A fully-mixed finite element method for the coupling of the Navier–Stokes and Darcy–Forchheimer equations*

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Abstract

In this work we present and analyse a fully-mixed formulation for the nonlinear model given by the coupling of the Navier–Stokes and Darcy–Forchheimer equations with the Beavers–Joseph–Saffman condition on the interface. Our approach yields non-Hilbertian normed spaces and a twofold saddle point structure for the corresponding operator equation. Furthermore, since the convective term in the Navier–Stokes equation forces the velocity to live in a smaller space than usual, we augment the variational formulation with suitable Galerkin type terms. The resulting augmented scheme is then written equivalently as a fixed point equation, so that the well-known Schauder and Banach theorems, combined with classical results on nonlinear monotone operators, are applied to prove the unique solvability of the continuous and discrete systems. In particular, given an integer $k \geq 0$, Raviart–Thomas spaces of order k, continuous piecewise polynomials of degree $\leq k+1$ and piecewise polynomials of degree $\leq k$ are employed in the fluid for approximating the pseudostress tensor, velocity and vorticity, respectively, whereas Raviart–Thomas spaces of order k and piecewise polynomials of degree $\leq k$ for the velocity and pressure, constitute a feasible choice in the porous medium. *A priori* error estimates and associated rates of convergence are derived, and several numerical examples illustrating the good performance of the method are reported.

Key words: Navier–Stokes equation, Darcy–Forchheimer equation, twofold saddle point, fixed point theory, augmented fully-mixed formulation, mixed finite element methods, *a priori* error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

The derivation of suitable mathematical and numerical models for the fluid flow between porous media and free-flow zones has been widely studied during the last decades, mostly due to its relevance in the fields of natural sciences, biology, and engineering branches. In particular, physical phenomena such as vuggy porous media appearing in petroleum extraction, groundwater system in karst aquifers, industrial filtrations, and blood motion in tumors and microvessels can be modelled by the Navier–Stokes/Darcy (or Stokes/Darcy) model (see, e.g., [4, 16, 31, 33]), which consists in a set of partial

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differential equations where the Navier–Stokes (or Stokes) problem is coupled with the Darcy model through a set of coupling equations acting on a common interface, which are given by mass conservation, balance of normal forces, and the so called Beavers–Joseph–Saffman condition. However, in applications such as the internal ventilation of a motorcycle helmet and reservoir wellbore (see, e.g., [7, 11, 3]), when the fluid velocity is higher and the porosity is nonuniform, which holds when the kinematic forces dominates over viscous forces, a better way to study this phenomenom is modifying the Darcy model in the porous medium by adding the Forchheimer term, which represents inertial effects, thus obtaining the Darcy–Forchheimer model (see [34, 32]).

In this context, and up to the authors' knowledge, one of the first works in analysing the Navier–Stokes/Darcy–Forchheimer coupled problem is [3]. In there, the authors study the coupling of a 2D reservoir model with a 1.5D vertical wellbore model, both written in axisymmetric form. The physical problems are described by the Darcy–Forchheimer and the compressible Navier–Stokes equations, respectively, together with an exhaustive energy equation. Later on, motivated by the study of the internal ventilation of a motorcycle helmet, a penalization approach, for both 2D and 3D domains, was introduced and analysed in [11]. In particular, the authors consider the velocity and pressure in the whole domain as the main unknowns of the system, and the corresponding Galerkin approximation employs piecewise quadratic and linear elements for the velocity and pressure, respectively. More recently, in [8] a primal-mixed formulation of the Navier–Stokes/Darcy–Forchheimer system is analyzed by means of a fixed-point argument and clasical results on nonlinear monotone operators (see [35, 36]). The corresponding mixed finite element scheme employs Bernardi–Raugel elements for the velocity in the free fluid region, Raviart–Thomas elements of lowest order for the filtration velocity in the porous media, and piecewise constant elements for the pressures and the Lagrange multiplier. Meanwhile, a fully-mixed finite element method is developed and analyzed for the coupling of the Stokes and Darcy–Forchheimer problems in [2]. This new approach yields non-Hilbertian normed spaces and a twofold saddle point structure for the corresponding operator equation, whose continuous and discrete solvabilities are analyzed by means of a suitable abstract theory developed for this purpose.

According to the above bibliographic discussion, the purpose of the present paper is to extend the results obtained in [8] and [2] to the coupling of the Navier–Stokes and Darcy–Forchheimer problems with constant density and viscosity, but unlike [8], by considering now dual-mixed formulations in both domains. We introduce the pseudostress tensor as in [6] and subsequently eliminate the pressure unknown using the incompressibility condition. In addition, and in order to impose the symmetry of the pseudostress tensor, similarly to [20, 2], the vorticity is introduced as an additional unknown. The transmission conditions consisting of mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law are imposed weakly, which yields the incorporation of additional Lagrange multipliers: the traces of the porous media pressure and the fluid velocity on the interface. Furthermore, the difficulty that the fluid velocity lives in H^1 instead of L^2 as usual, is resolved as in [6] by augmenting the variational formulation with residuals arising from the constitutive and equilibrium equations for the fluid flow, and the formula for the vorticity tensor. The resulting augmented variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness and uniqueness of both the continuous and discrete formulation is proved employing a fixed point argument and an abstract theory for twofold saddle point problems (see [18, 20, 2]). In particular, for the continuous formulation, and under a smallness data assumption, we prove existence and uniqueness of solution by means of a fixed-point strategy where the Schauder (for existence) and Banach (for uniqueness) fixed-point theorems are employed. In addition, an *a priori* error analysis is performed, and while it is possible to prove that the finite element method is convergent with a sub-optimal rate, the numerical results suggest that the method is optimally convergent provided the exact solutions are smooth enough. In particular, given an integer $k \geq 0$, we find that the interior Navier–Stokes variables: pseudostress tensor, velocity and vorticity, can be approximated using Raviart–Thomas spaces of order k, continuous piecewise polynomials of degree $\leq k+1$ and piecewise polynomials of degree $\leq k$, respectively, while the interior Darcy–Forchheimer variables: velocity and pressure, can be approximated using Raviart–Thomas spaces of order k and piecewise polynomials of degree $\leq k$.

The rest of this paper is organized as follows. The remainder of this section describes standard notations and functional spaces to be employed along the paper. In Section 2 we introduce the modelling equations for the free-flow zone, the porous medium and the interface, to then in Section 3, derive an augmented fully-mixed variational formulation that will be written as a nonlinear operator equation with a twofold saddle point structure. In addition, a suitable abstract theory for this type of problem is developed here, which includes the proper hypotheses on the spaces and involved operators to be imposed in order to guarantee the well-posedness of the continuous problem in rather general Banach spaces. Then, in Section 4 we use a fixed-point strategy to establish that our variational formulation is well posed. Next, in Section 5 we define the Galerkin scheme and derive general hypotheses on the discrete subspaces ensuring that, on the one hand, the discrete scheme becomes well posed, and on the other hand, it satisfies a Céa's estimate. A specific choice of finite element subspaces satisfying these assumptions as well as a sub-optimal rate of convergence are introduced in Section 6. Finally, several numerical examples illustrating the performance of the method, confirming the theoretical sub-optimal order of convergence, but at the same time suggesting an optimal rate of convergence, are reported in Section 7.

We end this section by introducing some definitions and fixing some notations. Let $\mathcal{O} \subseteq \mathbb{R}^n$, $n \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$ we denote by $\mathrm{L}^{\mathrm{p}}(\mathcal{O})$ and $\mathrm{W}^{\mathrm{s,p}}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^{\mathrm{p}}(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{s,p};\mathcal{O}}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\mathcal{O}) = \mathrm{L}^{\mathrm{p}}(\mathcal{O})$. If $\mathrm{p}=2$, we write $\mathrm{H}^{\mathrm{s}}(\mathcal{O})$ in place of $\mathrm{W}^{\mathrm{s},2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{\mathrm{s},\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{\mathrm{s},\mathcal{O}}$. In addition, we denote by $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$ the trace space of $\mathrm{W}^{1,\mathrm{p}}(\mathcal{O})$, and let $\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Gamma)$ be the dual space of $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/\mathrm{q},\mathrm{p};\Gamma}$ and $\|\cdot\|_{-1/\mathrm{q},\mathrm{q};\Gamma}$, respectively, with $\mathrm{p},\mathrm{q} \in (1,+\infty)$ satisfying $1/\mathrm{p}+1/\mathrm{q}=1$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij} \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Additionally, we recall that

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}\boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O}) \Big\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\mathcal{O}} := \|\boldsymbol{\tau}\|^2_{0,\mathcal{O}} + \|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\mathcal{O}},$$

is a standard Hilbert space in the realm of mixed problems. On the other hand, the following symbol for the $\mathrm{L}^2(\Gamma)$ inner product

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\, \lambda \quad \forall\, \xi, \lambda \in \mathrm{L}^2(\Gamma),$$

will also be employed for their respective extension as the duality parity between $\mathrm{W}^{-1/\mathrm{q},\mathrm{q}}(\Gamma)$ and $\mathrm{W}^{1/\mathrm{q},\mathrm{p}}(\Gamma)$. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^n$, $\mathrm{P}_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, and coherently with previous notations, we set $\mathbf{P}_k(S) := [\mathrm{P}_k(S)]^n$ and $\mathbb{P}_k(S) := [\mathrm{P}_k(S)]^{n\times n}$. Finally, we end this section by mentioning that, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The model problem

In order to describe the geometry under consideration we let Ω_S and Ω_D be bounded and simply connected open polyhedral domains in $\mathbb{R}^n$, $n = \{2, 3\}$, such that $\partial\Omega_\mathrm{S} \cap \partial\Omega_\mathrm{D} = \Sigma \neq \emptyset$ and $\Omega_\mathrm{S} \cap \Omega_\mathrm{D} = \emptyset$. Then, we let $\Gamma_\mathrm{S} := \partial\Omega_\mathrm{S} \setminus \overline{\Sigma}$, $\Gamma_\mathrm{D} := \partial\Omega_\mathrm{D} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_\mathrm{S} \cup \Sigma \cup \Omega_\mathrm{D}$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider a set of unit tangent vectors, which is given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Figure 2.1 below), and by $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$. The problem we are interested in consists of the movement of an incompressible viscous fluid occupying Ω_S which flows towards and from a porous medium Ω_D through Σ, where Ω_D is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In the free fluid domain Ω_S, the governing equations are those of the Navier–Stokes problem with constant density and viscosity, which are written in the following nonstandard pseudostress-velocity-pressure formulation:

$$\begin{gathered}\boldsymbol{\sigma}_\mathrm{S} = -p_\mathrm{S}\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_\mathrm{S}) - \rho(\mathbf{u}_\mathrm{S} \otimes \mathbf{u}_\mathrm{S}) \quad \text{in} \quad \Omega_\mathrm{S}, \quad \operatorname{div} \mathbf{u}_\mathrm{S} = 0 \quad \text{in} \quad \Omega_\mathrm{S}, \\ -\mathbf{div}\boldsymbol{\sigma}_\mathrm{S} = \mathbf{f}_\mathrm{S} \quad \text{in} \quad \Omega_\mathrm{S}, \quad \mathbf{u}_\mathrm{S} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{S},\end{gathered} \tag{2.1}$$

where $\boldsymbol{\sigma}_\mathrm{S}$ is the nonlinear pseudostress tensor, $\mathbf{u}_\mathrm{S}$ is the fluid velocity and p_S is the pressure. In addition, $\mathbf{e}(\mathbf{u}_\mathrm{S}) := \dfrac{1}{2}\Big\{\nabla\mathbf{u}_\mathrm{S} + (\nabla\mathbf{u}_\mathrm{S})^\mathrm{t}\Big\}$ stands for the strain tensor of small deformations, μ is the viscosity of the fluid, ρ is the density, and $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$ is a given external force.

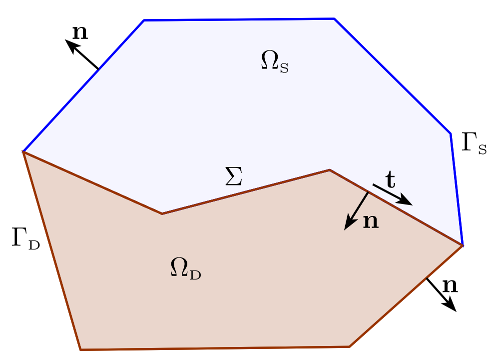

Figure 2.1: Sketch of a 2D geometry of our Navier–Stokes/Darcy–Forchheimer model

Now, in order to derive our fully-mixed formulation, we first observe, owing to the fact that $\operatorname{tr}\mathbf{e}(\mathbf{u}_S) = \operatorname{div}\mathbf{u}_S$, that the first two equations in (2.1) are equivalent to

$$\boldsymbol{\sigma}_S = -p_S\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_S) - \rho(\mathbf{u}_S\otimes\mathbf{u}_S), \quad p_S = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}_S + \rho(\mathbf{u}_S\otimes\mathbf{u}_S)) \quad \text{in} \quad \Omega_S, \tag{2.2}$$

and hence, eliminating the pressure p_S (which anyway can be approximated later on by the post-processed formula suggested by the second equation of (2.2)), the Navier–Stokes problem (2.1) can be rewritten as

$$\boldsymbol{\sigma}_S^{\mathrm{d}} = 2\mu\mathbf{e}(\mathbf{u}_S) - \rho(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathrm{d}} \quad \text{in} \quad \Omega_S, \quad -\mathbf{div}\boldsymbol{\sigma}_S = \mathbf{f}_S \quad \text{in} \quad \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S. \tag{2.3}$$

Next, in order to impose weakly the symmetry of the pseudostress tensor and employ the integration by parts formula, we introduce the additional unknown

$$\boldsymbol{\gamma}_S := \frac{1}{2}\Big\{\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathrm{t}}\Big\} \quad \text{in} \quad \Omega_S, \tag{2.4}$$

which represents the vorticity. In this way, instead of (2.3), in the sequel we consider the set of equations with unknowns $\boldsymbol{\sigma}_S$, $\boldsymbol{\gamma}_S$ and $\mathbf{u}_S$, given by

$$\frac{1}{2\mu}\boldsymbol{\sigma}_S^{\mathrm{d}} = \nabla\mathbf{u}_S - \boldsymbol{\gamma}_S - \frac{\rho}{2\mu}(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathrm{d}} \quad \text{in} \quad \Omega_S, \quad -\mathbf{div}\boldsymbol{\sigma}_S = \mathbf{f}_S \quad \text{in} \quad \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S, \tag{2.5}$$

where $\boldsymbol{\sigma}_S$ is a symmetric tensor in Ω_S.

In the porous medium Ω_D we consider a nonlinear version of the Darcy problem to approximate the velocity $\mathbf{u}_D$ and the pressure p_D, which is considered when the fluid velocity is higher and the porosity is nonuniform. More precisely, we consider the Darcy–Forchheimer equations [34, 32]:

$$\frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_D + \frac{\mathtt{F}}{\rho}|\mathbf{u}_D|\mathbf{u}_D + \nabla p_D = \mathbf{f}_D \quad \text{in} \quad \Omega_D, \quad \operatorname{div}\mathbf{u}_D = g_D \quad \text{in} \quad \Omega_D, \quad \mathbf{u}_D\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D, \tag{2.6}$$

where $\mathtt{F}$ represents the Forchheimer number of the porous medium, and $\mathbf{K}\in\mathbb{L}^\infty(\Omega_D)$ is a symmetric tensor in Ω_D representing the intrinsic permeability $\boldsymbol{\kappa}$ of the porous medium divided by the viscosity μ of the fluid. Throughout the paper we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\mathbf{w}\cdot\mathbf{K}^{-1}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}}|\mathbf{w}|^2, \tag{2.7}$$

for almost all $\mathbf{x}\in\Omega_D$, and for all $\mathbf{w}\in\mathbb{R}^n$. In turn, as will be explained below, $\mathbf{f}_D$ and g_D are given functions in $\mathbf{L}^{3/2}(\Omega_D)$ and $\mathrm{L}^2(\Omega_D)$, respectively. In addition, according to the compressibility conditions, the boundary conditions on $\mathbf{u}_D$ and $\mathbf{u}_S$, and the principle of mass conservation (cf. (2.8) below), g_D must satisfy the compatibility condition:

$$\int_{\Omega_D} g_D = 0.$$

Finally, the transmission conditions that couple the Navier–Stokes and the Darcy–Forchheimer models through the interface Σ are given by

$$\mathbf{u}_S\cdot\mathbf{n} = \mathbf{u}_D\cdot\mathbf{n} \quad \text{on} \quad \Sigma \quad \text{and} \quad \boldsymbol{\sigma}_S\mathbf{n} + \sum_{i=1}^{n-1}\omega_i^{-1}(\mathbf{u}_S\cdot\mathbf{t}_i)\mathbf{t}_i = -p_D\mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.8}$$

where $\{\omega_1, ..., \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.8) corresponds to mass conservation on Σ, whereas the second one establishes the balance of the normal forces and a Beavers–Joseph–Saffman law.

3 The continuous formulation

In this section we proceed analogously to [8, Section 2] (see also [2, 9, 21, 20]) and derive a weak formulation of the coupled problem given by (2.5), (2.6), and (2.8).

3.1 Preliminaries

We first introduce further notations and definitions. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, we set

$$(p,q)_\star := \int_{\Omega_\star} p\, q, \quad (\mathbf{u},\mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

In addition, in the sequel we will employ the following Banach space,

$$\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D}) := \Big\{ \mathbf{v}_\mathrm{D} \in \mathbf{L}^3(\Omega_\mathrm{D}) : \quad \mathrm{div}\,\mathbf{v}_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D}) \Big\},$$

endowed with the norm

$$\|\mathbf{v}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} := \Big(\|\mathbf{v}_\mathrm{D}\|^3_{\mathbf{L}^3(\Omega_\mathrm{D})} + \|\mathrm{div}\,\mathbf{v}_\mathrm{D}\|^3_{0,\Omega_\mathrm{D}} \Big)^{1/3}$$

and the following subspaces of $\mathbb{L}^2(\Omega_\mathrm{S})$, $\mathbf{H}^1(\Omega_\mathrm{S})$ and $\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})$, respectively

$$\begin{aligned}
\mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) &:= \Big\{ \boldsymbol{\eta}_\mathrm{S} \in \mathbb{L}^2(\Omega_\mathrm{S}) : \quad \boldsymbol{\eta}_\mathrm{S}^\mathrm{t} = -\boldsymbol{\eta}_\mathrm{S} \Big\}, \\
\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) &:= \Big\{ \mathbf{v}_\mathrm{S} \in \mathbf{H}^1(\Omega_\mathrm{S}) : \quad \mathbf{v}_\mathrm{S} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{S} \Big\}, \\
\mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D}) &:= \Big\{ \mathbf{v}_\mathrm{D} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D}) : \quad \mathbf{v}_\mathrm{D}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_\mathrm{D} \Big\}.
\end{aligned}$$

Notice that $\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D}) = \mathbf{H}(\mathrm{div}\,;\Omega_\mathrm{D}) \cap \mathbf{L}^3(\Omega_\mathrm{D})$, which guarantees that $\mathbf{v}_\mathrm{D}\cdot\mathbf{n}$ is well defined for $\mathbf{v}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$ (see [8, Section 2.2] for details). In addition, analogously to [20] (see also [9]) we need to introduce the space of traces $\mathbf{H}^{1/2}_{00}(\Sigma) := \Big[\mathrm{H}^{1/2}_{00}(\Sigma)\Big]^n$, where

$$\mathrm{H}^{1/2}_{00}(\Sigma) := \Big\{ v|_\Sigma : \ v \in \mathrm{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \Big\}.$$

Observe that, if $\mathrm{E}_{0,\mathrm{S}} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega_\mathrm{S})$ is the extension operator defined by

$$\mathrm{E}_{0,\mathrm{S}}(\psi) := \begin{cases} \psi & \text{on} \quad \Sigma \\ 0 & \text{on} \quad \Gamma_\mathrm{S} \end{cases} \qquad \forall\, \psi \in \mathrm{H}^{1/2}(\Sigma),$$

we have that

$$\mathrm{H}^{1/2}_{00}(\Sigma) = \Big\{ \psi \in \mathrm{H}^{1/2}(\Sigma) : \ \mathrm{E}_{0,\mathrm{S}}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_\mathrm{S}) \Big\},$$

which is endowed with the norm $\|\psi\|_{1/2,00;\Sigma} := \|\mathrm{E}_{0,\mathrm{S}}(\psi)\|_{1/2,\partial\Omega_\mathrm{S}}$. The dual space of $\mathbf{H}^{1/2}_{00}(\Sigma)$ is denoted by $\mathbf{H}^{-1/2}_{00}(\Sigma)$.

3.2 The augmented fully-mixed variational formulation

We now proceed with the derivation of our augmented fully-mixed variational formulation for the Navier–Stokes/Darcy–Forchheimer coupled problem. To this end, we begin by introducing two additional unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_{\mathrm{S}}|_\Sigma \in \mathbf{H}^{1/2}_{00}(\Sigma), \quad \lambda := p_{\mathrm{D}}|_\Sigma \in \mathrm{W}^{1/3,3/2}(\Sigma).$$

Then, similarly to [22, 20] and [8], we test the first equations of (2.5) and (2.6) with arbitrary $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ and $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}})$, respectively, integrate by parts, utilize the fact that $\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}} : \boldsymbol{\tau}_{\mathrm{S}} = \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}} : \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}$, and impose the remaining equations weakly, as well as the symmetry of $\boldsymbol{\sigma}_{\mathrm{S}}$ and the transmission conditions (2.8) to obtain the variational problem: Find $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}), \boldsymbol{\gamma}_{\mathrm{S}} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}), \boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{00}(\Sigma), \mathbf{u}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}), p_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}), \lambda \in \mathrm{W}^{1/3,3/2}(\Sigma)$ and $\mathbf{u}_{\mathrm{S}}$ in a suitable space (to be specified below), such that

$$\begin{aligned}
\frac{1}{2\mu}(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}\rangle_\Sigma + (\boldsymbol{\gamma}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + \frac{\rho}{2\mu}((\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} &= 0,\\
\frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (p_{\mathrm{D}}, \mathrm{div}\,\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - \langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \lambda\rangle_\Sigma &= (\mathbf{f}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}},\\
-(\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} &= (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}},\\
(q_{\mathrm{D}}, \mathrm{div}\,\mathbf{u}_{\mathrm{D}})_{\mathrm{D}} &= (g_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}},\\
(\boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} &= 0,\\
-\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi\rangle_\Sigma - \langle \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n}, \xi\rangle_\Sigma &= 0,\\
\langle \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda\rangle_\Sigma &= 0,
\end{aligned} \tag{3.1}$$

for all $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}), \boldsymbol{\eta}_{\mathrm{S}} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}), \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma), \mathbf{v}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}), q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}), \xi \in \mathrm{W}^{1/3,3/2}(\Sigma)$ and $\mathbf{v}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, where

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} := \sum_{i=1}^{n-1} \omega_i^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}_i, \boldsymbol{\psi} \cdot \mathbf{t}_i\rangle_\Sigma \,.$$

Notice here that the term $\langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi\rangle_\Sigma$ is well-defined for all $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$ and for all $\xi \in \mathrm{W}^{1/3,3/2}(\Sigma)$ (see [2, Lemma 2.2] for details). Notice also that the fifth term in the first equation of (3.1) requires $\mathbf{u}_{\mathrm{S}}$ to live in a smaller space than $\mathbf{L}^2(\Omega_{\mathrm{S}})$. In fact, by applying the Cauchy–Schwarz and Hölder inequalities and then the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega_{\mathrm{S}})$ into $\mathbf{L}^4(\Omega_{\mathrm{S}})$ (see, e.g., [1, Theorem 6.3]), we find that there holds

$$\left|((\mathbf{u}_{\mathrm{S}} \otimes \mathbf{w}_{\mathrm{S}})^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}}\right| \le \|\mathbf{u}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})}\|\mathbf{w}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \le \|\mathbf{i}_c\|^2\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}, \tag{3.2}$$

for all $\mathbf{u}_{\mathrm{S}}, \mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ and $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$. According to this, we propose to look for the unknown $\mathbf{u}_{\mathrm{S}}$ in $\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ and to restrict the set of corresponding test functions $\mathbf{v}_{\mathrm{S}}$ to the same space.

