Transitivity and minimality in the context of Turing machine topological models

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Abstract

Turing machines are studied as dynamical systems since two decades ago. Several results concerning topological properties such as equicontinuity, periodicity, mortality, entropy, etc. has been established. These properties strongly depend on the topological model that one consider, and for Turing machines two main models are pertinent. Here we focus on *transitivity*, *minimality* and other adjacent properties. In the context of Turing machines, transitivity means that there exists a configuration whose evolution contains every possible configuration over any finite window. Minimality means that every configuration fulfills this, and is a very strong property. This paper establishes the exact relations existing between four main properties: transitivity, minimality, existence of blocking words, and reversibility; showing examples on each non empty class at the intersection of these properties. It also develops the *embedding technique*, that combines two Turing machines to produce a third one that, under some particular conditions, will inherit the properties of one of the original machines. This powerful tool is used to establish undecidability results and also to multiply the number of examples that we have found.

1 Introduction

The study of Turing machines from a dynamical system point of view was opened by Moore [15] and Kůrka [14], formalizing three dynamical models: Generalized Shifts (GS), Turing machine with Moving Head (TMH) and Turing machine with moving Tape (TMT); the first one presented by Moore, and the last two by Kůrka.

In the TMT model, the head is fixed at position 0, and the tape is shifted. Generalized Shifts results to be equivalent to TMT. On the other hand, the TMH model inserts the head state in the tape, as an additional symbol that can move, treating the system as a cellular automaton over a sofic subshift. It can be said that TMH gives predominance to the cells around the position 0 of the tape, while TMT gives relevancy to the cells around the head. These two different topologies are not equivalent and a given machine may have different properties depending on the considered topological model, as Kůrka established. The seminal work

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of Kůrka inspired a rich line of research that consider properties such as immortality [10, 13, 11], entropy [14, 17, 12, 9], equicontinuity [6], periodicity [14, 1, 13, 2] and, recently, transitivity and minimality [2, 8].

Also, the column factor of the TMT model, the *trace-shift*, was deeply studied. This one-way subshift consists in the sequence of states and symbols that the head reads through the time. Previous works consider properties such as soficity, complexity, transitivity, periodicity and surjectivity [7, 4, 6, 18, 19].

This research is focused on *transitivity* and *minimality* which have been proved undecidable in [8]. Transitivity is a necessary condition to Devaney chaos [3], it means that there exists a point in the dynamical system whose orbit is dense, *i.e.*, it passes as near as desired to every other point. Minimality implies that every point is transitive, which is equivalent to say that the system has no proper subsystem.

In the context of Turing machines with TMT model, a transitive configuration will produce an evolution that will make appear every combination of tape symbols and head state on every finite window around the head. A machine with a minimal TMT system has this property for every initial configuration. The first example of Turing machine with such a strong property was the SMART machine [2], it was used to prove the undecidability of both, minimality and transitivity, properties by exploiting the *embedding technique*.

This research establishes the relation between four properties: reversibility, existence of blocking words, transitivity and minimality. Reversibility is equivalent to surjectivity in Turing machines, which is kept in TMT and TMH dynamical models, thus it is a necessary condition for transitivity. Nevertheless, the trace-shift can be surjective even if the machine is not, but only when the machine has what we call *blocking words* [18]. Both properties are strongly related with transitivity and minimality, and the exact relations between them are established in this work.

We present a classification of Turing machines based on how these four properties are distributed on the three dynamical models: TMT, TMH and the trace-shift. The SMART machine, as a base, is used to construct several of the presented examples. Also, using the embedding technique, it is possible to prove that every class of the classification has infinitely many members. We prove that, under some hypothesis, the embedding technique preserves most of the mentioned dynamical properties, providing a better understanding of this tool.

The present paper is structured as follows: In section 2 the basic definitions are given. Section 3 describes the embedding technique and their associated tools. The main results are exposed in section 4, where relationships between the above mentioned properties are established, intersection classes are defined and examples inside each class are given. In section 5 we establish conditions to preserve the studied properties through embedding, in this way we prove that each class contains infinitely many Turing machines. Finally, in section 6 we discuss the repercussions and the future work related to this research.

2 Definitions

2.1 Dynamical systems

A *topological dynamical system* is a pair (X, T), where X is a topological space called *phase space* and $T : X \to X$ is a continuous function called *global transition function*. For each $x \in X$, an infinite sequence, called *orbit of* x, is associated: $\mathcal{O}(x) = (T^n(x))_{n\in\mathbb{N}}$. We say that x is *pre-periodic* if there exists $t, n \in \mathbb{N}$ such that $T^{t+n}(x) = T^t(x)$, and if $t = 0$ we say that the point x is periodic. A system (Y, T) is a *subsystem* of (X, T) if $Y \subseteq X$ is a closed and T-invariant set. The smallest subsystem that contains a given point x is the closure of its orbit $\overline{\mathcal{O}(x)}$. A system (X, T) is *transitive* if there exists some point x such that $\overline{\mathcal{O}(x)} = X$, and a point satisfying this is called a *transitive point*. A system where every point is transitive is a *minimal* system. If (Z, F) is another topological dynamical system and there exists a continuous and onto function $\varphi : X \to Z$ such that $F \circ \varphi = \varphi \circ T$, then (Z, F) is a *factor* of (X, T) and it inherits several of its properties.

2.2 Symbolic systems

Given a finite set of symbols Σ, called *alphabet*, the set $\Sigma^{\mathbb{Z}}$ is the *two-sided full shift*, while $\Sigma^{\mathbb{N}}$ is the *one-sided full shift*. Let $\mathbb{T}$ denote either $\mathbb{Z}$ or $\mathbb{N}$. An element x of $\Sigma^{\mathbb{T}}$ is an ordered sequence $x = (x_i)_{i\in\mathbb{T}}$. The *shift* function σ is defined in $\Sigma^{\mathbb{T}}$ by $\sigma(y)_i = y_{i+1}$, and it is a bijective function if $\mathbb{T} = \mathbb{Z}$. The Cantor metric of the full shift is: $d(x, y) = 2^{-i}$, where $i = \min\{|n| \ : \ x_n \neq y_n\}$. With this metric, $\Sigma^{\mathbb{T}}$ is compact and $(\Sigma^{\mathbb{T}}, \sigma)$ is a topological dynamical system. The subsystems of $(\Sigma^{\mathbb{T}}, \sigma)$ are called *subshifts*. Frequently, we will prefer to denote $\Sigma^{\mathbb{N}}$ by Σ^{ω}, the set of right-infinite sequences; and symmetrically, we will use ${}^{\omega}\Sigma$ to denote the left-infinite sequences. Distinguishing the origin in a sequence in $\Sigma^{\mathbb{Z}}$ is important, thus we will add a *dot* before the symbol with coordinate 0, for example: $x = ...x_{-2}x_{-1}.x_0x_1x_2...$

2.3 Languages

Finite sequences of symbols of Σ are called *words* and the set of all words is denoted by Σ^*; this set includes the *empty word* which we denote by ϵ. A finite word v is said to be a *subword*[1] of another (finite or infinite) word z, if there exist two indexes i and j, such that $v = z_i z_{i+1}...z_j$. In this case, we write: $v \sqsubseteq z$. The length of a word u is denoted by $|u|$. Given a subshift S, a formal language is defined by $\mathcal{L}(S) = \{u \in \Sigma^* \ : \ \exists z \in S, \ u \sqsubseteq z\}$. Reciprocally, given a formal language L, a set of right-infinite sequences can be defined: $\mathcal{S}_L = \{z \in \Sigma^{\mathbb{N}} \ : \ \forall u \sqsubseteq z, \ u \in L\}$. It is a result that S is a subshift if and only if $\mathcal{S}_{\mathcal{L}(S)} = S$.

2.4 Turing machines

A Turing machine (TM) is determined by a 3-tuple $M = (Q, \Sigma, \delta)$, where Q is a finite set of states, Σ is an alphabet, and δ is the local transition function. The usual model includes also an initial and a final state, and a blank symbol, but we omit them here because we are interested in properties of the whole set of states. A *configuration* is a tuple $(r, i, w) \in X = Q \times \mathbb{Z} \times \Sigma^{\mathbb{Z}}$, where r is the *current state*, i is the *head position*, and w represents the contents of a *bi-infinite tape*, which contains one symbol of Σ on each cell. The transition function δ specifies the movement and actions of the head. It can be considered in two forms, the *quintuple model* and the *quadruple model* (as seen in [16]), but here we focuses in the quintuple model due to reasons that we justify later.

Quintuple model. $\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$. The machine transforms a configuration (r, i, w) into $(r', i + c, w')$, if there exists the instruction $\delta(r, w_i) = (r', \alpha', c)$, with w' defined by $w'_i = \alpha'$ and $w'_j = w_j$ for all $j \neq i$.

In this way, the *global transition function* $T : X \to X$ is defined and it may be a partial function.

If δ is not defined over some pair (r, α), we say that this is an *error pair* and that r is an *error state*. When there is no error pair, the Turing machine is *complete* (c-TM), and T is a total function. When T is onto, we say that the machine is *backward complete*. The only way a machine is not onto is due to the existence of pairs (r, α) which are not present in any tuple of the image of δ, such pairs are called *defective pairs*, the state r is called *defective state* and the symbol α is called a *defective symbol* for r. If every configuration has at most one pre-image, the machine is *injective*. The properties of backward completeness, completeness and injectivity have the peculiarity that in the presence of one of them, the other two are equivalent, in other words, either none, only one, or all the three properties are true.

Definition 1. *The* position *function* $\mathcal{P} : X \to \mathbb{Z}$ *is defined by* $\mathcal{P}(r, i, w) = i$.
The support *function* $\mathcal{S} : X \to \wp(\mathbb{Z})$ *is defined by*

$$\mathcal{S}(r, i, w) = \{\mathcal{P}(T^n(r, i, w)) \mid n \in \mathbb{N}\}.$$

The state *function* $\mathcal{Q} : X \to Q$ *is defined by* $\mathcal{Q}(r, i, w) = r$.

These three functions define the current position of the head, the set of cells that the head visits during the evolution of the system and the current state of the configuration.

We also consider *finite configurations* $(r, i, u.v)$, where $u, \ v \in \Sigma^*$ and $-|u| \leq i \leq |v| - 1$; here we interpret u as a word to the right: $u = u_{-|u|} \dots u_{-1}$. T can also act over these configurations if the head

[1]this notion is usually called "factor", but here the word "factor" is already used.

does not try to move outside the limits of $u.v$. With this spatial limitation, we can consider the dynamics of T until the head exits the limits of the configuration, and take the time $m(r,i,u.v)$ when this happen, *i. e.*, such that at $T^{m(r,i,u.v)}(r,i,u.v)$ the head is on the border and will exit at next iteration. We will write $x \vdash y$ if $y = T(x)$ and $x \vdash^* y$ if there exists a time t such that $y = T^t(x)$, this for both, finite and infinite configurations. Position, state and support functions can also be defined for finite configurations by $\mathcal{P}(r,i,u.v) = i$, $\mathcal{Q}(r,i,u.v) = r$ and $\mathcal{S}(r,i,u.v) = \{\mathcal{P}(T^n(r,i,u.v)) \mid n \in [0, m(r,i,u.v)]\}$. Translations of global or partial configurations are obtained by overloading the shift function σ, which we define in X by $\sigma(r,i,w) = (r, i-1, \sigma(w))$.

