DIVERGENCE-FREE FINITE ELEMENTS FOR THE NUMERICAL SOLUTION OF A HYDROELASTIC VIBRATION PROBLEM *

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Abstract. In this paper we analyze a divergence-free finite element method to solve a fluid-structure interaction spectral problem in the three-dimensional case. The unknowns of the resulting formulation are the displacements for the fluid and the solid, and the pressure of the fluid on the interface separating both media. The resulting mixed eigenvalue problem is approximated by using appropriate basis of the divergence-free lowest order Raviart–Thomas elements for the fluid, piecewise linear elements for the solid and piecewise constant elements for the interface pressure. It is proved that eigenvalues and eigenfunctions are efficiently approximated and some numerical results are presented in order to assess the performance of the method.

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1. Introduction

Fluid-structure interaction involves the motion of a deformable structure in contact with an internal or surrounding fluid. These kind of interactions are a crucial consideration in the design of many engineering systems, e.g., aircraft, engines and bridges. Another example where it has also to be taken into account is for the analysis of aneurysms in large arteries and artificial heart valves. As a result, much effort has gone into the development of general finite element methods for fluid-structure systems.

In this paper we are concerned with the interaction between a fluid and an elastic structure. We will consider the problem that consists of a bounded domain completely filled by the fluid and limited by the solid. Different formulations have been proposed to solve this problem (see, for instance, [12] and references therein).

Pure displacement formulations are very much used in applications. Indeed, they lead to simple well-posed generalized eigenvalue problems and are convenient to handle more complex interactions between fluids and structures (for instance, in presence of wall dissipation [11]). However, it is well-known that standard discretizations with Lagrange elements lead to spurious vibration frequencies interspersed among the physical ones [20]. In [9] (see also [8]) a finite element method for a 2D (two-dimensional) domain that does not present spurious modes has been introduced. It is based on using displacement variables for both the fluid and the

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solid. The pressure of the fluid is also used for the theoretical analysis. The displacements are approximated by piecewise linear elements for the solid and Raviart–Thomas elements of lowest order for the fluid. The interface coupling of both discretization is achieved in a non-conforming way. In the case of an incompressible fluid, the fluid displacement variables can be eliminated by using the so-called *added mass* formulation (see, for instance, [25]). This consists of taking into account the effect of the fluid by means of a Neumann-to-Dirichlet operator (also called Steklov-Poincaré operator) on the fluid-solid interface. The finite element discretization of this problem has been treated, for instance, in [6], [13].

Another strategy was presented in [9, 10]. It consists of using divergence-free fields for describing the fluid displacements. This is theoretically analyzed and conveniently implemented in the 2D case by means of curls of piecewise linear elements. However, its extension to 3D (three-dimensional case) is not straightforward. In particular, in order to use a similar approach in 3D, we would need to construct a basis of the divergence-free discrete fields. Two approaches have been recently proposed in [1] to this end. The first one consists in finding a suitable selection of curls of Nédélec finite elements. The second one, is based on an efficient algebraic procedure which is inspired by [3, 4] and consists of writing the basis functions as linear combinations of lowest-order Raviart–Thomas elements. We perform the numerical analysis of the proposed method and report numerical results by considering both sets of basis functions which illustrate the good performance of the method.

The outline of the paper is as follows. In Section 2 we introduce the model problem. Next, in Section 3, we recall the mixed formulation proposed in [9], introduce the solution operator and characterize its spectrum. In Section 4 we introduce the numerical approach based on standard piecewise linear finite elements for the displacement in the solid, divergence-free Raviart–Thomas elements for the displacement of the fluid and piecewise constant functions for the pressure on the interface. In addition, we introduce the discrete solution operator. Next, in Section 5 we prove convergence of the method and optimal-order error estimates for eigenfunctions and eigenvalues. Finally, in Section 6, we report some numerical experiments.

2. The model problem

We consider the problem of determining the vibration modes of a linear elastic structure containing an incompressible, inviscid and homogeneous fluid. We focus on the three-dimensional (3D) case and assume polygonal boundaries and interfaces.

Let Ω_F and Ω_S be polyhedral Lipschitz bounded domains occupied by the fluid and the solid, respectively, as shown in Figure 1. We assume Ω_F to be a connected domain with boundary $\Gamma_I = \overline{\Omega_S} \cap \overline{\Omega_F}$ and $\boldsymbol{\nu}$ its unit normal vector pointing toward Ω_S. We assume that $\Gamma_I = \bigcup_{j=1}^{J} \Gamma_j$, where Γ_j, $j = 1, ..., J$ are the faces of the polyhedral interface Γ_I.

The exterior boundary of the solid is the union of polyhedral surfaces Γ_D and Γ_N, the structure being fixed on Γ_D and free of stress on Γ_N. The outward unit normal vector to $\partial\Omega_S \setminus \Gamma_I$ is denoted by $\boldsymbol{n}$.

Thought this paper we use the standard notation for Sobolev spaces, norms and seminorms. The space $H(\operatorname{div};\Omega_F) := \{\mathbf{u} \in [L^2(\Omega_F)]^3 : \operatorname{div}\mathbf{u} \in L^2(\Omega_F)\}$ is endowed with the norm defined by

$$\|\mathbf{u}\|^2_{\operatorname{div},\Omega_F} := \|\mathbf{u}\|^2_{0,\Omega_F} + \|\operatorname{div}\mathbf{u}\|^2_{0,\Omega_F}.$$

We also introduce the spaces:

$$\begin{aligned} H(\operatorname{div}^0;\Omega_F) &:= \{\mathbf{u} \in H(\operatorname{div};\Omega_F) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_F\}, \\ H_0(\operatorname{div};\Omega_F) &:= \{\mathbf{u} \in H(\operatorname{div},\Omega_F) : \mathbf{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma_I\}, \\ H_0(\operatorname{div}^0;\Omega_F) &:= H(\operatorname{div}^0;\Omega_F) \cap H_0(\operatorname{div},\Omega_F). \end{aligned}$$

In what follows, we employ $\mathbf{0}$ to denote a generic null vector and C, with or without subscripts, bars, tildes or hats, to denote a generic positive constant independent of the discretization parameters, which may take different values at different places.

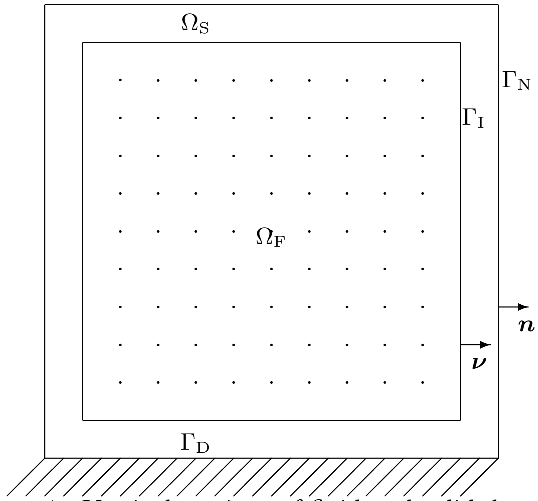


FIGURE 1. Vertical sections of fluid and solid domains.

In the case of an incompressible fluid, the classical elastoacoustics approximation for small amplitude motions yields the following eigenvalue problem for the vibration modes of the coupled system and their corresponding frequencies ω (see, for instance, [25]).

Problem 1. Find $\omega \geq 0$, $\mathbf{u} \in H(\mathrm{div}; \Omega_F)$, $\mathbf{v} \in H^1(\Omega_S)^3$ and $p \in H^1(\Omega_F)$, $(\mathbf{u}, \mathbf{v}, p) \neq (\mathbf{0}, \mathbf{0}, 0)$, such that

$$\nabla p - \omega^2 \rho_F \mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega_F, \tag{1a}$$

$$\mathrm{div}\, \mathbf{u} = 0 \quad \text{in} \quad \Omega_F, \tag{1b}$$

$$\mathrm{div}\, \boldsymbol{\sigma}(\mathbf{v}) + \omega^2 \rho_S \mathbf{v} = \mathbf{0} \quad \text{in} \quad \Omega_S, \tag{1c}$$

$$\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} + p\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_I, \tag{1d}$$

$$\mathbf{u} \cdot \boldsymbol{\nu} - \mathbf{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_I, \tag{1e}$$

$$\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N, \tag{1f}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_D, \tag{1g}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the displacements in the fluid and the solid, respectively, and ρ_F and ρ_S are the respective densities. The variable $\boldsymbol{\sigma}$ is the 3×3 stress tensor defined as follows

$$\boldsymbol{\sigma}(\mathbf{v}) := \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{v}),$$

where the elasticity operator $\mathcal{C} : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}^{3\times 3}$ is given by Hooke's law,

$$\mathcal{C}\boldsymbol{\tau} := \lambda_S (\mathrm{tr}\, \boldsymbol{\tau})\, I + 2\mu_S \boldsymbol{\tau}$$

with λ_S and μ_S being the Lamé coefficients, and $\boldsymbol{\varepsilon}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^t)$ is the linearized strain tensor. In spite of the fact that equations (1c)-(1f) of Problem 1 must be understood in the sense of distributions, since $p \in H^1(\Omega_F)$ and $\mathbf{v} \in [H^1(\Omega_S)]^3$, these interface conditions are valid in the $L^2(\Gamma_I)$ sense. Let us remark that the actual vibration modes of the coupled system are the solutions of Problem 1 with $\omega > 0$. However, $\omega = 0$ is an eigenvalue of Problem 1 with corresponding eigenspace $H_0(\mathrm{div}^0; \Omega_F) \times \{\mathbf{0}\} \times \{0\}$, which, from the physical point of view, is a spurious solution of the model given by equations (1).

3. Variational formulation and spectral characterization

Let us define some function spaces that we will use to pose a variational formulation of Problem 1. Let $P := L^2(\Gamma_I)$, $\mathbf{H} := [L^2(\Omega_F)]^3 \times [L^2(\Omega_S)]^3$, $\mathbf{X} := H(\mathrm{div};\Omega_F) \times [H^1_{\Gamma_D}(\Omega_S)]^3$ where $H^1_{\Gamma_D}(\Omega_S) := \{v \in H^1(\Omega_S) : v|_{\Gamma_D} = 0\}$ and

$$\mathbf{Y} := \left\{(\mathbf{u},\mathbf{v}) \in \mathbf{X} : \quad \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega_F \text{ and } \mathbf{u}\cdot\boldsymbol{\nu}\big|_{\Gamma_I} \in L^2(\Gamma_I)\right\}.$$

We denote by $\|\cdot\|_{\mathbf{X}}$ the natural norm on $\mathbf{X}$, by $\|\cdot\|_{\mathbf{H}}$ the L^2 norm on $\mathbf{H}$ and by $\|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{Y}} := \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{X}} + \|\mathbf{u}\cdot\boldsymbol{\nu}\|^2_{0,\Gamma_I}$ the norm on $\mathbf{Y}$.

Let us now define an additional unknown, $\mu := p|_{\Gamma_I}$. Multiplying by $\boldsymbol{\phi}$ in (1a) and $\boldsymbol{\psi}$ in (1c) such that $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{Y}$ and by $\zeta \in P$ in (1e), we obtain the following variational formulation of Problem 1, where $\lambda = \omega^2$:

Problem 2. Find $\lambda \in \mathbb{R}$ and $(\mathbf{u},\mathbf{v},\mu) \in \mathbf{Y} \times P$, $(\mathbf{u},\mathbf{v},\mu) \neq (\mathbf{0},\,\mathbf{0},\,0)$, such that

$$\int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) + \int_{\Gamma_I} \mu\,(\boldsymbol{\phi}\cdot\boldsymbol{\nu} - \boldsymbol{\psi}\cdot\boldsymbol{\nu}) = \lambda\left(\int_{\Omega_F} \rho_F \mathbf{u}\cdot\boldsymbol{\phi} + \int_{\Omega_S} \rho_S \mathbf{v}\cdot\boldsymbol{\psi}\right), \tag{2a}$$

$$\int_{\Gamma_I} \zeta\,(\mathbf{u}\cdot\boldsymbol{\nu} - \mathbf{v}\cdot\boldsymbol{\nu}) = 0, \tag{2b}$$

for all $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$, where $\boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) := \sum_{i,j=1}^3 \boldsymbol{\sigma}_{ij}(\mathbf{v})\boldsymbol{\varepsilon}_{ij}(\boldsymbol{\psi})$ denotes the usual inner product. In principle, for Problem 1 and 2 to be equivalent, in the latter λ should be sought in $[0,+\infty)$. However, it is easy to check that, for any solution of Problem 2, $\lambda \geq 0$.

