An augmented fully-mixed finite element method for a coupled flow-transport problem*

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Abstract

In this paper we analyze the coupling of the Stokes equations with a transport problem modelled by a scalar nonlinear convection-diffusion problem, where the viscosity of the fluid and the diffusion coefficient depend on the solution to the transport problem and its gradient, respectively. An augmented mixed variational formulation for both the fluid flow and the transport model is proposed. As a consequence, no discrete inf-sup conditions are required for the stability of the associated Galerkin scheme, and therefore arbitrary finite element subspaces can be used, which constitutes one of the main advantages of the present approach. In particular, the resulting fully-mixed finite element method can employ Raviart-Thomas spaces of order k for the Cauchy stress, continuous piecewise polynomials of degree $k+1$ for the velocity and for the scalar field, and discontinuous piecewise polynomial approximations for the gradient of the concentration. In turn, the Lax-Milgram lemma, monotone operators theory, and the classical Schauder and Brouwer fixed point theorems are utilized to establish existence of solution of the continuous and discrete formulations. In addition, suitable estimates, arising from the combined use of a regularity assumption with the Sobolev embedding and Rellich-Kondrachov compactness theorems, are also required for the continuous analysis. Then, sufficiently small data allow us to prove uniqueness of solution and to derive optimal a priori error estimates. Finally, several numerical tests, illustrating the performance of our method and confirming the predicted rates of convergence, are reported.

Key words: Stokes equations, nonlinear transport problem, augmented fully–mixed formulation, fixed point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 76R05, 76D07, 65N15 35Q79, 80A20,

1 Introduction

In recent years there has been an increasing interest in studying finite element approximations to simulate the transport of a species density in an immiscible fluid. In particular, the continuous and discrete solvability of a flow-transport model given by the coupling of the Stokes equations with a scalar nonlinear convection-diffusion equation, in which the viscosity of the fluid and the effective diffusivity depend on the solution to the transport problem and its gradient, respectively, was recently analyzed in [2] by using a mixed-primal variational approach. Regarding the underlying coupled

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model, and while the original unknowns of it are the velocity of the flow, the pressure, and the local solids concentration, it is well known that other variables, such as stress tensors, vorticity, and the aforementioned gradient, are also of great interest in applications, which include natural and thermal convection, sedimentation-consolidation processes, and granular flows, among others. According to this motivation, the model is reformulated in [2] as an augmented dual-mixed formulation for the fluid flow, coupled with the usual primal method for the transport model. As a consequence, the Cauchy stress and the velocity of the fluid are sought in $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively, whereas the concentration lies in $\mathrm{H}^1(\Omega)$. In this way, each row of the stress tensor is approximated with Raviart-Thomas elements of order k, whereas the other two unknowns are approximated with continuous piecewise polynomials of degree $\leq k+1$. Furthermore, fixed point arguments, suitable regularity hypotheses, the well-know Lax-Milgram theorem, classical results on monotone operators, the Sobolev embedding and Rellich-Kondrachov compactness theorems, and sufficiently small data assumptions, constitute the main tools yielding well-posedness of the continuous and Galerkin schemes, and the associated optimal a priori error estimates.

Other contributions concerning the solvability of flow-transport problems are certainly available in the literature as well. For example, the technique of parabolic regularization has been employed in [8] for the case of large fluid viscosity, whereas the existence of solutions to a model of chemically reacting non-Newtonian fluid with the effective diffusivity depending also on the gradient of the concentration, has been established in [7]. In turn, the extension of the approach from [2] to the more realistic case of steady sedimentation-consolidation systems, in which both the viscosity and the diffusivity depend only on the scalar value of the concentration, and hence neither of them on the concentration gradient (as in [2] and [7] for the latter), was developed in [3]. In this case, the model consist in the Brinkman problem with variable viscosity, written in terms of Cauchy pseudo-stresses and bulk velocity of the mixture, coupled with a nonlinear advection – diffusion equation describing the transport of the solids volume fraction. Then, similarly to [2], the variational formulation is based on an augmented mixed approach for the Brinkman equations and the usual primal approach for the transport equation. In addition, the solvability analyses make use of basically the same arguments from [2], the finite element subspaces employed are exactly those from [2], and suitable Strang-type inequalities are utilized to derive optimal error estimates in the natural norms.

On the other hand, it is worth mentioning that flow-transport models, and specially those involving sedimentation-consolidation processes, share some analytical similarities with Boussinesq and related problems, for which several mixed-primal and fully-mixed formulations have been proposed in recent years (see, e.g. [11], [12], [13], [15], [16], and [24]). In particular, the mixed finite element method for the Boussinesq problem developed in [15] introduces the gradient of velocity as an auxiliary unknown. In turn, following [9], the approach from [11] employs the nonlinear pseudostress tensor linking the pseudostress and the convective term, and then augment the resulting mixed formulation of the stationary Boussinesq problem with suitable Galerkin type terms. Furthermore, the technique of [12] proceeds similarly to [11], but in contrast to the latter, an augmented mixed formulation for the equation modelling the temperature is also proposed. More precisely, a new auxiliary vector unknown, involving the temperature, its gradient and the velocity, is introduced, and then the resulting new mixed formulation for the convection–diffusion equation is augmented with alternative testings of the constitutive and equilibrium temperature equations. In this way, classical fixed point theorems, together with the Lax-Milgram lemma and the Babuška-Brezzi theory, are applied to prove the well-posedness of the continuous and discrete formulations in [11] and [12]. However, up to our knowledge, fully-mixed formulations specifically designed for flow-transport models, and aiming to introduce further unknows of physical interest, are not yet available in the literature.

According to the previous bibliographic discussion, the purpose of the present paper is to keep contributing in the direction of [2] and [3] by applying now an augmented mixed variational formulation to both the fluid flow and the transport model. In this way, and besides the incorporation of other unknowns of physical interest, such as the gradient of concentration, the resulting decoupled problems yield a strongly elliptic bilinear form and a strongly monotone operator equation, respectively, and hence arbitrary finite element subspaces can be employed for defining the associated discrete schemes. The contents of the paper are organized as follows. The remainder of this section introduces some standard notation and functional spaces. In Section 2 we first describe the boundary value problem of interest, then slightly simplify it by eliminating the pressure unknown in the fluid and defining the gradient of the concentration as a new unknown variable. Next, in Section 3 we introduce and analyze the continuous formulation, which is defined by an augmented mixed approach in both media. The necessity of augmentation is clearly justified, and the solvability analysis is based on a fixed point strategy that makes use of the Lax-Milgran lemma, the Schauder theorem, and a well-known result on strongly monotone operators. We prove existence of solution and for sufficiently small data we derive uniqueness. The associated Galerkin scheme is introduced in Section 4 by employing Raviart-Thomas elements for the stress, continuous piecewise polynomial approximations for the velocity and concentration, and discontinuous piecewise polynomial approximations for the gradient of the concentration. Here the solvability is established by applying now the Brouwer fixed point theorem and analogue arguments to those employed in Section 3. In Section 5 we assume again sufficiently small data and, using a suitable Strang-type estimate for nonlinear problems, provide optimal a priori error estimates. Finally, in Section 6 we present numerical examples illustrating the good performance of the fully-mixed method and confirming the theoretical rates of convergence.

Preliminary notations

Let $\Omega \subseteq \mathrm{R}^n$, $n = 2, 3$, be a given bounded domain with polyhedral boundary $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, with $|\Gamma_D|, |\Gamma_N| > 0$, $\Gamma_D \cap \Gamma_N = \emptyset$ and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. A standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. In addition, given $\Gamma_* \subseteq \Gamma$ with $* \in \{D, N\}$, denote by $\langle\cdot,\cdot\rangle_{\Gamma_*}$ the duality pairing between $\mathrm{H}^{1/2}(\Gamma_*)$ and $\mathrm{H}^{-1/2}(\Gamma_*)$. Also, we let $\mathbf{M}$ and $\mathbb{M}$ be the vectorial and tensorial counterparts of a generic scalar functional space M. In turn, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes both the euclidean norm in R^n and the Frobenius norm in $\mathrm{R}^{n\times n}$. On the other hand, for any vector field $\boldsymbol{v} = (v_i)_{i=1,n}$ we set $\nabla\boldsymbol{v} := \left(\dfrac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}$ and $\operatorname{div}\boldsymbol{v} := \displaystyle\sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}$. Additionaly, for any tensor fields $\boldsymbol{\tau} = (\tau_{i,j})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{\mathrm{i}=1}^{\mathrm{n}} \tau_{\mathrm{ii}}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{\mathrm{i,j}=1}^{\mathrm{n}} \tau_{\mathrm{ij}}\zeta_{\mathrm{ij}} \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{\mathrm{n}}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

Furthermore, we recall that the space

$$\mathbb{H}(\mathbf{div},\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\Big\}\,,$$

equipped with the usual norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div}\,;\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\Omega}$, is a Hilbert space.

2 The model problem

The following system of partial differential equations, written as to apply a fully-mixed approach, describes the stationary state of the transport of species in an immiscible fluid occupying the domain $\Omega \subseteq \mathrm{R}^n$:

$$
\begin{array}{rcll}
\boldsymbol{\sigma} & = & \mu(\phi)\nabla\mathbf{u} - p\mathbb{I}, & \text{in} \quad \Omega\,, \\
-\mathbf{div}\,\boldsymbol{\sigma} & = & \mathbf{f}\phi, & \text{in} \quad \Omega\,, \\
\mathrm{div}\,\mathbf{u} & = & 0 & \text{in} \quad \Omega\,, \\
\mathbf{p} & = & \theta(|\nabla\phi|)\nabla\phi - \phi\mathbf{u} - \gamma(\phi)\mathbf{k} & \text{in} \quad \Omega\,, \\
\mathbf{div}\,\mathbf{p} & = & -g & \text{in} \quad \Omega\,, \\
\mathbf{u} & = & \mathbf{u}_D & \text{on} \quad \Gamma_D\,, \\
\boldsymbol{\sigma\nu} & = & \mathbf{0} & \text{on} \quad \Gamma_N\,, \\
\phi & = & \phi_D & \text{on} \quad \Gamma_D\,, \\
\mathbf{p}\cdot\boldsymbol{\nu} & = & 0 & \text{on} \quad \Gamma_N\,,
\end{array}
\tag{2.1}
$$

where the sought quantities are the Cauchy fluid stress $\boldsymbol{\sigma}$, the local volume-average velocity of the fluid $\mathbf{u}$, the pressure p, and the local concentration of species ϕ. Regarding this study, we will restrict ourselves to a specific physical scenario corresponding to the process of sedimentation-consolidation of a mixture. Also, $\mu : \mathrm{R}^+ \to \mathrm{R}^+$ is the kinematic effective viscosity, $\theta : \mathrm{R}^+ \to \mathrm{R}^+$ is the diffusion term modelling e.g. sediment compressibility, and $\gamma : \mathrm{R}^+ \to \mathrm{R}$ is the one dimensional flux describing hindered settling, all them nonlinear functions. In addition, $\mathbf{k}$ is a constant vector pointing in the direction of gravity, and $\mathbf{f} \in \mathbf{L}^\infty(\Omega)$, $g \in \mathrm{L}^2(\Omega)$, $\mathbf{u}_D \in \mathrm{H}^{1/2}(\Gamma_D)$ and $\phi_D \in \mathrm{H}^{1/2}(\Gamma_D)$ are given functions. We assume that:

i) there exist $\mu_1,\, \mu_2,\, \gamma_1,\, \gamma_2 > 0$ such that

$$
\mu_1 \le \mu(\phi) \le \mu_2 \quad \textit{and} \quad \gamma_1 \le \gamma(\phi) \le \gamma_2 \quad \forall\, \phi \in \mathrm{R}^+\,, \tag{2.2}
$$

ii) $\theta \in C^1(\mathrm{R}^+)$ and there exist $\theta_1, \theta_2 > 0$ such that

$$
\theta_1 \le \theta(s) \le \theta_2 \quad \textit{and} \quad \theta_1 \le \theta(s) + s\theta'(s) \le \theta_2 \quad \forall\, s \in \mathrm{R}^+\,, \tag{2.3}
$$

iii) there exist $L_\mu,\, L_\gamma > 0$ such that

$$
|\mu(s) - \mu(t)| \le L_\mu |s - t| \quad \textit{and} \quad |\gamma(s) - \gamma(t)| \le L_\gamma |s - t| \qquad \forall\, s,\, t\, \in \mathrm{R}^+\,. \tag{2.4}
$$

Now, following the approach employed in [2] y [3] , it can be seen from the first and third equations of (2.1) that

$$
p = -\frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\sigma}) \quad \mathrm{in}\Omega, \tag{2.5}
$$

which allows us to eliminate the pressure. Next, introducing the auxiliary unknown $\mathbf{t} := \nabla\phi$ in Ω, the fourth equation of (2.1) is rewritten as

$$
\mathbf{p} = \theta(|\mathbf{t}|)\,\mathbf{t} - \phi\mathbf{u} - \gamma(\phi)\mathbf{k} \quad \text{in} \quad \Omega\,,
$$

and hence, the coupled problem (2.1) becomes

$$
\begin{array}{rcll}
\dfrac{1}{\mu_(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} & = & \nabla\mathbf{u} & \text{in} \quad \Omega\,, \\
-\mathbf{div}\,\boldsymbol{\sigma} & = & \mathbf{f}\phi, & \text{in} \quad \Omega\,, \\
\mathbf{t} & = & \nabla\phi & \text{in} \quad \Omega\,, \\
\mathbf{p} & = & \theta(|\mathbf{t}|)\,\mathbf{t} - \phi\mathbf{u} - \gamma(\phi)\mathbf{k} & \text{in} \quad \Omega\,, \\
\mathbf{div}\,\mathbf{p} & = & -g & \text{in} \quad \Omega\,, \\
\mathbf{u} & = & \mathbf{u}_D & \text{on} \quad \Gamma_D\,, \\
\boldsymbol{\sigma\nu} & = & \mathbf{0} & \text{on} \quad \Gamma_N\,, \\
\phi & = & \phi_D & \text{on} \quad \Gamma_D\,, \\
\mathbf{p}\cdot\boldsymbol{\nu} & = & 0 & \text{on} \quad \Gamma_N\,.
\end{array}
\tag{2.6}
$$

We remark here that the incompressibility constraint $\operatorname{div}\mathbf{u} = 0 \in \Omega$ is implicitly present in the first equation of (2.6), that is in the constitutive equation relating $\boldsymbol{\sigma}$ and $\mathbf{u}$. Also, we observe that the pressure can be approximated later on through the post-process suggested by (2.5).

3 The continuous formulation

3.1 The augmented fully-mixed formulation

We begin by observing that the homogeneous Neumann boundary conditions for $\boldsymbol{\sigma}$ and $\mathbf{p}$ in Γ_N suggest the introduction of the following spaces

$$\mathbb{H}_N(\mathbf{div},\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div},\Omega) : \quad \boldsymbol{\tau}\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_N\Big\},$$

$$\mathbf{H}_N(\operatorname{div},\Omega) := \Big\{\mathbf{q} \in \mathbf{H}(\operatorname{div},\Omega) : \quad \mathbf{q}\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_N\Big\}.$$

Then, multiplying the first equation of (2.6) by $\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div},\Omega)$, integrating by parts, and using the Dirichlet boundary condition for $\mathbf{u}$, we obtain

$$\int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^d : \boldsymbol{\tau}^d + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle_{\Gamma_D} \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div},\Omega). \tag{3.1}$$

In addition, the equilibrium equation, that is the second equation of (2.6), is rewritten as

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = -\int_\Omega \mathbf{f}\phi\cdot\boldsymbol{v} \quad \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega). \tag{3.2}$$

Similarly for deriving the weak formulation of the transport equation we multiply by $\mathbf{q} \in \mathbf{H}_N(\operatorname{div},\Omega)$ the third equation of (2.6), integrate by parts, and use the Dirichlet boundary condition for ϕ, to yield

$$\int_\Omega \mathbf{t}\cdot\mathbf{q} + \int_\Omega \phi\operatorname{div}\mathbf{q} = \langle\mathbf{q}\cdot\boldsymbol{\nu},\phi_D\rangle_{\Gamma_D} \qquad \forall\,\mathbf{q} \in \mathbf{H}_N(\operatorname{div},\Omega). \tag{3.3}$$

Also, the corresponding equilibrium equation is stated as

$$\int_\Omega \varphi\operatorname{div}\mathbf{p} = -\int_\Omega g\,\varphi \quad \forall\,\varphi \in \mathrm{L}^2(\Omega). \tag{3.4}$$

Finally, multiplying by $\mathbf{s} \in \mathbf{L}^2(\Omega)$ the fourth equation of (2.1) and integrating, we arrive at

$$\int_\Omega \theta(|\mathbf{t}|)\mathbf{t}\cdot\mathbf{s} - \int_\Omega \mathbf{p}\cdot\mathbf{s} - \int_\Omega \phi\mathbf{u}\cdot\mathbf{s} = \int_\Omega \gamma(\phi)\mathbf{k}\cdot\mathbf{s} \quad \forall\mathbf{s} \in \mathbf{L}^2(\Omega). \tag{3.5}$$

Summarizing, given $\phi \in \mathrm{L}^2(\Omega)$, we obtain form (3.1) and (3.2) the following mixed formulation for the flow equations: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{H}_N(\operatorname{div},\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} a_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\mathbf{u}) &= \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle_{\Gamma_D} &\qquad \forall\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div},\Omega), \\ b(\boldsymbol{\sigma},\boldsymbol{v}) &= -\int_\Omega \mathbf{f}\phi\cdot\boldsymbol{v} &\qquad \forall\boldsymbol{v} \in \mathbf{L}^2(\Omega), \end{aligned} \tag{3.6}$$

where $a_\phi : \mathbb{H}_N(\mathbf{div},\Omega) \times \mathbb{H}_N(\mathbf{div},\Omega) \to \mathrm{R}$ and $b : \mathbb{H}_N(\mathbf{div},\Omega) \times \mathbf{L}^2(\Omega) \to \mathrm{R}$ are the bounded bilinear forms defined by

$$a_\phi(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\zeta}^d : \boldsymbol{\tau}^d \quad \text{and} \quad b(\boldsymbol{\tau},\boldsymbol{v}) := \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\tau},$$

for $\boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div}, \Omega)$ and $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$.

