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Extensions of the standard quadratic optimization problem: strong duality, optimality, hidden convexity and S-lemma

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Abstract Many formulations of quadratic allocation problems, portfolio optimization problems, the maximum weight clique problem, among others, take the form as the well-known standard quadratic optimization (StQO) problem, which consists in minimizing a homogeneous quadratic function on the usual simplex in the non negative orthant. We propose to analyze the same problem when the simplex is substituted by a convex and compact base of any pointed, closed, convex cone (so, the cone of positive semidefinite matrices or the cone of copositive matrices are particular instances). Three main duals (for which a semi-infinite formulation of the primal problem is required) are associated, and we establish some characterizations of strong duality with respect to each of the three duals in terms of copositivity of the Hessian of the quadratic objective function on suitable cones. Such a problem reveals a hidden convexity and the validity of S-lemma. In case of bidimensional quadratic optimization problems, copositivity of the Hessian of the objective function is characterized, and the case when every local solution is global.

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1 Introduction

Let A be any real symmetric matrix of order n, $C \subseteq \mathbb{R}^n$ be a pointed, closed, convex cone having non-empty (topological) interior. In this paper we will deal with the following optimization problem :

$$\mu_q \doteq \min \Big\{ \frac{1}{2} x^\top A x : \ e^\top x = 1, \ x \in C \Big\}, \tag{1}$$

where $e \in \text{int } C^*$. Here, C^* is the non-negative polar cone of C, defined by $C^* \doteq \{y \in \mathbb{R}^n : y^\top x \geq 0, \forall\, x \in C\}$. The feasible set to (1), denoted by $K \doteq \{x \in C : \ e^\top x = 1\}$, is a convex and compact base of C, i.e., $C = \bigcup_{t \geq 0} tK$. Notice that, due to the structure of K, any quadratic objective function may be reduced to an homogeneous one, since

$$\frac{1}{2} x^\top A x + a^\top x = \frac{1}{2} x^\top (A + e a^\top + a e^\top) x, \quad \forall\, a \in \mathbb{R}^n.$$

The standard quadratic optimization (StQO) problem introduced in Bomze [7] and further developed in [8–12] and references therein, corresponds to the case $C = \mathbb{R}^n_+$, $e = \mathbf{1} \doteq (1 \ \dots \ 1)^\top \in \text{int } \mathbb{R}^n$:

$$\min_{x \in \Delta} \frac{1}{2} x^\top A x, \tag{2}$$

with Δ being the simplex $\{x \in \mathbb{R}^n : \ \mathbf{1}^\top x = 1, \ x \geq 0\}$. Such a formulation has been proved to be very important in applications since it models quadratic allocation problems [30], portfolio optimization problems [34,35], the maximum weight clique problem [37,27], among others. In addition, this problem retains, as asserted in [12], most of the complexity of the general quadratic case having a polyhedron as a feasible set.

We will show in Section 3 that, as occurs for the special problem (2) (see [16]), problem (1) can be re-written as

$$\mu_q = \sup_{\lambda \in \mathbb{R}} \Big\{ -\lambda : \ A + 2\lambda e e^\top \text{ is copositive on } C \Big\}, \tag{3}$$

where we say that a symmetric matrix M is copositive on C, if $x^\top M x \geq 0$ for all $x \in C$. From (3), one obtains a linear programming representation of (2) by using [26] (see [16]). The previous formulation (3) may be seen as the (Lagrangian) dual problem when (1) is written equivalently as

$$\mu_q = \inf \Big\{ \frac{1}{2} x^\top A x : \ x^\top e e^\top x - 1 = 0, \ x \in C \Big\}.$$

Its (Lagrangian) dual problem, following Section 3, is:

$$\sup_{\lambda\in\mathbb{R}} \inf_{x\in C} \Big\{\frac{1}{2}x^\top Ax + \lambda(x^\top ee^\top x - 1)\Big\},$$

from which, (3) is derived (see Section 3 for details).

Problem (3) belongs to a class of problems termed "copositive optimization". This approach arises for analyzing the optimization problem where the objective function is quadratic but nonconvex, and the constraint set is a polyhedron, see [14] for a nice introduction to this topic. Instead, we will be interested in the qualitative study of the image space associated to the original problem (1).