Notation. For clearer presentation, we sometimes use the convenient notation $(\cdots\, w_{-1}\, .w_0 \,\cdots\, \underset{r}{w_i}\, w_{i+1} \,\cdots)$ to represent the configuration (r,i,w) in this research.

2.4.1 Reversibility in the quintuple model

The reverse T^{-1} of the global transition function of an injective Turing machine in quintuple model does not correspond, in general, to the transition function of a Turing machine, because in T^{-1} the head should move before reading. Traditionally, the theory of reversible Turing machines has been developed in the quadruple model, where for each step of the machine is allowed to move or write, but no both at the same time. Any machine can be expressed in this way, but there is a loss of velocity, and the dynamics is affected. Therefore, we adopt another, but equivalent, way to define a proper reverse machine, as introduced in [2].

In fact, even if T^{-1} is not a Turing machine, it is conjugated to one, as we will explain. Any reversible machine can be characterized by a pair (ρ, μ), where $\rho : Q \times \Sigma \to Q \times \Sigma$ is a partial injective function and a partial function $\mu : Q \to \{-1, 0, +1\}$, such that for every non error pair (r,s) we have $\delta(r,s) = (r', s', \mu(r'))$, where $r \in Q$, $s \in \Sigma$ and $(r', s') = \rho(r,s)$.

We define the *reverse machine* of M as $M^{-1} = (Q, \Sigma, \delta^{-1})$, where δ^{-1} is defined by $\delta^{-1}(r', s') = (r, s, -\mu(r))$ if and only if $r \in Q$, $s \in \Sigma$, and $(r', s') = \rho(r,s)$. Let us call T' the transition function associated to M^{-1}. If we define also the function $\varphi : (\Sigma^{\mathbb{Z}}, \mathbb{Z}, Q) \to (\Sigma^{\mathbb{Z}}, \mathbb{Z}, Q)$ by $\varphi(w,i,r) = (w, i-\mu(r), r)$, we obtain that $T^{-1} = \varphi^{-1} \circ T' \circ \varphi$. In this way, a *starting configuration* of M is a halting configuration of M^{-1} and viceversa. From these considerations, we will say that an injective Turing machine is *reversible*.

2.5 Turing machines seen as dynamical systems

In order to define a dynamical system, we have chosen to restrict our scope to complete machines. Providing $X = Q \times \mathbb{Z} \times \Sigma^{\mathbb{Z}}$ with the product topology, the pair (X, T) is a topological dynamical system, but X is not a compact set. In [14], Kůrka reformulates X in two ways in order to overcome with this problem.

2.5.1 Turing machine with moving head (TMH)

In this model, the head is added as an element of the tape; then, the phase space is the set $X_h \subset (\Sigma \cup Q)^{\mathbb{Z}}$, defined by $X_h = \{x \in (\Sigma \cup Q)^{\mathbb{Z}} \mid |\{i \in \mathbb{Z} : x_i \in Q\}| \leq 1\}$. The transition function T_h naturally consists in one application of the local transition function δ, taking the consideration that the head position is at the right of the unique cell that contains a state on the tape. Configurations with no state in the tape are headless configurations and are treated as fixed points. The natural function $\psi : X \to X_h$ transforms (r,i,w) into $\psi(r,i,w) = x$, where x is defined by $x_j = w_j$ if $j < i$, $x_i = r$ and $x_j = w_{j-1}$ if $j > i$.

As defined, X_h is a subshift of $(\Sigma \cup Q)^{\mathbb{Z}}$, and it has the same metrics and topology of this set. With this metrics, ψ results to be a continuous and one-to-one function, but $\psi(X) \neq X_h = \overline{\psi(X)}$, its topological closure. Configurations of $X_h \setminus \psi(X)$ are headless configurations, on which T_h is defined as the identity. The transition function T_h results to be continuous too.

Finite configurations are pointed words of the form $v.v'$, where the dot indicates the origin position. If v or v' has a symbol in Q, the word contains the 'head'; if this symbol is not the last character of v', T_h can be applied to $v.v'$. A configuration x is an *extension* of $v.v'$ if $x_{-|v|} \ldots x_{-1} = v$ and $x_0 \ldots x_{|v'|-1} = v'$, and it is noted as $v.v' \sqsubseteq x$.

The position, state and support functions can also be defined in $\psi(X)$ naturally, and we will have that $\mathcal{P}(x) = \mathcal{P}(\psi^{-1}(x))$ and it is equal to the unique index i such that $x_i \in Q$. Of course, they cannot be defined in $X_h \setminus \psi(X)$. We will consider also their extension to finite configurations as before.

If $x = \psi(r, i, w)$ is a periodic point in TMH with period n, then $T^n(r, i, w) = (r, i, w)$. This means that the evolution of the Turing machine in the time n returns to the same position, having the same state and the same tape. In this case we call it *static periodic point*.

2.5.2 Turing machine with moving tape (TMT)

Here the head is fixed at the origin and it is the tape which moves. The phase space is $X_t = {}^{\omega}\Sigma \times Q \times \Sigma^{\omega}$ and T_t consists in one application of δ by moving the tape instead of the head, as follows.

$$\begin{aligned}
\delta(r, \alpha) = (r', \alpha', +1) &\Rightarrow T_t(u, r, \alpha v) = (u\alpha', r', v) \\
\delta(r, \alpha) = (r', \alpha', 0) &\Rightarrow T_t(u, r, \alpha v) = (u, r', \alpha' v) \\
\delta(r, \alpha) = (r', \alpha', -1) &\Rightarrow T_t(u\beta, r, \alpha v) = (u, r', \beta\alpha' v)
\end{aligned}$$

The function $\gamma : X \to X_t$ to transform from configurations in X to TMT configurations is described below:

$$(r, i, w) \xrightarrow{\gamma} (...w_{i-2}w_{i-1}, r, w_i w_{i+1}...)$$

If X_t is provided with the natural metrics

$$d((u, r, u'), (w, r', w')) = \begin{cases} 1 & \text{if } r \neq r' \\ 2^{-min\{|n| \ : \ u_n \neq w_n \vee u'_n \neq w'_n\}} & \text{in other case,} \end{cases}$$

the function γ results to be continuous as well as T_t. Unhappily, γ is onto but not one-to-one. In particular, applying γ to a configuration, we loss information about the original head position. We will call *canonical pre-image* $\gamma^{-1}(w, r, w')$ to the configuration $(r, 0, w.w')$, which is the only one to have the head at the origin. Kůrka defines the notion of *relative position* of a configuration (w, r, w') at time t as follows. It represents the total movement of the head from time 0 to t over the configuration (w, r, w').

$$\mathcal{P}((w, r, w'), t) = \mathcal{P}(T^t(\gamma^{-1}(w, r, w')))$$

The support is again defined in terms of $\mathcal{P}$ as before by $\mathcal{S}(w, r, w') = \{\mathcal{P}((w, r, w'), t) \ : \ t \in \mathbb{N}\}$.

We also consider *finite configurations* in this model $(v, r, v') \in \Sigma^* \times Q \times \Sigma^*$. The map T_t acts on finite configurations while it is allowed, *i. e.*, if $|v'| \geq 2$ when the head moves to the right and $|v| \geq 1$ when the head moves to the left. Let us recall the time $m(v, r, v') \in \mathbb{Z}$ at which the head violates these conditions.A configuration (w, r, w') is an *extension* of (v, r, v') if v is a suffix of w and v' is a prefix of w', which we also noted as $(v, r, v') \sqsubseteq (w, r, w')$. The relative position and support are defined as well for finite configurations of this model, meanwhile the state function is easily defined in both finite and normal configurations.

If $x = \gamma(r, i, w)$ is a periodic point in TMT with period n, then $\exists d \in \mathbb{Z} : T^n(r, i, w) = (r, i + d, \sigma^d(w))$. This means that we observe the same configuration (r, i, w) in the evolution of the Turing machine, in time n, but shifted in d positions. In this case, we call it *shifted periodic point*. Note that this kind of periodic points includes the static ones, when $d = 0$.

2.5.3 The trace-shift

The column shift associated to these two systems can be considered [6], but we focus only on the shift associated to the moving tape model, which we call *trace-shift* and denote by S_t. It is obtained from the projection $\pi : X_t \to Q \times \Sigma$ defined by $\pi(u, r, \alpha v) = (r, \alpha)$. The *trace-shift* is the image of the factor map $\tau : X_t \to S_t$, defined as $\tau(x) = (\pi(T_t^n(x)))_{n \in \mathbb{N}}$. As before, we extend the map τ to finite configurations by $\tau(u, r, u') = \left(\pi(T_t^j(u, r, u'))\right)_{j \in \{0, \dots, m(u, r, u')\}}$.

The map τ is not invertible, but given a semi-infinite word $w \in S_t$, its pre-image x is uniquely defined over the set of visited cells, *i. e.*, if $(u, r, u'), (v, r, v') \in \tau^{-1}(w)$, then $\mathcal{S}(u, q, u') = \mathcal{S}(v, q, v')$ and $u|_{\mathcal{S}(u,r,u')} = v|_{\mathcal{S}(u,r,u')}$ and $u'|_{\mathcal{S}(u,q,u')} = v'|_{\mathcal{S}(v,q,v')}$. Analogously, given a finite word $w \in \mathcal{L}(S_t)$, all of its pre-images have a common support where they coincides. For a given word w either in S_t or in $\mathcal{L}(S_t)$, we define its *canonical pre-image* (u, r, u') as the smallest finite (or infinite) configuration whose image by τ is w; u and u' will only take values over the support of (u, r, u').

It is noteworthy to remark that when T is a reversible machine, S_t can be defined as a subshift of $\Sigma^{\mathbb{Z}}$ by redefining $\tau(u, r, u') = (\pi(T_t^n(x)))_{n\in\mathbb{Z}}$, and all of the results that we will establish in this paper remain true.

$$\begin{tikzcd}
S_t \arrow[d, "\sigma"] & X_t \arrow[l, "\tau"'] \arrow[d, "T_t"] & X \arrow[l, "\gamma"'] \arrow[r, "\psi"] \arrow[d, "T"] & X_h \arrow[d, "T_h"] \\
S_t & X_t \arrow[l, "\tau"'] & X \arrow[l, "\gamma"'] \arrow[r, "\psi"] & X_h
\end{tikzcd}$$

Figure 1: Commuting diagram of the systems associated to a Turing machine. The map ψ is injective but not surjective. The maps γ and τ are surjective but not injective.

2.6 Topological properties in Turing machines

Here we recall the two topological properties which are the subject of this article: *transitivity* and *minimality*. These strongly depends on the considered topological model, thus we rewrite them in each case.

In order to present a more clean description, we will exclude two classes of machines, which are less interesting and for which some of our characterizations fails. First we suppose that the alphabet has at least 2 symbols, otherwise the dynamics is too simple. Next we exclude those machines whose trace-shift consists in a *single finite orbit*. [2] These machines have a transitive (and minimal when reversible) trace-shift, but they do not share other properties of transitive machines, and several of our results fails on these machines.