In turn, given $\mathbf{u} \in \mathbf{L}^2(\Omega)$, at first instance we get from (3.3), (3.4) and (3.5) the following mixed formulation for the transport equations: Find $(\mathbf{t}, \mathbf{p}, \phi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_N(\mathrm{div}, \Omega) \times \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mathbf{t} \cdot \mathbf{q} + \int_\Omega \phi \,\mathrm{div}\, \mathbf{q} \quad &= \quad \langle \mathbf{q} \cdot \nu, \phi_D \rangle_{\Gamma_D} \qquad \forall\, \mathbf{q} \in \mathbf{H}_N(\mathrm{div}, \Omega), \\ \int_\Omega \theta(|\mathbf{t}|)\mathbf{t} \cdot \mathbf{s} - \int_\Omega \mathbf{p} \cdot \mathbf{s} - \int_\Omega \phi \mathbf{u} \cdot \mathbf{s} \quad &= \quad \int_\Omega \gamma(\phi) \mathbf{k} \cdot \mathbf{s} \qquad \forall \mathbf{s} \in \mathbf{L}^2(\Omega)\,, \\ \int_\Omega \varphi \,\mathrm{div}\, \mathbf{p} \quad &= \quad -\int_\Omega g\,\varphi \qquad \forall\, \varphi \in \mathrm{L}^2(\Omega)\,. \end{aligned} \tag{3.7}$$

Then, we observe that the assumption on μ given by (2.2) and the Babuska-Brezzi theory suffice to show that (3.6) is well-possed (see, e.g. [18, Thm. 2.1] for details). However, in order to deal with the analysis of (3.7), particularly to handle the third term of the second equation, it is required that actually $\mathbf{u}$ and ϕ belong to $\mathbf{H}^1(\Omega)$ and $\mathrm{H}^1(\Omega)$ respectively. In fact, using Cauchy-Schwarz's inequality and the continuous injections $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and $\mathbf{i} : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, we have that

$$\left| \int_\Omega \varphi \boldsymbol{v} \cdot \mathbf{s} \right| \le c(\Omega) \|\boldsymbol{v}\|_{1,\Omega} \, \|\varphi\|_{1,\Omega} \, \|\mathbf{s}\|_{0,\Omega} \quad \forall\, (\boldsymbol{v}, \varphi, \mathbf{s}) \in \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathbf{L}^2(\Omega)\,, \tag{3.8}$$

with $c(\Omega) := \|i\| \, \|\mathbf{i}\|$. Furthermore, while the exact solutions of (3.6) and (3.7) satisfy $\dfrac{1}{\mu(\phi)} \boldsymbol{\sigma}^{\mathrm{d}} = \nabla \mathbf{u}$ in $D'(\Omega)$ and $\mathbf{t} = \nabla \phi$ in $D'(\Omega)$, which implies that $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $\phi \in \mathrm{H}^1(\Omega)$, these distributional identities do not necessarily extend to the discrete cases of (3.6) and (3.7). Therefore, proceeding as in [2], we now incorporate the following redundant Galerkin terms

$$\begin{aligned} k_1 \int_\Omega \left(\nabla \mathbf{u} - \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d \right) : \nabla \boldsymbol{v} \quad &= \quad 0 \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,, \\ k_2 \int_\Omega \mathbf{div}\, \boldsymbol{\sigma} \cdot \mathbf{div}\, \boldsymbol{\tau} \quad &= \quad -k_2 \int_\Omega \mathbf{f} \phi \cdot \mathbf{div}\, \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div}, \Omega)\,, \\ k_3 \int_{\Gamma_D} \mathbf{u} \cdot \boldsymbol{v} \quad &= \quad k_3 \int_{\Gamma_D} \mathbf{u}_D \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,, \end{aligned} \tag{3.9}$$

where (k_1, k_2, k_3) is a vector of positive parameters to be specified later on. Notice that the first and third equations in (3.9) implicitly require the velocity $\mathbf{u}$ to belong to $\mathbf{H}^1(\Omega)$. In this way, instead of (3.6), from now on we consider the following augmented mixed formulation: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in H_1 := \mathbb{H}_N(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega)$ such that

$$B_\phi((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \boldsymbol{v})) = F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in H_1\,, \tag{3.10}$$

where

$$\begin{aligned} B_\phi((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \quad &:= \quad a_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) - b(\boldsymbol{\sigma}, \boldsymbol{v}) + k_1 \int_\Omega \left(\nabla \mathbf{u} - \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d \right) : \nabla \boldsymbol{v} \\ &+ \quad k_2 \int_\Omega \mathbf{div}\, \boldsymbol{\sigma} \cdot \mathbf{div}\, \boldsymbol{\tau} + k_3 \int_{\Gamma_D} \mathbf{u} \cdot \boldsymbol{v} \end{aligned} \tag{3.11}$$

and

$$F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \quad := \quad \langle \boldsymbol{\tau}, \mathbf{u}_D \rangle_{\Gamma_D} + \int_\Omega \mathbf{f} \phi \cdot \boldsymbol{v} - k_2 \int_\Omega \mathbf{f} \phi \cdot \mathbf{div}\, \boldsymbol{\tau} + k_3 \int_{\Gamma_D} \mathbf{u}_D \cdot \boldsymbol{v}\,. \tag{3.12}$$

Similarly, the transport formulation (3.7) is augmented with the following redundant Galerkin terms

$$
\begin{aligned}
l_1 \int_\Omega (\mathbf{p} - \theta(|\mathbf{t}|)\mathbf{t} + \phi\mathbf{u}) \cdot \mathbf{q} &= -l_1 \int_\Omega \gamma(\phi)\,\mathbf{k}\cdot\mathbf{q} && \forall\, \mathbf{q} \in \mathbf{H}_N(\mathrm{div},\Omega)\,,\\
l_2 \int_\Omega \mathrm{div}\,\mathbf{p}\,\mathrm{div}\,\mathbf{q} &= -l_2 \int_\Omega g\,\mathrm{div}\,\mathbf{q} && \forall\, \mathbf{q} \in \mathbf{H}_N(\mathrm{div},\Omega)\,,\\
l_3 \int_\Omega (\nabla\phi - \mathbf{t})\cdot\nabla\varphi &= 0 && \forall\, \varphi \in \mathrm{H}^1(\Omega)\,,\\
l_4 \int_{\Gamma_D} \phi\varphi &= l_4 \int_{\Gamma_D} \phi_D\,\varphi && \forall\, \varphi \in \mathrm{H}^1(\Omega)\,,
\end{aligned} \tag{3.13}
$$

where (l_1, l_2, l_3, l_4) is a vector of positive parameters to be specified later on. Analogously as before, the third and fourth equations of (3.13) require that ϕ belongs to $\mathrm{H}^1(\Omega)$. In this way, instead of (3.7), we consider from now on the following augmented mixed formulation: Find $(\mathbf{t}, \mathbf{p}, \phi) \in H_2 := \mathbf{L}^2(\Omega) \times \mathbf{H}_N(\mathrm{div},\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$
[(A + \widetilde{B}_{\mathbf{u}})(\mathbf{t},\mathbf{p},\phi), (\mathbf{s},\mathbf{q},\varphi)] = \widetilde{F}_\phi(\mathbf{s},\mathbf{q},\varphi) \qquad \forall\, (\mathbf{s},\mathbf{q},\varphi) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_N(\mathrm{div},\Omega) \times \mathrm{H}^1(\Omega) \tag{3.14}
$$

where $[\cdot,\cdot]$ stands for the duality pairing between H_2 and H_2', $A : H_2' \to H_2$ and $\widetilde{B}_{\mathbf{u}} : H_2' \to H_2$ are the nonlinear and linear operators, respectively, given by

$$
\begin{aligned}
[A(\mathbf{t},\mathbf{p},\phi), (\mathbf{s},\mathbf{q},\varphi)] :=& \int_\Omega \theta(|\mathbf{t}|)\mathbf{t}\cdot\mathbf{s} - \int_\Omega \mathbf{p}\cdot\mathbf{s} + \int_\Omega \mathbf{t}\cdot\mathbf{q} + \int_\Omega \phi\,\mathrm{div}\,\mathbf{q} - \int_\Omega \varphi\,\mathrm{div}\,\mathbf{p}\\
&+ l_1 \int_\Omega (\mathbf{p} - \theta(|\mathbf{t}|)\mathbf{t})\cdot\mathbf{q} + l_2 \int_\Omega \mathrm{div}\,\mathbf{p}\,\mathrm{div}\,\mathbf{q} + l_3 \int_\Omega (\nabla\phi - \mathbf{t})\cdot\nabla\varphi + l_4 \int_{\Gamma_D} \phi\varphi\,,
\end{aligned} \tag{3.15}
$$

and

$$
[\widetilde{B}_{\mathbf{u}}(\mathbf{t},\mathbf{p},\phi), (\mathbf{s},\mathbf{q},\varphi)] := \int_\Omega \phi\mathbf{u}\cdot(l_1\mathbf{q} - \mathbf{s})\,, \tag{3.16}
$$

and $\widetilde{F}_\phi \in H_2'$ is defined by

$$
\widetilde{F}_\phi(\mathbf{s},\mathbf{q},\varphi) := \langle \mathbf{q}\cdot\nu, \phi_D\rangle_{\Gamma_D} + \int_\Omega \gamma(\phi)\mathbf{k}\cdot(\mathbf{s} - l_1\mathbf{q}) + \int_\Omega \varphi\, g - l_2 \int_\Omega g\mathrm{div}\,\mathbf{q} + l_4 \int_{\Gamma_D} \phi_D\varphi \tag{3.17}
$$

for all $(\mathbf{s},\mathbf{q},\varphi) \in H_2$. The well-posedness of (3.10) and (3.14) is proved below in Section 3.3. Consequently, the augmented fully mixed formulation of the coupled problem (2.6) reduces to: Find $((\boldsymbol{\sigma},\mathbf{u},),(\mathbf{t},\mathbf{p},\phi)) \in H_1 \times H_2$ such that

$$
\begin{aligned}
B_\phi((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\boldsymbol{v})) &= F_\phi(\boldsymbol{\tau},\boldsymbol{v}) && \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in H_1\,,\\
\Big[(A + \widetilde{B}_{\mathbf{u}})(\mathbf{t},\mathbf{p},\phi), (\mathbf{s},\mathbf{q},\varphi)\Big] &= \widetilde{F}_\phi(\mathbf{s},\mathbf{q},\varphi) && \forall\,(\mathbf{s},\mathbf{q},\varphi) \in H_2\,.
\end{aligned} \tag{3.18}
$$

3.2 A fixed point strategy

According to the alternative formulations (3.10) and (3.14), and proceeding as in [2] and [3] (se also, [11] and [12]), we suggest a fixed point strategy to analyze (3.18). Indeed, let $\mathbf{S} : \mathrm{H}^1(\Omega) \to H_1$ be the operator defined by

$$
\mathbf{S}(\psi) = (\mathbf{S}_1(\psi), \mathbf{S}_2(\psi)) := (\boldsymbol{\sigma},\mathbf{u}) \in H_1 \quad \forall\,\psi \in \mathrm{H}^1(\Omega)\,,
$$

where $(\boldsymbol{\sigma},\mathbf{u})$ is the unique solution of (3.10) with the given $\phi = \psi$. In turn, let $\widetilde{\mathbf{S}} : \mathrm{H}^1(\Omega)\times\mathbf{H}^1(\Omega) \to H_2$ be the operator defined by

$$
\widetilde{\mathbf{S}}(\psi,\mathbf{u}) = (\widetilde{\mathbf{S}}_1(\psi,\mathbf{u}), \widetilde{\mathbf{S}}_2(\psi,\mathbf{u}), \widetilde{\mathbf{S}}_3(\psi,\mathbf{u})) := (\mathbf{t},\mathbf{p},\phi) \in H_2\,,
$$

where $(\mathbf{t},\mathbf{p},\phi)$ is the unique solution of (3.14) with $\phi=\psi$ and $\mathbf{u}$ given. Then, we define the operator $\mathbf{T}:\mathrm{H}^1(\Omega)\to\mathrm{H}^1(\Omega)$ by

$$\mathbf{T}(\psi):=\widetilde{\mathbf{S}}_3(\psi,\mathbf{S}_2(\psi)) \tag{3.19}$$

and realize that solving (3.18) is equivalent to seeking a fixed point of $\mathbf{T}$, that is : Find $\psi\in\mathrm{H}^1(\Omega)$ such that

$$\mathbf{T}(\psi)=\psi\,. \tag{3.20}$$

3.3 Well-posedness of the uncoupled problems

In this section, we show that the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are well defined, that is that the uncoupled problems (3.10) and (3.14) are in fact well-posed. We begin by recalling (see, e.g. [6]) that

$$\mathbb{H}(\mathbf{div},\Omega)=\mathbb{H}_0(\mathbf{div},\Omega)\oplus\mathrm{R}\mathbb{I}\,,\quad\text{where}\quad\mathbb{H}_0(\mathbf{div},\Omega):=\Big\{\zeta\in\mathbb{H}(\mathbf{div},\Omega):\int_\Omega\mathrm{tr}\,(\zeta)=0\Big\}\,.$$

More precisely, for each $\zeta\in\mathbb{H}(\mathbf{div},\Omega)$ there exists unique $\zeta_0:=\zeta-\Big\{\frac{1}{n|\Omega|}\int_\Omega\mathrm{tr}\,(\zeta)\Big\}\mathbb{I}\in\mathbb{H}_0(\mathbf{div},\Omega)$ and $d:=\frac{1}{n|\Omega|}\int_\Omega\mathrm{tr}\,(\zeta)\in\mathrm{R}$ such that $\zeta=\zeta_0+d\mathbb{I}$. The following three lemmas from [6], [19] and [17], which concern the above decomposition and an equivalence of norm, will be employed to show the well-posedness of (3.10) and (3.14).

Lemma 3.1. *There exists $c_1=c_1(\Omega)>0$ such that*

$$c_1\|\boldsymbol{\tau}_0\|_{0,\Omega}^2\le\|\boldsymbol{\tau}^d\|_{0,\Omega}^2+\|\mathbf{div}\,(\boldsymbol{\tau})\|_{0,\Omega}^2\quad\forall\,\boldsymbol{\tau}\,=\,\boldsymbol{\tau}_0+c\mathbb{I}\,\in\,\mathbb{H}(\mathbf{div},\Omega)\,,$$

with $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div},\Omega)$ and $c\in\mathrm{R}$.

Proof. See [6, Proposition 3.1]. □

Lemma 3.2. *There exists $c_2=c_2(\Omega,\Gamma_N)>0)$ such that*

$$c_2\|\boldsymbol{\tau}\|_{\mathbf{div}\,;\Omega}^2\le\|\boldsymbol{\tau}_0\|_{\mathbf{div}\,;\Omega}^2\quad\forall\,\boldsymbol{\tau}=\boldsymbol{\tau}_0+c\mathbb{I}\,\in\,\mathbb{H}_N(\mathbf{div},\Omega)$$

with $\boldsymbol{\tau}_0\,\in\,\mathbb{H}_0(\mathbf{div},\Omega)$ and $c\,\in\,\mathrm{R}$.

Proof. See [19, Lemma 2.2]. □

Lemma 3.3. *There exists $c_i=c_i(\Omega,\gamma_D)>0$, with $i\in\{3,4\}$ such that*

$$|\boldsymbol{v}|_{1,\Omega}^2+\|\boldsymbol{v}\|_{0,\Gamma_D}^2\ge c_3\|\boldsymbol{v}\|_{1,\Omega}^2\quad\forall\,\boldsymbol{v}\,\in\,\mathbf{H}^1(\Omega)$$

$$|\varphi|_{1,\Omega}^2+\|\varphi\|_{0,\Gamma_D}^2\ge c_4\|\varphi\|_{1,\Omega}^2\quad\forall\,\varphi\,\in\,\mathrm{H}^1(\Omega)$$

Proof. It corresponds to a slight modification of the proof of [17, Lemma 3.3]. □

On the other hand, the following results refers to the nonlinear term forming part of A (cf. (3.15)).

Lemma 3.4. *Let $\widetilde{\theta}_2:=max\{\theta_2,2\theta_2-\theta_1\}$ (cf. (2.3)). Then*

$$\|\theta(|\mathbf{r}|)\mathbf{r}-\theta(|\mathbf{s}|)\mathbf{s}\|_{0,\Omega}\le\widetilde{\theta}_2\|\mathbf{r}-\mathbf{s}\|_{0,\Omega}$$

$$\int_\Omega\{\theta(|\mathbf{r}|)\mathbf{r}-\theta(|\mathbf{s}|)\mathbf{s}\}\cdot(\mathbf{r}-\mathbf{s})\ge\theta_1\|\mathbf{r}-\mathbf{s}\|_{0,\Omega}{}^2$$

for all $\mathbf{r},\mathbf{s}\,\in\,\mathbf{L}^2(\Omega)$.

Proof. See [20, Theorem 3.8] for details. $\square$

In what follows, we consider

$$\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{H_1} := \left\{ \|\boldsymbol{\sigma}\|^2_{\mathbf{div}\,;\Omega} + \|\boldsymbol{v}\|^2_{1,\Omega} \right\}^{1/2} \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in H_1$$

and

$$\|(\mathbf{s}, \mathbf{q}, \varphi)\|_{H_2} := \left\{ \|\mathbf{s}\|^2_{0,\Omega} + \|\mathbf{q}\|^2_{\mathbf{div}\,;\Omega} + \|\varphi\|^2_{1,\Omega} \right\}^{1/2} \qquad \forall\, (\mathbf{s}, \mathbf{q}, \varphi) \in H_2\,.$$

We now prove the well-definiteness of $\mathbf{S}$.

Lemma 3.5. *Assume that* $k_1 \in \left(0, \dfrac{2\delta\mu_1}{\mu_2}\right)$ *with* $\delta \in (0, 2\mu_1)$*, and that* $0 < k_2, k_3$*. Then, for each* $\phi \in \mathrm{H}^1(\Omega)$ *the problem (3.10) has a unique solution* $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \mathbf{u}) \in H_1$*. Moreover, there exists* $C_{\mathbf{S}} > 0$*, independent of* ϕ*, such that*

$$\|\mathbf{S}(\phi)\|_{H_1} \;=\; \|(\boldsymbol{\sigma}, \mathbf{u})\|_{H_1} \leq C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega} \|\phi\|_{1,\Omega} \right\} \quad \forall\, \phi \,\in\, \mathrm{H}^1(\Omega)\,. \tag{3.21}$$

Proof. It reduces to show that, under the stipulated ranges for the parameters κ_1, κ_2, κ_3, and δ, the bilinear form B_ϕ becomes H_1-elliptic with an ellipticity constant independent of $\phi \in \mathrm{H}^1(\Omega)$. We omit details and refer to [2, Lemma 3.4]. $\square$

Throughout the rest of the paper, a regularity assumption will be made for the problem defining the operator $\mathbf{S}$. More precisely, we assume that $\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}$ for some

$$\varepsilon \in \left\{ \begin{array}{lll} (0,1) & if & n = 2\,, \\ (\frac{1}{2},1) & if & n = 3\,, \end{array} \right. \tag{3.22}$$

and that for each $\phi \in \mathrm{H}^1(\Omega)$ with $\|\phi\|_{1,\Omega} \leq r$, $r > 0$, there holds $\mathbf{S}(\phi) = (\zeta, \mathbf{s}) \in (\mathbb{H}_N(\mathbf{div}, \Omega) \cap \mathbb{H}^\varepsilon(\varepsilon)) \times \big(\mathbf{H}^1(\Omega) \cap \mathbb{H}^{1+\varepsilon}(\Omega)\big)$ and

$$\|\zeta\|_{\varepsilon,\Omega} + \|\mathbf{s}\|_{1+\varepsilon,\Omega} \leq \widetilde{C}_{\widetilde{\mathbf{S}}}(r) \left\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega} \|\phi\|_{0,\Omega} \right\}, \tag{3.23}$$

with a positive constant $\widetilde{C}_{\widetilde{\mathbf{S}}}(r)$ independent of the given ϕ but depending on the upper bound r of its H^1-norm. We remark that the reason of the stipulated ranges for ε will be clarified in the forthcoming analysis (see below proof of Lemma 3.11). For more details see [2].