Let us now consider the following continuous bilinear forms:

$$\begin{aligned}
a((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) &:= \int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}), \quad (\mathbf{u},\mathbf{v}),\,(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{Y},\\
b((\mathbf{u},\mathbf{v}),\zeta) &:= \int_{\Gamma_I} \zeta\,(\mathbf{u}\cdot\boldsymbol{\nu} - \mathbf{v}\cdot\boldsymbol{\nu}), \quad (\mathbf{u},\mathbf{v}) \in \mathbf{Y}, \quad \zeta \in P,\\
d((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) &:= \int_{\Omega_F} \rho_F \mathbf{u}\cdot\boldsymbol{\phi} + \int_{\Omega_S} \rho_S \mathbf{v}\cdot\boldsymbol{\psi}, \quad (\mathbf{u},\mathbf{v}),\,(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{Y}.
\end{aligned}$$

Next step would be to prove Brezzi's conditions for the source problem associated to Problem 2, i.e.,

H1: there exists $\alpha > 0$ such that

$$a((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) \geq \alpha\|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{Y}} \qquad \forall\,(\mathbf{u},\mathbf{v}) \in \mathbf{W},$$

where

$$\mathbf{W} := \{(\mathbf{u},\mathbf{v}) \in \mathbf{Y} : b((\mathbf{u},\mathbf{v}),\zeta) = 0 \;\; \forall\,\zeta \in P\} = \{(\mathbf{u},\mathbf{v}) \in \mathbf{Y} : \mathbf{u}\cdot\boldsymbol{\nu} = \mathbf{v}\cdot\boldsymbol{\nu} \text{ on } \Gamma_I\}.$$

H2: b satisfies the inf-sup condition

$$\inf_{\zeta\in P}\left[\sup_{\substack{(\mathbf{u},\mathbf{v})\in\mathbf{Y}\\(\mathbf{u},\mathbf{v})\neq(\mathbf{0},\mathbf{0})}} \frac{b((\mathbf{u},\mathbf{v}),\zeta)}{\|(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}}\|\zeta\|_{0,\Gamma_I}}\right] \geq \beta.$$

For the proof of **H2**, we refer to [9, Lemma 8.1] where the condition is proved in 2D; its extension to 3D is straightforward. In particular, as in that reference, the proof relies on the fact that for each $\zeta \in P$, there exists $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}$, satisfying

$$\mathbf{u}\cdot\boldsymbol{\nu} - \mathbf{v}\cdot\boldsymbol{\nu} = \zeta \quad \text{on} \quad \Gamma_I, \qquad \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_D \cup \Gamma_N \qquad \text{and} \qquad \|(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}} \leq C\|\zeta\|_{0,\Gamma_I}. \tag{3}$$

Concerning **H1**, it is clear that the form a is not coercive on $\mathbf{W}$, since for all $\mathbf{u} \in H_0(\operatorname{div}^0;\Omega_F)$, $(\mathbf{u},\mathbf{0}) \in \mathbf{W}$ and $a((\mathbf{u},\mathbf{0}),(\mathbf{u},\mathbf{0})) = 0$. However, following the ideas from [9] and considering $a^* := a + d$ instead of a, we obtain a coercive bilinear form on $\mathbf{W}$:

Lemma 3.1. *There exists $\alpha > 0$ such that*

$$a^*((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) \geq \alpha \|(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}} \qquad \forall\,(\mathbf{u},\mathbf{v}) \in \mathbf{W}.$$

Proof. Notice that for all $(\mathbf{u},\mathbf{v}) \in \mathbf{X}$ such that $\operatorname{div}\mathbf{u} = 0$,

$$a^*((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) = \int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \int_{\Omega_F} \rho_F |\mathbf{u}|^2 + \int_{\Omega_S} \rho_S |\mathbf{v}|^2 \geq \alpha \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{X}},$$

with $\alpha = \min\{\rho_F, \kappa\}$ where κ is the constant in Korn's inequality (see, for instance, [19]). Hence, from the fact that for all $(\mathbf{u},\mathbf{v}) \in \mathbf{W}$, $\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{0,\Gamma_I} = \|\mathbf{v}\cdot\boldsymbol{\nu}\|_{0,\Gamma_I} \leq C\|\mathbf{v}\|_{1,\Omega_S} \leq \widetilde{C}\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}$ we conclude the claimed result. □

Thus, we consider this modified eigenvalue problem:

Problem 3. Find $\widehat{\lambda} \in \mathbb{R}$ and $(\mathbf{u},\mathbf{v},\mu) \in \mathbf{Y} \times P$, $(\mathbf{u},\mathbf{v},\mu) \neq (\mathbf{0},\,\mathbf{0},\,0)$, such that

$$a^*((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) + b((\boldsymbol{\phi},\boldsymbol{\psi}),\mu) = \widehat{\lambda}\, d((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})), \tag{4a}$$

$$b((\mathbf{u},\mathbf{v}),\zeta) = 0, \tag{4b}$$

for all $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$. Notice that λ is an eigenvalue of Problem 2 if and only if $\widehat{\lambda} = 1 + \lambda$ is an eigenvalue of Problem 3 and the eigenfunctions for both problems coincide.

Theorem 3.2. *Problems 1 and 3 are equivalents. In particular, $\mu = p|_{\Gamma_I}$.*

Proof. We have just shown that if $(\lambda,\mathbf{u},\mathbf{v},p)$ is a solution of Problem 1, then $(\widehat{\lambda} = \lambda + 1, \mathbf{u}, \mathbf{v}, p|_{\Gamma_I})$ is a solution of Problem 3. Now, let $(\widehat{\lambda},\mathbf{u},\mathbf{v},\mu)$ be an eigenpair of Problem 3. First, it is immediate to check that $\widehat{\lambda} \geq 1$, so that $\lambda \geq 0$. Since $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}$ and satisfies (4b), then (1b), (1g) and (1e) are automatically satisfied. Moreover, taking $(\mathbf{0},\boldsymbol{\psi})$, with $\boldsymbol{\psi} \in [\mathcal{D}(\Omega_S)]^3$ in (4a), we obtain (1c), while considering $\boldsymbol{\psi} \in [H^1_{\Gamma_D}(\Omega_S)]^3$ together with (1c) it follows that

$$\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} + \mu\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_I \tag{5}$$

and (1f).

On the other hand, let $p \in H^1(\Omega_F)$ be the solution of the following compatible Neumann problem:

$$\Delta p = 0 \quad \text{in} \quad \Omega_F, \qquad \nabla p \cdot \boldsymbol{\nu} = \lambda\rho_F \mathbf{u}\cdot\boldsymbol{\nu} \quad \text{on} \quad \Gamma_I, \qquad \int_{\Gamma_I} p = \int_{\Gamma_I} \mu. \tag{6}$$

Hence, from (1b), we have that $(\nabla p - \lambda\rho_F\mathbf{u}) \in H_0(\operatorname{div}^0;\Omega_F)$. Now, taking as test function $(\boldsymbol{\phi},\mathbf{0})$ with $\boldsymbol{\phi} = \nabla p - \lambda\rho_F\mathbf{u}$ in (4a), it follows that $\lambda \int_{\Omega_F} \rho_F\mathbf{u}\cdot(\nabla p - \lambda\rho_F\mathbf{u}) = 0$. Subtracting this equation to $\int_{\Omega_F} \nabla p\cdot(\nabla p - \lambda\rho_F\mathbf{u}) = 0$, we conclude (1a).

Now, let $s \in L^2_0(\Gamma_I) := \{r \in L^2(\Gamma_I) \ : \ \int_{\Gamma_I} r = 0\}$. Let $r \in H^1(\Omega)/\mathbb{R}$ be the solution of the compatible Neumann problem

$$\Delta r = 0 \quad \text{in} \quad \Omega_F, \qquad \nabla r \cdot \boldsymbol{\nu} = s \quad \text{on} \quad \Gamma_I. \tag{7}$$

Then, testing (4a) with $(\boldsymbol{\phi},\boldsymbol{\psi}) = (\nabla r, \mathbf{0})$ we obtain

$$\int_{\Gamma_I} \mu s = \lambda \int_{\Omega_F} \rho_F \mathbf{u}\cdot\nabla r.$$

On the other hand, using again that $(\nabla p - \lambda\rho_F \mathbf{u}) \in H_0(\mathrm{div}^0;\Omega_F)$, we have that

$$0 = \int_{\Omega_F} (\nabla p - \lambda\rho_F\mathbf{u}) \cdot \nabla r = \int_{\Gamma_I} p\nabla r \cdot \boldsymbol{\nu} - \lambda \int_{\Omega_F} \rho_F \mathbf{u} \cdot \nabla r.$$

Therefore,

$$\int_{\Gamma_I} \mu\, s = \int_{\Gamma_I} p\nabla r \cdot \boldsymbol{\nu} = \int_{\Gamma_I} p\, s$$

for all $s \in L^2_0(\Gamma_I)$. Hence, $p = \mu + C$, with C a constant. However, thanks to the third equation of problem (6), we conclude that $p = \mu$ on Γ_I, which together with (5) yields (1d). This fact completes the proof. □

The following theorem gives a characterization of the eigenspace associated to the eigenvalue $\widehat{\lambda} = 1$ in Problem 3.

Theorem 3.3. *The eigenspace corresponding to $\widehat{\lambda} = 1$ in Problem 3 is $\mathbf{K} \times \{0\}$ with $\mathbf{K} := H_0(\mathrm{div}^0;\Omega_F) \times \{\mathbf{0}\}$.*

Proof. Let $(\mathbf{u}, \mathbf{0}) \in \mathbf{K}$. Clearly, $(\mathbf{u}, \mathbf{0}) \in \mathbf{W}$ and $(\mathbf{u}, \mathbf{0}, 0)$ satisfies Problem 3 with $\widehat{\lambda} = 1$. Therefore, $(\mathbf{u}, \mathbf{0}, 0)$ is an eigenfunction of Problem 3 with associated eigenvalue $\widehat{\lambda} = 1$. Conversely, let $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$, $(\mathbf{u}, \mathbf{v}, \mu) \neq (\mathbf{0}, \mathbf{0}, 0)$, such that

$$\begin{aligned} a((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) + b((\boldsymbol{\phi}, \boldsymbol{\psi}), \mu) &= 0, \\ b((\mathbf{u}, \mathbf{v}), \zeta) &= 0, \end{aligned}$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$. From the second equation we obtain $\mathbf{u}\cdot\boldsymbol{\nu} = \mathbf{v}\cdot\boldsymbol{\nu}$ on Γ_I. By using $(\boldsymbol{\phi}, \boldsymbol{\psi}) = (\mathbf{u}, \mathbf{v})$ in the first equation and Korn's inequality, we obtain that $\mathbf{v} = \mathbf{0}$ in Ω_S. Since $\mathrm{div}\,\mathbf{u} = 0$ in Ω_F and $\mathbf{u}\cdot\boldsymbol{\nu} = \mathbf{v}\cdot\boldsymbol{\nu} = 0$ on Γ_I, then $\mathbf{u} \in H_0(\mathrm{div}^0;\Omega_F)$. Finally, the inf-sup condition **H2** allow us to conclude that $\mu = 0$ on Γ_I. □

In order to obtain a spectral characterization of Problem 3, we introduce the solution operator:

$$\begin{aligned} \mathbf{T} : \mathbf{H} &\longrightarrow \mathbf{Y} \subseteq \mathbf{H}, \\ (\mathbf{f}, \mathbf{g}) &\longmapsto \mathbf{T}(\mathbf{f}, \mathbf{g}) := (\mathbf{u}, \mathbf{v}) \end{aligned}$$

with $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$ being the solution of the following source problem: Find $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$ such that

$$a^*((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) + b((\boldsymbol{\phi}, \boldsymbol{\psi}), \mu) = d((\mathbf{f}, \mathbf{g}), (\boldsymbol{\phi}, \boldsymbol{\psi})), \tag{8a}$$

$$b((\mathbf{u}, \mathbf{v}), \zeta) = 0, \tag{8b}$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$. By virtue of the following theorem, $\mathbf{T}$ is well-defined and bounded. In addition, $(\gamma, (\mathbf{u}, \mathbf{v}))$ with $\gamma \neq 0$ is an eigenpair of $\mathbf{T}$ if and only if there exists $\mu \in P$ such that $(1/\gamma, (\mathbf{u}, \mathbf{v}, \mu))$ is a solution of Problem 3.