The spaces X_t and X_h are topologically *perfect*, *i. e.*, they have not isolated points, and their structure does not depend on the machine transition function. On the other hand, the trace-shift topology strongly depends on the machine dynamics, it may be non perfect, but its isolated points corresponds exactly to the static pre-periodic points of the machine [7].

2.6.1 Transitivity

A transitive system is, as we said earlier, a system with a transitive point. Nevertheless, in the case of the three Turing machine systems it can be also characterized through open sets as follows, this is because both X_t and X_h are perfect spaces and isolated points of S_t are just the periodic points.

Proposition 1. *A dynamical system (X, T) is transitive if and only if for every pair of open sets U, V, there exists a time t such that $T^t(U) \cup V \neq \emptyset$.*

Based on this proposition, transitivity can be characterized on each of the three Turing machine dynamical models as follows.

Remark 1. *Given a Turing machine $M = (Q, \Sigma, \delta)$, the next equivalences hold.*

[2] Such a behavior is possible, for example, if the machine has only one state, two symbols, the head does not move, and the symbol is just flipped every time.

- *(X_h, T_h) is transitive if and only if for every pair of finite configurations $(u.u')$ and $(v.v')$, there exists $w \sqsupseteq (u.u')$, and a time $n \in \mathbb{N}$ such that $T_h^n(w) \sqsupseteq (v.v')$.*

- *(X_t, T_t) is transitive if and only if for every pair of finite configurations (u, r, u') and (v, r', v'), there exists $(w, r, w') \sqsupseteq (u, r, u')$ and a time $n \in \mathbb{N}$ such that $T_t^n(w, r, w') \sqsupseteq (v, r', v')$.*

- *(S_t, σ) is transitive if and only if for every two finite words u, v of $\mathcal{L}(S_t)$, there exists a third word $w \in \mathcal{L}(S_t)$ such that $vwu \in \mathcal{L}(S_t)$.*

2.6.2 Minimality

A minimal system is a system where every point is a transitive point, as stated before, i.e., a point whose orbit is dense. This can be stated as follows.

Remark 2. *A dynamical system (X, T) is minimal if and only if for every point $x \in X$ and for every open set U, there exists a time $n \in \mathbb{N}$ such that $T^n(x) \in U$.*

Remark 3. *It is important to note that (X_h, T_h) is cannot be minimal, because headless configurations are fixed points.*

Next proposition characterizes minimality in TMT and in the trace-shift.

Proposition 2. *Given a Turing machine $M = (Q, \Sigma, \delta)$, the next equivalences hold.*

- *(X_t, T_t) is minimal if and only if for every configuration $x \in X_t$ and for every finite configuration (u, r, u'), there exists a time $n \in \mathbb{N}$ such that $T_t^n(x) \sqsupseteq (u, r, u')$.*

- *(S_t, σ) is minimal if and only if for every infinite word $w \in S_t$ and every finite word $u \in \mathcal{L}(S_t)$, $u \sqsubseteq w$.*

3 Techniques

This research uses some construction techniques that modify and mix one or several Turing machines into a new one. They are presented below.

Reversing the time
This technique is used in [13]. It takes a reversible Turing machine $M = (Q, \Sigma, \delta)$, and creates two new reversible machines $M+ = (Q \times \{+\}, \Sigma, \delta^+)$, and $M- = (Q \times \{-\}, \Sigma, \delta^-)$, where $(r, +)$ and $(r, -)$ states represent M in state r running forwards or backwards in time, respectively. More specifically, $\delta^+((q, +), s) = \delta(q, s)$, and $\delta^-((q, -), s) = \delta^{-1}(q, s)$, where δ^{-1} is the function defined in section 2.4.1. In other words, $M+$ and $M-$ corresponds to M and M^{-1} but with disjoint set states.
Time-reflection
A useful construction consists in to connect the error pairs of $M+$ with the defective pairs of $M-$ (see figure 2) to form a new reversible machine that we will call I_M. More specifically, the alphabet of I_M is $Q \times \{+, -\}$ and its transition function in the union of δ^+, δ^-, and for each error pair of $M+$, $((q, +), s)$, we add the instruction $\delta^+((q, +), s) = ((q, -), s, -\mu(q))$, where μ is the function defined in section 2.4.1. This machine has the gentle property of *innocuousness*.

Definition 2. *A reversible machine is* innocuous *if:*

- *every defective pair (r, β) it corresponds a unique error pair (r', β), in such a way that*

- *for every defective configuration (r, i, w), where $w_i = \beta$, its evolution is either infinite or it leads to the halting configuration $(r', i - \mu(r), w)$.*

In other words, if it halts, it does it at the initial position and with the initial tape contents.

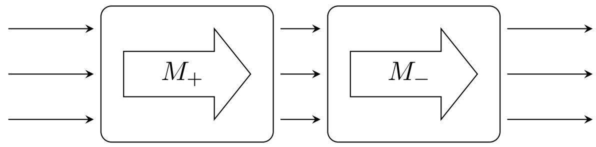

Figure 2: Technique of *time-reflection*, the non-complete machine M is connected with its inverse, producing an innocuous machine I_M.

The defective pairs of I_M correspond to the defective pairs of $M+$ and are associated to the error pairs of $M-$. If the machine starts on a defective state of $M+$, all the work that $M+$ do is undone by $M-$, thus the tape is unchanged when the machine finally stops (if this happens). In fact, this composition computes the function $\varphi^{-1} = T' \circ \varphi \circ T$ as seen in 2.4.1.

Other interesting constructions also produces innocuous machines [2], but here we only use this one.

Embedding

This technique, used in [8, 2], serves to insert an innocuous machine inside another to produce a new machine. We distinguish the *host* machine H and the *invited* innocuous machine I. We assume that the alphabet of I is contained in the alphabet of H. The embeeding uses the defective and error pairs of the invited machine to connect it with the host. The connection is done by replacing one of the transitions of the host machine with the whole invited machine. The new machine shares one or more properties with the host, depending on the particularities of the invited. The new machine is complete, thus this techinque can be used to complete the machine I.

Formally, I will be embedded in H in the following way: Let us suppose that the defective pairs of I are $(r_1, \beta_1), \ldots, (r_n, \beta_n)$, and that they are associated to the error pairs $(r'_1, \beta_1), \ldots, (r'_n, \beta_n)$. We select an instruction from H, lets say $\delta(p, s) = (p', s', c)$.

This instruction is deleted, and we add the following instructions:

- $\delta(p, s) = (r_1, \beta_1, \mu(r_1))$.
- $\forall i \in \{1, ..., n-1\} : \delta(r'_i, \beta_i) = (r_{i+1}, \beta_{i+1}, \mu(r_{i+1}))$.
- $\delta(r'_n, \beta_n) = (p', s', c)$.

Please refer to figure 3 for a graphical description.

If we start at a state of H in the resulting machine, we will see the evolution of H, alternated with some intervals of time in which it is the machine I that evolves; except if we fall into an infinite orbit of I. We can always suppose that we start at H, because evolving I backward will carry us to a defective state, and finally to H, except if we are over a backward infinite orbit of I.

The embedding will be particularly useful when H is minimal and I is mortal, because in this case, a single initial configuration of H will allow to test the behavior of I over all its defective configurations. The new machine formed by H and I is called H_I.

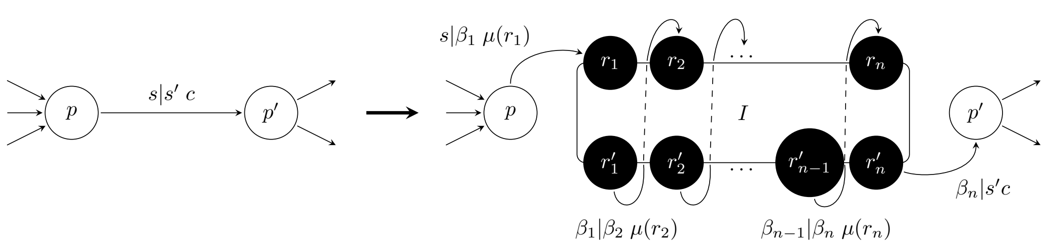

Figure 3: Embedding technique. The Invited machine I is inserted in the transition $\delta(p, s) = (p', s', c)$ of the host machine H.

4 Classification of Turing machines

In this section, a classification of Turing machines with a topological transitive dynamical system and its surroundings is presented, according to the properties which are present in the different topological models. We exhibit concrete examples inside each class, proving in this way that no class is empty. Let us start with some preliminary results.

Proposition 3. *Given a Turing machine M, the next implications hold.*

$$(X_h, T_h) \text{ transitive } \underset{1}{\Rightarrow} (X_t, T_t) \text{ transitive } \underset{2}{\Rightarrow} (S_t, \sigma) \text{ transitive}$$

Proof. ($\underset{1}{\Rightarrow}$) Any finite configuration of X_t corresponds to several finite configurations of X_h, thus if a point exists that visits any finite configuration of X_h, the same point will visit any finite configuration of X_t.

($\underset{2}{\Rightarrow}$) (S_t, σ) is a factor of (X_t, T_t) thus it inherits its transitivity. □

There exists a relation between static periodic points (that is, periodic points for T_h) and transitivity in Turing machine dynamical models. The existence of static periodic points avoids transitivity in the three dynamical systems of Turing machines, as these points are topologically isolated.

Proposition 4 ([7])**.** *If a machine M has a periodic point in TMH, its dynamical systems (X_h, T_h), (X_t, T_t) and (S_t, σ) are not transitive.*

A transitive system needs to be surjective (onto), thus if T_h or T_t are transitive, the machine needs to be reversible. But, if the trace-shift is surjective, the machine do not need to be surjective, in fact, a non backward complete machine can have a surjective trace-shift. This is possible only in the quintuple model and when the machine admits *blocking words* [8, 9]. The idea of a "blocking" configuration that avoids the head from going beyond some limit appears in several contexts, and it is related to stability and information travel. Blocking words are also relevant to transitivity and minimality by themselves, as the next propositions show.

Definition 3. *A finite configuration $(r, 0, .u)$ is a* blocking word to the left *if for every $w \sqsupseteq .u$ and every time n, $\mathcal{P}(T^n(r, 0, w)) \geq 0$. Analogously, we say that $(r, |u| - 1, .u)$ is a* blocking word to the right *if for every $w \sqsupseteq .u$ and every time n, $\mathcal{P}(T^n(r, |u| - 1, w)) \leq |u| - 1$.*

Proposition 5. *If M has a blocking word, then (X_h, T_h) is not transitive.*

Proof. If $M = (Q, \Sigma, \delta)$ has a blocking word to the left $(r, 0, .u)$, then no extension of $.ru$ can visit a finite configuration of the form $(r'\alpha.u')$, at any time, for any $r' \in Q$, $\alpha \in \Sigma$, $u' \in \Sigma^*$. This means that (X_h, T_h) is not transitive. □

Proposition 6. *If M has no blocking word, then the next equivalences hold.*

1. *(X_t, T_t) is transitive $\Leftrightarrow (S_t, \sigma)$ is transitive*
2. *(X_t, T_t) is minimal $\Leftrightarrow (S_t, \sigma)$ is minimal*

Proof. We only include the proof of 1), because the proof of 2) is equivalent.

