n} = \frac{3}{16}(y-4)(8-y),$$

respectively, as well as the compatible vorticity $\boldsymbol{\omega}_\mathrm{B} = \frac{1}{2}(y-6)$ and $\boldsymbol{\omega}_\mathrm{B} = \frac{3}{8}(y-6)$, respectively. Regarding the transport equation, on the left side of the Brinkman domain (denoted by Γ_0) we impose a maximum solute concentration $\phi = \phi_{\max} = 0.99$, whereas zero total flux is considered on $\Gamma_\mathrm{N} = \partial\Omega \setminus \Gamma_0$. The nonlinear diffusion assumes the form $\vartheta(\phi) = \exp(-\frac{1}{4}\phi)$ and the flux is simply linear $f_\mathrm{bk}(\phi) = 0.001\phi$. We take $\boldsymbol{k} = (0,-1)^\mathtt{t}$ and assume that an external source modulates the

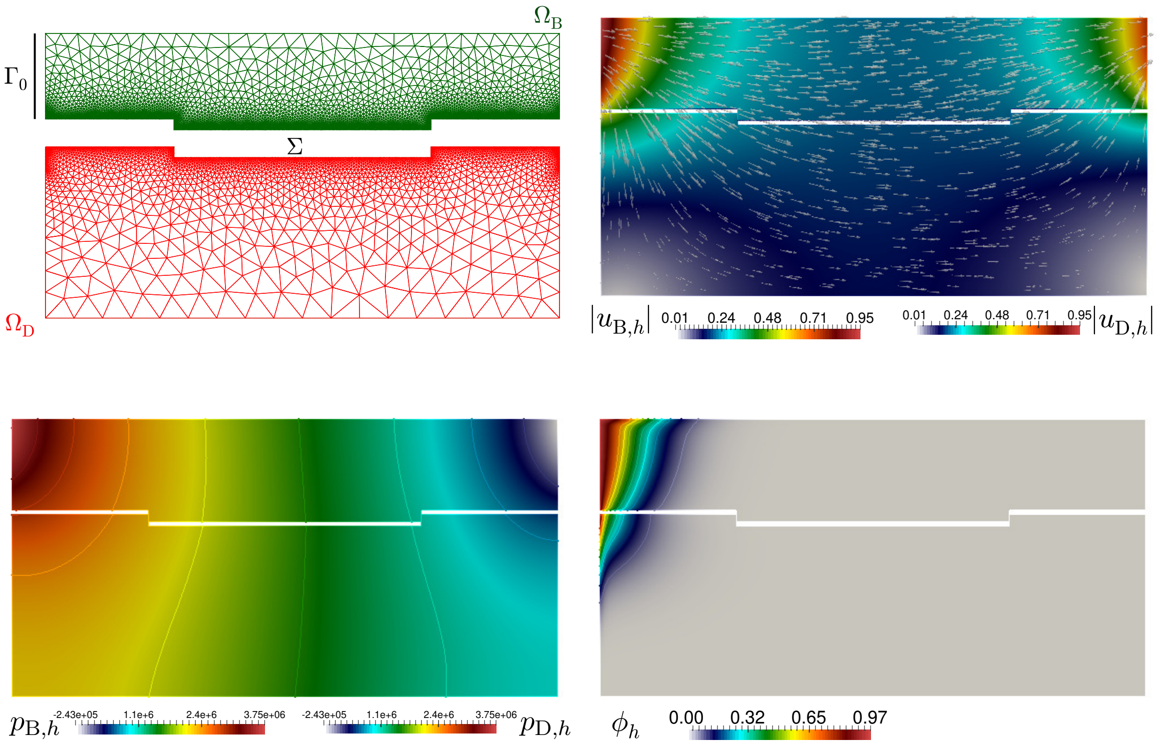

Figure 6.4: Test 2. Sample mesh and domain configuration (top left), and approximate velocity, pressure, and solute concentration produced with a second-order scheme.