Theorem 3.4. *Given $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}$, there exists a unique solution $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$ of the source problem*

$$a^*((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) + b((\boldsymbol{\phi}, \boldsymbol{\psi}), \mu) = d((\mathbf{f}, \mathbf{g}), (\boldsymbol{\phi}, \boldsymbol{\psi})),$$

$$b((\mathbf{u}, \mathbf{v}), \zeta) = 0,$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$. Moreover, there exists $C > 0$ such that

$$\|(\mathbf{u}, \mathbf{v})\|_{\mathbf{Y}} + \|\mu\|_{0,\Gamma_I} \leq C\|(\mathbf{f}, \mathbf{g})\|_{\mathbf{H}}. \tag{10}$$

Proof. Since the bilinear forms a^* and b satisfy the Brezzi's conditions, the mixed source problem above is well-posed. See, for instance, [15]. □

Notice that it is easy to prove that Problem 3 is equivalent to the following:
Problem 3*. Find $\widehat{\lambda} \in \mathbb{R}$ and $(\mathbf{u}, \mathbf{v}) \in \mathbf{W}$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$, such that

$$a^*((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = \widehat{\lambda}\, d((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})), \quad \forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{W}. \tag{11}$$

In fact, any solution of Problem 3 gives a solution of Problem 3*. Conversely, let $(\widehat{\lambda}, (\mathbf{u}, \mathbf{v}))$ be an eigenpair of Problem 3*. Thanks to Theorem 3.4, we know that there exists a unique solution $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}, \widetilde{\mu}) \in \mathbf{Y} \times P$ of the following source problem:

$$a^*((\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}), (\boldsymbol{\phi}, \boldsymbol{\psi})) + b((\boldsymbol{\phi}, \boldsymbol{\psi}), \widetilde{\mu}) = d(\widehat{\lambda}(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})),$$

$$b((\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}), \zeta) = 0,$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{Y}$ and $\zeta \in P$. Hence, $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) \in \mathbf{W}$ and it satisfies $a^*((\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = d(\widehat{\lambda}(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi}))$ for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{W}$. Since a^* is elliptic in $\mathbf{W}$, because of (11) $(\mathbf{u}, \mathbf{v})$ is the unique solution of this equation. Then, $(\widehat{\lambda}, (\mathbf{u}, \mathbf{v}, \mu))$ is an eigenpair of Problem 3.

Because of (8b), notice that $\mathbf{T}(\mathbf{H})$ is actually included into $\mathbf{W} \subseteq \mathbf{Y}$. We will focus on studying $\mathbf{T}|_{\mathbf{W}}$. Since the bilinear forms a^* and d are symmetric, $\mathbf{T}$ is self-adjoint with respect to both. Hence all of its eigenvalues are real and by virtue of (11) non negative. In addition, because of Theorem 3.3, $\mathbf{T}|_{\mathbf{K}}$ is the identity on the infinite dimensional subspace $\mathbf{K} \subseteq \mathbf{W}$. Therefore, $\mathbf{T}|_{\mathbf{W}}$ is not compact. However, the restriction of $\mathbf{T}$ to the orthogonal complement of $\mathbf{K}$ is compact. We will use this to characterize the spectrum of $\mathbf{T}$. To this end, we start recalling the classical Helmholtz decomposition (cf. [19, Theorem I.2.7]):

$$[L^2(\Omega_F)]^3 = H_0(\mathrm{div}^0; \Omega_F) \overset{\perp}{\oplus} \nabla H^1(\Omega_F).$$

As a consequence of this result and Theorem 3.3, we obtain that

$$\mathbf{K}^{\perp_{\mathbf{H}}} := \left\{ (\nabla q, \mathbf{v}) \,:\, q \in H^1(\Omega_F),\, \mathbf{v} \in [L^2(\Omega_S)]^3 \right\}.$$

In addition, it can be easily proved that $\mathbf{K}^{\perp_{\mathbf{W}}} = \mathbf{K}^{\perp_{\mathbf{H}}} \cap \mathbf{W}$ and that $\mathbf{K}$ and $\mathbf{K}^{\perp_{\mathbf{W}}}$ are also orthogonal with respect to a^*. Moreover, we have the following result.

Lemma 3.5. *The operator $\mathbf{T}$ satisfies $\mathbf{T}(\mathbf{K}^{\perp_{\mathbf{H}}}) \subseteq \mathbf{K}^{\perp_{\mathbf{W}}}$. Moreover, there exist $s > 1/2$, $t > 0$ and $C > 0$ such that if $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$ is the solution of problem (8) with $(\mathbf{f}, \mathbf{g}) \in \mathbf{K}^{\perp_{\mathbf{W}}}$, then $\mathbf{u} \in [H^s(\Omega_F)]^3$, $\mathbf{v} \in [H^{1+t}(\Omega_S)]^3$, $\mu \in H^{1/2+s}(\Gamma_I)$ and*

$$\|\mathbf{u}\|_{s,\Omega_F} + \|\mathbf{v}\|_{1+t,\Omega_S} + \|\mu\|_{1/2+s,\Gamma_I} \leq C \|(\mathbf{f}, \mathbf{g})\|_{\mathbf{X}}. \tag{12}$$

Proof. The proof of $\mathbf{T}(\mathbf{K}^{\perp_{\mathbf{H}}}) \subseteq \mathbf{K}^{\perp_{\mathbf{W}}}$ is similar to that of [9, Lemma 4.1]. We omit further details. Let $(\mathbf{f}, \mathbf{g}) \in \mathbf{K}^{\perp_{\mathbf{W}}}$ and let $(\mathbf{u}, \mathbf{v}, \mu) \in \mathbf{Y} \times P$ be the solution of problem (8). Since $(\mathbf{u}, \mathbf{v}) \in \mathbf{K}^{\perp_{\mathbf{W}}}$, there exists $\varphi \in H^1(\Omega_F)$ such that $\mathbf{u} = \nabla \varphi$ and, in addition, $\mathrm{div}\, \mathbf{u} = 0$ in Ω_F and $\mathbf{u} \cdot \boldsymbol{\nu} = \mathbf{v} \cdot \boldsymbol{\nu}$ on Γ_I. Thus, φ is a solution of the compatible Neumann problem

$$\Delta \varphi = 0 \quad \text{in} \quad \Omega_F, \qquad \nabla \varphi \cdot \boldsymbol{\nu} = \mathbf{v} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma_I.$$

As it is well-known (see [16, Corollary 23.5]), $\varphi \in H^{1+s}(\Omega_F)$ with $s > 1/2$. In addition,

$$\|\mathbf{u}\|_{s,\Omega_F} = \|\nabla \varphi\|_{s,\Omega_F} \leq C \sum_{j=1}^{J} \|\mathbf{v} \cdot \boldsymbol{\nu}\|_{1/2,\Gamma_j} \leq C \|(\mathbf{f}, \mathbf{g})\|_{\mathbf{X}},$$

where the last inequality follows from (10).

Now, since $(\mathbf{f},\mathbf{g}) \in \mathbf{K}^{\perp_\mathbf{W}}$, there exists $z \in H^1(\Omega_F)$ such that $\mathbf{f} = \nabla z$ and, in particular, z satisfies $\Delta z = \operatorname{div}\mathbf{f} = 0$ in Ω_F and $\nabla z \cdot \boldsymbol{\nu} = \mathbf{g}\cdot\boldsymbol{\nu}$ on Γ_I. Since $\mathbf{g} \in [H^1(\Omega_S)]^3$, z also belongs to $H^{1+s}(\Omega_F)$. Replacing $\mathbf{f}$ and $\mathbf{u}$ by ∇z and $\nabla\varphi$, respectively, in (8a) and considering as test function $(\boldsymbol{\phi},\boldsymbol{\psi}) = (\nabla r, \mathbf{0})$ with $r \in H^1(\Omega_F)$ being the solution of $\Delta r = 0$ in Ω_F and $\nabla r \cdot \boldsymbol{\nu} = s$ on Γ_I with $s \in L^2_0(\Gamma_I)$, we obtain

$$\int_{\Omega_F} \rho_F \nabla\varphi \cdot \nabla r + \int_{\Gamma_I} \mu\, s = \int_{\Omega_F} \rho_F \nabla z \cdot \nabla r.$$

Integrating by parts we get $\mu = \rho_F z - \rho_F \varphi \in H^{1/2+s}(\Gamma_I)/\mathbb{R}$ and, in addition,

$$\|\mu\|_{H^{1/2+s}(\Gamma_I)/\mathbb{R}} \leq C\|(\mathbf{f},\mathbf{g})\|_{\mathbf{X}}.$$

On the other hand, its is well known that there exists $C > 0$ such that

$$\|\mu\|_{1/2+s,\Gamma_I} \leq C\left(\|\mu\|_{H^{1/2+s}(\Gamma_I)/\mathbb{R}} + \|\mu\|_{0,\Gamma_I}\right).$$

From the above inequality and (10), we obtain that $\mu \in H^{1/2+s}(\Gamma_I)$ and

$$\|\mu\|_{1/2+s,\Gamma_I} \leq C\|(\mathbf{f},\mathbf{g})\|_{\mathbf{X}}.$$

Now, proceeding as in the proof of Theorem 3.2, we obtain that $\mathbf{v}$ is the solution, in the sense of distributions, of the following problem:

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{v}) + \rho_S\mathbf{v} = \rho_S\mathbf{g} \quad & \text{in} \quad \Omega_S, \\ \boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} = -\mu\boldsymbol{\nu} \quad & \text{on} \quad \Gamma_I, \\ \boldsymbol{\sigma}(\mathbf{v})\boldsymbol{n} = \mathbf{0} \quad & \text{on} \quad \Gamma_N, \\ \mathbf{v} = \mathbf{0} \quad & \text{on} \quad \Gamma_D. \end{aligned}$$

Then, there exists $t > 0$ such that $\mathbf{v} \in [H^{1+t}(\Omega_S)]^3$ and satisfies

$$\|\mathbf{v}\|_{1+t,\Omega_S} \leq C\left(\|\mathbf{g}\|_{0,\Omega_S} + \|\mu\|_{1/2+s,\Gamma_I}\right) \leq C\|(\mathbf{f},\mathbf{g})\|_{\mathbf{X}},$$

which completes the proof.

□

Theorem 3.6. $\mathbf{T}(\mathbf{K}^{\perp_\mathbf{W}}) \subseteq \mathbf{K}^{\perp_\mathbf{W}}$ *and the operator* $\mathbf{T}\big|_{\mathbf{K}^{\perp_\mathbf{W}}} : \mathbf{K}^{\perp_\mathbf{W}} \longrightarrow \mathbf{K}^{\perp_\mathbf{W}}$ *is compact.*

Proof. Because of Lemma 3.5, $\mathbf{T}(\mathbf{K}^{\perp_\mathbf{W}}) \subseteq \left\{[H^s(\Omega_F)]^3 \times [H^{1+t}(\Omega_S)]^3\right\} \cap \mathbf{K}^{\perp_\mathbf{W}}$, with s and t positive. On the other hand, $\left\{[H^s(\Omega_F)]^3 \times [H^{1+t}(\Omega_S)]^3\right\} \cap \mathbf{W}$ is compactly imbedded in $\mathbf{W}$. Hence, $\mathbf{T}\big|_{\mathbf{K}^{\perp_\mathbf{W}}}$ is compact. □

Since the eigenvalues of Problem 3 are the reciprocal of the nonvanishing eigenvalues of $\mathbf{T}|_\mathbf{W}$ and the associated eigenfunctions coincide, the above Theorem yields a spectral characterization of Problem 3.

