A fully-mixed finite element method for the coupling of the Stokes and Darcy-Forchheimer problems*

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Abstract

In this paper we introduce and analyze a fully-mixed formulation for the nonlinear problem given by the coupling of the Stokes and Darcy-Forchheimer equations with the Beavers-Joseph-Saffman condition on the interface. This new approach yields non-Hilbert normed spaces and a twofold saddle point structure for the corresponding operator equation, whose continuous and discrete solvabilities are analyzed by means of a suitable abstract theory developed for this purpose. In particular, feasible choices of finite element subspaces include PEERS of lowest order for the stress of the fluid, Raviart-Thomas of lowest order for the Darcy velocity, piecewise constants for the pressures, and continuous piecewise linear elements for the vorticity. A priori error estimates and associated rates of convergence are derived, and several numerical results illustrating the good performance of the method are reported.

Key words: Stokes equation, Darcy-Forchheimer equation, fully-mixed formulation, twofold saddle point, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q35, 76S05, 76D07

1 Introduction

The derivation of suitable mathematical and numerical models for the fluid flow between porous media and free-flow zones has received a growing interest, due to its many applications in engineering and biology, to name a few (cf., e.g., [7, 8, 9, 10, 11, 15, 17, 24, 25] and the references therein). For instance, filter design (cf. [17]) and reservoir models (cf. [9]) are just a couple of examples where models of this kind are applied. The system can be viewed as a coupled problem with two physical systems interacting across an interface: the first one being the free movement of a fluid, while the second one being the flow in a permeable material. In the free-flow zone, the Navier-Stokes equations have

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become the traditional way to model the movement of this fluid, although when dealing with creeping flows, or simply as a first step to treat the problem, the linear version of these equations, namely the Stokes equations, may be considered. On the other hand, even when the Darcy's law does describe the flow through a porous medium, the complexity of this phenomenon has lead the introduction of modifications to this model. One of them is the addition of a term which represents inertial effects, known as the Forchheimer term, thus obtaining the Darcy-Forchheimer model. In this way, having a model for both zones in the system, it only remains to specify proper conditions for the fluid flow when crossing the interface. Indeed, the Beavers-Joseph, Saffman and Jones conditions (cf. [5, 33, 31], respectively) are some examples of well-accepted methods to accomplish this purpose.

In this context, several finite element methods have been proposed. For instance, in [8, 17, 25] the authors propose several fully-mixed formulations for the Stokes/Darcy problem, including an augmented-type formulation in [8] and the treatment of a nonlinear Stokes equation in [17]. More recently, using fixed-point arguments, in [11, 15] finite element methods for the Navier-Stokes/Darcy problem have been developed. In all these works, the formulations are treated using tools for Hilbert spaces, but the nonlinearity of the Darcy-Forchheimer model motivates the introduction of nonconventional Banach spaces, as shown in recent work [10] where the authors analyze a primal-mixed formulation of the Navier-Stokes/Darcy-Forchheimer system by means of a fixed point argument, which albeit a useful technique, it requires that the boundary data and source terms are small, a requirement that may transcend the theory and manifest in the numerical computations.

According to the above, the purpose of this work is to extend the results available in [24] for the analysis of a fully-mixed formulation of the Stokes/Darcy problem for quasi-Newtonian fluids to the coupling of the Stokes and Darcy-Forchheimer problems. We stress that, differently from [10], where primal and mixed formulations, respectively, are applied to the fluid and the porous medium, in this paper we employ dual-mixed approaches in both domains. Moreover, our analysis will be carried out by means of a modified abstract theory for twofold saddle point problems, which on the one hand, does not require any smallness-of-data assumption, and on the other hand, it allows us to pose the variational formulation in terms of just Banach spaces. In addition, an *a priori* analysis is performed, and while it is possible to prove that the finite element method is convergent with a sub-optimal rate, the numerical results suggest that the method is optimally convergent provided the exact solutions are smooth enough. In particular, we find that the interior Stokes variables can be approximated using either PEERS or AFW elements, while the interior Darcy-Forchheimer variables can be approximated using a combination of Raviart-Thomas elements and constant piecewise functions.

The rest of this work is organized as follows. First, we end this Section by introducing some notation that will be used throughout the paper. Next, in Section 2 we introduce the modeling equations for the free-flow zone, the porous medium and the interface, to then construct a fully-mixed variational formulation that will be written as a nonlinear system with a twofold saddle point structure. Then, in Section 3, we develop an abstract theory for this type of problems, and we analyze which are the proper hypotheses on the spaces and involved operators to be imposed in order to guarantee the well-posedness of both the continuous and discrete problems in rather general Banach spaces. Finally, in Section 4, we apply this theory to the present context and specify a particular combination of finite element subspaces that results in a well-posed fully-mixed finite element method, to then in Section 5 present some numerical examples that show its good performance.

Preliminary notations

Let us denote by $\Omega \subset \mathrm{R}^2$ a given bounded domain with polygonal boundary Γ. Given $p > 1$ and $s > 0$, standard notation will be adopted for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces (cf. [1]) $W^{s,p}(\Omega)$ with norm $\|\cdot\|_{s,p;\Omega}$ and seminorm $|\cdot|_{s,p;\Omega}$. In addition, given an open subset Γ_0 of Γ, with $|\Gamma_0| > 0$, we let $W^{s-1/p,p}(\Gamma_0)$ be the usual space of traces of $W^{s,p}(\Omega)$ on Γ_0, with norm $\| \cdot \|_{s-1/p,p;\Gamma_0}$. On the other hand, by $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\| \cdot \|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$, we set the gradient and divergence operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,2} \qquad \text{and} \qquad \operatorname{div} \mathbf{v} := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,2}$, we let $\mathbf{div}\, \boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,2}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij} \zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau}) \mathbf{I},$$

where $\mathbf{I}$ stands for the identity tensor in $\mathrm{R}^{2\times 2}$. Furthermore, we set for $1 < p < \infty$

$$L_0^p(\Omega) := \left\{ q \in L^p(\Omega) : \int_{\Omega} q = 0 \right\},$$

and for $2 \leq r < \infty$ we define the Sobolev space

$$\mathbf{W}^{0,r}(\operatorname{div}; \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^r(\Omega) : \operatorname{div} \mathbf{v} \in L^r(\Omega) \right\}$$

endowed with the norm

$$\| \mathbf{v} \|_{r,\operatorname{div};\Omega} := \| \mathbf{v} \|_{0,r;\Omega} + \| \operatorname{div} \mathbf{v} \|_{0,r;\Omega}.$$

In particular, for $r = 2$, we denote $\mathbf{W}^{0,2}(\operatorname{div}, \Omega) =: \mathbf{H}(\operatorname{div}, \Omega)$, a standard Hilbert space in the realm of mixed problems. Finally, in what follows, $| \cdot |$ denotes the Euclidean norm in R^2, and we employ Θ or $\mathbf{0}$ to denote a generic null element of a vector space, and we use C or c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The continuous problem

We begin by formally describing the movement of a Newtonian fluid back and forth between a free-flow zone and a porous medium saturated with the same fluid.

2.1 The model problem

Let Ω_S and Ω_D be two bounded and simply connected polygonal domains in R^2 such that $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \emptyset$ and $\Omega_S \cap \Omega_D = \emptyset$. Then, let $\Gamma_S := \partial\Omega_S \backslash \overline{\Sigma}$, $\Gamma_D := \partial\Omega_D \backslash \overline{\Sigma}$ and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward on $\Gamma_S \cup \Gamma_D$ and inward to Ω_D on Σ, where we also consider a unit tangent vector $\mathbf{t}$ (see Figure 2.1). Then, the model is constructed using equations from both theoretical and experimental origin: the Stokes equations posed in the free-flow zone Ω_S, the Darcy-Forchheimer model in the porous medium Ω_D, and a variation of the Beavers-Joseph condition (1967) (cf. [5]) done by Saffman (1971) (cf. [33]) on the interface Σ. We give details on these equations next, but before we continue, we introduce some additional notation.

Given $\bigstar \in \{S, D\}$, $1 \le p \le \infty$ and $1 \le q \le \infty$ such that $p^{-1} + q^{-1} = 1$ we denote by $(\cdot,\cdot)_\bigstar$ the inner product between two scalars, vectors, or tensors in compatible Lebesgue spaces, i.e.,

$$\begin{aligned}
(w, v)_\bigstar &:= \int_{\Omega_\bigstar} w\, v, \qquad \forall\, w \in L^p(\Omega_\bigstar),\ \forall\, v \in L^q(\Omega_\bigstar),\\
(\mathbf{w}, \mathbf{v})_\bigstar &:= \int_{\Omega_\bigstar} \mathbf{w} \cdot \mathbf{v}, \quad \forall\, \mathbf{w} \in \mathbf{L}^p(\Omega_\bigstar),\ \forall\, \mathbf{v} \in \mathbf{L}^q(\Omega_\bigstar),\\
(\boldsymbol{\zeta}, \boldsymbol{\tau})_\bigstar &:= \int_{\Omega_\bigstar} \boldsymbol{\zeta} : \boldsymbol{\tau}, \quad \forall\, \boldsymbol{\zeta} \in \mathbb{L}^p(\Omega_\bigstar),\ \forall\, \boldsymbol{\tau} \in \mathbb{L}^q(\Omega_\bigstar).
\end{aligned}$$

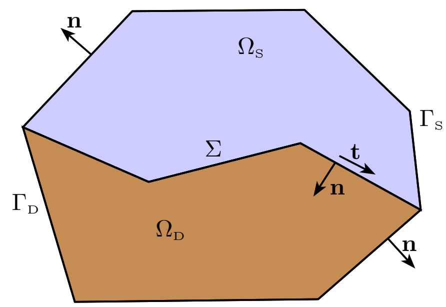

Figure 2.1: Sketch of the domain for the coupled problem.

On other hand, given $\Gamma_0 \subset \partial\Omega_\bigstar$, we recall that $W^{1/q,p}(\Gamma_0)$ is the space of traces of $W^{1,p}(\Omega_\bigstar)$ on Γ_0, and define the continuation of a given $\xi \in W^{1/q,p}(\Gamma_0)$ by zero to the rest of the boundary as

$$E^\bigstar_{\Gamma_0}(\xi) := \begin{cases} \xi & \text{on } \Gamma_0, \\ 0 & \text{on } \partial\Omega_\bigstar \backslash \Gamma_0. \end{cases} \tag{2.1}$$

Then, we introduce the space

$$\widetilde{W}^{1/q,p}_\bigstar(\Gamma_0) := \Big\{ \xi \in W^{1/q,p}(\Gamma_0) : \ E^\bigstar_{\Gamma_0}(\xi) \in W^{1/q,p}(\partial\Omega_\bigstar) \Big\} \tag{2.2}$$

endowed with the norm

$$\|\xi\|_{1/q,p;\Gamma_0} := \|E^\bigstar_{\Gamma_0}(\xi)\|_{1/q,p;\partial\Omega_\bigstar} \qquad \forall\, \xi \in \widetilde{W}^{1/q,p}_\bigstar(\Gamma_0)\,.$$

2.1.1 Free-flow zone

For each arbitrary velocity field $\mathbf{v}$, let $\mathbf{e}(\mathbf{v})$ be the symmetric part of the velocity gradient tensor $\nabla\mathbf{v}$. Then, given a source term $\mathbf{f}_S$, we seek in Ω_S a velocity field $\mathbf{u}_S$ and a pressure field p_S such that

$$\begin{aligned}
-\mu\Delta\mathbf{u}_S + \nabla p_S = \mathbf{f}_S &\quad \text{in } \Omega_S,\\
\operatorname{div}\mathbf{u}_S = 0 &\quad \text{in } \Omega_S,\\
\mathbf{u}_S = \mathbf{0} &\quad \text{on } \Gamma_S,
\end{aligned}$$

where the last equation corresponds to a non-slip condition on the non-interfacial boundary Γ_S and μ denotes the viscosity of the fluid. In this way, defining the stress and vorticity tensors as

$$\boldsymbol{\sigma}_S := -p_S\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u}_S) \quad \text{and} \quad \boldsymbol{\gamma}_S := \frac{1}{2}\Big(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}\Big)\,,$$

respectively, the pressure can be eliminated from the system above as

$$p_S := -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_S), \tag{2.3}$$

and the Stokes system can now be written as: Find $(\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_S)$ such that

$$\nabla \mathbf{u}_S - \boldsymbol{\gamma}_S = \frac{1}{2\mu}\boldsymbol{\sigma}_S^{\mathtt{d}} \quad \text{in } \Omega_S, \tag{2.4a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_S = \mathbf{f}_S \quad \text{in } \Omega_S, \tag{2.4b}$$

$$\mathbf{u}_S = 0 \quad \text{on } \Gamma_S. \tag{2.4c}$$

2.1.2 Porous medium

When kinematic effects surpass viscous effects in a porous medium, the Darcy velocity $\mathbf{u}_D$ and the pressure gradient ∇p_D do not satisfy a linear relation. Instead, a non-linear approximation, known as the Darcy-Forchheimer model, is considered. When it is imposed on the porous medium Ω_D with Neumann boundary conditions on Γ_D, the equations read: Find $(\mathbf{u}_D, p_D)$ such that

$$\frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_D + \frac{\mathtt{f}}{\rho}|\mathbf{u}_D|\mathbf{u}_D + \nabla p_D = \mathbf{g}_D \quad \text{in } \Omega_D, \tag{2.5a}$$

$$\operatorname{div}\mathbf{u}_D = f_D \quad \text{in } \Omega_D, \tag{2.5b}$$

$$\mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D, \tag{2.5c}$$

where ρ is the density of the fluid, $\mathtt{f}$ stands for the Forchheimer number of the porous medium, and $\mathbf{g}_D$ and f_D are source terms. Here, $\mathbf{K} \in \mathbb{L}^\infty(\Omega_D)$ is a symmetric and uniformly elliptic tensor describing the permeability of this medium, so that for each $\mathbf{x} \in \Omega_D$ we define $\varrho_{\min}(\mathbf{x}) := \min\{\varrho : \varrho \text{ is an eigenvalue of } \mathbf{K}(\mathbf{x})\}$, and deduce from the assumptions on $\mathbf{K}$ that there exists $\varrho_0 > 0$ such that

$$\varrho_{\min}(\mathbf{x}) \geq \varrho_0 > 0 \quad \forall\, \mathbf{x} \in \overline{\Omega}_D. \tag{2.6}$$

We recall in advance that one of the conditions that will be imposed on Σ is the continuity of normal velocities, and therefore an integration by parts of the compressibility condition (2.5b) leads us to ask for f_D to be a zero-mean function in Ω_D, that is, $\int_{\Omega_D} f_D = 0$.

2.1.3 Transmission conditions

At the interface between the free-flow zone and the porous media, the conservation of mass and balance of normal forces are well-accepted conditions in literature (cf. [17, 25, 36]). On the other hand, there are experimental conditions that must be satisfied, such as the Beavers-Joseph conditions (cf. [5]), which relate the jump of the velocity field across the interface Σ with the tangential component of the normal stress (known as the traction). Among the different ways to simplify these conditions (cf. [31]), we consider in this work those by Saffman (cf. [33]) coined as the Beavers-Joseph-Saffman (BJS) conditions, which can be obtained by neglecting the tangential velocity in the porous medium at the Beavers-Joseph conditions. In summary, we impose the following boundary conditions at the interface Σ:

1. Conservation of mass: $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$,

2. Balance of normal forces: $(\boldsymbol{\sigma}_S \mathbf{n}) \cdot \mathbf{n} = -p_D$,

3. The Beavers-Joseph-Saffman condition: $(\boldsymbol{\sigma}_S\mathbf{n})\cdot\mathbf{t} = -\mu\kappa^{-1}(\mathbf{u}_S\cdot\mathbf{t})\,,$

where κ is the friction coefficient. Considering that $(\mathbf{n},\mathbf{t})$ is a local orthonormal basis on Σ, we can rewrite 2 and 3 as a single equation, so that the transmission conditions become

$$\mathbf{u}_S\cdot\mathbf{n} = \mathbf{u}_D\cdot\mathbf{n} \quad \text{on } \Sigma, \tag{2.7a}$$

$$\boldsymbol{\sigma}_S\mathbf{n} + \mu\kappa^{-1}(\mathbf{u}_S\cdot\mathbf{t})\mathbf{t} = -p_D\mathbf{n} \quad \text{on } \Sigma. \tag{2.7b}$$

In this way, the Stokes/Darcy-Forchheimer (SDF) coupled problem consists of equations (2.4), (2.5) and (2.7). We end this section by mentioning that, although only homogeneous boundary conditions have been considered on $\Gamma_D\cup\Gamma_S$, it is possible to establish more general boundary conditions under little modifications (cf. [17]).

2.2 A fully-mixed formulation

We now focus on developing mixed formulations for each one of the previous sets of equations in a similar way to [25], where the authors introduce $\mathbf{u}_S|_\Sigma$ and $p_D|_\Sigma$ as additional unknowns of physical interest, which play the role of suitable Lagrange multipliers as well.

2.2.1 Stokes problem

First, we address the mixed formulation of the Stokes problem. On the one hand, uniqueness of the pressure is ensured whenever $p_S\in L^2_0(\Omega_S)$, which according to (2.3) suggests the introduction of the space

$$\mathbb{H}_0(\mathbf{div};\Omega_S) := \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega_S):\ \int_{\Omega_S}\mathrm{tr}(\boldsymbol{\tau}) = 0\right\}.$$

On the other hand, since the vorticity $\boldsymbol{\gamma}_S$ is a skew-symmetric tensor, this leads us to define the space

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega_S) := \left\{\boldsymbol{\eta}\in\mathbb{L}^2(\Omega_S):\ \boldsymbol{\eta}+\boldsymbol{\eta}^t = \mathbf{0}\right\}.$$

Therefore, proceeding as in [25], we first define the linear operator $\mathcal{A}_S:\mathbb{H}_0(\mathbf{div};\Omega_S)\to[\mathbb{H}_0(\mathbf{div};\Omega_S)]'$ as

$$\mathcal{A}_S(\boldsymbol{\sigma}_S) := \frac{1}{2\mu}\boldsymbol{\sigma}_S^{\mathtt{d}} \quad \forall\ \boldsymbol{\sigma}_S\in\mathbb{H}_0(\mathbf{div};\Omega_S), \tag{2.8}$$

and then we test the constitutive equation (2.4a) with $\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div};\Omega_S)$ and integrate by parts, the equilibrium equation (2.4b) with $\mathbf{v}_S\in\mathbf{L}^2(\Omega_S)$, and impose the symmetry of the stress tensor $\boldsymbol{\sigma}_S$ in a weak sense, which results in the weak problem: Find $(\boldsymbol{\sigma}_S,\mathbf{u}_S,\boldsymbol{\gamma}_S,\boldsymbol{\varphi})\in\mathbb{H}_0(\mathbf{div};\Omega_S)\times\mathbf{L}^2(\Omega_S)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega_S)\times\widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)$ such that

$$(\mathcal{A}_S(\boldsymbol{\sigma}_S),\boldsymbol{\tau}_S)_S + (\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{u}_S)_S + \langle\,\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\varphi}\,\rangle_\Sigma + (\boldsymbol{\gamma}_S,\boldsymbol{\tau}_S)_S = 0, \tag{2.9a}$$

$$(\mathbf{div}\,\boldsymbol{\sigma}_S,\mathbf{v}_S)_S + (\boldsymbol{\sigma}_S,\boldsymbol{\eta}_S) = -(\mathbf{f}_S,\mathbf{v}_S)_S, \tag{2.9b}$$

for all $(\boldsymbol{\tau}_S,\mathbf{v}_S,\boldsymbol{\eta}_S)\in\mathbb{H}_0(\mathbf{div};\Omega_S)\times\mathbf{L}^2(\Omega_S)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega_S)$, where $\boldsymbol{\varphi} := -\mathbf{u}_S\big|_\Sigma$,

$$\widehat{\mathbf{H}}^{1/2}_{00}(\Sigma) := \left\{\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma):\quad \langle\,\boldsymbol{\psi}\cdot\mathbf{n},1\,\rangle_\Sigma = 0\right\}, \tag{2.10}$$

and $\mathbf{H}^{1/2}_{00}(\Sigma) = \widetilde{\mathbf{W}}^{1/2,2}_S(\Sigma)$. Actually, under the assumption that $\mathbf{u}_S\in\mathbf{W}^{1,2}(\Omega_S)$ with $\mathbf{u}_S = \mathbf{0}$ on Γ_S, the natural search space for $\boldsymbol{\varphi}$ is $\mathbf{H}^{1/2}_{00}(\Sigma)$. However, the no-slip condition (2.4c) and the incompressibility constraint $\mathrm{div}\,\mathbf{u}_S = 0$ yields the belonging to $\widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)$.

2.2.2 Darcy-Forchheimer problem

Similarly to [18, 20], here we define the nonlinear operator $\mathcal{A}_D : \mathbf{L}^3(\Omega_D) \to \mathbf{L}^{3/2}(\Omega_D)$ given by

$$\mathcal{A}_D(\mathbf{u}_D) := \frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_D + \frac{\mathtt{f}}{\rho}|\mathbf{u}_D|\mathbf{u}_D \quad \forall\, \mathbf{u}_D \in \mathbf{L}^3(\Omega_D). \tag{2.11}$$

In this case, the Neumman condition (2.5c) motivates the introduction of the space for $2 \le r < \infty$

$$\mathbf{W}^{0,r}_{\Gamma_D}(\mathrm{div};\Omega_D) := \Big\{\mathbf{v} \in \mathbf{W}^{0,r}(\mathrm{div};\Omega_D) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_D\Big\},$$

where the precise meaning of the statement "$\mathbf{v}\cdot\mathbf{n} = 0$ on Γ_D" will be specified shortly. Hence, we test the equilibrium equation (2.5a) with $\mathbf{v}_D \in \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D)$, integrate by parts, and introduce the auxiliary unknown $\lambda := p_D|_\Sigma \in W^{1/3,3/2}(\Sigma)$. In turn, the compressibility equation (2.5b) is tested against $q_D \in L^{3/2}(\Omega_D)/R$. In this way, we arrive at the weak formulation of the Darcy-Forchheimer equations: Find $(\mathbf{u}_D, p_D, \lambda) \in \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D) \times L^{3/2}_0(\Omega_D) \times W^{1/3,3/2}(\Sigma)$ such that

$$(\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D)_D - (\mathrm{div}\,\mathbf{v}_D, p_D)_D - \langle\, \mathbf{v}_D\cdot\mathbf{n}, \lambda\,\rangle_\Sigma = (\mathbf{g}_D, \mathbf{v}_D)_D, \tag{2.12a}$$

$$-(\mathrm{div}\,\mathbf{u}_D, q_D)_D = -(f_D, q_D)_D, \tag{2.12b}$$

for all $(\mathbf{v}_D, q_D) \in \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D) \times L^{3/2}_0(\Omega_D)$. We now proceed to give a precise sense to the parity $\langle\, \mathbf{v}_D\cdot\mathbf{n}, \lambda\,\rangle_\Sigma$, and to explain the reason behind the election of the space where λ lives, but first, we will need some previous results.