Non negativity of a quadratic function (and so copositivity of a symmetric matrix) on a lower (upper) level set determined by a single quadratic function have been investigated in [42]. This allowed the authors to establish, under a Slater condition, S-lemma in both situations: as being either an inequality constraint ([42, Corollary 5]); or equality ([42, Corollary 6]). Only recently, without using any result on convexity of joint-range for a pair of quadratic functions, necessary and sufficient conditions for the validity of S-lemma, in the equality case, have been given in [43]; whereas the convexity of the same joint-range was completely characterized in [25]. However, an S-lemma related to problem (1) that reads as: (a) and (b) are equivalent, where

(a) $x \in C$, $g(x) \doteq e^\top x - 1 = 0 \Longrightarrow f(x) \doteq \dfrac{1}{2}x^\top Ax \geq 0$;
(b) there exists $\lambda \in \mathbb{R}$ such that $f(x) + \lambda g(x) \geq 0$ for all $x \in C$,

cannot be obtained from those results in [42] nor from [39,31].

Precisely, one of the goals of the present paper is to establish an S-lemma, and contrary to the approach developed in [42], we will first state an equivalence to the fulfillment to the strong duality for (1) (with respect to a suitable dual problem) in terms of the convexity of $(g, f)(C)+\mathbb{R}_+(0, 1)$, without passing by a copositive representation scheme.

A second issue we will deal with is the study of the validity of strong duality for the primal problem (1) with respect to each of its three possible Lagrangian dual problems: we characterize that property via the copositivity of A on suitable subsets of $\mathbb{R}^n$. One of the dual will require the formulation of (1) as a semi-infinite optimization problem, and will apply some of the main results from [19]. Our approach is similar to that carried out in [8], where various Lagrangians or semi-Lagrangians are considered. However, the main results of the present paper cannot be obtained from those in [8].

We must point out that a class of pairs of linear primal/dual copositive programs was analyzed, via semi-infinite optimization, in [1] with different purposes.

It is worth-while mentioning that C in problem (1) may be equal (after an isomorphism) either to the cone of (real) symmetric matrices of order n that are positive semidefinite, or the cone of copositive matrices on $\mathbb{R}^n_+$. In the first situation $C = C^*$ and int C^* coincides with the set of symmetric positive definite matrices; whereas, in the second case, one obtains

$$C^* = \left\{ Q \in \mathbb{R}^{n\times n} : \ Q = Q^\top, Q = \sum_{i=1}^{k} z_i z_i^\top, \ z_i \in \mathbb{R}^n_+, \ k \in \mathbb{N} \right\}$$
$$= \mathrm{co}\{zz^\top : \ z \in \mathbb{R}^n_+\},$$

which is the so called completely positive cone. The first equality is taken from [1], whereas the second may be found in [11]. See also [10]. For some characterizations of int C^*, we refer to [20,17]. Copositivity of quadratic forms on general sets has been studied in [21] with different perspective than that in [42] and here.

We analyze the cases $\mu_q = 0$ and $\mu_q > 0$. Obviously:

$$\mu_q = 0 \iff A \text{ is copositive but not strictly copositive on } C;$$
$$\mu_q > 0 \iff A \text{ is strictly copositive on } C.$$

This paper is organized as follows. Section 2 is devoted to state some basic definitions and preliminaries. In light of [24,23], the Lagrangian duality scheme is revisited in Section 3for the general problem with one single equality constraint. Section 4 presents our main results related to problem (1). They are concerned with some characterizations of strong duality with respect to each of the three dual problems in terms of copositivity of A on $\mathbb{R}^n$ (positive semidefiniteness), on $e^\perp$, and on C, respectively. This will reveal the hidden convexity, and the fulfillment of S-lemma; the case $n = 2$ is also analyzed in detail. Some sufficient and/or necessary conditions for local or global optimality for problem (1) are presented in Section 5.

2 Some basic definitions, notations and preliminaries

In what follows, given any nonempty set M in $\mathbb{R}^m$, its closure, topological interior, convex hull, closed convex hull, are denoted, respectively, by $\overline{M}$, int M, co M, $\overline{\mathrm{co}}$ M. In addition, by ri M and bd M we denote the relative interior of M and the boundary of M, respectively. Moreover, cone M is the smallest cone containing M, i. e., cone $M = \bigcup_{t\geq 0} tM$. The nonnegative polar cone of M is defined by

$$M^* \doteq \{z \in \mathbb{R}^m : \ \langle z, y\rangle \geq 0 \ \ \forall\, y \in M\}.$$

Here, $\langle z, y\rangle = z^\top y$ stands for the scalar product between two vectors z and y in $\mathbb{R}^m$, where $z^\top$ means the transpose of the vector z, which is considered a column vector. More generally, if A is a real matrix in $\mathbb{R}^{m\times n}$, $A^\top$ is the transpose of A belonging to $\mathbb{R}^{n\times m}$.