Theorem 3.7. *The spectrum of* $\mathbf{T}|_\mathbf{W} : \mathbf{W} \longrightarrow \mathbf{W}$ *consists of the eigenvalue* $\gamma = 1$ *and a sequence of finite multiplicity eigenvalues* $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ *converging to 0.* $\mathbf{K}$ *is the eigenspace associated to* $\gamma = 1$.

Proof. The spectral characterization is a direct consequence of the fact that $\mathbf{T}|_\mathbf{K}$ is the identity and Theorem 3.6. Moreover, it is easy to check that $\gamma_n \in (0,\,1)$. □

4. Finite element discretization

As it was proved in the previous section, $\mathbf{T}|_{\mathbf{K}}^{\perp \mathbf{W}}$ is compact. However, it is not clear at all how this space could be discretized, since the fluid displacement ought to be gradients with divergence in $L^2(\Omega)$. Because of this, we will deal with the full non-compact operator $\mathbf{T}$ instead.

Let $\{\mathcal{T}_h\}$ be a family of regular triangulation of $\Omega_F \cup \Omega_S$ such that each tetrahedron is completely contained either in Ω_F or in Ω_S compatible with the partition $\Gamma_D \cup \Gamma_N$ of the exterior boundary.

For each component of the displacements in the solid we use the standard piecewise linear finite element space

$$L_h(\Omega_S) := \left\{ v \in H^1(\Omega_S) \,:\, v|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h, \quad T \subseteq \overline{\Omega}_S \right\},$$

where $\mathbb{P}_1$ is the set of polynomials of degree not greater than 1.

For any $T \in \mathcal{T}_h$ lying in Ω_F, let

$$\mathcal{R}_0(T) := \left\{ \mathbf{u} \in [\mathbb{P}_1(T)]^3 \,:\, \mathbf{u}(x,y,z) = (a+dx, b+dy, c+dz), \;\; a,\, b,\, c,\, d \in \mathbb{R} \right\}.$$

The corresponding global space to approximate the fluid is the well-known Raviart–Thomas space defined as follows:

$$\mathbf{R}_h(\Omega_F) := \left\{ \mathbf{u} \in H(\mathrm{div};\Omega_F) \,:\, \mathbf{u}|_T \in \mathcal{R}_0(T) \quad \forall T \in \mathcal{T}_h, \quad T \subseteq \overline{\Omega}_F \right\}.$$

Given $\delta \in (0,1]$, let $\mathbf{I}_h^{\mathcal{R}} : [H^\delta(\Omega_F)]^3 \cap H(\mathrm{div};\Omega_F) \longrightarrow \mathbf{R}_h(\Omega_F)$ be the standard Raviart–Thomas interpolant, which for a sufficiently smooth function $\mathbf{u}$ is characterized by the identity:

$$\int_F \mathbf{I}_h^{\mathcal{R}}(\mathbf{u}) \cdot \boldsymbol{\nu}_F = \int_F \mathbf{u} \cdot \boldsymbol{\nu}_F, \tag{13}$$

for all faces F of elements $T \in \mathcal{T}_h$, $T \subseteq \overline{\Omega}_F$, with $\boldsymbol{\nu}_F$ being a unit vector normal to the face F.

It is well-known that $\mathbf{I}_h^{\mathcal{R}}$ is a bounded linear operator satisfying the following commuting diagram property (cf. [15]):

$$\mathrm{div}\left(\mathbf{I}_h^{\mathcal{R}} \mathbf{u}\right) = \mathcal{P}_h\left(\mathrm{div}\mathbf{u}\right) \qquad \forall\, \mathbf{u} \in [H^\delta(\Omega_F)]^3 \cap H(\mathrm{div};\Omega_F), \tag{14}$$

where $\mathcal{P}_h : L^2(\Omega_F) \longrightarrow \mathcal{Z}_h$ is the $L^2(\Omega_F)$-orthogonal projector onto $\mathcal{Z}_h := \{v \in L^2(\Omega_F) \,:\, v|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h\}$. In addition, following the arguments from [22, Theorem 3.16], it can be proved that there exists $C > 0$, independent of h, such that

$$\|\mathbf{u} - \mathbf{I}_h^{\mathcal{R}} \mathbf{u}\|_{0,\Omega_F} \leq C h^\delta \left\{ \|\mathbf{u}\|_{\delta,\Omega_F} + \|\mathrm{div}\mathbf{u}\|_{0,\Omega_F} \right\} \qquad \forall \mathbf{u} \in [H^\delta(\Omega_F)]^3 \cap H(\mathrm{div};\Omega_F). \tag{15}$$

On the other hand, let

$$\mathbf{X}_h := \left\{ (\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{R}_h(\Omega_F) \times [L_h(\Omega_S)]^3 \,:\, \mathbf{v}_h|_{\Gamma_D} = \mathbf{0} \right\}$$

be the finite element discretization of $\mathbf{X}$.

Finally, the discrete analogue of $\mathbf{Y}$ and P are

$$\mathbf{Y}_h := \{(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{X}_h \,:\, \mathrm{div}\, \mathbf{u}_h = 0 \text{ in } \Omega_F\} \qquad \text{and}$$
$$P_h := \left\{ \mu_h \in L^2(\Gamma_I) \,:\, \mu_h|_F \in \mathbb{P}_0(F)\; \forall\, F \subseteq \Gamma_I,\; F \text{ face of } T,\, T \in \mathcal{T}_h \right\},$$

respectively. Let us remark that the divergence-free constraint in the definition of $\mathbf{Y}_h$ will be imposed below by choosing appropiate basis functions of this space.

Notice that, according to (13), for all faces $F \subseteq \Gamma_I$,

$$\mathbf{I}_h^{\mathcal{R}}(\mathbf{u}) \cdot \boldsymbol{\nu}_F = \mathcal{P}_{h,\Gamma_I}(\mathbf{u} \cdot \boldsymbol{\nu}_F), \tag{16}$$

where $\mathcal{P}_{h,\Gamma_I} : L^2(\Gamma_I) \to P_h$ is the orthogonal projector, which satisfies (see, for instance, [18])

$$\|v - \mathcal{P}_{h,\Gamma_I} v\|_{0,F} \leq C h_F^{\min\{1/2+\delta,1\}} \|v\|_{1/2+\delta,F}, \qquad \forall\, v \in H^{1/2+\delta}(F). \tag{17}$$

Now, we are in position to introduce the following finite element discretization of Problem 3.

Problem 4. Find $\lambda_h \in \mathbb{R}$ and $(\mathbf{u}_h, \mathbf{v}_h, \mu_h) \in \mathbf{Y}_h \times P_h$, $(\mathbf{u}_h, \mathbf{v}_h, \mu_h) \neq (\mathbf{0}, \mathbf{0}, 0)$, such that

$$a^*((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)) + b((\boldsymbol{\phi}_h, \boldsymbol{\psi}_h), \mu_h) = \widehat{\lambda}_h\, d((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)), \tag{18a}$$

$$b((\mathbf{u}_h, \mathbf{v}_h), \zeta_h) = 0, \tag{18b}$$

for all $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \in \mathbf{Y}_h$ and $\zeta_h \in P_h$.

In what follows we will show that the bilinear forms a^* and b satisfy both of Brezzi's conditions on the finite element spaces $\mathbf{Y}_h$ and P_h. We include its proof since, in this case, we cannot proceed as in that of [9, Lemma 8.2], because some of its arguments hold only in 2D.

Lemma 4.1. *The bilinear forms a^* and b satisfy:*

H1$_h$: *There exists $\widehat{\alpha} > 0$, independent of h, such that*

$$a^*((\mathbf{u}_h, \mathbf{v}_h), (\mathbf{u}_h, \mathbf{v}_h)) \geq \widehat{\alpha} \|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{Y}}^2 \qquad \forall\, (\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h,$$

where $\mathbf{W}_h := \{(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{Y}_h : b((\mathbf{u}_h, \mathbf{v}_h), \zeta_h) = 0 \quad \forall\, \zeta_h \in P_h\}$.

H2$_h$: *There exists $\widehat{\beta} > 0$, independent of h, such that*

$$\inf_{\zeta_h \in P_h} \left[\sup_{\substack{(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{Y}_h \\ (\mathbf{u}_h, \mathbf{v}_h) \neq (\mathbf{0}, \mathbf{0})}} \frac{b((\mathbf{u}_h, \mathbf{v}_h), \zeta_h)}{\|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{Y}} \|\zeta_h\|_{0,\Gamma_I}} \right] \geq \widehat{\beta}.$$

Proof. Since

$$\mathbf{W}_h = \left\{ (\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{Y}_h \ : \ \int_F (\mathbf{u}_h \cdot \boldsymbol{\nu} - \mathbf{v}_h \cdot \boldsymbol{\nu}) = 0 \ \ \forall\, F \subseteq \Gamma_I,\ F \text{ face of } T,\ T \in \mathcal{T}_h \right\},$$

the first argument from the proof of Lemma 3.1 holds true for $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h$, so that

$$a^*((\mathbf{u}_h, \mathbf{v}_h), (\mathbf{u}_h, \mathbf{v}_h)) \geq \widehat{\alpha} \|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{X}}^2.$$

Hence, **H1**$_h$ is a consequence of the above inequality and the fact that $\mathbf{u}_h \cdot \boldsymbol{\nu}|_{\Gamma_I}$ is the $L^2(\Gamma_I)$ projection of $\mathbf{v}_h \cdot \boldsymbol{\nu}|_{\Gamma_I}$ onto P_h and, therefore, $\|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_I} \leq \|\mathbf{v}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_I} \leq C \|\mathbf{v}_h\|_{1,\Omega_S} \leq C \|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{X}}$.

The proof of **H2**$_h$ follows from an adaptation of that of [9, Lemma 8.2] to 3D. In fact, let $\zeta_h \in P_h \subseteq L^2(\Gamma_I)$. Thanks to (3) we know that there exists $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$ such that

$$(\mathbf{u} \cdot \boldsymbol{\nu} - \mathbf{v} \cdot \boldsymbol{\nu}) = \zeta_h \ \text{ on } \Gamma_I, \qquad \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma_D \cup \Gamma_N \qquad \text{and} \qquad \|(\mathbf{u}, \mathbf{v})\|_{\mathbf{Y}} \leq C \|\zeta_h\|_{0,\Gamma_I}. \tag{19}$$

Now, let $\mathbf{v}_h \in [L_h(\Omega_S)]^3$ be a Clément's type interpolant of $\mathbf{v}$, vanishing on $\Gamma_D \cup \Gamma_N$ which is defined at the nodes $B \in \Gamma_I$ as follows:

$$\mathbf{v}_h(B) := \frac{1}{|\omega_B|} \int_{\omega_B} \mathbf{v},$$

where ω_B is the union of all the faces on Γ_I sharing B. Notice that [29, Corollary 4.1] and (19) imply that

$$\|\mathbf{v}_h\|_{1,\Omega_S} \leq C \|\mathbf{v}\|_{1,\Omega_S} \leq \widetilde{C} \|\zeta_h\|_{0,\Gamma_I}.$$

On the other hand, by construction, $\int_{\Gamma_I} \mathbf{v}_h = \int_{\Gamma_I} \mathbf{v}$ and, hence, using (19) again,

$$\int_{\Gamma_I} \mathbf{v}_h \cdot \boldsymbol{\nu} = \int_{\Gamma_I} \mathbf{v} \cdot \boldsymbol{\nu} = \int_{\Gamma_I} \mathbf{u} \cdot \boldsymbol{\nu} - \int_{\Gamma_I} \zeta_h.$$

Since $\operatorname{div} \mathbf{u} = 0$ in Ω_F, then by using Gauss' Theorem, we obtain that $\int_{\Gamma_I} \mathbf{v}_h \cdot \boldsymbol{\nu} = -\int_{\Gamma_I} \zeta_h$. Hence, the following Neumann problem is compatible:

$$\Delta\varphi = 0 \quad \text{in} \quad \Omega_F, \qquad \nabla\varphi \cdot \boldsymbol{\nu} = \mathbf{v}_h \cdot \boldsymbol{\nu} + \zeta_h \quad \text{on} \quad \Gamma_I.$$

In addition, since $\mathbf{v}_h \cdot \boldsymbol{\nu} + \zeta_h \in L^2(\Gamma_I)$, [24, Theorem 3.17] ensures that $\varphi \in H^{3/2}(\Omega_F)$ and, therefore, $\nabla\varphi \in [H^{1/2}(\Omega_F)]^3 \cap H(\operatorname{div}^0; \Omega_F)$, so that we can define its Raviart–Thomas interpolant $\mathbf{u}_h = \mathbf{I}_h^{\mathcal{R}}(\nabla\varphi) \in \mathbf{R}_h(\Omega_F)$. Then, from (14) and (13) we have that

$$\operatorname{div} \mathbf{u}_h = 0 \quad \text{in} \quad \Omega_F, \qquad \int_F \mathbf{u}_h \cdot \boldsymbol{\nu} = \int_F (\mathbf{v}_h \cdot \boldsymbol{\nu} + \zeta_h) \quad \text{for all } F \subseteq \Gamma_I$$

and $\|\mathbf{u}_h\|_{\operatorname{div},\Omega_F} \leq C\|\nabla\varphi\|_{1/2,\Omega_F} \leq \widetilde{C}\|\mathbf{v}_h \cdot \boldsymbol{\nu} + \zeta_h\|_{0,\Gamma_I} \leq \widehat{C}\|\zeta_h\|_{0,\Gamma_I}$.