Lemma 2.1. *Given $p \in]1,2[$ and $q \in]2,+\infty[$ satisfying $p^{-1} + q^{-1} = 1$, the parity $\langle\, \mathbf{v}\cdot\mathbf{n}, \xi\,\rangle_{\partial\Omega_D}$ is well-defined for all $(\mathbf{v},\xi) \in \mathbf{W}^{0,q}_{\Gamma_D}(\mathrm{div};\Omega_D) \times W^{1/q,p}(\partial\Omega_D)$.*

Proof. It is enough to see that $\mathbf{v}\cdot\mathbf{n} \in W^{-1/q,q}(\partial\Omega_D)$, for all $\mathbf{v} \in \mathbf{W}^{0,q}(\mathrm{div};\Omega_D)$. □

Then, under the same ranges of p and q indicated in Lemma 2.1, we now define the parity $\langle\, \mathbf{v}\cdot\mathbf{n}, \xi\,\rangle_\Sigma$ for any $\mathbf{v} \in \mathbf{W}^{0,q}(\mathrm{div};\Omega_D)$ and $\xi \in W^{1/q,p}(\Sigma)$ as

$$\langle\, \mathbf{v}\cdot\mathbf{n}, \xi\,\rangle_\Sigma := \big\langle\, \mathbf{v}\cdot\mathbf{n}, E^D_\Sigma(\xi)\,\big\rangle_{\partial\Omega_D}, \tag{2.13}$$

where $E^D_\Sigma(\xi)$ is the continuation of ξ by zero on $\partial\Omega_D\backslash\Sigma = \Gamma_D$ defined in (2.1). This parity is indeed well-defined since [28, Theorem 1.5.2.3] guarantees that $\widetilde{W}^{1/q,p}(\Sigma) = W^{1/q,p}(\Sigma)$ (cf. (2.2) with $\Gamma_0 = \Sigma$ and $\bigstar = D$), that is $E^D_\Sigma(\xi) \in W^{1/q,p}(\partial\Omega_D)$ for all $\xi \in W^{1/q,p}(\Sigma)$, thus fulfilling the hypotheses of the preceding Lemma. In this way, we can properly define the condition $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ_D. Indeed, for $\mathbf{v} \in \mathbf{W}^{0,q}(\mathrm{div};\Omega_D)$ this condition is understood in the sense that

$$\big\langle\, \mathbf{v}\cdot\mathbf{n}, E^D_{\Gamma_D}(\xi)\,\big\rangle_{\partial\Omega_D} = 0 \quad \forall\, \xi \in \widetilde{W}^{1/q,p}(\Gamma_D),$$

which, in accordance to the notation and foregoing analysis, but now with $\Gamma_0 = \Gamma_D$ and $\bigstar = D$ in (2.2), means

$$\langle\, \mathbf{v}\cdot\mathbf{n}, \xi\,\rangle_{\Gamma_D} = 0 \quad \forall\, \xi \in W^{1/q,p}(\Gamma_D)\,.$$

2.2.3 Transmission conditions in weak form

Although we have considered the same transmission conditions as in [17, 25], it is necessary to clarify in what sense they will be imposed, since the corresponding spaces are different. Hence, we first test the conservation of mass condition (2.7a) with an arbitrary function $\xi \in W^{1/3,3/2}(\Sigma)$, to then test the traction constraint (2.7b) with an arbitrary $\boldsymbol{\psi} \in \widehat{\mathbf{H}}_{00}^{1/2}(\Sigma)$, which yields the following weak form of the transmission conditions on Σ:

$$\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma + \langle \mathbf{u}_D \cdot \mathbf{n}, \xi \rangle_\Sigma = 0 \quad \forall\, \xi \in W^{1/3,3/2}(\Sigma), \tag{2.14a}$$

$$\langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma - \mu\kappa^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\varphi} \cdot \mathbf{t} \rangle_\Sigma = 0 \quad \forall\, \boldsymbol{\psi} \in \widehat{\mathbf{H}}_{00}^{1/2}(\Sigma). \tag{2.14b}$$

We show next that the duality pairings in the first equation are indeed well-defined.

Lemma 2.2. *The duality pairing $\langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma$ is well-defined for any $(\boldsymbol{\psi}, \lambda) \in \mathbf{H}_{00}^{1/2}(\Sigma) \times W^{1/3,3/2}(\Sigma)$.*

Proof. According to the trace theorem, given $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, there exists $\widehat{\boldsymbol{\psi}} \in \mathbf{W}_{\Gamma_S}^{1,2}(\Omega_S)$ and $C_0 > 0$ such that

$$\gamma_0(\widehat{\boldsymbol{\psi}})\big|_\Sigma = \boldsymbol{\psi} \quad \text{and} \quad \|\widehat{\boldsymbol{\psi}}\|_{1,2;\Omega_S} \le C_0 \, \|\boldsymbol{\psi}\|_{1/2,2,\Sigma} \,.$$

On the other hand, since $\mathbf{W}^{1,2}(\Omega_S)$ is continuously embedded into $\mathbf{L}^p(\Omega_S)$ (with boundedness constant C_i) for $p > 2$ (cf. [1, Theorem 5.4 (6)]), we have that $\widehat{\boldsymbol{\psi}} \in \mathbf{L}^p(\Omega_S)$, which together with the fact that $\operatorname{div} \widehat{\boldsymbol{\psi}} \in L^2(\Omega_S)$, yields in a similar way to [30, Lemma 3.15] that

$$\begin{aligned}\langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma := \int_\Sigma (\widehat{\boldsymbol{\psi}} \cdot \mathbf{n}) \lambda \, &\le \, C \left\{ \|\widehat{\boldsymbol{\psi}}\|_{0,3;\Omega_S} \, + \, \|\operatorname{div} \widehat{\boldsymbol{\psi}}\|_{0,2;\Omega_S} \right\} \|\lambda\|_{1/3,3/2;\Sigma} \\ &\le \, C\,(C_i + 1)\, \|\widehat{\boldsymbol{\psi}}\|_{1,2;\Omega_S} \, \|\lambda\|_{1/3,3/2;\Sigma} \, \le \, C\,(C_i+1)\, C_0 \, \|\boldsymbol{\psi}\|_{1/2,2;\Sigma} \, \|\lambda\|_{1/3,3/2;\Sigma} \,.\end{aligned}$$

□

2.2.4 Resulting variational formulation

Following [23, 25], we put the weak forms of the Stokes, Darcy-Forchheimer and transmission conditions (that is, (2.9), (2.12) and (2.14)) together to form a nonlinear system with a twofold saddle point structure. In this way, denoting by

$$\begin{gathered}\vec{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_S, \mathbf{u}_D), \quad \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D), \quad \vec{\boldsymbol{\zeta}} = (\boldsymbol{\zeta}_S, \mathbf{w}_D), \quad \vec{\mathbf{u}} := (\boldsymbol{\varphi}, \lambda)\,, \\ \vec{\mathbf{v}} := (\boldsymbol{\psi}, \xi), \quad \vec{\mathbf{w}} := (\boldsymbol{\phi}, \boldsymbol{\rho}), \quad \vec{\boldsymbol{\gamma}} := (\mathbf{u}_S, p_D, \boldsymbol{\gamma}_S), \quad \vec{\boldsymbol{\eta}} := (\mathbf{v}_S, q_D, \boldsymbol{\eta}_S)\,,\end{gathered} \tag{2.15}$$

and

$$\begin{gathered}X := \mathbb{H}_0(\mathbf{div}; \Omega_S) \times \mathbf{W}_{\Gamma_D}^{0,3}(\operatorname{div}; \Omega_D), \quad Y := \widehat{\mathbf{H}}_{00}^{1/2}(\Sigma) \times W^{1/3,3/2}(\Sigma), \\ Z := \mathbf{L}^2(\Omega_S) \times L_0^{3/2}(\Omega_D) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega_S),\end{gathered}$$

the fully-mixed formulation of the SDF problem is given by: Find $(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{\gamma}}) \in X \times Y \times Z$ such that

$$\big[\mathbb{A}(\vec{\boldsymbol{\sigma}}), \vec{\boldsymbol{\tau}}\big] + \big[\mathbb{B}_1(\vec{\boldsymbol{\tau}}), \vec{\mathbf{u}}\big] + \big[\mathbb{B}(\vec{\boldsymbol{\tau}}), \vec{\boldsymbol{\gamma}}\big] = \big[F, \vec{\boldsymbol{\tau}}\big], \tag{2.16a}$$

$$\big[\mathbb{B}_1(\vec{\boldsymbol{\sigma}}), \vec{\mathbf{v}}\big] - \big[\mathbb{C}(\vec{\mathbf{u}}), \vec{\mathbf{v}}\big] = \big[G_1, \vec{\mathbf{v}}\big], \tag{2.16b}$$

$$\big[\mathbb{B}(\vec{\boldsymbol{\sigma}}), \vec{\boldsymbol{\eta}}\big] = \big[G, \vec{\boldsymbol{\eta}}\big], \tag{2.16c}$$

for all $(\vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{\eta}}) \in X \times Y \times Z$, where the nonlinear operator $\mathbb{A} : X \to X'$ and the linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$ and $\mathbb{C} : Y \to Y'$ are defined as

$$\big[\mathbb{A}(\vec{\boldsymbol{\zeta}}), \vec{\boldsymbol{\tau}}\big] := (\mathcal{A}_S(\boldsymbol{\zeta}_S), \boldsymbol{\tau}_S)_S + (\mathcal{A}_D(\mathbf{w}_D), \mathbf{v}_D)_D \qquad \forall\, \vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\tau}} \in X, \tag{2.17}$$

$$\big[\mathbb{B}(\vec{\boldsymbol{\tau}}), \vec{\boldsymbol{\eta}}\big] := (\mathbf{div}\, \boldsymbol{\tau}_S, \mathbf{v}_S)_S + (\boldsymbol{\tau}_S, \boldsymbol{\eta}_S)_S - (\operatorname{div} \mathbf{v}_D, q_D)_D \qquad \forall\, \vec{\boldsymbol{\tau}} \in X,\ \vec{\boldsymbol{\eta}} \in Z, \tag{2.18}$$

$$\big[\mathbb{B}_1(\vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}\big] := \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma \qquad \forall\, \vec{\boldsymbol{\tau}} \in X,\ \vec{\mathbf{v}} \in Y, \tag{2.19}$$

$$\big[\mathbb{C}(\vec{\mathbf{w}}), \vec{\mathbf{v}}\big] := \mu\kappa^{-1} \langle \boldsymbol{\psi} \cdot \mathbf{t}, \boldsymbol{\phi} \cdot \mathbf{t} \rangle_\Sigma - \langle \boldsymbol{\psi} \cdot \mathbf{n}, \boldsymbol{\rho} \rangle_\Sigma + \langle \boldsymbol{\phi} \cdot \mathbf{n}, \xi \rangle_\Sigma \qquad \forall\, \vec{\mathbf{w}}, \vec{\mathbf{v}} \in Y, \tag{2.20}$$

and the functionals $F \in X'$, $G_1 \in Y'$ and $G \in Z'$ are given by:

$$\big[F, \vec{\boldsymbol{\tau}} \big] := (\mathbf{g}_D, \mathbf{v}_D)_D \qquad \forall\, \vec{\boldsymbol{\tau}} \in X, \tag{2.21}$$

$$\big[G_1, \vec{\mathbf{v}} \big] := 0 \qquad \forall\, \vec{\mathbf{v}} \in Y, \tag{2.22}$$

$$\big[G, \vec{\boldsymbol{\eta}} \big] := -(\mathbf{f}_S, \mathbf{v}_S)_S - (f_D, q_D)_D \quad \forall\, \vec{\boldsymbol{\eta}} \in Z. \tag{2.23}$$

Notice that $\mathbb{B}$ and $\mathbb{B}_1$ show a block-diagonal structure, and that $\mathbb{C}$ is positive semi-definite. In the next section, we develop an extension of the Babuška-Brezzi theory to twofold saddle point problems of the form given by (2.16), which will allow us to conveniently analyze that problem.

3 A modified abstract theory for a twofold saddle point problem

3.1 The continuous setting

Let X, Y and Z be separable and reflexive Banach spaces with duals X', Y' and Z' also separable and reflexive Banach spaces. Additionally, consider bounded linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$, $\mathbb{C} : Y \to Y'$ (with $\mathbb{C}$ assumed to be positive semi-definite) and a nonlinear operator $\mathbb{A} : X \to X'$. Hence, given $(F, G_1, G) \in X' \times Y' \times Z'$, we are interested in the solvability of the following variational problem: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X \times Y \times Z$ such that

$$[\,\mathbb{A}(\mathbf{t}), \mathbf{s}\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}\,] + [\,\mathbb{B}^*(\mathbf{u}), \mathbf{s}\,] = [\,F, \mathbf{s}\,], \tag{3.1a}$$

$$[\,\mathbb{B}_1(\mathbf{t}), \boldsymbol{\tau}\,] - [\,\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}\,] = [\,G_1, \boldsymbol{\tau}\,], \tag{3.1b}$$

$$[\,\mathbb{B}(\mathbf{t}), \mathbf{v}\,] = [\,G, \mathbf{v}\,], \tag{3.1c}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in X \times Y \times Z$. In what follows, we adapt the analysis developed in [22, 23] to derive sufficient conditions under which (3.1) is well-posed. We first let $\mathbb{V}$ be the kernel of $\mathbb{B}$, and observe that in order to guarantee the existence of a unique preimage $\mathbf{t}_G \in X$ such that $\mathbb{B}(\mathbf{t}_G) = G$ and $\|\mathbf{t}_G\|_X = \|[\mathbf{t}_G]\|_{X/\mathbb{V}}$ (where $[\mathbf{t}_G]$ stands for the equivalence class of $\mathbf{t}_G$ in the quotient space $X/\mathbb{V}$), we require X to be uniformly convex (cf. [32, Remark A.1]). Therefore, from now on we assume that:

(i) X is uniformly convex.

(ii) $\mathbb{B} : X \to Z'$ is surjective, that is, there exists $\beta > 0$ such that

$$\sup_{\substack{\mathbf{s}\in X \\ \mathbf{s}\neq \mathbf{0}}} \frac{[\,\mathbb{B}^*(\mathbf{v}), \mathbf{s}\,]}{\|\,\mathbf{s}\,\|_X} \geq \beta \,\|\,\mathbf{v}\,\|_Z \quad \forall\, \mathbf{v} \in Z.$$

This condition is called "inf-sup condition" for $\mathbb{B}$. Note that this gives an upper bound β^{-1} for the norm of the pseudoinverse of $\mathbb{B}$.

As a consequence of these assumptions, we first observe that $\mathbb{B}$ has a continuous pseudoinverse $\widetilde{\mathbb{B}}^{-1}$ (cf. [34, Lemme 1.3 B.]). In addition, from the inf-sup condition for $\mathbb{B}$ we conclude that $\mathbb{B}^*$ is injective, and hence bijective onto $\mathcal{R}(\mathbb{B}^*) = {}^\circ\mathbb{V}$. Since $\mathbb{V}$ is also uniformly convex, the third row of (3.1) is always satisfied, and therefore it is possible to analyze the same problem with one less variable, as shown in the following results.

Lemma 3.1. *Under the assumptions* (i) *and* (ii)*, the following problems are equivalent:*

$$(P)\begin{cases} \textit{Find } (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X \times Y \times Z \textit{ such that} \\ [\,\mathbb{A}(\mathbf{t}), \mathbf{s}\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}\,] + [\,\mathbb{B}^*(\mathbf{u}), \mathbf{s}\,] = [\,F, \mathbf{s}\,], \\ \textit{for all } \mathbf{s} \in X. \end{cases} \qquad (\widetilde{P})\begin{cases} \textit{Find } (\mathbf{t}, \boldsymbol{\sigma}) \in X \times Y \textit{ such that} \\ [\,\mathbb{A}(\mathbf{t}), \mathbf{s}_0\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\,] = [\,F, \mathbf{s}_0\,], \\ \textit{for all } \mathbf{s}_0 \in \mathbb{V} = \mathcal{R}(\mathbb{B}^*)^\circ. \end{cases}$$

More precisely, if $(\mathbf{t},\boldsymbol{\sigma}) \in X \times Y$ *is a solution for* $(\widetilde{P})$ *and we define* $\mathbf{u} \in Z$ *as the unique solution of the problem: Find* $\mathbf{u} \in Z$ *such that*

$$[\,\mathbb{B}^*(\mathbf{u}),\mathbf{s}\,] = [\,F - (\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})),\mathbf{s}\,] \quad \forall\, \mathbf{s} \in X, \tag{3.2}$$

then $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ *is a solution for* (P)*. Conversely, if* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X \times Y \times Z$ *is a solution of* (P)*, then* $(\mathbf{t},\boldsymbol{\sigma})$ *is a solution of* $(\widetilde{P})$ *and* $\mathbf{u}$ *solves* (3.2)*.*

Proof. Given $(\mathbf{t},\boldsymbol{\sigma}) \in X \times Y$ solution of $(\widetilde{P})$, we first show that (3.2) has a unique solution. Indeed, if $(\mathbf{t},\boldsymbol{\sigma})$ solves $(\widetilde{P})$, then $F - (\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})) \in {}^\circ\mathbb{V} = \mathcal{R}(\mathbb{B}^*)$, and therefore there exists a unique $\mathbf{u} \in Z$ such that such that $\mathbb{B}^*(\mathbf{u}) = F - (\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma}))$, which is equivalent to $\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma}) + \mathbb{B}^*(\mathbf{u}) = F$, i.e., $\mathbf{u}$ is a solution to (3.2) and $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ solves (P). The converse implication follows by taking, in particular, $\mathbf{s} \in \mathbb{V} = \mathcal{R}(\mathbb{B}^*)^\circ$, and by using that, given $(\mathbf{t},\boldsymbol{\sigma}) \in X \times Y$, there exists a unique $\mathbf{u} \in Z$ solution to (3.2). □

Lemma 3.2. *Under the assumptions* (i) *and* (ii)*, problem* (3.1) *is equivalent to: Given* $\mathbf{t}_G \in X$ *such that* $\mathbb{B}(\mathbf{t}_G) = G$*, find* $(\mathbf{t}_0,\boldsymbol{\sigma}) \in \mathbb{V} \times Y$ *such that*

$$[\,\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G),\mathbf{s}_0\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}),\mathbf{s}_0\,] = [\,F,\mathbf{s}_0\,], \tag{3.3a}$$

$$[\,\mathbb{B}_1(\mathbf{t}_0 + \mathbf{t}_G),\boldsymbol{\tau}\,] - [\,\mathbb{C}(\boldsymbol{\sigma}),\boldsymbol{\tau}\,] = [\,G_1,\boldsymbol{\tau}\,], \tag{3.3b}$$

for all $(\mathbf{s}_0,\boldsymbol{\tau}) \in \mathbb{V} \times Y$*. More precisely, if* $(\mathbf{t}_0,\boldsymbol{\sigma})$ *is a solution of* (3.3)*, and* $\mathbf{u}$ *is a solution to* (3.2) *with* $\mathbf{t} = \mathbf{t}_G + \mathbf{t}_0$ *and* $\boldsymbol{\sigma}$*, then* $(\mathbf{t}_G + \mathbf{t}_0,\boldsymbol{\sigma},\mathbf{u})$ *solves* (3.1)*. Conversely, if* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X \times Y \times Z$ *is a solution to* (3.1)*, then* $(\mathbf{t} - \mathbf{t}_G,\boldsymbol{\sigma})$ *(with* $\mathbf{t}_G \in X$ *such that* $\mathbb{B}(\mathbf{t}_G) = G$*) solves* (3.3)*.*

Proof. Given $\mathbf{t}_G \in X$ such that $\mathbb{B}(\mathbf{t}_G) = G$ (whose existence is guaranteed by the surjectivity of $\mathbb{B}$), $(\mathbf{t}_0,\boldsymbol{\sigma}) \in \mathbb{V} \times Y$ solution to (3.3) and $\mathbf{u}$ solution to (3.2) with $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_G$, then using that $(\widetilde{P}) \Rightarrow (P)$ in Lemma 3.1, we see that $(\mathbf{t}_0 + \mathbf{t}_G,\boldsymbol{\sigma},\mathbf{u})$ indeed satisfies (3.1). The converse implication holds trivially by taking, in particular, $\mathbf{s} \in \mathbb{V} = \mathcal{R}(\mathbb{B}^*)^\circ$ in (3.1a). □

Lemma 3.3. *Under the assumptions* (i) *and* (ii)*, problem* (3.1) *has a unique solution if and only if* (3.3) *has a unique solution too.*

Proof. It follows from Lemma 3.2 by noticing that given $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ a solution of (3.1), we let $\mathbf{t}_0 := \mathbf{t} - \mathbf{t}_G$, with $\mathbf{t}_G$ being the unique element in X such that $\mathbb{B}(\mathbf{t}_G) = G$ and $\|\mathbf{t}_G\|_X = \|[\mathbf{t}_G]\|_{X/\mathbb{V}}$ (which follows from (i) and (ii)), and we see that $(\mathbf{t}_0,\boldsymbol{\sigma})$ indeed solves (3.3). □

According to the analysis above, our next goal is to study the solvability of (3.3), for which we adapt the approach from [23]. Hence, we first need to show the well-posedness of the problem: given $\mathbf{t}_G \in X$ and $\boldsymbol{\tau} \in Y$, find $\mathbf{t}_0 \in \mathbb{V}$ such that

$$[\,\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G),\mathbf{s}_0\,] = [\,F - \mathbb{B}_1^*(\boldsymbol{\tau}),\mathbf{s}_0\,] \quad \forall\, \mathbf{s}_0 \in \mathbb{V}. \tag{3.4}$$

To this end, under some additional hypotheses we are going to prove that, given $\mathbf{t}_G$ and $\boldsymbol{\tau}$, there exists a unique $\mathbf{t}_0 \in \mathbb{V}$ solution to the previous problem, i.e., $\mathbb{A}(\cdot + \mathbf{t}_G) : \mathbb{V} \to \mathbb{V}'$ is bijective, and also that, if we consider $\boldsymbol{\tau}_1 \neq \boldsymbol{\tau}_2$, then $F - \mathbb{B}_1^*(\boldsymbol{\tau}_1) \neq F - \mathbb{B}_1^*(\boldsymbol{\tau}_2)$.

Henceforth, and motivated by the subsequent application to the model introduced in Section 2, we assume that $X = X_1 \times X_2$, where X_1 and X_2, as well as its duals X_1' and X_2', are all separable and reflexive Banach spaces. Nevertheless, we stress in advance that the forthcoming analysis can be easily adapted to the case of a product space $X_1 \times X_2 \times \cdots \times X_N$, with $N \in \mathbb{N}$, which certainly includes the particular case of a single space X. According to the above, we now introduce the following assumptions:

(A_0) X_1 and X_2 are uniformly convex and separable Banach spaces,

(A_1) there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{s}_0 \in \mathbb{V} \\ \mathbf{s}_0 \neq \mathbf{0}}} \frac{[\mathbb{B}_1^*(\boldsymbol{\tau}), \mathbf{s}_0\,]}{\|\,\mathbf{s}_0\,\|_X} \,\geq\, \beta_1 \,\|\,\boldsymbol{\tau}\,\|_Y \quad \forall\, \boldsymbol{\tau} \in Y\,, \tag{3.5}$$

(A_2) there exist $\gamma_1,\, \gamma_2 > 0$, $\varsigma_1,\, \varsigma_2 > 0$ and $p_1,\, p_2 \geq 2$, such that

$$\|\,\mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{r})\,\|_{X'} \,\leq\, \sum_{j=1}^{2} \left\{ \varsigma_j \|\, s_j - r_j\,\|_{X_j} + \gamma_j \|\, s_j - r_j\,\|_{X_j} \Big(\|\, s_j\,\|_{X_j} + \|\, r_j\,\|_{X_j} \Big)^{p_j - 2} \right\},$$

for all $\mathbf{s} := (s_1, s_2)$, $\mathbf{r} := (r_1, r_2) \in X := X_1 \times X_2$, and

(A_3) for each $\mathbf{t}_G \in X$, $\mathbb{A}(\cdot + \mathbf{t}_G) : \mathbb{V} \to \mathbb{V}'$ is a uniformly strictly monotone mapping. More precisely, there exists $\alpha > 0$, independent of $\mathbf{t}_G$, such that

$$[\,\mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{r} + \mathbf{t}_G), \mathbf{s} - \mathbf{r}\,] \,\geq\, \alpha \Big\{ \|\, s_1 - r_1\,\|_{X_1}^{p_1} + \|\, s_2 - r_2\,\|_{X_2}^{p_2} \Big\}$$

for all $\mathbf{s} = (s_1, s_2), \mathbf{r} = (r_1, r_2) \in \mathbb{V} \subseteq X$.

As a direct consequence of (A_2)-(A_3), for each $\mathbf{t}_G \in X$ the operator $\mathbb{A}(\cdot + \mathbf{t}_G) : \mathbb{V} \to \mathbb{V}'$ is bijective (cf. [35]), and therefore, given $\boldsymbol{\tau} \in Y$, there exists a unique $\mathbf{t}_0 \in \mathbb{V}$ solution of (3.4). From this fact, for each $\boldsymbol{\tau} \in Y$ it is now possible to define $\mathbf{t}_0(\boldsymbol{\tau})$ as the unique element in $\mathbb{V}$ such that

$$[\,\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G), \mathbf{s}_0\,] = [\,F - \mathbb{B}_1^*(\boldsymbol{\tau}), \mathbf{s}_0\,] \quad \forall\, \mathbf{s}_0 \in \mathbb{V}. \tag{3.6}$$

Notice from this identity that for any $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$ there holds

$$[\,\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{s}_0\,] = [\,\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s}_0\,] \quad \forall\, \mathbf{s}_0 \in \mathbb{V}. \tag{3.7}$$

In particular, choosing $\mathbf{s}_0 = \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)$, we get

$$[\,\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\,] = [\,\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1, \mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2))\,]. \tag{3.8}$$

In this way, we first prove that a different choice of $\boldsymbol{\tau} \in Y$ in (3.6) will give rise to a different solution $\mathbf{t}_0(\boldsymbol{\tau}) \in \mathbb{V}$, and then we show an upper bound for $\|\,\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\,\|_X$ in terms of $\|\,\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\,\|$.

Lemma 3.4. *Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$ such that $\mathbf{t}_0(\boldsymbol{\tau}_1) = \mathbf{t}_0(\boldsymbol{\tau}_2)$. Then, necessarily $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$.*

Proof. Under the hypotheses on $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, (3.7) becomes

$$[\,\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s}_0\,] = 0 \quad \forall\, \mathbf{s}_0 \in \mathbb{V}\,,$$

which, together with the inf-sup condition (3.5), implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. □

Lemma 3.5. *There holds*

$$\begin{aligned} &\|\,\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\,\|_X \\ &\quad \leq\, 2\,\max\left\{ \left(\frac{2}{\alpha}\|\,\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\,\|\right)^{1/(p_1-1)}, \left(\frac{2}{\alpha}\|\,\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\,\|\right)^{1/(p_2-1)} \right\} \end{aligned} \tag{3.9}$$

for all $\boldsymbol{\tau}_1,\, \boldsymbol{\tau}_2 \in Y$.