Next, in order to demonstrate that S is well-posed, we need the following two previous Lemmas.

Lemma 3.6. *For each* $\mathbf{u} \in \mathbf{H}^1(\Omega)$*,* $A + \widetilde{B}_{\mathbf{u}}$ *is Lipschitz-continuous.*

Proof. Given $(\mathbf{t}, \mathbf{p}, \phi)$, $(\mathbf{r}, \mathbf{o}, \psi)$ and $(\mathbf{s}, \mathbf{q}, \varphi) \in H_2$, we first notice that

$$\begin{aligned} \big|[A(\mathbf{t}, \mathbf{p}, \phi) \;&-\; A(\mathbf{r}, \mathbf{o}, \psi), (\mathbf{s}, \mathbf{q}, \varphi)]\big| = \Big| \int_\Omega \{\theta(|\mathbf{t}|)\mathbf{t} - \theta(|\mathbf{r}|)\mathbf{r}\} \cdot \mathbf{s} + \int_\Omega (\mathbf{o} - \mathbf{p}) \cdot \mathbf{s} + \int_\Omega (\mathbf{t} - \mathbf{r}) \cdot \mathbf{q} \\ &+\; \int_\Omega (\phi - \psi) \cdot \mathbf{q} + \int_\Omega \varphi \,\mathrm{div}\,(\mathbf{o} - \mathbf{p}) + l_1 \int_\Omega (\mathbf{o} - \mathbf{p}) \cdot \mathbf{q} + l_1 \int_\Omega \{\theta(|\mathbf{t}|)\mathbf{t} - \theta(|\mathbf{r}|)\mathbf{r}\} \cdot \mathbf{q} \\ &+\; l_2 \int_\Omega \mathrm{div}\,(\mathbf{p} - \mathbf{q})\,\mathrm{div}\,\mathbf{q} + l_3 \int_\Omega \nabla(\phi - \psi) \cdot \nabla\varphi + l_3 \int_\Omega (\mathbf{r} - \mathbf{t}) \cdot \nabla\varphi + l_4 \int_{\Gamma_D} (\phi - \psi)\,\varphi \Big| \end{aligned}$$

Now, using the Cauchy-Schwarz inequality, the Lipschitz-continuity of the operator induced by θ (cf. Lemma 3.4) and the trace theorem (with constant c_0), we deduce from the foregoing equation that

$$
\begin{aligned}
&\big|[A(\mathbf{t},\mathbf{p},\phi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)]\big| \le \|\theta(|\mathbf{t}|)\mathbf{t}-\theta(|\mathbf{r}|)\mathbf{r}\|_{0,\Omega}\|\mathbf{s}\|_{0,\Omega}+\|\mathbf{o}-\mathbf{p}\|_{0,\Omega}\|\mathbf{s}\|_{0,\Omega}\\
&\quad+\|\mathbf{t}-\mathbf{r}\|_{0,\Omega}\|\mathbf{q}\|_{0,\Omega}+\|\phi-\psi\|_{0,\Omega}\|\mathbf{q}\|_{0,\Omega}+\|\varphi\|_{0,\Omega}\|\mathrm{div}\,(\mathbf{o}-\mathbf{p})\|_{0,\Omega}\\
&\quad+l_1\|\mathbf{o}-\mathbf{p}\|_{0,\Omega}\|\mathbf{q}\|_{0,\Omega}+l_1\|\theta(|\mathbf{t}|)\mathbf{t}-\theta(|\mathbf{r}|)\mathbf{r}\|_{0,\Omega}\|\mathbf{q}\|_{0,\Omega}+l_2\|\mathrm{div}\,(\mathbf{p}-\mathbf{q})\|_{0,\Omega}\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega}\\
&\quad+l_3|\phi-\psi|_{1,\Omega}|\varphi|_{1,\Omega}+l_3\|\mathbf{r}-\mathbf{t}\|_{0,\Omega}|\varphi|_{1,\Omega}+l_4\|\phi-\psi\|_{0,\Gamma_D}\|\varphi\|_{0,\Gamma_D}\\
&\quad\le \widetilde{L}_A(4\|\mathbf{t}-\mathbf{r}\|_{0,\Omega}^2+2\|\mathbf{p}-\mathbf{o}\|_{0,\Omega}^2+2\|\phi-\psi\|_{1,\Omega}^2\\
&\quad+2\|\mathrm{div}\,(\mathbf{p}-\mathbf{o})\|_{0,\Omega}^2+|\phi-\psi|_{1,\Omega}^2)^{1/2}(2\|\mathbf{s}\|_{0,\Omega}^2+4\|\mathbf{q}\|_{0,\Omega}^2+\|\varphi\|_{0,\Omega}^2+\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega}^2\\
&\quad+2|\varphi|_{1,\Omega}^2+\|\varphi\|_{1,\Omega}^2)^{1/2}\,,
\end{aligned}
$$

with $\widetilde{L}_A=\max\{\widetilde{\theta}_2,1,l_1,l_1\widetilde{\theta}_2,l_2,l_3,l_4c_0\}$, which yields

$$
\big|[A(\mathbf{t},\mathbf{p},\phi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)]\big|\le L_A\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{r},\mathbf{o},\psi)\|_{H_2}\|(\mathbf{s},\mathbf{q},\varphi)\|_{H_2} \tag{3.24}
$$

for all $(\mathbf{t},\mathbf{p},\phi)$, $(\mathbf{r},\mathbf{o},\psi)$, $(\mathbf{s},\mathbf{q},\varphi)\in H_2$, with $L_A:=4\widetilde{L}_A$. In turn, it readily follows from (3.8) and (3.16) that

$$
\begin{aligned}
|[\widetilde{B}_{\mathbf{u}}(\mathbf{s},\mathbf{q},\varphi),(\mathbf{r},\mathbf{o},\psi)]| &= \Big|\int_\Omega \varphi\mathbf{u}\cdot(l_1\mathbf{o}-\mathbf{r})\Big|\\
&\le \|\varphi\|_{\mathrm{L}^4(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}(l_1^2+1)^{1/2}\Big\{\|\mathbf{r}\|_{0,\Omega}^2+\|\mathbf{o}\|_{\mathbf{div}\,;\Omega}^2\Big\}^{1/2}\\
&\le c(\Omega)(l_2^2+1)^{1/2}\|\mathbf{u}\|_{1,\Omega}\|\varphi\|_{1,\Omega}\|(\mathbf{r},\mathbf{o},\psi)\|_{H_2}\\
&\le c(\Omega)(l_2^2+1)^{1/2}\|\mathbf{u}\|_{1,\Omega}\|(\mathbf{s},\mathbf{q},\varphi)\|_{H_2}\|(\mathbf{r},\mathbf{o},\psi)\|_{H_2}\,,
\end{aligned} \tag{3.25}
$$

which, thanks to the linearity of $\widetilde{B}_{\mathbf{u}}$, and together with (3.24), confirms that $A+\widetilde{B}_{\mathbf{u}}$ is Lipschitz-continuous with constant $L_C:=L_A+c(\Omega)(l_2^2+1)^{1/2}\|\mathbf{u}\|_{1,\Omega}$. ☐

The strong monotonicity of the operator $A+\widetilde{B}_{\mathbf{u}}$ is established next.

Lemma 3.7. *Assume that* $l_1\in\Big(0,\frac{2\theta_1\delta}{\widetilde{\theta}_2}\Big)$ *and* $l_3\in\big(0,2\widetilde{\delta}\big(\theta_1-\frac{\widetilde{\theta}_2 l_1}{2\delta}\big)\big)$, *with* $\delta\in\Big(0,\frac{2}{\widetilde{\theta}_2}\Big)$ *and* $\widetilde{\delta}\in(0,2)$, *and that* $l_2,l_4>0$. *Then, for each* $\mathbf{u}\in\mathbf{H}^1(\Omega)$ *such that* $\|\mathbf{u}\|_{1,\Omega}\le\dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$, $A+\widetilde{B}_{\mathbf{u}}$ *is strongly monotone.*

Proof. Given $(\mathbf{s},\mathbf{q},\varphi),(\mathbf{r},\mathbf{o},\psi)\in H_2$, we first observe that

$$
\begin{aligned}
&[A(\mathbf{s},\mathbf{q},\varphi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)]=\int_\Omega\Big\{\theta(|\mathbf{s}|)\mathbf{s}-\theta(|\mathbf{r}|)\mathbf{r}\Big\}\cdot(\mathbf{s}-\mathbf{r})+l_1\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2\\
&\quad-l_1\int_\Omega(\theta(|\mathbf{s}|)\mathbf{s}-\theta(|\mathbf{r}|)\mathbf{r})\cdot(\mathbf{q}-\mathbf{o})+l_2\|\mathbf{div}\,(\mathbf{q}-\mathbf{o})\|_{0,\Omega}^2+l_3|\varphi-\psi|_{1,\Omega}^2\\
&\quad-l_3\int_\Omega(\mathbf{s}-\mathbf{r})\cdot\nabla(\varphi-\psi)+l_4\|\varphi-\psi\|_{0,\Gamma_D}^2\,.
\end{aligned}
$$

Then, thanks to Lemma 3.4 and the Cauchy-Schwarz and Young inequalities, we obtain

$$
\begin{aligned}
&[A(\mathbf{s},\mathbf{q},\varphi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)] \geq \theta_1\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2+l_1\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2+l_2\|\mathrm{div}\,(\mathbf{q}-\mathbf{o})\|_{0,\Omega}^2 \\
&\quad -l_1\|\theta(|\mathbf{s}|)\mathbf{s}-\theta(|\mathbf{r}|)\mathbf{r}\|_{0,\Omega}\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}+l_3|\varphi-\psi|_{1,\Omega}^2-l_3\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}|\varphi-\psi|_{1,\Omega}+l_4\|\varphi-\psi\|_{0,\Gamma_D} \\
&\quad \geq \theta_1\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2+l_1\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2-l_1\widetilde{\theta}_2\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}+l_2\|\mathrm{div}\,(\mathbf{q}-\mathbf{o})\|_{0,\Omega}^2+l_3|\varphi-\psi|_{1,\Omega}^2 \\
&\quad -l_3\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}|\varphi-\psi|_{1,\Omega}+l_4\|\varphi-\psi\|_{0,\Gamma_D} \\
&\quad \geq \theta_1\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2+l_1\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2-l_1\widetilde{\theta}_2\frac{1}{2\delta}\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2-l_1\widetilde{\theta}_2\frac{\delta}{2}\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2+l_2\|\mathrm{div}\,(\mathbf{q}-\mathbf{o})\|_{0,\Omega}^2 \\
&\quad +l_3|\varphi-\psi|_{1,\Omega}^2-l_3\frac{1}{2\widetilde{\delta}}\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2-l_3\frac{\widetilde{\delta}}{2}|\varphi-\psi|_{1,\Omega}^2+l_4\|\varphi-\psi\|_{0,\Gamma_D}\,,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&[A(\mathbf{s},\mathbf{q},\varphi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)] \geq \Big(\theta_1-l_1\widetilde{\theta}_2\tfrac{1}{2\delta}-l_3\tfrac{1}{2\widetilde{\delta}}\Big)\|\mathbf{s}-\mathbf{r}\|_{0,\Omega}^2 \\
&\quad +l_1\Big(1-\tfrac{\widetilde{\theta}_2\delta}{2}\Big)\|\mathbf{q}-\mathbf{o}\|_{0,\Omega}^2+l_2\|\mathrm{div}\,(\mathbf{q}-\mathbf{o})\|_{0,\Omega}^2+l_3\Big(1-\tfrac{\widetilde{\delta}}{2}\Big)|\varphi-\psi|_{1,\Omega}^2+l_4\|\varphi-\psi\|_{0,\Gamma_D}
\end{aligned}
\tag{3.26}
$$

In this way, assuming the stipulated hypotheses on δ, l_1, l_2, l_3, l_4, we can define the positive constants

$$
\alpha_0(\Omega):=\Big(\theta_1-l_1\widetilde{\theta}_2\frac{1}{2\delta}-l_3\frac{1}{2\widetilde{\delta}}\Big)\,,\qquad \alpha_1(\Omega):=\min\Big\{l_1\Big(1-\frac{\widetilde{\theta}_2\delta}{2}\Big),l_2\Big\}\,,
$$

and

$$
\alpha_2(\Omega):=\min\Big\{l_3\Big(1-\frac{\widetilde{\delta}}{2}\Big),l_4\Big\}\,,
$$

which, together with (3.26), imply that

$$
[A(\mathbf{s},\mathbf{q},\varphi)-A(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)] \geq \alpha(\Omega)\|(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)\|_{H_2}^2\,, \tag{3.27}
$$

with

$$
\alpha(\Omega):=\min\Big\{\alpha_0(\Omega),\alpha_1(\Omega),c_4\alpha_2(\Omega)\Big\}\,. \tag{3.28}
$$

Moreover, by combining (3.25) and (3.27), we obtain

$$
\begin{aligned}
&[(A+\widetilde{B}_{\mathbf{u}})(\mathbf{s},\mathbf{q},\varphi)-(A+\widetilde{B}_{\mathbf{u}})(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)] \\
&\quad \geq \Big\{\alpha(\Omega)-(1+l_1^2)^{1/2}c(\Omega)\|\mathbf{u}\|_{1,\Omega}\Big\}\,\|(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)\|_{H_2}^2\,,
\end{aligned}
\tag{3.29}
$$

and assuming that $\|\mathbf{u}\|_{1,\Omega}\leq\dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$ we conclude that

$$
[(A+\widetilde{B}_{\mathbf{u}})(\mathbf{s},\mathbf{q},\varphi)-(A+\widetilde{B}_{\mathbf{u}})(\mathbf{r},\mathbf{o},\psi),(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)] \geq \frac{\alpha(\Omega)}{2}\|(\mathbf{s},\mathbf{q},\varphi)-(\mathbf{r},\mathbf{o},\psi)\|_{H_2}^2\,, \tag{3.30}
$$

which shows the strong monotonicity of $A+\widetilde{B}_{\mathbf{u}}$ with constant $\frac{\alpha(\Omega)}{2}$. □

Having proved the properties of $A+\widetilde{B}_{\mathbf{u}}$ given by the previous Lemmas 3.6 and 3.7, we are now in a position to show the well-posedness of the operator $\widetilde{\mathbf{S}}$.

Lemma 3.8. *Assume that* $l_1 \in \Big(0, \frac{2\theta_1\delta}{\widetilde{\theta}_2}\Big)$ *and* $l_3 \in \big(0, 2\widetilde{\delta}(\theta_1 - \frac{\widetilde{\theta}_2 l_1}{2\delta})\big)$*, with* $\delta \in \Big(0, \frac{2}{\widetilde{\theta}_2}\Big)$ *and* $\widetilde{\delta} \in (0,2)$*, and that* $l_2, l_4 > 0$*. Then given* $\phi \in \mathrm{H}^1(\Omega)$ *and* $\mathbf{u} \in \mathbf{H}^1(\Omega)$ *such that* $\|\mathbf{u}\|_{1,\Omega} \le \dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$*, there exists a unique* $\widetilde{\mathbf{S}}(\phi,\mathbf{u}) := (\mathbf{t},\mathbf{p},\phi) \in H_2$ *solution of* (3.14) *and there hols*

$$\|\widetilde{\mathbf{S}}(\phi,\mathbf{u})\|_{H_2} = \|(\mathbf{t},\mathbf{p},\phi)\|_{H_2} \le C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}, \tag{3.31}$$

where $C_{\widetilde{\mathbf{S}}} = \dfrac{2}{\alpha(\Omega)} C_{\widetilde{F}_\phi}$ *and* $C_{\widetilde{F}_\phi} = \max\Big\{1, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\| l_1, l_2, c_0 l_4\Big\}$.

Proof. Given $\phi \in \mathrm{H}^1(\Omega)$, we begin by noticing that the Cauchy-Schwarz inequality and the trace theorems imply

$$\begin{aligned}
|\widetilde{F}_\phi(\mathbf{s},\mathbf{q},\varphi)| \;&\le\; |\langle \mathbf{q}\cdot\nu, \phi_D\rangle_{\Gamma_D}| + \left|\int_\Omega \gamma(\phi)\mathbf{k}\cdot(\mathbf{s} - l_1\mathbf{q})\right| + \left|\int_\Omega \varphi g\right| + l_2\left|\int_\Omega g\,\mathrm{div}\,\mathbf{q}\right| + l_4\left|\int_{\Gamma_D}\phi_D\varphi\right| \\
&\le\; \|\mathbf{q}\|_{-1/2,\Gamma_D}\|\phi_D\|_{1/2,\Gamma_D} + \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|\|\mathbf{s} - l_1\mathbf{q}\|_{0,\Omega} + \|g\|_{0,\Omega}\|\varphi\|_{0,\Omega} \\
&+\; l_2\|g\|_{0,\Omega}\|\mathrm{div}\,\mathbf{q}\|_{0,\Omega} + l_4\|\phi_D\|_{0,\Gamma_D}\|\varphi\|_{0,\Gamma_D} \\
&\le\; \|\mathbf{q}\|_{\mathrm{div};\Omega}\|\phi_D\|_{1/2,\Gamma_D} + \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|(\|\mathbf{s}\|_{0,\Omega} + l_1\|\mathbf{q}\|_{\mathrm{div};\Omega}) + \|g\|_{0,\Omega}\|\varphi\|_{1,\Omega} \\
&+\; l_2\|g\|_{0,\Omega}\|\mathbf{q}\|_{\mathrm{div};\Omega} + l_4 c_0\|\phi_D\|_{0,\Gamma_D}\|\varphi\|_{1,\Omega} \\
&\le\; C_{\widetilde{F}_\phi}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}\|(\mathbf{s},\mathbf{q},\varphi)\|_{H_2},
\end{aligned} \tag{3.32}$$

with $C_{\widetilde{F}_\phi} = \max\Big\{1, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\| l_1, l_2, c_0 l_4\Big\}$, which shows that $\widetilde{F}_\phi \in H_2'$. In this way, knowing from Lemmas 3.6 and 3.7 that for each $\mathbf{u} \in \mathrm{H}^1(\Omega)$ the operator $A + \widetilde{B}_{\mathbf{u}}$ is Lipschitz-continuous and strongly monotone, a classical result on the bijectivity of monotone operators (see e.g. [23, Theorem 3.3.23]) allows us to conclude that there exists a unique solution $\widetilde{\mathbf{S}}(\phi,\mathbf{u}) := (\mathbf{t},\mathbf{p},\phi) \in H_2$ of (3.14). Then, by applying (3.30) with $(\mathbf{s},\mathbf{q},\varphi) = (\mathbf{t},\mathbf{p},\phi)$ and $(\mathbf{r},\mathbf{o},\psi) = (\mathbf{0},\mathbf{0},0)$, we obtain

$$\begin{aligned}
\frac{\alpha(\Omega)}{2}\|(\mathbf{t},\mathbf{p},\phi)\|_{H_2}^2 &\le [(A + \widetilde{B}_{\mathbf{u}})(\mathbf{t},\mathbf{p},\phi), (\mathbf{t},\mathbf{p},\phi)] \le |\widetilde{F}_\phi(\mathbf{t},\mathbf{p},\phi)| \\
&\le C_{\widetilde{F}_\phi}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}\|(\mathbf{t},\mathbf{p},\phi)\|_{H_2},
\end{aligned}$$

which yields

$$\|(\mathbf{t},\mathbf{p},\phi)\|_{H_2} \le C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}.$$

□

We end this section by remarking that a suitable constant $\alpha_0(\Omega)$ can be computed by taking the parameters δ, $l_1, \widetilde{\delta}$ and l_3 as the middle points of their feasible ranges. Then we choose l_2 and l_4 so that the minima defining $\alpha_1(\Omega)$ and $\alpha_2(\Omega)$ are maximized. More precisely, we simply take $\delta = \dfrac{1}{\widetilde{\theta}_2}$ and $\widetilde{\delta} = 1$, which implies $l_1 = \dfrac{\theta_1}{\widetilde{\theta}_2^2}$, and $l_3 = \dfrac{\theta_1}{2}$, and then we set $l_2 = l_1\Big(1 - \dfrac{\widetilde{\theta}_2\delta}{2}\Big) = \frac{l_1}{2}$, and $l_4 = l_3\Big(1 - \dfrac{\widetilde{\delta}}{2}\Big) = \frac{l_3}{2}$, whence

$$\alpha_0(\Omega) = \Big(\theta_1 - l_1\widetilde{\theta}_2\frac{1}{2\delta} - l_3\frac{1}{2\widetilde{\delta}}\Big) = \frac{\theta_1}{2},$$

$$\alpha_1(\Omega) = \min\{l_1\Big(1-\frac{\widetilde{\theta}_2\delta}{2}\Big), l_2\} = \frac{l_1}{2} = \frac{\theta_1}{4\widetilde{\theta}_2^2}\,,$$

$$\alpha_2(\Omega) = \min\{l_3\Big(1-\frac{\widetilde{\delta}}{2}\Big), l_4\} = \frac{l_3}{2} = \frac{\theta_1}{4}\,,$$

and

$$\alpha(\Omega) = \min\Big\{\alpha_0(\Omega), \alpha_1(\Omega), c_4\alpha_2(\Omega)\Big\} = \min\Big\{\frac{\theta_1}{2}, \frac{\theta_1}{4\widetilde{\theta}_2^2}, c_4\frac{\theta_1}{4}\Big\}\,.$$

The foregoing explicit values of the stabilization parameters l_i, $i \in \{1,...,4\}$, will be employed in Section 5 for the corresponding numerical experiments.