Thus, $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{Y}_h$ and $\|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{Y}} = \|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{X}} + \|\mathbf{u}_h \cdot \boldsymbol{\nu}\|_{0,\Gamma_I} \leq C\|\zeta_h\|_{0,\Gamma_I}$. □

As a consequence of this lemma and the standard theory of mixed methods, we obtain an analogous result to Theorem 3.4 for the discrete problem.

Theorem 4.2. *Given $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}$, there exists a unique solution $(\mathbf{u}_h, \mathbf{v}_h, \mu_h) \in \mathbf{Y}_h \times P_h$ of the following source problem:*

$$a^*((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)) + b((\boldsymbol{\phi}_h, \boldsymbol{\psi}_h), \mu_h) = d((\mathbf{f}, \mathbf{g}), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)), \tag{20a}$$

$$b((\mathbf{u}_h, \mathbf{v}_h), \zeta_h) = 0, \tag{20b}$$

for all $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \in \mathbf{Y}_h$ and $\zeta_h \in P_h$. Moreover, there exists $C > 0$, independent of h, such that

$$\|(\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{Y}} + \|\mu_h\|_{0,\Gamma_I} \leq C\|(\mathbf{f}, \mathbf{g})\|_{\mathbf{H}}. \tag{21}$$

Proof. See, for instance, [15]. □

As for the continuous problem, we define the corresponding discrete solution operator:

$$\begin{aligned} \mathbf{T}_h : \mathbf{H} &\longrightarrow \mathbf{Y}_h \subseteq \mathbf{H}, \\ (\mathbf{f}, \mathbf{g}) &\longmapsto \mathbf{T}_h(\mathbf{f}, \mathbf{g}) := (\mathbf{u}_h, \mathbf{v}_h) \end{aligned}$$

with $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{Y}_h$ such that there exists $\mu_h \in P_h$ satisfying (20).

It is easy to check that $\mathbf{T}_h$ is self-adjoint with respect to a^* and d. Clearly, as a consequence of Theorem 4.2, $\mathbf{T}_h$ is a well-defined bounded linear operator. Moreover, $(\gamma_h, (\mathbf{u}_h, \mathbf{v}_h))$ with $\gamma_h \neq 0$, is an eigenpair of $\mathbf{T}_h$ if and only if there exists $\mu_h \in P_h$ such that $(1/\gamma_h, (\mathbf{u}_h, \mathbf{v}_h, \mu_h))$ is a solution of Problem 4.

Similarly to the continuous case, it can be proved that Problem 4 is equivalent to the following:

Problem 4*. Find $\widehat{\lambda}_h \in \mathbb{R}$ and $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h$, $(\mathbf{u}_h, \mathbf{v}_h) \neq (\mathbf{0}, \mathbf{0})$, such that

$$a^*((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)) = \widehat{\lambda}_h\, d((\mathbf{u}_h, \mathbf{v}_h), (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)) \quad \forall\ (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \in \mathbf{W}_h.$$

5. Spectral approximation

Analogously to the continuous case, (20b) implies that $\mathbf{T}_h(\mathbf{H}) \subseteq \mathbf{W}_h$. In addition, notice that $\mathbf{W}_h \not\subseteq \mathbf{W}$ because, for $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h$, it does not hold $\mathbf{u}_h \cdot \boldsymbol{\nu} = \mathbf{v}_h \cdot \boldsymbol{\nu}$ on Γ_I, so that we are dealing with a non-conforming approximation of the spectral problem. In order to prove that the eigenvalues and eigenfunctions of $\mathbf{T}|_{\mathbf{W}}$ are well-approximated by those of $\mathbf{T}_h|_{\mathbf{W}_h}$, we will use the theory developed in [17] for noncompact operators and, following the ideas in [9], will adapt this theory to our nonconforming case. In fact, let $\mathrm{sp}(\cdot)$ denote the spectrum of an operator. Since $\mathbf{W}_h \subseteq \mathbf{Y} \subseteq \mathbf{X}$, then $\mathbf{T}_h|_{\mathbf{W}_h}$ can be seen as a conforming discretization of $\mathbf{T}|_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathbf{X}$. It is easy to prove that $\mathrm{sp}(\mathbf{T}|_{\mathbf{X}}) = \mathrm{sp}(\mathbf{T}|_{\mathbf{W}}) \cup \{0\}$ (see, for instance, [8, Lemma 4.1]). Thus, it is enough to prove the following two approximations properties to be able to apply the theory from [17]:

P1. For each eigenfunction $(\mathbf{u}, \mathbf{v})$ of $\mathbf{T}$ associated with an eigenvalue $\gamma \in (0, 1)$, there holds

$$\lim_{h\to 0} \mathrm{dist}\ \left((\mathbf{u}, \mathbf{v}), \mathbf{W}_h\right) = 0,$$

where dist is the distance measured in the norm $\|\cdot\|_{\mathbf{X}}$;

P2. There holds

$$\lim_{h\to 0} \|(\mathbf{T} - \mathbf{T}_h)|_{\mathbf{W}_h}\|_{\mathbf{X}} = 0.$$

Before proving **P1** and **P2**, we will introduce the following result.

Theorem 5.1. *$\gamma_h = 1$ is an eigenvalue of $\mathbf{T}_h$ with eigenspace $\mathbf{K}_h = \mathbf{K} \cap \mathbf{W}_h$.*

Proof. The proof is similar to that in 2D. See, for instance, [8, Theorem 4.2]. □

In what follows, we establish some properties and definitions that will be used in the sequel.

First, we denote by $\boldsymbol{\pi}_h : [H^1(\Omega_S)]^3 \longrightarrow [L_h(\Omega_S)]^3$ the orthogonal projector with respect to the $H^1(\Omega_S)$-norm, which, for any $\epsilon \in (0, 1]$, satisfies

$$\|\mathbf{v} - \boldsymbol{\pi}_h\mathbf{v}\|_{1,\Omega_S} \leq Ch^{\epsilon}\|\mathbf{v}\|_{1+\epsilon,\Omega_S} \qquad \forall\, \mathbf{v} \in [H^{1+\epsilon}(\Omega_S)]^3. \tag{22}$$

On the other hand, for any $\mathbf{v} \in [H^1(\Omega_S)]^3$ such that $\int_{\Gamma_I} \mathbf{v} \cdot \boldsymbol{\nu} = 0$, we proceed as in [23] and define $\mathcal{E}(\mathbf{v}) \in H(\mathrm{div}^0; \Omega_F)$ as follows:

$\mathcal{E}(\mathbf{v}) := \nabla q$ where q is the solution of the following compatible Neumann problem:

$$\Delta q = 0 \quad \text{in} \quad \Omega_F, \qquad \nabla q \cdot \boldsymbol{\nu} = \mathbf{v} \cdot \boldsymbol{\nu} \quad \text{on} \quad \Gamma_I.$$

Note that for all $\mathbf{v} \in [H^1_{\Gamma_D}(\Omega_S)]^3$, $\mathbf{v} \cdot \boldsymbol{\nu}|_{\Gamma_j} \in H^{1/2}(\Gamma_j)$, $j = 1, \ldots, J$. Therefore, $q \in H^{1+s}(\Omega_F)$ with $s > 1/2$, so that

$$\mathcal{E}(\mathbf{v}) \in [H^s(\Omega_F)]^3 \cap H(\mathrm{div}^0; \Omega_F) \qquad \text{and} \qquad \|\mathcal{E}(\mathbf{v})\|_{s,\Omega_F} \leq C\|\mathbf{v}\|_{1,\Omega_S}. \tag{23}$$

Consequently, $\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\mathbf{v})$ is well defined and the following result holds true.

Lemma 5.2. *There exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\mathbf{v})\|_{0,\Omega_F} \leq C\|\mathbf{v}\|_{1,\Omega_S} \qquad \forall\, \mathbf{v} \in [H^1(\Omega_S)]^3.$$

Now, we are in a position to define an appropriate divergence-free approximation of smooth functions in $\mathbf{W}$.

Lemma 5.3. *For $(\mathbf{u}, \mathbf{v}) \in \mathbf{W} \cap \left\{[H^s(\Omega_F)]^3 \times [H^{1+t}(\Omega_S)]^3\right\}$ with $s \in (1/2,\ 1]$ and $t \in (0,\ 1]$, let*

$$(\mathbf{u}_h, \mathbf{v}_h) := \left(\mathbf{I}_h^{\mathcal{R}}\mathbf{u} + \left(\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\right), \boldsymbol{\pi}_h\mathbf{v}\right).$$

Then, $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h$ and

$$\|(\mathbf{u}, \mathbf{v}) - (\mathbf{u}_h, \mathbf{v}_h)\|_{\mathbf{Y}} \leq Ch^r \left\{\|\mathbf{u}\|_{s,\Omega_F} + \|\mathbf{v}\|_{1+t,\Omega_S}\right\}, \tag{24}$$

where $r := min\,\{s,t\}$.

Proof. Notice that $\mathbf{u}_h$ is divergence-free in Ω_F thanks to (14), the fact that $\mathbf{u}$ is divergence-free in Ω_F and the definition of the operator $\mathcal{E}$. In addition, from the definition of $\mathcal{E}$, (13) and the fact that $\mathbf{u}\cdot\boldsymbol{\nu} = \mathbf{v}\cdot\boldsymbol{\nu}$ on Γ_I, we easily obtain that

$$\int_F \mathbf{u}_h \cdot \boldsymbol{\nu}_F = \int_F \mathbf{v}_h \cdot \boldsymbol{\nu}_F \qquad \forall\, F \subseteq \Gamma_I.$$

Hence, $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbf{W}_h$. On the other hand, by using triangle inequality, the fact that $\mathbf{u}$ and $\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})$ are divergence-free and (14), we obtain

$$\begin{aligned} \|(\mathbf{u},\mathbf{v}) - (\mathbf{u}_h,\mathbf{v}_h)\|_{\mathbf{Y}} \leq \|\mathbf{u} - \mathbf{I}_h^{\mathcal{R}}\mathbf{u}\|_{0,\Omega_F} + \|\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\|_{0,\Omega_F} + \|\mathbf{v} - \boldsymbol{\pi}_h\mathbf{v}\|_{1,\Omega_S} \\ + \|\left(\mathbf{u} - \mathbf{I}_h^{\mathcal{R}}\mathbf{u}\right)\cdot\boldsymbol{\nu}\|_{0,\Gamma_I} + \|\left(\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\right)\cdot\boldsymbol{\nu}\|_{0,\Gamma_I}. \end{aligned} \tag{25}$$

From Lemma 5.2 and (22), we have that

$$\|\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\|_{0,\Omega_F} \leq C\|\mathbf{v} - \boldsymbol{\pi}_h\mathbf{v}\|_{1,\Omega_S} \leq Ch^t\|\mathbf{v}\|_{1+t,\Omega_S}. \tag{26}$$

On the other hand, since $\mathbf{u}\cdot\boldsymbol{\nu} = \mathbf{v}\cdot\boldsymbol{\nu}$ on Γ_I, we have that the fourth term of (25) can be rewritten as $\|\left(\mathbf{v} - \mathbf{I}_h^{\mathcal{R}}\mathbf{v}\right)\cdot\boldsymbol{\nu}\|_{0,\Gamma_I}$. Hence, from (16) and (17), we obtain that

$$\|\left(\mathbf{u} - \mathbf{I}_h^{\mathcal{R}}\mathbf{u}\right)\cdot\boldsymbol{\nu}\|_{0,\Gamma_I} \leq Ch^{\min\{1/2+t,1\}}\|\mathbf{v}\|_{1+t,\Omega_S}. \tag{27}$$

To estimate the last term in (25) we use that $(\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v}))\cdot\boldsymbol{\nu}_F = \mathcal{P}_{h,\Gamma_I}((\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v}))\cdot\boldsymbol{\nu}_F)$, for all $F \subseteq \Gamma_I$. Then, from the fact that this projector is bounded, the definition of $\mathcal{E}$ and (22) we write

$$\|\left(\mathbf{I}_h^{\mathcal{R}}\mathcal{E}(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\right)\cdot\boldsymbol{\nu}\|_{0,\Gamma_I} \leq C\sum_{F\subseteq\Gamma_I}\|(\boldsymbol{\pi}_h\mathbf{v} - \mathbf{v})\cdot\boldsymbol{\nu}_F\|_{0,F} \leq Ch^t\|\mathbf{v}\|_{1+t,\Omega_S}. \tag{28}$$

Thus, (24) follows from (25), (15), (22), (26), (27) and (28). □

Now, we are in a position to prove property **P1**.