Next, analogously to [20] (see also [9]), it is not difficult to see that the system (3.1) is not uniquely solvable since, given any solution $(\boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\gamma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}}, \lambda)$ in the indicated spaces, and given any constant $c \in \mathbb{R}$, the vector defined by $(\boldsymbol{\sigma}_{\mathrm{S}} - c\mathbb{I}, \boldsymbol{\gamma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\varphi}, \mathbf{u}_{\mathrm{D}}, p_{\mathrm{D}} + c, \lambda + c)$ also becomes a solution. As a consequence of the above, from now on we require the Darcy–Forchheimer pressure p_{D} to be in $\mathrm{L}^2_0(\Omega_{\mathrm{D}})$, where

$$\mathrm{L}^2_0(\Omega_{\mathrm{D}}) := \Big\{ q_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}}) : \ (q_{\mathrm{D}}, 1)_{\mathrm{D}} = 0 \Big\}.$$

In turn, due to the decomposition $\mathrm{L}^2(\Omega_{\mathrm{D}}) = \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \oplus \mathbb{R}$, the boundary conditions $\mathbf{u}_{\mathrm{S}} = \mathbf{0}$ on Γ_{S} and $\mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} = 0$ on Γ_{D}, the first transmission condition in (2.8), and the fact that $(g_{\mathrm{D}}, 1)_{\mathrm{D}} = 0$, guarantee that the fourth equation of (3.1) is equivalent to requiring it for all $q_{\mathrm{D}} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})$.

On the other hand, for convenience of the subsequent analysis, we consider the decomposition (see, for instance, [5, 17])

$$\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) = \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \oplus \mathbb{R}\mathbb{I},$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) : \ (\mathrm{tr}\,\boldsymbol{\tau}, 1)_{\mathrm{S}} = 0 \Big\}$$

and redefine the pseudostress tensor as $\boldsymbol{\sigma}_{\mathrm{S}} := \boldsymbol{\sigma}_{\mathrm{S}} + \ell\mathbb{I}$, with the new unknowns $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$ and $\ell \in \mathbb{R}$. In this way the first and the seventh equations of (3.1) are rewritten, equivalently, as

$$\begin{aligned}
\frac{1}{2\mu}(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}\rangle_{\Sigma} + (\gamma_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + \frac{\rho}{2\mu}((\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}}, \boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} &= 0, \\
\jmath\, \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1\rangle_{\Sigma} &= 0, \\
\langle \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda\rangle_{\Sigma} + \ell\, \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1\rangle_{\Sigma} &= 0,
\end{aligned} \tag{3.3}$$

for all $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}})$, $\jmath \in \mathbb{R}$ and $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, respectively. Finally, we augment the resulting system through the incorporation of the following redundant Galerkin-type terms:

$$\begin{aligned}
\kappa_1(\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} &= -\kappa_1(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} \quad \forall\, \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}), \\
\kappa_2 \left(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \frac{\rho}{2\mu}(\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}}, \mathbf{e}(\mathbf{v}_{\mathrm{S}}) \right)_{\mathrm{S}} &= 0 \quad \forall\, \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \\
\kappa_3 \left(\gamma_{\mathrm{S}} - \frac{1}{2}\Big\{ \nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}} \Big\}, \boldsymbol{\eta}_{\mathrm{S}} \right)_{\mathrm{S}} &= 0 \quad \forall\, \boldsymbol{\eta}_{\mathrm{S}} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}),
\end{aligned} \tag{3.4}$$

where κ_1, κ_2 and κ_3 are positive parameters to be specified later. Notice that the foregoing terms are nothing but consistent expressions, arising from the equilibrium and constitutive equations, and the definition of the vorticity in terms of the velocity gradient (cf. (2.4)). It is easy to see that each solution of the original system is also a solution of the resulting augmented one, and hence by solving the latter we find all the solutions of the former.

Now, it is clear that there are many different ways of ordering the augmented mixed variational formulation described above, but for the sake of the subsequent analysis, we proceed as in [23] (see also [20, 9]), and adopt one leading to a doubly-mixed structure. For this purpose, we group the spaces, unknowns, and test functions as follows:

$$\begin{gathered}
\mathbf{X}_1 := \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}), \quad \mathbf{X}_2 := \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}), \\
\mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2, \quad \mathbf{Y} := \mathbf{H}^{1/2}_{00}(\Sigma) \times \mathrm{W}^{1/3,3/2}(\Sigma), \\
\mathbb{H} := \mathbf{X} \times \mathbf{Y} \quad \text{and} \quad \mathbb{Q} := \mathrm{L}^2_0(\Omega_{\mathrm{D}}) \times \mathbb{R},
\end{gathered}$$

$$\begin{gathered}
\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \gamma_{\mathrm{S}}) \in \mathbf{X}_1, \quad \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{X}_1, \\
\underline{\mathbf{t}} := (\underline{\boldsymbol{\sigma}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda) \in \mathbf{Y}, \quad \underline{\mathbf{p}} = (p_{\mathrm{D}}, \ell) \in \mathbb{Q}, \\
\underline{\mathbf{r}} := (\underline{\boldsymbol{\tau}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{Y}, \quad \underline{\mathbf{q}} = (q_{\mathrm{D}}, \jmath) \in \mathbb{Q},
\end{gathered}$$

where $\mathbf{X}_1$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbb{H}$ and $\mathbb{Q}$ are respectively endowed with the norms

$$\begin{aligned}
\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} &:= \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}, \\
\|\underline{\mathbf{r}}\|_{\mathbf{X}} &:= \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}, \\
\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} &:= \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\xi\|_{1/3,3/2;\Sigma}, \\
\|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}} &:= \|\underline{\mathbf{r}}\|_{\mathbf{X}} + \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \\
\|\underline{\mathbf{q}}\|_{\mathbb{Q}} &:= \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |\jmath|.
\end{aligned}$$

Hence, the augmented fully-mixed variational formulation for the system (3.1), (3.3) and (3.4) reads: Find $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}) \in \mathbb{H}\times\mathbb{Q}$ such that

$$\begin{aligned}
[\mathbf{A}(\mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{p}}] &= [\mathbf{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] \quad \forall\,(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\
[\mathbf{B}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{q}}] &= [\mathbf{G},\underline{\mathbf{q}}] \qquad \forall\,\underline{\mathbf{q}} \in \mathbb{Q},
\end{aligned} \tag{3.5}$$

where, given $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, the operator $\mathbf{A}(\mathbf{w}_{\mathrm{S}}) : \mathbb{H} \to \mathbb{H}'$ is defined by

$$[\mathbf{A}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] := [\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}] + [\mathbf{b}(\underline{\mathbf{r}}),\underline{\boldsymbol{\varphi}}] + [\mathbf{b}(\underline{\mathbf{t}}),\underline{\boldsymbol{\psi}}] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}], \tag{3.6}$$

with the operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}}) : \mathbf{X} \to \mathbf{X}'$ given by

$$[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}}),\underline{\mathbf{r}}] := [\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}] + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}], \tag{3.7}$$

with

$$\begin{aligned}
[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}] &:= \frac{1}{2\mu}(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}},\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} + \kappa_1(\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}},\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \\
&\quad + (\boldsymbol{\gamma}_{\mathrm{S}},\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} - (\boldsymbol{\sigma}_{\mathrm{S}},\boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} + \kappa_2\left(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}},\mathbf{e}(\mathbf{v}_{\mathrm{S}})\right)_{\mathrm{S}} \\
&\quad + \kappa_3\left(\boldsymbol{\gamma}_{\mathrm{S}} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^t),\boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}}, \\
[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}] &:= \frac{\rho}{2\mu}\Big((\mathbf{w}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}},\boldsymbol{\tau}_{\mathrm{S}} - \kappa_2\mathbf{e}(\mathbf{v}_{\mathrm{S}})\Big)_{\mathrm{S}}, \\
[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}] &:= \frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}},
\end{aligned} \tag{3.8}$$

whereas the operators $\mathbf{b} : \mathbf{X} \to \mathbf{Y}'$ and $\mathbf{c} : \mathbf{Y} \to \mathbf{Y}'$ are given, respectively, by

$$[\mathbf{b}(\underline{\mathbf{r}}),\underline{\boldsymbol{\psi}}] := \langle\boldsymbol{\tau}_{\mathrm{S}}\mathbf{n},\boldsymbol{\psi}\rangle_{\Sigma} - \langle\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma}, \tag{3.9}$$

$$[\mathbf{c}(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}] := \langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle\boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_{\Sigma} - \langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_{\Sigma}, \tag{3.10}$$

and the operator $\mathbf{B} : \mathbb{H} \to \mathbb{Q}'$ is defined by

$$[\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{q}}] := -(q_{\mathrm{D}},\mathrm{div}\,\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \jmath\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_{\Sigma}. \tag{3.11}$$

In turn, the functionals $\mathbf{F}$ and $\mathbf{G}$ are set as

$$[\mathbf{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] := -\kappa_1(\mathbf{f}_{\mathrm{S}},\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \text{and} \quad [\mathbf{G},\underline{\mathbf{q}}] := -(g_{\mathrm{D}},q_{\mathrm{D}})_{\mathrm{D}}. \tag{3.12}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathbf{b}' : \mathbf{Y} \to \mathbf{X}'$ and $\mathbf{B}' : \mathbb{Q} \to \mathbb{H}'$ be the adjoint of $\mathbf{b}$ and $\mathbf{B}$, respectively, which satisfy

$[\mathbf{b}'(\underline{\boldsymbol{\psi}}), (\underline{\mathbf{r}})] = [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]$ and $[\mathbf{B}'(\underline{\mathbf{q}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] = [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}]$ for all $\underline{\mathbf{r}} \in \mathbf{X}, \underline{\boldsymbol{\psi}} \in \mathbf{Y}$ and $\underline{\mathbf{q}} \in \mathbb{Q}$. Then, it is clear that (3.5) can be written equivalently as

$$\left[\begin{array}{cc|c} \mathbf{a}(\mathbf{w}_{\mathrm{S}}) & \mathbf{b}' & \mathbf{B}' \\ \mathbf{b} & -\mathbf{c} & \\ \hline \multicolumn{2}{c|}{\mathbf{B}} & \mathbf{O} \end{array}\right] \left[\begin{array}{c} (\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) \\ \underline{\mathbf{p}} \end{array}\right] = \left[\begin{array}{c} \mathbf{F} \\ \mathbf{G} \end{array}\right],$$

from which the twofold saddle point structure is evident.

3.3 An abstract theory for twofold saddle point problems

In this section we develop and analyze an abstract theory motivated by the twofold saddle point problem (3.5). To this end, a modification of what was done in [2] will be employed (which is already a modification of what was done in [18] and [29]). First we introduce some definitions that will be utilized next. Let X and Y be reflexive Banach spaces. Then, we say that a nonlinear operator $T : X \to Y$ is bounded if $T(S)$ is bounded for each bounded set $S \subseteq X$. In addition, we say that a nonlinear operator $T : X \to X'$ is of *type M* if $u_n \rightharpoonup u$, $Tu_n \rightharpoonup f$ and $\limsup\, [Tu_n, u_n] \le f(u)$ imply $Tu = f$. In turn, we say that T is coercive if

$$\frac{[Tu, u]}{\|u\|} \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Now, let X_1, X_2, Y, Q be separable and reflexive Banach spaces, set $X := X_1 \times X_2$ and $H := X \times Y$, and let $X_1', X_2', Y', Q', X' := X_1' \times X_2'$, and $H' := X' \times Y'$, be their respective duals. Let $a : X \to X'$ be a nonlinear operator and $b : X \to Y'$, $c : Y \to Y'$, and $B : H \to Q'$ be linear bounded operators. We also let $b' : Y \to X'$ and $B' : Q \to H'$ be the corresponding adjoints, and define the nonlinear operator $A : H \to H'$ as:

$$A(\mathbf{r}, \boldsymbol{\psi}) := \left[\begin{array}{cc} a & b' \\ b & -c \end{array}\right] \left[\begin{array}{c} \mathbf{r} \\ \boldsymbol{\psi} \end{array}\right] \in H' \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in H.$$

Then, we are interested in the following nonlinear variational problem: Given $(F, G) \in H' \times Q'$, find $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$ such that

$$\begin{aligned} [A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] &= [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in H, \\ [B(\mathbf{t}, \boldsymbol{\varphi}), \mathbf{q}] &= [G, \mathbf{q}] \qquad \forall\, \mathbf{q} \in Q. \end{aligned} \tag{3.13}$$

In what follows we proceed as in [2, 18, 20] to derive sufficient conditions under which (3.13) is well-posed. We first let V be the kernel of B, that is

$$V := \Big\{ (\mathbf{r}, \boldsymbol{\psi}) \in H : \quad [B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}] = 0 \quad \forall\, \mathbf{q} \in Q \Big\},$$

and assume that:

(B_0) H is uniformly convex.

(B_1) $B : H \to Q'$ is surjective, which means that there exists $\beta > 0$ such that

$$\sup_{\substack{(\mathbf{r}, \boldsymbol{\psi}) \in H \\ (\mathbf{r}, \boldsymbol{\psi}) \neq \mathbf{0}}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_H} \ge \beta\, \|\mathbf{q}\|_Q \quad \forall\, \mathbf{q} \in Q.$$

Then, given $G \in Q'$ there exists a unique $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ such that (see [29, Lemma A.1] for details):

$$B(\mathbf{t}_G, \boldsymbol{\varphi}_G) = G \quad \text{and} \quad \|(\mathbf{t}_G, \boldsymbol{\varphi}_G)\|_H = \|[\mathbf{t}_G, \boldsymbol{\varphi}_G]\|_{H/V} \leq \frac{1}{\beta}\,\|G\|_{Q'}, \tag{3.14}$$

where $[\mathbf{t}_G, \boldsymbol{\varphi}_G] := \Big\{(\mathbf{r}, \boldsymbol{\psi}) \in H : (\mathbf{t}_G - \mathbf{r}, \boldsymbol{\varphi}_G - \boldsymbol{\psi}) \in V\Big\}$ is the equivalence class in the quotient space H/V. Under the previous assumptions, we can show the following preliminary result.

Lemma 3.1 *Assume that hypotheses* (B_0) *and* (B_1) *hold. Then, the following problems are equivalent:*

$$(P)\left\{\begin{array}{l} \textit{Find } ((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q \textit{ such that} \\ {[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})],} \\ \textit{for all } (\mathbf{r}, \boldsymbol{\psi}) \in H. \end{array}\right. \qquad (\widetilde{P})\left\{\begin{array}{l} \textit{Find } (\mathbf{t}, \boldsymbol{\varphi}) \in H \textit{ such that} \\ {[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})],} \\ \textit{for all } (\mathbf{r}, \boldsymbol{\psi}) \in V. \end{array}\right.$$

More precisely, if $(\mathbf{t}, \boldsymbol{\varphi}) \in H$ is solution of $(\widetilde{P})$, we can define $\mathbf{p} \in Q$ as the unique solution of the following problem: Find $\mathbf{p} \in Q$ such that

$$[B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] = [F - A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in H. \tag{3.15}$$

Then $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})$ is solution of (P). Conversely, if $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$ is a solution of (P) then $(\mathbf{t}, \boldsymbol{\varphi})$ is solution of $(\widetilde{P})$ and $\mathbf{p}$ is solution of (3.15).

Proof. First, let $(\mathbf{t}, \boldsymbol{\varphi}) \in H$ solution of $(\widetilde{P})$ and notice that (3.15) has a unique solution. In fact, since $(\mathbf{t}, \boldsymbol{\varphi})$ is solution of $(\widetilde{P})$ then $F - A(\mathbf{t}, \boldsymbol{\varphi}) \in {}^{\circ}V := \Big\{G \in H' : \; G(\mathbf{r}, \boldsymbol{\psi}) = 0 \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in V\Big\}$. Hence, since assumption (B_1) also guarantees that the adjoint operator B' is an isomorphism from Q into ${}^{\circ}V$, we deduce that there exists a unique $\mathbf{p} \in Q$ solution of (3.15) satisfying

$$\|\mathbf{p}\|_Q \leq \frac{1}{\beta}\,\|B'(\mathbf{p})\|_{H'} = \frac{1}{\beta}\,\|F - A(\mathbf{t}, \boldsymbol{\varphi})\|_{H'}, \tag{3.16}$$

and therefore $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})$ is solution of (P). The second assertion is evident. □

Next, for the subsequent analysis, we let $\widetilde{X}$ and $\widetilde{Y}$ subspaces of X and Y, respectively, such that $V := \widetilde{X} \times \widetilde{Y}$. In addition, since F is linear we are able to define two functionals, F_1 and F_2, such that $[F, (\mathbf{r}, \boldsymbol{\psi})] = [F_1, \mathbf{r}] + [F_2, \boldsymbol{\psi}]$. In this way, we can state the following lemma.

Lemma 3.2 *Assume that hypotheses* (B_0) *and* (B_1) *hold, and let $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying* (3.14)*. Then, problem* (3.13) *is equivalent to: Find $(\mathbf{t}_0, \boldsymbol{\varphi}_0) \in V$ such that*

$$[A(\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in V, \tag{3.17}$$

or equivalently, such that

$$\begin{aligned} [a(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{r}] + [b'(\boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \mathbf{r}] \quad &= \quad [F_1, \mathbf{r}] \quad \forall\, \mathbf{r} \in \widetilde{X}, \\ [b(\mathbf{t}_0 + \mathbf{t}_G), \boldsymbol{\psi}] - [c(\boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \boldsymbol{\psi}] \quad &= \quad [F_2, \boldsymbol{\psi}] \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}. \end{aligned} \tag{3.18}$$

Moreover, the problem (3.13) *has a unique solution if and only if the problem* (3.17) *has a unique solution.*

Proof. It follows from a slight adaptation of [2, Lemma 3.2] taking in account now (3.15), in conjunction with [2, Lemma 3.3]. We omit further details. □

According to the previous analysis, we focus now on analyzing the solvability of (3.18). To that end, let us first assume the following assumptions:

(A_0) X_1, X_2 and Y are uniformly convex.

(A_1) there exists constant $\gamma > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \geq 2$, such that

$$\|a(\mathbf{t}) - a(\mathbf{r})\|_{X'} \leq \gamma \sum_{j=1}^{2} \left\{ \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} + \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} \Big(\|\mathbf{t}_j\|_{X_j} + \|\mathbf{r}_j\|_{X_j} \Big)^{\mathrm{p}_j - 2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in X$.

(A_2) for each $\mathbf{s} \in X$, the operator $a(\cdot + \mathbf{s}) : \widetilde{X} \to \widetilde{X}'$ is strictly monotone in the sense that there exist $\alpha > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \geq 2$, such that

$$[a(\mathbf{t} + \mathbf{s}) - a(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \geq \alpha \left\{ \|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{\mathrm{p}_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{\mathrm{p}_2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in \widetilde{X}$.

(A_3) there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{r} \in \widetilde{X} \\ \mathbf{r} \neq \mathbf{0}}} \frac{[b(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_X} \geq \beta_1 \|\boldsymbol{\psi}\|_Y \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}.$$

Notice that hypothesis (A_0) implies (B_0). Then, we can state the following preliminary lemma.