The left to right implication was already proved, thus let us prove the converse. Let us assume that (S_t, σ) is transitive and that M has no blocking word. Let (u, r, u') and (v, r', v') be two finite configurations of X_t. Since M has no blocking words, there exist finite extensions $(su, r, u's')$ and $(wv, r', v'w')$ such that the head visits all the cells of (u, r, u') and (v, r', v') respectively. In other words, $\mathcal{S}(su, r, u's') \supseteq [-|u|, |u'| - 1]$ and $\mathcal{S}(wv, r', v'w') \supseteq [-|v|, |v'| - 1]$. Let us consider now the words $\tau(su, r, u's')$ and $\tau(wv, r', v'w')$. Since (S_t, σ) is transitive, there exists a point $(x, r, y) \in X_t$ such that $\tau(x, r, y)$ starts with $\tau(su, r, u's')$ and contains $\tau(wv, r', v'w')$ afterward. As we have commented in section 2.5.3, $\tau(su, r, u's')$ determines the state of all the cells that the head visits when producing it, then $(x, r, y) \sqsupseteq (u, r, u')$. By the same reason, there exist a time t such that $T_t^t(x, r, y) \sqsupseteq (v, r', v')$, which concludes the proof. □

Proposition 7. *If the trace-subshift S_t is minimal, then the machine has no blocking words.*

Proof. Let us suppose that the machine T has a blocking word $(r, 0, .u)$, and let $w_b = \tau(\epsilon, r, u)$ be its trace. Since S_t is minimal, there exists $s \in S_t$ that contains w_b with bounded gaps N, that is, there exists $(x, r, y) \in X_t$ such that

- $(\epsilon, r, u) \sqsubseteq (x, r, y)$,
- $s = \tau(x, r, y)$, and
- $\exists (t_i)_{i\in\mathbb{N}}, \forall i \in \mathbb{N}, T_t^{t_i}(x, r, y) \sqsupseteq (\epsilon, r, u) \wedge 0 \leq t_{i+1} - t_i \leq N$.

Since $(r, 0, .u)$ is blocking, we can assert that $0 \leq \mathcal{P}(T_t^{t_i}(x, r, y), t) \leq N$, for every $t \in [t_i, t_{i+1}[$. But the set $W = \{w \in \Sigma^{\{0,\ldots,N\}} | (\exists r' \in Q)(\epsilon, r', w) \sqsubseteq T_t^{t_i}(x, r, y)\}$ is clearly finite, so there exists a pair $i < j$ such that there exists $w' \in W, r' \in Q : (\epsilon, r', w') \sqsubseteq T_t^{t_i}(x, r, y) \wedge (\epsilon, r', w') \sqsubseteq T_t^{t_j}(x, r, y)$. We will suppose, without loss of generality, that $i = 0$ and $t_0 = 0$, thus $T_t^{t_i}(x, r, y) = (x, r, y)$.

Let $p = \mathcal{P}((x, r, y), t_j)$ be the position of the head at time t_j, which is larger than or equal to 0 because $(r, 0, .u)$ is blocking. Until iteration t_j only the cells between 0 and $p + N$ have been visited, *i. e.*, $0 \leq \mathcal{P}((x, r, y), t) \leq p + N$, for every $t \in [0, t_j[$. Let us call $u' = y_{[|u|,N[}$ and $v = y_{[N,p+N[}$.

Now let us consider the configuration $(x, r, uu'v^\omega)$. The machine behaves over this configuration and over (x, r, y) in the same way, at least until time t_j. At time t_j the head position is p, and the N cells to its right contain the configuration $y_{[0,N[} = uu'$. The cells at $[p + N, \infty[$, has not yet been visited and thus they contain the configuration v^ω. In other words, $T_t^{t_j}(x, r, uu'v^\omega) = (z, r, uu'v^\omega)$, for some $z \in {}^\omega\Sigma$. Since the cells to the left of the head position at time t_j are not visited any more, the machine will periodically repeat the same movements, proving that $\tau(x, r, uu'v^\omega)$ is a periodical trace; but this is not possible when S_t is minimal, unless the machine has a single finite orbit, but we have excluded these machines from our study. □

With these results, we can depict the universe of Turing machines by figure 4. All the inclusions are strict, and in the next sections we show an example inside each class.

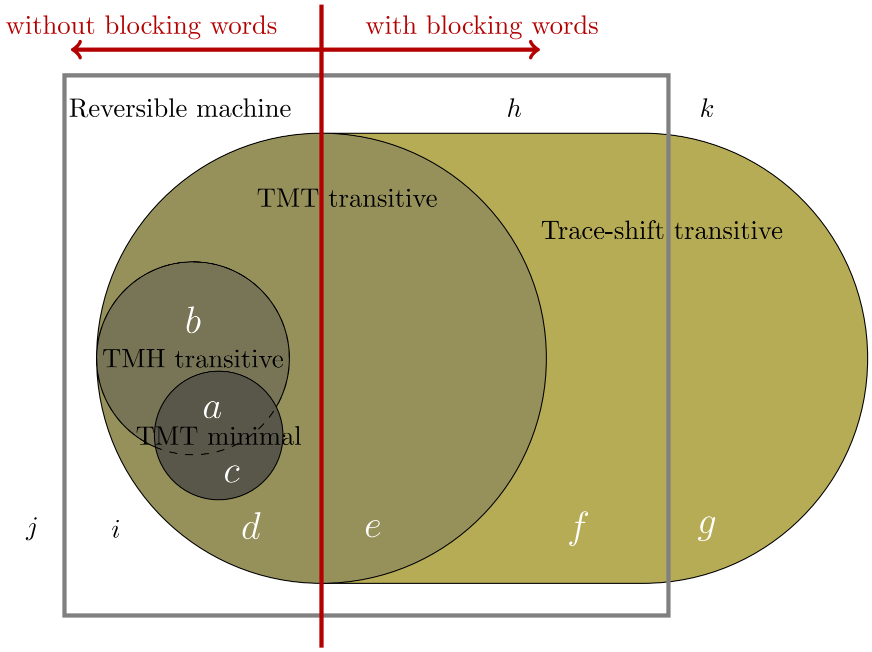

Figure 4: The universe of Turing machines depicted in relation with reversibility, transitivity, blocking words, and minimality.

4.1 Machine of type a: Transitive in TMH and minimal in TMT

The machine of figure 5 is called *SMART*, it has several strong properties as it is shown in [2]. In particular, its TMT system is minimal and its trace-shift is substitutive. As it is also proved in [8], it is topologically transitive in TMH. We use this machine in next sections to construct other examples. Its state set is $Q_S = \{p, q, b, d\}$, its symbol set is $\Sigma_S = \{0, 1, 2\}$ and we call δ_S its transition function, which is described in figure 5.

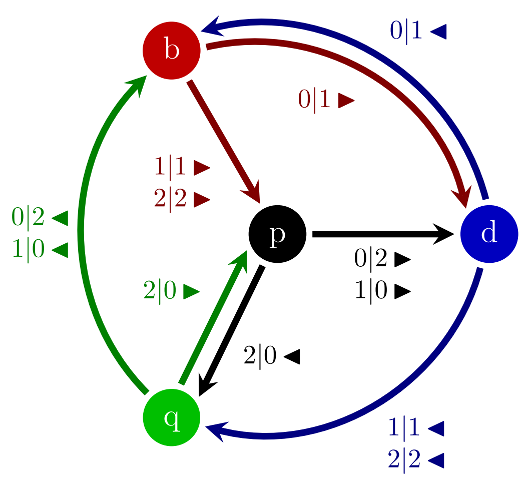

Figure 5: The SMART machine.

4.2 Machine of type *b*: Transitive on TMH, but not minimal in TMT

Figure 6 represents $SMART_{Shift}$, which is an embedding of the Shift machine of two symbols (figure 8 of section 4.5) inside SMART (fig. 5). The idea here is to have an infinite evolution in the invited machine. This evolution will be interrupted each time the head finds a 2 on the tape.

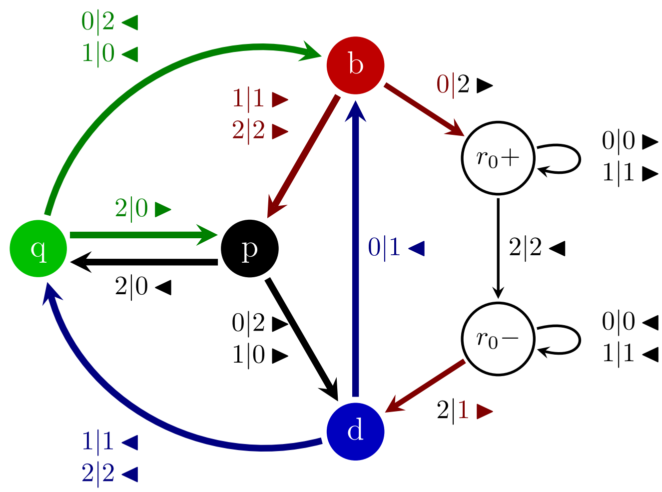

Figure 6: The embedding of the Shift machine, inside SMART, it produces SMART$_{Shift}$

Theorem 1. *$SMART_{Shift}$ is transitive in TMH, but it is not minimal in TMT.*

Proof. As the Shift machine was embedded inside SMART using *reversing the time* and *time-reflection*, it is innocuous. Since SMART is transitive for TMH, the set of transitive points is dense in X_h, this imply in particular that there exists a transitive point that contains an infinite set of cells with symbol 2 to the right, let x be such point. When SMART$_{shift}$ works on x, each interruption of the Shift machine takes a finite number of iterations, therefore, the transitivity of x is kept, and SMART$_{Shift}$ is also transitive in the TMH model.

On the other hand, the configuration $({}^{\omega}0, r_0+, 0^{\omega})$ is clearly not a transitive point, then SMART$_{Shift}$ is not minimal in TMT. □

4.3 Machine of type *c*: Minimal on TMT, but not transitive in TMH.

The machine presented for this example is based on the original SMART (fig. 5) but with twice the number of states and a bipartite transition graph such that the state alternates between even and odd type. We call it $SMART_P$.

Definition 4. *The machine $SMART_P$ is defined by $(Q_S \times \{e, o\}, \{0, 1, 2\}, \delta_P)$, where δ_P is given by $\forall r, r' \in Q_S, a \in \{e, o\}, \forall s, s' \in \Sigma, \forall c \in \{-1, +1\}$:*

$$\delta_P((r, a), s) = ((r', \neg a), s', c) \iff \delta_S(r, s) = (r', s', c),$$

and $\neg e = o$ and $\neg o = e$.

SMART$_P$ has the same dynamics than SMART, but if we start in an even (odd) position of the tape with a state say (r, e), then on every even (odd) position its state will be of the form $(*, e)$, and on every odd (even) position its state will be of the form $(*, o)$, as stated in the next straightforward lemma.

Lemma 1. *$SMART_P$ satisfies that $\forall a \in \{e, o\}, \forall t \in \mathbb{N}, \forall x \in X_h : \mathcal{P}(x) - \mathcal{P}(T_h^{2t}(x))$ is even $\wedge\, [\mathcal{Q}(x) \in Q \times \{a\} \implies \mathcal{Q}(T_h^{2t}(x)) \in Q \times \{a\}]$.*

Now, we are ready to prove that the machine SMART$_P$ has a minimal TMT model but its TMH model is not transitive.