Proof. Recalling that $\mathbb{V} \subseteq X = X_1 \times X_2$, we split $\mathbf{t}_0(\boldsymbol{\tau}_1)$, $\mathbf{t}_0(\boldsymbol{\tau}_2) \in \mathbb{V}$ as $\mathbf{t}_0(\boldsymbol{\tau}_1) =: \mathbf{s} = (s_1, s_2)$ and $\mathbf{t}_0(\boldsymbol{\tau}_2) =: \mathbf{r} = (r_1, r_2)$, respectively. Hence

$$\| \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2) \|_X = \| s_1 - r_1 \|_{X_1} + \| s_2 - r_2 \|_{X_2} ,$$

and using (3.8), we find that

$$[\, \mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{r} + \mathbf{t}_G), \mathbf{s} - \mathbf{r} \,] = [\, \mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s} - \mathbf{r} \,] \leq \| \mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1) \| \| \mathbf{s} - \mathbf{r} \| .$$

On the other hand, the outermost left hand side of the previous expression can be bounded below using the strict monotonicity of $\mathbb{A}$ (cf. (A$_3$)), thus obtaining

$$\alpha \Big\{ \| s_1 - r_1 \|_{X_1}^{p_1} + \| s_2 - r_2 \|_{X_2}^{p_2} \Big\} \leq \| \mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1) \| \Big\{ \| s_1 - r_1 \|_{X_1} + \| s_2 - r_2 \|_{X_2} \Big\} ,$$

which, after simple algebraic manipulations, and separating the cases $\| s_1 - r_1 \|_{X_1} \leq \| s_2 - r_2 \|_{X_2}$ and $\| s_2 - r_2 \|_{X_2} \leq \| s_1 - r_1 \|_{X_1}$, yields

$$\| s_1 - r_1 \|_{X_1} + \| s_2 - r_2 \|_{X_2} \leq 2 \max \left\{ \left(\frac{2}{\alpha} \| \mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1) \| \right)^{1/(p_1 - 1)} , \left(\frac{2}{\alpha} \| \mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1) \| \right)^{1/(p_2 - 1)} \right\} ,$$

thus finishing the proof. □

In light of the foregoing analysis, problem (3.3) is now equivalent to: given $\mathbf{t}_G \in X$ such that $\mathbb{B}(\mathbf{t}_G) = G$, find $\boldsymbol{\sigma} \in Y$ such that

$$[\, \mathbb{T}(\boldsymbol{\sigma}), \boldsymbol{\tau} \,] := [\, -\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\sigma})), \boldsymbol{\tau} \,] + [\, \mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau} \,] = \Big[\widetilde{G}_1, \boldsymbol{\tau} \Big] \quad \forall\, \boldsymbol{\tau} \in Y, \tag{3.10}$$

where $\widetilde{G}_1 := \mathbb{B}_1(\mathbf{t}_G) - G_1$. Therefore, we now focus on proving that the operator $\mathbb{T}$ given by the previous expression is bijective.

Lemma 3.6. *The operator $\mathbb{T}$ defined by (3.10) is injective.*

Proof. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$ such that $\mathbb{T}(\boldsymbol{\tau}_1) = \mathbb{T}(\boldsymbol{\tau}_2)$. it follows that

$$[\, \mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau} \,] = 0 \quad \forall\, \boldsymbol{\tau} \in Y,$$

whence, according to the definition of $\mathbb{T}$, and taking in particular $\boldsymbol{\tau} = \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1$, results in

$$[\, \mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)), \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1 \,] + [\, \mathbb{C}(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2), \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1 \,] = 0.$$

The foregoing equation, and the fact that $\mathbb{C}$ is positive semi-definite, allows us to deduce that

$$0 \leq [\, \mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)), \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1 \,],$$

which, thanks to the identity (3.8), gives

$$[\, \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2) \,] \leq 0.$$

Since $\mathbb{A}(\cdot + \mathbf{t}_G)$ is a strictly monotone mapping (cf. (A$_3$)), we deduce from the previous inequality that $\mathbf{t}_0(\boldsymbol{\tau}_1) = \mathbf{t}_0(\boldsymbol{\tau}_2)$, and by Lemma 3.4, $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. □

Next, we show the surjectivity of $\mathbb{T}$ by means of classical results from nonlinear functional analysis. More precisely, we will see that, under the hypotheses that have been assumed for the solvability of (3.3), the operator $\mathbb{T}$ is continuous and monotone, and therefore, of type M (cf. [35, Lemma 2.1]). Then, by also proving that $\mathbb{T}$ is bounded and coercive [35, Corollary 2.2], the surjectivity of $\mathbb{T}$ is ensured. We recall here that $\mathbb{T}$ is said to be bounded if it transforms bounded sets of Y into bounded sets of Y.

Lemma 3.7. *The operator* $\mathbb{T}$ *defined by* (3.10) *is continuous.*

Proof. Let $\{\boldsymbol{\tau}_n\}_{n=1}^{\infty} \subseteq Y$ and $\boldsymbol{\tau} \in Y$ such that $\| \boldsymbol{\tau}_n - \boldsymbol{\tau} \| \to 0$ as $n \to \infty$. Thus, from the definition of $\mathbb{T}$, we get

$$\begin{aligned} \| \mathbb{T}(\boldsymbol{\tau}_n) - \mathbb{T}(\boldsymbol{\tau}) \|_{Y'} &= \| \mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\boldsymbol{\tau}_n)) + \mathbb{C}(\boldsymbol{\tau}_n - \boldsymbol{\tau}) \|_{Y'} \\ &\leq \| \mathbb{B}_1 \| \, \| \mathbf{t}_0(\boldsymbol{\tau}_n) - \mathbf{t}_0(\boldsymbol{\tau}) \|_X + \| \mathbb{C} \| \, \| \boldsymbol{\tau}_n - \boldsymbol{\tau} \|_Y . \end{aligned}$$

Then, by applying (3.9) in the right-hand side of the previous inequality, we effectively see that $\| \mathbb{T}(\boldsymbol{\tau}_n) - \mathbb{T}(\boldsymbol{\tau}) \|_{Y'} \to 0$ as $n \to \infty$. □

Lemma 3.8. *The operator* $\mathbb{T}$ *defined by* (3.10) *is monotone.*

Proof. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$. According to the definition of $\mathbb{T}$, we have

$$\frac{[\mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2]}{\| \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \|} = \frac{[\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2]}{\| \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \|} + \frac{[\mathbb{C}(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2]}{\| \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \|} .$$

Notice that, since $\mathbb{C}$ is positive semi-definite, we can discard the last term in the previous equation. Then, using the identity (3.8) and the strict monotonicity of $\mathbb{A}$ (cf. (A_3)), we see that

$$\frac{[\mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2]}{\| \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \|} \geq \frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)]}{\| \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \|} \geq 0 ,$$

thus proving that $\mathbb{T}$ is monotone. □

Lemma 3.9. *The operator* $\mathbb{T}$ *defined by* (3.10) *is bounded.*

Proof. Let $\boldsymbol{\tau} \in Y$. According to the triangle inequality and the definition of $\mathbb{T}$, we see that

$$\begin{aligned} \| \mathbb{T}(\boldsymbol{\tau}) \|_{Y'} &\leq \| \mathbb{T}(\boldsymbol{\tau}) - \mathbb{T}(\mathbf{0}) \|_{Y'} + \| \mathbb{T}(\mathbf{0}) \|_{Y'} \\ &\leq \| \mathbb{B}_1 \| \, \| \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0}) \|_X + \| \mathbb{C} \| \, \| \boldsymbol{\tau} \|_Y + \| \mathbb{B}_1(\mathbf{t}_0(\mathbf{0})) \| . \end{aligned}$$

In turn, from (3.9), we have

$$\| \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0}) \|_X \leq 2 \max \left\{ \left(\frac{2}{\alpha} \| \mathbb{B}_1^*(\boldsymbol{\tau}) \| \right)^{1/(p_1 - 1)} , \left(\frac{2}{\alpha} \| \mathbb{B}_1^*(\boldsymbol{\tau}) \| \right)^{1/(p_2 - 1)} \right\} ,$$

and from the foregoing inequalities, we conclude that $\mathbb{T}$ is bounded. □

Lemma 3.10. *The operator* $\mathbb{T}$ *defined by* (3.10) *is coercive.*

Proof. Let $\boldsymbol{\tau} \in Y$. Similarly as in Lemma 3.8, we have

$$\begin{aligned} \frac{[\mathbb{T}(\boldsymbol{\tau}), \boldsymbol{\tau}]}{\| \boldsymbol{\tau} \|} \geq \frac{[-\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau})), \boldsymbol{\tau}]}{\| \boldsymbol{\tau} \|} &= \frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau})]}{\| \boldsymbol{\tau} \|} \\ &= \frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0})]}{\| \boldsymbol{\tau} \|} \\ &\quad + \frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\mathbf{0})]}{\| \boldsymbol{\tau} \|} . \end{aligned} \tag{3.11}$$

Next, we show that (3.11) diverges when $\| \boldsymbol{\tau} \| \to \infty$. In fact, we will prove that the first term on the right-hand side of (3.11) diverges while the second one remains bounded. First, thanks to the inf-sup condition for $\mathbb{B}_1$ (cf. (A_1)), and the identity (3.7), we see that

$$\beta_1 \| \boldsymbol{\tau} \|_Y \leq \| \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G) \| \quad \forall\, \boldsymbol{\tau} \in Y. \tag{3.12}$$

Now, we set $(s_1, s_2) := \mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G$ and $(r_1, r_2) := \mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G$, so that applying the boundedness of $\mathbb{A}$ (cf. (A_2)), the triangle inequality, and the fact that (cf. [1, Lemma 2.2])

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \quad \forall\, a, b \geq 0,\ \forall\, p \geq 1\,, \tag{3.13}$$

we get

$$\begin{aligned} \beta_1 \| \boldsymbol{\tau} \|_Y &\leq \| \mathbb{A}(s_1, s_2) - \mathbb{A}(r_1, r_2) \| \\ &\leq \sum_{j=1}^{2} \left\{ \varsigma_j \| s_j - r_j \|_{X_j} + \gamma_j \| s_j - r_j \|_{X_j} \Big(\| s_j \|_{X_j} + \| r_j \|_{X_j} \Big)^{p_j - 2} \right\} \\ &\leq \sum_{j=1}^{2} \left\{ \varsigma_j \| s_j - r_j \|_{X_j} + 2^{p_j - 3} \gamma_j \| s_j - r_j \|_{X_j}^{p_j - 1} + 2^{p_j - 2} \gamma_j \| s_j - r_j \|_{X_j} \| r_j \|_{X_j}^{p_j - 2} \right\}. \end{aligned} \tag{3.14}$$

Then, it follows that $\| \mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G \| = \| s_1 \|_{X_1} + \| s_2 \|_{X_2} \to \infty$ as $\| \boldsymbol{\tau} \|_Y \to \infty$. In this way, rewriting the first term on the right-hand side of (3.11) in terms of (s_1, s_2) and (r_1, r_2), and applying the strict-monotone property of $\mathbb{A}$ (cf. (A_3)) and the foregoing inequality, we find that

$$\begin{aligned} &\frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0})]}{\| \boldsymbol{\tau} \|} \\ &\quad = \frac{[\mathbb{A}(s_1, s_2) - \mathbb{A}(r_1, r_2), (s_1 - r_1, s_2 - r_2)]}{\| \boldsymbol{\tau} \|} \\ &\quad \geq \frac{\beta_1 \alpha \big(\| s_1 - r_1 \|_{X_1}^{p_1} + \| s_2 - r_2 \|_{X_2}^{p_2} \big)}{\displaystyle\sum_{j=1}^{2} \left\{ \varsigma_j \| s_j - r_j \|_{X_j} + 2^{p_j - 3} \gamma_j \| s_j - r_j \|_{X_j}^{p_j - 1} + 2^{p_j - 2} \gamma_j \| s_j - r_j \|_{X_j} \| r_j \|_{X_j}^{p_j - 2} \right\}}\,, \end{aligned} \tag{3.15}$$

which tends to infinity as $\| s_1 \|_{X_1} + \| s_2 \|_{X_2} \to \infty$, since (r_1, r_2) is fixed (it does not depend on $\boldsymbol{\tau}$) and $p_1,\, p_2 \geq 2$. On the other hand, for the second term on the right-hand side of (3.11), it suffices to see that

$$\frac{[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\mathbf{0})]}{\| \boldsymbol{\tau} \|} \geq -\beta_1 \| \mathbf{t}_0(\mathbf{0}) \|\,. \tag{3.16}$$

In this way, the coercivity of $\mathbb{T}$ follows from (3.11) and the estimates (3.15) and (3.16). □

It follows straightforwardly from the foregoing analysis that $\mathbb{T}$ is bijective, whence the problem (3.10) has a unique solution $\boldsymbol{\sigma} \in Y$. Thus, according to Lemma 3.3, $(\mathbf{t}_0(\boldsymbol{\sigma}), \boldsymbol{\sigma})$ is the unique solution of (3.3), and thanks to Lemma 3.1, there exists a unique $\mathbf{u} \in Z$ such that $(\mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G, \boldsymbol{\sigma}, \mathbf{u})$ is the unique solution to (3.1). In what follows we prove an *a priori* bound for the solution of (3.3), to then use this result to show an *a priori* bound for (3.1). To this end, and for further use along the paper, we now recall Young's inequality, which says that

$$ab \leq \frac{\delta^p\, a^p}{p} + \frac{b^q}{\delta^q\, q} \qquad \forall\, \delta,\, a,\, b > 0, \tag{3.17}$$

with $p,\, q \in]1, +\infty[$ conjugates, that is $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 3.11. *Given $\mathbf{t}_G \in X$, the solution $(\mathbf{t}_0(\boldsymbol{\sigma}), \boldsymbol{\sigma})$ of the problem (3.3) is bounded above by data and $\| \mathbf{t}_G \|$.*

Proof. We begin by noticing, thanks to Lemma 3.5, that it suffices to bound $\| \boldsymbol{\sigma} \|_Y$. Proceeding similarly as in [23, Lemma 2.1], using the fact that $\mathbb{C}$ is positive semi-definite, and according to the notation introduced in (3.6) and (3.10), we have

$$[\,\mathbb{A}(\mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\sigma}) - \mathbf{t}_0(\mathbf{0})\,] \le [\,\mathbb{T}(\boldsymbol{\sigma}) - \mathbb{T}(\mathbf{0}), \boldsymbol{\sigma}\,] = \Big[\,\widetilde{G}_1 - \mathbb{T}(\mathbf{0}), \boldsymbol{\sigma}\,\Big],$$

where $\widetilde{G}_1 = \mathbb{B}_1(\mathbf{t}_G) - G_1$. Now, we set again $(s_1, s_2) := \mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G$ and $(r_1, r_2) := \mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G$, and then, according to the strict-monotone property of $\mathbb{A}$ (cf. (A_3)), we obtain

$$\alpha\Big(\| s_1 - r_1 \|_{X_1}^{p_1} + \| s_2 - r_2 \|_{X_2}^{p_2}\Big) \le \|\widetilde{G}_1 - \mathbb{T}(\mathbf{0})\| \, \| \boldsymbol{\sigma} \|_Y ,$$

from which we deduce that

$$\| s_1 - r_1 \|_{X_1} \le \mathcal{M}^{1/p_1} \| \boldsymbol{\sigma} \|_Y^{1/p_1} \quad \text{and} \quad \| s_2 - r_2 \|_{X_2} \le \mathcal{M}^{1/p_2} \| \boldsymbol{\sigma} \|_Y^{1/p_2} , \tag{3.18}$$

where $\mathcal{M} := \alpha^{-1}\|\widetilde{G}_1 - \mathbb{T}(\mathbf{0})\|$. On the other hand, using (3.12) and (3.14) we find that

$$\begin{aligned}\beta_1 \| \boldsymbol{\sigma} \|_Y &\le \| \mathbb{A}(\mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G) \| = \| \mathbb{A}(s_1, s_2) - \mathbb{A}(r_1, r_2) \| \\ &\le \sum_{j=1}^{2} \Big\{ \varsigma_j \| s_j - r_j \|_{X_j} + 2^{p_j - 3} \gamma_j \| s_j - r_j \|_{X_j}^{p_j - 1} + 2^{p_j - 2} \gamma_j \| s_j - r_j \|_{X_j} \| r_j \|_{X_j}^{p_j - 2} \Big\} ,\end{aligned}$$

which, using (3.18), results in

$$\beta_1 \| \boldsymbol{\sigma} \|_Y \le \sum_{j=1}^{2} \Big\{ \Big(\varsigma_j \mathcal{M}^{1/p_j} + 2^{p_j - 2} \gamma_j \| r_j \|_{X_j}^{p_j - 2} \Big) \| \boldsymbol{\sigma} \|_Y^{1/p_j} + \Big(2^{p_j - 3} \gamma_j \, \mathcal{M}^{(p_j - 1)/p_j} \Big) \| \boldsymbol{\sigma} \|_Y^{(p_j - 1)/p_j} \Big\} .$$

Then, suitably applying Young's inequality (3.17) to each one of the products within the sum in the foregoing inequality, and denoting by p_j' the conjugate of p_j, that is $p_j' := p_j/(p_j - 1)$, we conclude that for each $j \in \{1, 2\}$ there exist positive constants c_j and $\widehat{c}_j$, depending on p_j and β_1, such that

$$\| \boldsymbol{\sigma} \|_Y \le \sum_{j=1}^{2} \Big\{ c_j \, \Big(\varsigma_j \mathcal{M}^{1/p_j} + 2^{p_j - 2} \gamma_j \| r_j \|_{X_j}^{p_j - 2} \Big)^{p_j'} + \widehat{c}_j \, \mathcal{M}^{p_j - 1} \Big\} ,$$

which ends the proof. □

Lemma 3.12. *The solution of (3.1) is bounded above by data.*

Proof. Let $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ be the solution to (3.1). We know that $\mathbf{t} = \mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G$, where $(\mathbf{t}_0(\boldsymbol{\sigma}), \boldsymbol{\sigma})$ is the unique solution of (3.3) and $\mathbf{t}_G$ is the unique element in X such that $\mathbb{B}(\mathbf{t}_G) = G$ and $\|\mathbf{t}_G\|_X = \|[\mathbf{t}_G]\|_{X/\mathbb{V}}$. In addition, according to the inf-sup condition for $\mathbb{B}$ (cf. (i)), we have

$$\| \mathbf{t}_G \|_X \le \frac{1}{\beta} \| G \| \quad \text{and} \quad \| \mathbf{u} \|_Y \le \frac{1}{\beta} \| F - \mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G) - \mathbb{B}_1^*(\boldsymbol{\sigma}) \|_{X'} .$$

Then, the boundedness of the solution of (3.1) follows from the boundedness of the solution of (3.3). □

We summarize the preceding analysis in the following theorem, whose proof is immediate from the recently exposed results.

Theorem 3.13. *Let X_1, X_2, Y and Z be separable and reflexive Banach spaces, and let $X := X_1 \times X_2$. In addition, let $\mathbb{A} : X \to X'$ be a nonlinear operator, and let $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$, and $\mathbb{C} : Y \to Y'$ be linear operators, such that $\mathbb{C}$ is positive semi-definite. Then, given $(F, G_1, G) \in X' \times Y' \times Z'$, consider the variational problem: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X \times Y \times Z$ such that*

$$
\begin{aligned}
[\,\mathbb{A}(\mathbf{t}), \mathbf{s}\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}\,] + [\,\mathbb{B}^*(\mathbf{u}), \mathbf{s}\,] &= [\,F, \mathbf{s}\,] \quad && \forall\, \mathbf{s} \in X, && (3.19a)\\
[\,\mathbb{B}_1(\mathbf{t}), \boldsymbol{\tau}\,] - [\,\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}\,] &= [\,G_1, \boldsymbol{\tau}\,] \quad && \forall\, \boldsymbol{\tau} \in Y, && (3.19b)\\
[\,\mathbb{B}(\mathbf{t}), \mathbf{v}\,] &= [\,G, \mathbf{v}\,] \quad && \forall\, \mathbf{v} \in Z\,. && (3.19c)
\end{aligned}
$$

Additionally, assume that

i) *X_1 and X_2 are uniformly convex,*

ii) *there exists $\beta > 0$ such that*

$$\sup_{\substack{\mathbf{s} \in X \\ \mathbf{s} \neq \mathbf{0}}} \frac{[\,\mathbb{B}^*(\mathbf{v}), \mathbf{s}\,]}{\|\, \mathbf{s}\,\|_X} \,\ge\, \beta \|\, \mathbf{v}\,\|_Z \quad \forall\, \mathbf{v} \in Z\,, \tag{3.20}$$

iii) *there exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\mathbf{s}_0 \in \mathbb{V} \\ \mathbf{s}_0 \neq \mathbf{0}}} \frac{[\,\mathbb{B}_1^*(\boldsymbol{\tau}), \mathbf{s}_0\,]}{\|\, \mathbf{s}_0\,\|_X} \,\ge\, \beta_1 \,\|\, \boldsymbol{\tau}\,\|_Y \quad \forall\, \boldsymbol{\tau} \in Y\,, \tag{3.21}$$

iv) *there exists constants γ_1, $\gamma_2 > 0$, ς_1, $\varsigma_2 > 0$ and p_1, $p_2 \ge 2$, such that*

$$\|\, \mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{r})\,\|_{X'} \,\le\, \sum_{j=1}^{2} \left\{ \varsigma_j \,\|\, s_j - r_j\,\|_{X_j} \,+\, \gamma_j \,\|\, s_j - r_j\,\|_{X_j} \left(\|\, s_j\,\|_{X_j} + \|\, r_j\,\|_{X_j} \right)^{p_j - 2} \right\}\,, \tag{3.22}$$

for all $\mathbf{s} := (s_1, s_2)$, $\mathbf{r} := (r_1, r_2) \in X$, and

v) *for each $\mathbf{t}_G \in X$, $\mathbb{A}(\cdot + \mathbf{t}_G) : \mathbb{V} \to \mathbb{V}'$ is a uniformly strictly monotone mapping, that is there exists $\alpha > 0$, independent of $\mathbf{t}_G$, such that*

$$[\,\mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{r} + \mathbf{t}_G), \mathbf{s} - \mathbf{r}\,] \,\ge\, \alpha \left\{ \|\, s_1 - r_1\,\|_{X_1}^{p_1} + \|\, s_2 - r_2\,\|_{X_2}^{p_2} \right\}, \tag{3.23}$$

for all $\mathbf{s} = (s_1, s_2)$, $\mathbf{r} = (r_1, r_2) \in \mathbb{V} \subseteq X$.

Then, the continuous problem (3.19) has a unique solution which is bounded in terms of data.

3.2 The discrete setting

We now consider a conforming finite element method for (3.19), and as usual, we require to impose certain restrictions on the finite-dimensional spaces to be chosen. Hence, let X_1, X_2, Y and Z be separable and reflexive Banach spaces with duals X_1', X_2', Y' and Z', respectively, and let $X := X_1 \times X_2$. Additionally, we consider bounded linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$, $\mathbb{C} : Y \to Y'$ (with $\mathbb{C}$ positive semi-definite) and a non-linear operator $\mathbb{A} : X \to X'$. Then, given $(F, G_1, G) \in X' \times Y' \times Z'$, and $X_{1,h}$, $X_{2,h}$, Y_h and Z_h finite-dimensional subspaces of X_1, X_2, Y and Z, respectively, let $X_h := X_{1,h} \times X_{2,h}$ and consider the discrete problem: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_h \times Y_h \times Z_h$ such that

$$
\begin{aligned}
&[\,\mathbb{A}(\mathbf{t}_h), \mathbf{s}_h\,] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma}_h), \mathbf{s}_h\,] + [\,\mathbb{B}^*(\mathbf{u}_h), \mathbf{s}_h\,] = [\,F, \mathbf{s}_h\,] \quad \forall\, \mathbf{s}_h \in X_h, && (3.24a)\\
&[\,\mathbb{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h\,] - [\,\mathbb{C}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h\,] = [\,G_1, \boldsymbol{\tau}_h\,] \quad \forall\, \boldsymbol{\tau}_h \in Y_h, && (3.24b)\\
&[\,\mathbb{B}(\mathbf{t}_h), \mathbf{v}_h\,] = [\,G, \mathbf{v}_h\,] \quad \forall\, \mathbf{v}_h \in Z_h. && (3.24c)
\end{aligned}
$$

In order to stablish a well-posedness result for this problem, we need to define the "discrete kernel" of $\mathbb{B}$ as follows

$$
\mathbb{V}_h \,:=\, \Big\{ \mathbf{s}_h \in X_h : \quad [\,\mathbb{B}(\mathbf{s}_h), \mathbf{v}_h\,] = 0 \quad \forall\, \mathbf{v}_h \in Z_h \Big\}. \tag{3.25}
$$

Then, an immediate application of Theorem 3.13 to the present discrete setting gives us the desired result.