Lemma 3.3 *Assume that hypotheses* (A_1) – (A_3) *hold. Then, given* $\boldsymbol{\psi} \in \widetilde{Y}$ *and* $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ *satisfying* (3.14) *there exists a unique* $\mathbf{t}_0(\boldsymbol{\psi}) \in \widetilde{X}$*, such that*

$$[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G), \mathbf{r}] = [F_1 - b'(\boldsymbol{\psi} + \boldsymbol{\varphi}_G), \mathbf{r}] \quad \forall\, \mathbf{r} \in \widetilde{X}. \tag{3.19}$$

Moreover, there exists $C_1 > 0$*, depending only on* $\alpha, \beta, \gamma, \mathrm{p}_1, \mathrm{p}_2$ *and* $\|b'\|$*, such that*

$$\|\mathbf{t}_0(\boldsymbol{\psi})\|_X \leq C_1 \max_{i \in \{1,2\}} \left\{ \Big(\|\boldsymbol{\psi}\|_Y + \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{\mathrm{p}_1 - 1} + \|G\|_{Q'}^{\mathrm{p}_2 - 1} + \|a(0)\|_{X'} \Big)^{1/(\mathrm{p}_i - 1)} \right\}. \tag{3.20}$$

In addition, given $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \widetilde{Y}$ *for which* $\mathbf{t}_0(\boldsymbol{\psi}_1)$ *and* $\mathbf{t}_0(\boldsymbol{\psi}_2)$ *satisfy* (3.19)*, there exists* $C_2 > 0$*, depending only on* $\alpha, \mathrm{p}_1, \mathrm{p}_2$ *and* $\|b'\|$*, such that*

$$\|\mathbf{t}_0(\boldsymbol{\psi}_1) - \mathbf{t}_0(\boldsymbol{\psi}_2)\|_X \leq C_2 \max_{i \in \{1,2\}} \left\{ \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_Y^{1/(\mathrm{p}_i - 1)} \right\}. \tag{3.21}$$

Proof. We begin by noting that hypothesis (A_1) implies that the nonlinear operator a is continuous, and hence obviously hemi-continuous. In this way, as a consequence of hypotheses (A_1) – (A_2) we deduce the well-posedness of the problem (3.19) (see [8, Theorem 3.1] for details). In turn, given $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \widetilde{Y}$ for which $\mathbf{t}_0(\boldsymbol{\psi}_1)$ and $\mathbf{t}_0(\boldsymbol{\psi}_2)$ satisfy (3.19), we deduce that

$$[a(\mathbf{t}_0(\boldsymbol{\psi}_1) + \mathbf{t}_G) - a(\mathbf{t}_0(\boldsymbol{\psi}_2) + \mathbf{t}_G), \mathbf{r}] = [b'(\boldsymbol{\psi}_2 - \boldsymbol{\psi}_1), \mathbf{r}] \quad \forall\, \mathbf{r} \in \widetilde{X}. \tag{3.22}$$

Then, if we assume that $\mathbf{t}_0(\boldsymbol{\psi}_1) = \mathbf{t}_0(\boldsymbol{\psi}_2)$, hypotheses ($A_1$) and ($A_3$), and (3.22), imply that $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2$. Equivalently, this shows that given $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \widetilde{Y}$ with $\boldsymbol{\psi}_1 \neq \boldsymbol{\psi}_2$ the solutions $\mathbf{t}_0(\boldsymbol{\psi}_1)$ and $\mathbf{t}_0(\boldsymbol{\psi}_2)$ of (3.19) are in fact different. Now, in order to obtain (3.20), we proceed similarly to [35, Proposition 2.3] (see also [8, Theorem 3.1]). In fact, given $\boldsymbol{\psi} \in \widetilde{Y}$, we take $\mathbf{r} = \mathbf{t}_0(\boldsymbol{\psi}) \in \widetilde{X}$ in (3.19), and observe that

$$[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G) - a(0 + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\psi}) - 0] = [F_1 - b'(\boldsymbol{\psi} + \boldsymbol{\varphi}_G) - a(\mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\psi})].$$

Then, combining hypotheses $(\mathrm{A}_1) - (\mathrm{A}_2)$ and (3.14), it is clear that

$$\alpha\Big\{\|(\mathbf{t}_0(\boldsymbol{\psi}))_1\|_{X_1}^{\mathrm{p}_1} + \|(\mathbf{t}_0(\boldsymbol{\psi}))_2\|_{X_2}^{\mathrm{p}_2}\Big\} \leq \Big\{\|F_1\|_{X'} + \|b'(\boldsymbol{\psi}+\boldsymbol{\varphi}_G)\|_{X'} + \|a(\mathbf{t}_G)\|_{X'}\Big\}\|\mathbf{t}_0(\boldsymbol{\psi})\|_X$$
$$\leq c_1\Big\{\|\boldsymbol{\psi}\|_Y + \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{\mathrm{p}_1-1} + \|G\|_{Q'}^{\mathrm{p}_2-1} + \|a(0)\|_{X'}\Big\}\|\mathbf{t}_0(\boldsymbol{\psi})\|_X,$$

with $c_1 > 0$ depending only on $\gamma, \beta, \mathrm{p}_1, \mathrm{p}_2$, and $\|b'\|$, which, after simple algebraic manipulation, yields (3.20). In turn, in order to derive (3.21), we take $\mathbf{r} = \mathbf{t}_0(\boldsymbol{\psi}_1) - \mathbf{t}_0(\boldsymbol{\psi}_2)$ in (3.22), to obtain

$$[a(\mathbf{t}_0(\boldsymbol{\psi}_1)+\mathbf{t}_G) - a(\mathbf{t}_0(\boldsymbol{\psi}_2)+\mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\psi}_1) - \mathbf{t}_0(\boldsymbol{\psi}_2)] = [b'(\boldsymbol{\psi}_2 - \boldsymbol{\psi}_1), \mathbf{t}_0(\boldsymbol{\psi}_1) - \mathbf{t}_0(\boldsymbol{\psi}_2)], \qquad (3.23)$$

and then proceed analogously to (3.20) (see [2, Lemma 3.5] for details). □

According to the above, and given $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (3.14), problem (3.18) is equivalent to: find $\boldsymbol{\varphi}_0 \in \widetilde{Y}$ such that

$$[\mathrm{L}(\boldsymbol{\varphi}_0), \boldsymbol{\psi}] := -[b(\mathbf{t}_0(\boldsymbol{\varphi}_0)), \boldsymbol{\psi}] + [c(\boldsymbol{\varphi}_0), \boldsymbol{\psi}] = [\widetilde{F}_2, \boldsymbol{\psi}] \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}, \qquad (3.24)$$

where $\widetilde{F}_2 := b(\mathbf{t}_G) - c(\boldsymbol{\varphi}_G) - F_2$. More precisely, given $\mathbf{t}_0(\boldsymbol{\varphi}_0) \in \widetilde{X}$ solution of (3.19) with $\boldsymbol{\varphi}_0 \in \widetilde{Y}$ solution of (3.24), the vector $(\mathbf{t}_0, \boldsymbol{\varphi}_0) := (\mathbf{t}_0(\boldsymbol{\varphi}_0), \boldsymbol{\varphi}_0) \in \widetilde{X} \times \widetilde{Y}$ solves (3.18). The converse is straightforward. Hence, we now focus on proving that L is bijective. To that end, we assume one more hypothesis:

(A_4) c is positive semi-definite on $\widetilde{Y}$, that is,

$$[c(\boldsymbol{\psi}), \boldsymbol{\psi}] \geq 0 \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}.$$

Then, the bijectivity of the operator L follows from a slight adaptation of [2, Section 3.1], which means, equivalently, that (3.24) has a unique solution $\boldsymbol{\varphi}_0 \in \widetilde{Y}$. In particular, we remark here that under the hypotheses that have been assumed for the solvability of (3.18), the operator L is continuous and monotone, and therefore, of *type M* (cf. [36, Lemma II.2.1]). Then, by also proving that L is bounded and coercive [36, Corollary II.2.2], the surjective of L is ensured. The main result of this section is established now.

Theorem 3.4 *Let X_1, X_2, Y, Q be separable and reflexive Banach spaces, being X_1, X_2 and Y uniformly convex, set $X := X_1 \times X_2$ and $H := X \times Y$, and let $X_1', X_2', Y', Q', X' := X_1' \times X_2'$, and $H' := X' \times Y'$, be their respective duals. In addition, let $a : X \to X'$ be a nonlinear operator, and let $b : X \to Y'$, $c : Y \to Y'$, and $B : H \to Q'$ be linear bounded operators. We also let $b' : Y \to X'$ and $B' : Q \to H'$ be the corresponding adjoints and define the nonlinear operator $A : H \to H'$ as*

$$[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] := [a(\mathbf{t}), \mathbf{r}] + [b'(\boldsymbol{\varphi}), \mathbf{r}] + [b(\mathbf{t}), \boldsymbol{\psi}] - [c(\boldsymbol{\varphi}), \boldsymbol{\psi}] \quad \forall\, (\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi}) \in H.$$

In turn, let V be the kernel of B, that is

$$V := \Big\{(\mathbf{r}, \boldsymbol{\psi}) \in H : \quad [B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}] = 0 \quad \forall\, \mathbf{q} \in Q\Big\},$$

and let $\widetilde{X}$ and $\widetilde{Y}$ be subspaces of X and Y, respectively, such that $V = \widetilde{X} \times \widetilde{Y}$. Assume that

(i) *there exists constant $\gamma > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \geq 2$, such that*

$$\|a(\mathbf{t}) - a(\mathbf{r})\|_{X'} \leq \gamma \sum_{j=1}^{2} \Big\{\|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} + \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j}\Big(\|\mathbf{t}_j\|_{X_j} + \|\mathbf{r}_j\|_{X_j}\Big)^{\mathrm{p}_j - 2}\Big\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in X$.

(ii) *for each* $\mathbf{s} \in X$*, the operator* $a(\cdot + \mathbf{s}) : \widetilde{X} \to \widetilde{X}'$ *is strictly monotone in the sense that there exist* $\alpha > 0$ *and* $\mathrm{p}_1, \mathrm{p}_2 \geq 2$*, such that*

$$[a(\mathbf{t}+\mathbf{s}) - a(\mathbf{r}+\mathbf{s}), \mathbf{t}-\mathbf{r}] \geq \alpha \left\{ \|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{\mathrm{p}_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{\mathrm{p}_2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in \widetilde{X}$.

(iii) c *is positive semi-definite on* $\widetilde{Y}$*, that is,*

$$[c(\boldsymbol{\psi}), \boldsymbol{\psi}] \geq 0 \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}.$$

(iv) b *satisfies an inf-sup condition on* $\widetilde{X} \times \widetilde{Y}$*, that is, there exists* $\beta_1 > 0$ *such that*

$$\sup_{\substack{\mathbf{r} \in \widetilde{X} \\ \mathbf{r} \neq \mathbf{0}}} \frac{[b(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_X} \geq \beta_1 \, \|\boldsymbol{\psi}\|_Y \quad \forall\, \boldsymbol{\psi} \in \widetilde{Y}.$$

(v) B *satisfies an inf-sup condition on* $H \times Q$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\substack{(\mathbf{r}, \boldsymbol{\psi}) \in H \\ (\mathbf{r}, \boldsymbol{\psi}) \neq \mathbf{0}}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_H} \geq \beta \, \|\mathbf{q}\|_Q \quad \forall\, \mathbf{q} \in Q.$$

Then, for each $(F, G) \in H' \times Q'$ *there exists a unique* $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$*, such that*

$$\begin{aligned} [A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] &= [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall\, (\mathbf{r}, \boldsymbol{\psi}) \in H, \\ [B(\mathbf{t}, \boldsymbol{\varphi}), \mathbf{q}] &= [G, \mathbf{q}] \qquad \forall\, \mathbf{q} \in Q. \end{aligned} \tag{3.25}$$

Moreover, there exists $C > 0$*, depending only on* $\alpha, \gamma, \beta, \beta_1, \mathrm{p}_1, \mathrm{p}_2, \|b\|, \|b'\|$*, and* $\|c\|$ *such that*

$$\|((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})\|_{H \times Q} \leq C \, \mathcal{M}(F, G), \tag{3.26}$$

where

$$\begin{aligned} \mathcal{M}(F,G) := \max \Big\{ &\mathcal{N}(F,G)^{1/(\mathrm{p}_1 - 1)}, \mathcal{N}(F,G)^{1/(\mathrm{p}_2 - 1)}, \mathcal{N}(F,G), \\ &\mathcal{N}(F,G)^{(\mathrm{p}_1 - 1)/(\mathrm{p}_2 - 1)}, \mathcal{N}(F,G)^{(\mathrm{p}_2 - 1)/(\mathrm{p}_1 - 1)} \Big\}, \end{aligned}$$

and $\mathcal{N}(F, G)$ *is defined below in* (3.29).

Proof. We begin by noting that the well-posedness of the problem (3.25) follows straightforwardly from Lemmas 3.1–3.3 and the fact that the operator L is bijective (cf. (3.24)). Now, in order to obtain (3.26), we proceed similarly to [8, Theorem 3.1]. To that end, we first recall that $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) = ((\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \mathbf{p}) \in H \times Q$ is solution of (3.25) with $(\mathbf{t}_0, \boldsymbol{\varphi}_0) \in V$ solution of (3.17) and $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (3.14). Next, applying (3.20) with $\boldsymbol{\psi} = 0$, we get

$$\|\mathbf{t}_0(0)\|_X \leq C_1 \max_{i \in \{1,2\}} \left\{ \mathcal{N}_1(F,G)^{1/(\mathrm{p}_i - 1)} \right\}, \tag{3.27}$$

where $\mathcal{N}_1(F,G) := \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{\mathrm{p}_1 - 1} + \|G\|_{Q'}^{\mathrm{p}_2 - 1} + \|a(0)\|_{X'}$. In turn, according to the definition of L (cf. (3.24)), and employing hypothesis (iii), the identity (3.23) with $\boldsymbol{\psi}_1 = \boldsymbol{\varphi}_0$ and $\boldsymbol{\psi}_2 = 0$, and the fact that $\mathbf{t}_0 = \mathbf{t}_0(\boldsymbol{\varphi}_0)$, we deduce that

$$[a(\mathbf{t}_0 + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G), \mathbf{t}_0 - \mathbf{t}_0(0)] \leq [\mathrm{L}(\boldsymbol{\varphi}_0) - \mathrm{L}(0), \boldsymbol{\varphi}_0] = [\widetilde{F}_2 - \mathrm{L}(0), \boldsymbol{\varphi}_0],$$

where $\widetilde{F}_2 = b(\mathbf{t}_G) - c(\boldsymbol{\varphi}_G) - F_2$. In this way, hypothesis (ii) and the definition of the operator L (cf. (3.24)), yield

$$\alpha\left\{\|(\mathbf{t}_0)_1 - (\mathbf{t}_0(0))_1\|_{X_1}^{\mathrm{p}_1} + \|(\mathbf{t}_0)_2 - (\mathbf{t}_0(0))_2\|_{X_2}^{\mathrm{p}_2}\right\} \le \|\widetilde{F}_2 - \mathrm{L}(0)\|_{Y'}\|\boldsymbol{\varphi}_0\|_Y \le C_2\,\mathcal{N}_2(F,G)\|\boldsymbol{\varphi}_0\|_Y,$$

where $\mathcal{N}_2(F,G) := \|F_2\|_{Y'} + \max\limits_{i\in\{1,2\}}\left\{\mathcal{N}_1(F,G)^{1/(\mathrm{p}_i-1)}\right\}$. Thus, it is clear that

$$\|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i} \le C\,\mathcal{N}_2(F,G)^{1/\mathrm{p}_i}\|\boldsymbol{\varphi}_0\|_Y^{1/\mathrm{p}_i} \quad \text{for} \quad i\in\{1,2\}.$$

Then, employing the identity (3.22) with $\boldsymbol{\psi}_1 = \boldsymbol{\varphi}_0$ and $\boldsymbol{\psi}_2 = 0$, hypotheses (i) and (iv), the foregoing inequality, and the fact that the upper bound in (3.27) can be bounded by $\mathcal{N}_2(F,G)$, we find that

$$\begin{aligned}
\|\boldsymbol{\varphi}_0\|_Y &\le \frac{1}{\beta_1}\,\|a(\mathbf{t}_0+\mathbf{t}_G) - a(\mathbf{t}_0(0)+\mathbf{t}_G)\|_{X'}\\
&\le c\sum_{i=1}^{2}\left\{\left(1 + \|(\mathbf{t}_0(0)+\mathbf{t}_G)_i\|_{X_i}^{\mathrm{p}_i-2}\right)\|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i} + \|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i}^{\mathrm{p}_i-1}\right\}\\
&\le C\sum_{i=1}^{2}\left\{\left(\mathcal{N}_2(F,G)^{1/\mathrm{p}_i} + \mathcal{N}_2(F,G)^{(\mathrm{p}_i-1)^2/\mathrm{p}_i}\right)\|\boldsymbol{\varphi}_0\|_Y^{1/\mathrm{p}_i} + \mathcal{N}_2(F,G)^{(\mathrm{p}_i-1)/\mathrm{p}_i}\|\boldsymbol{\varphi}_0\|_Y^{(\mathrm{p}_i-1)/\mathrm{p}_i}\right\},
\end{aligned}$$

with C depending on $\alpha,\gamma,\beta,\beta_1,\mathrm{p}_1,\mathrm{p}_2,\|b\|,\|b'\|$, and $\|c\|$. In turn, applying Young's inequality conveniently allows us to deduce that

$$\|\boldsymbol{\varphi}_0\|_Y \le C_3\,\mathcal{N}(F,G), \tag{3.28}$$

where

$$\mathcal{N}(F,G) := \max\left\{\mathcal{N}_2(F,G)^{1/(\mathrm{p}_1-1)}, \mathcal{N}_2(F,G)^{1/(\mathrm{p}_2-1)}, \mathcal{N}_2(F,G)^{\mathrm{p}_1-1}, \mathcal{N}_2(F,G)^{\mathrm{p}_2-1}\right\}. \tag{3.29}$$

Therefore, using that $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G$, and combining (3.14) and (3.28), we conclude that

$$\|\boldsymbol{\varphi}\|_Y \le \|\boldsymbol{\varphi}_0\|_Y + \|\boldsymbol{\varphi}_G\|_Y \le c_1\,\mathcal{N}(F,G), \tag{3.30}$$

with $c_1 > 0$ depending only on $\alpha,\gamma,\beta,\beta_1,\mathrm{p}_1,\mathrm{p}_2,\|b\|,\|b'\|$, and $\|c\|$. Similarly, recalling that $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_G$, and employing (3.14), (3.20), and (3.28), we conclude that

$$\|\mathbf{t}\|_X \le \|\mathbf{t}_0\|_X + \|\mathbf{t}_G\|_X \le c_2\,\max_{i\in\{1,2\}}\left\{\mathcal{N}(F,G)^{1/(\mathrm{p}_i-1)}\right\}, \tag{3.31}$$

with $c_2 > 0$ depending only on $\alpha,\gamma,\beta,\beta_1,\mathrm{p}_1,\mathrm{p}_2,\|b\|,\|b'\|$, and $\|c\|$. On the other hand, from (3.16) and (i), we deduce that

$$\|\mathbf{p}\|_Q \le C\left\{\|F\|_{H'} + \|\mathbf{t}\|_X + \|\boldsymbol{\varphi}\|_Y + \|\mathbf{t}_1\|_{X_1}^{\mathrm{p}_1-1} + \|\mathbf{t}_2\|_{X_2}^{\mathrm{p}_2-1} + \|a(0)\|_{X'}\right\},$$

which, together with (3.30) and (3.31), conclude the proof. □

We remark that when $\mathrm{p}_1 = \mathrm{p}_2 = 2$ and $\|a(0)\|_{X'}$ is equal to zero, the previous analysis leads to the classical estimate

$$\|((\mathbf{t},\boldsymbol{\varphi}),\mathbf{p})\|_{H\times Q} \le C\left\{\|F\|_{H'} + \|G\|_{Q'}\right\},$$

with $C > 0$, depending only on $\alpha,\gamma,\beta,\beta_1,\|b\|,\|b'\|$, and $\|c\|$.

4 Analysis of the continuous formulation

In this section we analyze the well-posedness of the continuous problem (3.5) by using a fixed-point strategy and the abstract theory on twofold saddle point problems developed in Section 3.3. We begin by collecting some previous results and notations that will serve for the forthcoming analysis.

4.1 Preliminaries

Concerning the stability properties of the operators in (3.8), (3.9), (3.10), and (3.11), we first observe that $\mathcal{A}_{\mathrm{S}}, \mathbf{b}, \mathbf{c}$, and $\mathbf{B}$ are all continuous, that is there exist positive constants $C_{\mathcal{A}_{\mathrm{S}}}$, $C_{\mathbf{b}}$, $C_{\mathbf{c}}$, and $C_{\mathbf{B}}$, such that

$$\begin{gathered}\left|[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}]\right| \le C_{\mathcal{A}_{\mathrm{S}}}\,\|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1}\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}, \quad \left|[\mathbf{b}(\underline{\mathbf{r}}),\underline{\boldsymbol{\psi}}]\right| \le C_{\mathbf{b}}\,\|\underline{\mathbf{r}}\|_{\mathbf{X}}\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \\ \left|[\mathbf{c}(\underline{\boldsymbol{\varphi}}),\underline{\boldsymbol{\psi}}]\right| \le C_{\mathbf{c}}\,\|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}}\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \quad \left|[\mathbf{B}((\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})),\underline{\mathbf{q}}]\right| \le C_{\mathbf{B}}\,\|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}\|\underline{\mathbf{q}}\|_{\mathbb{Q}}\,,\end{gathered} \tag{4.1}$$

whereas from the definition of $\mathcal{B}_{\mathrm{S}}$ (cf. (3.8)) and (3.2) we easily obtain that

$$\left|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}]\right| \le \frac{\rho}{2\mu}(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|\|\mathbf{w}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} \le C_{\mathcal{B}_{\mathrm{S}}}\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1}\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}, \tag{4.2}$$

with $C_{\mathcal{B}_{\mathrm{S}}} := \dfrac{\rho}{2\mu}(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|^2$. In turn, from the definition of $\mathcal{A}_{\mathrm{D}}$ (cf. (3.8)), (2.7), and the triangle and Hölder inequalities, we deduce that there exists $L_{\mathcal{A}_{\mathrm{D}}} > 0$, depending only on $\mu, \rho, \mathtt{F}, \mathbf{K}$ and Ω_{D}, such that

$$\begin{aligned}&\|\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}) - \mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\|_{(\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}))'} \\ &\quad \le L_{\mathcal{A}_{\mathrm{D}}}\Big\{\|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big(\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big)\Big\}\end{aligned} \tag{4.3}$$

for all $\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})$. In addition, using the Cauchy–Schwarz and Young's inequalities, it is not difficult to see that $\mathbf{F}$ and $\mathbf{G}$ are bounded (cf. (3.12)), that is, there exist constants $C_{\mathbf{F}}, C_{\mathbf{G}} > 0$, such that

$$\|\mathbf{F}\|_{\mathbb{H}'} \le C_{\mathbf{F}}\left\{\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{\mathbf{L}^{3/2}(\Omega_{\mathrm{D}})}\right\} \tag{4.4}$$

and

$$\|\mathbf{G}\|_{\mathbb{Q}'} \le C_{\mathbf{G}}\,\|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}\,, \tag{4.5}$$

which confirm the announced smoothness of $\mathbf{f}_{\mathrm{D}}$. On the other hand, from the definition of $\mathcal{A}_{\mathrm{D}}$ (cf. (3.8)), inequality (2.7) and [25, Lemma 5.1] (see [8, Section 2.3] for details), we deduce that there exists $\alpha_{\mathrm{D}} > 0$, depending only on $\rho, \mathtt{F}$ and Ω_{D}, such that for each $\mathbf{t}_{\mathrm{D}} \in \mathbf{L}^3(\Omega_{\mathrm{D}})$ there holds

$$[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}} + \mathbf{t}_{\mathrm{D}}) - \mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}} + \mathbf{t}_{\mathrm{D}}), \mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}] \ge \alpha_{\mathrm{D}}\,\|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{L}^3(\Omega_{\mathrm{D}})} \quad \forall\,\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}} \in \mathbf{L}^3(\Omega_{\mathrm{D}})\,. \tag{4.6}$$

Finally, we recall that there exist positive constants $C_{\mathrm{d}}(\Omega_{\mathrm{S}})$ and C_{Ko}, such that (see, [5, Proposition IV.3.1] and [5, 24], respectively, for details)

$$C_{\mathrm{d}}(\Omega_{\mathrm{S}})\,\|\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \le \|\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \quad \forall\,\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \tag{4.7}$$

and

$$C_{\mathrm{Ko}}\,\|\mathbf{v}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}} \le \|\mathbf{e}(\mathbf{v}_{\mathrm{S}})\|^2_{0,\Omega_{\mathrm{S}}} \quad \forall\,\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}). \tag{4.8}$$

Notice that, in particular, (4.8) is known as Korn's inequality. Then, we establish next the ellipticity of the operator $\mathcal{A}_{\mathrm{S}}$.