Theorem 2. *SMART$_P$ has a TMT dynamical system which is minimal, but its TMH dynamical system is not transitive.*

Proof. (X_h, T_h) is not transitive because X_h can be divided in three sets: a) configurations without head, b) configurations with the head on a state (r, a) where a matches with the parity of the head position, and c) configurations with the head on a state (r, a) where a does not match with the parity of the head position. Lemma 1 asserts that configurations of type b) cannot be approached from configurations of type c) and vice versa, which avoids transitivity.

Now we prove that SMART$_P$ is TMT minimal, that is, given any configuration $(x, (r, a), x') \in X_t$, and any finite configuration $(v, (r', a'), v')$, we must find a time $t \in \mathbb{N}$ such that $T_t^t(x, (r, a), x') \sqsupseteq (v, (r', a'), v')$.

Let us take $((r, a), 0, x.x')$ as the canonical pre-image of $(x, (r, a), x')$ in X and $((r', a'), 0, v.v')$ a partial configuration corresponding to $(v, (r', a'), v')$. Let us consider the configurations $(r, 0, x.x')$ and $(r', 0, v.v')$, of the phase space of SMART. We will work with SMART and prove that (x, r, x') reaches both $\sigma^i(r', 0, v.v')$ and $\sigma^{i+1}(r', 0, v.v')$, for some i that we will fix later.

Let us take now a point $(\overline{r}, 0, \overline{z}.\overline{z}')$ such that $\Psi((\overline{r}, 0, \overline{z}.\overline{z}')) \in X_h$ is transitive for the TMH model, whose existence is proved in [8]. There exist two time steps t and t' such that $T^t(\overline{r}, 0, \overline{z}.\overline{z}') \sqsupseteq (r', 0, v.v')$ and $T^{t'}(\overline{r}, 0, \overline{z}.\overline{z}') \sqsupseteq \sigma(r', 0, v.v')$. Thanks to lemma 1, it is clear that t is even and t' is odd.

Now let us take a finite restriction $(\overline{r}, 0, \overline{w}.\overline{w}')$ of $(\overline{r}, 0, \overline{z}.\overline{z}')$ so that $m(\overline{r}, 0, \overline{w}.\overline{w}') \geq \max\{t, t'\}$. Now we use the minimality of the TMT model of SMART to assert that there exists a time n such that $T_t^n(x, r, x') \sqsupseteq (\overline{w}, \overline{r}, \overline{w}')$. We now fix $i = \mathcal{P}(T^n(x, r, x'))$, and we obtain that $T^n(r, 0, x.x') \sqsupseteq \sigma^i(\overline{r}, 0, \overline{w}.\overline{w}')$.

In this way, we have that $T^{t+n}(r, 0, x.x') \sqsupseteq \sigma^i(r', 0, v.v')$ and also that $T^{t'+n}(r, 0, x.x') \sqsupseteq \sigma^{i+1}(r', 0, v.v')$, which implies that $T_t^{t+n}(x, r, x') \sqsupseteq (v, r', v')$ and $T_t^{t'+n}(x, r, x') \sqsupseteq (v, r', v')$.

We come back now to the machine SMART$_P$. Let us suppose without loss of generality that n is even (the other case is analogous). We can distinguish two cases.

- $a' = a$: Since $n + t$ is even, we can assert that $T_t^{t+n}(x, (r, a), x') \sqsupseteq (v, (r', a'), v')$.
- $a' = \neg a$: Since $n + t'$ is odd, we can assert that $T_t^{t'+n}(x, (r, a), x') \sqsupseteq (v, (r', a'), v')$.

Which completes the proof. □

Remark 4. *It can be proved that every configuration with head in the TMH model of a TMT minimal Turing will "see" every pattern at an infinite set of positions, here we have prove this for position 0 and 1, but the same argument can be repeated for a growing set of positions.*

4.4 Machine of type *d*: Transitive on TMT, but not in TMH, not minimal in TMT, and without blocking words

In order to have an example of this class, it is enough to take again SMART (or any machine of class a) and to multiply its movements by two, *i. e.*, instead of moving one cell to the left or right, to move twice. We call this machine SMART$_{Skip}$ (see figure 7).

It will not be transitive in TMH, because if the head starts with state p on a even cell, it will not be able to modify the content of any odd cell, thus many configurations will be unreachable. It is neither minimal on TMT, as a configuration filled with 2s on every odd cell, for example $({}^{\omega}2, p, 2^{\omega})$, can never reach a configuration with two consecutive 1s, for example $(11, p, 11)$.

Theorem 3. *SMART$_{Skip}$ is Transitive on TMT and has not blocking words.*

Proof. It is easy to see that SMART$_{Skip}$ has not blocking words, because if it start with a state in $\{p, q, d, b\}$ and if we focus only on the even cells, we will see SMART in action, and we know that SMART has no blocking words thanks to its transitivity on the TMH model.

Let $(\hat{X}_t, \hat{T}_t)$ be the dynamical system with moving tape of SMART$_{Skip}$. We need to prove the next claim.

$$\forall (u, r, u'), (v, r', v'), \exists t \in \mathbb{N}, \exists x \sqsupseteq (u, r, u') : F^t(x) \sqsupseteq (v, r', v')$$

Let us take two finite configurations of $\hat{X}_t$: (u, r, u'), (v, r', v'). We can assume that r, $r' \in \{p, q, d, b\}$, otherwise we just do one step forward. We can also assume that u, u', v and v' are of even length. Let z be a TMH transitive point of SMART such that $z \sqsupseteq (u_{-|u|}u_{-|u|+2}...u_{-2}ru'_0u'_2...u'_{|u'|-2})$. If we consider:

$$x = (...0z_{-|u|-2}0u, r, u'z_{|u'|}v_{-|v|+1}z_{|u'|+2}...v_{-1}z_{|u'v|}v'_1z_{|u'v|+2}...v_{|v|-1}z_{|u'vv'|}...)$$

In other words, the point x consists in z at the even positions, and the odd positions are arranged in a way such that when the head is positioned at $|u'v|$, the odd positions surrounded are exactly the odd positions of vv'. In this way, as z is a TMH transitive point for SMART, $\exists t' \in \mathbb{N} : T_h^{t'}(z) \sqsupseteq (v_{-|v|}v_{-|v|+2}...v_{-2}r'v'_0v'_2...v'_{|v'|-2})$ and as SMART$_{Skip}$ works at the even positions as SMART, establishing $t = 2t'$ and our x, we can proof our claim. $\square$

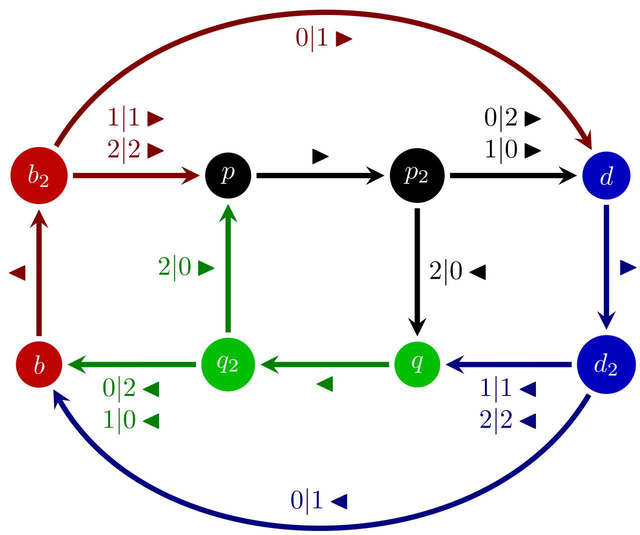

Figure 7: SMART$_{Skip}$: The SMART machine with twice its movement.

4.5 Machine of type e: Transitive on TMT, with blocking words

The following is a machine that is transitive on TMT, but not on TMH, with the blocking word to the left: $(r_0, 0, .\epsilon)$.

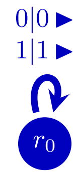

Figure 8: Shift machine

We call this machine the *shift machine*, because its TMT model is conjugated to the fullshift on two symbols. As such, this machine is clearly transitive in TMT.

4.6 Machine of type f: Transitive just for the trace-shift and reversible

Now we present a machine that we call *Lexicographical Ant* (LA) that has a transitive trace-shift, which is not transitive neither in TMT, nor in TMH, and that has a blocking word. Its transition function is depicted in figure 9. It has the particularity that it is always "counting", when it starts from configuration $(\rightarrow, 0, {}^{\omega}0.1^{\omega})$, it will persistently comeback to the position 0 and we will see all the binary sequences appearing in increasing order at the left part of the tape. . The configuration $(\rightarrow, 0, .1)$ is a blocking word to the right. It avoids transitivity in the TMT model because the finite configuration $\left(u\ \underset{\rightarrow}{1}\ v \right)$ cannot reach the finite configuration $\left(u\ \underset{\rightarrow}{1}\ w \right)$, if $v \neq w$.

Given two words $v = v_0 \ldots v_{n-1}$, $v' = v'_0 \ldots v'_{n-1} \in \{0,1\}^n$, let us define the lexicographical order by $v < v'$ if $\sum_{i=0}^{n-1} v_i 2^{n-i-1} < \sum_{i=0}^{n-1} v'_i 2^{n-i-1}$.

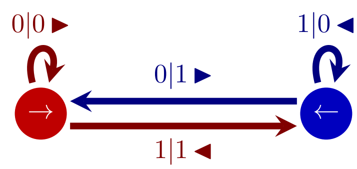

Figure 9: The lexicographical ant machine.

Lemma 2. *The finite configuration* $\left(0^n\ \underset{\rightarrow}{1} \right)$ *will produce the sequence of finite configurations of the form* $\left(v\ \underset{\rightarrow}{1} \right)$, *in lexicographical order, for every* $v \in \{0,1\}^n$, *without exiting the interval* $[-n, 0]$.

Proof. Proof by induction on n.

Basis ($n = 1$): $\left(0\ \underset{\rightarrow}{1} \right) \vdash \left(\underset{\leftarrow}{0}\ 1 \right) \vdash \left(1\ \underset{\rightarrow}{1} \right)$

Induction hypothesis ($n = k$): $\left(0^k\ \underset{\rightarrow}{1} \right) \vdash^* \left(v\ \underset{\rightarrow}{1} \right) \vdash^* \left(v'\ \underset{\rightarrow}{1} \right) \vdash^* \left(1^k\ \underset{\rightarrow}{1} \right)$, for all $v, v' \in \{0,1\}^k$ such that $v < v'$.

Induction thesis ($n = k+1$): $\left(0^{k+1}\ \underset{\rightarrow}{1} \right) \vdash^* \left(v''\ \underset{\rightarrow}{1} \right) \vdash^* \left(v'''\ \underset{\rightarrow}{1} \right) \vdash^* \left(1^{k+1}\ \underset{\rightarrow}{1} \right)$, for all $v'', v''' \in \{0,1\}^{k+1}$ such that $v'' < v'''$.