Theorem 3.14. *Assume that*

i) *there exists $\beta_h > 0$ such that*

$$
\sup_{\substack{\mathbf{s}_h \in X_h \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{[\,\mathbb{B}^*(\mathbf{v}_h), \mathbf{s}_h\,]}{\|\,\mathbf{s}_h\,\|_X} \,\geq\, \beta_h \,\|\,\mathbf{v}_h\,\|_Z \quad \forall\, \mathbf{v}_h \in Z_h\,, \tag{3.26}
$$

ii) *there exists $\beta_{1,h} > 0$ such that*

$$
\sup_{\substack{\mathbf{s}_{0,h} \in \mathbb{V}_h \\ \mathbf{s}_{0,h} \neq \mathbf{0}}} \frac{[\,\mathbb{B}_1^*(\boldsymbol{\tau}_h), \mathbf{s}_{0,h}\,]}{\|\,\mathbf{s}_{0,h}\,\|_X} \,\geq\, \beta_{1,h} \,\|\,\boldsymbol{\tau}_h\,\|_Y \quad \forall\, \boldsymbol{\tau}_h \in Y_h\,, \tag{3.27}
$$

iii) *there exist constants γ_1, $\gamma_2 > 0$, ς_1, $\varsigma_2 > 0$ and p_1, $p_2 \geq 2$, such that*

$$
\begin{aligned}
&\|\,\mathbb{A}(\mathbf{s}_h) - \mathbb{A}(\mathbf{r}_h)\,\|_{X_h'} \\
&\quad \leq \sum_{j=1}^{2} \left\{ \varsigma_j \|\,s_{j,h} - r_{j,h}\,\|_{X_j} + \gamma_j \|\,s_{j,h} - r_{j,h}\,\|_{X_j} \Big(\|\,s_{j,h}\,\|_{X_j} + \|\,r_{j,h}\,\|_{X_j} \Big)^{p_j - 2} \right\},
\end{aligned} \tag{3.28}
$$

for all $\mathbf{s}_h := (s_{1,h}, s_{2,h})$, $\mathbf{r}_h := (r_{1,h}, r_{2,h}) \in X_h := X_{1,h} \times X_{2,h}$, and

iv) *there exists $\alpha_h > 0$ such that for each $\mathbf{t}_{G,h} \in X_h$ there holds*

$$
[\,\mathbb{A}(\mathbf{s}_h + \mathbf{t}_{G,h}) - \mathbb{A}(\mathbf{r}_h + \mathbf{t}_{G,h}), \mathbf{s}_h - \mathbf{r}_h\,] \,\geq\, \alpha_h \left\{ \|\,s_{1,h} - r_{1,h}\,\|_{X_1}^{p_1} + \|\,s_{2,h} - r_{2,h}\,\|_{X_2}^{p_2} \right\}, \tag{3.29}
$$

for all $\mathbf{s}_h := (s_{1,h}, s_{2,h})$, $\mathbf{r}_h := (r_{1,h}, r_{2,h}) \in \mathbb{V}_h \subseteq X_{1,h} \times X_{2,h}$.

Then, the discrete problem (3.24) has a unique solution which is bounded in terms of data.

3.3 A priori error estimate

Let $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ and $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)$ be the solutions of the continuous and discrete problems (3.19) and (3.24). Then, at a discrete level, we have

$$
\begin{aligned}
\mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u}_h) &= \mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma}) + \mathbb{B}^*(\mathbf{u}) \quad \text{in } X_h', && (3.30)\\
\mathbb{B}_1(\mathbf{t}_h) - \mathbb{C}(\boldsymbol{\sigma}_h) &= \mathbb{B}_1(\mathbf{t}) - \mathbb{C}(\boldsymbol{\sigma}) \quad \text{in } Y_h', && (3.31)\\
\mathbb{B}(\mathbf{t}_h) &= \mathbb{B}(\mathbf{t}) \quad \text{in } Z_h'. && (3.32)
\end{aligned}
$$

We begin with a preliminary bound for $\|\,\mathbf{u} - \mathbf{u}_h\,\|_Z$.

Lemma 3.15. *There exists $\widetilde{C}_1 > 0$, depending on data, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$ and β_h, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_Z \,\leq\, \widetilde{C}_1 \Big\{ \|\mathbf{t} - \mathbf{t}_h\|_X \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \,+\, \operatorname{dist}(\mathbf{u}, Z_h) \Big\} . \tag{3.33}$$

Proof. According to (3.30), for each $\mathbf{z}_h \in Z_h$ there holds

$$\mathbb{B}^*(\mathbf{u}_h - \mathbf{z}_h) = \mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{z}_h) \quad \text{in} \quad X_h' .$$

Then, using the discrete inf-sup condition for $\mathbb{B}$ (cf. (3.26)), and the boundedness properties of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{B}_1$, we obtain that

$$\begin{aligned} \beta_h \|\mathbf{u}_h - \mathbf{z}_h\|_Z \,&\leq\, \|\mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h)\| + \|\mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| + \|\mathbb{B}^*(\mathbf{u} - \mathbf{z}_h)\| \\ &\leq\, C_A \|\mathbf{t} - \mathbf{t}_h\|_X + \|\mathbb{B}_1^*\| \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y + \|\mathbb{B}^*\| \|\mathbf{u} - \mathbf{z}_h\|_Z , \end{aligned} \tag{3.34}$$

where, according to (3.22), C_A depends on $\|\mathbf{t}\|$ and $\|\mathbf{t}_h\|$, both bounded in terms of data, so that C_A can be replaced by the resulting bound. Next, observing by triangle inequality that

$$\|\mathbf{u} - \mathbf{u}_h\|_Z \,\leq\, \|\mathbf{u} - \mathbf{z}_h\|_Z \,+\, \|\mathbf{u}_h - \mathbf{z}_h\|_Z , \tag{3.35}$$

and taking the infimum over all $\mathbf{z}_h \in Z_h$ in the inequality resulting from the combination of (3.35) and (3.34), we obtain (3.33) and end the proof. □

In a similar way, we present a bound for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y$.

Lemma 3.16. *There exists $\widetilde{C}_2 > 0$, depending on data, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$ and β_h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \,\leq\, \widetilde{C}_2 \Big\{ \|\mathbf{t} - \mathbf{t}_h\|_X \,+\, \operatorname{dist}(\boldsymbol{\sigma}, Y_h) \,+\, \operatorname{dist}(\mathbf{u}, Z_h) \Big\} . \tag{3.36}$$

Proof. According to (3.30), for each $\boldsymbol{\tau}_h \in Y_h$ there holds

$$\mathbb{B}_1^*(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = \mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\tau}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{u}_h) \quad \text{in} \quad X_h' ,$$

which, using that $[\mathbb{B}^*(\mathbf{z}_h - \mathbf{u}_h), \mathbf{s}_{0,h}] = 0 \quad \forall\, (\mathbf{s}_{0,h}, \mathbf{z}_h) \in \mathbb{V}_h \times Z_h$, yields

$$\mathbb{B}_1^*(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = \mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\tau}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{z}_h) \quad \text{in} \quad \mathbb{V}_h' .$$

Then, using the discrete inf-sup condition for $\mathbb{B}_1$ (3.27) and the boundedness properties of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{B}_1$, we obtain

$$\beta_{1,h} \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \,\leq\, C_A \|\mathbf{t} - \mathbf{t}_h\|_X + \|\mathbb{B}_1^*\| \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\| \|\mathbf{u} - \mathbf{z}_h\|_Z , \tag{3.37}$$

which, upon another application of the triangle inequality, implies

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \leq \frac{C_A}{\beta_{1,h}} \|\mathbf{t} - \mathbf{t}_h\|_X + \left(1 + \frac{\|\mathbb{B}_1^*\|}{\beta_{1,h}}\right) \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \frac{\|\mathbb{B}^*\|}{\beta_{1,h}} \|\mathbf{u} - \mathbf{z}_h\|_Z .$$

In this way, taking infimum over all $\boldsymbol{\tau}_h \in Y_h$ and $\mathbf{z}_h \in Z_h$ in the foregoing inequality, we arrive at (3.36) and conclude the proof. □

We now aim to bound $\|\mathbf{t} - \mathbf{t}_h\|_X$, for which we first split each $\mathbf{r}_h \in X_h$ as $\bar{\mathbf{r}}_h + \mathbf{r}_h^\perp$, with $\bar{\mathbf{r}}_h \in \mathbb{V}_h$ and $\mathbf{r}_h^\perp \in \mathbb{V}_h^\perp$, where, given any scalar product $\langle \cdot, \cdot \rangle_{X_h}$ in X_h, we set as usual

$$\mathbb{V}_h^\perp \,:=\, \Big\{ \mathbf{s}_h \in X_h : \quad \langle \mathbf{s}_h, \mathbf{r}_h \rangle_{X_h} = 0 \quad \forall\, \mathbf{r}_h \in \mathbb{V}_h \Big\} . \tag{3.38}$$

In this way, thanks to the triangle inequality, we can write

$$\|\mathbf{t} - \mathbf{t}_h\| \leq \|\mathbf{t} - (\bar{\mathbf{r}}_h + \mathbf{t}_h^\perp)\| + \|\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\| \quad \forall\, \mathbf{r}_h \in X_h . \tag{3.39}$$

Next, we bound $\|\mathbf{t} - (\bar{\mathbf{r}}_h + \mathbf{t}_h^\perp)\|$ independently of $\mathbf{t}_h$. More precisely, we have the following result.

Lemma 3.17. *There exists $\widetilde{C}_3 > 0$, depending on $\|\mathbb{B}\|$ and β_h, such that for each $\mathbf{r}_h \in X_h$ there holds*

$$\|\mathbf{t} - (\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp})\| \,\leq\, \widetilde{C}_3 \,\|\mathbf{t} - \mathbf{r}_h\| \,. \tag{3.40}$$

Proof. We first observe from (3.32) that for each $\mathbf{r}_h \in X_h$ there holds

$$\mathbb{B}(\mathbf{t} - \mathbf{r}_h) = \mathbb{B}(\mathbf{t}_h - \mathbf{r}_h) = \mathbb{B}(\mathbf{t}_h^{\perp} - \mathbf{r}_h^{\perp}) \quad \text{in} \quad Z_h' \,. \tag{3.41}$$

In addition, we recall that the discrete inf-sup condition (3.26) can be stated equivalently as

$$\sup_{\substack{\mathbf{v}_h \in Z_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\,\mathbb{B}(\mathbf{s}_h), \mathbf{v}_h\,]}{\|\,\mathbf{v}_h\,\|_Z} \,\geq\, \beta_h \,\|\,\mathbf{s}_h\,\|_X \quad \forall\, \mathbf{s}_h \in X_h \,. \tag{3.42}$$

Then, applying (3.42) to $\mathbf{s}_h = \mathbf{t}_h^{\perp} - \mathbf{r}_h^{\perp}$, and using (3.41), we get

$$\beta_h \|\mathbf{t}_h^{\perp} - \mathbf{r}_h^{\perp}\| \,\leq\, \|\mathbb{B}(\mathbf{t}_h - \mathbf{r}_h)\| = \|\mathbb{B}(\mathbf{t} - \mathbf{r}_h)\| \,\leq\, \|\,\mathbb{B}\,\|\, \|\mathbf{t} - \mathbf{r}_h\| \quad \forall\, \mathbf{r}_h \in X_h \,,$$

which yields (3.40), with $\widetilde{C}_3 := \left(1 + \dfrac{\|\,\mathbb{B}\,\|}{\beta_h}\right)$, after a simple application of the triangle inequality. □

In order to complete the estimate for $\|\,\mathbf{t} - \mathbf{t}_h\,\|$, it only remains to bound $\|\,\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\,\|$. In fact, denoting $\bar{\mathbf{t}}_h := (\vartheta_1, \vartheta_2)$ and $\bar{\mathbf{r}}_h := (\rho_1, \rho_2)$, we first observe, thanks to the strict monotonicity property of $\mathbb{A}(\cdot + \mathbf{t}_h^{\perp})$ on $\mathbb{V}_h$ (cf. (3.29)) that there holds

$$\begin{aligned} \alpha_h \left\{ \|\vartheta_1 - \rho_1\|_{X_1}^{p_1} + \|\vartheta_2 - \rho_2\|_{X_2}^{p_2} \right\} \,&\leq\, [\mathbb{A}(\bar{\mathbf{t}}_h + \mathbf{t}_h^{\perp}) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h] \\ &=\, [\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h] \,. \end{aligned} \tag{3.43}$$

In turn, using (3.30) we obtain

$$\mathbb{A}(\mathbf{t}_h) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}) \,=\, \mathbb{A}(\mathbf{t}) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{u}_h) \,, \tag{3.44}$$

whence, as suggested by (3.43), in what follows we aim to bound

$$[\mathbb{A}(\mathbf{t}) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h] + [\,\mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\,] + [\,\mathbb{B}^*(\mathbf{u} - \mathbf{u}_h), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\,]. \tag{3.45}$$

The following lemmas provide preliminary bounds for each one of the three terms in (3.45).

Lemma 3.18. *Let $\mathcal{F}_A : \mathrm{R}^+ \to \mathrm{R}^+$ be the function defined by $\mathcal{F}_A(x) = x + x^{p_1 - 1} + x^{p_2 - 1} \;\; \forall\, x \in \mathrm{R}^+$. Then, for each $\mathbf{r}_h \in X_h$ there holds*

$$[\mathbb{A}(\mathbf{t}) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}), \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h] \,\leq\, C_A \,\mathcal{F}_A\big(\|\,\mathbf{t} - \mathbf{r}_h\,\|\big)\, \|\,\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\,\| \,, \tag{3.46}$$

where C_A is a positive constant depending on data, p_1, p_2, $\|\mathbb{B}\|$, and β_h.

Proof. Simple algebraic manipulations based on (3.22) (cf. Theorem 3.13 - iv)), the triangle inequality, and the inequality (3.13) show that for each $\mathbf{s}$, $\mathbf{r} \in X$ there exists a positive constant $\widetilde{C}_A$, depending on $\mathbf{s}$, p_1, and p_2 such that

$$\|\,\mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{r})\,\| \,\leq\, \widetilde{C}_A \left\{ \|\,\mathbf{s} - \mathbf{r}\,\| + \|\,\mathbf{s} - \mathbf{r}\,\|^{p_1 - 1} + \|\,\mathbf{s} - \mathbf{r}\,\|^{p_2 - 1} \right\},$$

that is

$$\|\,\mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{r})\,\| \,\leq\, \widetilde{C}_A \,\mathcal{F}_A\big(\|\,\mathbf{s} - \mathbf{r}\,\|\big) \,,$$

which explains the reason for defining $\mathcal{F}_A$ in that way. Hence, applying the foregoing inequality to $\mathbf{s} = \mathbf{t}$ and $\mathbf{r} = \bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp}$, and then employing the estimate (3.40) and the non-decreasing character of $\mathcal{F}_A$, we find that

$$\|\mathbb{A}(\mathbf{t}) - \mathbb{A}(\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp})\| \,\leq\, \widetilde{C}_A \,\mathcal{F}_A\big(\|\mathbf{t} - (\bar{\mathbf{r}}_h + \mathbf{t}_h^{\perp})\|\big) \,\leq\, C_A \,\mathcal{F}_A\big(\|\mathbf{t} - \mathbf{r}_h\|\big) \,,$$

with C_A as announced, which yields (3.46) and finishes the proof. □

Lemma 3.19. *For each* $\mathbf{r}_h \in X_h$ *there holds*

$$\begin{aligned} [\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,&\le\, \|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\,\|\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h\| \\ &+\,\left\{\|\,\mathbb{B}_1\,\|\left(1+\frac{\|\,\mathbb{B}\,\|}{\beta_h}\right)\|\,\mathbf{t}-\mathbf{r}_h\,\|+\|\mathbb{C}\|\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\right\}\|\,\boldsymbol{\sigma}_h-\boldsymbol{\tau}_h\,\| \quad \forall\,\boldsymbol{\tau}_h\in Y_h\,. \end{aligned} \tag{3.47}$$

Proof. Given $\mathbf{r}_h \in X_h$ and $\boldsymbol{\tau}_h \in Y_h$, we first observe that

$$[\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,=\, [\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\tau}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,+\, [\mathbb{B}_1^*(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h]\,,$$

from which it follows that

$$[\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,\le\, \|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\,\|\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h\| \,+\, [\mathbb{B}_1^*(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h]\,. \tag{3.48}$$

In turn, recalling that $\mathbf{t}_h = \bar{\mathbf{t}}_h + \mathbf{t}_h^{\perp}$, and using (3.31), we deduce that

$$\begin{aligned} [\mathbb{B}_1^*(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,&=\, [\mathbb{B}_1(\mathbf{t}_h-(\bar{\mathbf{r}}_h+\mathbf{t}_h^{\perp})),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h] \\ =\, [\mathbb{B}_1(\mathbf{t}-(\bar{\mathbf{r}}_h+\mathbf{t}_h^{\perp})),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h] \,&-\, [\mathbb{C}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h]\,, \end{aligned} \tag{3.49}$$

whereas the positive semi-definiteness of $\mathbb{C}$ implies that

$$[\mathbb{C}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h] \,\ge\, [\mathbb{C}(\boldsymbol{\sigma}-\boldsymbol{\tau}_h),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h]\,. \tag{3.50}$$

In this way, using now the boundedness of $\mathbb{B}_1$ and $\mathbb{C}$, we deduce from (3.49) and (3.50) that

$$\begin{aligned} [\mathbb{B}_1^*(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,&\le\, [\mathbb{B}_1(\mathbf{t}-(\bar{\mathbf{r}}_h+\mathbf{t}_h^{\perp})),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h] - [\mathbb{C}(\boldsymbol{\sigma}-\boldsymbol{\tau}_h),\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h] \\ &\le\, \Big(\|\,\mathbb{B}_1\,\|\,\|\mathbf{t}-(\bar{\mathbf{r}}_h+\mathbf{t}_h^{\perp})\| \,+\, \|\mathbb{C}\|\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\Big)\,\|\boldsymbol{\sigma}_h-\boldsymbol{\tau}_h\|\,, \end{aligned}$$

which, combined with (3.40) and (3.48), leads to (3.47), thus ending the proof. □

Lemma 3.20. *For each* $\mathbf{r}_h \in X_h$ *there holds*

$$[\mathbb{B}^*(\mathbf{u}-\mathbf{u}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,\le\, \|\mathbb{B}^*\|\,\|\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h\|\,\mathrm{dist}\,(\mathbf{u},Z_h)\,. \tag{3.51}$$

Proof. Given $\mathbf{r}_h \in X_h$ and $\mathbf{z}_h \in Z_h$, we have $[\mathbb{B}^*(\mathbf{z}_h-\mathbf{u}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,=\, [\mathbb{B}(\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h),\mathbf{z}_h-\mathbf{u}_h] \,=\, 0$, and hence

$$[\mathbb{B}^*(\mathbf{u}-\mathbf{u}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,=\, [\mathbb{B}^*(\mathbf{u}-\mathbf{z}_h),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,\le\, \|\mathbb{B}^*\|\,\|\mathbf{u}-\mathbf{z}_h)\|\,\|\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h\|\,,$$

from which the required estimate follows. □

In this way, as a consequence of the previous three lemmas, we conclude the following bound for the expression on the right-hand side of (3.43).

Lemma 3.21. *There exist positive constants* $\widetilde{C}_4$ *(depending on data,* p_1*,* p_2*,* $\|\mathbb{B}\|$*,* $\|\mathbb{B}_1\|$ *and* β_h*) and* $\widetilde{C}_5$ *(depending on* $\|\mathbb{B}\|$*,* $\|\mathbb{B}_1\|$*,* $\|\mathbb{C}\|$ *and* β_h*), such that for each* $\mathbf{r}_h \in X_h$ *there holds*

$$\begin{aligned} [\mathbb{A}(\mathbf{t}_h)-\mathbb{A}(\bar{\mathbf{r}}_h+\mathbf{t}_h^{\perp}),\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h] \,&\le\, \widetilde{C}_4\Big\{\mathcal{M}_1(\mathbf{r}_h,\boldsymbol{\tau}_h) \,+\, \mathrm{dist}\,(\mathbf{u},Z_h)\Big\}\,\|\,\bar{\mathbf{t}}_h-\bar{\mathbf{r}}_h\,\| \\ &+\quad \widetilde{C}_5\,\mathcal{M}_2(\mathbf{r}_h,\boldsymbol{\tau}_h)\,\|\boldsymbol{\sigma}_h-\boldsymbol{\tau}_h\| \qquad \forall\,\boldsymbol{\tau}_h\in Y_h\,, \end{aligned} \tag{3.52}$$

where

$$\mathcal{M}_1(\mathbf{r}_h,\boldsymbol{\tau}_h) := \mathcal{F}_A\big(\|\mathbf{t}-\mathbf{r}_h\|\big) \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\,,$$

and

$$\mathcal{M}_2(\mathbf{r}_h,\boldsymbol{\tau}_h) := \|\mathbf{t}-\mathbf{r}_h\| \,+\, \|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|\,.$$

Our next goal is to provide a bound for $\|\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\|$ that does not depend on $\mathbf{t}_h$. For this purpose, we now recall that, given $N \in \mathbb{N}$, $r \in]0,1[$, and $a_j \geq 0$ for all $j \in \{1, ..., N\}$, there holds

$$\left\{\sum_{j=1}^{N} a_j\right\}^{1/r} \leq \sum_{j=1}^{N} a_j^{1/r}. \tag{3.53}$$

Then, we have the following result.