Lemma 5.4. *For each eigenfunction* $(\mathbf{u},\mathbf{v})$ *of* $\mathbf{T}$ *associated to an eigenvalue* $\gamma \in (0,1)$*, with* $\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}} = 1$*, there exists a constant* $C > 0$*, independent of* h*, such that*

$$\operatorname{dist}((\mathbf{u},\mathbf{v}),\mathbf{W}_h) \leq Ch^r, \tag{29}$$

where $r := min\,\{s,t\}$ *with* s *and* t *as in Lemma 3.5.*

Proof. Let $(\mathbf{u},\mathbf{v}) \in \mathbf{W}$ with $\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}} = 1$ such that $\mathbf{T}(\mathbf{u},\mathbf{v}) = \gamma(\mathbf{u},\mathbf{v})$. Then, there exists $\mu \in P$ such that $(\mathbf{u},\mathbf{v},\mu)$ is an eigenfunction of Problem 3 with eigenvalue $1/\gamma > 1$. In particular, from Lemma 3.5, $\mathbf{u} \in [H^s(\Omega_F)]^3$ for some $s > 1/2$ and $\mathbf{v} \in [H^{1+t}(\Omega_S)]^3$ for some $t > 0$. Hence, from Lemma 5.3 we obtain (29). □

On the other hand, the first part of the proof of property **P2** follows very closely that of [9, Theorem 6.3]. We include a sketch of this proof.

Lemma 5.5. *There exists a constant* $C > 0$*, independent of* h*, such that for all* $(\mathbf{f}_h,\mathbf{g}_h) \in \mathbf{W}_h$*,*

$$\|(\mathbf{T} - \mathbf{T}_h)(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}} \leq Ch^r\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}},$$

where $r := min\,\{s,t\}$ *with* s *and* t *as in Lemma 3.5.*

Proof. First, note that it is enough to prove the theorem for $(\mathbf{f}_h,\mathbf{g}_h)\in\mathbf{K}_h^{\perp\mathbf{W}_h}$ since $\mathbf{T}$ and $\mathbf{T}_h$ coincide on $\mathbf{K}_h$. So, let $(\mathbf{f}_h,\mathbf{g}_h)\in\mathbf{K}_h^{\perp\mathbf{W}_h}$. Since $(\mathbf{f}_h,\mathbf{g}_h)\in\mathbf{W}_h$, we have that $\int_{\Gamma_I}\mathbf{f}_h\cdot\boldsymbol{\nu}=\int_{\Gamma_I}\mathbf{g}_h\cdot\boldsymbol{\nu}$ and the following Neumann problem is compatible:

$$\Delta\varphi\,=\,\operatorname{div}\mathbf{f}_h\,=\,0\quad\text{in}\quad\Omega_F,\qquad\nabla\varphi\cdot\boldsymbol{\nu}=\mathbf{g}_h\cdot\boldsymbol{\nu}\quad\text{on}\quad\Gamma_I.$$

As it is well-known, φ belongs to $H^{1+s}(\Omega_F)$ with $s>1/2$ and, moreover, satisfies $\|\nabla\varphi\|_{s,\Omega_F}\le C\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}$. Let $\boldsymbol{\chi}:=\mathbf{f}_h-\nabla\varphi$. Then,

$$(\mathbf{f}_h,\mathbf{g}_h)=(\boldsymbol{\chi},\mathbf{0})+(\nabla\varphi,\mathbf{g}_h)\,,\tag{30}$$

$$\operatorname{div}(\boldsymbol{\chi})\,=\,0\quad\text{in }\Omega_F\quad\text{and}\quad\int_{\Gamma_I}\boldsymbol{\chi}\cdot\boldsymbol{\nu}\,=\,0.\tag{31}$$

Therefore,

$$\|(\mathbf{T}-\mathbf{T}_h)(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}\le\|(\mathbf{T}-\mathbf{T}_h)(\boldsymbol{\chi},\mathbf{0})\|_{\mathbf{X}}+\|(\mathbf{T}-\mathbf{T}_h)(\nabla\varphi,\mathbf{g}_h)\|_{\mathbf{X}}.\tag{32}$$

Since $\mathbf{T}$ and $\mathbf{T}_h$ are bounded uniformly on h, the first term on the right hand side can be controlled as follows:

$$\|(\mathbf{T}-\mathbf{T}_h)(\boldsymbol{\chi},\mathbf{0})\|_{\mathbf{X}}\le(\|\mathbf{T}\|+\|\mathbf{T}_h\|)\|(\boldsymbol{\chi},\mathbf{0})\|_{\mathbf{X}}\le C\|\boldsymbol{\chi}\|_{0,\Omega_F}.\tag{33}$$

In order to estimate $\|\boldsymbol{\chi}\|_{0,\Omega_F}$, we first notice that $(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)\in\mathbf{R}_h(\Omega_F)$. Hence,

$$\operatorname{div}(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)|_T\in\mathbb{P}_0(T)\quad\forall\,T\subseteq\Omega_F\quad\text{and}\quad(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)|_F\cdot\boldsymbol{\nu}\in\mathbb{P}_0(F)\quad\forall\,F\subseteq\Gamma_I.$$

Then, from the Gauss Theorem, (13) and (31), we easily obtain that for each $T\subseteq\Omega_F$, $\operatorname{div}(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)|_T=0$ and for every face $F\subseteq\Gamma_I$, $(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)|_F\cdot\boldsymbol{\nu}=0$.

So, $((\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h),\mathbf{0})\in\mathbf{K}\cap\mathbf{W}_h=\mathbf{K}_h$. Since, in addition, $(\mathbf{f},\mathbf{g})\in\mathbf{K}_h^{\perp\mathbf{W}_h}$, we easily obtain that

$$\begin{aligned}\|\boldsymbol{\chi}\|_{0,\Omega_F}^2&=\int_{\Omega_F}(\nabla\varphi-\mathbf{f}_h)\cdot(\nabla\varphi-\mathbf{I}_h^{\mathcal{R}}\nabla\varphi)+\int_{\Omega_F}(\nabla\varphi-\mathbf{f}_h)\cdot(\mathbf{I}_h^{\mathcal{R}}\nabla\varphi-\mathbf{f}_h)\\&=\int_{\Omega_F}(\nabla\varphi-\mathbf{f}_h)\cdot(\nabla\varphi-\mathbf{I}_h^{\mathcal{R}}\nabla\varphi).\end{aligned}$$

Now, by using the previous equality, the approximation property (15) and the a priori estimate $\|\nabla\varphi\|_{s,\Omega_F}\le C\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{Y}}$, we obtain

$$\|\boldsymbol{\chi}\|_{0,\Omega_F}^2\le\|\boldsymbol{\chi}\|_{0,\Omega_F}\|\nabla\varphi-\mathbf{I}_h^{\mathcal{R}}\nabla\varphi\|_{0,\Omega_F}^2\le\|\boldsymbol{\chi}\|_{0,\Omega_F}Ch^s\|\nabla\varphi\|_{0,\Omega_F}^2\le Ch^s\|\boldsymbol{\chi}\|_{0,\Omega_F}\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}.$$

Hence,

$$\|\boldsymbol{\chi}\|_{0,\Omega_F}\le Ch^s\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}.\tag{34}$$

Thus, from (33) and (34) we obtain

$$\|(\mathbf{T}-\mathbf{T}_h)(\boldsymbol{\chi},\mathbf{0})\|_{\mathbf{X}}\le Ch^s\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}\,.\tag{35}$$

On the other hand, let $(\mathbf{u},\mathbf{v}):=\mathbf{T}(\nabla\varphi,\mathbf{g}_h)$ and $(\mathbf{u}_h,\mathbf{v}_h):=\mathbf{T}_h(\nabla\varphi,\mathbf{g}_h)$. Then, there exist μ and μ_h such that $(\mathbf{u},\mathbf{v},\mu)\in\mathbf{Y}\times P$ and $(\mathbf{u}_h,\mathbf{v}_h,\mu_h)\in\mathbf{Y}_h\times P_h$ are the solutions of the source problems (8) and (20), respectively, with $\mathbf{f}$ substituted by $\nabla\varphi$. Now, since $\mathbf{Y}_h\subseteq\mathbf{Y}$, (20) is a conforming approximation of (8). Therefore, since a^* and b satisfy the continuity, discrete ellipticity in the kernel and inf-sup conditions, the standard theory (see, for instance, [15]) yields

$$\|(\mathbf{u},\mathbf{v})-(\mathbf{u}_h,\mathbf{v}_h)\|_{\mathbf{X}}\le\|(\mathbf{u},\mathbf{v})-(\mathbf{u}_h,\mathbf{v}_h)\|_{\mathbf{Y}}\le C\left\{\operatorname{dist}((\mathbf{u},\mathbf{v}),\mathbf{W}_h)+\operatorname{dist}(\mu,P_h)\right\}$$

with a constant C independent of h. Hence, from Lemmas 3.5, 5.4 and estimate (17), we obtain

$$\|(\mathbf{T}-\mathbf{T}_h)(\nabla\varphi,\mathbf{g}_h)\|_{\mathbf{X}} = \|(\mathbf{u},\mathbf{v})-(\mathbf{u}_h,\mathbf{v}_h)\|_{\mathbf{X}} \le Ch^{\min\{s,t\}}\|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}},$$

which together with (35) allow us to conclude the proof. □

Next theorem, which shows that there are no spurious eigenvalues for h small enough, has been proved in [17].

Theorem 5.6. *Let J be a closed interval such that $J\cap\operatorname{sp}(\mathbf{T})=\emptyset$. There exists a strictly positive constant h_J such that if $h\le h_J$, then $J\cap\operatorname{sp}(\mathbf{T}_h)=\emptyset$.*

For an open interval $I\subseteq(0,\,1)$, let $\mathbf{E}_I$ be the direct sum of the eigenspaces of $\mathbf{T}$ associated with eigenvalues lying in I. Let us denote by $\mathbf{E}_I^h$ the analogue for $\mathbf{T}_h$. We have the following error estimates to approximate eigenfunctions.

Theorem 5.7. *There exist strictly positive constants C and h_I such that, if $h\le h_I$, then*

(i) *for each $(\mathbf{u}_h,\mathbf{v}_h)\in\mathbf{E}_I^h$ with $\|(\mathbf{u}_h,\mathbf{v}_h)\|_{\mathbf{X}}=1$, $\operatorname{dist}((\mathbf{u}_h,\mathbf{v}_h),\mathbf{E}_I)\le Ch^r$,*

(ii) *for each $(\mathbf{u},\mathbf{v})\in\mathbf{E}_I$ with $\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}=1$, $\operatorname{dist}((\mathbf{u},\mathbf{v}),\mathbf{E}_I^h)\le Ch^r$,*

where $r:=$ min $\{s,t\}$ and s and t as in Lemma 3.5.