Case 1. $v'' = 0u''$ and $v''' = 0u'''$

$$\begin{aligned}
\left(0\ 0^k\ \underset{\rightarrow}{1} \right) & \vdash^{H.I.} \left(0\ u''\ \underset{\rightarrow}{1} \right) \\
& \vdash^{H.I.} \left(0\ u'''\ \underset{\rightarrow}{1} \right) \vdash^{H.I.} \left(0\ 1^k\ \underset{\rightarrow}{1} \right) \vdash^* \left(\underset{\leftarrow}{0}\ 0^k\ 1 \right) \vdash \left(1\ \underset{\rightarrow}{0^k}\ 1 \right) \vdash^* \left(1\ 0^k\ \underset{\rightarrow}{1} \right) \\
& \vdash^{H.I.} \left(1\ 1^k\ \underset{\rightarrow}{1} \right)
\end{aligned}$$

Case 2. $v'' = 0u''$ and $v''' = 1u'''$

$$\begin{aligned}
\left(\,0\;0^k\;\underset{\rightarrow}{1}\,\right) &\vdash^{H.I.} \left(\,0\;u''\;\underset{\rightarrow}{1}\,\right)\\
&\vdash^{H.I.} \left(\,0\;1^k\;\underset{\rightarrow}{1}\,\right) \vdash^{*} \left(\,\underset{\leftarrow}{0}\;0^k\;1\,\right) \vdash \left(\,1\;\underset{\rightarrow}{0^k}\;1\,\right) \vdash^{*} \left(\,1\;0^k\;\underset{\rightarrow}{1}\,\right)\\
&\vdash^{H.I.} \left(\,1\;u'''\;\underset{\rightarrow}{1}\,\right) \vdash^{H.I.} \left(\,1\;1^k\;\underset{\rightarrow}{1}\,\right)
\end{aligned}$$

Case 3. $v'' = 1u''$ and $v''' = 1u'''$

$$\begin{aligned}
\left(\,0\;0^k\;\underset{\rightarrow}{1}\,\right) &\vdash^{H.I.} \left(\,0\;1^k\;\underset{\rightarrow}{1}\,\right) \vdash^{*} \left(\,\underset{\leftarrow}{0}\;0^k\;1\,\right) \vdash \left(\,1\;\underset{\rightarrow}{0^k}\;1\,\right) \vdash^{*} \left(\,1\;0^k\;\underset{\rightarrow}{1}\,\right)\\
&\vdash^{H.I.} \left(\,1\;u''\;\underset{\rightarrow}{1}\,\right) \vdash^{H.I.} \left(\,1\;u'''\;\underset{\rightarrow}{1}\,\right) \vdash^{H.I.} \left(\,1\;1^k\;\underset{\rightarrow}{1}\,\right)
\end{aligned}$$

□

Corollary 1. *The finite configuration* $\underset{\rightarrow}{1}$ *is a blocking word to the right.*

Proof. By lemma 2, we have that starting from any configuration of the form $^{v}\,\underset{\rightarrow}{1}$, where $v \in \{0,1\}^n$, the machine will arrive to $^{1^n}\,\underset{\rightarrow}{1}$, without going to the right of cell 0. But then again it is on a configuration of this form, for a larger 'n', thus it will never go to the right of cell 0. □

Let us take the following function:

$$a(n) = \begin{cases} 1+a(\frac{n-1}{2}) & \text{if } n = \text{odd} \\ 0 & \text{if not} \end{cases} . \tag{1}$$

This function describes the length of the first block of 1s to the right of the binary extension of n.

It is easy to see that, if we start counting from $x = \left(\,^{\omega}0\;\underset{\rightarrow}{1}\,\right)$, we have that the corresponding sequence in the trace-shift is the following.

$$\tau(x) = \prod_{i\in\mathbb{N}} \left(\,\underset{\rightarrow}{1}\;\underset{\leftarrow}{1^{a(i)}}\;\underset{\leftarrow}{0}\;\underset{\rightarrow}{0^{a(i)}}\,\right) \tag{2}$$

The head of the machine will start at $\underset{\rightarrow}{1}$. As we already explained, it will switch all the 1s next to this position. The amount of 1s to convert is given by the $a()$ function.

Lemma 3. *The trace-shift of the Lexicographical Ant is described by:*

$$S_{LA} = \mathcal{S}_{\mathcal{L}(\{u\})} = \overline{\mathcal{O}(u)},\ \textit{with } u = \prod_{i\in\mathbb{N}} \left(\,\underset{\rightarrow}{1}\;\underset{\leftarrow}{1^{a(i)}}\;\underset{\leftarrow}{0}\;\underset{\rightarrow}{0^{a(i)}}\,\right) \tag{3}$$

Proof. :

$\overline{\mathcal{O}(u)} \subseteq S_{LA}$. It is clear from equation 2 and the fact that S_{LA} is closed.

$S_{LA} \subseteq \overline{\mathcal{O}(u)}$. It is enough to prove that $\mathcal{L}(S_{LA}) \subseteq \mathcal{L}(u)$. Let $w \in \mathcal{L}(S_{LA})$, and let (v, r, v') be its canonical pre-image by τ, *i. e.*, $\tau(v, r, v') = w$ and every coordinate of $(v.v')$ is visited during the evolution of LA on (v, r, v'). Let us suppose that $v = v_0...v_{m-1}$ and $v' = v'_0...v'_{n-1}$. We need to prove that $x = (^{\omega}0, \rightarrow, 1) \vdash^{*} (v, r, v')$. Four cases appear.

$r = \rightarrow$:

$v'_0 = 1$. By corollary 1, $v' = v_0$, and by lemma 2 $\left(\,^{\omega}0\;\underset{\rightarrow}{1}\,\right) \vdash^{*} \left(\,^{\omega}0\;v\;\underset{\rightarrow}{1}\,\right)$ which proves that w is a subword of u.

$v'_0 = 0$. The machine will move to the right until it finds a symbol 1. If v' has a 1, then by corollary 1 it is the last symbol of v'. Now let us suppose that j is the last coordinate of v such that $v_j = 1$ (if $v = 0^m$ we are already in a pre-image of u). By lemma 2 $\left({}^{\omega}0\ \underset{\rightarrow}{1} \right) \vdash^* \left({}^{\omega}0\ v_0 \,..\, v_{j-1}\ 0\ 1 \,..\, 1\ \underset{\rightarrow}{1} \right) \vdash^* \left({}^{\omega}0\ v_0 \,..\, v_{j-1}\ \underset{\leftarrow}{0}\ 0 \,..\, 0\ 1 \right) \vdash^* \left({}^{\omega}0\ v_0 \,..\, v_{j-1}\ 1\ 0\ \underset{\rightarrow}{..}\ 0\ 1 \right)$ which contains (v, r, v'), proving that w is a subword of u.
If v' is equal to 0^n, then the head exit (v, r, v') at iteration $n+1$, $v = \epsilon$, and $w = \left(\underset{\rightarrow}{0} \right)^n$ which is clearly a subword of u.

$r = \leftarrow$:

$v'_0 = 0$. In this case, $\left(v\ \underset{\leftarrow}{0}\ v'_1 \,..\, v'_{n-1} \right) \vdash^* \left(v\ 1\ \underset{\rightarrow}{v'_1} \,..\, v'_{n-1} \right)$. This configuration fits in one of the former cases, thus it is attained by x. Since the machine is reversible, its predecessor is also attained by x.

$v'_0 = 1$. Let j be the last coordinate of v such that $v_j = 0$ (if $v = 1^m$, $v' = \epsilon$ and $w = \left(\underset{\leftarrow}{1} \right)^m$, which is clearly a subword of u), then $\left(v_0 \,..\, v_{j-1}\ 0\ 1 \,..\, 1\ \underset{\leftarrow}{1}\ v'_1 \,..\, v'_n \right) \vdash^* \left(v_0 \,..\, v_{j-1}\ \underset{\leftarrow}{0}\ 0 \,..\, 0\ 0\ v'_1 \,..\, v'_n \right)$. Which is proved to be in the orbit of x in the last case, and again this implies, by reversibility, that the original configuration is reached by x.

□

We can note that, in fact, the Lexicographical Ant is not transitive in TMT. This is because from configuration $\left(1\ 0\ 0\ 1\ \underset{\rightarrow}{1}\ 1 \right)$ T_t can never get to $\left(1\ 0\ 0\ 1\ \underset{\rightarrow}{1}\ 0 \right)$, due to Corollary 1. Although, Lexicographical Ant has a transitive trace-shift, because this is described as the closure of the orbit of a unique infinite word.

Theorem 4. *The trace-shift of the Lexicographical Ant is transitive.*

4.7 Machine of type g: Transitive for the trace-shift and non reversible

Changing a little bit the "shift machine", we obtain a machine that is not reversible but whose trace-shift is still transitive.

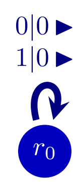

Figure 10: Erasing Shift machine.

This machine "erases" the tape, that is why it is not backward complete: the configuration $\left(1\ \underset{r_0}{1}\ 1 \right)$ has no preimage. The state r_0 is blocking to the left by itself. Its trace-shift coincides with the trace-shift of the shift machine, it is thus transitive.

4.8 Machine of type h: Reversible, non-transitive in the trace-shift and with blocking words.

Any reversible machine with static periodic points is in this class, for example the machine that moves over the 0s and rebounds over the 1s is not transitive in none of the three systems (figure 11). Any of its periods contains blocking words to the left and right, for example the blocking word to the left $(\leftarrow, 0, .1001)$. Blocking words to the left alternates through the time with blocking words to the right.

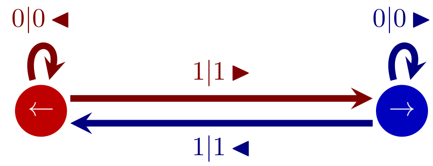

Figure 11: Bouncing machine.

Other interesting example is the so called “one-dimensional Langton’s ant”, studied in [5] (figure 12). This reversible machine has no periodic points for the TMH model, and it has one blocking word to the left $(\rightarrow, 0, .0)$ and one blocking word to the right $(\leftarrow, 0, .0)$. It is not transitive because if the blocking word to the left is found, then only this kind of blocking word will be found in the future and this will systematically happen (every 1 or 3 steps). The set of configurations S_t can be divided in two sets, one with the blocking word to the left and the other with the blocking word to the right.

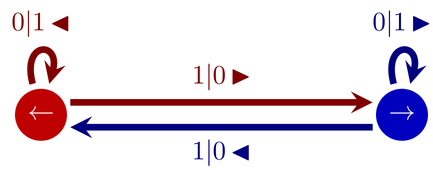

Figure 12: One-dimensional Langton’s ant.

4.9 Machine of type *i*: Reversible, non-transitive in the trace-shift and without blocking words.

In order to have an example of machines without a transitive trace-shift, reversible and without blocking words, it is enough to consider a machine composed by two disconnected copies of SMART. Such machine cannot have a blocking word by definition and it is clearly reversible, but no state from one copy can reach a state from the other copy through the evolution of the machine, therefore it is not transitive in any dynamical model.

We present a more interesting example, where the states are all reachable form each other. The Turing machine called *Double Lexicographical Ant* (*Double LA*), depicted in figure 14, is based on the Lexicographical Ant (LA) machine (fig. 9), but glued with its mirrored form; that is, same computation but in the opposite direction. We call this mirrored form *Right Lexicographical Ant* ($\vec{LA}$, figure 13).