Lemma 3.22. *Let $\mathcal{G} : \mathrm{R}^+ \to \mathrm{R}^+$ be the function defined by $\mathcal{G}(x) = x^{p_1'/p_1} + x^{p_1'/p_2} + x^{p_2'/p_1} + x^{p_2'/p_2}$ $\forall\, x \in \mathrm{R}^+$. Then, there exists a positive constant $\widetilde{C}_6$, depending on data, p_1, p_2, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$, $\|\mathbb{C}\|$, α_h and β_h, such that for each $\mathbf{r}_h \in X_h$ there holds*

$$\begin{aligned} \| \bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h \| \,\leq\, \widetilde{C}_6 \Big\{ & \mathcal{G}\big(\mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h)\big) \,+\, \mathcal{G}\big(\mathrm{dist}\,(\mathbf{u}, Z_h)\big) \\ & + \sum_{j=1}^{2} \big(\mathcal{M}_2(\mathbf{r}_h, \boldsymbol{\tau}_h)\big)^{1/p_j} \,\| \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \|^{1/p_j} \Big\} \qquad \forall\, \boldsymbol{\tau}_h \in Y_h\,. \end{aligned} \tag{3.54}$$

Proof. Recalling that $\bar{\mathbf{t}}_h := (\vartheta_1, \vartheta_2)$ and $\bar{\mathbf{r}}_h := (\rho_1, \rho_2)$, we first deduce from (3.43) and (3.52) (cf. Lemma 3.21) that

$$\begin{aligned} \alpha_h \Big\{ \| \vartheta_1 - \rho_1 \|_{X_1}^{p_1} + \| \vartheta_2 - \rho_2 \|_{X_2}^{p_2} \Big\} \,\leq\, & \widetilde{C}_4 \Big\{ \mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h) \,+\, \mathrm{dist}\,(\mathbf{u}, Z_h) \Big\} \, \|\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\| \\ & + \quad \widetilde{C}_5 \, \mathcal{M}_2(\mathbf{r}_h, \boldsymbol{\tau}_h) \, \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\| \qquad \forall\, \boldsymbol{\tau}_h \in Y_h\,, \end{aligned}$$

where $\|\bar{\mathbf{t}}_h - \bar{\mathbf{r}}_h\| := \|\vartheta_1 - \rho_1\|_{X_1} + \|\vartheta_2 - \rho_2\|_{X_2}$. Then, in order to isolate $\| \vartheta_1 - \rho_1 \|_{X_1}^{p_1} + \| \vartheta_2 - \rho_2 \|_{X_2}^{p_2}$, we appropriately use Young's inequality (3.17) in the first two terms of the right-hand side of the foregoing inequality, thus getting

$$\begin{aligned} \|\vartheta_1 - \rho_1\|_{X_1}^{p_1} + \|\vartheta_2 - \rho_2\|_{X_2}^{p_2} \,\leq\, \widetilde{C} \Big\{ & \big(\mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h)\big)^{p_1'} + \big(\mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h)\big)^{p_2'} \\ + \big(\mathrm{dist}\,(\mathbf{u}, Z_h)\big)^{p_1'} \,+\, & \big(\mathrm{dist}\,(\mathbf{u}, Z_h)\big)^{p_2'} + \mathcal{M}_2(\mathbf{r}_h, \boldsymbol{\tau}_h) \, \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\| \Big\}, \end{aligned} \tag{3.55}$$

where p_j' is the conjugate of p_j, that is $p_j' := p_j/(p_j - 1)$ for $j \in \{1, 2\}$, and $\widetilde{C}$ is a positive constant depending on data, p_1, p_2, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$, $\|\mathbb{C}\|$, α_h and β_h. Finally, noting that for each $j \in \{1, 2\}$, $\|\vartheta_j - \rho_j\|_{X_1}$ is certainly bounded by the right-hand side of (3.55) to the power $1/p_j$, and then using (3.53) on the latter expression, we arrive to (3.54), thus ending the proof. $\square$

We are now in a position to derive a preliminary estimate for $\|\mathbf{t} - \mathbf{t}_h\|$. In fact, starting from (3.39), employing the bounds given by Lemmas 3.17 and 3.22, and applying Young's inequality (3.17) (with an arbitrary parameter $\delta > 0$) to each term within the sum, we find that

$$\begin{aligned} \| \mathbf{t} - \mathbf{t}_h \| \,\leq\, \widetilde{C}_3 \, \|\mathbf{t} - \mathbf{r}_h\| \,+\, \widetilde{C}_6 \Bigg\{ & \mathcal{G}\big(\mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h)\big) \,+\, \mathcal{G}\big(\mathrm{dist}\,(\mathbf{u}, Z_h)\big) \\ + \sum_{j=1}^{2} \frac{1}{\delta^{p_j'} p_j'} \, & \big(\mathcal{M}_2(\mathbf{r}_h, \boldsymbol{\tau}_h)\big)^{p_j'/p_j} \,+\, \left(\sum_{j=1}^{2} \frac{\delta^{p_j}}{p_j} \right) \| \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \| \Bigg\}, \end{aligned} \tag{3.56}$$

from which a simple application of the triangle inequality to the expression $\| \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \|$ yields

$$\begin{aligned} \| \mathbf{t} - \mathbf{t}_h \| \,\leq\, \widetilde{C}_7 \Bigg\{ & \|\mathbf{t} - \mathbf{r}_h\| \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\| \,+\, \mathcal{G}\big(\mathcal{M}_1(\mathbf{r}_h, \boldsymbol{\tau}_h)\big) \,+\, \mathcal{G}\big(\mathrm{dist}\,(\mathbf{u}, Z_h)\big) \\ + \sum_{j=1}^{2} & \big(\mathcal{M}_2(\mathbf{r}_h, \boldsymbol{\tau}_h)\big)^{p_j'/p_j} \Bigg\} \,+\, \widetilde{C}_6 \left(\sum_{j=1}^{2} \frac{\delta^{p_j}}{p_j} \right) \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \| \qquad \forall\, (\mathbf{r}_h, \boldsymbol{\tau}_h) \in X_h \times Y_h\,, \end{aligned} \tag{3.57}$$

with a positive constant $\widetilde{C}_7$, depending on data, p_1, p_2, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$, $\|\mathbb{C}\|$, α_h, β_h and δ. Then, taking the infimum over all those $\mathbf{r}_h \in X_h$ and $\boldsymbol{\tau}_h \in Y_h$, using the non-decreasing character of all the polynomial functions involved, applying whenever necessary either (3.13) or (3.53), and performing some algebraic manipulations, we obtain

$$
\begin{aligned}
\|\,\mathbf{t}-\mathbf{t}_h\,\| \,\le\, \widetilde{C}_8 \Big\{ & \operatorname{dist}(\mathbf{t}, X_h) \;+\; \operatorname{dist}(\boldsymbol{\sigma}, Y_h) \;+\; \mathcal{G}\big(\operatorname{dist}(\mathbf{t}, X_h)\big) \\
& +\; \sum_{j=1}^{2} \mathcal{G}\big((\operatorname{dist}(\mathbf{t}, X_h))^{p_j-1}\big) \;+\; \mathcal{G}\big(\operatorname{dist}(\boldsymbol{\sigma}, Y_h)\big) \\
& +\; \mathcal{G}\big(\operatorname{dist}(\mathbf{u}, Z_h)\big)\Big\} \;+\; \widetilde{C}_6\left(\sum_{j=1}^{2}\frac{\delta^{p_j}}{p_j}\right)\|\,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,\|\,,
\end{aligned}
\tag{3.58}
$$

where $\widetilde{C}_8$ is another positive constant with the same parameter dependence of $\widetilde{C}_7$. In particular, note that the expressions $\sum_{j=1}^{2}\big(\operatorname{dist}(\mathbf{t}, X_h)\big)^{p_j'/p_j}$ and $\sum_{j=1}^{2}\big(\operatorname{dist}(\boldsymbol{\sigma}, Y_h)\big)^{p_j'/p_j}$, which arise from the first sum on the right-hand side of (3.57), are dominated by $\mathcal{G}\big(\operatorname{dist}(\mathbf{t}, X_h)\big)$ and $\mathcal{G}\big(\operatorname{dist}(\boldsymbol{\sigma}, Y_h)\big)$, respectively.

Having establised (3.58), we are able to provide next the a priori error estimate for the full error. More precisely, the main result of this section is stated as follows.

Theorem 3.23. *There exists a positive constant $\widetilde{C}_9$, depending on data, p_1, p_2, $\|\mathbb{B}\|$, $\|\mathbb{B}_1\|$, $\|\mathbb{C}\|$, α_h and β_h, such that*

$$
\begin{aligned}
\|\,\mathbf{t}-\mathbf{t}_h\,\| + \|\,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,\| + \|\,\mathbf{u}-\mathbf{u}_h\,\| \,\le\, \widetilde{C}_9 \Big\{ & \operatorname{dist}(\mathbf{t}, X_h) \;+\; \operatorname{dist}(\boldsymbol{\sigma}, Y_h) \;+\; \operatorname{dist}(\mathbf{u}, Z_h) \\
+\; \mathcal{G}\big(\operatorname{dist}(\mathbf{t}, X_h)\big) \;+\; \sum_{j=1}^{2} \mathcal{G}\big((\operatorname{dist}(\mathbf{t}, X_h))^{p_j-1}\big) \;+\; & \mathcal{G}\big(\operatorname{dist}(\boldsymbol{\sigma}, Y_h)\big) \;+\; \mathcal{G}\big(\operatorname{dist}(\mathbf{u}, Z_h)\big)\Big\}\,.
\end{aligned}
\tag{3.59}
$$

Proof. The a priori error estimate (3.59) for $\|\,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,\|$ follows from (3.36) (cf. Lemma 3.16) and (3.58) by choosing a sufficiently small δ. In turn, the corresponding upper bound for $\|\,\mathbf{t}-\mathbf{t}_h\,\|$ is obtained from (3.58) and the one for $\|\,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,\|$, whereas the estimate for $\|\,\mathbf{u}-\mathbf{u}_h\,\|$ is consequence of (3.33) (cf. Lemma 3.15) and the previous ones. We omit further details. □

4 Analysis of the coupled problem

We now turn to the analysis of the SDF problem using the theory developed in the previous section. Recall that the variational formulation for this problem was proposed at the end of Section 2.

4.1 The continuous formulation

We first analyze the properties of the spaces and operators associated with the continuous formulation of the SDF problem (cf. (2.16)) with the aim of applying Theorem 3.13. Recall that

$$
\begin{gathered}
X_1 := \mathbb{H}_0(\mathbf{div};\Omega_S), \quad X_2 := \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D), \quad X := X_1 \times X_2, \\
Y := \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma) \times W^{1/3,3/2}(\Sigma) \quad \text{and} \quad Z := \mathbf{L}^2(\Omega_S) \times L^{3/2}_0(\Omega_D) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega_S).
\end{gathered}
$$

Observe here that X is a uniformly convex space. Indeed, this property follows from Hanner's inequality (which implies that $L^p(\Omega)$ is uniformly convex when $1 < p < \infty$), from the fact that every closed subspace of a uniformly convex Banach space is uniformly convex, the continuity of the trace and normal trace operators, and the orthogonal decomposition theorem. In turn, using similar arguments, all of the other spaces are uniformly convex and separable Banach spaces.

Next, we verify that the rest of the assumptions of Theorem 3.13 are satisfied. More precisely, we take advantage of the block-diagonal structure shown by $\mathbb{B}$ and $\mathbb{B}_1$ and prove that these inf-sup conditions can be reduced, equivalently, to four simpler inf-sup conditions. For this purpose, we first introduce some technical results.

Lemma 4.1. *Given $1 < p < \infty$ and $f \in L^p(\Omega)$, define the sets $\Omega_0 := \big\{x \in \Omega : \quad f(x) = 0\big\}$, $\Omega_1 := \Omega \backslash \Omega_0$ and the function given by*

$$\widehat{f}(x) := \begin{cases} |f|^{p-2} f & \text{for } x \in \Omega_1, \\ 0 & \text{for } x \in \Omega_0. \end{cases}$$

Then, for $1 \leq q \leq \infty$ such that $p^{-1} + q^{-1} = 1$ there holds $\widehat{f} \in L^q(\Omega)$, $f = |\widehat{f}|^{q-2}\widehat{f}$ in Ω_1 a.e., and

$$\int_\Omega f\widehat{f} \,=\, \|f\|^p_{0,p;\Omega} = \|\widehat{f}\|^q_{0,q;\Omega} \,=\, \|f\|_{0,p;\Omega}\, \|\widehat{f}\|_{0,q;\Omega}\,.$$

Proof. It is direct from the definition of $\widehat{f}$. We omit details. □

Lemma 4.2. *For each $1 < p < \infty$, let $\mathcal{V} := \big\{\mathbf{v} \in \mathbf{W}^{0,p}(\mathrm{div};\Omega_D) : \quad \mathrm{div}\,\mathbf{v} \in P_0(\Omega_D)\big\}$ with $P_0(\Omega_D)$ being the space of constant functions in Ω_D. Then, there exists $C = C(\Omega_D, p) > 0$ such that*

$$\|\,\mathrm{div}\,\mathbf{v}\,\|_{0,p;\Omega_D} \,\leq\, C\,\|\,\mathbf{v}\,\|_{0,p;\Omega_D} \quad \forall\, \mathbf{v} \in \mathcal{V}\,, \tag{4.1}$$

and hence, with $\widehat{C} := 1 + C$, there holds

$$\|\,\mathbf{v}\,\|_{p,\mathrm{div};\Omega_D} := \|\,\mathbf{v}\,\|_{0,p;\Omega_D} + \|\,\mathrm{div}\,\mathbf{v}\,\|_{0,p;\Omega_D} \,\leq\, \widehat{C}\,\|\,\mathbf{v}\,\|_{0,p;\Omega_D} \quad \forall\, \mathbf{v} \in \mathcal{V}. \tag{4.2}$$

Proof. Given $\mathbf{v} \in \mathcal{V}$ and $w \in W^{1,q}(\Omega_D)$, with $1 < q < \infty$ such that $p^{-1} + q^{-1} = 1$, we have from Green's formula that

$$\langle\,\mathbf{v}\cdot\mathbf{n}, w\,\rangle_{W^{-1/p,p}(\partial\Omega_D),W^{1/p,q}(\partial\Omega_D)} \,=\, (\mathbf{v},\nabla w)_D + (\mathrm{div}\,\mathbf{v}, w)_D = (\mathbf{v},\nabla w)_D + \mathrm{div}\,\mathbf{v}\,(1,w)_D\,.$$

In particular, taking a cut-off function $w \in \mathcal{C}_0^\infty(\Omega_D)$ such that $\int_{\Omega_D} \widetilde{w} = 1$, we obtain

$$|\mathrm{div}\,\mathbf{v}| \,=\, |(\mathbf{v},\nabla w)_D| \,\leq\, \|\nabla w\|_{0,q;\Omega_D}\, \|\mathbf{v}\|_{0,p;\Omega_D}\,,$$

which yields (4.1) with $C = |\Omega_D|^{1/p}\,\|\nabla w\|_{0,q;\Omega_D}$. □

Lemma 4.3. *The operator $\mathbb{B} : X \to Z'$ defined by (2.18) satisfies the inf-sup condition (3.20), that is, there exists $\beta > 0$ such that*

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in X\\ \vec{\boldsymbol{\tau}}\neq\vec{\mathbf{0}}}} \frac{\big[\mathbb{B}(\vec{\boldsymbol{\tau}}),\vec{\boldsymbol{\eta}}\,\big]}{\|\,\vec{\boldsymbol{\tau}}\,\|_X} \,\geq\, \beta\,\|\,\vec{\boldsymbol{\eta}}\,\|_Z \quad \forall\,\vec{\boldsymbol{\eta}} \in Z\,. \tag{4.3}$$

Proof. Recall that for any $\vec{\boldsymbol{\tau}} = (\boldsymbol{\tau}_S, \mathbf{v}_D) \in X$, $\vec{\boldsymbol{\eta}} = (\mathbf{v}_S, q_D, \boldsymbol{\eta}_S) \in Z$,

$$\big[\,\mathbb{B}(\vec{\boldsymbol{\tau}}),\vec{\boldsymbol{\eta}}\,\big] = (\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{v}_S)_S + (\boldsymbol{\tau}_S,\boldsymbol{\eta}_S)_S - (\mathrm{div}\,\mathbf{v}_D, q_D)_D.$$

Then, according to the quasi-diagonal structure shown by $\mathbb{B}$, (4.3) is equivalent to the existence of positive constants $\widetilde{\beta}$ and $\widehat{\beta}$, such that the following separate inf-sup conditions hold:

$$\sup_{\substack{\boldsymbol{\tau}_S\in X_1\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{v}_S)_S+(\boldsymbol{\tau}_S,\boldsymbol{\eta}_S)_S}{\|\boldsymbol{\tau}_S\|_{X_1}} \,\geq\, \widetilde{\beta}\,\|(\mathbf{v}_S,\boldsymbol{\eta}_S)\| \qquad \forall\,(\mathbf{v}_S,\boldsymbol{\eta}_S)\in\mathbf{L}^2(\Omega_S)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega_S) \tag{4.4}$$

and

$$\sup_{\substack{\mathbf{v}_D\in X_2\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_D,q_D)_D}{\|\mathbf{v}_D\|_{X_2}} \,\geq\, \widehat{\beta}\,\|q_D\|_{0,3/2;\Omega_D} \qquad \forall\,q_D\in L^{3/2}_0(\Omega_D)\,. \tag{4.5}$$

To prove the foregoing inequalities, we proceed as usual by introducing suitable auxiliary problems. Indeed, (4.4) holds by the analysis done in [21, Section 2.4.3.1] in the context of linear elasticity (nonetheless, applicable in this case), whereas for (4.5), given $q_D\in L^{3/2}_0(\Omega_D)$, we consider the Neumann problem: Find $w\in W^{1,3}(\Omega_D)$ such that

$$\Delta\,w \,=\, \widetilde{q}_D \quad\text{in}\quad \Omega_D,\quad \nabla w\cdot\mathbf{n}\,=\,0 \quad\text{on}\quad \partial\Omega_D,\quad\text{and}\quad \int_{\Omega_D} w=0, \tag{4.6}$$

where

$$\widetilde{q}_D := \widehat{q}_D - \frac{1}{|\Omega_D|}\int_{\Omega_D}\widehat{q}_D \qquad\text{and}\qquad \widehat{q}_D := \begin{cases} |q_D|^{-1/2}\,q_D & \text{if } q_D\neq 0,\\ 0 & \text{if } q_D=0\,.\end{cases}$$

Existence and uniqueness of a solution $w\in W^{1,3}(\Omega_D)$ of (4.6) is guaranteed by [13]. Hence, defining $\mathbf{z}_D := \nabla w$ and applying Lemma 4.1 with $p=3/2$, we get

$$(\mathrm{div}\,\mathbf{z}_D,q_D)_D \,=\, (\Delta\,w,q_D)_D \,=\, (\widetilde{q}_D,q_D)_D \,=\, (\widehat{q}_D,q_D)_D \,=\, \|q_D\|^{3/2}_{0,3/2;\Omega_D}\,. \tag{4.7}$$

In turn, denoting by C_D the continuous dependence constant for (4.6), which is independent of w, and employing again Lemma 4.1, we find that

$$\|\mathbf{z}_D\|_{0,3;\Omega_D} \,\leq\, \|w\|_{1,3;\Omega_D} \,\leq\, C_D\,\|\widetilde{q}_D\|_{0,3;\Omega_D} \,\leq\, C_D\,\|\widehat{q}_D\|_{0,3;\Omega_D} \,=\, C_D\|q_D\|^{1/2}_{0,3/2;\Omega_D}\,,$$

and

$$\|\mathrm{div}\,\mathbf{z}_D\|_{0,3;\Omega_D} \,=\, \|\widetilde{q}_D\|_{0,3;\Omega_D} \,\leq\, \|\widehat{q}_D\|_{0,3;\Omega_D} \,=\, \|q_D\|^{1/2}_{0,3/2,\Omega_D}\,,$$

whence $\mathbf{z}_D\in\mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D)$ and there holds

$$\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D} \,\leq\, (C_D+1)\,\|q_D\|^{1/2}_{0,3/2;\Omega_D}\,. \tag{4.8}$$

In this way, bounding below with $\mathbf{v}_D=\mathbf{z}_D$, we deduce from (4.7) and (4.8) that

$$\sup_{\substack{\mathbf{v}_D\in X_2\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_D,q_D)_D}{\|\mathbf{v}_D\|_{X_2}} \,\geq\, \frac{(\mathrm{div}\,\mathbf{z}_D,q_D)_D}{\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D}} \,\geq\, \frac{1}{C_D+1}\,\|q_D\|_{0,3/2;\Omega_D}\,,$$

which gives (4.5) with $\widehat{\beta}=\frac{1}{C_D+1}$. □

Using similar techniques, we prove next that the inf-sup condition (3.21) holds for $\mathbb{B}_1$. To this end, we first notice from the definition of $\mathbb{B}$ (cf. (2.18)) that its kernel becomes $\mathbb{V} \,=\, \widetilde{X}_1\times\widetilde{X}_2$, where

$$\widetilde{X}_1 \,:=\, \Big\{\boldsymbol{\tau}_S\in X_1:\quad \boldsymbol{\tau}_S=\boldsymbol{\tau}_S^{\mathtt{t}}\quad\text{and}\quad \mathbf{div}\,\boldsymbol{\tau}_S=\mathbf{0}\Big\} \tag{4.9}$$

and

$$\widetilde{X}_2 \,:=\, \Big\{\mathbf{v}_D\in X_2:\quad \mathrm{div}\,\mathbf{v}_D\in P_0(\Omega_D)\Big\}\,. \tag{4.10}$$

Then, the announced result is stated as follows.

Lemma 4.4. *There exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\vec{\boldsymbol{\tau}}\in\mathbb{V}\\ \vec{\boldsymbol{\tau}}\neq\vec{\mathbf{0}}}} \frac{\left[\mathbb{B}_1(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}\right]}{\|\,\vec{\boldsymbol{\tau}}\,\|_X} \geq \beta_1 \,\|\,\vec{\mathbf{v}}\,\|_Y \qquad \forall\, \vec{\mathbf{v}} \in Y. \tag{4.11}$$

Proof. We begin by recalling from the definition of $\mathbb{B}_1$ (cf. (2.19)) that for any $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D) \in X := \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D)$ and $\vec{\mathbf{v}} := (\boldsymbol{\psi}, \xi) \in Y := \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma) \times W^{1/3,3/2}(\Sigma)$, we have

$$\left[\mathbb{B}_1(\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}\right] = \langle\, \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\,\rangle_\Sigma - \langle\, \mathbf{v}_D\cdot\mathbf{n}, \xi\,\rangle_\Sigma\,.$$

Then, according to the diagonal structure of $\mathbb{B}_1$, establishing (4.11) is equivalent to proving the existence of positive constants $\widetilde{\beta}_1$ and $\widehat{\beta}_1$, such that

$$\sup_{\substack{\boldsymbol{\tau}_S\in\widetilde{X}_1\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\, \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\,\rangle_\Sigma}{\|\,\boldsymbol{\tau}_S\,\|_{X_1}} \;\geq\; \widetilde{\beta}_1 \,\|\,\boldsymbol{\psi}\,\|_{1/2,00;\Sigma} \quad \forall\, \boldsymbol{\psi} \in \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)\,, \tag{4.12}$$

and

$$\sup_{\substack{\mathbf{v}_D\in\widetilde{X}_2\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\langle\, \mathbf{v}_D\cdot\mathbf{n}, \xi\,\rangle_\Sigma}{\|\,\mathbf{v}_D\,\|_{X_2}} \;\geq\; \widehat{\beta}_1 \,\|\,\xi\,\|_{1/3,3/2;\Sigma} \quad \forall\, \xi \in W^{1/3,3/2}(\Sigma)\,. \tag{4.13}$$

Similarly to [17, 21], we introduce auxiliary problems to show that these inf-sup conditions hold. Indeed, for the first one, given $\boldsymbol{\psi} \in \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)$, we consider the problem: Find $\mathbf{z} \in \mathbf{H}^1(\Omega_S)$ such that

$$\mathbf{div}\,\mathbf{e}(\mathbf{z}) = \mathbf{0} \quad \text{in } \Omega_S, \quad \mathbf{e}(\mathbf{z})\mathbf{n} = \mathcal{R}_{00}^{-1}(\boldsymbol{\psi}) \quad \text{on } \Sigma, \quad \text{and} \quad \mathbf{z} = \mathbf{0} \quad \text{on } \Gamma_S\,, \tag{4.14}$$

where $\mathcal{R}_{00} : \left[\widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)\right]' \to \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)$ is the Riesz application (cf. [21, Section 2.4.2]). The variational formulation for (4.14) reduces to (cf. [21, Eq. (2.64)]): Find $\mathbf{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that

$$\int_{\Omega_S} \mathbf{e}(\mathbf{z}) : \mathbf{e}(\mathbf{w}) = \left\langle\, \mathcal{R}_{00}^{-1}(\boldsymbol{\psi}), \mathbf{w}\,\right\rangle_\Sigma \quad \forall\, \mathbf{w} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)\,.$$

In this case, the Korn inequality, the trace theorem and the Lax-Milgram lemma allow to prove that this problem has a unique solution $\mathbf{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ and that there exists $C_S > 0$ such that

$$\|\mathbf{z}\|_{1,2,\Omega_S} \;\leq\; C_S\,\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}\,. \tag{4.15}$$

Then, defining $\boldsymbol{\zeta}_S := \mathbf{e}(\mathbf{z}) - \left(\frac{1}{2|\Omega_S|}\int_\Sigma \mathbf{z}\cdot\mathbf{n}\right)\mathbf{I}$, and employing the Gauss theorem, we readily get that

$$\boldsymbol{\zeta}_S = \boldsymbol{\zeta}_S^{\mathrm{t}}, \quad \mathbf{div}\,\boldsymbol{\zeta}_S = 0, \quad \text{and} \quad \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\zeta}_S) = 0\,,$$

which implies that $\boldsymbol{\zeta}_S \in \widetilde{X}_1$. In turn, recalling from the definition of $\widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)$ (cf. (2.10)) that $\langle\, \boldsymbol{\psi}\cdot\mathbf{n}, 1\,\rangle_\Sigma = 0$, we also obtain

$$\langle\, \boldsymbol{\zeta}_S\mathbf{n}, \boldsymbol{\psi}\,\rangle_\Sigma = \langle\, \mathbf{e}(\mathbf{z})\mathbf{n}, \boldsymbol{\psi}\,\rangle_\Sigma = \left\langle\, \mathcal{R}_{00}^{-1}(\boldsymbol{\psi}), \boldsymbol{\psi}\,\right\rangle_\Sigma \;=\; \|\boldsymbol{\psi}\|^2_{1/2,00;\Sigma}\,. \tag{4.16}$$

Hence, bounding below the supremum in (4.12) with $\boldsymbol{\tau}_S = \boldsymbol{\zeta}_S$, using the identity (4.16), and employing (4.15) to get un upper bound of $\|\boldsymbol{\zeta}_S\|_{X_1}$ in terms of $\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}$, we arrive at the required inf-sup condition (4.12). On the other hand, in order to prove (4.13), we now consider $\xi \in W^{1/3,3/2}(\Sigma)$. Then, because of the identity $\|\xi\|_{1/3,3/2;\Sigma} \,=\, \sup\limits_{\substack{\Lambda\in W^{-1/3,3}(\Sigma)\\ \Lambda\neq 0}} \dfrac{\langle\Lambda,\xi\rangle_\Sigma}{\|\Lambda\|_{-1/3,3;\Sigma}}$, there must exist $\Lambda\in W^{-1/3,3}(\Sigma)$ such that

$$\langle\Lambda,\xi\rangle_\Sigma \,\geq\, \frac{1}{2}\,\|\xi\|_{1/3,3/2;\Sigma}\,\|\Lambda\|_{-1/3,3;\Sigma}\,. \tag{4.17}$$

Next, we introduce an extension $\widetilde{\Lambda}$ of Λ onto $W^{-1/3,3}(\partial\Omega_D)$, which is defined by

$$\langle\widetilde{\Lambda},\eta\rangle_{\partial\Omega_D} \,:=\, \langle\Lambda,\eta|_\Sigma\rangle_\Sigma \quad \forall\,\eta\in W^{1/3,3/2}(\partial\Omega_D)\,, \tag{4.18}$$

whence it follows that

$$\|\widetilde{\Lambda}\|_{-1/3,3;\partial\Omega_D} \,\leq\, \|\Lambda\|_{-1/3,3;\Sigma}\,.$$

Having this extension in mind, we now consider the Neumann problem: Find $w\in W^{1,3}(\Omega_D)$ such that

$$\Delta\,w \,=\, \frac{1}{|\Omega_D|}\,\langle\widetilde{\Lambda},1\rangle_{\partial\Omega_D} \quad\text{in}\quad \Omega_D\,,\quad \nabla w\cdot\mathbf{n}=\widetilde{\Lambda} \quad\text{on}\quad \partial\Omega_D,\quad\text{and}\quad \int_{\Omega_D} w=0. \tag{4.19}$$

From [26] we know that (4.19) has a unique solution $w\in W^{1,3}(\Omega_D)$, and that there exists $C_D>0$, depending on Ω_D, such that $\|w\|_{1,3;\Omega_D}+\|\Delta w\|_{0,3;\Omega_D} \,\leq\, C_D\,\|\widetilde{\Lambda}\|_{-1/3,3;\partial\Omega_D}$. Then, defining $\mathbf{z}_D \,:=\, \nabla w$, we find that

$$\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D} \,\leq\, C_D\,\|\widetilde{\Lambda}\|_{-1/3,3;\partial\Omega_D} \,\leq\, C_D\,\|\Lambda\|_{-1/3,3;\Sigma}\,. \tag{4.20}$$

In turn, according to (2.13), and using (4.17), (4.18), and (4.20), we get

$$\begin{aligned}\langle\mathbf{z}_D\cdot\mathbf{n},\xi\rangle_\Sigma \,&:=\, \langle\mathbf{z}_D\cdot\mathbf{n},E^D_\Sigma(\xi)\rangle_{\partial\Omega_D} \,=\, \langle\widetilde{\Lambda},E^D_\Sigma(\xi)\rangle_{\partial\Omega_D}\\ &=\, \langle\Lambda,\xi\rangle_\Sigma \,\geq\, \frac{1}{2\,C_D}\,\|\xi\|_{1/3,3/2;\Sigma}\,\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D}\,.\end{aligned} \tag{4.21}$$

In this way, bounding below the supremum in (4.13) with $\mathbf{v}_D=\mathbf{z}_D$, and employing (4.21), we obtain this inf-sup condition with the constant $\widehat{\beta}_1 \,=\, \dfrac{1}{2\,C_D}$, which finishes the proof of the lemma. □

We now proceed to show that $\mathbb{A}$ does satisfy hypotheses iv) - v) of Theorem 3.13. We begin with the first of them.