3.4 Solvability analysis of the fixed point equation

Having established in the previous section the well-posedness of the uncoupled problems (3.10) and (3.14), thus showing that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$ and $\mathbf{T}$ are well defined, we now address the solvability analysis of the fixed point equation (3.20). For this purpose, in what follows we verify the hypotheses of the Schauder fixed point theorem, which is recalled next (cf. [10, Theorem. 9.12-1(b)]).

Theorem 3.9. *Let $\mathbf{W}$ be a closed and convex subset of a Banach space X and let $\mathbf{T}:\mathbf{W}\to\mathbf{W}$ be a continuous mapping such that $\overline{\mathbf{T}(\mathbf{W})}$ is compact. Then $\mathbf{T}$ has at least one fixed point.*

We start with the following result.

Lemma 3.10. *Given $r>0$, set $\mathbf{W} := \Big\{\phi \in \mathrm{H}^1(\Omega) : \|\phi\|_{1,\Omega} \le r\Big\}$, and assume that the data satisfy*

$$C_{\mathbf{S}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega}\Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{3.33}$$

and

$$C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\} \le r\,. \tag{3.34}$$

Then $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$.

Proof. Given $\phi \in \mathbf{W}$, we get from (3.21) (cf. Lemma 3.5) that

$$\|\mathbf{S}(\phi)\|_{H_1} \;=\; \|(\boldsymbol{\sigma},\mathbf{u})\|_{H_1} \le C_{\mathbf{S}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{1,\Omega}\Big\}$$

and hence, thanks to the restriction (3.33), we observe that $\mathbf{u} = \mathbf{S}_2(\phi)$ satisfies the assumption requested in the statement of Lemma 3.8. Moreover, the corresponding estimate (3.31) gives

$$\begin{aligned}\|\mathbf{T}(\phi)\|_{1,\Omega} &= \|\widetilde{\mathbf{S}}_3(\phi,\mathbf{u})\|_{1,\Omega} \le \|\widetilde{\mathbf{S}}(\phi,\mathbf{u})\|_{H_2} = \|(\mathbf{t},\mathbf{p},\phi)\|_{H_1}\\ &\le C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}\,,\end{aligned} \tag{3.35}$$

which, according to the hypothesis (3.34), guarantees that $\|\mathbf{T}(\phi)\|_{1,\Omega} \le r$, and hence $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$. □

Next, we will demonstrate the continuity and compactness of $\mathbf{T}$, which will be consequence of the following Lemmas proving the continuity of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$.

Lemma 3.11. *There exists a constant $C>0$, depending on μ_1, l_1, l_2, L_μ the ellipticity constant α of B_ϕ (cf. [2, eq. (3.19)]), and the regularity parameter ε (cf. (3.23)), such that*

$$\|\mathbf{S}(\phi)-\mathbf{S}(\varphi)\|_{H_1} \le C\Big\{\|\mathbf{f}\|_{\infty,\Omega}\|\phi-\varphi\|_{0,\Omega} + \|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi-\varphi\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\Big\} \quad \forall \phi,\varphi \in \mathrm{H}^1(\Omega)\,. \tag{3.36}$$

Proof. It follows exactly as in ([2, Lemma 3.9]) irrespective of the fact that now $\phi \in \mathrm{H}^1(\Omega)$. $\square$

Lemma 3.12. *There exists* $\widetilde{C} := \dfrac{2}{\alpha(\Omega)}(1+l_1^2)^{1/2} \max\big\{c(\Omega), L_\gamma\big\}$ *(cf. (3.8), (3.28)) such that for all* $(\phi_1, \mathbf{u}_1), (\phi_2, \mathbf{u}_2) \in \mathrm{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ *with* $\|\mathbf{u}_1\|_{1,\Omega}, \|\mathbf{u}_2\|_{1,\Omega} \le \dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$, *there holds*

$$\|\widetilde{\mathbf{S}}(\phi_1, \mathbf{u}_1) - \widetilde{\mathbf{S}}(\phi_2, \mathbf{u}_2)\|_{H_2} \le \widetilde{C}\Big\{\|\widetilde{\mathbf{S}}_{\mathbf{3}}(\phi_2, \mathbf{u}_2)\|_{1,\Omega}\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} + \|\mathbf{k}\|\|\phi_1 - \phi_2\|_{0,\Omega}\Big\}. \tag{3.37}$$

Proof. Given $(\phi_1, \mathbf{u}_1), (\phi_2, \mathbf{u}_2) \in \mathrm{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ such that $\|\mathbf{u}_1\|_{1,\Omega}$ and $\|\mathbf{u}_2\|_{1,\Omega}$ are bounded above by (3.21), we let

$$(\mathbf{t}_1, \mathbf{p}_1, \phi_1) = \widetilde{\mathbf{S}}(\phi_1, \mathbf{u}_1) = (\widetilde{\mathbf{S}}_1(\phi_1, \mathbf{u}_1), \widetilde{\mathbf{S}}_2(\phi_1, \mathbf{u}_1), \widetilde{\mathbf{S}}_3(\phi_1, \mathbf{u}_1)) \in H_2$$

and

$$(\mathbf{t}_2, \mathbf{p}_2, \phi_2) = \widetilde{\mathbf{S}}(\phi_2, \mathbf{u}_2) = (\widetilde{\mathbf{S}}_1(\phi_2, \mathbf{u}_2), \widetilde{\mathbf{S}}_2(\phi_2, \mathbf{u}_2), \widetilde{\mathbf{S}}_3(\phi_2, \mathbf{u}_2)) \in H_2\,,$$

which means

$$[(A + \widetilde{B}_{\mathbf{u}_1})(\mathbf{t}_1, \mathbf{p}_1, \phi_1), (\mathbf{s}, \mathbf{q}, \varphi)] = \widetilde{F}_{\phi_1}(\mathbf{s}, \mathbf{q}, \varphi)$$

and

$$[(A + \widetilde{B}_{\mathbf{u}_2})(\mathbf{t}_2, \mathbf{p}_2, \phi_2), (\mathbf{s}, \mathbf{q}, \varphi)] = \widetilde{F}_{\phi_2}(\mathbf{s}, \mathbf{q}, \varphi)$$

for all $(\mathbf{s}, \mathbf{q}, \varphi) \in H_2$. Then, thanks to the strong monotonicity of $A + \widetilde{B}_{\mathbf{u}_1}$, we have

$$\begin{aligned} &\frac{\alpha(\Omega)}{2}\|(\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)\|_{H_2}^2 \\ &\quad \le [(A + \widetilde{B}_{\mathbf{u}_1})(\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (A + \widetilde{B}_{\mathbf{u}_1})(\mathbf{t}_2, \mathbf{p}_2, \phi_2), (\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)]\,, \end{aligned} \tag{3.38}$$

from which, adding and subtracting $\widetilde{B}_{\mathbf{u}_2}(\mathbf{t}_2, \mathbf{p}_2, \phi_2)$, we find that

$$\begin{aligned} &\frac{\alpha(\Omega)}{2}\|(\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)\|_{H_2}^2 \\ &\le [(A + \widetilde{B}_{\mathbf{u}_1})(\mathbf{t}_1, \mathbf{p}_1, \phi_1) + \widetilde{B}_{\mathbf{u}_2}(\mathbf{t}_2, \mathbf{p}_2, \phi_2) - \widetilde{B}_{\mathbf{u}_2}(\mathbf{t}_2, \mathbf{p}_2, \phi_2) \\ &- (A + \widetilde{B}_{\mathbf{u}_1})(\mathbf{t}_2, \mathbf{p}_2, \phi_2), (\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)] \\ &\le \widetilde{F}_{\phi_1}((\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)) - \widetilde{F}_{\phi_2}((\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)) \\ &+ [\widetilde{B}_{\mathbf{u}_2 - \mathbf{u}_1}(\mathbf{t}_2, \mathbf{p}_2, \phi_2), (\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)]\,. \end{aligned}$$

In this way, using the injections of $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and $\mathbf{i} : \mathbf{H^1(\Omega)} \to \mathbf{L^4}$, and denoting again $c(\Omega) := \|i\|\|\mathbf{i}\|$ as we did in (3.8), we get

$$\begin{aligned} &|[\widetilde{B}_{\mathbf{u}_2 - \mathbf{u}_1}(\mathbf{t}_2, \mathbf{p}_2, \phi_2), (\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)]| = \Big|\int_\Omega \phi_2(\mathbf{u}_1 - \mathbf{u}_2)\cdot\{l_1(\mathbf{p}_1 - \mathbf{p}_2) - (\mathbf{t}_1 - \mathbf{t}_2)\}\Big| \\ &\quad \le \|\phi_2\|_{\mathrm{L}^4(\Omega)}\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^4(\Omega)}(1+l_1^2)^{1/2}\Big\{\|\mathbf{p}_1 - \mathbf{p}_2\|_{0,\Omega}^2 + \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,\Omega}^2\Big\}^{1/2} \\ &\quad \le c(\Omega)(1+l_1^2)^{1/2}\|\phi_2\|_{1,\Omega}\|\mathbf{u}_2 - \mathbf{u}_1\|_{1,\Omega}\|(\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)\|_{H_2} \\ &\quad = c(\Omega)(1+l_1^2)^{1/2}\|\widetilde{\mathbf{S}}_3(\phi_2, \mathbf{u}_2)\|_{1,\Omega}\|\mathbf{u}_2 - \mathbf{u}_1\|_{1,\Omega}\|(\mathbf{t}_1, \mathbf{p}_1, \phi_1) - (\mathbf{t}_2, \mathbf{p}_2, \phi_2)\|_{H_2}\,, \end{aligned} \tag{3.39}$$

whereas the Lipchitz continuity of γ (cf. (2.4)) yields

$$
\begin{aligned}
&|\widetilde{F}_{\phi_1}((\mathbf{t}_1,\mathbf{p}_1,\phi_1)-(\mathbf{t}_2,\mathbf{p}_2,\phi_2))-\widetilde{F}_{\phi_2}((\mathbf{t}_1,\mathbf{p}_1,\phi_1)-(\mathbf{t}_2,\mathbf{p}_2,\phi_2))|\\
&\quad=\Big|\int_\Omega(\gamma(\phi_1)-\gamma(\phi_2))\mathbf{k}\cdot((\mathbf{t}_1-\mathbf{t}_2)-l_1(\mathbf{p}_1-\mathbf{p}_2))\Big|\\
&\quad\le L_\gamma(1+l_1^2)^{1/2}\|\mathbf{k}\|\|\phi_1-\phi_2\|_{0,\Omega}\|(\mathbf{t}_1,\mathbf{p}_1,\phi_1)-(\mathbf{t}_2,\mathbf{p}_2,\phi_2)\|_{H_2}\,.
\end{aligned}
\tag{3.40}
$$

In this way, it follows from (3.39) and (3.40) that

$$
\begin{aligned}
&\|\widetilde{\mathbf{S}}(\phi_1,\mathbf{u}_1)-\widetilde{\mathbf{S}}(\phi_2,\mathbf{u}_2)\|_{H_2}=\|(\mathbf{t}_1,\mathbf{p}_1,\phi_1)-(\mathbf{t}_2,\mathbf{p}_2,\phi_2)\|_{H_2}\\
&\quad\le\frac{2}{\alpha(\Omega)}(1+l_1^2)^{1/2}\Big\{c(\Omega)\|\widetilde{\mathbf{S}}_{\mathbf{3}}(\phi_2,\mathbf{u}_2)\|_{1,\Omega}\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}+L_\gamma\|\mathbf{k}\|\|\phi_1-\phi_2\|_{0,\Omega}\Big\}\\
&\quad\le\widetilde{C}\Big\{\|\widetilde{\mathbf{S}}_{\mathbf{3}}(\phi_2,\mathbf{u}_2)\|_{1,\Omega}\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}+\|\mathbf{k}\|\|\phi_1-\phi_2\|_{0,\Omega}\Big\}\,,
\end{aligned}
$$

with $\widetilde{C}$ as indicated, which finishes the proof. □

As a consequence of Lemmas 3.11 and 3.12, we obtain the Lipschitz-continuity of $\mathbf{T}$. More precisely, we have the following result.

Lemma 3.13. *Given $r>0$, let $\mathbf{W}:=\big\{\phi\in\mathrm{H}^1(\Omega):\|\phi\|_{1,\Omega}\le r\big\}$, and assume that the data satisfy*

$$
C_S\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D}+r\|\mathbf{f}\|_{\infty,\Omega}\Big\}\le\frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}
\tag{3.41}
$$

and

$$
C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D}+2\|\mathbf{k}\|+2\|g\|_{0,\Omega}+\|\phi_D\|_{0,\Gamma_D}\Big\}\le r\,.
\tag{3.42}
$$

Then, with the constants C and $\widetilde{C}$ from Lemmas 3.11 and 3.12, there holds

$$
\begin{aligned}
&\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega}\\
&\quad\le\widetilde{C}(C\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega}+\|\mathbf{k}\|)\|\phi-\varphi\|_{0,\Omega}+C\widetilde{C}\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi-\varphi\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}
\end{aligned}
\tag{3.43}
$$

for all $\phi,\varphi\in\mathbf{W}$.

Proof. Given ϕ and φ in $\mathbf{W}$, we first recall from (3.20) that $\mathbf{T}(\phi):=\widetilde{\mathbf{S}}_{\mathbf{3}}(\phi,\mathbf{S}_2(\phi))$ and $\mathbf{T}(\varphi):=\widetilde{\mathbf{S}}_{\mathbf{3}}(\varphi,\mathbf{S}_2(\varphi))$. Then, using Lemmas 3.11 and 3.12, we deduce that

$$
\begin{aligned}
&\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega}=\|\widetilde{\mathbf{S}}_{\mathbf{3}}(\phi,\mathbf{S}_2(\phi))-\widetilde{\mathbf{S}}_{\mathbf{3}}(\varphi,\mathbf{S}_2(\varphi))\|_{1,\Omega}\\
&\quad\le\|\widetilde{\mathbf{S}}(\phi,\mathbf{S}_2(\phi))-\widetilde{\mathbf{S}}(\varphi,\mathbf{S}_2(\varphi))\|_{H_2}\\
&\quad\le\widetilde{C}\Big\{\|\widetilde{\mathbf{S}}_{\mathbf{3}}(\varphi,\boldsymbol{v})\|_{1,\Omega}\|\mathbf{u}-\boldsymbol{v}\|_{1,\Omega}+\|\mathbf{k}\|\|\phi-\varphi\|_{0,\Omega}\Big\}\\
&\quad=\widetilde{C}\Big\{\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_2(\phi)-\mathbf{S}_2(\varphi)\|_{1,\Omega}+\|\mathbf{k}\|\|\phi-\varphi\|_{0,\Omega}\Big\}\\
&\quad\le\widetilde{C}\Big\{\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_2(\phi)-\mathbf{S}_2(\varphi)\|_{H_1}+\|\mathbf{k}\|\|\phi-\varphi\|_{0,\Omega}\Big\}\\
&\quad\le\widetilde{C}\Big\{\|\mathbf{T}(\varphi)\|_{1,\Omega}C\Big\{\|\mathbf{f}\|_{\infty,\Omega}\|\phi-\varphi\|_{0,\Omega}+\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi-\varphi\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\Big\}+\|\mathbf{k}\|\|\phi-\varphi\|_{0,\Omega}\Big\}\\
&\quad=\widetilde{C}\Big\{C\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega}+\|\mathbf{k}\|\Big\}\|\phi-\varphi\|_{0,\Omega}+C\widetilde{C}\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi-\varphi\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\,,
\end{aligned}
$$

which is the required estimate, thus completing the proof. □

Next, we prove the required compactness property of $\mathbf{T}$.

Lemma 3.14. *Given $r > 0$, let $\mathbf{W} := \{\phi \in \mathrm{H}^1(\Omega) : \|\phi\|_{1,\Omega} \le r\}$, and assume that the data satisfy*

$$C_S\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega}\Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{3.44}$$

and

$$C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\} \le r\,. \tag{3.45}$$

Then $\mathbf{T} : \mathbf{W} \to \mathbf{W}$ is continuous and $\overline{\mathbf{T}(\mathbf{W})}$ is compact.