Proof. From [28, Theorem 2.1], we have that

$$\operatorname{dist}((\mathbf{u}_h,\mathbf{v}_h),\mathbf{E}_I)+\operatorname{dist}((\mathbf{u},\mathbf{v}),\mathbf{E}_I^h)\le C\delta_h$$

where

$$\delta_h := \sup_{\substack{(\mathbf{u},\mathbf{v})\in\mathbf{E}_I\\ \|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}=1}}\left(\inf_{(\mathbf{f}_h,\mathbf{g}_h)\in\mathbf{W}_h}\|(\mathbf{u},\mathbf{v})-(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}\right)+\sup_{\substack{(\mathbf{f}_h,\mathbf{g}_h)\in\mathbf{W}_h\\ \|(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}=1}}\|(\mathbf{T}-\mathbf{T}_h)(\mathbf{f}_h,\mathbf{g}_h)\|_{\mathbf{X}}.$$

Hence, the proof follows from Lemmas 5.4 and 5.5. □

Let $I\subseteq(0,\,1)$ be an interval containing a unique eigenvalue γ of $\mathbf{T}$. As a consequence of this theorem, for h sufficiently small, the dimension of the linear space $\mathbf{E}_h^I$ must coincide with that of $\mathbf{E}_I$ (let us say n). This implies the convergence to γ of exactly n eigenvalues of the discrete problem $\gamma_h^{(1)},\dots,\gamma_h^{(n)}$. Moreover, the following double-order error estimate holds true.

Theorem 5.8. *There exist strictly positive constants C and h_I such that, if $h\le h_I$, then*

$$\left|\gamma-\gamma_h^{(i)}\right|\le Ch^{2r},\quad i=1,...,n,$$

with $r:=$ min $\{s,t\}$, s and t as in Lemma 3.5, and C depending on γ.

Proof. From [28, Theorem 2.3] and the fact that $\mathbf{T}$ is self-adjoint with respect to a^*, we have that

$$\max_{i=1,\dots,n}\left|\gamma-\gamma_h^{(i)}\right|\le C(\delta_h^2+M_h), \tag{36}$$

where δ_h is as in Theorem 5.7 and

$$M_h := \sup_{\substack{(\mathbf{u},\mathbf{v})\in\mathbf{E}_I\\ \|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}=1}}\ \sup_{\substack{(\boldsymbol{\phi},\boldsymbol{\psi})\in\mathbf{E}_I\\ \|(\boldsymbol{\phi},\boldsymbol{\psi})\|_{\mathbf{X}}=1}}\left(a^*(\mathbf{T}(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi}))-d((\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi}))\right)$$

with Π_h being the projection onto $\mathbf{W}_h$ with respect to a^*.

We focus on estimating M_h. In order to put our problem in the context of [28], we consider the continuous and discrete Problems 3* and 4*, respectively, in which case we are dealing with a non-conforming scheme. Let $(\mathbf{u},\mathbf{v}) \in \mathbf{E}_I$, so that $\mathbf{T}(\mathbf{u},\mathbf{v}) = \gamma(\mathbf{u},\mathbf{v})$. Let $\widehat{\lambda} = 1/\gamma$. Arguing as in Theorem 3.2, it can be proved that $(\mathbf{u},\mathbf{v})$ is the solution of the following problem:

$$\begin{aligned} \nabla p + \rho_F \mathbf{u} &= \widehat{\lambda}\rho_F \mathbf{u} \quad \text{in} \quad \Omega_F, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in} \quad \Omega_F, \\ -\operatorname{div} \boldsymbol{\sigma}(\mathbf{v}) + \rho_S \mathbf{v} &= \widehat{\lambda}\rho_S \mathbf{v} \quad \text{in} \quad \Omega_S, \\ \boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} + p\boldsymbol{\nu} &= \mathbf{0} \quad \text{on} \quad \Gamma_I, \\ \mathbf{u}\cdot\boldsymbol{\nu} - \mathbf{v}\cdot\boldsymbol{\nu} &= 0 \quad \text{on} \quad \Gamma_I, \\ \boldsymbol{\sigma}(\mathbf{v})\boldsymbol{n} &= \mathbf{0} \quad \text{on} \quad \Gamma_N, \\ \mathbf{v} &= \mathbf{0} \quad \text{on} \quad \Gamma_D. \end{aligned}$$

Hence, from Lemma 3.5, we observe that $p \in H^{1+s}(\Omega_F)$, with $s > 1/2$.

Let $(\mathbf{u},\mathbf{v})$, $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{E}_I$ and $(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) := \Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})$. Integrating by parts, we obtain that

$$a^*(\mathbf{T}(\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})) - d((\mathbf{u},\mathbf{v}),\Pi_h(\boldsymbol{\phi},\boldsymbol{\psi})) = \int_{\Gamma_I} p(\boldsymbol{\phi}_h\cdot\boldsymbol{\nu} - \boldsymbol{\psi}_h\cdot\boldsymbol{\nu}).$$

For $(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) \in \mathbf{W}_h$ we have that $\boldsymbol{\phi}_h\cdot\boldsymbol{\nu} = \mathcal{P}_{h,\Gamma_I}(\boldsymbol{\psi}\cdot\boldsymbol{\nu})$, where, once more, $\mathcal{P}_{h,\Gamma_I}$ is the $L^2(\Gamma_I)$ projection onto P_h. Therefore,

$$\begin{aligned} \left|\int_{\Gamma_I} p(\boldsymbol{\phi}_h\cdot\boldsymbol{\nu} - \boldsymbol{\psi}_h\cdot\boldsymbol{\nu})\right| &= \left|\int_{\Gamma_I} (p - \mathcal{P}_{h,\Gamma_I}(p))\,(\mathcal{P}_{h,\Gamma_I}(\boldsymbol{\psi}_h\cdot\boldsymbol{\nu}) - \boldsymbol{\psi}_h\cdot\boldsymbol{\nu})\right| \\ &\le \|p - \mathcal{P}_{h,\Gamma_I}(p)\|_{0,\Gamma_I}\|\mathcal{P}_{h,\Gamma_I}(\boldsymbol{\psi}_h\cdot\boldsymbol{\nu}) - \boldsymbol{\psi}_h\cdot\boldsymbol{\nu}\|_{0,\Gamma_I}. \end{aligned}$$

Since, $p|_{\Gamma_I} \in H^{1/2+s}(\Gamma_I)$ with $s > 1/2$, we have that $\|p - \mathcal{P}_{h,\Gamma_I} p\|_{0,\Gamma_I} \le Ch\|p\|_{1,\Gamma_I} \le Ch\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}$. For the other term, we proceed as in the proof of [28, Theorem 3.1] to derive that $\|\mathcal{P}_{h,\Gamma_I}(\boldsymbol{\psi}_h\cdot\boldsymbol{\nu}) - \boldsymbol{\psi}_h\cdot\boldsymbol{\nu}\| \le Ch^r$. Therefore, $M_h \le Ch^{1+r}\|(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}$ which allows us to conclude the proof.

□

Remark 5.9. *Notice that $\mathbf{T}_h$ can also be seen as a conforming discretization of $\mathbf{T}|_{\mathbf{Y}} : \mathbf{Y} \longrightarrow \mathbf{Y}$ which would allow us to do an alternative analysis. However, in such a case, only a suboptimal order of convergence can be proved in Theorems 5.7 and 5.8.*

6. Numerical experiments

In this section, we report numerical results for a couple of tests obtained with a Matlab code based on [5,27]. For the incompressible fluid, we have used two approaches to construct basis for the divergence-free lowest-order Raviart–Thomas finite element space. Let $\mathcal{T}_h^F$ be a mesh of Ω_F. The first approach, that we call [A], has been proposed in [1, Sec. 3.1]. For a topologically trivial domain Ω_F, this method consists of the following steps:

(1) consider the graph with nodes the vertices of $\mathcal{T}_h^F$ and arcs the edges of $\mathcal{T}_h^F$;
(2) construct a spanning tree of the graph;
(3) consider the classical basis of the lowest-order Nédélec finite element space;

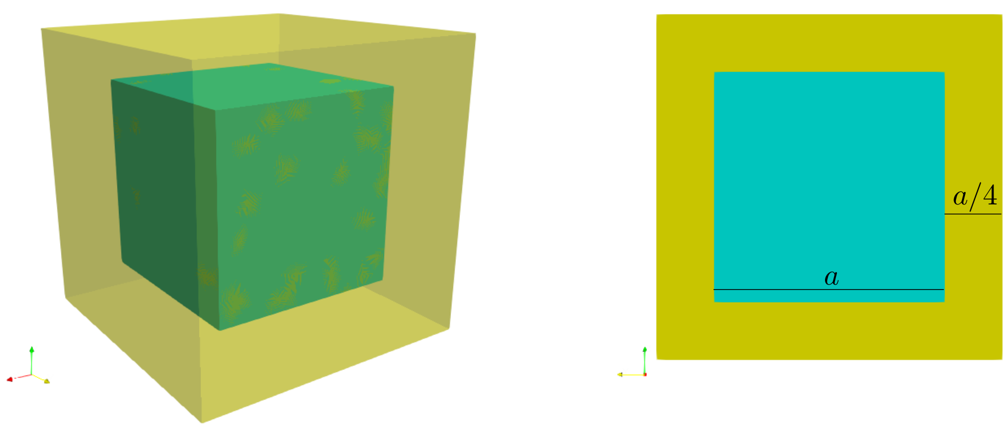
(4) the proposed divergence-free basis is formed by the curl of the basis functions associated to those arcs that are not in the spanning tree.

For topologically non trivial domains, an alternative approach was proposed in [1, Sec. 3.1].

The second approach, that we call [B], has been proposed in [1, Sec. 3.2]. It consists of obtaining a basis of the divergence-free lowest-order Raviart–Thomas elements, written as appropriate linear combinations of the classical Raviart–Thomas basis functions. The coefficients of each linear combination are determined by using a spanning tree of the *dual graph* (a graph where the arcs are the faces of $\mathcal{T}_h^F$ and the nodes are the tetrahedra of $\mathcal{T}_h^F$, plus one extra node for the exterior of the fluid domain). See [1, Sec. 3.2] for details and [2] for an efficient algorithm.

6.1. **Cubic vessel**

For our first test, we have considered a cubic vessel completely filled with a fluid and clamped by its bottom. We use the geometrical setting shown in Fig. 2.



(A) Cubic vessel. (B) Vertical section of the cubic vessel: $a = 1\,\mathrm{m}$.

Figure 2. Topologically trivial fluid domain.

Physical parameters of steel have been used for the solid: density $\rho_S = 7700\,\mathrm{kg/m^3}$, Young's modulus $= 1.44 \times 10^{11}\,\mathrm{Pa}$, Poisson's ratio $= 0.35$. For the fluid, we have used 'perfectly incompressible' water; i.e., an incompressible liquid with the same density as that of water: $\rho_F = 1000\,\mathrm{kg/m^3}$.

In order to validate our code, we have resorted to the results from [9, Sec. 7] which show that the vibration frequencies for an incompressible fluid are limits of the corresponding ones for a compressible fluid as the acoustic speed going to infinity. Therefore, we have also solved the problem for a compressible fluid with increasing values of the acoustic speed. In particular, we have taken successive multiples of the acoustic speed of water: $c = 1430\,\mathrm{m/s}$. Table 1 shows the vibration frequencies obtained in a same mesh with 88,262 elements for several values of the sound speed in the fluid and for the incompressible case. Let us remark that, for the latter, we have solved the problem applying the two strategies described above to construct the lowest-order divergence-free Raviart–Thomas basis.

It can be seen from Table 1 that both strategies lead to identical results (as the theory in [1] predicts). It can also be seen that the vibration frequencies computed with the compressible fluid clearly converge to those of the incompressible one as the acoustic speed goes to infinity. This agrees with the theory from [9, 10] and allows us to validate our code. Figures 3 and 4 show two vibration modes of the coupled system. These modes qualitatively agree with those reported in [7] (see Figures 6 and 8 from this reference).