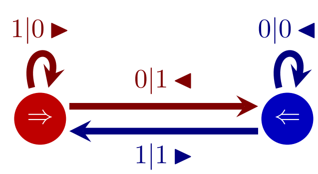

Figure 13: $\vec{LA}$: Right Lexicographical Ant.

The original Lexicographical Ant adds 1 to the binary number expressed at the left of its blocking word to the right $(\rightarrow, 0, .1)$, starting and ending that computation at the position of that blocking word, in an infinite adding cycle. The Double Lexicographical Ant does the same, but every time it returns to the position of the once blocking word, it moves to the right and it adds 1s to the binary number represented backwards at the right, mirroring the computation.

Another good way to explain this new machine is considering an embedding using LA as the Host, and replacing the instruction $\delta_{LA}(\rightarrow, 1) = (\leftarrow, 1, ◀)$ with the machine $(\vec{LA})$ without the mirrored instruction, as the Invited machine. Even when the invited machine is not innocuous in this case, it only modifies cells that the Host LA never visits, therefore the embedded technique works as intended.

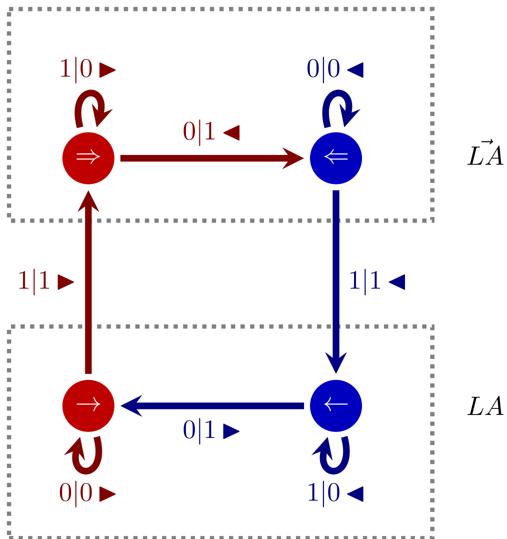

Figure 14: Double LA: An example of a reversible, non-transitive in the trace-shift and without blocking words machine.

Theorem 5. *The machine Double LA is reversible, non-transitive in the trace-shift and without blocking words.*

Proof. The reversibility of the machine can be easily seen in its transition function represented in figure 14. Also, this machine has no blocking word, as the unique blocking word of the LA machine, which is a blocking word to the right, is replaced with a computation that has the capacity to go arbitrarily far to the right.

Finally, the Double LA machine is not transitive in its trace-shift; to see this, let us remark that the machine counts at the left and the right of $(\rightarrow, 0, .1)$ every time in an infinite adding cycle, thus the relative parity of the numbers is preserved each time the head is at an extension of the configuration $(\rightarrow, 0, .1)$; this imply that, for example, it is impossible to find the trace of $(\rightarrow, 0, 0.10)$ in the trace of any extension of $(\rightarrow, 0, 0.11)$.

□

4.10 Machine of type j: Non-transitive in the trace-shift, non-reversible and without blocking words.

As we stated in section 4, any machine with a non-surjective trace-shift is non-transitive in any dynamical model. An example can be seen in figure 15, the Oscillating machine. This machine just rebounds on 0s by the right and on 1s by the left, while flipping all the symbols every time, its oscillation amplitude is always growing. It is not reversible, and then it has garden-of-eden (without pre-image) configurations.

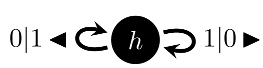

Figure 15: The Oscillating machine.

In [18], it is proved that a non-surjective Turing machine can be surjective in its trace-shift if and only if the configuration with no pre-image has a blocking word that avoids the head from visiting the conflictive cells in the future. The finite configuration $\left(\begin{smallmatrix} 1 & .0 & 0 \\ & h & \end{smallmatrix}\right)$ has no pre-image, because no configuration coming from the left can produce 1 at position -1, and also no configuration coming from the right can produce 0 at position 1. Also $(h, 0, .0)$ and $(h, 1, .10)$ are clearly not blocking to the left and right respectively, therefore the trace-shift of the Oscillating machine is not surjective. Moreover, the Oscillating machine has no blocking words, as next proposition establishes.

Proposition 8. *The Oscillating machine has no blocking word.*

Proof. Let us suppose that $(h, 0, .v)$ is a blocking word to the left. If so, v_0 has to be 1, otherwise the machine immediately goes to the left. But this implies that v_1 has to be 1 as well, as $(h, 1, .00)$, in two steps, reaches position -1. The same goes for every v_i. As v has to be finite, we can always put a 0 after v making the head to reach position -1 in $|v|$ steps. Therefore, $(h, 0, .v)$ is not a blocking word to the left.

The same argument can be given to assure that $(h, |v| - 1, .v)$ is not a blocking word to the right, but exchanging 1 and 0. □

4.11 Machine of type k: Non-transitive in the trace-shift, non-reversible and with a blocking word.

Now, in proposition 4 it is established that a Turing machine with a periodic points in T_h cannot have a transitive dynamical system. As periodic points in T_h are blocking, we have also examples for non-transitive Turing machines with blocking words. As an example, we present the simple machine in figure 16.

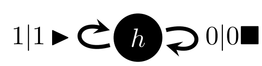

Figure 16: A simple machine with periodic points in T_h. The movement 0 is represented by the symbol ■.

5 Enriching the family: Embedding preserving properties

In [8] and [2], a particular embedding was constructed to prove the undecidability of *transitivity* and *aperiodicity*, respectively. The idea was to embed a reversible machine without periodic points into SMART in such a way that the resulting machine is either in class (b) or class (i), depending on the characteristics of the invited machine. This construction shows the stability of these classes under embedding when the invited machine satisfies certain hypothesis.

Minimality is also proved undecidable with a similar technique but using a simpler embedding, based on *time-reflection*. In this section, we will use this embedding technique on the 11 machines presented in the last section. We will study which properties are preserved through embedding and we will conclude that each of the 11 classes contains infinitely many members. For this purpose, let us introduce *Mortal machines* [10] and *Aperiodic machines* [13], whose definitions are the following.

Definition 5. *A Turing machine M is called* mortal *if from every configuration it eventually halts. This results, by compactness, to be equivalent to the existence of a global time $t_M \in \mathbb{N}$ such that for all $x \in X$: $T^{t_M}(x)$ is not defined. We call t_M the* mortality constant*. If M is not mortal, it is called* immortal*. It is important to note that the reverse of a mortal machine is also mortal.*

Definition 6. *A Turing machine M is called* static aperiodic *if it does not possess any static periodic point. Note that in this case, we not need that M has a dynamical system (*i.e. *it can be incomplete). In the same way, a Turing machine M is called* shifted aperiodic *if it does not posses any shifted periodic point.*

For the following propositions, consider a reversible non-complete Turing machine $M = (Q_M, \Sigma_M, \delta_M)$ that we use to form an innocuous machine I_M by using time-reflection. Later embed this machine into some Host Turing machine $H = (Q_H, \Sigma_H, \delta_H)$, to form H_{I_M}. Also, consider $(X_h, T_h), (X_t, T_t)$ and S_t the dynamical systems for H and $(\hat{X}_h, \hat{T}_h), (\hat{X}_t, \hat{T}_t)$ and $\hat{S}_t$ the dynamical systems for H_{I_M}.

Proposition 9. *H_{I_M} is reversible $\Leftrightarrow$ H is reversible.*

Proof. Reversibility is a local property. Embedding and time-reflection preserves, for every modified transition, the written symbol and the moving direction. Also added transitions respect the proceeding direction of each state, and the written symbols corresponds to defective ones, thus no ambiguity in the proceeding configuration is introduced. □

Proposition 10. *Let M be shifted aperiodic, the following assertions hold:*

1. *For every $x \in X_h$, x is periodic for $\hat{T}_h \Rightarrow x$ is periodic for T_h.*
2. *For every $x \in X_t$, x is periodic for $\hat{T}_t \Rightarrow x$ is periodic for T_t.*
3. *H is static aperiodic $\Rightarrow H_{I_M}$ is static aperiodic.*
4. *H is shifted aperiodic $\Rightarrow H_{I_M}$ is shifted aperiodic.*

Proof.

1. Let $x \in X_h$ be a periodic point for $\hat{T}_h$. Since its state is in Q_H, the orbit of x stays only for finite intervals of time in the states Q_{I_M}. Cutting out this intervals we obtain the orbit of x through I_M, which is also periodic.
2. This case is analogous to the last one.
3. Let us suppose that H is static aperiodic, and let us take a point x static periodic for H_{I_M}. Since I_M is shifted aperiodic, the orbit of x contains states of H. Let us suppose, without loss of generality, that x has a state in H. Now we apply item 1 to x, and we conclude that x is a periodic point of H in TMH, which is a contradiction.
4. This case is analogous to the last one.

□

Let us remark that the reciprocal in the last two items does not hold, since an appropriate embedding can destroy periodic points.

Proposition 11. *Let M be shifted aperiodic and $\Sigma_M \subset \Sigma_H$, the following assertions hold:*

1. *For every $x \in X_h$, with infinitely many symbols in $\Sigma_H \setminus \Sigma_M$, x is a transitive point in $(X_h, T_h) \Leftrightarrow x$ is a transitive point in $(\hat{X}_h, \hat{T}_h)$.*
2. *For every $x \in X_t$, with infinitely many symbols in $\Sigma_H \setminus \Sigma_M$, x is a transitive point in $(X_t, T_t) \Leftrightarrow x$ is a transitive point in $(\hat{X}_t, \hat{T}_t)$.*
3. *H_{I_M} is transitive in TMH $\Leftrightarrow H$ is transitive in TMH.*
4. *H_{I_M} is transitive in TMT $\Leftrightarrow H$ is transitive in TMT.*

Proof.

1. Let x be a point in X_h, with infinitely many symbols in $\Sigma_H \setminus \Sigma_M$. Since x has infinitely many symbols in $\Sigma_H \setminus \Sigma_M$, and M is shifted aperiodic, the orbit of x under the machine H_{I_M} cannot stay infinitely many iterations in the states of I_M, because it will eventually find a symbol outside the alphabet of M. Therefore, the orbit of x under H_{I_M} can be described as the orbit of x under H, interrupted by intervals of times where is I_M that acts.

(⇒) Let us suppose that x is transitive in (X_h, T_h), and let $(u.u')$ be a partial configuration of $(\hat{X}_h, \hat{T}_h)$. If $(u.u')$ has a state in Q_H, the transitivity of x assures that its orbit will contain $(u.u')$ at some iteration. If not, let us extend it to $(\alpha u.u'\alpha)$, with $\alpha \in \Sigma_H \setminus \Sigma_M$, and make $\hat{T}_h$ to run backward until it reaches a state in Q_H at finite configuration v; the set of visited cells during this time is contained in the finite interval $[-|u|, |u'|]$. The transitivity of x assures that the orbit of x contains the finite configuration v at some iteration, after which $(u.u')$ also appears in the orbit of x, which ends the proof.

(⇐) Now let us assume that x is transitive in $(\hat{X}_h, \hat{T}_h)$. The orbit of x under $\hat{T}_h$ is equal to the orbit of x under T_h where the iterations with a state in I_M are cut out. Thus, x is also transitive for (X_h, T_h).

2. If x is a point of X_t, and we consider its dynamic trough T_t or $\hat{T}_t$, the analysis is similar to the last one, we can conclude the same.