Proof. We first recall, thanks now to the Rellich-Kondrachov compactness Theorem (cf. [1, Theorem 3.7]) that the injection $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$ is compact, and hence continuous, for each $s \ge 1$ (when $n = 2$), and for each $s \in [1, 6)$ (when $n = 3$). Then, according to the assumptions on the further regularity ε (cf. 3.23), that is $\varepsilon \in (0, 1)$ in R^2 and $\varepsilon \in (\frac{1}{2}, 1)$ in R^3, we realize that $\frac{n}{\varepsilon}$ belongs to the indicated ranges for s. It follows that $\mathrm{H}^1(\Omega)$ is compactly, and hence continuously, embedded in $\mathrm{L}^{n/\varepsilon}(\Omega)$, with constant $\widehat{C}$, which together with (3.43) imply

$$\begin{aligned}&\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega}\\ &\quad\le \widetilde{C}(C\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega} + \|\mathbf{k}\|)\|\phi - \varphi\|_{0,\Omega} + C\widetilde{C}\widehat{C}\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi - \varphi\|_{1,\Omega}\,,\end{aligned} \tag{3.46}$$

from which the continuity of $\mathbf{T}$ is obtained. In turn, let $\{\phi_k\}_{k\in\mathbb{N}}$ a sequence that live in $\mathbf{W}$, which is clearly bounded. It follows that there exist a subsequence $\{\phi_k^{(1)}\}_{k\in\mathbb{N}} \subseteq \{\phi_k\}_{k\in\mathbb{N}}$ and $\phi \in \mathrm{H}^1(\Omega)$ such that $\phi_k^{(1)} \xrightarrow{w} \phi$. Then, since the injections $i : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Omega)$ and $\tilde{i} : \mathrm{H}^1(\Omega) \to \mathrm{L}^{n/\varepsilon}$ are compact, we deduce that $\phi_k^{(1)} \to \phi$ in $\mathrm{L}^2(\Omega)$ and in $\mathrm{L}^{n/\varepsilon}$, which thanks again to (3.43), implies that $\mathbf{T}(\phi_k^{(1)}) \to \mathbf{T}(\phi)$ in $\mathrm{H}^1(\Omega)$. This proves the compactness of $\overline{\mathbf{T}(\mathbf{W})}$ and finishes the proof. ☐

Finally, the main result of the section is given as follows.

Theorem 3.15. *Given $r > 0$, let $\mathbf{W} := \Big\{\phi \in \mathrm{H}^1(\Omega) : \|\phi\|_{1,\Omega} \le r\Big\}$, and assume that the data satisfy*

$$C_S\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega}\Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{3.47}$$

and

$$C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\} \le r\,. \tag{3.48}$$

Then, the augmented fully mixed problem (3.18) has at least one solution $\big((\boldsymbol{\sigma}, \mathbf{u}), (\mathbf{t}, \mathbf{p}, \phi)\big) \in H_1 \times H_2$ with $\phi \in \mathbf{W}$, and there holds

$$\|(\mathbf{t}, \mathbf{p}, \phi)\|_{H_2} \le C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\} \tag{3.49}$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_{H_1} \le C_{\mathbf{S}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{1,\Omega}\Big\}\,. \tag{3.50}$$

Moreover, if the data $\mathbf{k}, \mathbf{f}$ and $\mathbf{u}_D$ are sufficiently small so that, with the constant C, $\widetilde{C}$ and $C_{\mathbf{S}}(r)$ from Lemmas 3.11 and 3.12 and the estimate (3.23), there holds

$$\widetilde{C}\Big\{C\, r\Big(\|\mathbf{f}\|_{\infty,\Omega} + \widehat{C}\widetilde{C}_{\widetilde{\mathbf{S}}}(r)\Big(\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{0,\Omega}\Big)\Big) + \|\mathbf{k}\|\Big\} < 1\,, \tag{3.51}$$

then the solution ϕ is unique in $\mathbf{W}$.

Proof. According to the equivalence between (3.18) and the fixed point equation (3.20), and thanks to Lemmas 3.10 and 3.14, the existence of a solution is just a straightforward application of the Schauder fixed point Theorem (cf. Theorem 3.9). In turn, the estimates (3.49) and (3.50) follow from (3.21) and (3.31), respectively. Furthermore, according to (3.46) we have

$$
\begin{aligned}
&\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega}\\
&\quad\le \widetilde{C}(C\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega}+\|\mathbf{k}\|)\|\phi-\varphi\|_{0,\Omega}+C\widetilde{C}\widehat{C}\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\|\phi-\varphi\|_{1,\Omega}\\
&\quad\le \widetilde{C}\Big\{C\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega}+\|\mathbf{k}\|+C\widehat{C}\|\mathbf{T}(\varphi)\|_{1,\Omega}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\Big\}\|\phi-\varphi\|_{1,\Omega}\\
&\quad= \widetilde{C}\Big\{C\|\mathbf{T}(\varphi)\|_{1,\Omega}(\|\mathbf{f}\|_{\infty,\Omega}+\widehat{C}\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega})+\|\mathbf{k}\|\Big\}\|\phi-\varphi\|_{1,\Omega}\,,
\end{aligned}
\tag{3.52}
$$

which, using from (3.35) and (3.23), that

$$
\|\mathbf{T}(\varphi)\|_{1,\Omega}\quad\le\quad\|\widetilde{\mathbf{S}}(\varphi,\mathbf{S}_2(\varphi))\|_{H_2}\le C_{\widetilde{\mathbf{S}}}\left\{\|\phi_D\|_{1/2,\Gamma_D}+2\|\mathbf{k}\|+2\|g\|_{0,\Omega}+\|\phi_D\|_{0,\Gamma_D}\right\}\le r\,,
$$

and

$$
\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}\le \widetilde{C}_{\widetilde{\mathbf{S}}}(r)\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}+\|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{0,\Omega}\}\,,
$$

leads to

$$
\begin{aligned}
&\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega}\\
&\quad\le \widetilde{C}\Big\{C\,r\Big(\|\mathbf{f}\|_{\infty,\Omega}+\widehat{C}\widetilde{C}_{\widetilde{\mathbf{S}}}(r)\Big(\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}+\|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{0,\Omega}\Big)\Big)+\|\mathbf{k}\|\Big\}\|\phi-\varphi\|_{1,\Omega}.
\end{aligned}
$$

The foregoing inequality shows that $\mathbf{T}$ is a contraction if the condition (3.51) is satisfied, and hence by the Banach fixed point Theorems we get that $\phi\in \mathrm{H}^1(\Omega)$ is unique. $\square$

We end this section by remarking that the foregoing theorem ensures that, under the assumptions (3.47), (3.48) and (3.51) on the data, there exists a unique solution $\big((\boldsymbol{\sigma},\mathbf{u}),(\mathbf{t},\mathbf{p},\phi)\big)\in H_1\times H_2$ of problem (3.18) such that $\phi\in\mathbf{W}$.

4 The Galerkin scheme

In this section we introduce the Galerkin scheme of the augmented fully mixed formulation (3.18), and analyze its solvability by employing a discrete version of the fixed point strategy developed in Section 3.2. To this end, we now let $\mathbb{H}_h^{\boldsymbol{\sigma}}\subseteq\mathbb{H}_N(\mathbf{div},\Omega)$, $\mathbf{X}_h^{\mathbf{u}}\subseteq\mathbf{H}^1(\Omega)$, $\mathbf{Y}_h^{\mathbf{t}}\subseteq\mathbf{L}^2(\Omega)$, $\mathbf{H}_h^{\mathbf{p}}\subseteq\mathbf{H}_N(\mathrm{div},\Omega)$, and $\mathrm{X}_h^{\phi}\subseteq\mathrm{H}^1(\Omega)$, be arbitrary finite element subspaces for approximating the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, $\mathbf{t}$, $\mathbf{p}$, and ϕ, respectively, and set $H_{1,h}:=\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{X}_h^{\mathbf{u}}$ and $H_{2,h}:=\mathbf{Y}_h^{\mathbf{t}}\times\mathbf{H}_h^{\mathbf{p}}\times\mathrm{X}_h^{\phi}$. In this way, the underlying Galerkin scheme, given by the discrete counterpart of (3.18), reads: Find $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\big)\in H_{1,h}\times H_{2,h}$ such that

$$
\begin{aligned}
B_{\phi_h}((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\tau}_h,\boldsymbol{v}_h)) &= F_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h) &&\forall\,(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\in H_{1,h}\,,\\
\Big[(A+\widetilde{B}_{\mathbf{u}_h})(\mathbf{t}_h,\mathbf{p}_h,\phi_h),(\mathbf{s}_h,\mathbf{q}_h,\varphi_h)\Big] &= \widetilde{F}_{\phi_h}(\mathbf{s}_h,\mathbf{q}_h,\varphi_h) &&\forall\,(\mathbf{s}_h,\mathbf{q}_h,\varphi_h)\in H_{2,h}.
\end{aligned}
\tag{4.1}
$$

Throughout the rest of this section, we adopt the discrete analogue of the fixed point strategy introduced in Section 3.2. In fact, we now let $\mathbf{S}_h:\mathrm{X}_h^{\phi}\to H_{1,h}$ be the operator defined by

$$
\mathbf{S}_h(\phi_h)=(\mathbf{S}_{1,h}(\phi_h),\mathbf{S}_{2,h}(\phi_h)):=(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in H_{1,h}\quad\forall\,\phi_h\in\mathrm{X}_h^{\phi}\,,
$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{2,h}$ is the unique solution of the problem

$$B_{\phi_h}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) = F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in H_{1,h}\,, \tag{4.2}$$

and B_{ϕ_h} and F_{ϕ_h} are defined by (3.11) and (3.12), respectively, with $\phi = \phi_h$. In addition, we let $\widetilde{\mathbf{S}}_h : \mathrm{X}_h^\phi \times \mathbf{X}_h^{\mathbf{u}} \to H_{2,h}$ be the operator defined by

$$\widetilde{\mathbf{S}}_h(\phi_h, \mathbf{u}_h) = (\widetilde{\mathbf{S}}_{1,h}(\phi_h, \mathbf{u}_h), \widetilde{\mathbf{S}}_{2,h}(\phi_h, \mathbf{u}_h), \widetilde{\mathbf{S}}_{3,h}(\phi_h, \mathbf{u}_h)) := (\mathbf{t}_h, \mathbf{p}_h, \widetilde{\phi_h}) \in H_{2,h} \quad \forall (\phi_h, \mathbf{u}_h) \in \mathbf{X}_h^{\mathbf{u}} \times \mathrm{X}_h^\phi\,,$$

where $(\mathbf{t}_h, \mathbf{p}_h, \widetilde{\phi}_h) \in H_{2,h}$ is the the unique solution of

$$\Big[(A + \widetilde{B}_{\mathbf{u}_h})(\mathbf{t}_h, \mathbf{p}_h, \widetilde{\phi_h}), (\mathbf{s}_h, \mathbf{q}_h, \varphi_h)\Big] = \widetilde{F}_{\phi_h}(\mathbf{s}_h, \mathbf{q}_h, \varphi_h) \qquad \forall\, (\mathbf{s}_h, \mathbf{q}_h, \varphi_h) \in H_{2,h}\,, \tag{4.3}$$

and $\widetilde{B}_{\mathbf{u}_h}$ and $\widetilde{F}_{\phi_h}$ are defined by (3.16) and (3.17), respectively, with $\mathbf{u} = \mathbf{u}_h$ and $\phi = \phi_h$. Finally, we define the operator $\mathbf{T}_h : \mathrm{X}_h^\phi \to \mathrm{X}_h^\phi$ by

$$\mathbf{T}_h(\phi_h) := \widetilde{\mathbf{S}}_{3,h}(\phi_h, \mathbf{S}_{2,h}(\phi_h)) \qquad \forall \phi_h \in \mathrm{X}_h^\phi\,, \tag{4.4}$$

and realize that (4.1) can be rewritten, equivalently, as the fixed point equation: Find $\phi_h \in \mathrm{X}_h^\phi$ such that

$$\mathbf{T}_h(\phi_h) = \phi_h\,. \tag{4.5}$$

At this point we remark that all the above makes sense if the discrete problems (4.2) and (4.3) are well-possed. Indeed, it is easy to see that the respective proofs are almost verbatim as the continuous versions provided in Section 3 (cf. Lemmas 3.5 and 3.8). More precisely, we obtain the following results.

Lemma 4.1. *Assume that* $k_1 \in \left(0, \frac{2\delta\mu_1}{\mu_2}\right)$ *with* $\delta \in (0, 2\mu_1)$*, and that* $k_2, k_3 > 0$*. Then, for each* $\phi_h \in \mathrm{X}_h^\phi$ *the problem (4.2) has a unique solution* $\mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_{1,h}$*. Moreover, with the same constant* $C_{\mathbf{S}} > 0$ *from Lemma 3.5, there holds*

$$\|\mathbf{S}_h(\phi_h)\|_{H_1} = \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_1} \leq C_{\mathbf{S}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi_h\|_{1,\Omega}\Big\} \quad \forall\, \phi_h \in \mathrm{X}_h^\phi\,.$$

Proof. Similarly to the proof of Lemma 3.5, it is consequence of the uniform $H_{1,h}$-ellipticity of the bilinear form B_{ϕ_h} for each $\phi_h \in \mathrm{X}_h^\phi$. We omit further details and refer to [2, Lemma 4.1]. □

Lemma 4.2. *Assume that* $l_1 \in \left(0, \frac{\theta_1\delta}{\theta_2}\right)$ *and* $l_3 \in (0, \theta_1\widetilde{\delta})$*, with* $\delta \in \left(0, \frac{2}{\theta_2}\right)$ *and* $\widetilde{\delta} \in (0, 2)$*, and that* $l_2, l_4 > 0$*. In addition, let* $\phi_h \in \mathrm{X}_h^\phi$ *and* $\mathbf{u}_h \in \mathbf{X}_h^{\mathbf{u}}$ *be such that* $\|\mathbf{u}_h\|_{1,\Omega} \leq \dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$*. Then there exists a unique* $\widetilde{\mathbf{S}}(\phi_h, \mathbf{u}_h) = (\mathbf{t}_h, \mathbf{p}_h, \phi_h) \in H_{2,h}$ *solution of (3.14), and there holds*

$$\|\widetilde{\mathbf{S}}(\phi_h, \mathbf{u}_h)\|_{H_2} = \|(\mathbf{t}, \mathbf{p}, \phi)\|_{H_2} \leq C_{\widetilde{\mathbf{S}}}\Big\{\|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D}\Big\}, \tag{4.6}$$

where $C_{\widetilde{\mathbf{S}}} = \dfrac{2}{\alpha(\Omega)} C_{\widetilde{F}_\phi}$ *and* $C_{\widetilde{F}_\phi} = \max\Big\{1, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|, \gamma_2|\Omega|^{1/2}\|\mathbf{k}\|l_1, l_2, c_0 l_4\Big\}$.

Proof. Similarly to the proof of Lemma 3.8, it basically follows by observing that, under the assumption on $\|\mathbf{u}_h\|_{1,\Omega}$, $A + \widetilde{B}_{\mathbf{u}_h} : H_{2,h} \to H_{2,h}'$ becomes Lipschitz-continuous and strongly monotone with the same constants $L_C := L_A + c(\Omega)(l_2^2+1)^{1/2}\|\mathbf{u}_h\|_{1,\Omega}$ and $\dfrac{\alpha(\Omega)}{2}$, respectively, given in the proofs of Lemmas 3.6 and 3.7. Further details are omitted. □

We now aim to show the solvability of (4.1) by analyzing the equivalent fixed point equation (4.5). To this end, in what follows we verify the hypotheses of the Brouwer fixed point theorem, which is given as follows (see e.g. [10, Theorem. 9.9-2]).

Theorem 4.3. *Let W be a compact and convex subset of a finite dimensional Banach space X and let $T : W \to W$ be a continuous mapping. Then T has at least one fixed point.*

Then, the discrete form of Lemma 3.10 is established next.

Lemma 4.4. *Given $r > 0$, let $\mathbf{W}_h := \Big\{ \phi_h \in \mathrm{X}_h^\phi : \|\phi_h\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy*

$$C_{\mathbf{S}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega} \Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{4.7}$$

and

$$C_{\widetilde{\mathbf{S}}} \Big\{ \|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D} \Big\} \le r\,. \tag{4.8}$$

Then $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$.

Proof. It follows directly from Lemmas 4.1 and 4.2. □

The discrete form of Lemma 3.11 is provided next. We notice in advance that, instead of the regularity assumption employed in the proof of that result, which actually is not needed nor could be applied in the present discrete case, we simply utilize a $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ factorization.

Lemma 4.5. *There exists a constant $C > 0$, depending on μ_1, l_1, l_2, L_μ the ellipticity constant α of B_ϕ (cf. [2, eq. (3.19)]), and the regularity parameter ε (cf. (3.23)), such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_{H_1} \le C \Big\{ \|\mathbf{f}\|_{\infty,\Omega}\|\phi_h - \varphi_h\|_{0,\Omega} + \|\mathbf{S}_{1,h}(\varphi_h)\|_{\mathbb{L}^4(\Omega)}\|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \Big\} \tag{4.9}$$

for all $\phi_h\,, \varphi_h \in \mathrm{X}_h^\phi$.

Proof. The proof is the same as in [2, Lemma 4.5] . □

Now, the discrete analogue of Lemma 3.12 is stated as follows.

Lemma 4.6. *There exists $\widetilde{C} := \dfrac{2}{\alpha(\Omega)}(1+l_1^2)^{1/2} \max\big\{c(\Omega), L_\gamma\big\}$ (cf. (3.8), (3.28)) such that for all $(\phi_{1,h}, \mathbf{u}_{1,h}), (\phi_{2,h}, \mathbf{u}_{2,h}) \in \mathrm{X}_h^\phi \times \mathbf{X}_h^{\mathbf{u}}$ with $\|\mathbf{u}_{1,h}\|_{1,\Omega}\,, \|\mathbf{u}_{2,h}\|_{1,\Omega} \le \dfrac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}}$, there holds*

$$\begin{aligned} &\|\widetilde{\mathbf{S}}_h(\phi_{1,h}, \mathbf{u}_{1,h}) - \widetilde{\mathbf{S}}_h(\phi_{2,h}, \mathbf{u}_{2,h})\|_{H_2} \\ &\quad \le \widetilde{C} \Big\{ \|\widetilde{\mathbf{S}}_{3,h}(\phi_{2,h}, \mathbf{u}_{2,h})\|_{1,\Omega}\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{1,\Omega} + \|\mathbf{k}\|\|\phi_{1,h} - \phi_{2,h}\|_{0,\Omega} \Big\}. \end{aligned} \tag{4.10}$$

Proof. The proof is analogous to the one of Lemma 3.12. □

Then, using Lemmas 4.5 and 4.6, the following result is proved.

Lemma 4.7. *Given $r > 0$, let $\mathbf{W}_h := \Big\{ \phi_h \in \mathrm{X}_h^\phi : \|\phi_h\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy*

$$C_S \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega} \Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{4.11}$$

and

$$C_{\widetilde{\mathbf{S}}} \Big\{ \|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D} \Big\} \le r\,. \tag{4.12}$$

Then, with the constants C and $\widetilde{C}$ from, Lemmas 4.5 and 4.6, there holds

$$\begin{aligned} \|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{1,\Omega} &\le \widetilde{C}(C\|\mathbf{T}_h(\varphi_h)\|_{1,\Omega}\|\mathbf{f}\|_{\infty,\Omega} + \|\mathbf{k}\|)\|\phi_h - \varphi_h\|_{0,\Omega} \\ &+ C\,\widetilde{C}\|\mathbf{T}_h(\varphi_h)\|_{1,\Omega}\|\mathbf{S}_{1,h}(\varphi_h)\|_{\mathbb{L}^4(\Omega)}\|\phi_h - \varphi_h\|_{\mathrm{L}^4(\Omega)} \end{aligned} \tag{4.13}$$

for all $\phi_h, \varphi_h \in \mathrm{X}_h^\phi$.