Lemma 4.1 *Assume that for* $\delta_1 \in (0, 4\mu)$ *and* $\delta_2 \in (0, 2)$ *we choose*

$$\kappa_1 \in (0, +\infty), \quad \kappa_2 \in (0, 2\delta_1) \quad \text{and} \quad \kappa_3 \in \left(0, 2C_{\mathrm{Ko}}\kappa_2\delta_2\left(1 - \frac{\delta_1}{4\mu}\right)\right).$$

Then, there exists a constant $\alpha_{\mathrm{S}} > 0$*, such that there holds*

$$[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] \geq \alpha_{\mathrm{S}} \, \|\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}_1} \quad \forall\, \underline{\boldsymbol{\tau}} \in \mathbf{X}_1. \tag{4.9}$$

Proof. Let $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}) \in \mathbf{X}_1$. Then from (3.8) we have that

$$\begin{aligned}[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] = & \frac{1}{2\mu}\|\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \kappa_1\|\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \kappa_2\|\mathbf{e}(\mathbf{v}_{\mathrm{S}})\|^2_{0,\Omega_{\mathrm{S}}} + \kappa_3\|\boldsymbol{\eta}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \\ & - \frac{\kappa_2}{2\mu}\big(\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}, \mathbf{e}(\mathbf{v}_{\mathrm{S}})\big)_{\mathrm{S}} - \frac{\kappa_3}{2}\left(\nabla\mathbf{v}_{\mathrm{S}} - (\nabla\mathbf{v}_{\mathrm{S}})^{\mathrm{t}}, \boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}}.\end{aligned}$$

Hence, we proceed similarly to the proof of [6, Lemma 3.4] and utilize the Cauchy–Schwarz and Young inequalities to find that for any $\delta_1, \delta_2 > 0$, and for all $\underline{\boldsymbol{\tau}} \in \mathbf{X}_1$, there holds

$$\begin{aligned}[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] \geq & \frac{1}{2\mu}\left(1 - \frac{\kappa_2}{2\delta_1}\right)\|\boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \kappa_1\|\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} \\ & + \kappa_2\left(1 - \frac{\delta_1}{4\mu}\right)\|\mathbf{e}(\mathbf{v}_{\mathrm{S}})\|^2_{0,\Omega_{\mathrm{S}}} - \frac{\kappa_3}{2\delta_2}\|\mathbf{v}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}} + \kappa_3\left(1 - \frac{\delta_2}{2}\right)\|\boldsymbol{\eta}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}}.\end{aligned}$$

Then, assuming the stipulated ranges on $\delta_1, \delta_2, \kappa_1, \kappa_2$, and κ_3, and applying the inequalities (4.7) and (4.8), we can define the positive constants

$$\alpha_0(\Omega_{\mathrm{S}}) := \min\left\{\frac{1}{2\mu}\left(1 - \frac{\kappa_2}{2\delta_1}\right), \frac{\kappa_1}{2}\right\}, \quad \alpha_1(\Omega_{\mathrm{S}}) := \min\Big\{C_{\mathrm{d}}(\Omega_{\mathrm{S}})\,\alpha_0(\Omega_{\mathrm{S}}), \frac{\kappa_1}{2}\Big\}$$

$$\alpha_2(\Omega_{\mathrm{S}}) := C_{\mathrm{Ko}}\kappa_2\left(1 - \frac{\delta_1}{4\mu}\right) - \frac{\kappa_3}{2\delta_2}, \quad \alpha_3(\Omega_{\mathrm{S}}) := \kappa_3\left(1 - \frac{\delta_2}{2}\right),$$

which allows us to conclude (4.9) with $\alpha_{\mathrm{S}} := \min\Big\{\alpha_1(\Omega_{\mathrm{S}}), \alpha_2(\Omega_{\mathrm{S}}), \alpha_3(\Omega_{\mathrm{S}})\Big\}$. □

We end this section by remarking that the explicit expressions yielding the computation of the ellipticity constant α_{S} of $\mathcal{A}_{\mathrm{S}}$ (cf. (4.9)), can be maximized by taking the parameters $\delta_1, \delta_2, \kappa_2$ and κ_3 as the middle points of their feasible ranges, and by choosing κ_1 so that it maximizes the minimum defining $\alpha_0(\Omega_{\mathrm{S}})$. More precisely, we simply take

$$\begin{gathered}\delta_1 = 2\mu, \quad \delta_2 = 1, \quad \kappa_2 = \delta_1 = 2\mu, \\ \kappa_3 = C_{\mathrm{Ko}}\kappa_2\delta_2\left(1 - \frac{\delta_1}{4\mu}\right) = C_{\mathrm{Ko}}\mu, \quad \kappa_1 = \frac{1}{\mu}\left(1 - \frac{\kappa_2}{2\delta_1}\right) = \frac{1}{2\mu},\end{gathered} \tag{4.10}$$

which yields

$$\alpha_0(\Omega_{\mathrm{S}}) = \frac{1}{4\mu}, \quad \alpha_1(\Omega_{\mathrm{S}}) = \frac{1}{4\mu}\min\big\{C_{\mathrm{d}}(\Omega_{\mathrm{S}}), 1\big\}, \quad \alpha_2(\Omega_{\mathrm{S}}) = \alpha_3(\Omega_{\mathrm{S}}) = \frac{C_{\mathrm{Ko}}\mu}{2},$$

and hence

$$\alpha_{\mathrm{S}} = \min\left\{\frac{1}{4\mu}\min\big\{C_{\mathrm{d}}(\Omega_{\mathrm{S}}), 1\big\}, \frac{C_{\mathrm{Ko}}\mu}{2}\right\}.$$

The explicit values of the stabilization parameters $\kappa_i, i \in \{1, 2, 3\}$, given by (4.10), will be employed in Section 7 for the corresponding numerical experiments.

4.2 A fixed-point approach

We begin the solvability analysis of (3.5) by defining the operator $\mathbf{T} : \mathbf{H}^1_{\Gamma_S}(\Omega_S) \to \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ by

$$\mathbf{T}(\mathbf{w}_S) := \mathbf{u}_S \quad \forall\, \mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S), \tag{4.11}$$

where $\underline{\mathbf{t}} := (\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_S, \mathbf{u}_D)$ is the first component of the unique solution (to be confirmed below) of the nonlinear problem: Find $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall\, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ [\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \qquad \forall\, \underline{\mathbf{q}} \in \mathbb{Q}. \end{aligned} \tag{4.12}$$

Hence, it is not difficult to see that $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ is a solution of (3.5) if and only if $\mathbf{u}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ is a fixed-point of $\mathbf{T}$, that is

$$\mathbf{T}(\mathbf{u}_S) = \mathbf{u}_S. \tag{4.13}$$

In this way, in what follows we focus on proving that $\mathbf{T}$ possesses a unique fixed-point. However, we remark in advance that the definition of $\mathbf{T}$ will make sense only in a closed ball of $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$.

Before continuing with the solvability analysis of (4.13) (equivalently of (3.5)), we provide the hypotheses under which $\mathbf{T}$ is well defined. To that end, we first note that, given $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, the problem (4.12) has the same structure of the one in Theorem 3.4. Therefore, in what follows we apply this abstract result to establish the well-posedness of (4.12), or equivalently, the well-definiteness of $\mathbf{T}$. We begin by observing that, thanks to the uniform convexity and separability of $\mathrm{L}^p(\Omega)$ for $p \in (1, +\infty)$, each space defining $\mathbb{H}$ and $\mathbb{Q}$ shares the same properties, which implies that $\mathbb{H}$ and $\mathbb{Q}$ are uniformly convex and separable as well.

We continue our analysis by proving that hypothesis (i) of Theorem 3.4 is verified with $p_1 = 2$ and $p_2 = 3$.

Lemma 4.2 *Let $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$. Then, there exists $\gamma > 0$, depending on $C_{\mathcal{A}_S}, C_{\mathcal{B}_S}$ and $L_{\mathcal{A}_D}$ (cf. (4.1), (4.2), (4.3)), such that*

$$\begin{aligned} \|\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{r}})\|_{\mathbf{X}'} \leq \gamma \Big\{ \Big(1 + \|\mathbf{w}_S\|_{1,\Omega_S}\Big) \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)} \\ + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)} \Big(\|\mathbf{u}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)} + \|\mathbf{v}_D\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)} \Big) \Big\}, \end{aligned} \tag{4.14}$$

for all $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_D), \underline{\mathbf{r}} = (\underline{\boldsymbol{\tau}}, \mathbf{v}_D) \in \mathbf{X}$.

Proof. The result follows straightforwardly from the definition of $\mathbf{a}(\mathbf{w}_S)$ (cf. (3.7)), the triangle inequality, and the stability properties (4.1), (4.2) and (4.3). We omit further details. □

Now, let us look at the kernel of the operator $\mathbf{B}$ (cf. (3.11)), which can be written, equivalently, as

$$\mathbb{V} = \Big\{ (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H} : \quad [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}] = 0 \quad \forall\, \underline{\mathbf{q}} \in \mathbb{Q} \Big\} = \widetilde{\mathbf{X}} \times \widetilde{\mathbf{Y}}, \tag{4.15}$$

where

$$\widetilde{\mathbf{X}} = \mathbf{X}_1 \times \widetilde{\mathbf{X}}_2 \quad \text{and} \quad \widetilde{\mathbf{Y}} = \widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \times \mathrm{W}^{1/3,3/2}(\Sigma),$$

with

$$\widetilde{\mathbf{X}}_2 = \Big\{ \mathbf{v}_D \in \mathbf{H}^3_{\Gamma_D}(\mathrm{div}\,;\Omega_D) : \quad \mathrm{div}\,(\mathbf{v}_D) \in \mathrm{P}_0(\Omega_D) \Big\}$$

and

$$\widetilde{\mathbf{H}}_{00}^{1/2}(\Sigma) := \Big\{ \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \Big\}.$$

In addition, from a slight adaptation of [2, Lemma 4.2], we deduce that there exist a constant $C_{\text{div}} > 0$ such that

$$C_{\text{div}}\,\|\mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}^3(\text{div}\,;\Omega_{\mathrm{D}})} \le \|\mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{L}^3(\Omega_{\mathrm{D}})} \quad \forall\,\mathbf{v}_{\mathrm{D}} \in \widetilde{\mathbf{X}}_2. \tag{4.16}$$

Thus, in the following result we provide the assumptions under which operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}})$ satisfies hypothesis (ii) of Theorem 3.4.

Lemma 4.3 *Suppose that the parameters $\kappa_1,\kappa_2,\kappa_3$, satisfy the conditions required by* Lemma 4.1*, and let $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ be such that $\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le r$ with $r\in(0,r_0)$, and*

$$r_0 := \frac{\alpha_{\mathrm{S}}\,\mu}{\rho(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|^2}\,, \tag{4.17}$$

where $\|\mathbf{i}_c\|$ is the constant in (3.2) *and α_{S} is the ellipticity constant of the operator $\mathcal{A}_{\mathrm{S}}$ (cf.* (4.9)*). Then, for each $\underline{\mathbf{s}}\in\mathbf{X}$, the nonlinear operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\,\cdot + \underline{\mathbf{s}})$ is strictly monotone on $\widetilde{\mathbf{X}}$ (cf.* (4.15)*).*

Proof. Let $\underline{\mathbf{s}} := (\underline{\boldsymbol{\zeta}}, \mathbf{s}_{\mathrm{D}}) \in \mathbf{X}$ fixed, and let $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ as indicated. Then, according to the definition of $\mathbf{a}(\mathbf{w}_{\mathrm{S}})$ (cf. (3.7)), the linearity of $\mathcal{A}_{\mathrm{S}}$ and $\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})$, and combining (4.9), (4.6) and (4.16), we deduce that

$$\begin{aligned}[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}}+\underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{r}}+\underline{\mathbf{s}}), \underline{\mathbf{t}}-\underline{\mathbf{r}}] \;\ge\; &\alpha_{\mathrm{S}}\,\|\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}_1}\\ &+ \alpha_{\mathrm{D}}\,C_{\text{div}}\,\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}^3(\text{div}\,;\Omega_{\mathrm{D}})} + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}), (\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}})],\end{aligned}$$

for all $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_{\mathrm{D}})$, $\underline{\mathbf{r}} = (\underline{\boldsymbol{\tau}}, \mathbf{v}_{\mathrm{D}}) \in \widetilde{\mathbf{X}}$. In addition, similarly to [9, Lemma 3.1], we know from the second inequality in (4.2) that

$$\Big|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}]\Big| \le \frac{\rho}{2\mu}(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|^2\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}_1},$$

which implies

$$\begin{aligned}&[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}}+\underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{r}}+\underline{\mathbf{s}}), \underline{\mathbf{t}}-\underline{\mathbf{r}}]\\ &\quad\ge \Big\{\alpha_{\mathrm{S}} - \frac{\rho}{2\mu}(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|^2\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\Big\}\|\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}_1} + \alpha_{\mathrm{D}}\,C_{\text{div}}\,\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}^3(\text{div}\,;\Omega_{\mathrm{D}})}.\end{aligned}$$

Consequently, by requiring $\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le r_0$, with r_0 defined by (4.17), the foregoing inequality imply

$$[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{t}}+\underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\underline{\mathbf{r}}+\underline{\mathbf{s}}), \underline{\mathbf{t}}-\underline{\mathbf{r}}] \ge \alpha\Big\{\|\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}_1} + \|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}^3(\text{div}\,;\Omega_{\mathrm{D}})}\Big\}, \tag{4.18}$$

for all $\underline{\mathbf{t}}, \underline{\mathbf{r}} \in \widetilde{\mathbf{X}}$, with $\alpha := \min\Big\{\dfrac{\alpha_{\mathrm{S}}}{2}, \alpha_{\mathrm{D}}C_{\text{div}}\Big\}$ independent of $\mathbf{w}_{\mathrm{S}}$. □

We end the verification of the hypotheses of Theorem 3.4, with the positive semi-definiteness of $\mathbf{c}$ and corresponding inf-sup conditions for the operators $\mathbf{b}$ and $\mathbf{B}$, respectively.

Lemma 4.4 *There holds*

$$[\mathbf{c}(\underline{\boldsymbol{\psi}}), \underline{\boldsymbol{\psi}}] \ge 0 \quad \forall\,\underline{\boldsymbol{\psi}}\in\mathbf{Y}, \tag{4.19}$$

and there exists positive constants β_1 and β, such that

$$\sup_{\substack{\underline{\mathbf{r}}\in\widetilde{\mathbf{X}}\\ \underline{\mathbf{r}}\neq\mathbf{0}}} \frac{[\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]}{\|\underline{\mathbf{r}}\|_{\mathbf{X}}} \ge \beta_1\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \quad \forall\,\underline{\boldsymbol{\psi}}\in\widetilde{\mathbf{Y}}, \tag{4.20}$$

and

$$\sup_{\substack{(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\in\mathbb{H}\\(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\neq\mathbf{0}}}\frac{[\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{q}}]}{\|(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})\|_{\mathbb{H}}}\geq\beta\|\underline{\mathbf{q}}\|_{\mathbb{Q}}\quad\forall\underline{\mathbf{q}}\in\mathbb{Q}.\tag{4.21}$$

Proof. For the proof of (4.19) we refer the reader to [9, Lemma 3.3]. On the other hand, by using the diagonal character of the operators $\mathbf{b}$ and $\mathbf{B}$, the proof of (4.21) follows from a slight adaptation of [22, Lemma 3.6], whereas combining [22, Lemma 3.8] and [8, Lemma 3.5] we deduce (4.20). We omit further details. □

We are now in position of establishing the well-definiteness of $\mathbf{T}$. To that end, and in order to simplify the subsequent analysis, we first note that, given $\mathbf{w}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, there holds $\|\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{0})\|_{\mathbf{X}'}=0$. Then, by considering $\mathrm{p}_1=2$ and $\mathrm{p}_2=3$ in Theorem 3.4, we introduce the following notation

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}):=\max\Big\{\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^{1/8},\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^{1/4},\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^{1/2},$$
$$\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}),\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^2,\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^4\Big\},$$

with

$$\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}):=\|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}+\|\mathbf{f}_{\mathrm{D}}\|_{\mathbf{L}^{3/2}(\Omega_{\mathrm{D}})}+\|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}+\|g_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}.$$

The main result of this section is established now.

Lemma 4.5 *Suppose that the parameters $\kappa_1,\kappa_2,\kappa_3$, satisfy the conditions required by* Lemma 4.1. *Let $r\in(0,r_0)$, with r_0 given by* (4.17) *and let $\mathbf{f}_{\mathrm{S}}\in\mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}}\in\mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}}\in\mathrm{L}^2_0(\Omega_{\mathrm{D}})$. Then, the problem* (4.12) *has a unique solution $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}})\in\mathbb{H}\times\mathbb{Q}$ for each $\mathbf{w}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that $\|\mathbf{w}_{\mathrm{S}}\|\leq r$. Moreover, there exists a constant $c_{\mathbf{T}}>0$, independent of $\mathbf{w}_{\mathrm{S}}$ and the data $\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}}$, and g_{D}, such that*

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}}=\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\leq\|((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}})\|_{\mathbb{H}\times\mathbb{Q}}\leq c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}).\tag{4.22}$$

Proof. It follows from Lemmas 4.2–4.4 and a straightforward application of Theorem 3.4. In turn, estimate (4.22) is a direct consequence of (3.26) (cf. Theorem 3.4) and (4.4)-(4.5). □

4.3 Solvability analysis of the fixed-point equation

In this section we proceed analogously to [6, Section 3.3] (see also [13, 9, 8]), and establish existence of a fixed point of the operator $\mathbf{T}$ (cf. (4.11)) by means of the well known Schauder fixed-point theorem. The uniqueness can then be established by means of the Banach fixed-point theorem by utilizing the same estimates derived for the existence. We begin by recalling the first of the aforementioned results (see, e.g., [10, Theorem 9.12-1(b)]).

Theorem 4.6 *Let W be a closed and convex subset of a Banach space X, and let $T:W\to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

The verification of the hypotheses of Theorem 4.6 is provided next.

Lemma 4.7 *Let $r\in(0,r_0)$, with r_0 given by* (4.17), *let $\mathbf{W}_r$ be the closed ball defined by*

$$\mathbf{W}_r:=\Big\{\mathbf{w}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}):\quad\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\leq r\Big\},\tag{4.23}$$

and assume that the data satisfy

$$c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})\leq r,\tag{4.24}$$

with $c_{\mathbf{T}}$ the positive constant satisfying (4.22). *Then there holds $\mathbf{T}(\mathbf{W}_r)\subseteq\mathbf{W}_r$.*

Proof. It is straightforward consequence of Lemma 4.5. $\square$

We continue with the following results providing an estimate needed to derive the required continuity and compactness properties of the operator $\mathbf{T}$ (cf. (4.11)).

Lemma 4.8 *Let $r \in (0, r_0)$, with r_0 given by* (4.17)*, and let $\mathbf{W}_r$ given by* (4.23)*. Then there exists a positive constant $C_{\mathbf{T}}$, depending on $\kappa_2, \|\mathbf{i}_c\|$, and $\alpha_{\rm S}$ (cf.* (4.9)*), such that*

$$\|\mathbf{T}(\mathbf{w}_{\rm S}) - \mathbf{T}(\widetilde{\mathbf{w}}_{\rm S})\|_{1,\Omega_{\rm S}} \le C_{\mathbf{T}} \|\mathbf{T}(\widetilde{\mathbf{w}}_{\rm S})\|_{1,\Omega_{\rm S}} \|\mathbf{w}_{\rm S} - \widetilde{\mathbf{w}}_{\rm S}\|_{\mathbf{L}^4(\Omega_{\rm S})} \quad \forall\, \mathbf{w}_{\rm S}, \widetilde{\mathbf{w}}_{\rm S} \in \mathbf{W}_r. \tag{4.25}$$

Proof. Given $\mathbf{w}_{\rm S}, \widetilde{\mathbf{w}}_{\rm S} \in \mathbf{W}_r$, we let $\mathbf{u}_{\rm S} := \mathbf{T}(\mathbf{w}_{\rm S})$ and $\widetilde{\mathbf{u}}_{\rm S} := \mathbf{T}(\widetilde{\mathbf{w}}_{\rm S})$. According to the definition of $\mathbf{T}$ (cf. (4.12)), it follows that

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_{\rm S})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) - \mathbf{A}(\widetilde{\mathbf{w}}_{\rm S})(\widetilde{\underline{\mathbf{t}}}, \widetilde{\underline{\boldsymbol{\varphi}}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}} - \widetilde{\underline{\mathbf{p}}}] &= 0 \quad \forall\, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ [\mathbf{B}(\underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}, \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}), \underline{\mathbf{q}}] &= 0 \quad \forall\, \underline{\mathbf{q}} \in \mathbb{Q}. \end{aligned}$$

In particular, taking $\underline{\mathbf{r}} = \underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}$, $\underline{\boldsymbol{\psi}} = \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}$, and $\underline{\mathbf{q}} = \underline{\mathbf{p}} - \widetilde{\underline{\mathbf{p}}}$ in the latter system, and recalling the definition of $\mathbf{A}$ (cf. (3.6)), we get

$$\begin{aligned} [\mathbf{a}(\mathbf{w}_{\rm S})(\underline{\mathbf{t}}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\rm S})(\widetilde{\underline{\mathbf{t}}}), \underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}] + [\mathbf{b}(\underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}), \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}] &= 0, \\ [\mathbf{b}(\underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}), \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}] - [\mathbf{c}(\underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}), \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}] &= 0, \end{aligned}$$

which clearly implies

$$[\mathbf{a}(\mathbf{w}_{\rm S})(\underline{\mathbf{t}}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\rm S})(\widetilde{\underline{\mathbf{t}}}), \underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}] = -[\mathbf{c}(\underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}), \underline{\boldsymbol{\varphi}} - \widetilde{\underline{\boldsymbol{\varphi}}}], \tag{4.26}$$

where $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_{\rm D})$ and $\widetilde{\underline{\mathbf{t}}} = (\widetilde{\underline{\boldsymbol{\sigma}}}, \widetilde{\mathbf{u}}_{\rm D})$. Hence, adding and subtracting $\mathcal{B}_{\rm S}(\mathbf{w}_{\rm S})(\widetilde{\underline{\boldsymbol{\sigma}}})$ in the second term of the left-hand side of (4.26), noting that $\underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}} \in \widetilde{\mathbf{X}}$, and using the strict monotonicity of $\mathbf{a}(\mathbf{w}_{\rm S})$ (cf. (4.18)) and the fact that $\mathbf{c}$ is positive semi-definite (cf. (4.19)), it follows that

$$\frac{\alpha_{\rm S}}{2} \|\underline{\boldsymbol{\sigma}} - \widetilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 \le [\mathbf{a}(\mathbf{w}_{\rm S})(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{w}_{\rm S})(\widetilde{\underline{\mathbf{t}}}), \underline{\mathbf{t}} - \widetilde{\underline{\mathbf{t}}}] \le [\mathcal{B}_{\rm S}(\widetilde{\mathbf{w}}_{\rm S} - \mathbf{w}_{\rm S})(\widetilde{\underline{\boldsymbol{\sigma}}}), \underline{\boldsymbol{\sigma}} - \widetilde{\underline{\boldsymbol{\sigma}}}].$$

In this way, by applying the first inequality in (4.2) we deduce that

$$\|\underline{\boldsymbol{\sigma}} - \widetilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 \le \frac{\rho(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|}{\alpha_{\rm S}\mu} \|\widetilde{\mathbf{u}}_{\rm S}\|_{1,\Omega_{\rm S}} \|\mathbf{w}_{\rm S} - \widetilde{\mathbf{w}}_{\rm S}\|_{\mathbf{L}^4(\Omega_{\rm S})} \|\underline{\boldsymbol{\sigma}} - \widetilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1},$$

which implies (4.25) with

$$C_{\mathbf{T}} := \frac{\rho(1+\kappa_2^2)^{1/2}\|\mathbf{i}_c\|}{\alpha_{\rm S}\mu}, \tag{4.27}$$

thus completing the proof. $\square$

Owing to the above analysis, we establish now the announced properties of the operator $\mathbf{T}$.

Lemma 4.9 *Given $r \in (0, r_0)$, with r_0 defined by* (4.17)*, we let $\mathbf{W}_r$ as in* (4.23) *and assume that the data $\mathbf{f}_{\rm S}, \mathbf{f}_{\rm D}$ and $g_{\rm D}$ satisfy* (4.24)*. Then, $\mathbf{T} : \mathbf{W}_r \to \mathbf{W}_r$ is continuous and $\overline{\mathbf{T}(\mathbf{W}_r)}$ is compact.*

Proof. The required result follows straightforwardly from estimate (4.25) and the compactness of $\mathbf{i}_c : \mathbf{H}^1(\Omega_{\rm S}) \to \mathbf{L}^4(\Omega_{\rm S})$. We omit further details and refer to [6, Lemma 3.8]. $\square$

We are now in position of establishing the main result of this section.