Lemma 4.5. *Let $\mathbb{A}: X\to X'$ be the operator defined by (2.17). Then, there exist constants $\varsigma_1,\varsigma_2>0$, $\gamma_1,\gamma_2>0$, and $p_1,\,p_2\geq 2$, depending only on the domains, such that*

$$\|\mathcal{A}_S(\boldsymbol{\zeta}_S)-\mathcal{A}_S(\boldsymbol{\tau}_S)\|_{X_1'} \,\leq\, \varsigma_1\,\|\boldsymbol{\zeta}_S-\boldsymbol{\tau}_S\|_{X_1}+\gamma_1\|\boldsymbol{\zeta}_S-\boldsymbol{\tau}_S\|_{X_1}\Big(\|\boldsymbol{\zeta}_S\|_{X_1}+\|\boldsymbol{\tau}_S\|_{X_1}\Big)^{p_1-2}\,, \tag{4.22}$$

and

$$\|\mathcal{A}_D(\mathbf{w}_D)-\mathcal{A}_D(\mathbf{v}_D)\|_{X_2'} \,\leq\, \varsigma_2\,\|\mathbf{w}_D-\mathbf{v}_D\|_{X_2}+\gamma_2\|\mathbf{w}_D-\mathbf{v}_D\|_{X_2}\Big(\|\mathbf{w}_D\|_{X_2}+\|\mathbf{v}_D\|_{X_2}\Big)^{p_2-2}\,, \tag{4.23}$$

for all $\vec{\boldsymbol{\zeta}}:=(\boldsymbol{\zeta}_S,\mathbf{w}_D)$, $\vec{\boldsymbol{\tau}}:=(\boldsymbol{\tau}_S,\mathbf{v}_D)\in X:=X_1\times X_2$.

Proof. Given $\vec{\boldsymbol{\zeta}}:=(\boldsymbol{\zeta}_S,\mathbf{w}_D)$, $\vec{\boldsymbol{\tau}}:=(\boldsymbol{\tau}_S,\mathbf{v}_D)\in X$, we easily see from the definition of $\mathcal{A}_S$ (cf. (2.8)) that there holds

$$\|\mathcal{A}_S(\boldsymbol{\zeta}_S)-\mathcal{A}_S(\boldsymbol{\tau}_S)\|_{X_1'} \,\leq\, \frac{1}{2\mu}\|\boldsymbol{\zeta}_S-\boldsymbol{\tau}_S\|_{X_1}\,,$$

which yields (4.22) with $\varsigma_1 = \dfrac{1}{2\mu}$, $\gamma_1 = 0$, and $p_1 = 2$. In turn, according to the definition of $\mathcal{A}_D$ (cf. (2.11)), and employing the hypotheses on $\mathbf{K}$ (cf. Section 2.1.2) and the fact, by Cauchy-Schwarz's inequality, that $\|\cdot\|_{0,3/2;\Omega_D} \le |\Omega_D|^{1/3}\,\|\cdot\|_{0,3;\Omega_D}$, we readily obtain

$$\begin{aligned}
&\|\mathcal{A}_D(\mathbf{w}_D) - \mathcal{A}_D(\mathbf{v}_D)\|_{X_2'}\\
&\quad\le \frac{\mu}{\rho}\|\mathbf{K}^{-1}\|_\infty\,\|\mathbf{w}_D - \mathbf{v}_D\|_{0,3/2;\Omega_D} + \frac{\mathtt{f}}{\rho}\|\mathbf{w}_D - \mathbf{v}_D\|_{0,3;\Omega_D}\Big(\|\mathbf{w}_D\|_{0,3;\Omega_D} + \|\mathbf{v}_D\|_{0,3;\Omega_D}\Big)\\
&\quad\le \frac{\mu}{\rho}\,\|\mathbf{K}^{-1}\|_\infty\,|\Omega_D|^{1/3}\,\|\mathbf{w}_D - \mathbf{v}_D\|_{0,3;\Omega_D} + \frac{\mathtt{f}}{\rho}\|\,\mathbf{w}_D - \mathbf{v}_D\,\|_{0,3;\Omega_D}\Big(\|\mathbf{w}_D\|_{0,3;\Omega_D} + \|\mathbf{v}_D\|_{0,3;\Omega_D}\Big),
\end{aligned}$$

from which (4.23) follows with $\varsigma_2 = \dfrac{\mu}{\rho}\,\|\mathbf{K}^{-1}\|_\infty\,|\Omega_D|^{1/3}$, $\gamma_2 = \dfrac{\mathtt{f}}{\rho}$, and $p_2 = 3$. □

Next, we show that $\mathbb{A}$ satisfies hypothesis v) of Theorem 3.13 with the same exponents $p_1 = 2$ and $p_2 = 3$ from the previous lemma.

Lemma 4.6. *There exists $\alpha > 0$ such that for each $\vec{\boldsymbol{\sigma}}_G \in X$ there holds*

$$\big[\,\mathbb{A}(\vec{\boldsymbol{\zeta}} + \vec{\boldsymbol{\sigma}}_G) - \mathbb{A}(\vec{\boldsymbol{\tau}} + \vec{\boldsymbol{\sigma}}_G), \vec{\boldsymbol{\zeta}} - \vec{\boldsymbol{\tau}}\,\big] \;\ge\; \alpha\Big\{\|\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S\|^2_{X_1} + \|\mathbf{w}_D - \mathbf{v}_D\|^3_{X_2}\Big\}, \tag{4.24}$$

for all $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_S, \mathbf{w}_D)$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D) \in \mathbb{V} \subseteq X := X_1 \times X_2$.

Proof. Let $\vec{\boldsymbol{\sigma}}_G := (\boldsymbol{\sigma}_{S,G}, \mathbf{u}_{D,G}) \in X$. Then, given $\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_S, \mathbf{w}_D)$, $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D) \in \mathbb{V}$, we certainly have $\mathbf{div}\,(\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S) = \mathbf{0}$ (cf. (4.9)) and $\int_{\Omega_S} \mathrm{tr}(\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S) = 0$, whence according to the definition of $\mathcal{A}_S$ (cf. (2.8)) and the lower bound from [6, Proposition IV.3.1] (see also [21, Lemma 2.3]), we find that

$$\begin{aligned}
&[\,\mathcal{A}_S(\boldsymbol{\zeta}_S + \boldsymbol{\sigma}_{S,G}) - \mathcal{A}_S(\boldsymbol{\tau}_S + \boldsymbol{\sigma}_{S,G}), \boldsymbol{\zeta}_S - \boldsymbol{\tau}_S\,] \;\ge\; \frac{1}{2\mu}\,\|(\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S)^{\mathtt{d}}\|^2_{0,2;\Omega_S}\\
&\quad\ge\; \frac{C_S}{2\mu}\,\|\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S\|^2_{0,2;\Omega_S} \;=\; \frac{C_S}{2\mu}\,\|\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S\|^2_{\mathbf{div};\Omega} \;=\; \frac{C_S}{2\mu}\,\|\boldsymbol{\zeta}_S - \boldsymbol{\tau}_S\|^2_{X_1},
\end{aligned}\tag{4.25}$$

where C_S is a positive constant depending only on Ω_S. On the other hand, starting from the definition of $\mathcal{A}_D$ (cf. (2.11)), and employing the properties of $\mathbf{K}$ (cf. (2.6)), the lower bound given by [27, Lemme 5.1], and the fact that $\mathrm{div}\,(\mathbf{w}_D - \mathbf{v}_D) \in P_0(\Omega_D)$ (cf. (4.10)) together with Lemma 4.2, we deduce that

$$\begin{aligned}
&[\,\mathcal{A}_D(\mathbf{w}_D + \mathbf{u}_{D,G}) - \mathcal{A}_D(\mathbf{v}_D + \mathbf{u}_{D,G}), \mathbf{w}_D - \mathbf{v}_D\,] \;=\; \frac{\mu}{\rho}\,[\,\mathbf{K}^{-1}(\mathbf{w}_D - \mathbf{v}_D), \mathbf{w}_D - \mathbf{v}_D\,]\\
&\qquad + \frac{\mathtt{f}}{\rho}\,[\,|\mathbf{w}_D + \mathbf{u}_{D,G}|(\mathbf{w}_D + \mathbf{u}_{D,G}) - |\mathbf{v}_D + \mathbf{u}_{D,G}|(\mathbf{v}_D + \mathbf{u}_{D,G}), \mathbf{w}_D - \mathbf{v}_D\,]\\
&\qquad \ge\; \frac{\mu}{\rho}\,\varrho_0\,\|\mathbf{w}_D - \mathbf{v}_D\|^2_{0,2;\Omega_D} + \frac{\mathtt{f}}{\rho}\,c\,\|\mathbf{w}_D - \mathbf{v}_D\|^3_{0,3;\Omega_D}\\
&\qquad \ge\; \frac{\mu}{\rho}\,\varrho_0\,\|\mathbf{w}_D - \mathbf{v}_D\|^2_{0,2;\Omega_D} + \frac{\mathtt{f}}{\rho}\,c\,\widehat{C}^{-1}\,\|\mathbf{w}_D - \mathbf{v}_D\|^3_{3,\mathrm{div};\Omega_D}\\
&\qquad \ge\; \frac{\mathtt{f}}{\rho}\,c\,\widehat{C}^{-1}\,\|\mathbf{w}_D - \mathbf{v}_D\|^3_{X_2}.
\end{aligned}\tag{4.26}$$

In this way, (4.25) and (4.26) lead to (4.24) with $\alpha := \min\Big\{\dfrac{C_S}{2\mu}, \dfrac{\mathtt{f}}{\rho}\,c\,\widehat{C}^{-1}\Big\}$. □

Therefore, the foregoing analysis and Theorem 3.13 guarantee the existence of a unique $(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{\gamma}}) \in X \times Y \times Z$ solution of the variational formulation (2.16).

4.2 The fully-mixed finite element method

In this section we introduce and analyze a generic Galerkin scheme for (2.16), and propose specific choices of finite element subspaces satisfying the required conditions for well-posedness.

4.2.1 Galerkin Scheme

We first consider arbitrary finite dimensional subspaces

$$\mathbf{H}_h(\Omega_S) \subseteq \mathbf{H}(\mathrm{div};\Omega_S), \quad \mathbf{H}_h(\Omega_D) \subseteq \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D), \quad \mathbf{\Lambda}^S_h(\Sigma) \subseteq \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma)\,,$$

$$\Lambda^D_h(\Sigma) \subseteq W^{1/3,3/2}(\Sigma), \quad L_h(\Omega_S) \subseteq L^2(\Omega_S), \quad L_h(\Omega_D) \subseteq L^{3/2}_0(\Omega_D), \quad \mathbb{L}_h(\Omega_S) \subseteq \mathbb{L}^2_{\mathtt{skew}}(\Omega_S)\,,$$

set

$$\begin{aligned}
\mathbb{H}_h(\Omega_S) \,&:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega_S) : \ \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall\, \mathbf{c} \in \mathrm{R}^2 \ \Big\},\\
\mathbb{H}_{h,0}(\Omega_S) \,&:=\, \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div};\Omega_S),\\
\mathbf{L}_h(\Omega_S) \,&:=\, [L_h(\Omega_S)]^2\,,
\end{aligned}$$

and define the global spaces

$$\begin{aligned}
&X_{S,h} := \mathbb{H}_{h,0}(\Omega_S), \quad X_{D,h} := \mathbf{H}_h(\Omega_D), \quad X_h := X_{S,h} \times X_{D,h},\\
&Y_h := \mathbf{\Lambda}^S_h(\Sigma) \times \Lambda^D_h(\Sigma), \quad Z_h := \mathbf{L}_h(\Omega_S) \times L_h(\Omega_D) \times \mathbb{L}_h(\Omega_S)\,.
\end{aligned}$$

Then, the Galerkin scheme associated to (2.16) reads: Find $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{\gamma}}_h) \in X_h \times Y_h \times Z_h$ such that:

$$\begin{aligned}
\big[\,\mathbb{A}(\vec{\boldsymbol{\sigma}}_h), \vec{\boldsymbol{\tau}}_h\,\big] + \big[\,\mathbb{B}_1(\vec{\boldsymbol{\tau}}_h), \vec{\mathbf{u}}_h\,\big] + \big[\,\mathbb{B}(\vec{\boldsymbol{\tau}}_h), \vec{\boldsymbol{\gamma}}_h\,\big] &= \big[\,F, \vec{\boldsymbol{\tau}}_h\,\big], &\text{(4.27a)}\\
\big[\,\mathbb{B}_1(\vec{\boldsymbol{\sigma}}_h), \vec{\mathbf{v}}_h\,\big] - \big[\,\mathbb{C}(\vec{\mathbf{u}}_h), \vec{\mathbf{v}}_h\,\big] &= \big[\,G_1, \vec{\mathbf{v}}_h\,\big], &\text{(4.27b)}\\
\big[\,\mathbb{B}(\vec{\boldsymbol{\sigma}}_h), \vec{\boldsymbol{\eta}}_h\,\big] &= \big[\,G, \vec{\boldsymbol{\eta}}_h\,\big], &\text{(4.27c)}
\end{aligned}$$

for all $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{\eta}}_h) \in X_h \times Y_h \times Z_h$, where, following (2.15), we employ the notation

$$\begin{aligned}
&\vec{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{D,h}), \quad \vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h}, \mathbf{v}_{D,h}), \quad \vec{\mathbf{u}}_h := (\boldsymbol{\varphi}_h, \lambda_h)\,,\\
&\vec{\mathbf{v}}_h := (\boldsymbol{\psi}_h, \xi_h), \quad \vec{\boldsymbol{\gamma}}_h := (\mathbf{u}_{S,h}, p_{D,h}, \boldsymbol{\gamma}_{S,h}), \quad \vec{\boldsymbol{\eta}}_h := (\mathbf{v}_{S,h}, q_{D,h}, \boldsymbol{\eta}_{S,h})\,.
\end{aligned} \tag{4.28}$$

Next, in order to apply Theorem 3.14 to prove the well-posedness of this scheme, we now specify suitable hypotheses on the finite element subspaces introduced here. More precisely, we first assume that $\mathbb{B}$ satisfies (3.26), which, noting again (as in the proof of Lemma 4.3) the quasi-diagonal structure of this operator, is equivalent to the following discrete inf-sup conditions:

(H.1) there exists $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S)\\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h})_S + (\boldsymbol{\tau}_{S,h}, \boldsymbol{\eta}_{S,h})_S}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div};\Omega}} \,\geq\, \widehat{\beta}\,\|(\mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h})\| \tag{4.29}$$

for all $(\mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}) \in \mathbf{L}_h(\Omega_S) \times \mathbb{L}_h(\Omega_S)$, and

$$\sup_{\substack{\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D)\\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{D,h}, q_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{3,\mathrm{div};\Omega_D}} \,\geq\, \widehat{\beta}\,\|q_{D,h}\| \tag{4.30}$$

for all $q_{D,h} \in L_h(\Omega_D)$.

Next, in order to give a more explicit definition of the discrete kernel $\mathbb{V}_h$ of $\mathbb{B}$, we assume that

(H.2) $\operatorname{div} \mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S)$ and $\operatorname{div} \mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D)$,

which yields $\mathbb{V}_h = \widetilde{X}_{S,h} \times \widetilde{X}_{D,h}$, with

$$\widetilde{X}_{S,h} := \Big\{ \boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) : \ (\boldsymbol{\tau}_{S,h}, \boldsymbol{\eta}_{S,h})_S = 0 \quad \forall\, \boldsymbol{\eta}_{S,h} \in \mathbb{L}_h(\Omega_S), \quad \text{and} \quad \mathbf{div}\, \boldsymbol{\tau}_{S,h} = \mathbf{0} \ \Big\},$$

and

$$\widetilde{X}_{D,h} := \Big\{ \mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D) : \ \operatorname{div} \mathbf{v}_{D,h} \in P_0(\Omega_D) \ \Big\}.$$

Similarly, due to the diagonal structure of $\mathbb{B}_1$, we can equivalently assume for the inf-sup condition (3.27) that

(H.3) there exists $\widehat{\beta}_1 > 0$, such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \widetilde{X}_{S,h} \\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{\langle\, \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\psi}_h \,\rangle_\Sigma}{\|\, \boldsymbol{\tau}_{S,h} \,\|_{\mathbf{div};\Omega_S}} \geq \widehat{\beta}_1 \|\, \boldsymbol{\psi}_h \,\|_{1/2,2;\Sigma} \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^S(\Sigma), \tag{4.31}$$

and

$$\sup_{\substack{\mathbf{v}_{D,h} \in \widetilde{X}_{D,h} \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\langle\, \mathbf{v}_{D,h} \cdot \mathbf{n}, \xi_h \,\rangle_\Sigma}{\|\, \mathbf{v}_{D,h} \,\|_{3,\operatorname{div};\Omega_D}} \geq \widehat{\beta}_1 \|\, \xi_h \,\|_{1/2,3/2;\Sigma} \quad \forall\, \xi_h \in \Lambda_h^D(\Sigma). \tag{4.32}$$

Hence, it remains to show that hypotheses iii) - iv) of Theorem 3.14 are satisfied. In fact, it is easy to see that the former follows from Lemma 4.5 and the straightforward inequalities

$$\|\mathcal{A}_S(\boldsymbol{\zeta}_{S,h}) - \mathcal{A}_S(\boldsymbol{\tau}_{S,h})\|_{X'_{S,h}} \ \leq \ \|\mathcal{A}_S(\boldsymbol{\zeta}_{S,h}) - \mathcal{A}_S(\boldsymbol{\tau}_{S,h})\|_{X'_1}$$

and

$$\|\mathcal{A}_D(\mathbf{w}_{D,h}) - \mathcal{A}_D(\mathbf{v}_{D,h})\|_{X'_{D,h}} \ \leq \ \|\mathcal{A}_D(\mathbf{w}_{D,h}) - \mathcal{A}_D(\mathbf{v}_{D,h})\|_{X'_2}$$

for all $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_{S,h}, \mathbf{w}_{D,h})$, $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h}, \mathbf{v}_{D,h}) \in X_h := X_{S,h} \times X_{D,h}$, whereas the latter is a direct consequence of Lemma 4.6 and the fact that $\mathbf{div}\,(\boldsymbol{\zeta}_{S,h} - \boldsymbol{\tau}_{S,h}) = \mathbf{0}$ and $\operatorname{div}(\mathbf{w}_{D,h} - \mathbf{v}_{D,h}) \in P_0(\Omega_D)$ for all $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_{S,h}, \mathbf{w}_{D,h})$, $\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h}, \mathbf{v}_{D,h}) \in \mathbb{V}_h := \widetilde{X}_{S,h} \times \widetilde{X}_{D,h}$.

4.2.2 A specific choice of finite element subspaces

Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be triangulations for Ω_S and Ω_D, respectively, both shape-regular in the sense of Ciarlet-Raviart (cf. [12, page 247]), and assume that they match on Σ, that is, $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. Furthermore, given an integer $k \geq 0$ and a subset $U \subseteq \mathrm{R}^2$, we let $P_k(U)$ be the space of polynomials defined on U of total degree $\leq k$, and denote its vector and tensor counterparts as $\mathbf{P}_k(U)$ and $\mathbb{P}_k(U)$, respectively. In addition, let b_T be the element-bubble function defined as the unique polynomial in $P_3(T)$ that vanishes on ∂T and $\int_T b_T = 1$, and let $\mathbf{x} \in \mathrm{R}^2$ be a generic vector. Then, we define for each $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ the local Raviart-Thomas and bubble spaces of order 0, respectively by

$$\mathbf{RT}_0(T) := \mathbf{P}_0(T) + \mathbf{x} P_0(T) \quad \text{and} \quad \mathbf{B}_0(T) := P_0(T) \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1} \right).$$

Then, we consider the following finite element spaces

$$\mathbf{H}_h(\Omega_S) := \Big\{ \boldsymbol{\tau}_{S,h} \in \mathbf{H}(\mathrm{div};\Omega_S): \ \boldsymbol{\tau}_{S,h}\big|_T \in \mathbf{RT}_0(T) \oplus \mathbf{B}_0(T) \quad \forall\, T \in \mathcal{T}_h^S \Big\}, \tag{4.33}$$

$$\mathbf{H}_h(\Omega_D) := \Big\{ \mathbf{v}_{D,h} \in \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div};\Omega_D): \ \mathbf{v}_{D,h}\big|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^D \Big\}, \tag{4.34}$$

$$L_h(\Omega_S) := \Big\{ v_{S,h} \in L^2(\Omega_S): \ v_{S,h}\big|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^S \Big\}, \tag{4.35}$$

$$L_h(\Omega_D) := \Big\{ q_{D,h} \in L^{3/2}_0(\Omega_D): \ q_{D,h}\big|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^D \Big\}, \tag{4.36}$$

$$\mathbb{L}_h(\Omega_S) := \Big\{ \eta_{S,h} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: \ \eta_{S,h} \in \mathcal{C}(\overline{\Omega}_S) \quad \text{and} \quad \eta_{S,h}\big|_T \in P_1(T) \quad \forall\, T \in \mathcal{T}_h^S \Big\}. \tag{4.37}$$

Note that the foregoing definitions mean that we are considering PEERS elements (cf. [2]) for the Stokes problem, whereas for the Darcy-Forchheimer part we use Raviart-Thomas elements (cf. [6, 3.12, Ch. III]. Now, denoting by Σ_h the partition of Σ inherited from the interior triangulations, we assume, without loss of generality, that the resulting number of edges in Σ_h is even. Then, we let Σ_{2h} be the partition of Σ that arises by joining pairs of adjacent edges of Σ_h, and denote the resulting edges still by e. Since Σ_h is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded), we deduce that so is Σ_{2h}. Hence, denoting by x_0 and x_1 the extreme points of Σ, we define,

$$\Lambda_h^S(\Sigma) := \Big\{ \psi_h \in \mathcal{C}(\Sigma): \ \psi_h\big|_e \in P_1(e) \quad \forall\, e \in \Sigma_{2h} \quad \text{and} \quad \psi_h(x_0) = \psi_h(x_1) = 0 \Big\}, \tag{4.38}$$

$$\boldsymbol{\Lambda}_h^S(\Sigma) := \big[\Lambda_h^S(\Sigma)\big]^2 \cap \widehat{\mathbf{H}}^{1/2}_{00}(\Sigma), \tag{4.39}$$

$$\Lambda_h^D(\Sigma) := \Big\{ \xi_h : \Sigma \to R: \quad \xi_h\big|_e \in P_0(e) \quad \forall\, e \in \Sigma_h \Big\}. \tag{4.40}$$

We stress here that $\Lambda_h^D(\Sigma)$ is in fact a finite element subspace of $W^{1/3,3/2}(\Sigma)$ since, as established in [28, Theorem 1.5.2.3-(a)], $\prod_{e\in\Sigma_h} W^{1-1/p,p}(e)$ coincides with $W^{1-1/p,p}(\Sigma)$, without extra conditions, when $1 < p < 2$ (in this case $p = 3/2$).