Therefore, using Lemma 4.7 and the continuous injection of $\mathrm{H}^1(\Omega)$ in $\mathrm{L}^4(\Omega)$, we deduce that T_h is continuous, and hence, thanks to the Brouwer fixed point theorem (cf. Theorem 4.3), and Lemmas 4.4 and 4.7, we obtain the main result of this section.

Theorem 4.8. *Given $r > 0$, let $\mathbf{W}_h := \Big\{ \phi_h \in \mathrm{X}_h^\phi : \|\phi_h\|_{1,\Omega} \le r \Big\}$ and assume that the data satisfy*

$$C_S \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma_D} + r\|\mathbf{f}\|_{\infty,\Omega} \Big\} \le \frac{\alpha(\Omega)}{2c(\Omega)(1+l_1^2)^{1/2}} \tag{4.14}$$

and

$$C_{\widetilde{\mathbf{S}}} \Big\{ \|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D} \Big\} \le r\,. \tag{4.15}$$

Then, the augment fully mixed scheme (4.1) has at least one solution $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\big) \in H_{1,h} \times H_{2,h}$ with $\phi_h \in \mathbf{W}_h$, and there holds

$$\|(\mathbf{t}_h, \mathbf{p}_h, \phi_h)\|_{H_2} \le C_{\widetilde{\mathbf{S}}} \Big\{ \|\phi_D\|_{1/2,\Gamma_D} + 2\|\mathbf{k}\| + 2\|g\|_{0,\Omega} + \|\phi_D\|_{0,\Gamma_D} \Big\} \tag{4.16}$$

and

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_1} \le C_{\mathbf{S}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{1,\Omega} \Big\}\,. \tag{4.17}$$

5 A priori error analysis

Let $\big((\boldsymbol{\sigma}, \mathbf{u}), (\mathbf{t}, \mathbf{p}, \phi)\big) \in H_1 \times H_2$, with $\phi \in \mathbf{W}$, and $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\big) \in H_{1,h} \times H_{2,h}$, with $\phi_h \in \mathbf{W}_h$, be solutions of (3.18) and (4.1), respectively, that is

$$\begin{array}{rcll} B_\phi((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \boldsymbol{v})) &=& F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) & \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in H_1\,, \\ B_{\phi_h}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) &=& F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) & \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in H_{1,h}\,, \end{array} \tag{5.1}$$

and

$$\begin{array}{rcll} [(A + \widetilde{B}_{\mathbf{u}})(\mathbf{t}, \mathbf{p}, \phi), (\mathbf{s}, \mathbf{q}, \varphi)] &=& \widetilde{F}_\phi(\mathbf{s}, \mathbf{q}, \varphi) & \forall\, (\mathbf{s}, \mathbf{q}, \varphi) \in H_2\,, \\ [(A + \widetilde{B}_{\mathbf{u}_h})(\mathbf{t}_h, \mathbf{p}_h, \phi_h), (\mathbf{s}_h, \mathbf{q}_h, \varphi_h)] &=& \widetilde{F}_{\phi_h}(\mathbf{s}_h, \mathbf{q}_h, \varphi_h) & \forall\, (\mathbf{s}_h, \mathbf{q}_h, \varphi_h) \in H_{2,h}\,. \end{array} \tag{5.2}$$

We now aim to derive a corresponding a priori error estimate. For this purpose, we recall from [21] a Strang-type lemma, which will be utilized in our subsequent analysis.

Lemma 5.1. *Let H be a Hilbert space, $F \in H'$, and $\mathbf{A} : H \to H'$ a nonlinear operator. In addition, let $\{H_n\}_{n\in N}$ be a sequence of finite dimensional subspaces of H, and for each $n \in N$ consider a nonlinear operator $\mathbf{A}_n : H_n \to H_n$ and a functional $F_n \in H_n'$. Assume that the family $\{\mathbf{A}\} \cup \{\mathbf{A}_n\}_{n\in N}$ is uniformly Lipschitz continuous and strongly monotone with constants Λ_{LC} and Λ_{SM}, respectively. In turn, let $u \in H$ and $u_n \in H_n$ such that*

$$[\mathbf{A}(u), v] = [F, v] \quad \forall\, v \in H \quad \text{and} \quad [\mathbf{A}_n(u_n), v_n] = [F_n, v_n] \quad \forall\, v_n \in H_n\,,$$

where $[\cdot,\cdot]$ denotes the duality pairings of both $H' \times H$ and $H_n' \times H_n$. Then for each $n \in N$ there holds

$$\begin{aligned}\|u - u_n\|_H \;\; &\le \;\; \Lambda_{ST}\Bigg\{ \sup_{\substack{w_n\in H_n\\ w_n\neq \mathbf{0}}} \frac{|[F, w_n] - [F_n, w_n]|}{\|w_n\|_H} \\ &\quad + \inf_{\substack{v_n\in H_n\\ v_n\neq \mathbf{0}}} \left(\|u - v_n\|_H + \sup_{\substack{w_n\in H_n\\ w_n\neq \mathbf{0}}} \frac{|[\mathbf{A}(v_n), w_n] - [\mathbf{A}_n(v_n), w_n]|}{\|w_n\|_H} \right) \Bigg\},\end{aligned}$$

with $\Lambda_{ST} := \Lambda_{SM}^{-1} \max\{1,\, \Lambda_{SM} + \Lambda_{LC}\}$.

Proof. It is a particular case of [21, Theorem. 6.4]. □

We begin our analysis defining and denoting as usual

$$\operatorname{dist}((\boldsymbol{\sigma}, \mathbf{u}), H_{1,h}) := \inf_{(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\in H_{1,h}} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{H_1}$$

$$\operatorname{dist}((\mathbf{t}, \mathbf{p}, \phi), H_{2,h}) := \inf_{(\mathbf{s}_h,\mathbf{q}_h,\varphi)\in H_{2,h}} \|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{s}_h, \mathbf{q}_h, \varphi)\|_{H_2}.$$

Then, we have the following result concerning the error $\|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\|_{H_1}$.

Lemma 5.2. *Let $\widetilde{C}_{ST} := \frac{2}{\alpha(\Omega)} \max\left\{1, \frac{\alpha(\Omega)}{2} + L_A\right\}$ (cf. Lemma 3.31). Then, there holds*

$$\begin{aligned}\|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\|_{H_1} &\le \widetilde{C}_{ST}\Big\{ L_\gamma \|\mathbf{k}\| (1 + l_1^2)^{1/2} \|\phi - \phi_h\|_{0,\Omega} \\ &+ \;\; (1 + l_1^2)^{1/2} c(\Omega) \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\phi\|_{1,\Omega} \\ &+ \;\; \Big(1 + (1 + l_1^2)^{1/2} c(\Omega) \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}\Big) \operatorname{dist}((\mathbf{t}, \mathbf{p}, \phi), H_{2,h}) \Big\}\,.\end{aligned} \tag{5.3}$$

Proof. We begin by observing, thanks to Lemmas 3.6, 3.7 and 4.2, that $A + \widetilde{B}_{\mathbf{u}}$ and $A + \widetilde{B}_{\mathbf{u}_h}$ are Lipschitz-continuous and strongly monotone with constants $L_A := 4 \max\{\widetilde{\theta}_2, 1, l_1, l_1\widetilde{\theta}_2, l_2, l_3, l_4 c_0\}$ and $\frac{\alpha(\Omega)}{2}$, respectively. Then, by applying the abstract Lemma 5.1 to the context given by (5.2) we have

$$\begin{aligned}\|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\|_{H_2} &\le \widetilde{C}_{ST}\Bigg\{ \sup_{\substack{(\mathbf{s}_h,\mathbf{q}_h,\varphi)\in H_{2,h}\\ w_n\neq \mathbf{0}}} \frac{|\widetilde{F}_\phi(\mathbf{s}_h, \mathbf{q}_h, \varphi) - \widetilde{F}_{\phi_h}(\mathbf{s}_h, \mathbf{q}_h, \varphi)|}{\|(\mathbf{s}_h, \mathbf{q}_h, \varphi)\|_{H_2}} \\ &+ \inf_{\substack{(\mathbf{r}_h,\mathbf{o}_h,\psi_h)\in H_{2,h}\\ (\mathbf{r}_h,\mathbf{o}_h,\psi_h)\neq \mathbf{0}}} \Bigg(\|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{r}_h, \mathbf{o}_h, \psi_h)\|_{H_2} \\ &+ \sup_{\substack{(\mathbf{s}_h,\mathbf{q}_h,\varphi)\in H_{2,h}\\ (\mathbf{s}_h,\mathbf{q}_h,\varphi)\neq \mathbf{0}}} \frac{|[(A + \widetilde{B}_{\mathbf{u}})(\mathbf{r}_h, \mathbf{o}_h, \psi_h), (\mathbf{s}_h, \mathbf{q}_h, \varphi)] - [(A + \widetilde{B}_{\mathbf{u}_h})(\mathbf{r}_h, \mathbf{o}_h, \psi_h), (\mathbf{s}_h, \mathbf{q}_h, \varphi)]|}{\|(\mathbf{s}_h, \mathbf{q}_h, \varphi)\|_{H_2}} \Bigg) \Bigg\},\end{aligned} \tag{5.4}$$

where $\widetilde{C}_{ST} := \frac{2}{\alpha(\Omega)} \max\left\{1, \frac{\alpha(\Omega)}{2} + L_A\right\}$. Then, using the Cauchy Schwarz inequality, we obtain that

$$\begin{aligned}
|\widetilde{F}_\phi((\mathbf{s}_h,\mathbf{q}_h,\varphi)) \quad &- \quad \widetilde{F}_{\phi_h}((\mathbf{s}_h,\mathbf{q}_h,\varphi))| = \Big|\int_\Omega (\gamma(\phi)-\gamma(\phi_h))\mathbf{k}\cdot(\mathbf{s}_h - l_1\mathbf{q}_h)\Big| \\
&\leq \quad L_\gamma \,\|\mathbf{k}\| \int_\Omega |\phi-\phi_h|\,|\mathbf{s}_h - l_1\mathbf{q}_h| \\
&\leq \quad L_\gamma \,\|\mathbf{k}\|\,(1+l_1^2)^{1/2}\,\|\phi-\phi_h\|_{0,\Omega}\,\|(\mathbf{s}_h,\mathbf{q}_h,\varphi)\|_{H_2},
\end{aligned}$$

and hence

$$\sup_{\substack{(\mathbf{s}_h,\mathbf{q}_h,\varphi)\in H_{2,h}\\ w_n\neq\mathbf{0}}} \frac{|\widetilde{F}_\phi((\mathbf{s}_h,\mathbf{q}_h,\varphi)) - \widetilde{F}_{\phi_h}((\mathbf{s}_h,\mathbf{q}_h,\varphi))|}{\|(\mathbf{s}_h,\mathbf{q}_h,\varphi)\|_{H_2}} \leq L_\gamma\|\mathbf{k}\|(1+l_1^2)^{1/2}\|\phi-\phi_h\|_{0,\Omega}. \tag{5.5}$$

In order to estimate the supreme, we notice that adding and subtracting $\widetilde{B}_{\mathbf{u}-\mathbf{u}_h}(\mathbf{t},\mathbf{p},\phi)$, we find that

$$\begin{aligned}
&|[(A+\widetilde{B}_{\mathbf{u}})(\mathbf{r}_h,\mathbf{o}_h,\psi_h),(\mathbf{s}_h,\mathbf{q}_h,\varphi)] - [(A+\widetilde{B}_{\mathbf{u}_h})(\mathbf{r}_h,\mathbf{o}_h,\psi_h),(\mathbf{s}_h,\mathbf{q}_h,\varphi)]| \\
&\quad = [\widetilde{B}_{\mathbf{u}-\mathbf{u}_h}(\mathbf{r}_h,\mathbf{o}_h,\psi_h),(\mathbf{s}_h,\mathbf{q}_h,\varphi)] \\
&\quad = [\widetilde{B}_{\mathbf{u}-\mathbf{u}_h}(\mathbf{r}_h,\mathbf{o}_h,\psi_h),(\mathbf{s}_h,\mathbf{q}_h,\varphi)] + [\widetilde{B}_{\mathbf{u}-\mathbf{u}_h}(\mathbf{t},\mathbf{p},\phi),(\mathbf{s}_h,\mathbf{q}_h,\varphi)] - [\widetilde{B}_{\mathbf{u}-\mathbf{u}_h}(\mathbf{t},\mathbf{p},\phi),(\mathbf{s}_h,\mathbf{q}_h,\varphi)] \\
&\quad = \int_\Omega (\psi_h-\phi)(\mathbf{u}-\mathbf{u}_h)\cdot(l_1\mathbf{q}-\mathbf{s}_h) + \int_\Omega \phi(\mathbf{u}-\mathbf{u}_h)\cdot(l_1\mathbf{q}-\mathbf{s}_h) \\
&\quad \leq c(\Omega)(1+l_1^2)^{1/2}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\Big\{\|\psi_h-\phi\|_{1,\Omega} + \|\phi\|_{1,\Omega}\Big\}\|(\mathbf{s}_h,\mathbf{q}_h,\varphi)\|_{H_2}.
\end{aligned} \tag{5.6}$$

In this way, replacing (5.5) and (5.6) back into (5.4), we arrive to (5.3) and conclude the proof. □

The following lemma provides a preliminary estimate for the error $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_1}$.

Lemma 5.3. *Let $C_{ST} := \alpha^{-1}\max\{1, \alpha + \|B\|\}$, where $\|B\|$ and α are the boundedness and ellipticity constants, respectively, of the bilinear forms B_ϕ (cf. [2, Lemma 3.9]). Then there holds*

$$\begin{aligned}
\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_1} \leq C_{ST}\Bigg\{&\Big(1+2\|B\|\Big)\,\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h}) \\
&+ (1+k_2^2)^{1/2}\|\mathbf{f}\|_{\infty,\Omega}\|\phi-\phi_h\|_{0,\Omega} + \frac{L_\mu(1+k_1^2)^{1/2}}{\mu_1^2}C_\varepsilon\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega}\|\phi-\phi_h\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\Bigg\}.
\end{aligned} \tag{5.7}$$

Proof. See the proof in [2, Lemma 5.3]. □

Now, combining the inequalities provided by Lemmas 5.2 and 5.3 we obtain the Cea estimate for the total error $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_1} + \|(\mathbf{t},\mathbf{p},\phi) - (\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2}$.

Theorem 5.4. *Assume that the data $\mathbf{k}$, $\mathbf{f}$ and $\mathbf{u}_D$ are sufficiently small so that*

$$C_1\|\mathbf{k}\| + \widehat{C}_2\|\mathbf{f}\|_{\infty,\Omega} + \widehat{C}_3\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D} < \frac{1}{2}, \tag{5.8}$$

where the constants C_1, $\widehat{C}_2$, and $\widehat{C}_3$ will be defined along the proof below. Then, there exist positive constants $\widehat{C}_4$ and $\widehat{C}_5$, depending only on parameters, data, and other constants, all them independent of h, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_1} + \|(\mathbf{t},\mathbf{p},\phi) - (\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} \\
&\quad \leq \widehat{C}_4\,\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h}) + \widehat{C}_5\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}).
\end{aligned} \tag{5.9}$$

Proof. In order to simplify the subsequent writing, we first introduce the following constants

$$C_1 := L_\gamma(1+l_1^2)^{1/2}\widetilde{C}_{ST}\,, \quad \text{and} \quad C_2 := (1+l_1^2)^{1/2}c(\Omega)\widetilde{C}_{ST}\,.$$

Therefore (5.3) becomes

$$\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} \le C_1\|\mathbf{k}\|\|\phi-\phi_h\|_{0,\Omega}$$
$$+\,C_2\|\phi\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega} + \widetilde{C}_{ST}\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}) + C_2\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}).$$

Now, replacing the second term $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ by the bound given by (5.7), and noticing that thanks to (3.23)

$$\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega} \le \widetilde{C}(r)\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D} + \|\mathbf{f}\|_{\infty,\Omega}\|\phi\|_{0,\Omega}\Big\}\,,$$

and that $\|\phi\|_{1,\Omega} \le r$, we get

$$\begin{aligned}
&\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} \le C_1\|\mathbf{k}\|\|\phi-\phi_h\|_{0,\Omega} + C_2 r C_{ST}\Bigg\{(1+2\|B\|)\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h})\\
&\quad+(1+k_2^2)^{1/2}\|\mathbf{f}\|_{\infty,\Omega}\|\phi-\phi_h\|_{0,\Omega} + \frac{L_\mu(1+k_1^2)^{1/2}}{\mu_1^2}C_\varepsilon\widetilde{C}(r)\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}\\
&\quad+r\|\mathbf{f}\|_{\infty,\Omega}\Big\}\|\phi-\phi_h\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\Bigg\} + \widetilde{C}_{ST}\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}) + C_2\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}).
\end{aligned}$$

Besides, since $\mathbf{u}$ and $\mathbf{u}_h$ are controled by the data according to (3.50) and (4.17), we obtain from the foregoing inequality that

$$\begin{aligned}
&\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} \le C_1\|\mathbf{k}\|\|\phi-\phi_h\|_{0,\Omega} + C_2 r C_{ST}\Bigg\{(1+2\|B\|)\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h})\\
&\quad+(1+k_2^2)^{1/2}\|\mathbf{f}\|_{\infty,\Omega}\|\phi-\phi_h\|_{0,\Omega} + \frac{L_\mu(1+k_1^2)^{1/2}}{\mu_1^2}C_\varepsilon\widetilde{C}(r)\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}\\
&\quad+r\|\mathbf{f}\|_{\infty,\Omega}\Big\}\|\phi-\phi_h\|_{\mathrm{L}^{n/\varepsilon}(\Omega)}\Bigg\} + \widetilde{C}_{ST}\,\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h}) + 2C_2C_{\mathbf{S}}\Big(\|\mathbf{u}_D\|_{1/2,\Gamma_D}\\
&\quad+r\|\mathbf{f}\|_{\infty,\Omega}\Big)\mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h})
\end{aligned} \tag{5.10}$$

Then, utilizing the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\varepsilon}(\Omega)$, with constant $\widetilde{C}_\varepsilon$, and defining the constants

$$C_3 := \frac{L_\mu(1+k_1^2)^{1/2}}{\mu_1^2}C_\varepsilon\widetilde{C}_\varepsilon\widetilde{C}(r) \quad \text{and} \quad C_4 := C_{ST}(1+2\|B\|)\,,$$

the estimate (5.10) yields

$$\begin{aligned}
&\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} \le \Big(C_1\|\mathbf{k}\| + rC_2C_{ST}\Big\{(1+k_2^2)^{1/2}+rC_3\Big\}\|\mathbf{f}\|_{\infty,\Omega}\\
&+rC_2C_3C_{ST}\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}\Big)\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2} + rC_2C_4\,\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h})\\
&+\Big(\widetilde{C}_{ST}+2C_2C_{\mathbf{S}}\{\|\mathbf{u}_D\|_{1/2,\Gamma_D}+r\|\mathbf{f}\|_{\infty,\Omega}\}\Big)\ \mathrm{dist}((\mathbf{t},\mathbf{p},\phi),H_{2,h})
\end{aligned} \tag{5.11}$$

On the other hand, the error estimate (5.7) can be rewritten as

$$\begin{aligned}
\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{H_1} \;\le\;& C_4\,\mathrm{dist}((\boldsymbol{\sigma},\mathbf{u}),H_{1,h}) + C_{ST}\Big(\Big\{1+k_2^2)^{1/2}+rC_3\Big\}\,\|\mathbf{f}\|_{\infty,\Omega}\\
+\;& C_3\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma_D}\Big)\|(\mathbf{t},\mathbf{p},\phi)-(\mathbf{t}_h,\mathbf{p}_h,\phi_h)\|_{H_2}.
\end{aligned} \tag{5.12}$$

Consequently, combining the foregoing inequalities and defining the constants

$$\widehat{C}_2 := C_{ST}\{(1+k_2^2)^{1/2}+rC_3\}(rC_2+1) \quad \text{and} \quad \widehat{C}_3 := C_{ST}C_3(1+rC_2),$$

we arrive at (5.9) and conclude the proof. $\square$

At this point we highlight that the well-posedness of the decoupled discrete problems (4.2) and (4.3) (cf. Lemmas 4.1 and 4.2), as well as the existence of solution of the resulting augmented fully-mixed scheme (4.1) (cf. Theorem 4.8), and the associated a priori error estimate provided by Theorem 5.4, are all valid for arbitrary finite element subspaces approximating the corresponding unknowns. As previously remarked, we stress once again that this fact is consequence of the properties satisfied by the continuous and discrete bilinear forms and nonlinear operators involved, thanks to which no discrete inf-sup conditions to be satisfied by the aforementioned subspaces are required.