Theorem 4.10 *Suppose that the parameters* $\kappa_1, \kappa_2, \kappa_3$ *satisfy the conditions required by* Lemma 4.1. *In addition, given* $r \in (0, r_0)$, *with* r_0 *defined by* (4.17), *we let* $\mathbf{W}_r$ *as in* (4.23), *and assume that the data* $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$ *and* g_{D} *satisfy* (4.24). *Then, the augmented fully-mixed formulation* (3.5) *has a unique solution* $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ *with* $\mathbf{u}_{\mathrm{S}} \in \mathbf{W}_r$, *and there holds*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{H} \times \mathbb{Q}} \le c_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}). \tag{4.28}$$

Proof. The equivalence between (3.5) and the fixed-point equation (4.11), together with Lemmas 4.7 and 4.9, confirm the existence of solution of (3.5) as a direct application of the Schauder fixed-point Theorem 4.6. In addition, it is clear that the estimate (4.28) follows straightforwardly from (4.22). On the other hand, from (4.25), the continuity of the compact injection $\mathbf{i}_c$ and the definition of $C_{\mathbf{T}}$ (cf. (4.27)), we obtain

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}}) - \mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \le \frac{r}{r_0} \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \quad \forall\, \mathbf{w}_{\mathrm{S}}, \widetilde{\mathbf{w}}_{\mathrm{S}} \in \mathbf{W}_r,$$

which, thanks to the Banach fixed-point theorem, implies that the solution is actually unique. □

5 The Galerkin scheme

In this section we introduce the Galerkin scheme of problem (3.5) and analyze its well-posedness by establishing suitable assumptions on the finite element subspaces involved.

5.1 Preliminaries

We first consider a set of arbitrary discrete subspaces, namely

$$\begin{gathered} \mathbf{H}_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}}), \quad \mathbf{H}^1_h(\Omega_{\mathrm{S}}) \subseteq \mathbf{H}^1(\Omega_{\mathrm{S}}), \quad \mathbb{L}_h(\Omega_{\mathrm{S}}) \subseteq \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}), \\ \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}), \quad \Lambda^{\mathrm{S}}_h(\Sigma) \subseteq \mathrm{H}^{1/2}_{00}(\Sigma), \quad \Lambda^{\mathrm{D}}_h(\Sigma) \subseteq \mathrm{W}^{1/3,3/2}(\Sigma), \quad \mathrm{L}_h(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}^2(\Omega_{\mathrm{D}}) \end{gathered} \tag{5.1}$$

and set

$$\begin{gathered} \mathbb{H}_h(\Omega_{\mathrm{S}}) := \Big\{ \boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}}) : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{c} \in \mathbb{R}^n \Big\}, \quad \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) := [\Lambda^{\mathrm{S}}_h(\Sigma)]^n, \\ \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) := \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}), \quad \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) := \mathbf{H}^1_h(\Omega_{\mathrm{S}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \\ \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) := \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}), \quad \mathrm{L}_{h,0}(\Omega_{\mathrm{D}}) := \mathrm{L}_h(\Omega_{\mathrm{D}}) \cap \mathrm{L}^2_0(\Omega_{\mathrm{D}}). \end{gathered} \tag{5.2}$$

Then, defining the global subspaces, unknowns, and test functions as follows

$$\begin{gathered} \mathbf{X}_{h,1} := \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}_h(\Omega_{\mathrm{S}}), \quad \mathbf{X}_{h,2} := \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}), \\ \mathbf{X}_h := \mathbf{X}_{h,1} \times \mathbf{X}_{h,2}, \quad \mathbf{Y}_h := \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) \times \Lambda^{\mathrm{D}}_h(\Sigma), \\ \mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \quad \text{and} \quad \mathbb{Q}_h := \mathrm{L}_{h,0}(\Omega_{\mathrm{D}}) \times \mathbb{R}, \end{gathered} \tag{5.3}$$

$$\begin{gathered} \underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}) \in \mathbf{X}_{h,1}, \quad \underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}) \in \mathbf{X}_{h,1}\,, \\ \underline{\mathbf{t}}_h := (\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{Y}_h, \quad \underline{\mathbf{p}}_h = (p_{\mathrm{D},h}, \ell_h) \in \mathbb{Q}_h, \\ \underline{\mathbf{r}}_h := (\underline{\boldsymbol{\tau}}_h, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{Y}_h, \quad \underline{\mathbf{q}}_h = (q_{\mathrm{D},h}, \jmath_h) \in \mathbb{Q}_h, \end{gathered}$$

the Galerkin scheme associated with problem (3.5) reads: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall\, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h, \\ [\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \qquad \forall\, \underline{\mathbf{q}}_h \in \mathbb{Q}_h. \end{aligned} \tag{5.4}$$

Now, we proceed similarly to [22] (see also [20, 9]), and derive suitable hypotheses on the spaces (5.1) ensuring the well-posedness of problem (5.4). We begin by noticing that, in order to have meaningful subspaces $\mathbb{H}_{h,0}(\Omega_{\mathrm{S}})$ and $\mathrm{L}_{h,0}(\Omega_{\mathrm{D}})$ we need to be able to eliminate multiplies of the identity matrix and constant polynomials from $\mathbb{H}_h(\Omega_{\mathrm{S}})$ and $\mathrm{L}_h(\Omega_{\mathrm{D}})$, respectively. This requirement is certainly satisfied if we assume:

(H.0) $\mathbf{P}_0(\Omega_{\mathrm{S}}) := [\mathrm{P}_0(\Omega_{\mathrm{S}})]^n \subseteq \mathbf{H}_h(\Omega_{\mathrm{S}})$ and $\mathrm{P}_0(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}_h(\Omega_{\mathrm{D}})$.

In particular, it follows that $\mathbb{I} \in \mathbb{H}_h(\Omega_{\mathrm{S}})$ for all h, and hence there holds the decomposition

$$\mathbb{H}_h(\Omega_{\mathrm{S}}) = \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) \oplus \mathrm{P}_0(\Omega_{\mathrm{S}})\, \mathbb{I}.$$

Next, we look at the discrete kernel of $\mathbf{B}$, which is given by

$$\mathbb{V}_h = \Big\{ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h : \quad [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h] = 0 \quad \forall\, \underline{\mathbf{q}}_h \in \mathbb{Q}_h \Big\}. \tag{5.5}$$

In order to have a more explicit definition of $\mathbb{V}_h$, we introduce the following assumption:

(H.1) $\operatorname{div} \mathbf{H}_h(\Omega_{\mathrm{D}}) \subseteq \mathrm{L}_h(\Omega_{\mathrm{D}})$.

Then, owing to **(H.1)** and recalling the definition of $\mathbf{B}$ (cf. (3.11)), it follows that $\mathbb{V}_h = \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h$, where

$$\widetilde{\mathbf{X}}_h = \mathbf{X}_{h,1} \times \widetilde{\mathbf{X}}_{h,2} \quad \text{and} \quad \widetilde{\mathbf{Y}}_h = \widetilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma) \times \Lambda_h^{\mathrm{D}}(\Sigma),$$

with

$$\widetilde{\mathbf{X}}_{h,2} := \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) : \quad \operatorname{div}(\mathbf{v}_{\mathrm{D},h}) \in \mathrm{P}_0(\Omega_{\mathrm{D}}) \Big\}$$

and

$$\widetilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma) := \Big\{ \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) : \quad \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \Big\}.$$

In particular, it readily follows that $\mathbb{V}_h \subseteq \mathbb{V}$.

On the other hand, for the subsequent analysis we need to ensure the discrete version of the inf-sup conditions (4.20) and (4.21) of $\mathbf{b}$ and $\mathbf{B}$ (cf. (3.9), (3.11)), respectively, namely the existence of constants $\widetilde{\beta}_1, \widetilde{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\underline{\mathbf{r}}_h \in \widetilde{\mathbf{X}}_h \\ \underline{\mathbf{r}}_h \neq \mathbf{0}}} \frac{[\mathbf{b}(\underline{\mathbf{r}}_h), \underline{\boldsymbol{\psi}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbf{X}}} \geq \widetilde{\beta}_1 \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}} \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \widetilde{\mathbf{Y}}_h \tag{5.6}$$

and

$$\sup_{\substack{(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h \\ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \neq \mathbf{0}}} \frac{[\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h]}{\|(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)\|_{\mathbb{H}}} \geq \widetilde{\beta}\, \|\underline{\mathbf{q}}_h\|_{\mathbb{Q}} \quad \forall\, \underline{\mathbf{q}}_h \in \mathbb{Q}_h. \tag{5.7}$$

For instance, applying the same diagonal argument utilized in [22, Section 3] (see also [20, 9, 8]), we deduce that $\mathbf{b}$ satisfies the discrete inf-sup condition (5.6) if and only if the following hypothesis holds:

(H.2) There exist $\widetilde{\beta}_{1,\mathrm{S}}, \widetilde{\beta}_{1,\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h}\in\mathbb{H}_{h,0}(\Omega_{\mathrm{S}})\\ \boldsymbol{\tau}_{\mathrm{S},h}\neq\mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}}} \geq \widetilde{\beta}_{1,\mathrm{S}}\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \quad \forall\,\boldsymbol{\psi}_h \in \widetilde{\boldsymbol{\Lambda}}_h^{\mathrm{S}}(\Sigma) \tag{5.8}$$

and

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h}\in\widetilde{\mathbf{X}}_{h,2}\\ \mathbf{v}_{\mathrm{D},h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} \geq \widetilde{\beta}_{1,\mathrm{D}}\|\xi_h\|_{1/3,3/2;\Sigma} \quad \forall\,\xi_h \in \Lambda_h^{\mathrm{D}}(\Sigma). \tag{5.9}$$

Similarly, employing the same arguments from [20, Section 5.2], we obtain that $\mathbf{B}$ satisfies the discrete inf-sup condition (5.7) provided that the following hypothesis holds:

(H.3) There exist $\widetilde{\beta}_{\mathrm{D}} > 0$, independent of h, and $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$, such that

$$\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}_h^{\mathrm{S}}(\Sigma) \quad \forall\,h \quad \text{and} \quad \langle \boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0\,, \tag{5.10}$$

and

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h}\in\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})\\ \mathbf{v}_{\mathrm{D},h}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{\mathrm{D},h}, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} \geq \widetilde{\beta}_{\mathrm{D}}\|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \quad \forall\,q_{\mathrm{D},h} \in \mathrm{L}_{h,0}(\Omega_{\mathrm{D}}). \tag{5.11}$$

5.2 Solvability analysis of the discrete problem

In what follows, we assume that hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)** hold, and, analogously to the analysis of the continuous problem, we apply a fixed-point argument to prove the well-posedness of the Galerkin scheme (5.4). To that end, we now let $\mathbf{T}_h : \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \to \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ be the discrete operator defined by

$$\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h}) := \mathbf{u}_{\mathrm{S},h} \quad \forall\,\mathbf{w}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \tag{5.12}$$

where $\underline{\mathbf{t}}_h := (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\gamma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h})$ is the first component of the unique solution (to be confirmed below) of the discrete nonlinear problem: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned}
[\mathbf{A}(\mathbf{w}_{\mathrm{S},h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall\,(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h,\\
[\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \qquad\quad \forall\,\underline{\mathbf{q}}_h \in \mathbb{Q}_h.
\end{aligned} \tag{5.13}$$

Then, the Galerkin scheme (5.4) can be rewritten, equivalently, as the fixed-point problem: Find $\mathbf{u}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}$ such that

$$\mathbf{T}_h(\mathbf{u}_{\mathrm{S},h}) = \mathbf{u}_{\mathrm{S},h}. \tag{5.14}$$

Next, similarly to the analysis developed in Section 4.2, in what follows we provide suitable assumptions under which problem (5.13) is well posed or equivalently $\mathbf{T}_h$ is well defined. For this purpose, we will require a discrete version of Theorem 3.4. In fact, we let $X_{h,1}, X_{h,2}, Y_h$ and Q_h be finite dimensional subspaces of X_1, X_2, Y and Q, respectively, and set $X_h = X_{h,1} \times X_{h,2} \subseteq X = X_1 \times X_2$ and $H_h = X_h \times Y_h \subseteq H = X \times Y$. Let $a_h : X_h \to X_h'$ be the discrete version of the nonlinear operator a. Thus, we define the nonlinear operator $A_h : H_h \to H_h'$, as:

$$A_h(\mathbf{r}_h, \boldsymbol{\psi}_h) := \begin{bmatrix} a_h & b' \\ b & -c \end{bmatrix} \begin{bmatrix} \mathbf{r}_h \\ \boldsymbol{\psi}_h \end{bmatrix} \in H_h' \quad \forall\,(\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h.$$

Next, we let V_h be the discrete kernel of B, that is,

$$V_h := \Big\{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h : \quad [B(\mathbf{r}_h, \boldsymbol{\psi}_h), \mathbf{q}_h] = 0 \quad \forall\, \mathbf{q}_h \in Q_h\Big\},$$

and let $\widetilde{X}_h$ and $\widetilde{Y}_h$ be subspaces of X_h and Y_h, respectively, such that $V_h = \widetilde{X}_h \times \widetilde{Y}_h$. Then, we establish the following preliminary result, which reduces to a simple application of Theorem 3.4 to the present discrete setting.

Theorem 5.1 *Assume that*

(i) *there exist constants* $\widetilde{\gamma} > 0$ *and* $\mathrm{p}_1, \mathrm{p}_2 \geq 2$*, such that*

$$\|a_h(\mathbf{t}_h) - a_h(\mathbf{r}_h)\|_{X'} \leq \widetilde{\gamma} \sum_{j=1}^{2} \left\{ \|\mathbf{t}_{j,h} - \mathbf{r}_{j,h}\|_{X_j} + \|\mathbf{t}_{j,h} - \mathbf{r}_{j,h}\|_{X_j} \Big(\|\mathbf{t}_{j,h}\|_{X_j} + \|\mathbf{r}_{j,h}\|_{X_j} \Big)^{r_j - 2} \right\},$$

for all $\mathbf{t}_h = (\mathbf{t}_{1,h}, \mathbf{t}_{2,h}), \mathbf{r}_h = (\mathbf{r}_{1,h}, \mathbf{r}_{2,h}) \in X_h$.

(ii) *for each* $\mathbf{s}_h \in X_h$*, the operator* $a_h(\cdot + \mathbf{s}_h) : \widetilde{X}_h \to \widetilde{X}_h'$ *is strictly monotone in the sense that there exist* $\widetilde{\alpha} > 0$ *and* $\mathrm{p}_1, \mathrm{p}_2 \geq 2$*, such that*

$$[a_h(\mathbf{t}_h + \mathbf{s}_h) - a_h(\mathbf{r}_h + \mathbf{s}_h), \mathbf{t}_h - \mathbf{r}_h] \geq \widetilde{\alpha} \Big\{ \|\mathbf{t}_{1,h} - \mathbf{r}_{1,h}\|_{X_1}^{\mathrm{p}_1} + \|\mathbf{t}_{2,h} - \mathbf{r}_{2,h}\|_{X_2}^{\mathrm{p}_2} \Big\},$$

for all $\mathbf{t}_h = (\mathbf{t}_{1,h}, \mathbf{t}_{2,h}), \mathbf{r}_h = (\mathbf{r}_{1,h}, \mathbf{r}_{2,h}) \in \widetilde{X}_h$.

(iii) c *is positive semi-definite on* $\widetilde{Y}_h$*, that is,*

$$[c(\boldsymbol{\psi}_h), \boldsymbol{\psi}_h] \geq 0 \quad \forall\, \boldsymbol{\psi}_h \in \widetilde{Y}_h.$$

(iv) b *satisfies an inf-sup condition on* $\widetilde{X}_h \times \widetilde{Y}_h$*, that is, there exists* $\widetilde{\beta}_1 > 0$ *such that*

$$\sup_{\substack{\mathbf{r}_h \in \widetilde{X}_h \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[b(\mathbf{r}_h), \boldsymbol{\psi}_h]}{\|\mathbf{r}_h\|_X} \geq \widetilde{\beta}_1 \|\boldsymbol{\psi}_h\|_Y \quad \forall\, \boldsymbol{\psi}_h \in \widetilde{Y}_h.$$

(v) B *satisfies an inf-sup condition on* $H_h \times Q_h$*, that is, there exists* $\widetilde{\beta} > 0$ *such that*

$$\sup_{\substack{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h \\ (\mathbf{r}_h, \boldsymbol{\psi}_h) \neq \mathbf{0}}} \frac{[B(\mathbf{r}_h, \boldsymbol{\psi}_h), \mathbf{q}_h]}{\|(\mathbf{r}_h, \boldsymbol{\psi}_h)\|_H} \geq \widetilde{\beta} \|\mathbf{q}_h\|_Q \quad \forall\, \mathbf{q}_h \in Q_h.$$

Then, for each $(F, G) \in H' \times Q'$ *there exists a unique* $((\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{p}_h) \in H_h \times Q_h$*, such that*

$$\begin{aligned} [A_h(\mathbf{t}_h, \boldsymbol{\varphi}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] + [B'(\mathbf{p}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] &= [F, (\mathbf{r}_h, \boldsymbol{\psi}_h)] \quad \forall\, (\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h, \\ [B(\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{q}_h] &= [G, \mathbf{q}_h] \qquad\quad \forall\, \mathbf{q}_h \in Q_h. \end{aligned}$$

Moreover, there exists $\widetilde{C} > 0$*, depending only on* $\widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\beta}, \widetilde{\beta}_1, \mathrm{p}_1, \mathrm{p}_2, \|b\|, \|b'\|$ *and* $\|c\|$*, such that*

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{p}_h)\|_{H \times Q} \leq \widetilde{C}\, \mathcal{M}(F_h, G_h),$$

where

$$\mathcal{M}(F_h, G_h) := \max \Big\{ \mathcal{N}(F_h, G_h)^{1/(\mathrm{p}_1 - 1)}, \mathcal{N}(F_h, G_h)^{1/(\mathrm{p}_2 - 1)}, \mathcal{N}(F_h, G_h),$$
$$\mathcal{N}(F_h, G_h)^{(\mathrm{p}_1 - 1)/(\mathrm{p}_2 - 1)}, \mathcal{N}(F_h, G_h)^{(\mathrm{p}_2 - 1)/(\mathrm{p}_1 - 1)} \Big\},$$

and $\mathcal{N}(F_h, G_h)$ *is defined in* (3.29)*, with* $F_h := F|_{H_h}$ *and* $G_h := G|_{Q_h}$.

The following lemma establishes the well-definiteness of operator $\mathbf{T}_h$ (cf. (5.12)).

Lemma 5.2 *Assume that hypotheses* **(H.0)**, **(H.1)**, **(H.2)** *and* **(H.3)** *hold. Assume further that the parameters* $\kappa_1, \kappa_2, \kappa_3$*, satisfy the conditions required by* Lemma 4.1. *Let* $r \in (0, r_0)$*, with* r_0 *defined by* (4.17)*, and let* $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S), \mathbf{f}_D \in \mathbf{L}^{3/2}(\Omega_D)$ *and* $g_D \in L^2_0(\Omega_S)$*. Then, problem* (5.13) *has a unique solution* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ *for each* $\mathbf{w}_{S,h} \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S)$ *such that* $\|\mathbf{w}_{S,h}\|_{1,\Omega_S} \le r$*. Moreover, there exists a constant* $\widetilde{c}_{\mathbf{T}}$ *independent of* $\mathbf{w}_{S,h}$ *and the data, such that*

$$\|\mathbf{T}_h(\mathbf{w}_{S,h})\|_{1,\Omega_S} \le \|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H}\times\mathbb{Q}} \le \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \tag{5.15}$$

Proof. Let $\mathbf{w}_{S,h} \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{1,\Omega_S} \le r$. Recalling that $\mathbb{H}_h \subseteq \mathbb{H}$, $\mathbb{Q}_h \subseteq \mathbb{Q}$ and $\mathbb{V}_h \subseteq \mathbb{V}$, a straightforward application of Lemmas 4.2 and 4.3 and (4.19), implies, respectively, that hypotheses (i), (ii) and (iii) in Theorem 5.1, hold. In turn, the inf-sup conditions (iv) and (v) follow from hypotheses **(H.2)** and **(H.3)**, respectively. Therefore, according to the above, a direct application of Theorem 5.1 allows us to conclude that there exists a unique $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ solution to (5.13) which satisfies (5.15), whit $\widetilde{c}_{\mathbf{T}}$ independent of $\mathbf{w}_{S,h}$ and h. □

We are now in position of establishing the well posedness of (5.4)

Theorem 5.3 *Assume that hypotheses* **(H.0)**, **(H.1)**, **(H.2)** *and* **(H.3)** *hold. Assume further that the parameters* $\kappa_1, \kappa_2, \kappa_3$ *satisfy the conditions required by* Lemma 4.1. *In addition, given* $r \in (0, r_0)$*, with* r_0 *defined by* (4.17)*, we let* $\mathbf{W}^h_r := \Big\{\mathbf{w}_{S,h} \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S) : \ \|\mathbf{w}_{S,h}\|_{1,\Omega_S} \le r\Big\}$*, and assume that the data* $\mathbf{f}_S, \mathbf{f}_D$ *and* g_D *satisfy*

$$\widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) \le r, \tag{5.16}$$

with $\widetilde{c}_{\mathbf{T}} > 0$ *the constant in* (5.15)*. Then, there exists a unique* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ *solution to* (5.4)*, which satisfies* $\mathbf{u}_{S,h} \in \mathbf{W}^h_r$*, and*

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H}\times\mathbb{Q}} \le \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \tag{5.17}$$

Proof. We first observe, owing to (5.15), that the assumption (5.16) guarantees that $\mathbf{T}_h(\mathbf{W}^h_r) \subseteq \mathbf{W}^h_r$. Next, analogously to the proof of Lemma 4.8, that is, applying the strict monotonicity of $\mathbf{a}(\mathbf{w}_{S,h})$, for each $\mathbf{w}_{S,h} \in \mathbf{W}^h_r$, we find that

$$\|\mathbf{T}_h(\mathbf{w}_{S,h}) - \mathbf{T}_h(\widetilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S} \le C_{\mathbf{T}}\|\mathbf{i}_c\|\|\mathbf{T}_h(\widetilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S}\|\mathbf{w}_{S,h} - \widetilde{\mathbf{w}}_{S,h}\|_{1,\Omega_S} \quad \forall\, \mathbf{w}_{S,h}, \widetilde{\mathbf{w}}_{S,h} \in \mathbf{W}^h_r,$$

which, together with (4.27), (5.15), (5.16) and (4.17), implies

$$\|\mathbf{T}_h(\mathbf{w}_{S,h}) - \mathbf{T}_h(\widetilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S} \le \frac{r}{r_0}\|\mathbf{w}_{S,h} - \widetilde{\mathbf{w}}_{S,h}\|_{1,\Omega_S} \quad \forall\, \mathbf{w}_{S,h}, \widetilde{\mathbf{w}}_{S,h} \in \mathbf{W}^h_r,$$

thus confirming that $\mathbf{T}_h : \mathbf{W}^h_r \to \mathbf{W}^h_r$ is a contraction mapping. Then, the Banach fixed-point theorem and the equivalence between (5.4) and (5.14) imply the well-posedness of (5.4). In turn, the estimate (5.17) follows directly from (5.15). □