Brinkman flow (for instance due to rain, to wind, or to the leakage of contaminants through the top portion of the boundary, that represents the surface) so we use $\boldsymbol{f}_{\rm B} = (0.1, 0.001)^{\tt t}$. We employ a second-order scheme resulting on a linear system of 1,517,352 unknowns for the Brinkman-Darcy subproblem and 327,707 D.o.f. for the transport equation. Seven fixed-point iterations were needed to reach the desired tolerance and only two Newton steps were required for the convergence of the inner linearisation (probably due to the fact that the nonlinear diffusion is in this case milder than that used in Tests 1 and 3). The results are collected in Figure 6.4, which shows flow patterns as well as the solute entering the domain and starting to propagate towards the right.

Test 3. We finalise this section presenting a 3D computation that illustrates the use of our method in the numerical simulation of filtering devices. Flow-rate conditions are taken similarly to those employed in [12, 5], namely a cylindrical geometry aligned with the $y-$axis, with varying cross section; where the Brinkman domain is the region with largest radius ($r = 4\,$cm and length $L = 6\,$cm), and the Darcy domain constitutes the two other sections of the device (of radii $r = 2\,$cm and $r = 3\,$cm and lengths of $L = 3\,$cm and $L = 5.1\,$cm, respectively). We assume that there is an inlet boundary belonging to $\Gamma_{\rm B}$ and an outlet disk at the end of the cylinder, on $\Gamma_{\rm D}$. The flow is driven basically by injection of fluid. A Poiseuille Brinkman velocity is prescribed at the inlet, as well a compatible vorticity

$$\boldsymbol{u}_{\rm B} \cdot \boldsymbol{n} = 2\big(1 - \frac{1}{4}(x^2 + z^2)\big), \quad \text{and} \quad \boldsymbol{\omega}_{\rm B} \times \boldsymbol{n} = \Big(-2x\big(1 - \frac{1}{4}(x^2 + z^2)\big), 0, -2x\big(1 - \frac{1}{4}(x^2 + z^2)\big)\Big)^{\tt t},$$

whereas on the outlet boundary we impose a constant Darcy pressure $p_{\mathrm{D}} = p_0 = 0.1$. On the remainder of the domain boundary we set slip conditions for velocity (and zero tangential vorticity on the curved Brinkman boundary). For the transport equation we impose a constant concentration on the inlet and assume zero total flux everywhere else, therefore the inlet (the disc of radius 4 m and centred at the origin) is the boundary Γ_0 and the remainder of the boundary is Γ_{N}. The interface conditions correspond to the ones stated in (2.4), and a depiction of the domain and boundary setup is presented in the first panel of Figure 6.5. The constitutive equations specifying the nonlinear diffusion ϑ and the unidirectional flux f_{bk} are simply taken as in Test 1 above, with $c = 0.4$. Other model parameters are chosen as

$$\mu = 0.01, \quad \mathbb{K}_{\mathrm{B}} = 0.01, \quad \mathbb{K}_{\mathrm{D}} = 0.00001, \quad \beta = \begin{cases} \frac{1}{2} & \text{in } \Omega_{\mathrm{B}}, \\ 10 & \text{in } \Omega_{\mathrm{D}} \end{cases}, \quad \boldsymbol{f}_{\mathrm{B}} = (0, \cos(xyz)\sin(\pi x)\cos(\pi z), 0)^{\mathtt{t}},$$
$$\boldsymbol{f}_{\mathrm{D}} = (\exp(-xy) + x\exp(-x^2), \cos(\pi y) - y\exp(-y^2), xyz - z\exp(-z^2))^{\mathtt{t}}, \quad \boldsymbol{k} = (0,1,0)^{\mathtt{t}},$$

where we note that the hydraulic conductivity is discontinuous across the interface. The domain has been discretised with an unstructured tetrahedral mesh of 74,108 elements, and we have employed a first-order scheme. The approximate solutions are shown in the remaining panels of Figure 6.5. The first observation from the velocity streamlines is that the non-symmetric external forces $\boldsymbol{f}_{\mathrm{B}}$ and $\boldsymbol{f}_{\mathrm{D}}$ rapidly disrupt the Poiseuille profile as the flow moves away from the inlet. We can also see that the Lagrange multiplier enforces correctly the continuity of pressure across the interface but that there exists a very large Brinkman pressure and a large pressure drop is then seen in the Darcy domain. Also, the tangential components of vorticity slowly decrease when approaching the the interface. As the flow patterns stabilise due to the interfacial conditions, the propagation of concentration also becomes very uniform.