Having said the above, we now let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles K (or tetrahedra in R^3) of diameter h_K, and define the mesh size $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, given a generic integer $\ell \geq 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_\ell(K)$ be the space of polynomial functions on K of degree $\leq \ell$, and define the corresponding local Raviart-Thomas space of order ℓ as $\mathrm{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\,\mathbf{x}$, where, according to the notations described in Section 1, $\mathbf{P}_\ell(K) = [\mathrm{P}_\ell(K)]^n$ and $\mathbf{x}$ is the generic vector in R^n. Then, given a particular integer $k \geq 0$, we introduce next the explicit finite element subspaces to be employed in the numerical results reported below in Section 6:

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_N(\mathbf{div},\Omega) : \quad \mathbf{c}^t\boldsymbol{\tau}_h|_K \in \mathrm{RT}_k(K) \quad \forall\,\mathbf{c}\in\mathrm{R}^n \quad \forall\,K \in \mathcal{T}_h\Big\}, \tag{5.13}$$

$$\mathbf{X}_h^{\mathbf{u}} := \Big\{\boldsymbol{v}_h \in \mathbf{C}(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_{k+1}(K) \quad \forall\,K \in \mathcal{T}_h\Big\}, \tag{5.14}$$

$$\mathbf{Y}_h^{\mathbf{t}} := \Big\{\mathbf{s}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{s}_h|_K \in \mathbf{P}_k(K) \quad \forall\,K \in \mathcal{T}_h\Big\}, \tag{5.15}$$

$$\mathbf{H}_h^{\mathbf{p}} := \Big\{\boldsymbol{\tau}_h \in \mathbf{H}_N(\mathrm{div},\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathrm{RT}_k(K) \quad \forall\,K \in \mathcal{T}_h\Big\}, \tag{5.16}$$

$$\mathrm{X}_h^{\phi} := \Big\{\varphi \in C(\Omega) : \quad \varphi|_K \in \mathrm{P}_{k+1}(K) \quad \forall\,K \in \mathcal{T}_h\Big\}. \tag{5.17}$$

In turn, the corresponding approximation properties are as follows:

$(\mathbf{AP}_{\mathbf{h}}^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_N(\mathbf{div},\Omega)$ with $\mathrm{div}\,\boldsymbol{\tau} \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \leq C h^s \Big\{\|\boldsymbol{\tau}\|_{s,\Omega} + \|\mathrm{div}\,\boldsymbol{\tau}\|_{s,\Omega}\Big\}.$$

$(\mathbf{AP}_{\mathbf{h}}^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{v} \in \mathbf{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{v}, \mathbf{X}_h^{\mathbf{u}}) \leq C h^s \|\boldsymbol{v}\|_{s+1,\Omega}.$$

$(\mathbf{AP}_{\mathbf{h}}^{\mathbf{t}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{r} \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{r}, \mathbf{Y}_h^{\mathbf{t}}) \leq C h^s \|\mathbf{r}\|_{s,\Omega}.$$

$(\mathbf{AP}_{\mathbf{h}}^{\mathbf{p}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{q} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_N(\mathrm{div},\Omega)$ with $\mathrm{div}\,\mathbf{q} \in \mathrm{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{q}, \mathbf{H}_h^{\mathbf{p}}) \leq C h^s \Big\{\|\mathbf{q}\|_{s,\Omega} + \|\mathrm{div}\,\mathbf{q}\|_{s,\Omega}\Big\}.$$

$(\mathbf{AP}_{\mathbf{h}}^{\phi})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\varphi \in \mathrm{H}^{s+1}(\Omega)$, there holds

$$\mathrm{dist}(\varphi, \mathrm{X}_h^{\phi}) \leq C h^s \|\varphi\|_{s+1,\Omega}.$$

In this way, the convergence rates of the Galerkin scheme are stated as follows.

Theorem 5.5. *In adition to the hypotheses of Theorems 3.15, 4.8 and 5.4, assume that there exists $s > 0$ such that $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\operatorname{div} \boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, $\mathbf{t} \in \mathbf{H}^s(\Omega)$, $\mathbf{p} \in \mathbf{H}^s(\Omega)$, $\operatorname{div} \mathbf{p} \in \mathrm{H}^s(\Omega)$ and $\phi \in \mathrm{H}^{s+1}(\Omega)$. Then, there exists $\widehat{C} > 0$, independent of h, such that, with the finite element subspaces defined by (5.13) – (5.17), there holds*

$$\begin{aligned} \|(\mathbf{t}, \mathbf{p}, \phi) - (\mathbf{t}_h, \mathbf{p}_h, \phi_h)\|_{H_1} + \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H_2} \leq \widehat{C} h^{\min\{s,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{s,\Omega} \\ + \|\operatorname{div} \boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{u}\|_{1+s,\Omega} + \|\mathbf{p}\|_{s,\Omega} + \|\operatorname{div} \mathbf{p}\|_{s,\Omega} + \|\mathbf{t}\|_{s,\Omega} + \|\phi\|_{1+s,\Omega} \Big\} \end{aligned} \tag{5.18}$$

Proof. It follows directly from the Cea estimate (5.8) and the above approximation properties. □

6 Numerical results

In this section we present some examples illustrating the performance of our augmented fully-mixed finite element 4.1 on a set of quasi-uniform triangulations of the corresponding domains and considering the finite element spaces introduced in Section 5. Our implementation is based on a FreeFem++ code (see [22]), in conjuntion with the direct nolinear solvers UMFPACK (see [14]) and MUMPS. A Newton algorithm with a fixed given tolerance *tol* has been used for the corresponding fixed-point problem (4.5) and the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates, say $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq tol\,,$$

where $\|\cdot\|$ stands for the usual euclidean norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{X}_h^{\mathbf{u}}$, $\mathbf{Y}_h^{\mathbf{t}}$, $\mathbf{H}_h^{\mathbf{p}}$ and X_h^{ϕ}. The stabilization parameters are chosen according to the ranges indicated in Lemmas 4.1 and 4.2 (see also Lemmas 3.5 and 3.7).

We now introduce some additional notation. The individual and total errors are denoted by:

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\operatorname{div};\Omega}\,, \quad e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}\,,$$

$$e(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \quad e(\mathbf{p}) := \|\mathbf{p} - \mathbf{p}_h\|_{\operatorname{div};\Omega}\,, \quad e(\phi) := \|\phi - \phi_h\|_{1,\Omega}\,.$$

Next, as usual, we let $r(\cdot)$ be the experimental rate of convergence given by

$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/\widehat{e}(\boldsymbol{\sigma}))}{\log(h/\widehat{h})}\,, \quad r(\boldsymbol{\sigma}) := \frac{\log(e(\mathbf{u})/\widehat{e}(\mathbf{u}))}{\log(h/\widehat{h})}\,,$$

$$r(\mathbf{t}) := \frac{\log(e(\mathbf{t})/\widehat{e}(\mathbf{t}))}{\log(h/\widehat{h})}\,, \quad r(\mathbf{p}) := \frac{\log(e(\mathbf{p})\widehat{e}(\mathbf{p}))}{\log(h/\widehat{h})}\,, \quad r(\phi) := \frac{\log(e(\phi)/\widehat{e}(\phi))}{\log(h/\widehat{h})}\,,$$

where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{e}$, respectively.

Example 1. In our first example we illustrate the accuracy of our method in 2D by considering a manufactured exact solution defined on $\Omega := (0,1)^2$. We introduce the coefficients $\mu(\phi) = (1 - c\phi)^{-2}$, $\gamma(\phi) = c\phi(1 - c\phi)^2$, $\vartheta(|\mathbf{t}|) = m_1 + m_2(1 + |\mathbf{t}|^2)^{m_3/2-1}$, and the source terms on the right hand sides are adjusted in such a way that the exact solutions are given by the smooth functions

$$\phi(x_1, x_2) = b - b\exp(-x_1(x_1 - 1)x_2(x_2 - 1))\,, \quad \mathbf{t} = \nabla\phi\,,$$

$$\mathbf{u}(x_1, x_2) = \begin{pmatrix} \sin(2\pi x_1)\cos(2\pi x_2) \\ -\cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix}\,, \quad \boldsymbol{\sigma} = \mu(\phi)\nabla\mathbf{u} - (x_1^2 - x_2^2)\mathbb{I}\,,$$

for $(x_1, x_2) \in \overline{\Omega}$. We take $b = 15$, $c = m_1 = m_2 = 1/2$, $m_3 = 3/2$ and set $\Gamma_D = \partial\Omega$, where ϕ vanishes and $\mathbf{u}_D$ is imposed accordingly to the exact solution. The mean value of $\operatorname{tr} \boldsymbol{\sigma}_h$ over Ω is fixed via Lagrange multiplier strategy. As defined above, the scalar field ϕ is bounded in Ω and so the coefficients are also bounded. In particular we have $\mu_1 = 0.99$, $\mu_2 = 3.35$, $\vartheta_1 = 0.81$, $\vartheta_2 = 1$ and $\tilde{\vartheta}_2 = 1.19$. Therefore, the stabilization constants are chosen as $\kappa_1 = \mu_1^2/\mu_2 = 0.2976$, $\kappa_2 = 1/\mu_2 = 0.2985$, $\kappa_3 = \kappa_1/2 = 0.1488$, $l_1 = \vartheta_1/\tilde{\vartheta}_2^2 = 0.5720$, $l_2 = l_1/2 = 0.2860$, $l_3 = \vartheta_1/2 = 0.4050$ and $l_4 = 0.2025$. The domain is partitioned into quasi-uniform meshes with $2^n + 3$, n=0,1,...,8 vertices on each side of the domain. Values and plots of errors and corresponding rates associated to $\mathbb{RT}_k - \mathbf{P_{k+1}} - \mathbf{P}_k - \mathbf{RT}_k - P_{k+1}$ approximations with $k = 0$ and $k = 1$ are summarized in Table 6.1 and Figure 6.1, respectively, where we observe convergence rates of $O(h^{k+1})$ for stresses, velocities, gradient of velocities and the scalar fields in the relevant norms. These findings are in agreement with the theoretical error bounds of Section 5 (cf. 5.18).

Example 2. Our second test focuses on the case where, under quasi-uniform mesh refinement the convergence rates are affected by a non-convex domain $\Omega = (0,1)^2 \setminus [0.5, 1]^2$, where an exact solution to (2.6) and the functions μ, ϑ and γ are given as in the previous test. In this case, $b = 3$, $c = m_1 = m_2 = 1/2$, $m_3 = 3/2$. Now the boundary is indeed split into $\Gamma_N = (0.5, 1) \times (0.5, 1)$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Values and plots of errors and corresponding rates associated to $\mathbb{RT}_k - \mathbf{P_{k+1}} - \mathbf{P}_k - \mathbf{RT}_k - P_{k+1}$ approximations with $k = 0$ and $k = 1$ are summarized in Table 6.2 and Figure 6.2. We can see that with respect to Example 1, a more refined mesh is required to reach the convergence orders indicated by the theory.

Example 3. In this example, we consider $\Omega = (0,1)^3$. The functions μ, ϑ and γ are established as in Example 1. With respect to boundary conditions, we impose Neumann conditions on $\Gamma_N := [0,1]^2 \times \{1\}$ and Dirichlet conditions on the rest of the boundary, that is, $\Gamma_D := \partial\Omega \setminus \Gamma_N$. We consider boundary data ϕ_D and source therms $\mathbf{f}$ and g such that the exact solution is given by

$$\phi(x_1, x_2, x_3) = b - b\exp(x_1(x_1-1)x_2(x_2-1)x_3(x_3-1))\,, \quad \mathbf{t} = \nabla\phi\,,$$

$$\mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)cos(\pi x_3) \\ -\cos(\pi x_1)\sin(\pi x_2)cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}\,, \quad \boldsymbol{\sigma} = \mu(\phi)\nabla\mathbf{u} - (x_1 - 0.5)^3\sin(x_3 + x_2)\mathbb{I}\,,$$

We take $b = 15$, $c = m_1 = m_2 = 1/2$, $m_3 = 3/2$. Concerning the stabilization parameters, we take them again as in Example 1. Part of the solution is show in Figure 6.3, and a convergence history for a set of quasi-uniform mesh refinements is shown in Table 6.3,thus showing also that, having the problem a smooth exact solution, this fully-mixed finite element method converges optimally with order $O(h)$ (when using a first order element).

Example 4. To conclude, we replicate Example 2 in a three-dimensional setting. The domain consists on the polyhedral region $\Omega = (0,1)^3 \setminus [0.5, 1]^3$, where we impose Neumann conditions on $\Gamma_N := [0.5, 1]^3$ and $\Gamma_D = \Omega \setminus \Gamma_N$. All parameters and functions are taken as in the previous test. Part of the solution is show in Figure 6.4, and a convergence history for a set of quasi-uniform mesh refinements is shown in Table 6.4. We can see that with respect to Example 3, the convergence rate of $\boldsymbol{\sigma}$ is affected.

We end the paper by announcing that the corresponding a posteriori error analysis of the fully-mixed finite element method proposed here, which follows the approach from [4] and [5] for the methods in [2] and [3], respectively, will be reported in a forthcoming work.

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Augmented $\mathbb{RT}_0 - \mathbf{P_1} - \mathbf{P}_0 - \mathbf{RT}_0 - P_1$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
307	58.1408	7.5339	0.6924	1.4120	0.8780
463	48.2946	6.0966	0.5832	1.1562	0.7111
871	35.2493	4.0013	0.2765	0.8584	0.5127
2071	22.6828	2.2054	0.1594	0.5616	0.3278
6007	13.1637	1.1324	0.0862	0.3291	0.1899
20023	7.1376	0.5799	0.1792	0.1792	0.1031
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.3536	-	-	-	-	-
0.2828	0.8315	0.9486	0.7691	0.8957	0.9451
0.2020	0.9358	1.2516	0.8973	0.8852	0.9718
0.1286	0.9753	1.3179	0.9827	0.9386	0.9899
0.0744	0.9956	1.2196	1.0082	0.9775	0.9983
0.0404	1.0019	1.0955	1.0050	0.9944	0.9999

Augmented $\mathbb{RT}_1 - \mathbf{P_2} - \mathbf{P}_1 - \mathbf{RT}_1 - P_2$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
821	19.8155	1.8648	0.1152	0.3691	0.1282
1245	13.5241	1.1568	0.0736	0.2709	0.0821
2357	7.2093	0.5596	0.0375	0.1557	0.0418
5637	2.9893	0.2150	0.0151	0.0692	0.0168
16421	1.0124	0.0694	0.0051	0.0232	0.0056
54885	0.2993	0.0201	0.0014	0.0067	0.0016
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.3536	-	-	-	-	-
0.2828	1.7118	2.1398	2.0101	1.3860	1.9978
0.2020	1.8697	2.1584	2.0047	1.6472	2.0094
0.1286	1.9477	2.1162	2.0057	1.7938	2.0105
0.0744	1.9810	2.0680	2.0009	2.0015	2.0056
0.0404	1.9985	2.0086	1.9999	1.9924	2.0004

Table 6.1: Convergence history for Example 1, with a quasi-uniform mesh refinement and a tolerance of 10^{-6}. For the first order approximation, the first and second simulations took 8 fixed-point iterations, the third took 7 fixed-point iterations and the last three simulations took 6 fixed-point iterations. For the second order approximation all the simulations took 6 fixed-point iterations.

Augmented $\mathbb{RT}_0 - \mathbf{P_1} - \mathbf{P}_0 - \mathbf{RT}_0 - P_1$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
297	33.0811	4.1740	0.1191	0.2584	0.1388
625	21.0794	2.6568	0.0759	0.1784	0.0774
1585	13.0911	1.4813	0.0481	0.1082	0.0487
4769	7.3657	0.7983	0.0278	0.0642	0.0281
16753	3.8631	0.4073	0.0145	0.0332	0.0145
60465	1.9959	0.2113	0.0075	0.0173	0.0075
230321	1.0120	0.1079	0.0038	0.0088	0.0038
907393	0.5099	0.0544	0.0019	0.0044	0.0019
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.3601	-	-	-	-	-
0.2372	1.0797	1.0823	1.0801	0.8871	1.3973
0.1491	1.0266	1.2589	0.9799	1.0785	0.9980
0.0855	1.0328	1.1102	0.9817	0.9369	0.9838
0.0452	1.9810	1.0578	1.0261	1.0338	1.0392
0.0264	1.2293	1.2218	1.2127	1.2112	1.2124
0.0152	1.2379	1.2247	1.2248	1.2301	1.2297
0.0072	0.9223	0.9196	0.9259	0.9299	0.9253

Augmented $\mathbb{RT}_1 - \mathbf{P_2} - \mathbf{P}_1 - \mathbf{RT}_1 - P_2$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
791	9.0327	1.0919	0.0169	0.0525	0.0166
1683	3.9097	0.4354	0.0075	0.0201	0.0074
4303	1.5381	0.1594	0.0028	0.0076	0.0027
13019	0.4787	0.0486	0.0009	0.0025	0.0008
45895	0.1304	0.0135	0.0002	0.0006	0.0002
165943	0.0364	0.0036	6.8348e-05	0.0001	6.5972e-05
632727	0.0094	0.0009	1.8102e-05	4.8961e-05	1.7185e-05
2494035	0.0024	0.0002	4.5159e-05	1.2450e-05	4.3366e-06
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.3601	-	-	-	-	-
0.2372	2.0062	2.2026	1.9655	2.2902	1.9391
0.1491	2.2547	2.2547	2.1678	2.1729	2.2159
0.0855	2.0640	2.0640	1.9845	1.9029	1.9925
0.0452	2.0137	2.0137	2.0442	2.0466	2.0291
0.0264	2.4374	2.4374	2.4015	2.4169	2.4171
0.0154	2.5301	2.5620	2.4866	2.5427	2.5177
0.0072	1.8050	1.8023	1.8330	1.8076	1.8178

Table 6.2: Convergence history for Example 2, with a quasi-uniform mesh refinement and a tolerance of 10^{-6}. For the first and second order approximation, all simulations took 7 fixed-point iterations.