5.3 *A priori* error analysis

In this section we establish the corresponding Céa estimate. For this purpose, we first introduce some notations and state a couple of previous results. We begin by recalling the discrete inf-sup condition of $\mathbf{B}$ (cf. (5.7)), and a classical result on mixed methods (see, for instance [17, Theorem 2.6]) ensuring the existence of a constant $c>0$, independent of h, such that:

$$\operatorname{dist}\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\mathbb{V}_h\big)\le c\operatorname{dist}\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\mathbb{H}_h\big). \tag{5.18}$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_{\underline{\boldsymbol{\sigma}}}=\underline{\boldsymbol{\sigma}}-\underline{\boldsymbol{\sigma}}_h$, $\mathbf{e}_{\mathbf{u}_{\rm D}}=\mathbf{u}_{\rm D}-\mathbf{u}_{{\rm D},h}$, $\mathbf{e}_{\underline{\boldsymbol{\varphi}}}=\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}}_h$ and $\mathbf{e}_{\underline{\mathbf{p}}}=\underline{\mathbf{p}}-\underline{\mathbf{p}}_h$. Then, proceeding similarly to [2, Section 3.3], we consider the unique decompositions $\underline{\boldsymbol{\sigma}}_h=\widetilde{\underline{\boldsymbol{\sigma}}}_h+\widetilde{\underline{\boldsymbol{\sigma}}}_h^{\perp}$, $\mathbf{u}_{{\rm D},h}=\widetilde{\mathbf{u}}_{{\rm D},h}+\widetilde{\mathbf{u}}_{{\rm D},h}^{\perp}$ and $\underline{\boldsymbol{\varphi}}_h=\widetilde{\underline{\boldsymbol{\varphi}}}_h+\widetilde{\underline{\boldsymbol{\varphi}}}_h^{\perp}$, with $((\widetilde{\underline{\boldsymbol{\sigma}}}_h,\widetilde{\mathbf{u}}_{{\rm D},h}),\widetilde{\underline{\boldsymbol{\varphi}}}_h)\in\mathbb{V}_h$ and $((\widetilde{\underline{\boldsymbol{\sigma}}}_h^{\perp},\widetilde{\mathbf{u}}_{{\rm D},h}^{\perp}),\widetilde{\underline{\boldsymbol{\varphi}}}_h^{\perp})\in\widetilde{\mathbf{X}}_h^{\perp}\times\widetilde{\mathbf{Y}}_h^{\perp}=:\mathbb{V}_h^{\perp}$, where

$$\mathbb{V}_h^{\perp}=\Big\{(\underline{\mathbf{s}}_h,\underline{\boldsymbol{\phi}}_h)\in\mathbb{H}_h:\quad\Big\langle(\underline{\mathbf{s}}_h,\underline{\boldsymbol{\phi}}_h),(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h)\Big\rangle=0\quad\forall\,(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h)\in\mathbb{V}_h\Big\}.$$

Next, given arbitrary $((\widetilde{\underline{\boldsymbol{\tau}}}_h,\widetilde{\mathbf{v}}_{{\rm D},h}),\widetilde{\underline{\boldsymbol{\psi}}}_h)\in\mathbb{V}_h$ and $\widetilde{\underline{\mathbf{q}}}_h\in\mathbb{Q}_h$, we decompose the errors into

$$\mathbf{e}_{\underline{\boldsymbol{\sigma}}}=\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}-\widetilde{\underline{\boldsymbol{\sigma}}}_h^{\perp}+\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}},\quad \mathbf{e}_{\mathbf{u}_{\rm D}}=\boldsymbol{\delta}_{\mathbf{u}_{\rm D}}-\widetilde{\mathbf{u}}_{{\rm D},h}^{\perp}+\boldsymbol{\eta}_{\mathbf{u}_{\rm D}},\quad \mathbf{e}_{\underline{\boldsymbol{\varphi}}}=\boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}-\widetilde{\underline{\boldsymbol{\varphi}}}_h^{\perp}+\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}\quad\text{and}\quad \mathbf{e}_{\underline{\mathbf{p}}}=\boldsymbol{\delta}_{\underline{\mathbf{p}}}+\boldsymbol{\eta}_{\underline{\mathbf{p}}}, \tag{5.19}$$

with

$$\begin{gathered}\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}=\underline{\boldsymbol{\sigma}}-\widetilde{\underline{\boldsymbol{\tau}}}_h,\quad \boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}=\widetilde{\underline{\boldsymbol{\tau}}}_h-\widetilde{\underline{\boldsymbol{\sigma}}}_h,\quad \boldsymbol{\delta}_{\mathbf{u}_{\rm D}}=\mathbf{u}_{\rm D}-\widetilde{\mathbf{v}}_{{\rm D},h},\quad \boldsymbol{\eta}_{\mathbf{u}_{\rm D}}=\widetilde{\mathbf{v}}_{{\rm D},h}-\widetilde{\mathbf{u}}_{{\rm D},h},\\ \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}=\underline{\boldsymbol{\varphi}}-\widetilde{\underline{\boldsymbol{\psi}}}_h,\quad \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}=\widetilde{\underline{\boldsymbol{\psi}}}_h-\widetilde{\underline{\boldsymbol{\varphi}}}_h,\quad \boldsymbol{\delta}_{\underline{\mathbf{p}}}=\underline{\mathbf{p}}-\widetilde{\underline{\mathbf{q}}}_h,\quad \boldsymbol{\eta}_{\underline{\mathbf{p}}}=\widetilde{\underline{\mathbf{q}}}_h-\underline{\mathbf{p}}_h.\end{gathered} \tag{5.20}$$

Consequently, the following Galerkin orthogonality property holds:

$$\begin{aligned}{}[\mathbf{A}(\mathbf{u}_{\rm S})(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}})-\mathbf{A}(\mathbf{u}_{{\rm S},h})(\underline{\mathbf{t}}_h,\underline{\boldsymbol{\varphi}}_h),(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h)]+[\mathbf{B}(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h),\mathbf{e}_{\underline{\mathbf{p}}}]&=0,\\ [\mathbf{B}((\mathbf{e}_{\underline{\boldsymbol{\sigma}}},\mathbf{e}_{\mathbf{u}_{\rm D}}),\mathbf{e}_{\underline{\boldsymbol{\varphi}}}),\underline{\mathbf{q}}_h]&=0,\end{aligned} \tag{5.21}$$

for all $(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h):=((\underline{\boldsymbol{\tau}}_h,\mathbf{v}_{{\rm D},h}),\underline{\boldsymbol{\psi}}_h)\in\mathbb{H}_h$ and $\underline{\mathbf{q}}_h\in\mathbb{Q}_h$.

We now establish the main result of this section.

Theorem 5.4 *Assume that the hypotheses* $(\mathbf{H.0}),(\mathbf{H.1}),(\mathbf{H.2})$*, and* $(\mathbf{H.3})$*, as well as the conditions on* $\kappa_1,\kappa_2,\kappa_3$ *required by* Lemma 4.1*, hold. Let* $r\in(0,r_0)$ *with* r_0 *defined by* (4.17)*, and assume further that the data* $\mathbf{f}_{\rm S},\mathbf{f}_{\rm D}$*, and* $g_{\rm D}$ *satisfy*

$$c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\rm S},\mathbf{f}_{\rm D},g_{\rm D})\le\frac{r}{2}, \tag{5.22}$$

with $c_{\mathbf{T}}$ *the constant satisfying* (4.22)*. Let* $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}):=((\underline{\boldsymbol{\sigma}},\mathbf{u}_{\rm D},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}})\in\mathbb{H}\times\mathbb{Q}$ *with* $\underline{\boldsymbol{\sigma}}:=(\boldsymbol{\sigma}_{\rm S},\mathbf{u}_{\rm S},\boldsymbol{\gamma}_{\rm S})$ *and* $\mathbf{u}_{\rm S}\in\mathbf{W}_r$*, and* $((\underline{\mathbf{t}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{p}}_h):=((\underline{\boldsymbol{\sigma}}_h,\mathbf{u}_{{\rm D},h},\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{p}}_h)\in\mathbb{H}_h\times\mathbb{Q}_h$ *with* $\underline{\boldsymbol{\sigma}}_h:=(\boldsymbol{\sigma}_{{\rm S},h},\mathbf{u}_{{\rm S},h},\boldsymbol{\gamma}_{{\rm S},h})$ *and* $\mathbf{u}_{{\rm S},h}\in\mathbf{W}_r^h$*, be the unique solutions of problems* (3.5) *and* (5.4)*, respectively. Then there exists* $C>0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned}&\big\|\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}\big)-\big((\underline{\mathbf{t}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{p}}_h\big)\big\|_{\mathbb{H}\times\mathbb{Q}}\\ &\quad\le C\Big\{\operatorname{dist}\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\mathbb{H}_h\big)^{1/2}+\operatorname{dist}\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\mathbb{H}_h\big)+\operatorname{dist}\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\mathbb{H}_h\big)^2+\operatorname{dist}\big(\underline{\mathbf{p}},\mathbb{Q}_h\big)\Big\}.\end{aligned} \tag{5.23}$$

Proof. Given $((\widetilde{\underline{\boldsymbol{\tau}}}_h,\widetilde{\mathbf{v}}_{{\rm D},h}),\widetilde{\underline{\boldsymbol{\psi}}}_h)\in\mathbb{V}_h$, and $\widetilde{\underline{\mathbf{q}}}_h\in\mathbb{Q}_h$, we define $\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}},\boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}},\boldsymbol{\delta}_{\underline{\mathbf{p}}},\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}},\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}$, and $\boldsymbol{\eta}_{\underline{\mathbf{p}}}$, as in (5.20). In turn, since $((\widetilde{\underline{\boldsymbol{\sigma}}}_h,\widetilde{\mathbf{u}}_{{\rm D},h}),\widetilde{\underline{\boldsymbol{\varphi}}}_h)\in\mathbb{V}_h$, it follows that $((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}),\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}})\in\mathbb{V}_h$. Thus, from the second equation in (5.21), (5.19), and the fact that $[\mathbf{B}((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}),\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}),\underline{\mathbf{q}}_h]=0$, it follows that

$$[\mathbf{B}((\widetilde{\underline{\boldsymbol{\sigma}}}_h^{\perp},\widetilde{\mathbf{u}}_{{\rm D},h}^{\perp}),\widetilde{\underline{\boldsymbol{\varphi}}}_h^{\perp}),\underline{\mathbf{q}}_h]=[\mathbf{B}((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}}),\boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}),\underline{\mathbf{q}}_h],$$

which together with the continuity and discrete inf-sup condition of $\mathbf{B}$ (cf. (4.1) and (5.7)), (5.19), (5.20), and the triangle inequality, yields

$$\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}) - ((\widetilde{\underline{\boldsymbol{\sigma}}}_h^\perp, \widetilde{\mathbf{u}}_{\mathrm{D},h}^\perp), \widetilde{\underline{\boldsymbol{\varphi}}}_h^\perp)\|_{\mathbb{H}} \leq \left(1 + \frac{C_\mathbf{B}}{\widetilde{\beta}}\right) \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}. \tag{5.24}$$

On the other hand, taking $(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) := ((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}) \in \mathbb{V}_h \subset \mathbb{H}_h$ in the first row of (5.21), recalling the definition of the operator $\mathbf{A}(\mathbf{u}_\mathrm{S})$ (cf. (3.6)), observing that thanks to $(\mathbf{H.1})$ we have that $\mathbb{V}_h \subseteq \mathbb{V}$, we deduce

$$\begin{aligned} [\mathbf{a}(\mathbf{u}_\mathrm{S})(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h), (\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})] + [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \mathbf{e}_{\underline{\boldsymbol{\varphi}}}] &= 0, \\ [\mathbf{b}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}, \mathbf{e}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}] - [\mathbf{c}(\mathbf{e}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}] &= 0. \end{aligned} \tag{5.25}$$

In this way, from (5.19) and the first equation of (5.25), we find that

$$\begin{aligned} [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}] = -\Big\{ & [\mathbf{a}(\mathbf{u}_\mathrm{S})(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{u}_\mathrm{S})(\underline{\mathbf{t}}_h), (\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})] + [\mathcal{B}_\mathrm{S}(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h})(\underline{\boldsymbol{\sigma}}_h), \boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}] \\ & + [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}} - \widetilde{\underline{\boldsymbol{\varphi}}}_h^\perp] \Big\}. \end{aligned}$$

Hence, noting that $(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}) \in \widetilde{\mathbf{X}}_h$, employing the discrete inf-sup condition of $\mathbf{b}$ (cf. (5.6)), inequality (4.14), and the continuity of $\mathbf{b}$ and $\mathbf{B}$ (cf. (4.1)), and then applying the first inequality in (4.2) and bounding $\|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|_{\mathbf{L}^4(\Omega_\mathrm{S})}$ by $\|\mathbf{i}_c\| \|\mathbf{e}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}$, we deduce that there exist a constant $C_1 > 0$, independent of h, such that

$$\begin{aligned} \widetilde{\beta}_1 \|\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}\|_\mathbf{Y} \leq C_1 \Big\{ & \Big(1 + \|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}\Big) \|\mathbf{e}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1} \\ & + \Big(1 + \|\mathbf{u}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + \|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big) \|\mathbf{e}_{\mathbf{u}_\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + \|\boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}} - \widetilde{\underline{\boldsymbol{\varphi}}}_h^\perp\|_\mathbf{Y} \Big\}. \end{aligned} \tag{5.26}$$

Then, recalling that both $\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}$ and $\|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}$ are bounded by r_0 (cf. (4.17)), as well as that both $\|\mathbf{u}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}$ and $\|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}$ are bounded by data (cf. Theorems 4.10 and 5.3), the estimate (5.26) together with (5.19), (5.20), and (5.24) allow us to conclude that

$$\|\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}\|_\mathbf{Y} \leq C_2 \Big\{ \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})\|_\mathbf{X} \Big\}, \tag{5.27}$$

with $C_2 > 0$ depending only on parameters, data and other constants, all of them independent of h. In turn, noting that $\underline{\mathbf{t}}_h = \widetilde{\underline{\mathbf{t}}}_h + \widetilde{\underline{\mathbf{t}}}_h^\perp$ with $\widetilde{\underline{\mathbf{t}}}_h = (\widetilde{\underline{\boldsymbol{\sigma}}}_h, \widetilde{\mathbf{u}}_{\mathrm{D},h}) \in \widetilde{\mathbf{X}}_h$ and $\widetilde{\underline{\mathbf{t}}}_h^\perp = (\widetilde{\underline{\boldsymbol{\sigma}}}_h^\perp, \widetilde{\mathbf{u}}_{\mathrm{D},h}^\perp) \in \widetilde{\mathbf{X}}_h^\perp$, and combining the first and second equation of (5.25), we are able to find that

$$\begin{aligned} [\mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\widetilde{\underline{\mathbf{r}}}_h + \widetilde{\underline{\mathbf{t}}}_h^\perp) - \mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\widetilde{\underline{\mathbf{t}}}_h + \widetilde{\underline{\mathbf{t}}}_h^\perp), (\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})] = {} & [\mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\widetilde{\underline{\mathbf{r}}}_h + \widetilde{\underline{\mathbf{t}}}_h^\perp) - \mathbf{a}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}), (\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}})] \\ & - [\mathcal{B}_\mathrm{S}(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h})(\underline{\boldsymbol{\sigma}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}] - [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \mathbf{e}_{\underline{\boldsymbol{\varphi}}}], \end{aligned}$$

where

$$\begin{aligned} [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \mathbf{e}_{\underline{\boldsymbol{\varphi}}}] = {} & [\mathbf{b}(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_\mathrm{D}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}} - \widetilde{\underline{\boldsymbol{\varphi}}}_h^\perp] - [\mathbf{b}((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}}) - (\widetilde{\underline{\boldsymbol{\sigma}}}_h^\perp, \widetilde{\mathbf{u}}_{\mathrm{D},h}^\perp), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}] \\ & + [\mathbf{c}(\boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}} - \widetilde{\underline{\boldsymbol{\varphi}}}_h^\perp), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}] + [\mathbf{c}(\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}]. \end{aligned}$$

Next, using the second inequality in (4.2), bounding $\|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}$ by $\|\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}} - \widetilde{\underline{\boldsymbol{\sigma}}}_h^\perp\|_{\mathbf{X}_1} + \|\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}$, recalling from (4.28) that $\|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1} \leq c_\mathbf{T} \mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})$, and employing assumption (5.22), we have

$$[\mathcal{B}_\mathrm{S}(\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h})(\underline{\boldsymbol{\sigma}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}] \leq \frac{\alpha_\mathrm{S}}{4} \Big\{ \|\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 + \|\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}} - \widetilde{\underline{\boldsymbol{\sigma}}}_h^\perp\|_{\mathbf{X}_1} \|\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1} \Big\}.$$

Hence, using the strict monotonicity of $\mathbf{a}(\mathbf{u}_{\mathrm{S},h})$ (cf. Lemma 4.3), the continuity of $\mathbf{b}, \mathbf{c}$, and $\mathbf{B}$ (cf. (4.1)), the inequalities (4.14) and (5.24), and the positive semi-definiteness property of $\mathbf{c}$ (cf. (4.19)), we deduce that there exist $C_3, C_4 > 0$ independents of h, such that

$$
\begin{aligned}
&\frac{\alpha_{\mathrm{S}}}{4}\|\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}_1} + \alpha_{\mathrm{D}} C_{\mathrm{div}}\,\|\boldsymbol{\eta}_{\mathbf{u}_{\mathrm{D}}}\|^3_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} \\
&\quad \le C_3\left\{\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^2_{\mathbb{H}}\right\}\|(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_{\mathrm{D}}})\|_{\mathbf{X}} + C_4\,\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}\|\boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}},
\end{aligned}
$$

which together with (5.27), Young's inequality and simple algebraic manipulations, yield

$$
\|(\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_{\mathrm{D}}})\|_{\mathbf{X}} \le \widetilde{c}\left\{\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^{1/2}_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^2_{\mathbb{H}}\right\}. \tag{5.28}
$$

In this way, employing (5.19), (5.24), (5.27), (5.28), and the triangle inequality, we obtain

$$
\begin{aligned}
\|((\mathbf{e}_{\underline{\boldsymbol{\sigma}}}, \mathbf{e}_{\mathbf{u}_{\mathrm{D}}}), \mathbf{e}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} &\le \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}) - ((\underline{\widetilde{\boldsymbol{\sigma}}}^{\perp}_h, \widetilde{\mathbf{u}}^{\perp}_{\mathrm{D},h}), \underline{\widetilde{\boldsymbol{\varphi}}}^{\perp}_h)\|_{\mathbb{H}} + \|((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} \\
&\le \widetilde{C}\left\{\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^{1/2}_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^2_{\mathbb{H}}\right\}.
\end{aligned} \tag{5.29}
$$

In turn, in order to estimate $\mathbf{e}_{\underline{\mathbf{p}}}$, we first observe from (5.19) and the first row of (5.21), that

$$
\begin{aligned}
[\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \boldsymbol{\eta}_{\underline{\mathbf{p}}}] = -\Big\{&[\mathbf{A}(\mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) - \mathbf{A}(\mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \\
&+ [\mathcal{B}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h})(\underline{\boldsymbol{\sigma}}_h), \underline{\boldsymbol{\tau}}_h] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \boldsymbol{\delta}_{\underline{\mathbf{p}}}]\Big\}.
\end{aligned}
$$

Then, proceeding similarly to (5.26), we employ again the discrete inf-sup condition of $\mathbf{B}$ (cf. (5.7)), the definition of $\mathbf{A}(\mathbf{u}_{\mathrm{S}})$ (cf. (3.6)), the inequality (4.14), and the continuity of $\mathbf{b}, \mathbf{c}, \mathbf{B}$, and $\mathcal{B}_{\mathrm{S}}$ (cf. (4.1), (4.2)), to obtain

$$
\begin{aligned}
\widetilde{\beta}\|\boldsymbol{\eta}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}} \le C_5\Big\{&\Big(1 + \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S},h}}\Big)\|\mathbf{e}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1} \\
&+ \Big(1 + \|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big)\|\mathbf{e}_{\mathbf{u}_{\mathrm{D}}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{e}_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}} + \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}}\Big\}.
\end{aligned}
$$

Thus, using again that both $\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}$ and $\|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}$ are bounded by r_0 (cf. (4.17)), as well as that both $\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}$ and $\|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}$ are bounded by data (cf. Theorems 4.10 and 5.3), the decomposition (5.19), the triangle inequality, the foregoing bound and (5.29), yields

$$
\begin{aligned}
\|\mathbf{e}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}} &\le \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}} + \|\boldsymbol{\eta}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}} \\
&\le \widehat{C}\left\{\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^{1/2}_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\mathbf{u}_{\mathrm{D}}}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|^2_{\mathbb{H}} + \|\boldsymbol{\delta}_{\underline{\mathbf{p}}}\|_{\mathbb{Q}}\right\}.
\end{aligned} \tag{5.30}
$$

Finally, recalling that $((\underline{\widetilde{\boldsymbol{\tau}}}_h, \widetilde{\mathbf{v}}_{\mathrm{D},h}), \underline{\widetilde{\boldsymbol{\psi}}}_h) \in \mathbb{V}_h$, and $\widetilde{\mathbf{q}}_h \in \mathbb{Q}_h$ are arbitrary, (5.29) and (5.30) give

$$
\begin{aligned}
&\big\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\big\|_{\mathbb{H}\times\mathbb{Q}} \\
&\quad \le C\left\{\mathrm{dist}\big((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h\big)^{1/2} + \mathrm{dist}\big((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h\big) + \mathrm{dist}\big((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h\big)^2 + \mathrm{dist}\,(\underline{\mathbf{p}}, \mathbb{Q}_h)\right\},
\end{aligned}
$$

which together with (5.18), concludes the proof. □

6 A particular choice of finite element subspaces

We now introduce specific discrete spaces satisfying hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**) in 2D and 3D. To this end, we let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles (in $\mathbb{R}^2$) or tetrahedra (in $\mathbb{R}^3$), and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). Then for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we set the local Raviart–Thomas space of order k as

$$\mathrm{RT}_k(T) := \mathbf{P}_k(T) \oplus \mathrm{P}_k(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, ..., x_n)^t$ is a generic vector of $\mathbb{R}^n$.