We now turn to state the assumptions under which it is possible to ensure the validity of **(H.1)**, **(H.2)** and **(H.3)**. Notice first that **(H.2)** holds trivially from the definitions given in (4.38)-(4.40). Next, for the Stokes terms of **(H.1)** and **(H.3)** (that is, (4.29) and (4.31)), following [25] we introduce the hypothesis of quasiuniformity in a neighbourhood Ω_Σ^S of the interface Σ on the Ω_S-side. More precisely, we assume that Ω_Σ^S has a Lipschitz-continuous boundary and that there exists $c > 0$, independent of h, such that

$$\max_{T\subset\Omega_\Sigma^S} h_T \le c \min_{T\subset\Omega_\Sigma^S} h_T \quad \forall\, h < h_0.$$

Under this new hypothesis, it is enough to prove the Darcy-Forchheimer part of **(H.1)** and **(H.3)** (that is, (4.30) and (4.32)). To this end, we introduce the Raviart-Thomas interpolation operator of lowest order in Ω_D. Indeed, given a sufficiently smooth vector field $\mathbf{v} : \Omega_D \to \mathrm{R}^2$, we define $\Pi_h^D(\mathbf{v})$ as the unique element in $\mathbf{H}_h(\Omega_D)$ such that

$$\int_e \Pi_h^D(\mathbf{v}) \cdot \mathbf{n} = \int_e \mathbf{v} \cdot \mathbf{n} \quad \forall\, e \in \mathcal{E}_h^D, \tag{4.41}$$

where $\mathcal{E}_h^D$ is the set of edges of $\mathcal{T}_h^D$. The main properties of this operator are:

(a) for each $p \in]2, +\infty[$, the interpolation operator Π_h^D is well-defined in $\mathbf{W}^{0,p}(\mathrm{div};\Omega_D)$ (cf. [6, III.3.3]),

(b) for each $p \in]2, +\infty[$, there holds

$$\big(\mathrm{div}\, \Pi_h^D(\mathbf{v}), q_h\big)_D = (\mathrm{div}\, \mathbf{v}, q_h)_D \quad \forall\, q_h \in L_h(\Omega_D), \quad \forall\, \mathbf{v} \in \mathbf{W}^{0,p}(\mathrm{div};\Omega_D),$$

(c) if $\mathbf{v}\cdot\mathbf{n}\in\Lambda_h^D(\Sigma)$, then $\Pi_h^D(\mathbf{v})\cdot\mathbf{n}=\mathbf{v}\cdot\mathbf{n}$,

(d) there exists $c_D>0$, independent of h, such that for each $p\in]2,+\infty[$ there holds

$$\left\| \Pi_h^D(\mathbf{v}) \right\|_{0,p;\Omega_D} \leq c_D \| \mathbf{v} \|_{3,\mathrm{div};\Omega_D} \quad \forall\, \mathbf{v}\in\mathbf{W}^{0,p}(\mathrm{div};\Omega_D)\,.$$

Notice that this estimate comes from [30, Lemmas 3.13 and 3.15], but in this case, the regularity $W^{\delta,p}(\Omega_D)$ is not necessary.

The following results provide us the fulfillment of the remaining hypotheses to establish the well-posedness of the fully-mixed finite element method given by (4.27) and the finite element spaces (4.33)-(4.40).

Lemma 4.7. *There exists $\widehat{\beta}>0$, independent of h, such that*

$$\sup_{\substack{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{D,h},q_{D,h})}{\|\mathbf{v}_{D,h}\|_{3,\mathrm{div};\Omega_D}} \geq \widehat{\beta}\,\|q_{D,h}\|_{0,3/2;\Omega_D} \quad \forall\, q_{D,h}\in L_h(\Omega_D).$$

Proof. Similarly as in [34, Example 3], we proceed locally on each triangle of $\mathcal{T}_h^D$. More precisely, we consider $q_{D,h}\in L_h(\Omega_D)$ and define $\widehat{q}_h\in L^3(\Omega_D)$ as

$$\widehat{q}_h\big|_T := \widehat{q_{D,h}\big|_T} \quad \forall\, T\in\mathcal{T}_h^D,$$

in agreement with the notation introduced in Lemma 4.1. Then, we set $\widetilde{q}_h\in L_0^3(\Omega_D)$ as

$$\widetilde{q}_h = \widehat{q}_h - \frac{1}{|\Omega_D|}\int_{\Omega_D}\widehat{q}_h.$$

Similarly to the second part of the proof of the continuous inf-sup condition for $\mathbb{B}$ in Lemma 4.3, we look for $\mathbf{z}_{D,h}\in\mathbf{H}_h(\Omega_D)$ such that $\mathrm{div}\,\mathbf{z}_{D,h}=\widetilde{q}_h$. To this end, we consider the problem: Find $w\in W^{1,3}(\Omega_D)$ such that

$$\Delta w=\widetilde{q}_h \quad \text{in} \quad \Omega_D, \quad \nabla w\cdot\mathbf{n}=0 \quad \text{on} \quad \partial\Omega_D, \quad \text{and} \quad \int_{\Omega_D} w = 0. \tag{4.42}$$

Since $\widetilde{q}_h\in L_0^3(\Omega_D)$, we deduce from the analysis in [13] that the foregoing problem has a unique solution $w\in W^{1+\delta,3}(\Omega_D)$ with $0<\delta<1/3$, and notice that $\nabla w\in\mathbf{W}^{0,3}(\mathrm{div};\Omega_D)$. Then, defining $\mathbf{z}_{D,h}:=\Pi_h^D(\nabla w)$, we find that the continuous dependence result for (4.42) and the properties of the Raviart-Thomas interpolator imply that

$$\|\mathbf{z}_{D,h}\|_{0,3;\Omega_D} \leq C_D\|\widehat{q}_h\|_{0,3;\Omega_D} \leq C_D\|q_{D,h}\|_{0,3/2;\Omega_D}^{1/2}$$

and

$$\|\mathrm{div}\,\mathbf{z}_{D,h}\|_{0,3;\Omega_D} = \left\|\mathrm{div}\,\Pi_h^D(\nabla w)\right\|_{0,3;\Omega_D} \leq \|\widehat{q}_h\|_{0,3;\Omega_D} \leq \|q_{D,h}\|_{0,3/2;\Omega_D}^{1/2}.$$

Therefore, bounding below by $\mathbf{z}_{D,h}\in\mathbf{H}_h(\Omega_D)$, we deduce that

$$\sup_{\substack{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{(\mathrm{div}\,\mathbf{v}_{D,h},q_{D,h})_D}{\|\mathrm{div}\,\mathbf{v}_{D,h}\|_{0,3;\Omega_D}} \geq \frac{1}{C_D+1}\|q_{D,h}\|_{0,3/2;\Omega_D} \quad \forall\, q_{D,h}\in L_h(\Omega_D)\,,$$

which ends the proof. □

Lemma 4.8. *There exists $\widehat{\beta}_1 > 0$, independent of h, such that*

$$\sup_{\substack{\mathbf{v}_{D,h}\in\widetilde{X}_{D,h}\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{3,\mathrm{div};\Omega}} \geq \widehat{\beta}_1 \|\xi_h\|_{1/3,3/2;\Sigma} \quad \forall\, \xi_h \in \Lambda_h^D(\Sigma)\,.$$

Proof. Given $\xi_h \in \Lambda_h^D(\Sigma) \subseteq W^{1/3,3/2}(\Sigma)$, we proceed exactly as in the second part of the proof of Lemma 4.4 to derive the existence of $\mathbf{z}_D \in \mathbf{W}^{0,3}(\mathrm{div};\Omega_D)$ such that $\mathrm{div}\,\mathbf{z}_D \in P_0(\Omega_D)$ and

$$\langle \mathbf{z}_D\cdot\mathbf{n}, \xi_h\rangle_\Sigma \;\geq\; \frac{1}{2\,C_D}\|\xi_h\|_{1/3,3/2;\Sigma}\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D}\,,$$

where C_D is the same constant from that proof. Then, according to the properties of Π_h^D, there holds $\mathrm{div}\,\Pi_h^D(\mathbf{z}_D) \in P_0(\Omega_D)$,

$$\langle \mathbf{z}_D\cdot\mathbf{n}, \xi_h\rangle_\Sigma \;=\; \int_\Sigma (\Pi_h^D(\mathbf{z}_D)\cdot\mathbf{n})\,\xi_h \quad\text{and}\quad \|\Pi_h^D(\mathbf{z}_D)\|_{0,3;\Omega_D} \;\leq\; c_D\|\mathbf{z}_D\|_{3,\mathrm{div};\Omega_D}\,.$$

In this way, bounding above with $\Pi_h^D(\mathbf{z}_D) \in \widetilde{X}_{D,h}$, we find that

$$\sup_{\substack{\mathbf{v}_{D,h}\in\widetilde{X}_{D,h}\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{3,\mathrm{div};\Omega}} \;\geq\; \frac{\langle \Pi_h^D(\mathbf{z}_D)\cdot\mathbf{n}, \xi_h\rangle_\Sigma}{\|\Pi_h^D(\mathbf{z}_D)\|_{3,\mathrm{div};\Omega_D}} \;\geq\; \widehat{\beta}_1\|\xi_h\|_{1/3,3/2;\Sigma}\,,$$

with $\widehat{\beta}_1 \;=\; \dfrac{1}{2\,c_D\,C_D}$, which finishes the proof. □

Consequently, the analysis of the present Section 4.2 and Theorem 3.14 imply the existence of a unique $(\vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{\gamma}}_h) \in X_h \times Y_h \times Z_h$ solution of the Galerkin scheme (4.27) with the finite element subspaces introduced in Section 4.2.2. Moreover, a straightforward application of the abstract result given by Theorem 3.23 to the present context yields the a priori error estimate

$$\begin{aligned}\|\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\| + \|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\| + \|\vec{\boldsymbol{\gamma}}-\vec{\boldsymbol{\gamma}}_h\| \;\leq\; C\Big\{&\mathrm{dist}\big(\vec{\boldsymbol{\sigma}}, X_h\big) \;+\; \mathrm{dist}\big(\vec{\mathbf{u}}, Y_h\big) \;+\; \mathrm{dist}\big(\vec{\boldsymbol{\gamma}}, Z_h\big)\\ + \mathcal{G}\big(\mathrm{dist}\,(\vec{\boldsymbol{\sigma}}, X_h)\big) \;+\; \sum_{j=1}^{2}\mathcal{G}\big((\mathrm{dist}\,(\vec{\boldsymbol{\sigma}}, X_h))^{p_j-1}\big) &\;+\; \mathcal{G}\big(\mathrm{dist}\,(\vec{\mathbf{u}}, Y_h)\big) \;+\; \mathcal{G}\big(\mathrm{dist}\,(\vec{\boldsymbol{\gamma}}, Z_h)\big)\Big\}\,,\end{aligned} \tag{4.43}$$

with a positive constant C independent of h. Therefore, in order to derive from (4.43) the theoretical rates of convergence of our discrete scheme, in what follows we recall from [10, Section 5], [16], and [19], the approximation properties of the finite element subspaces involved, which are named after the unknowns to which they are applied later on:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}_S})$ there exists $C > 0$, independent of h, such that for each $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div};\Omega_S)\cap\mathbb{H}^1(\Omega_S)$ with $\mathbf{div}\,\boldsymbol{\tau}_S \in H^1(\Omega_S)$ there holds

$$\inf_{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_{h,0}(\Omega_S)}\|\boldsymbol{\tau}_S - \boldsymbol{\tau}_{S,h}\|_{\mathbf{div};\Omega_S} \;\leq\; C\,h\Big\{\|\boldsymbol{\tau}_S\|_{1,\Omega_S} + \|\mathbf{div}\,\boldsymbol{\tau}_S\|_{1,\Omega_S}\Big\}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_D})$ there exists $C > 0$, independent of h, such that for each $\mathbf{v}_D \in \mathbf{W}_{\Gamma_D}^{0,3}(\mathrm{div};\Omega_D)\cap\mathbf{W}^{1,3}(\Omega_D)$ with $\mathrm{div}\,\mathbf{v}_D \in H^1(\Omega_D)$ there holds

$$\inf_{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)}\|\mathbf{v}_D - \mathbf{v}_{D,h}\|_{3,\mathrm{div};\Omega_D} \;\leq\; C\,h\Big\{\|\mathbf{v}_D\|_{1,3;\Omega_D} + \|\mathrm{div}\,\mathbf{v}_D\|_{1,\Omega_D}\Big\}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\varphi}})$ there exists $C > 0$, independent of h, such that for each $\boldsymbol{\psi} \in \widehat{\mathbf{H}}_{00}^{1/2}(\Sigma) \cap \mathbf{H}^{3/2}(\Sigma)$ there holds

$$\inf_{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^S(\Sigma)} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1/2,\Sigma} \;\leq\; C\, h\, \|\boldsymbol{\psi}\|_{3/2,\Sigma}\,.$$

$(\mathbf{AP}_h^{\lambda})$ there exists $C > 0$, independent of h, such that for each $\xi \in W^{1,3/2}(\Sigma)$ there holds

$$\inf_{\xi_h \in \Lambda_h^D(\Sigma)} \|\xi - \xi_h\|_{1/3,3/2;\Sigma} \;\leq\; C\, h^{2/3}\, \|\xi\|_{1,3/2;\Sigma}\,.$$

$(\mathbf{AP}_h^{\mathbf{u}_S})$ there exists $C > 0$, independent of h, such that for each $\mathbf{v}_S \in \mathbf{H}^1(\Omega_S)$ there holds

$$\inf_{\mathbf{v}_{S,h} \in \mathbf{L}_h(\Omega_S)} \|\mathbf{v}_S - \mathbf{v}_{S,h}\|_{0,\Omega_S} \;\leq\; C\, h\, \|\mathbf{v}_S\|_{1,\Omega_S}\,.$$

$(\mathbf{AP}_h^{p_D})$ there exists $C > 0$, independent of h, such that for each $q_D \in L_0^{3/2}(\Omega_D) \cap W^{1,3/2}(\Omega_D)$ there holds

$$\inf_{q_{D,h} \in L_h(\Omega_D)} \|q_D - q_{D,h}\|_{0,3/2;\Omega_D} \;\leq\; C\, h\, \|q_D\|_{1,3/2;\Omega_D}\,.$$

$(\mathbf{AP}_h^{\gamma_S})$ there exists $C > 0$, independent of h, such that for each $\boldsymbol{\eta}_S \in \mathbb{L}^2_{\mathtt{skew}}(\Omega_S) \cap \mathbb{H}^1(\Omega_S)$ there holds

$$\inf_{\boldsymbol{\eta}_{S,h} \in \mathbb{L}_h(\Omega_S)} \|\boldsymbol{\eta}_S - \boldsymbol{\eta}_{S,h}\|_{0,\Omega_S} \;\leq\; C\, h\, \|\boldsymbol{\eta}_S\|_{1,\Omega_S}\,.$$

It follows that there exist positive constants $C(\vec{\boldsymbol{\sigma}})$, $C(\vec{\mathbf{u}})$, and $C(\vec{\boldsymbol{\gamma}})$, depending on the extra regularity assumptions for $\vec{\boldsymbol{\sigma}}$, $\vec{\mathbf{u}}$, and $\vec{\boldsymbol{\gamma}}$, respectively, and whose explicit expressions are obtained from the right hand side of the foregoing approximation properties, such that

$$\operatorname{dist}\big(\vec{\boldsymbol{\sigma}}, X_h\big) \;\leq\; C(\vec{\boldsymbol{\sigma}})\, h\,, \quad \operatorname{dist}\big(\vec{\mathbf{u}}, Y_h\big) \;\leq\; C(\vec{\mathbf{u}})\, h^{2/3}\,, \quad \text{and} \quad \operatorname{dist}\big(\vec{\boldsymbol{\gamma}}, Z_h\big) \;\leq\; C(\vec{\boldsymbol{\gamma}})\, h\,.$$

In turn, we recall from Lemmas 4.5 and 4.6 that $p_1 = 2$ and $p_2 = 3$, which yields the conjugates $p_1' = 2$ and $p_2' = 3/2$, and hence the real function $\mathcal{G}$ introduced in the statement of Lemma 3.22 becomes $\mathcal{G}(x) = x + x^{2/3} + x^{3/4} + x^{1/2} \quad \forall\, x \in \mathrm{R}^+$. In this way, replacing the above expressions into (4.43), choosing only the dominant powers of h in the resulting expressions involving $\mathcal{G}$, and performing some minor algebraic manipulations, we derive the existence of a positive constant $C(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{\gamma}})$, depending on $C(\vec{\boldsymbol{\sigma}})$, $C(\vec{\mathbf{u}})$, and $C(\vec{\boldsymbol{\gamma}})$, such that

$$\big\|\, \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \,\big\| + \big\|\, \vec{\mathbf{u}} - \vec{\mathbf{u}}_h \,\big\| + \big\|\, \vec{\boldsymbol{\gamma}} - \vec{\boldsymbol{\gamma}}_h \,\big\| \;\leq\; C(\vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{\gamma}}) \Big\{ h + h^{2/3} + h^{1/2} + h^{1/3} \Big\}\,, \tag{4.44}$$

from which at least a sub-optimal rate of convergence of order $O(h^{1/3})$ is confirmed for our Galerkin scheme. Nevertheless, the numerical results to be reported below in Section 5 show more generous experimental rates of convergence than expected, which certainly suggests that eventually some technical difficulties of the analysis might be stopping us of proving better theoretical results.

5 Numerical Results

In this section, we present some numerical examples that illustrate the performance of the fully-mixed finite element method that has been presented in this work. The computational implementation was carried out in two codes, each one using a different finite element for the Stokes part of the problem: one based on MATLAB (Examples 1 and 2) with PEERS elements, and one based on `FreeFem++` (cf.

[29]) with AFW elements (Examples 3 and 4), and whose characterization is given by the following set of spaces, which respectively approximate the pseudostress $\boldsymbol{\sigma}_S$, velocity $\mathbf{u}_S$ and vorticity $\boldsymbol{\gamma}_S$:

$$\begin{aligned}\mathbf{H}_h(\Omega_S) &:= \Big\{\boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega_S): \ \boldsymbol{\tau}_h\big|_T \in \mathbf{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^S\Big\},\\ L_h(\Omega_S) &:= \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega_S): \ \mathbf{v}_h\big|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^S\Big\},\\ \mathbb{L}_h(\Omega_S) &:= \Big\{\eta_h \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}: \ \eta_h \in L^2(\Omega_S) \quad \text{and} \quad \eta_h\big|_T \in P_0(T) \quad \forall\, T \in \mathcal{T}_h^S\Big\}.\end{aligned}$$

For the sake of briefness, we only mention based on [24] (in addition to the original works by Arnold, Falk and Winther, cf. [3, 4]) that this finite element does satisfy the inf-sup conditions (4.29) and (4.31) and that it yields the same approximation properties as those mentioned for PEERS.

Concerning the nonlinear system (4.27), this is linearized using Newton's method. The resulting iterative scheme is: Given $\mathbf{u}_{D,h}^0 \neq \mathbf{0}$, for $m \geq 1$, find $(\vec{\boldsymbol{\sigma}}_h^m, \vec{\mathbf{u}}_h^m, \vec{\boldsymbol{\gamma}}_h^m) \in X_h \times Y_h \times Z_h$ such that

$$\Big[\widetilde{\mathbb{A}}(\mathbf{u}_{D,h}^{m-1})(\vec{\boldsymbol{\sigma}}_h^m), \vec{\boldsymbol{\tau}}_h\Big] + \Big[\mathbb{B}_1(\vec{\boldsymbol{\tau}}_h), \vec{\mathbf{u}}_h^m\Big] + \Big[\mathbb{B}(\vec{\boldsymbol{\tau}}_h), \vec{\boldsymbol{\gamma}}_h^m\Big] = \Big[\widetilde{F}^{m-1}, \vec{\boldsymbol{\tau}}_h\Big], \tag{5.1a}$$

$$\Big[\mathbb{B}_1(\vec{\boldsymbol{\sigma}}_h^m), \vec{\mathbf{v}}_h\Big] - \Big[\mathbb{C}(\vec{\mathbf{u}}_h^m), \vec{\mathbf{v}}_h\Big] = \big[G, \vec{\mathbf{v}}_h\big], \tag{5.1b}$$

$$\Big[\mathbb{B}(\vec{\boldsymbol{\sigma}}_h^m), \vec{\boldsymbol{\eta}}_h\Big] = \big[E, \vec{\boldsymbol{\eta}}_h\big], \tag{5.1c}$$

for all $(\vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{\eta}}_h) \in X_h \times Y_h \times Z_h$, where

$$\begin{aligned}\Big[\widetilde{\mathbb{A}}(\mathbf{u}_{D,h}^{m-1})(\vec{\boldsymbol{\sigma}}_h^m), \vec{\boldsymbol{\tau}}_h\Big] &:= \big(\mathcal{A}_S(\boldsymbol{\sigma}_{S,h}^m), \boldsymbol{\tau}_{S,h}\big)_S + \frac{\mu}{\rho}\big(\mathbf{K}^{-1}\mathbf{u}_{D,h}^m, \mathbf{v}_{D,h}\big)_D\\ &\quad + \frac{\mathtt{f}}{\rho}\Big(|\mathbf{u}_{D,h}^{m-1}|\,\mathbf{u}_{D,h}^m, \mathbf{v}_{D,h}\Big)_D + \frac{\mathtt{f}}{\rho}\left(\frac{\mathbf{u}_{D,h}^{m-1}\cdot\mathbf{u}_{D,h}^m}{|\mathbf{u}_{D,h}^{m-1}|}, \mathbf{u}_{D,h}^{m-1}\cdot\mathbf{v}_{D,h}\right)_D,\end{aligned}$$

and

$$\Big[\widetilde{F}^{m-1}, \vec{\boldsymbol{\tau}}_h\Big] := \big[F, \vec{\boldsymbol{\tau}}_h\big] + \frac{\mathtt{f}}{\rho}\Big(|\mathbf{u}_{D,h}^{m-1}|\mathbf{u}_{D,h}^{m-1}, \mathbf{v}_{D,h}\Big)_D.$$

The linear systems (5.1) are solved by the multifrontal method from [14]. Then, iterations are simply stopped whenever the relative error between two consecutive steps of the complete coefficient vector measured in the discrete ℓ^2 norm is sufficiently small, that is,

$$\frac{\|\,\mathbf{coeff}^{\,m+1} - \mathbf{coeff}^{\,m}\,\|_{\ell^2}}{\|\,\mathbf{coeff}^{\,m+1}\,\|_{\ell^2}} < \mathtt{tol},$$

where `tol` is a specified tolerance. We also recall that the pressure p_S is post-processed as suggested by (2.3). In this way, we define the error per variable

$$\begin{aligned}&e(\boldsymbol{\sigma}_S) := \|\,\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\,\|_{\mathbf{div};\Omega_S}, \quad e(\mathbf{u}_D) := \|\,\mathbf{u}_D - \mathbf{u}_{D,h}\,\|_{3,\mathrm{div};\Omega_D}, \quad e(\boldsymbol{\varphi}) := \|\,\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\,\|_{1/2,2,\mathtt{int};\Sigma},\\ &e(\lambda) := \|\,\lambda - \lambda_h\,\|_{0,3/2;\Sigma}, \quad e(\mathbf{u}_S) := \|\,\mathbf{u}_S - \mathbf{u}_{S,h}\,\|_{0,2;\Omega_S}, \quad e(p_D) := \|\,p_D - p_{D,h}\,\|_{0,3/2;\Omega_D},\end{aligned}$$

$$e(\boldsymbol{\gamma}_S) := \|\,\boldsymbol{\gamma}_S - \boldsymbol{\gamma}_{S,h}\,\|_{0,2;\Omega_S}, \quad e(p_S) := \|\,p_S - p_{S,h}\,\|_{0,2;\Omega_S};$$

as well as their corresponding rates of convergence

$$\begin{aligned}&r(\boldsymbol{\sigma}_S) := \frac{\log(e(\boldsymbol{\sigma}_S)/e'(\boldsymbol{\sigma}_S))}{\log(h_S/h'_S)}, \quad r(\mathbf{u}_D) := \frac{\log(e(\mathbf{u}_D)/e'(\mathbf{u}_D))}{\log(h_D/h'_D)}, \quad r(\boldsymbol{\varphi}) := \frac{\log(e(\boldsymbol{\varphi})/e'(\boldsymbol{\varphi}))}{\log(h_\Sigma/h'_\Sigma)},\\ &r(\lambda) := \frac{\log(e(\lambda)/e'(\lambda))}{\log(h_\Sigma/h'_\Sigma)}, \quad r(\mathbf{u}_S) := \frac{\log(e(\mathbf{u}_S)/e'(\mathbf{u}_S))}{\log(h_S/h'_S)}, \quad r(p_D) := \frac{\log(e(p_D)/e'(p_D))}{\log(h_D/h'_D)},\end{aligned}$$

$$r(\boldsymbol{\gamma}_S) := \frac{\log(e(\boldsymbol{\gamma}_S)/e'(\boldsymbol{\gamma}_S))}{\log(h_S/h'_S)}, \quad r(p_S) := \frac{\log(e(p_S)/e'(p_S))}{\log(h_S/h'_S)},$$

where $h_\star$ and $h'_\star$ ($\star \in \{S, D, \Sigma\}$) denote two consecutive meshes with errors e and e', respectively. Notice that, since the natural norms to measure the error of the interface unknowns $\|\lambda - \lambda_h\|_{1/3,3/2;\Sigma}$ and $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,2;\Sigma}$ are not computable, we have decided to replace them respectively by $\|\cdot\|_{0,3/2;\Sigma}$ and $\|\cdot\|_{1/2,2,\mathtt{int};\Sigma}$, where the last one is defined based on the fact that $\mathbf{H}^{1/2}(\Sigma)$ is the interpolating space between $\mathbf{L}^2(\Sigma)$ and $\mathbf{H}^1(\Sigma)$:

$$\|\boldsymbol{\psi}\|_{1/2,2,\mathtt{int};\Sigma} := \|\boldsymbol{\psi}\|_{0,2;\Sigma}^{1/2} \|\boldsymbol{\psi}\|_{1,2;\Sigma}^{1/2} \quad \forall\, \boldsymbol{\psi} \in \mathbf{H}^1(\Sigma).$$

Additionally, regarding the conditions $\int_{\Omega_\star} p_{\star,h} = 0$ for $\star \in \{S, D\}$, and $\langle \boldsymbol{\varphi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0$, these are imposed via a penalization strategy. In what follows, we consider $\mu = \rho = \kappa = 1$, $\mathbf{K} = \mathbf{I}$, and $\mathtt{tol} = 10^{-8}$, and take $\mathbf{u}_{D,h}^0$ as the discrete solution of the linear problem arising after the elimination of the nonlinearity in the Darcy-Forchheimer equations, that is the second term on the left-hand side of (2.5a).