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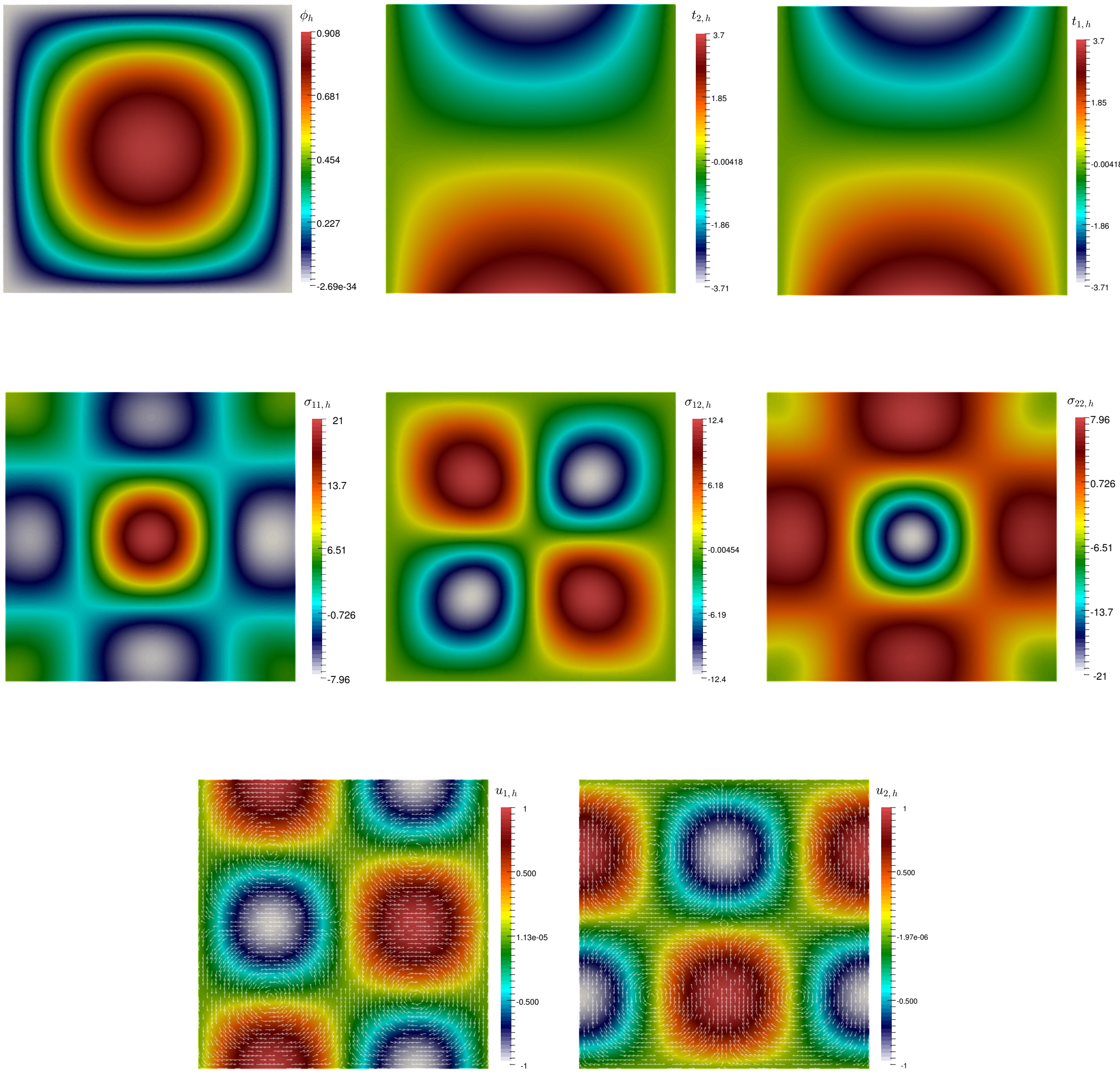

Figure 6.1: Numerical Results for Example 1: From left to right and from up to down: approximation of scalar field concentration ϕ_h , gradient component of concentration $\mathbf{t}_h$, stress components $\boldsymbol{\sigma}_h$ and velocity components $\mathbf{u}_h$. Snapshots obtained from a simulation with 54 885 DOF.

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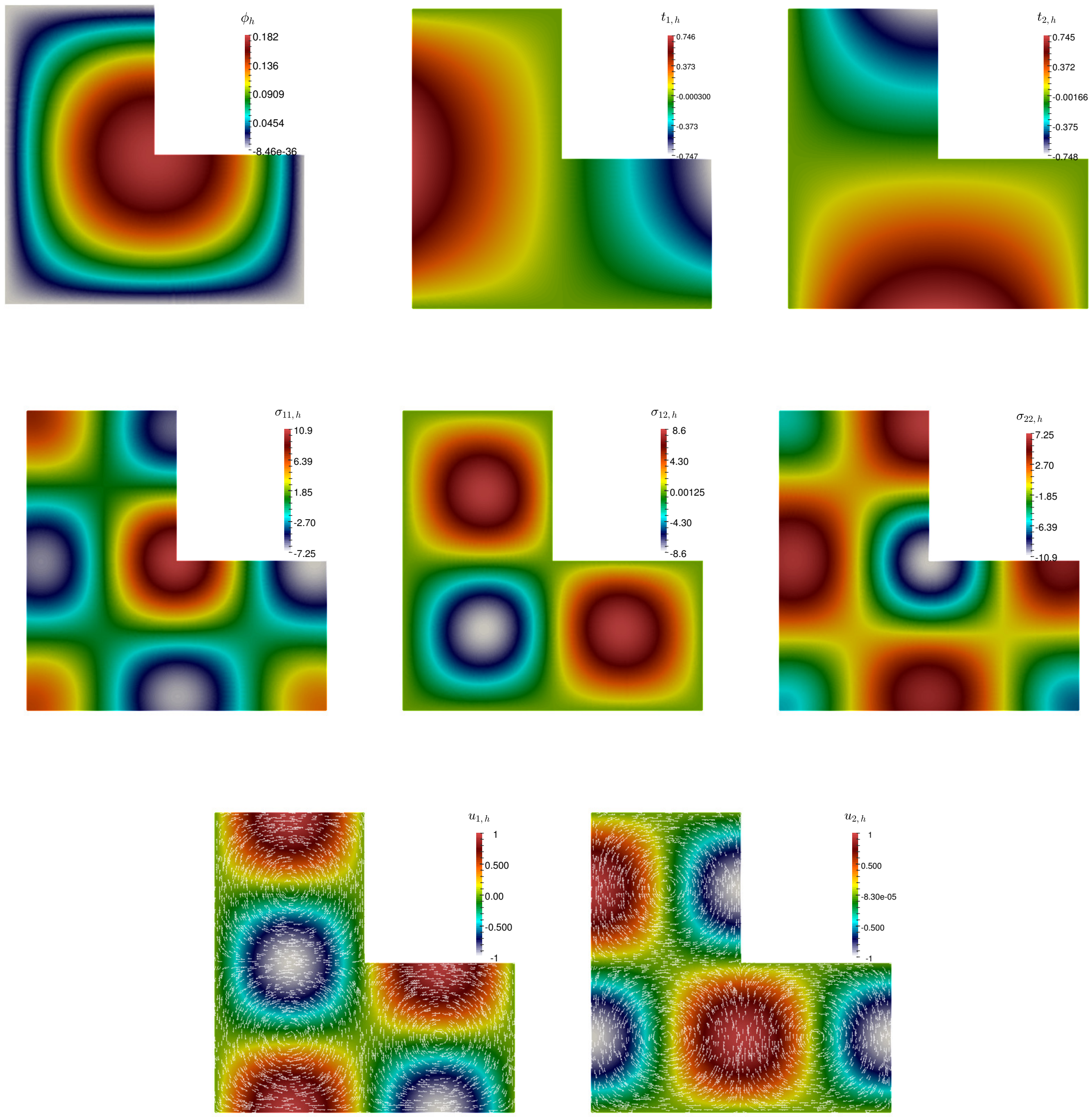

Figure 6.2: Numerical Results for Example 2: From left to right and from up to down: approximation of scalar field concentration ϕ_h , gradient component of concentration $\mathbf{t}_h$, stress components $\boldsymbol{\sigma}_h$ and velocity components $\mathbf{u}_h$. Snapshots obtained from a simulation with 230 321 DOF.

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Augmented $\mathbb{RT}_0 - \mathbf{P_1} - \mathbf{P}_0 - \mathbf{RT}_0 - P_1$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
5108	18.9903	2.5832	0.1557	0.5325	0.2445
9714	16.7408	2.0745	0.1265	0.4308	0.1997
25862	13.2246	1.4643	0.0917	0.3106	0.1452
97662	9.0988	0.9101	0.0589	0.1987	0.0935
493358	5.5202	0.5155	0.0342	0.1153	0.0544
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.3601	-	-	-	-	-
0.2372	0.5650	0.9828	0.9282	0.9497	0.9081
0.1491	0.7007	1.0352	0.9562	0.9722	0.9454
0.0855	0.8273	1.0522	0.9788	0.9877	0.9741
0.0452	0.9143	1.0398	0.9919	0.9901	0.9901

Table 6.3: Convergence history for Example 3, with a quasi-uniform mesh refinement and first order approximation. The first simulatons took 9 fixed-point iterations and the rest took 8 fixed-point iterations with a tolerance *tol*$= 10^{-8}$.

Augmented $\mathbb{RT}_0 - \mathbf{P_1} - \mathbf{P}_0 - \mathbf{RT}_0 - P_1$					
DOF	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(\mathbf{t})$	$e(\mathbf{p})$	$e(\phi)$
2032	19.8915	2.7683	0.1885	0.6237	0.3009
4548	16.7019	2.1680	0.1468	0.4790	0.2369
14602	12.6054	1.4720	0.1006	0.3255	0.1620
64974	8.4005	0.8881	0.0613	0.1970	0.0984
369094	5.0242	0.4943	0.0342	0.1098	0.0549
h	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(\mathbf{t})$	$r(\mathbf{p})$	$r(\phi)$
0.4714	-	-	-	-	-
0.3535	0.6075	0.8496	0.8699	0.9174	0.8305
0.2357	0.6940	0.9549	0.9314	0.9530	0.9378
0.1414	0.7944	0.9890	0.9696	0.9821	0.9757
0.0785	0.8744	0.9968	0.9892	0.9941	0.9919

Table 6.4: Convergence history for Example 4, with a quasi-uniform mesh refinement and first order approximation. All simulatons took 6 fixed-point iterations with a tolerance *tol*$= 10^{-8}$.

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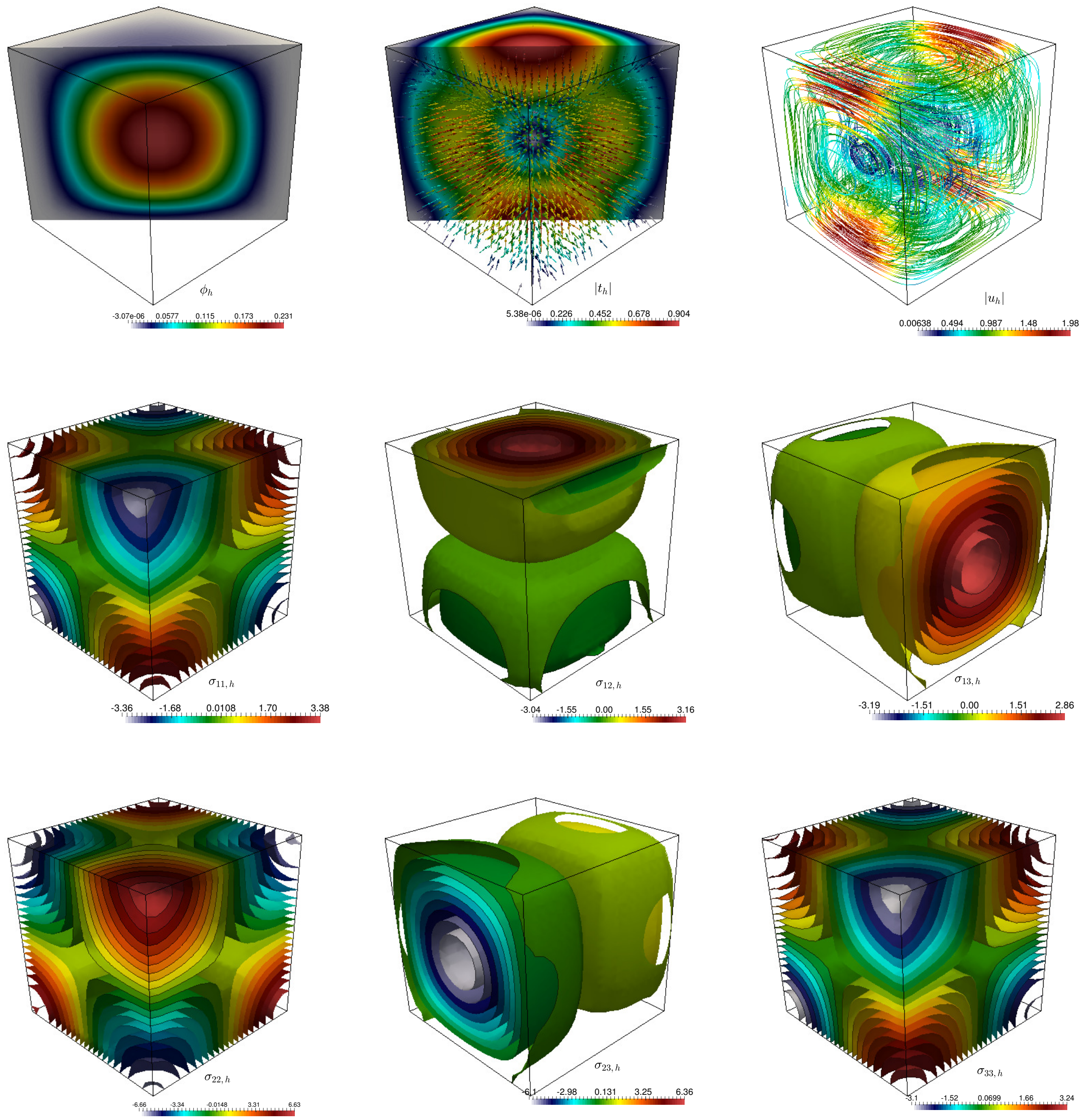

Figure 6.3: Numerical Results for Example 3: From left to right and from up to down: approximation of scalar field concentration ϕ_h , gradient of concentration $\mathbf{t}_h$, velocity $\mathbf{u}_h$ and stress components $\boldsymbol{\sigma}_h$. Snapshots obtained from a simulation with 493 358 DOF.

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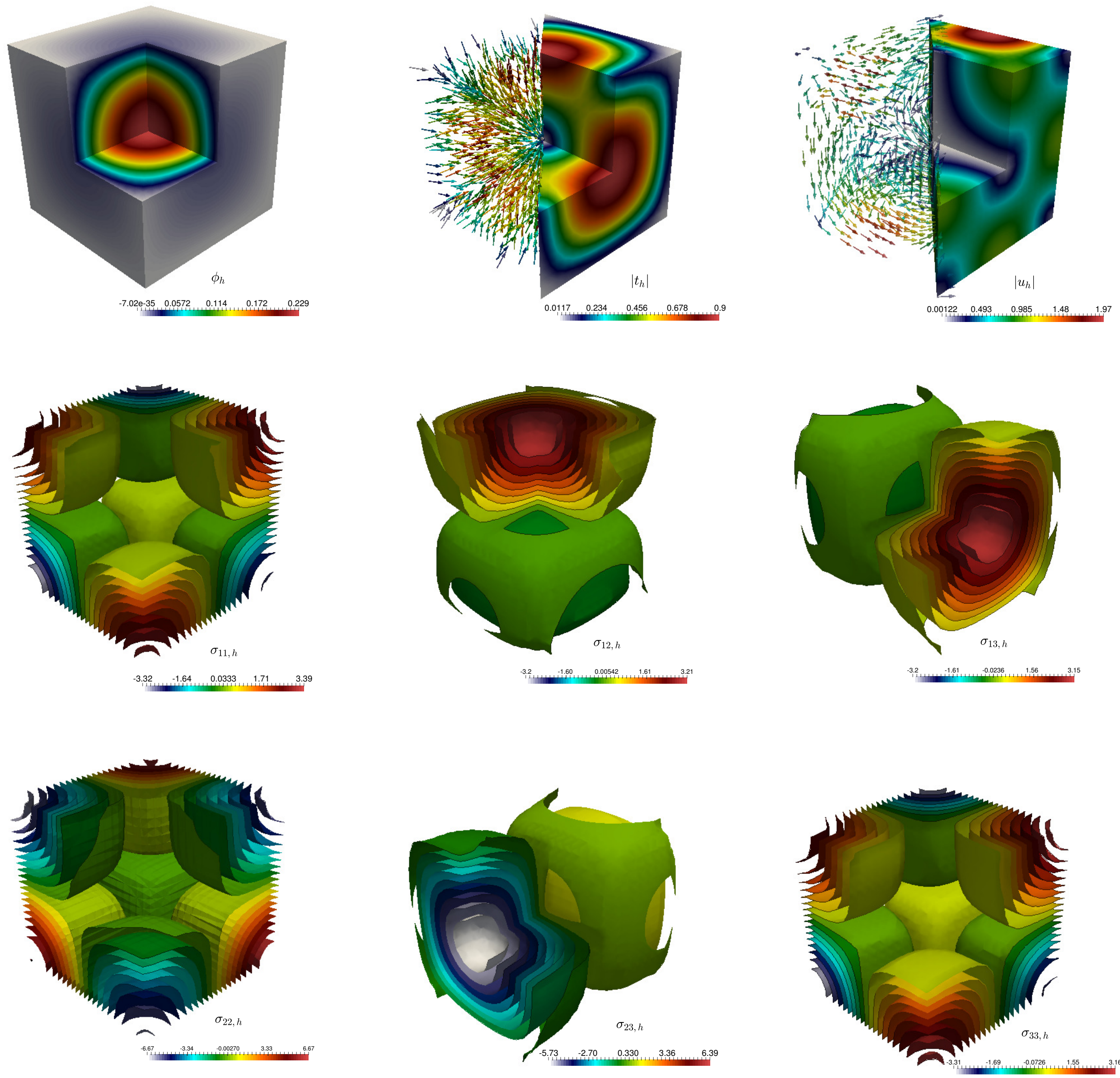

Figure 6.4: Numerical Results for Example 4: From left to right and from up to down: approximation of scalar field concentration ϕ_h , gradient of concentration $\mathbf{t}_h$, velocity components $\mathbf{u}_h$, stress components $\boldsymbol{\sigma}_h$. Snapshots obtained from a simulation with 369 094 DOF.

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