6.1 Raviart–Thomas elements in 2D

We define the discrete subspaces in (5.1) as follows:

$$\begin{aligned}
\mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \Big\{ \tau_{\mathrm{S},h} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}}) : \quad \tau_{\mathrm{S},h}|_T \in \mathrm{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathbf{H}_h^1(\Omega_{\mathrm{S}}) &:= \Big\{ \mathbf{v}_{\mathrm{S},h} \in [\mathcal{C}(\overline{\Omega}_{\mathrm{S}})]^2 : \quad \mathbf{v}_{\mathrm{S},h}|_T \in \mathbf{P}_{k+1}(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathbb{L}_h(\Omega_{\mathrm{S}}) &:= \Big\{ \boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}) : \quad \boldsymbol{\eta}_{\mathrm{S},h}|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}} \Big\}, \\
\mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D},h}|_T \in \mathrm{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \Big\}, \\
\mathrm{L}_h(\Omega_{\mathrm{D}}) &:= \Big\{ q_{\mathrm{D},h} \in \mathrm{L}^2(\Omega_{\mathrm{D}}) : \quad q_{\mathrm{D},h}|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}} \Big\}.
\end{aligned} \tag{6.1}$$

In addition, in order to introduce the particular subspaces $\Lambda_h^{\mathrm{S}}(\Sigma)$ and $\Lambda_h^{\mathrm{D}}(\Sigma)$, we follow the simplest approach suggested in [22] and [8], respectively. In fact, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ that arises by joining pairs of adjacent edges of Σ_h, and denote the resulting edges still by e. Since Σ_h is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h is odd, we simply reduce it to even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this modified partition. Hence, denoting by x_0 and x_N the extreme points of Σ, we define

$$\Lambda_h^{\mathrm{S}}(\Sigma) := \Big\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_e \in \mathrm{P}_{k+1}(e) \quad \forall \text{ edge } e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \Big\}. \tag{6.2}$$

In turn, since the space $\prod_{e \in \Sigma_h} \mathrm{W}^{1-1/\mathrm{p},\mathrm{p}}(e)$ coincides with $\mathrm{W}^{1-1/\mathrm{p},\mathrm{p}}(\Sigma)$, without extra conditions when $1 < \mathrm{p} < 2$ (in this case $\mathrm{p} = 3/2$) [27, Theorem 1.5.2.3-(a)], it can be readily seen that a conforming finite element subspace for $\mathrm{W}^{1/3,3/2}(\Sigma)$ can be defined by

$$\Lambda_h^{\mathrm{D}}(\Sigma) := \Big\{ \xi_h : \Sigma \to \mathbb{R} : \quad \xi_h|_e \in \mathrm{P}_k(e) \quad \forall \text{ edge } e \in \Sigma_h \Big\}. \tag{6.3}$$

Then, we define the global spaces $\mathbb{H}_h$ and $\mathbb{Q}_h$ (cf. (5.3)), by combining (5.2), (5.3), (6.1), (6.2), and (6.3). Now, concerning hypotheses (**H.0**)-(**H.3**), we start mentioning that (**H.0**) and (**H.1**) are straightforward from the definitions in (6.1). In turn, the discrete inf-sup condition (5.8) in (**H.2**) can be derived by combining the results in [30, Theorem A.1] and [22, Section 5.2]. In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ satisfying (5.10) in (**H.3**) follows as explained in [23, Section 3.2] (see also [22, Section 5.3]). Finally, the inf-sup condition (5.9) in (**H.2**) follows from [8, Lemma 4.5], whereas (5.11) in (**H.3**) follows from a slight adaptation of [8, Lemma 4.6].

6.2 Raviart–Thomas elements in 3D

Let us now consider the discrete spaces:

$$
\begin{aligned}
\mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \Big\{\tau_{\mathrm{S},h} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\mathrm{S}}): \quad \tau_{\mathrm{S},h}|_T \in \mathrm{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbf{H}^1_h(\Omega_{\mathrm{S}}) &:= \Big\{\mathbf{v}_{\mathrm{S},h} \in [\mathcal{C}(\overline{\Omega}_{\mathrm{S}})]^3: \quad \mathbf{v}_{\mathrm{S},h}|_T \in \mathbf{P}_{k+1}(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbb{L}_h(\Omega_{\mathrm{S}}) &:= \Big\{\boldsymbol{\eta}_{\mathrm{S},h} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_{\mathrm{S}}): \quad \boldsymbol{\eta}_{\mathrm{S},h}|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \Big\{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}): \quad \mathbf{v}_{\mathrm{D},h}|_T \in \mathrm{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\},\\
\mathrm{L}_h(\Omega_{\mathrm{D}}) &:= \Big\{q_{\mathrm{D},h} \in \mathrm{L}^2(\Omega_{\mathrm{D}}): \quad q_{\mathrm{D},h}|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{D}}\Big\}.
\end{aligned} \tag{6.4}
$$

Now, proceeding analogously to the 2D case, we introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}$, and define $\widehat{h}_\Sigma := \big\{\widehat{h}_K : K \in \Sigma_{\widehat{h}}\big\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$
\Lambda_h^{\mathrm{S}}(\Sigma) := \Big\{\psi_h \in \mathcal{C}(\Sigma): \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \psi_h = 0 \quad \text{on} \quad \partial\Sigma\Big\}. \tag{6.5}
$$

In turn, similarly to (6.3), we deduce now from [26, Section 2] that a conforming finite element subspace for $\mathrm{W}^{1/3,3/2}(\Sigma)$ can be defined by

$$
\Lambda_h^{\mathrm{D}}(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R}: \quad \xi_h|_e \in \mathrm{P}_k(K) \quad \forall \text{ face } K \in \Sigma_h\Big\}. \tag{6.6}
$$

Then, we define the global spaces $\mathbb{H}_h$ and $\mathbb{Q}_h$ (cf. (5.3)), by combining (5.2), (5.3), (6.4), (6.5) and (6.6). Now, concerning hypotheses (**H.0**), (**H.1**), (**H.2**) and (**H.3**), we first observe that applying the same arguments as for the 2D case, it follows that (**H.0**), (**H.1**), and (**H.3**) hold. However, for the inf-sup conditions in (**H.2**) we employ [19, Lemma 7.5] to conclude that there exists $C_0 \in (0,1)$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \le C_0\,\widehat{h}_\Sigma$, the inf-sup condition (5.8) hold, whereas (5.9) follows analogously to the 2D case taking in account now (6.6).

6.3 Rate of convergence

Now, for both cases 2D and 3D domains, we derive the theoretical rate of convergence of our discrete scheme (5.4). To that end, we first recall recall from [8, Section 5], [17, 24] and [15], the approximation properties of the finite element subspaces involved, which are named after the unknowns to which they are applied later on.

($\mathbf{AP}_h^{\boldsymbol{\sigma}_{\mathrm{S}}}$) For each $\delta \in (0, k+1]$ and for each $\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \cap \mathbb{H}^\delta(\Omega_{\mathrm{S}})$ with $\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}})$, there holds

$$
\mathrm{dist}\Big(\boldsymbol{\tau}_{\mathrm{S}}, \mathbb{H}_{h,0}(\Omega_{\mathrm{S}})\Big) := \inf_{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_{h,0}(\Omega_{\mathrm{S}})} \|\boldsymbol{\tau}_{\mathrm{S}} - \boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div};\Omega_{\mathrm{S}}} \le C\,h^\delta \Big\{\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}}\Big\}.
$$

($\mathbf{AP}_h^{\mathbf{u}_{\mathrm{S}}}$) For each $\delta \in (0, k+1]$ and for each $\mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \cap \mathbf{H}^{1+\delta}(\Omega_{\mathrm{S}})$, there holds

$$
\mathrm{dist}\Big(\mathbf{v}_{\mathrm{S}}, \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\Big) := \inf_{\mathbf{v}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})} \|\mathbf{v}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \le C\,h^\delta\,\|\mathbf{v}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}}.
$$

($\mathbf{AP}_h^{\gamma_S}$) For each $\delta\in(0,k+1]$ and for each $\boldsymbol{\eta}_S\in\mathbb{L}^2_{\mathtt{skew}}(\Omega_S)\cap\mathbb{H}^\delta(\Omega_S)$, there holds

$$\mathrm{dist}\Big(\boldsymbol{\eta}_S,\mathbb{L}_h(\Omega_S)\Big):=\inf_{\boldsymbol{\eta}_{S,h}\in\mathbb{L}_h(\Omega_S)}\|\boldsymbol{\eta}_S-\boldsymbol{\eta}_{S,h}\|_{0,\Omega_S}\le C\,h^\delta\,\|\boldsymbol{\eta}_S\|_{\delta,\Omega_S}.$$

($\mathbf{AP}_h^{\mathbf{u}_D}$) For each $\delta\in(0,k+1]$ and for each $\mathbf{v}_D\in\mathbf{H}^3_{\Gamma_D}(\mathrm{div}\,;\Omega_D)\cap\mathbf{W}^{\delta,3}(\Omega_D)$ with $\mathrm{div}\,\mathbf{v}_D\in\mathrm{H}^\delta(\Omega_D)$, there holds

$$\mathrm{dist}\Big(\mathbf{v}_D,\mathbf{H}_{h,\Gamma_D}(\Omega_D)\Big):=\inf_{\mathbf{v}_{D,h}\in\mathbf{H}_{h,\Gamma_D}(\Omega_D)}\|\mathbf{v}_D-\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)}\le C\,h^\delta\Big\{\|\mathbf{v}_D\|_{\delta,3;\Omega_D}+\|\mathrm{div}\,\mathbf{v}_D\|_{\delta,\Omega_D}\Big\}.$$

($\mathbf{AP}_h^{\boldsymbol{\varphi}}$) For each $\delta\in(0,k+1]$ and for each $\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)\cap\mathbf{H}^{1/2+\delta}(\Sigma)$, there holds

$$\mathrm{dist}\Big(\boldsymbol{\psi},\boldsymbol{\Lambda}_h^S(\Sigma)\Big):=\inf_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h^S(\Sigma)}\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1/2,00;\Sigma}\le C\,h^\delta\,\|\boldsymbol{\psi}\|_{1/2+\delta,\Sigma}.$$

($\mathbf{AP}_h^{\lambda}$) For each $\delta\in(0,k+1]$ and for each $\xi\in\mathrm{W}^{\delta,3/2}(\Sigma)$, there holds

$$\mathrm{dist}\Big(\xi,\Lambda_h^D(\Sigma)\Big):=\inf_{\xi_h\in\Lambda_h^D(\Sigma)}\|\xi-\xi_h\|_{1/3,3/2;\Sigma}\le C\,h^{\delta-1/3}\,\|\xi\|_{\delta,3/2;\Sigma}.$$

($\mathbf{AP}_h^{p_D}$) For each $\delta\in(0,k+1]$ and for each $q_D\in\mathrm{L}^2_0(\Omega_D)\cap\mathrm{H}^\delta(\Omega_D)$, there holds

$$\mathrm{dist}\Big(q_D,\mathrm{L}_{h,0}(\Omega_D)\Big):=\inf_{q_{D,h}\in\mathrm{L}_{h,0}(\Omega_D)}\|q_D-q_{D,h}\|_{0,\Omega_D}\le C\,h^\delta\,\|q_D\|_{\delta,\Omega_S}.$$

We remark here, similarly to [8, Section 5], that the sub-optimal approximation property ($\mathbf{AP}_h^{\lambda}$) follows from the fact that $\mathrm{W}^{1/3,3/2}(\Sigma)$ is the interpolation space with index $1/(3\delta)$ between $\mathrm{W}^{\delta,3/2}(\Sigma)$ and $\mathrm{L}^{3/2}(\Sigma)$, and from the estimate $\|\xi-\xi_h\|_{\mathrm{L}^{3/2}(\Sigma)}\le Ch^\delta\|\xi\|_{\delta,3/2;\Sigma}$, which is valid for all $\xi\in\mathrm{W}^{\delta,3/2}(\Sigma)$ and $\xi_h:=\mathcal{P}_\Sigma(\xi)$, with $\mathcal{P}_\Sigma$ being the $\mathrm{L}^2(\Sigma)$-orthogonal projection onto $\Lambda_h^D(\Sigma)$ (cf. [14, Proposition 1.135]). In fact, given $\xi\in\mathrm{W}^{\delta,3/2}(\Sigma)$ there exists a constant $C>0$, depending on Σ, such that

$$\|\xi-\xi_h\|_{1/3,3/2;\Sigma}\le c\,\|\xi-\xi_h\|^{1-1/(3\delta)}_{\mathrm{L}^{3/2}(\Sigma)}\|\xi\|^{1/(3\delta)}_{\delta,3/2;\Sigma}\le C\,h^{\delta-1/3}\|\xi\|_{\delta,3/2;\Sigma},$$

where we have used the fact that ξ_h is piecewise polynomial of degree $\le k$ and then for each $\delta\in(0,k+1]$ there holds $\|\xi-\xi_h\|_{\delta,3/2;\Sigma}\le C\,\|\xi\|_{\delta,3/2;\Sigma}$.

It follows that there exist positive constants $C(\underline{\mathbf{t}})$, $C(\underline{\boldsymbol{\varphi}})$, and $C(\underline{\mathbf{p}})$, depending on the extra regularity assumptions for $\underline{\mathbf{t}}$, $\underline{\boldsymbol{\varphi}}$, and $\underline{\mathbf{p}}$, respectively, and whose explicit expressions are obtained from the right-hand side of the foregoing approximation properties, such that

$$\mathrm{dist}\,(\underline{\mathbf{t}},\mathbf{X}_h)\le C(\underline{\mathbf{t}})\,h^\delta,\quad \mathrm{dist}\,(\underline{\boldsymbol{\varphi}},\mathbf{Y}_h)\le C(\underline{\boldsymbol{\varphi}})\,h^{\delta-1/3}\quad\text{and}\quad \mathrm{dist}\,(\underline{\mathbf{p}},\mathbb{Q}_h)\le C(\underline{\mathbf{p}})\,h^\delta.$$

Then, we stablish the theoretical rate of convergence of our Galerkin scheme (5.4). Notice that, defining $s:=\min\{\delta,k+1\}$, at least a sub-optimal rate of convergence of order $O(h^{(s-1/3)/2})$ is confirmed.

Theorem 6.1 *Assume that the hypotheses of* Theorem 5.4 *hold. Let* $\big((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}\big)\in\mathbb{H}\times\mathbb{Q}$ *with* $\mathbf{u}_S\in\mathbf{W}_r$ *and* $\big((\underline{\mathbf{t}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{p}}_h\big)\in\mathbb{H}_h\times\mathbb{Q}_h$ *with* $\mathbf{u}_{S,h}\in\mathbf{W}_r^h$ *be the unique solutions of the continuous and discrete problems* (3.5) *and* (5.4)*, respectively. Assume further that there exists* $\delta>0$*, such that* $\boldsymbol{\sigma}_S\in\mathbb{H}^\delta(\Omega_S)$*,* $\mathbf{div}\boldsymbol{\sigma}_S\in\mathbf{H}^\delta(\Omega_S)$*,* $\mathbf{u}_S\in\mathbf{H}^{1+\delta}(\Omega_S)$*,* $\boldsymbol{\gamma}_S\in\mathbb{H}^\delta(\Omega_S)$*,* $\mathbf{u}_D\in\mathbf{W}^{\delta,3}(\Omega_D)$*,* $\mathrm{div}\,\mathbf{u}_D\in\mathrm{H}^\delta(\Omega_D)$*,* $\boldsymbol{\varphi}\in\mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\lambda\in\mathrm{W}^{\delta,3/2}(\Sigma)$ *and* $p_D\in\mathrm{H}^\delta(\Omega_D)$*. Then, defining* $s:=\min\{\delta,k+1\}$*, there exists a positive constant*

$C((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}})$ *depending on* $C(\underline{\mathbf{t}}), C(\underline{\boldsymbol{\varphi}})$*, and* $C(\underline{\mathbf{p}})$*, all them independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned}&\big\|((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{p}}_h)\big\|_{\mathbb{H}\times\mathbb{Q}}\\ &\quad\le C((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}})\left\{h^{(s-1/3)/2} + h^{s/2} + h^{s-1/3} + h^{s} + h^{2(s-1/3)} + h^{2s}\right\}.\end{aligned}$$

Proof. It follows from a direct application of Theorem 5.4 and the approximation properties of the discrete subspaces. We omit further details. □

7 Numerical Results

In this section we present two examples illustrating the performance of our augmented mixed finite element scheme (5.4) on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a FreeFem++ code [28], in conjunction with the direct linear solver UMFPACK [12].

In order to solve the nonlinear problem (5.4), given $\mathbf{0} \neq \mathbf{w}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}})$ we introduce the Gâteaux derivate associated to $\mathcal{A}_{\mathrm{D}}$ (cf. (3.8)):

$$\mathcal{D}\mathcal{A}_{\mathrm{D}}(\mathbf{w}_{\mathrm{D}})(\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}) := \frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}(|\mathbf{w}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}\left(\frac{\mathbf{w}_{\mathrm{D}}\cdot\mathbf{u}_{\mathrm{D}}}{|\mathbf{w}_{\mathrm{D}}|}, \mathbf{w}_{\mathrm{D}}\cdot\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}},$$

for all $\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}})$. In this way, we propose the Newton-type strategy: Given $\underline{\mathbf{t}}_h^0 := (\underline{\boldsymbol{\sigma}}_h^0, \mathbf{u}_{\mathrm{D},h}^0) \in \mathbf{X}_h$ with $\mathbf{u}_{\mathrm{D},h}^0 \neq \mathbf{0}$, for $m \ge 1$, find $((\underline{\mathbf{t}}_h^m,\underline{\boldsymbol{\varphi}}_h^m),\underline{\mathbf{p}}_h^m) \in \mathbb{H}_h\times\mathbb{Q}_h$ such that

$$\begin{aligned}&[\mathcal{A}_{\mathrm{S}}(\underline{\boldsymbol{\sigma}}_{\mathrm{S},h}^m),\underline{\boldsymbol{\tau}}_h] + [\mathcal{B}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S},h}^{m-1})(\underline{\boldsymbol{\sigma}}_{\mathrm{S},h}^m),\underline{\boldsymbol{\tau}}_h] + [\mathcal{B}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S},h}^{m})(\underline{\boldsymbol{\sigma}}_{\mathrm{S},h}^{m-1}),\underline{\boldsymbol{\tau}}_h] + \mathcal{D}\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D},h}^{m-1})(\mathbf{u}_{\mathrm{D},h}^m,\mathbf{v}_{\mathrm{D},h})\\ &\quad + [\mathbf{b}(\underline{\mathbf{r}}_h),\underline{\boldsymbol{\varphi}}_h^m] + [\mathbf{b}(\underline{\mathbf{t}}_h^m),\underline{\boldsymbol{\psi}}_h] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}_h^m),\underline{\boldsymbol{\psi}}_h] + [\mathbf{B}(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{p}}_h^m]\\ &\quad = [\mathcal{B}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S},h}^{m-1})(\underline{\boldsymbol{\sigma}}_{\mathrm{S},h}^{m-1}),\underline{\boldsymbol{\tau}}_h] + \frac{\mathtt{F}}{\rho}(|\mathbf{u}_{\mathrm{D},h}^{m-1}|\mathbf{u}_{\mathrm{D},h}^{m-1},\mathbf{v}_{\mathrm{D},h})_{\mathrm{D}} + [\mathbf{F},(\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h)],\\ &[\mathbf{B}(\underline{\mathbf{t}}_h^m,\underline{\boldsymbol{\varphi}}_h),\underline{\mathbf{q}}_h] = [\mathbf{G},\underline{\mathbf{q}}_h],\end{aligned} \tag{7.1}$$

for all $((\underline{\mathbf{r}}_h,\underline{\boldsymbol{\psi}}_h),\underline{\mathbf{q}}_h) \in \mathbb{H}_h\times\mathbb{Q}_h$.

In all the numerical experiments below, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates, say $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \le tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h$ and $\mathbb{Q}_h$, and tol is a fixed tolerance chosen as $tol = 1E-06$. As usual, the individual errors are denoted by:

$$\mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}}) := \|\boldsymbol{\sigma}_{\mathrm{S}} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}},\quad \mathrm{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}},\quad \mathrm{e}(\boldsymbol{\gamma}_{\mathrm{S}}) := \|\boldsymbol{\gamma}_{\mathrm{S}} - \boldsymbol{\gamma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}},$$

$$\mathrm{e}(p_{\mathrm{S}}) := \|p_{\mathrm{S}} - p_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}},\quad \mathrm{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})},\quad \mathrm{e}(p_{\mathrm{D}}) := \|p_{\mathrm{D}} - p_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}},$$

$$\mathrm{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1),\Sigma},\quad \mathrm{e}(\lambda) := \|\lambda - \lambda_h\|_{\mathrm{L}^{3/2}(\Sigma)},$$

where $p_{\mathrm{S},h}$ is the postprocessed pressure given by

$$p_{\mathrm{S},h} := -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h}\otimes\mathbf{u}_{\mathrm{S},h})) - \ell_h \quad \text{in} \quad \Omega_{\mathrm{S}}.$$

Notice that, since the natural norms to measure the error of the interface unknowns $\|\lambda-\lambda_h\|_{1/3,3/2;\Sigma}$ and $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,00;\Sigma}$ are not computable, we have decided to replace them respectively by $\|\cdot\|_{\mathrm{L}^{3/2}(\Sigma)}$ and $\|\cdot\|_{(0,1),\Sigma}$, where the last one is defined based on the fact that $\mathbf{H}^{1/2}(\Sigma)$ is the interpolation space with index $1/2$ between $\mathbf{H}^1(\Sigma)$ and $\mathbf{L}^2(\Sigma)$:

$$\|\boldsymbol{\psi}\|_{(0,1),\Sigma} := \|\boldsymbol{\psi}\|_{0,\Sigma}^{1/2}\,\|\boldsymbol{\psi}\|_{1,\Sigma}^{1/2} \quad \forall\,\boldsymbol{\psi}\in\mathbf{H}^1(\Sigma).$$

Next, we define the experimental rates of convergence

$$r(\boldsymbol{\sigma}_{\mathrm{S}}) := \frac{\log(e(\boldsymbol{\sigma}_{\mathrm{S}})/e'(\boldsymbol{\sigma}_{\mathrm{S}}))}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})}, \quad r(\mathbf{u}_{\mathrm{S}}) := \frac{\log(e(\mathbf{u}_{\mathrm{S}})/e'(\mathbf{u}_{\mathrm{S}}))}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})}, \quad r(\boldsymbol{\gamma}_{\mathrm{S}}) := \frac{\log(e(\boldsymbol{\gamma}_{\mathrm{S}})/e'(\boldsymbol{\gamma}_{\mathrm{S}}))}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})},$$
$$r(p_{\mathrm{S}}) := \frac{\log(e(p_{\mathrm{S}})/e'(p_{\mathrm{S}}))}{\log(h_{\mathrm{S}}/h'_{\mathrm{S}})}, \quad r(\mathbf{u}_{\mathrm{D}}) := \frac{\log(e(\mathbf{u}_{\mathrm{D}})/e'(\mathbf{u}_{\mathrm{D}}))}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})}, \quad r(p_{\mathrm{D}}) := \frac{\log(e(p_{\mathrm{D}})/e'(p_{\mathrm{D}}))}{\log(h_{\mathrm{D}}/h'_{\mathrm{D}})},$$
$$r(\boldsymbol{\varphi}) := \frac{\log(e(\boldsymbol{\varphi})/e'(\boldsymbol{\varphi}))}{\log(\hat{h}_\Sigma/\hat{h}'_\Sigma)}, \quad r(\lambda) := \frac{\log(e(\lambda)/e'(\lambda))}{\log(h_\Sigma/h'_\Sigma)},$$

where $h_\star$ and $h'_\star$ ($\star\in\{\mathrm{S},\mathrm{D},\Sigma\}$) denote two consecutive mesh sizes with their respective errors e and e', respectively. In turn, we take $\hat{h}_\Sigma$ as two times h_Σ, which comes from the restriction on the mesh sizes $h_\Sigma \leq C_0\widehat{h}_\Sigma$ when considering the constant $C_0 = 1/2$. The numeric results confirm that this choice is suitable. The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we choose the parameters $\mu=1$, $\rho=1$, $\omega=1$ and $\mathbf{K}=\mathbb{I}$, and according to (4.10), the stabilization parameters are taken as $\kappa_1 = 1/(2\mu), \kappa_2 = 2\mu$ and $\kappa_3 = C_{\mathrm{Ko}}\mu$, where, similarly to [9, Section 7] we choose heuristically $C_{\mathrm{Ko}} = 1/2$. Additionally, regarding the conditions $(\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{S},h},1)_{\mathrm{S}} = 0$ and $(p_{\mathrm{D},h},1)_{\mathrm{D}} = 0$, these are imposed via a penalization strategy.