Mode	$10^0 \times c$	$10^1 \times c$	$10^2 \times c$	$10^3 \times c$	[A]	[B]
ω_1	1560.443	1563.435	1563.461	1563.462	1563.462	1563.462
ω_2	1560.624	1563.616	1563.642	1563.642	1563.642	1563.642
ω_3	2495.252	2495.312	2495.313	2495.313	2495.313	2495.313
ω_4	3454.110	3578.650	3581.309	3581.337	3581.338	3581.338
ω_5	4152.016	4416.885	4417.155	4417.158	4417.158	4417.158

Table 1. Lowest vibration frequencies for a compressible fluid with different values of acoustic speed, and for an incompressible fluid in a mesh $\mathcal{T}_h$ with $88,262$ elements.

In order to appreciate the convergence of our proposed scheme, we have solved the same problem on several meshes with different degrees of refinement. We report in Table 2 the five smallest computed vibration frequencies, which allows us to appreciate the convergence of all of them.

N	$53,447$	$111,433$	$150,741$	$217,260$	$256,006$	$298,547$
ω_1	1573.05	1559.70	1555.81	1551.52	1549.40	1548.22
ω_2	1573.60	1560.25	1555.98	1551.64	1549.53	1548.26
ω_3	2506.28	2490.63	2485.86	2480.40	2478.02	2476.59
ω_4	3614.97	3567.39	3551.04	3534.02	3525.98	3521.20
ω_5	4501.34	4381.43	4341.64	4300.28	4280.97	4269.29

Table 2. Convergence of the smallest vibration frequencies for different meshes $\mathcal{T}_h$ with N elements.

We appreciate from Table 2 the convergence of each vibration frequency.

6.2. **Hollow ring**

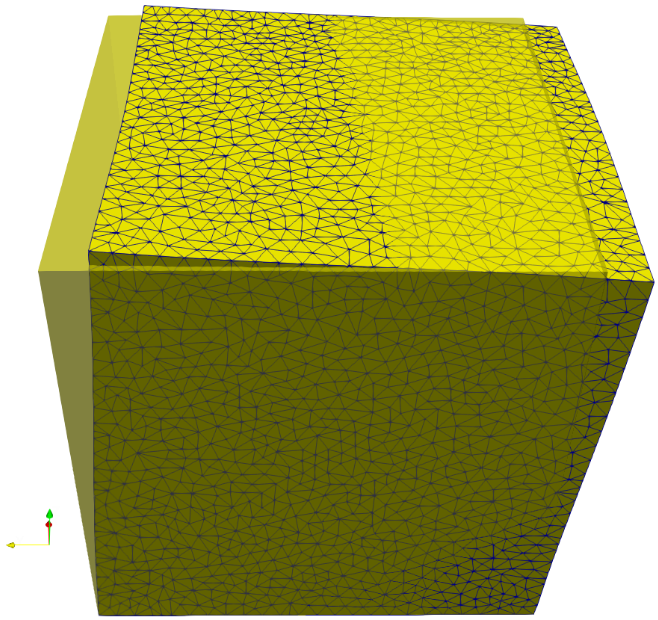
In order to test our code with a topologically non trivial fluid domain, we have considered a hollow ring filled with a fluid. We have used a hollow circular rod of square section with the radius of the centroid line taken as $R = 0.5\,\mathrm{m}$. We have used the geometrical setting shown in Fig. 5 and the same physical parameters as in Section 6.1.

First, as in the previous test, we have validated the proposed methodology by solving the same problem for a compressible fluid with sound speed going to infinity. We do not report here the computed values since their behavior is similar to that shown in Table 1 for the cubic vessel. Secondly, we have solved the problem on several meshes with different degrees of refinement. We report in Table 3 the ten smallest computed vibration frequencies.

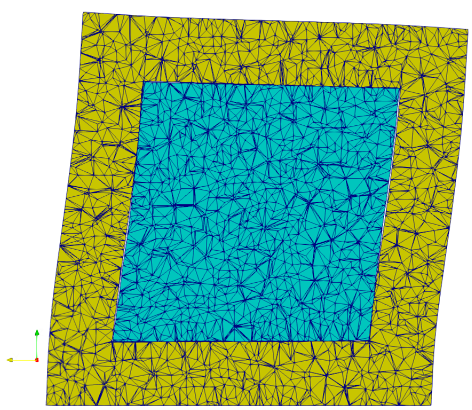
N	$213,743$	$244,777$	$274,079$	$349,383$
ω_1	1973.83	1968.98	1966.54	1962.18
ω_2	1974.43	1969.20	1966.92	1962.49
ω_3	2145.59	2142.21	2140.40	2136.37
ω_4	2145.93	2142.53	2141.23	2136.66
ω_5	5190.65	5177.57	5171.43	5160.60
ω_6	5190.97	5178.32	5172.28	5161.13
ω_7	5534.93	5524.93	5520.34	5507.62
ω_8	5536.67	5526.38	5521.29	5507.76
ω_9	6122.00	6111.01	6104.86	6090.66
ω_{10}	7899.31	7880.38	7869.53	7844.82

Table 3. Convergence of the smallest vibration frequencies for different meshes $\mathcal{T}_h$ with N elements.

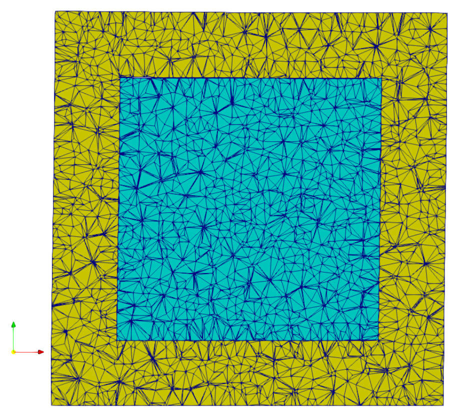
We appreciate from Table 3 the convergence behavior for each vibration frequency. Finally, we show in Figure 6 the deformed structures corresponding to three of the vibration modes: ω_2, ω_3 and ω_9. Let us remark that they look similar to those of a solid ring reported in Figures 15, 16 and 17 from [21].



(A) Deformed structure.



(B) Section x constant, at the middle of the cube.



(C) Section y constant, at the middle of the cube.

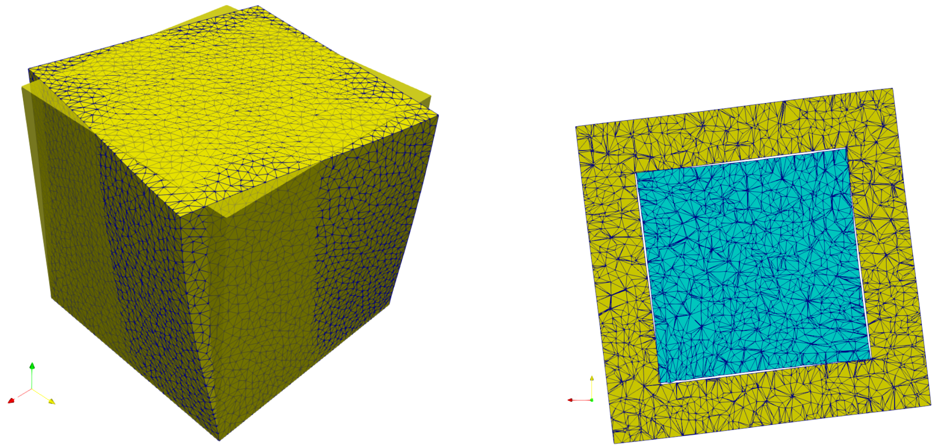
FIGURE 3. Mode ω_1: deformed structure.

7. CONCLUSIONS

We propose a finite element method to solve the vibration problem of a coupled system which consists of an elastic structure in contact with an incompressible fluid. With this aim, we extend to the 3D case, the analysis in [9, 10], where the divergence-free displacements are written as curls of a stream function.

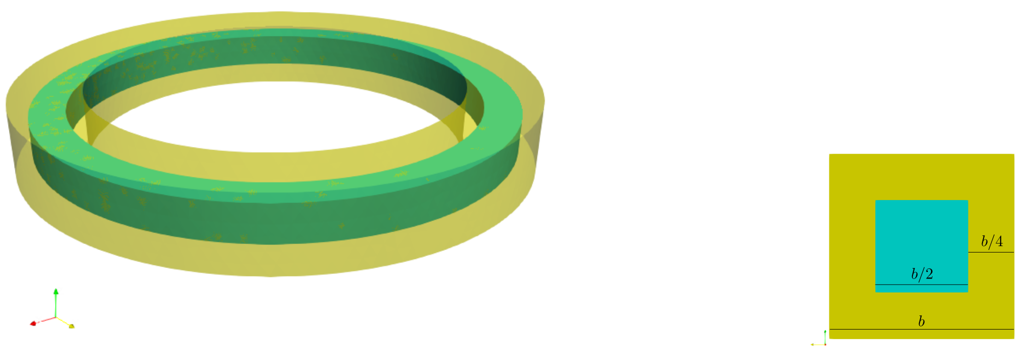
The extension to 3D is not trivial at all, since the kernel of the curl operator is much more complicated in 3D than in 2D. In particular, we follow the approach from [1, 2] to construct appropriate basis for the space of the divergence-free fields. The proposed strategy holds for topologically non-trivial domains, too.

We prove spectral convergence with optimal-order error estimates and report results for some numerical tests.



(A) Deformed structure. (B) Section z constant, at the middle of the cube.

FIGURE 4. Mode ω_3: deformed structure.



(A) Hollow ring filled with a fluid. (B) Vertical section of the hollow ring: $b = 0.05\pi$ m.

FIGURE 5. Topologically non trivial fluid domain.

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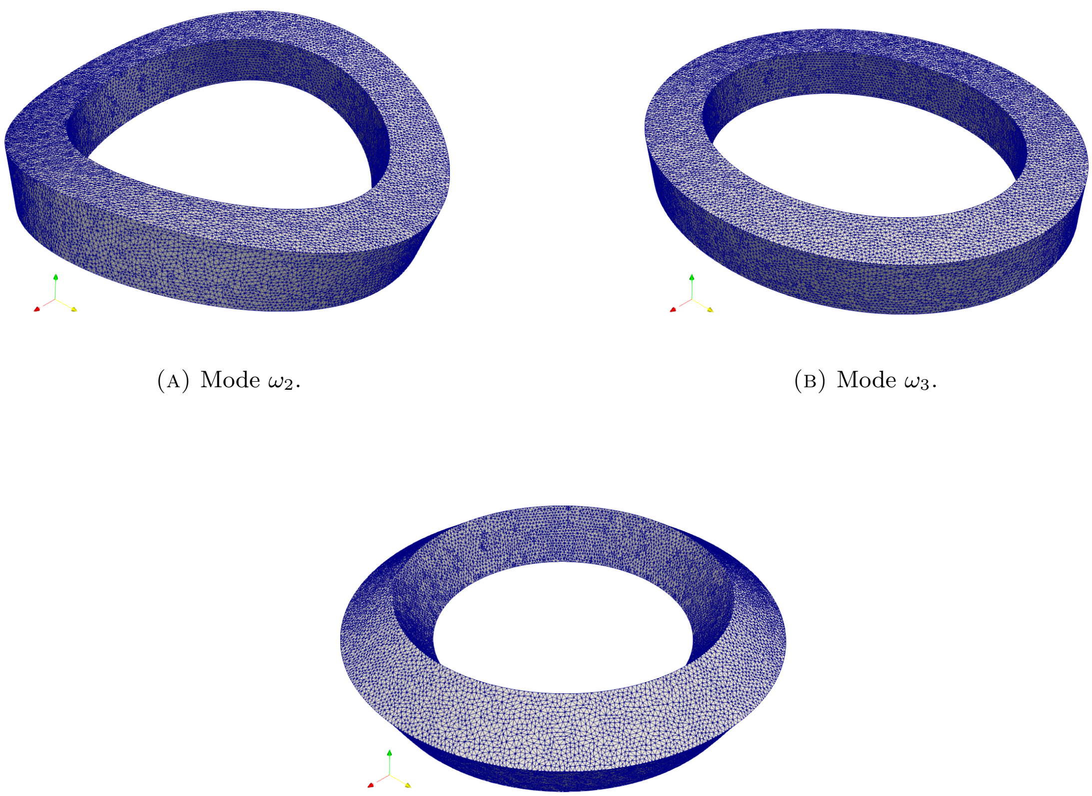

(A) Mode ω_2.

(B) Mode ω_3.

(C) Mode ω_9.

FIGURE 6. Deformed structure for the vibration modes corresponding to ω_2, ω_3 and ω_9.

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