We will see in the subsequent examples that the order $\mathcal{O}(h^{1/3})$ is easily achieved in all variables. Moreover, they suggest that we are in the presence of a first order numerical method, i.e., $\mathcal{O}(h)$, leaving open the question whether this can be theoretically proved or not.

5.1 Example 1

First, we consider the two-dimensional domain given by $\Omega_S :=]0,1[\times]0.5,1[$, $\Omega_D :=]0,1[\times]0,0.5[$ and $\Sigma :=]0,1[\times\{0.5\}$. Then we take $\mathtt{f} = 1$ and let the source terms $\mathbf{f}_S$, $\mathbf{g}_D$ and f_D be such that the exact solution to the SDF problem is defined by

$$\mathbf{u}_S := \begin{pmatrix} -x\,\sin(2\pi x)\,(x-1)\,(y-1)\,\exp(xy)\,(2-x+xy) \\[1ex] (y-1)^2\,\exp(xy)\Big(2x\,\sin(2\pi x) - \sin(2\pi x) - 2\pi x\,\cos(2\pi x) \\ +\, 2\pi x^2\,\cos(2\pi x) - xy\,\sin(2\pi x) + yx^2\,\sin(2\pi x)\Big) \end{pmatrix},$$

$$p_S := -\pi\,\cos(\pi x/2)\Big(y + 0.5 - 2\cos(\pi(y+0.5)/2)^2\Big)/4 + 4.433\cdot 10^{-1},$$

$$\boldsymbol{\sigma}_S := \mu\Big(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathtt{t}}\Big) - p_S\,\mathbf{I}, \quad \boldsymbol{\gamma}_S := \frac{1}{2}\Big(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}\Big), \quad \boldsymbol{\varphi} := -\mathbf{u}_S\big|_\Sigma,$$

$$\mathbf{u}_D := \begin{pmatrix} -y\,\sin(\pi y)\Big(-\sin(2\pi x) + 2x\,\sin(2\pi x) - 2\pi x\,\cos(2\pi x) + 2\pi x^2\,\cos(2\pi x)\Big) \\ -x(x-1)\,\sin(2\pi x)\,\Big(\sin(\pi y) + \pi y\,\cos(\pi y)\Big) \end{pmatrix},$$

$$p_D := xy\,(1-x)\,\sin(2\pi x)\,\sin(\pi y), \quad \lambda := p_D\big|_\Sigma.$$

Notice that this solution does not meet all boundary conditions, hence the right hand side of (4.27) must be modified properly. See Table 5.1 for the corresponding convergence history of a sequence of quasi-uniform mesh refinements.

5.2 Example 2

Next, we consider the three-dimensional domain defined by $\Omega_S :=]0,1[^2\times]0.5,1[$, $\Omega_D :=]0,1[^2\times]0,0.5[$ and $\Sigma :=]0,1[^2\times\{0.5\}$. Then we take again $\mathtt{f} = 1$ and let the source terms $\mathbf{f}_S$, $\mathbf{g}_D$ and f_D be such that the exact solution to the SDF problem is given by

$$\mathbf{u}_S := \nabla \times \begin{pmatrix} x^2\,(1-x^2)\,y^2\,(1-y)^2\,(1-z)^2\,\sin(\pi x) \\ x^2\,(1-x^2)\,y^2\,(1-y)^2\,(1-z)^2\,\sin(\pi y) \\ x^2\,(1-x^2)\,y^2\,(1-y)^2\,(1-z)^2\,\sin(\pi z) \end{pmatrix}, \quad p_S := (x^3+y^3)\,\exp(z) - 1.069\,,$$

$$\boldsymbol{\sigma}_S := \mu\left(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathtt{t}}\right) - p_S\,\mathbf{I}, \quad \boldsymbol{\gamma}_S := \frac{1}{2}\left(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}\right), \quad \boldsymbol{\varphi} := -\mathbf{u}_S\big|_\Sigma\,,$$

$$\mathbf{u}_D := \begin{pmatrix} yz\,(1-y)\,\sin(2\pi y)\,\sin(\pi z)\Big((1-x)\,\sin(2\pi x) - x\,\sin(2\pi x) + 2\pi x\,(1-x)\,\cos(2\pi x)\Big) \\ xz\,(1-x)\,\sin(2\pi x)\,\sin(\pi z)\Big((1-y)\,\sin(2\pi y) - y\,\sin(2\pi y) + 2\pi y\,(1-y)\,\cos(2\pi y)\Big) \\ xy\,(1-x)\,(1-y)\,\sin(2\pi x)\,\sin(2\pi y)\Big(\sin(\pi z) + \pi z\,\cos(\pi z)\Big) \end{pmatrix},$$

$$p_D := (x-x^2)\,(y-y^2)\,(z-z^2) - 4.629\cdot 10^{-3}\,, \quad \lambda := p_D\big|_\Sigma\,.$$

Here, we have assumed that the foregoing analysis can be extended to the three-dimensional case with some minor changes on the interface conditions. See Table 5.2 for the corresponding convergence history.

5.3 Example 3

Here, we test our method with $\Omega_S :=]-0.5, 0.5[\times]0, 0.5[$, $\Omega_D :=]-0.5, 0.5[\times]-0.5, 0[$, $\Sigma :=]-0.5, 0.5[\times\{0\}$ and $\mathtt{f} = 1$. Then we consider source terms $\mathbf{f}_S$, $\mathbf{g}_D$ and f_D such that the exact solution to the SDF problem is defined by

$$\mathbf{u}_S := \begin{pmatrix} 16y\cos(\pi x)^2(y^2-0.25) \\ 8\pi\cos(\pi x)\sin(\pi x)(y^2-0.25)^2 \end{pmatrix}, \quad p_S := \exp(y)\sin(x),$$

$$\boldsymbol{\sigma}_S := \mu\left(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathtt{t}}\right) - p_S\,\mathbf{I}, \quad \boldsymbol{\gamma}_S := \frac{1}{2}\left(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}\right), \quad \boldsymbol{\varphi} := -\mathbf{u}_S\big|_\Sigma,$$

$$\mathbf{u}_D := \begin{pmatrix} -2y\cos(\pi x)^2 \\ -2\pi\cos(\pi x)\sin(\pi x)(y^2-0.25) \end{pmatrix}, \quad p_D := \exp(y)\sin(x), \quad \lambda := p_D\big|_\Sigma.$$

This solution (which was also considered in [15]) does meet the boundary conditions of the problem, except for the BJS condition (2.7b), hence, the right-hand side of (4.27) must be modified accordingly. In Figure 5.1 we show part of the obtained numerical solution with 215,023 DOF using the AFW-based finite element, whereas in Table 5.3 we show the convergence history of a sequence of quasi-uniform mesh refinements.

5.4 Example 4

Finally, and inspired by [10], we focus on the performance of the numerical method with respect to the number of Newton iterations required to achieve certain tolerance given different Forchheimer numbers. Hence, we consider $\mathtt{f} \in \{10^0, 10^1, \dots, 10^6\}$, the tombstone-shaped domain described by $\Omega_S := \big\{(x,y) : x^2 + (y-0.5)^2 < 1,\ y > 0.5\big\}$, $\Omega_D :=]-0.5, 0.5[^2$ and $\Sigma :=]-0.5, 0.5[\times\{0.5\}$, and source terms $\mathbf{f}_S$, $\mathbf{g}_D$ and f_D such that the exact solution is given by

$$\mathbf{u}_S := \begin{pmatrix} \pi\cos(\pi x)\sin(\pi y) \\ -\pi\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p_S := \sin(\pi x)\sin(\pi y),$$

$$\boldsymbol{\sigma}_S := \mu\left(\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathtt{t}}\right) - p_S\,\mathbf{I}, \quad \boldsymbol{\gamma}_S := \frac{1}{2}\left(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^{\mathtt{t}}\right), \quad \boldsymbol{\varphi} := -\mathbf{u}_S\big|_\Sigma,$$

$$\mathbf{u}_D := \begin{pmatrix} -\pi\cos(\pi x)\sin(\pi y) \\ -\pi\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p_D := \sin(\pi x)\sin(\pi y), \quad \lambda := p_D\big|_\Sigma.$$

As in the previous example, not all boundary conditions are met. In addition to the source term appearing in the BJS condition (2.7b), the velocity on Γ_S is not zero, thus appearing the term $\langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \mathbf{u}_S \rangle_{\Gamma_S}$ in the right-hand side of (4.27a). In Table 5.4 we appreciate the robustness of the present mixed finite element method, as varying the Forchheimer number from 10^0 to 10^6 only increases the number of Newton iterations from 5 to 14. For the case when $\mathtt{f} = 10$, we have depicted part of the solution in Figure 5.2, while in Table 5.5 a convergence history is shown.

Finite Element: PEERS-based							
DOF	$e(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$e(\mathbf{u}_D)$	$e(p_D)$	$e(\boldsymbol{\varphi})$	$e(\lambda)$
705	1.32527	0.03714	0.30565	0.49549	0.00374	0.52250	0.05398
2,685	0.65868	0.01658	0.11955	0.24963	0.00164	0.15914	0.01176
5,945	0.43919	0.01098	0.07406	0.16668	0.00106	0.08301	0.00535
10,485	0.32946	0.00822	0.05358	0.12508	0.00079	0.05298	0.00318
23,405	0.21969	0.00548	0.03446	0.08342	0.00052	0.02844	0.00160
92,885	0.10986	0.00274	0.01662	0.04172	0.00026	0.00995	0.00054
257,205	0.06592	0.00164	0.00983	0.02503	0.00016	0.00461	0.00025
h	$r(\boldsymbol{\sigma}_S)$	$r(\mathbf{u}_S)$	$r(\boldsymbol{\gamma}_S)$	$r(\mathbf{u}_D)$	$r(p_D)$	$r(\boldsymbol{\varphi})$	$r(\lambda)$
0.12500	-	-	-	-	-	-	-
0.06250	1.00863	1.16389	1.35427	0.98908	1.19169	1.71515	2.19906
0.04167	0.99964	1.01522	1.18108	0.99611	1.06945	1.60508	1.93986
0.03125	0.99924	1.00507	1.12509	0.99803	1.03674	1.56067	1.80589
0.02083	0.99946	1.00216	1.08839	0.99899	1.01977	1.53483	1.69143
0.01042	0.99979	1.00069	1.05186	0.99964	1.00747	1.51459	1.58061
0.00625	0.99996	1.00019	1.02836	0.99990	1.00231	1.50534	1.52885

Table 5.1: Convergence history for Example 1 showing the degrees of freedom (DOF) in the entire domain, mesh sizes, errors and rates of convergence, on a set of quasi-uniform mesh refinements. Here, $h := \max\{h_S, h_D\}$.

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Finite Element: PEERS-based							
DOF	$e(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$e(\gamma_S)$	$e(\mathbf{u}_D)$	$e(p_D)$	$e(\boldsymbol{\varphi})$	$e(\lambda)$
4,550	0.51956	0.00878	0.05928	0.22385	0.00127	0.01972	0.00707
34,596	0.26126	0.00287	0.02200	0.11631	0.00061	0.01047	0.00347
114,914	0.17363	0.00141	0.01209	0.07848	0.00041	0.00563	0.00197
270,272	0.12982	0.00084	0.00782	0.05912	0.00030	0.00359	0.00127
525,438	0.10361	0.00056	0.00556	0.04739	0.00024	0.00253	0.00091
905,180	0.08619	0.00040	0.00419	0.03954	0.00020	0.00190	0.00069
h	$r(\boldsymbol{\sigma}_S)$	$r(\mathbf{u}_S)$	$r(\gamma_S)$	$r(\mathbf{u}_D)$	$r(p_D)$	$r(\boldsymbol{\varphi})$	$r(\lambda)$
0.25000	-	-	-	-	-	-	-
0.12500	0.99181	1.61242	1.42993	0.94461	1.04795	0.91282	1.02621
0.08333	1.00764	1.74790	1.47774	0.97025	1.01802	1.52985	1.39845
0.06250	1.01093	1.80488	1.51203	0.98463	1.01224	1.56381	1.52434
0.05000	1.01072	1.84195	1.53185	0.99068	1.01049	1.56763	1.50496
0.04167	1.00965	1.86284	1.54380	0.99376	1.00863	1.56932	1.48874

Table 5.2: Convergence history for Example 2 showing the degrees of freedom (DOF) in the entire domain, mesh sizes, errors and rates of convergence, on a set of quasi-uniform mesh refinements. Here, $h := \max\{h_S, h_D\}$.

Finite Element: AFW-based									
DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$e(\gamma_S)$	$e(p_S)$	$r(\boldsymbol{\sigma}_S)$	$r(\mathbf{u}_S)$	$r(\gamma_S)$	$r(p_S)$
958	0.1875	4.4741	0.0802	0.5187	0.3396	-	-	-	-
3,552	0.1085	2.1383	0.0366	0.2201	0.1020	1.3491	1.4357	1.5667	2.1989
13,736	0.0500	1.0405	0.0178	0.1068	0.0336	0.9305	0.9321	0.9336	1.4323
53,802	0.0274	0.5210	0.0089	0.0529	0.0112	1.1525	1.1464	1.1726	1.8400
215,023	0.0131	0.2572	0.0044	0.0261	0.0041	0.9510	0.9483	0.9518	1.3338
h_D	h_Σ	$e(\mathbf{u}_D)$	$e(p_D)$	$e(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\mathbf{u}_D)$	$r(p_D)$	$r(\boldsymbol{\varphi})$	$r(\lambda)$
0.2001	0.2500	0.1066	0.0188	0.5028	0.0416	-	-	-	-
0.0938	0.1250	0.0522	0.0080	0.2219	0.0169	0.9405	1.1321	1.1797	1.2957
0.0494	0.0625	0.0255	0.0039	0.1060	0.0080	1.1182	1.1238	1.0658	1.0806
0.0262	0.0312	0.0129	0.0019	0.0523	0.0039	1.0736	1.0869	1.0183	1.0194
0.0141	0.0156	0.0064	0.0010	0.0260	0.0020	1.1267	1.1305	1.0108	1.0065

Table 5.3: Convergence history for Example 3 showing the degrees of freedom (DOF) in the entire domain, mesh sizes, errors and rates of convergence, on a set of quasi-uniform mesh refinements.

Finite Element: AFW-based						
f \ h	0.3827	0.2481	0.1335	0.0664	0.0344	0.0172
10^0	5	6	6	6	6	6
10^1	7	8	9	9	9	10
10^2	7	9	10	11	12	12
10^3	9	9	10	11	13	14
10^4	9	10	11	12	13	14
10^5	9	10	11	12	14	14
10^6	9	10	11	12	14	14

Table 5.4: Performance of the iterative method (number of Newton iterations) upon variations of the Forchheimer number. Here, the global mesh size was calculated as $h = \max\{h_S, h_D\}$.

Finite Element: AFW-based									
DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$e(p_S)$	$r(\boldsymbol{\sigma}_S)$	$r(\mathbf{u}_S)$	$r(\boldsymbol{\gamma}_S)$	$r(p_S)$
230	0.3827	8.8506	0.4362	1.1581	0.7893	-	-	-	-
842	0.2481	4.8830	0.2402	0.6452	0.3772	1.3721	1.3761	1.3497	1.7037
3,003	0.1335	2.4767	0.1225	0.3271	0.1179	1.0961	1.0874	1.0969	1.8778
11,443	0.0664	1.2393	0.0616	0.1631	0.0413	0.9910	0.9842	0.9964	1.5018
44,480	0.0344	0.6171	0.0307	0.0825	0.0148	1.0581	1.0582	1.0332	1.5595
177,960	0.0172	0.3065	0.0153	0.0405	0.0056	1.0098	1.0070	1.0271	1.4025
h_D	h_Σ	$e(\mathbf{u}_D)$	$e(p_D)$	$e(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\mathbf{u}_D)$	$r(p_D)$	$r(\boldsymbol{\varphi})$	$r(\lambda)$
0.3727	0.2500	2.8412	0.4139	1.0074	0.4076	-	-	-	-
0.1901	0.1250	1.3973	0.1517	0.5125	0.2260	1.0542	1.4904	0.9749	0.8506
0.0978	0.0625	0.7094	0.0447	0.2301	0.0670	1.0192	1.8366	1.1552	1.7542
0.0535	0.0312	0.3558	0.0174	0.1111	0.0235	1.1458	1.5635	1.0509	1.5142
0.0249	0.0156	0.1769	0.0078	0.0556	0.0098	0.9151	1.0501	0.9973	1.2581
0.0145	0.0078	0.0879	0.0038	0.0285	0.0046	1.2838	1.3390	0.9654	1.1080

Table 5.5: Convergence history for Example 4 showing the degrees of freedom (DOF) in the entire domain, mesh sizes, errors and rates of convergence, on a set of quasi-uniform mesh refinements with Forchheimer number $\mathtt{f} = 10$.

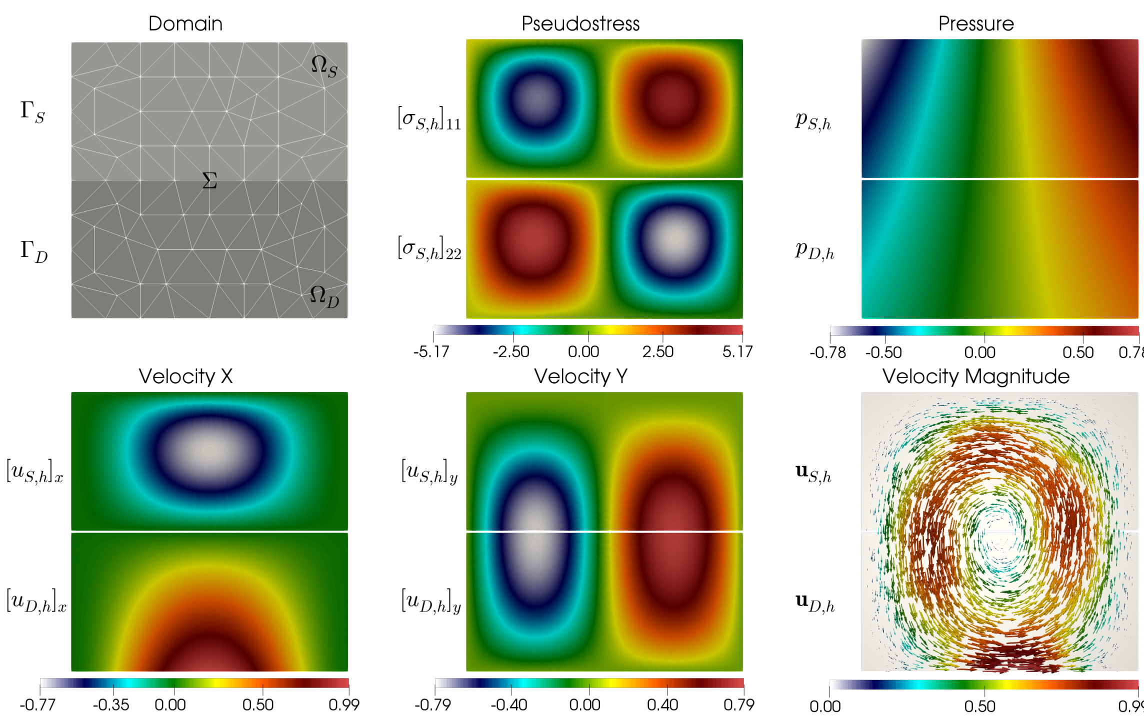

Figure 5.1: Part of the solution to Example 3. Results calculated with 215,023 DOF and an AFW-based finite element.

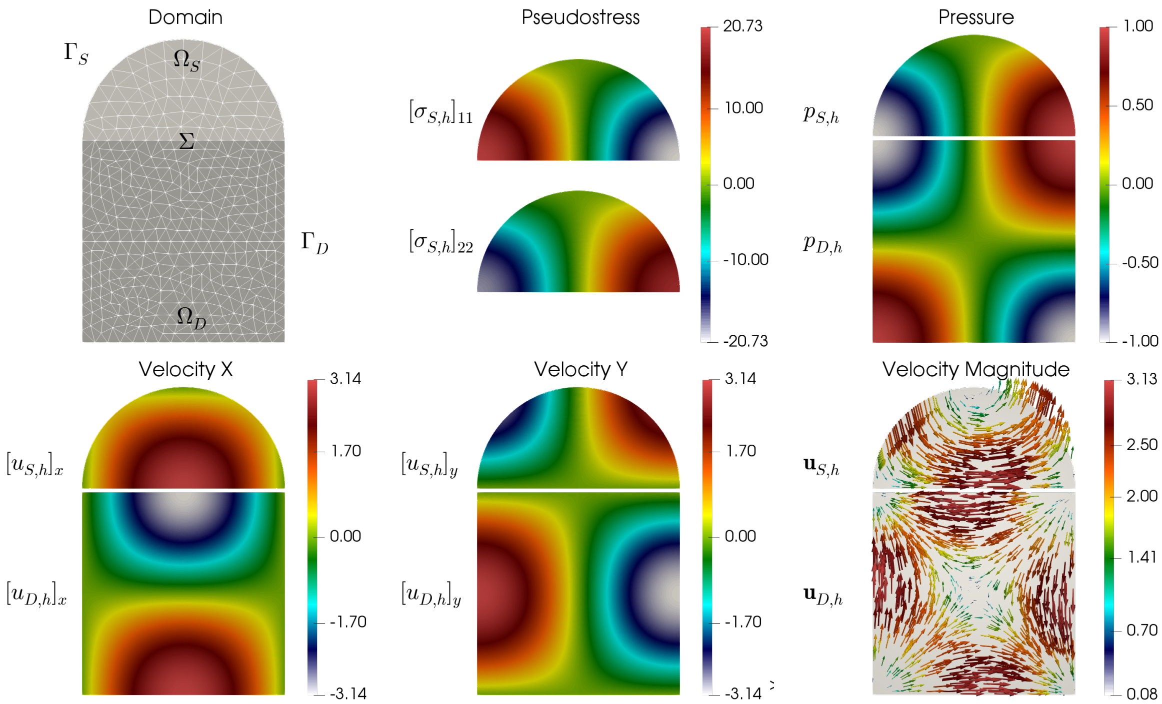

Figure 5.2: Part of the solution to Example 4. Results calculated with 177,960 DOF and an AFW-based finite element.

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