Example 1: Inverted-L-shaped domain coupled with a square domain.

In our first example, we consider an inverted-L-shaped domain coupled with a square, which yields a porous medium partially surrounded by a fluid. More precisely, we consider the domain $\Omega = \Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$, with $\Omega_{\mathrm{D}} := (-1,0)^2$, $\Omega_{\mathrm{S}} := (-1,1)^2\backslash\Omega_{\mathrm{D}}$ and $\Sigma := (-1,0)\times\{0\}\cup\{0\}\times(-1,0)$. The Forchheimer number is chosen as $\mathtt{F}=1$ and the data $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$, and g_{D}, are adjusted so that the exact solution in the square Ω is given by the smooth functions

$$\mathbf{u}_{\mathrm{S}} = \begin{pmatrix} -\pi\sin(\pi x_1)\cos(\pi x_2) \\ \pi\cos(\pi x_1)\sin(\pi x_2)\end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{D}} = \begin{pmatrix} \sin(\pi x_1)\exp(x_2) \\ \exp(x_1)\sin(\pi x_2)\end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{D}},$$
$$p_\star = \cos(\pi x_1)\cos(\pi x_2) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star\in\{\mathrm{S},\mathrm{D}\}.$$

Example 2: 2D helmet-shaped domain with different Forchheimer numbers.

In our second example, and inspired by [8], we focus on the performance of the numerical method (7.1) with respect to the number of Newton iterations required to achieve certain tolerance given different Forchheimer numbers. Hence, we consider $\mathtt{F}\in\{0,10^0,10^1,10^2,10^3,10^4,10^5\}$, the 2D helmet-shaped domain described by $\Omega = \Omega_{\mathrm{S}}\cup\Sigma\cup\Omega_{\mathrm{D}}$, where $\Omega_{\mathrm{D}} := (-1,1)\times(-0.5,0)$, $\Sigma := (-1,1)\times\{0\}$, and $\Omega_{\rm S} := (-1,1) \times (0,1.25)\backslash\widetilde{\Omega}_{\rm s}$ with $\widetilde{\Omega}_{\rm S} := (-0.75,0.75) \times (0.25,1.25)$. The data $\mathbf{f}_{\rm S}, \mathbf{f}_{\rm D}$, and $g_{\rm D}$, are adjusted so that the exact solution in the 2D helmet-shaped domain Ω is given by the smooth functions

$$\mathbf{u}_{\rm S} = \begin{pmatrix} -\sin(2\pi x_1)\cos(2\pi x_2) \\ \cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\rm S}, \quad \mathbf{u}_{\rm D} = \begin{pmatrix} \sin(2\pi x_1)\exp(x_2) \\ \exp(x_1)\sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\rm D},$$

$$p_\star = \sin(\pi x_1)\exp(x_2) \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{\mathrm{S},\mathrm{D}\}.$$

Notice that, in both examples, the solutions satisfy $\mathbf{u}_{\rm S}\cdot\mathbf{n} = \mathbf{u}_{\rm D}\cdot\mathbf{n}$ on Σ and $\mathbf{u}_{\rm D}\cdot\mathbf{n} = 0$ on $\Gamma_{\rm D}$. However, the Beavers–Joseph–Saffman condition (cf. (2.8)) is not satisfied and the Dirichlet boundary condition for the Navier–Stokes velocity on $\Gamma_{\rm S}$ is non-homogeneous and therefore the right-hand side of the resulting system must be modified accordingly.

In Tables 7.1 and 7.3 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the finite element spaces introduced in Section 6.1 with $k = 0$, and solving the nonlinear problem (7.1), which requires around four and nine Newton iterations for the Examples 1 and 2, respectively. We observe that the sub-optimal rate of convergence $O(h^{(k+2/3)/2})$ provided by Theorem 6.1 (when $\delta = k+1$) is attained in all the cases (with $k = 0$). Even more, the numerical results suggest that perhaps only technical difficulties stop us of proving optimal rate of convergence $O(h^{k+1})$. In Table 7.2 we show the behaviour of the iterative method (7.1) as a function of the Forchheimer number $\mathtt{F}$, considering different mesh sizes $h := \max\{h_{\rm S}, h_{\rm D}\}$, and a tolerance $tol = 1E-06$. Here we observe that the higher the parameter $\mathtt{F}$ the higher the number of iterations as it occurs also in the Newton method for the Navier–Stokes/Darcy–Forchheimer coupled problem. Notice also that when $\mathtt{F} = 0$ the Darcy–Forchheimer equations reduce to the classical linear Darcy equations and as expected the iterative Newton method (7.1) is faster.

On the other hand, the approximated spectral norm of the pseudostress tensor components, the skew-symmetric part of the Navier–Stokes velocity gradient, the velocity streamlines, the velocity components on the whole domain, the geometry configuration and the pressure field in the whole domain of the approximate solutions for the two examples are displayed in Figures 7.1 and 7.2. All the figures were obtained with 657612 and 1076768 degrees of freedom for the Examples 1 and 2, respectively. In particular, we can observe in Figure 7.1 the continuity of the normal components of the velocities on Σ since the first components of $\mathbf{u}_{\rm S}$ and $\mathbf{u}_{\rm D}$ coincide on $\{0\}\times(-1,0)$, whereas their second components coincide on $(-1,0)\times\{0\}$. Moreover, it can be seen that the pressure is continuous in the whole domain and preserves the sinusoidal behaviour. Finally, similarly to Figure 7.1, in Figure 7.2 we can also observe that the continuity of the normal components of the velocities on Σ is preserved since the second components of $\mathbf{u}_{\rm S}$ and $\mathbf{u}_{\rm D}$ coincide on Σ as expected. In turn, we can see that the velocity streamlines are higher in the Darcy–Forchheimer domain.

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*. Second edition. Pure and Applied Mathematics (Amsterdan), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] J.A. Almonacid, H.S. Díaz, G.N. Gatica, and A. Márquez, *A fully-mixed finite element method for the Darcy–Forchheimer/Stokes coupled problem*. Submitted. [preliminary version available as Preprint 2018-29 in `http://www.ci2ma.udec.cl/publicaciones/prepublicaciones/en.php`].

DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$r(\boldsymbol{\gamma}_S)$	$e(p_S)$	$r(p_S)$
756	0.375	23.6030	–	5.3592	–	4.6970	–	3.3911	–
2847	0.195	11.3037	1.223	2.7705	1.006	2.5019	0.961	1.7970	0.969
10644	0.096	5.6685	0.977	1.3917	0.974	1.3373	0.886	0.8695	1.027
42043	0.052	2.7945	1.152	0.6885	1.146	0.6898	1.078	0.4231	1.173
165156	0.029	1.4040	1.198	0.3452	1.201	0.3468	1.197	0.2097	1.221
657612	0.015	0.7003	0.993	0.1728	0.988	0.1723	0.998	0.1057	0.978

h_D	$\widehat{h}_\Sigma$	h_Σ	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\lambda)$	iter
0.373	1/2	1/4	0.4384	–	0.1273	–	2.0677	–	0.2560	–	4
0.190	1/4	1/8	0.1974	1.185	0.0583	1.159	1.1721	0.819	0.1151	1.154	4
0.093	1/8	1/16	0.1002	0.946	0.0306	0.900	0.5560	1.076	0.0556	1.049	4
0.051	1/16	1/32	0.0491	1.183	0.0150	1.184	0.2832	0.973	0.0277	1.007	4
0.025	1/32	1/64	0.0247	0.982	0.0074	1.004	0.1418	0.997	0.0138	1.003	4
0.014	1/64	1/128	0.0124	1.153	0.0037	1.147	0.0701	1.017	0.0069	1.001	4

Table 7.1: Example 1, Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $\mathtt{F} = 1$.

F \ h	0.200	0.100	0.050	0.026	0.014	0.007
0	4	3	3	3	3	3
10^0	5	5	5	5	5	5
10^1	7	8	8	9	9	9
10^2	8	9	10	10	11	11
10^3	9	9	10	11	11	12
10^4	9	9	10	11	12	12
10^5	9	9	10	11	12	12

Table 7.2: Example 2, Performance of the iterative method (number of Newton iterations) upon variations of the Forchheimer number $\mathtt{F}$.

[3] M. Amara, D. Capatina, and L. Lizaik, *Coupling of Darcy–Forchheimer and compressible Navier–Stokes equations with heat transfer*. SIAM J. Sci. Comput. 31 (2008/09), no. 2, 1470–1499.

[4] T. Arbogast and H.L. Lehr, *Homogenization of a Darcy–Stokes system modeling vuggy porous media*. Comput. Geosci. 10 (2006), no. 3, 291–302.

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[6] J. Camaño, G.N. Gatica, R. Oyarzúa, and G. Tierra, *An augmented mixed finite element method for the Navier–Stokes equations with variable viscosity*. SIAM J. Numer. Anal. 54 (2016), no. 2, 1069–1092.

[7] C. Canuto and F. Cimolin, *A sweating model for the internal ventilation of a motorcycle helmet*. Comput. Fluids 43 (2011), no. 1, 29–37.

DOF	h_{S}	$\mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$
1264	0.188	14.7840	–	2.1893	–	1.5363	–	1.4364	–
4833	0.100	7.1648	1.153	1.0943	1.104	0.7868	1.065	0.6581	1.243
17586	0.050	3.6068	0.991	0.5431	1.011	0.4149	0.923	0.3027	1.120
69327	0.026	1.7936	1.054	0.2718	1.045	0.2087	1.038	0.1477	1.083
269604	0.014	0.8992	1.183	0.1350	1.200	0.1044	1.187	0.0738	1.190
1076768	0.007	0.4490	0.932	0.0674	0.932	0.0523	0.926	0.0368	0.935

h_{D}	$\widehat{h}_{\Sigma}$	h_{Σ}	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathrm{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathrm{e}(\lambda)$	$r(\lambda)$	iter
0.200	1/4	1/8	1.2761	–	0.1131	–	1.1300	–	0.2569	–	7
0.095	1/8	1/16	0.6135	0.984	0.0388	1.438	0.4481	1.335	0.0744	1.787	8
0.049	1/16	1/32	0.3115	1.037	0.0151	1.447	0.2129	1.074	0.0302	1.300	8
0.026	1/32	1/64	0.1566	1.081	0.0067	1.283	0.1003	1.086	0.0142	1.089	9
0.013	1/64	1/128	0.0784	0.968	0.0033	0.999	0.0502	0.997	0.0070	1.026	9
0.007	1/128	1/256	0.0393	1.204	0.0016	1.221	0.0249	1.011	0.0035	1.010	9

Table 7.3: Example 2, Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $\mathtt{F} = 10$.

[8] S. Caucao, M. Discacciati, G.N. Gatica, and R. Oyarzúa, *A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem.* Submitted. [preliminary version available as Preprint 2017-29 in `http://www.ci2ma.udec.cl/publicaciones/prepublicaciones/en.php`].

[9] S. Caucao, G.N. Gatica, R. Oyarzúa, and I. Šebestová, *A fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity.* J. Numer. Math. 25 (2017), no. 2, 55–88.

[10] P.G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[11] F. Cimolin and M. Discacciati, *Navier–Stokes/Forchheimer models for filtration through porous media.* Appl. Numer. Math. 72 (2013), 205–224.

[12] T. Davis, *Algorithm 832: UMFPACK V4.3 - an unsymmetric-pattern multifrontal method.* ACM Trans. Math. Software 30 (2004), no. 2, 196–199.

[13] M. Discacciati and R. Oyarzúa, *A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem.* Numer. Math. 135 (2017), no. 2, 571–606.

[14] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements.* Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[15] M. Farhloul, *A mixed finite element method for a nonlinear Dirichlet problem.* IMA J. Numer. Anal. 18 (1998), no. 1, 121–132.

[16] J. Faulkner, B.X. Hu, S. Kish, and F. Hua, *Laboratory analog and numerical study of groundwater flow and solute transport in a karst aquifer with conduit and matrix domains.* J. Contaminant Hydrology 110 (2009), no. 1-2, 34–44.

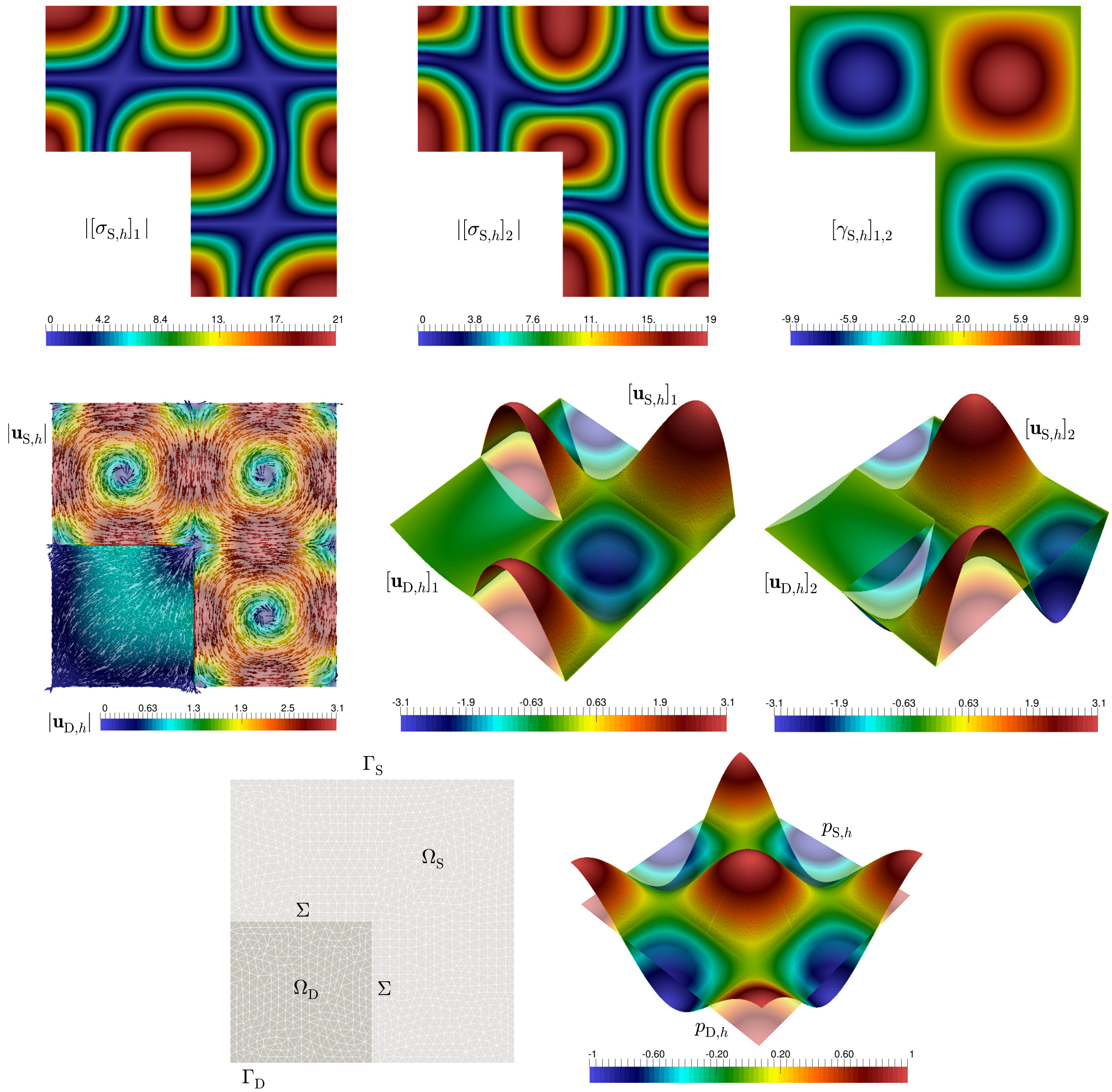

Figure 7.1: Example 1. Approximated spectral norm of the nonlinear pseudostress tensor components and the skew-symmetric part of the Navier–Stokes velocity gradient (top panel), velocity streamlines and velocity components on the whole domain (middle panel), and geometry configuration and pressure field in the whole domain (bottom panel).

[17] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* Springer Briefs in Mathematics. Springer, Cham, 2014.

[18] G.N. Gatica, N. Heuer, and S. Meddahi, *On the numerical analysis of nonlinear twofold saddle point problems.* IMA J. Numer. Anal. 23 (2003), no. 2, 301–330.

[19] G.N. Gatica, G.C. Hsiao, and S. Meddahi, *A coupled mixed finite element method for the*

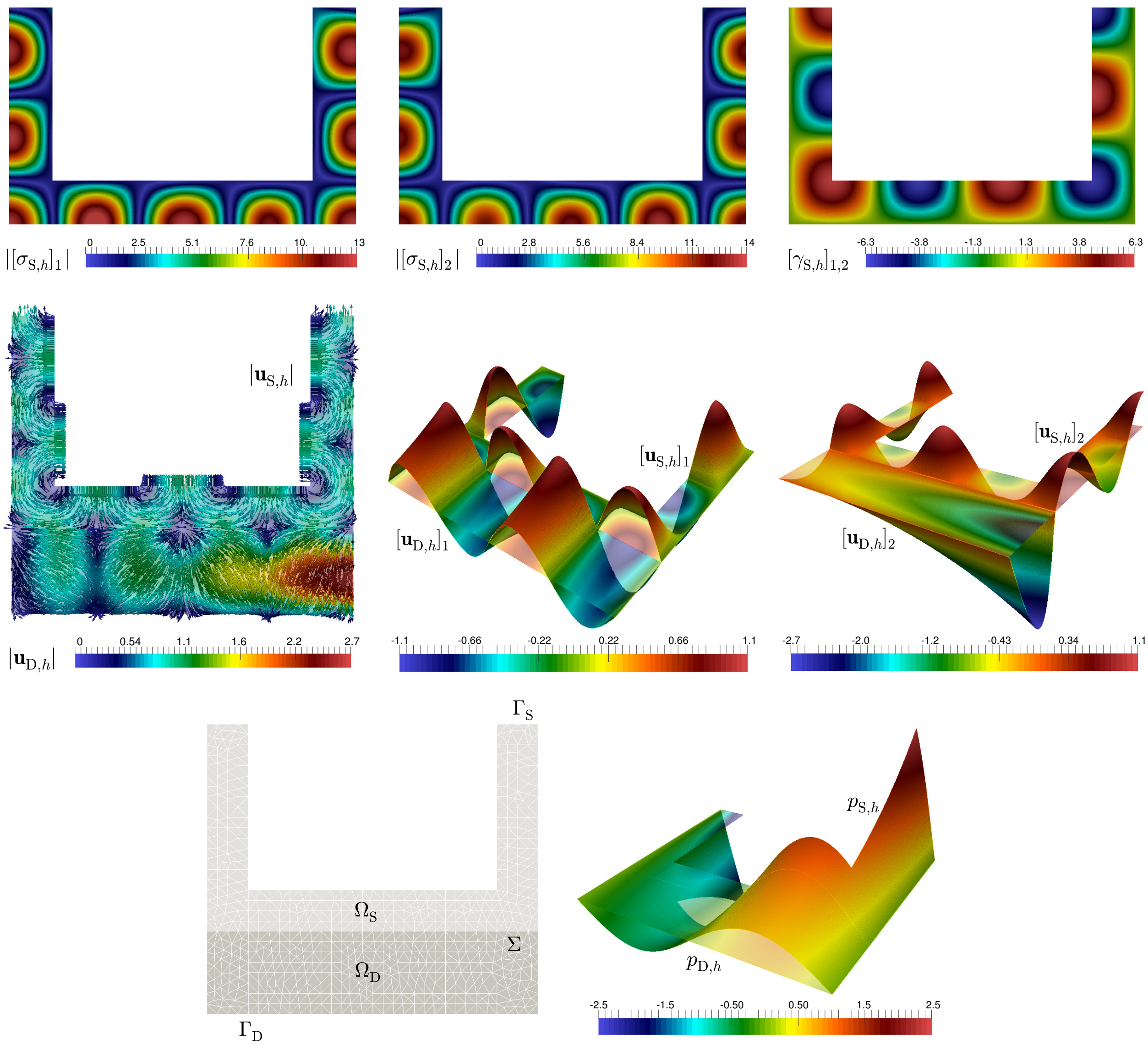

Figure 7.2: Example 2. Approximated spectral norm of the nonlinear pseudostress tensor components and the skew-symmetric part of the Navier–Stokes velocity gradient (top panel), velocity streamlines and velocity components on the whole domain (middle panel), and geometry configuration and pressure field in the whole domain (bottom panel).

interaction problem between electromagnetic field and elastic body. SIAM J. Numer. Anal. 48 (2010), 1338–1368.

[20] G.N. Gatica, A. Márquez, R. Oyarzúa, and R. Rebolledo, *Analysis of an augmented fully-mixed approach for the coupling of quasi-Newtonian fluids and porous media*. Comput. Methods Appl. Mech. Engrg. 270 (2014), 76–112.

[21] G.N. Gatica, S. Meddahi, and R. Oyarzúa, *A conforming mixed finite-element method for the coupling of fluid flow with porous media flow*. IMA J. Numer. Anal. 29 (2009), no. 1, 86–108.

[22] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas, *Analysis of fully-mixed finite element methods for the Stokes–Darcy coupled problem*. Math. Comp. 80 (2011), no. 276, 1911–1948.

[23] G.N. Gatica, R. Oyarzúa, and F.-J. Sayas, *A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow*. IMA J. Numer. Anal. 32 (2012), no. 3, 845–887.

[24] V. Girault and P.A. Raviart, *Finite Element Methods for Navier–Stokes Equations. Theory and algorithms*. Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[25] R. Glowinski and A. Marroco, *Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires*. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. 9 (1975), no. R-2, 41–76.

[26] P. Grisvard, *Théorèmes de traces relatifs à un polyèdre*. C. R. Acad. Sci. Paris Sr. A 278 (1974), 1581–1583.

[27] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[28] F. Hecht, *New development in FreeFem++*. J. Numer. Math. 20 (2012), no. 3-4, 251–265.

[29] H. Manouzi and M. Farhloul, *Mixed finite element analysis of a non-linear three-fields Stokes model*. IMA J. Numer. Anal. 21 (2001), no. 1, 143–164.

[30] A. Márquez, S. Meddahi and F.-J. Sayas, *Strong coupling of finite element methods for the Stokes–Darcy problem* IMA J. Numer. Anal. 35 (2015), 969–988.

[31] V. Nassehi, *Modelling of combined Navier–Stokes and Darcy flows in crossflow membrane filtration*. Chem. Eng. Sci. 53 (1998), no. 6, 1253–1265.

[32] H. Pan and H. Rui, *Mixed element method for two-dimensional Darcy–Forchheimer model*. J. Sci. Comput. 52 (2012), no. 3, 563–587.

[33] C. Pozrikidis and D.A. Farrow, *A model of fluid flow in solid tumors*. Ann. Biomed. Eng. 31 (2003), 181–194.

[34] D. Ruth and H. Ma, *On the derivation of the Forchheimer equation by means of the averaging theorem*. Transp. Porous Media 7 (1992), no. 3, 255–264.

[35] B. Scheurer, *Existence et approximation de points selles pour certains problèmes non linéaires*. RAIRO Anal. Numér. 11 (1977), no. 4, 369–400.

[36] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Mathematical Surveys and Monographs, 49. American Mathematical Society, Providence, RI, 1997.