3. (⇐) If H is transitive in the TMH model, then its set of transitive points x is dense in X_h. We can choose a point x with infinitely many symbols in $\Sigma_H \setminus \Sigma_M$, and apply item 1 to conclude that H_{I_M} is transitive too.

   (⇒) If H_{I_M} is transitive in the TMH model, as before we can choose a point x with infinitely many symbols in $\Sigma_H \setminus \Sigma_M$, but the state of x may be a state of I_M. We take the first preimage y of x with a state in H, and apply 1 to point y and conclude that H_{I_M} is transitive for the TMH model. We can do this because transitivity is inherited by preimage.

4. This case is analogous to the last one.

□

Proposition 12. *Let M be a mortal machine, the following assertions hold:*

1. *x is a transitive point in (X_t, T_t) ⇔ x is a transitive point in $(\hat{X}_t, \hat{T}_t)$.*
2. *H_{I_M} is minimal in TMT ⇔ H is minimal in TMT.*
3. *For every $x \in X_h$, x is periodic for T_h ⇔ x is periodic for $\hat{T}_h$.*
4. *For every $x \in X_t$, x is periodic for T_t ⇔ x is periodic for $\hat{T}_t$.*
5. *H_{I_M} is static aperiodic ⇔ H is static aperiodic.*

Proof. 1. Since M is mortal, the orbit of x under the machine H_{I_M} cannot stay infinitely many iterations in the states of I_M. Therefore, the orbit of x under H_{I_M} can be described as the orbit of x under H, interrupted by intervals of times where is I_M that acts.

(⇒) Let us suppose that x is transitive in (X_t, T_t), and let (u, q, u') be a partial configuration of $(\hat{X}_t, \hat{T}_t)$. If $q \in Q_H$, the transitivity of x assures that its orbit will contain (u, q, u') at some iteration. If not, assuming w.l.g. that $|u| + |u'| > 2t_M$, we know that for some $t < 2t_m$, $\hat{T}_t^{-t}(u, q, u')$ will have a state in Q_H.

(⇐) Now let us assume that x is transitive in $(\hat{X}_t, \hat{T}_t)$. The orbit of x under $\hat{T}_t$ is equal to the orbit of x under T_t where the iterations with a state in I_M are cut out. Thus, x is also transitive for (X_t, T_t).

2. (⇐) If H is minimal in TMT, each of its points $x \in X_t$ is transitive for T_t. By using item 1, we know that every $x \in X_t$ is also transitive for $\hat{T}_t$. Now if we take a point $y \in \hat{X}_t$, there exists a point $x \in X_t$ in its orbit, because I_M is mortal. Since x is transitive for $\hat{T}_t$, y is also transitive for $\hat{T}_t$, and then $(\hat{X}_t, \hat{T}_t)$ is minimal.

   (⇒) If H_{I_M} is minimal, every $y \in \hat{X}_t$ is transitive for $\hat{T}_t$, in particular, every $y \in X_t$ is transitive for $\hat{T}_t$. By using item 1, we know that y is also transitive for T_t, thus (X_t, T_t) is minimal.

3. One direction was already proved in proposition 10. Let x be a periodic point for T_h. Since I_M is mortal, the orbit of x in $\hat{T}_h$ is equal to its orbit in T_h interrupted with finite intervals where the state is in I_M, this means that x is also periodic for $\hat{T}_h$.

4. This proof is analogous to the last one.

5. This is directly implied by item 3.

□

Proposition 13. *If M is immortal, then H_{I_M} cannot be minimal.*

Proof. There exists a point x such that its orbits in I_M is infinite, therefore that orbit cannot reach any configuration with states in H. □

Corollary 2. *Minimality is undecidable, even if we know that the machine is transitive in TMT or in TMH.*

For the next proposition, we need to consider a modification of H_{I_M}. First, let us consider M a mortal machine with t_M its mortality constant. Also, consider a set of auxiliary states $q_1, ..., q_{t_M}$ that we add between H and I_M, that are used to move the head t_M cells to the right, in such a way that no computation inside I_M would be able to surpass the position of the head after exiting H. Also consider another set of auxiliary states $q_1^-, ..., q_{t_M}^-$ that we add after the last error state of I_M to return the head to its original position before re-entering H. We call this new machine $H_{+t_M I_M}$ and its left version is called $H_{-t_M I_M}$ (move the head to the left before entering I_M). Finally, consider $(X_h, T_h), (X_t, T_t)$ and S_t the dynamical systems for H and $(\vec{X_h}, \vec{T_h}), (\vec{X_t}, \vec{T_t})$ and $\vec{S_t}$ the dynamical systems for $H_{+t_M I_M}$

Proposition 14. *Consider a mortal machine M and the embedding $H_{+t_M I_M}$ and $H_{-t_M I_M}$. Given a finite configuration u of H in the TMH model, the following assertions hold:*

1. *u is a blocking word to the left (right) in $H \Leftrightarrow u$ is a blocking word to the left (right) in $H_{+t_M I_M}$ ($H_{-t_M I_M}$).*

2. *H has a transitive trace-shift $\Leftrightarrow H_{+t_M I_M}$ or $H_{-t_M I_M}$ have a transitive trace-shift.*

Proof. 1. If $u = (r, 0, .x)$ is a finite word of H, the orbit of $x \sqsupseteq u$ over $H_{+t_M I_M}$ is the same that the orbit of x over H, but interrupted by finite executions of $+t_M I_M$, in which the head do not goes to the left of the point where this execution started. Therefore, u is a blocking word to the left for H if and only if it is a blocking word to the left for $H_{+t_M I_M}$. The same happens with blocking words to the right for $H_{-t_M I_M}$.

2. We first recall that if H has a transitive trace-shift and it has a blocking word to the left (or right), then all of its blocking words are to the left (or right). Then we will assume, without loss of generality, that when transitivity is present, H has only blocking words to the left (if any). We will concentrate on machines with at least one blocking word, because if these words are absent, the transitivity of the trace-shift is equivalent to the transitivity in the TMT model, which has been already considered in Proposition 10 and Proposition 11.

   (⇒) Let $(r, 0, .u)$ be a blocking word of H to the left, and let $\overline{u} = \tau_H(r, 0, .u)$. Let us suppose that S_t is transitive, and let $w \in S_t$ be a transitive point for (S_t, σ). Let x be the canonical pre-imagen of w. Now, let $w' = \tau_{H_{+t_M I_M}}(x)$. We will prove that w' is transitive in $\vec{S_t}$. Let $v' \in \mathcal{L}(\vec{S_t})$. We can assume without loss of generality that $v'_0 = (q, s)$ for some $q \in Q_H$ and $s \in \Sigma_H$; if this were not the case, we use reversibility of $+t_M I_M$ to extend v' backward until to find a state of H, this happen in finite time, because I_M is mortal. Let V' be the canonical pre-image of v' in $\vec{X_t}$.

   Now, let us take $v = \tau_H(V')$. The word v will be very similar to v', except that the machine $+t_M I_M$ will not be involved. This will imply $\mathcal{S}_H(v) \subseteq \mathcal{S}_{H_{+t_M I_M}}(v')$, being the possible difference at the righter cells. If they are not equal, let V be the canonical pre-image of a word of the form

$vz\overline{u}z' \in \mathcal{L}(S_t)$ (where $z \in \mathcal{L}(S_t)$ exist thanks to the transitivity of S_t). As u is a blocking word to the left, we can select z' as big as we need, such that $V \sqsupseteq V'$.

Now, $\tau_H(V) \sqsubseteq w$ as w is a transitive point of H, and since the support of V is completely visited by the head, this imply that there exists $n \in \mathbb{N}$ such that $T_t^n(x) \sqsupseteq V$. As I_M is mortal, we can conclude that $\vec{T_t}^n(x) \sqsupseteq V'$, and then $v' \sqsubseteq w'$, which concludes the proof.

($\Leftarrow$) Let us suppose that $\vec{S_t}$ is transitive, and let $u, w \in \mathcal{L}(S_t)$. Let U and W be the canonical pre-image of u and w in X_t, respectively. If we execute $H_{+t_M I_M}$ over U and W maybe some additional space will be needed in order to allow $+t_M I_M$ to run between the steps of H that produce u and w. Let us take any symbol $\alpha \in \Sigma_M$, and let $U' = U\alpha^{t_M}$ and $W' = W\alpha^{t_M}$. Now we can define $u' = \tau_{H_{+t_M I_M}}(U')$ and $w' = \tau_{H_{+t_M I_M}}(W')$. By the transitivity of $\vec{S_t}$, we know that there exists a word v' such that $z' = u'v'w' \in \mathcal{L}(\vec{S_t})$. This word z' is produced by an infinite configuration $x \in \vec{X_t}$, but since its state is in the first symbol of u' (which is equal to the first symbol of u), then $x \in X_t$. We see that $\tau_H(x) = uzw$, for some $z \in \mathcal{L}(S_t)$, which concludes the proof.

□

With the previous results, we can assure that embedding keeps the properties expressed in the classification of figure 4. Therefore, we can prove the following.

Theorem 6. *The classes $a, b, c, d, e, f, g, h, i, j$ and k are infinitely countable.*

Proof. Using embedding H_{I_M} we can preserve the properties of the machines without blocking words a, b, c, d, i and j, as the last propositions assure. In the same way, using the embedding $H_{+t_M I_M}$ or $H_{-t_M I_M}$, we can preserve the properties of the machines with blocking words e, f, g, h and k. □

6 Discussion

We explored transitivity and some properties which are related with it. We obtained a chart that shows the relations between each notion and we have found examples that proves that all the regions of the chart have "inhabitants". The *embedding technique* showed its power helping in the constructions of several examples. A promising application of this technique is insinuated in the last section, where it is shown how some properties are preserved by embedding. This proves that each of the presented examples can be multiplied with this technique, establishing that we have, in fact, an infinite family of examples. Moreover, it shows that some classes are undecidable even within a bigger class.

It is tempting to enlarge this study, considering other properties, and other embedding techniques, as the one used in [2]. Such a study should reveal all the potential applications of the tool, and it may help to find strong results, for example, one could expect to prove a Rice theorem of dynamical properties of Turing machine, if this exists.

Other related properties has been left to further studies, for example, *mixing* property, already known undecidable, the *coded family*, explored in [19], and the relation of all of these properties with the properties of *periodicity*, *aperiodicity*, *time-symmetry*, *soficity*, and the value of the *entropy*.

Another field of research is the arithmetical hierarchy of all of these classes, as the exact classification is unknown. Actually, Topological Minimality and Transitivity (in the three models) are known to be Σ_1^0-hard and Π_1^0-hard, respectively, and both belong to Π_2^0 [19]. Nevertheless, it is unknown whether they are complete in these classes. Finally, Topological Mixing is proved to be Π_3^0 and Π_1^0-hard, but again its exact classification is yet to be determined.

We leave an intriguing open problem: the existence of *single orbit machines*: machines with a trace-shift with a single orbit. It is not difficult to construct such a machine if no movement is allowed, in which case the set of accessible cells (the support) has size 1. But we were unable to construct a single orbit machine with a larger support. The difficulty comes from the need to produce the whole set of configurations relative to the "head vision" of the machine. But a proof of non existence was elusive too.

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