For this problem, the convergence of the Picard algorithm occurred after nine iterations and the inner Newton iterations for the transport problem converged after four steps.

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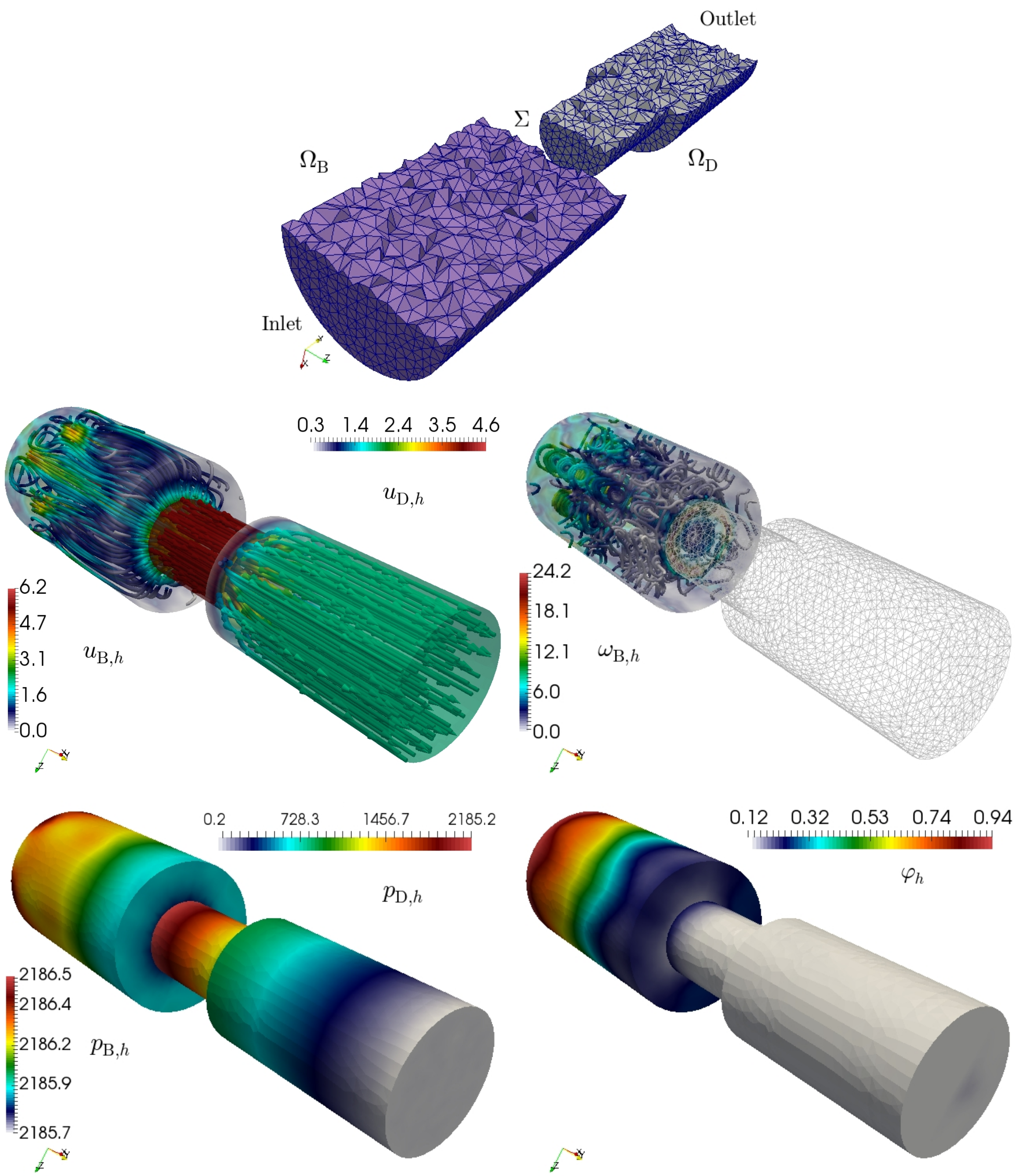

Figure 6.5: Test 3. Sample (coarse) mesh and domain configuration (top), and approximate velocity, vorticity, pressure, and solute concentration produced with a first-order scheme.

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