Let $h : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, $\overline{h}$ and $\overline{\mathrm{co}}$ h stand for the greatest lower semicontinuous function not larger than h and for the greatest convex and lower semicontinuous function not larger than h, respectively. Just for convenience, we need the following definition of epigraph of a function: epi $h \doteq \{(y,t) \in \mathbb{R}^m \times \mathbb{R} : \ h(y) \le t\}$. It is known that

$$\text{epi } \overline{h} = \overline{\text{epi } h}; \quad \overline{\text{co}}(\text{epi } h) = \text{epi } \overline{\text{co}}\ h.$$

Moreover,

$$\overline{\text{co}}\ h(y) > -\infty \ \ \forall\ y \in \mathbb{R}^m \Longrightarrow \overline{\text{co}}\ h(y) = h^{**}(y) \ \ \forall\ y \in \mathbb{R}^m, \tag{4}$$

where $h^{**} = (h^*)^*$ is the bipolar or biconjugate of h, that is, the conjugate (or polar) of h^* defined by

$$h^*(z) \doteq \sup_{y \in \mathbb{R}^m} \{\langle z, y\rangle - h(y)\}.$$

In addition, δ_M stands for the indicator function of the set M, defined by 0 on M, and $+\infty$ on the complementary of M.
There are examples showing the assumption $\overline{\text{co}}\ h(y) > -\infty$ for all $y \in \mathbb{R}^m$ is necessary to get the equality $h^{**} = \overline{\text{co}}\, h$. In general we have $h^{**} \le \overline{\text{co}}\ h \le h$. For details see [41].

In the subsequent sections, we set $\mathbb{R}_+ \doteq [0, +\infty[$; $\mathbb{R}_{++} \doteq\ ]0, +\infty[$. Given a vector $a \in \mathbb{R}^m \setminus \{0\}$, $\mathbb{R}_+ a$ stands for the set $\{ta : \ t \ge 0\}$, which is the ray starting from the origin and direction a; and $a^\perp$ is the orthogonal subspace to a, which is a hyperplane.

3 The Lagrange duality theory revisited: the general case with one single equality and geometric constraints

We now deal with the general minimization problem under one single equality constraint and a geometric constraint. Its presentation follows the abstract framework employed in [22]. Let $f, g : X \to \mathbb{R}$ be any finite-valued functions, with X to be a normed vector space, and $C \subseteq X$ be any nonempty set. Let us consider the problem

$$\mu \doteq \inf\{f(x) : \ \ g(x) = 0, \ x \in C\}, \tag{5}$$

whose (Lagrangian) dual problem is defined by

$$\nu \doteq \sup_{\lambda \in \mathbb{R}} \inf_{x \in C} [f(x) + \lambda g(x)]. \tag{6}$$

We say that there is no duality gap, or the duality gap is zero, between (5) and (6) if $\nu = \mu$. It is said that (5) has the strong duality property with respect to (6), or simply that strong duality holds for (5) with respect to (6), if $\mu = \nu$ and problem (6) admits some solution.

One can infer that, if $\mu = -\infty$ then there is no duality gap since $\nu = -\infty$ as well, and we conclude that any element in $\mathbb{R}$ is a solution for the problem (6), and so, we always have strong duality for (5) whenever $\mu = -\infty$.

Set $F(x) \doteq (g(x), f(x))$. Notice that $F = (f, g)$ was used in [23] instead. Assuming that $\mu \in \mathbb{R}$, we obtain

$$(F(C) - \mu(0,1)) \cap -(\{0\} \times \mathbb{R}_{++}) = \emptyset,$$

which amounts to writing

$$(F(C) + (\{0\} \times \mathbb{R}_+) - \mu(0,1)) \cap -(\{0\} \times \mathbb{R}_{++}) = \emptyset, \tag{7}$$

or, equivalently,

$$\text{cone}(F(C) + \mathbb{R}_+(0,1) - \mu(0,1)) \cap -(\{0\} \times \mathbb{R}_{++}) = \emptyset. \tag{8}$$

We will show, next, that the zero duality gap and strong duality can be characterized by reinforcing (7) or (8).

The optimal value function $\psi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ to problem (5) is defined by

$$\psi(a) = \begin{cases} \inf\{f(x) : \; x \in K(a)\} & \text{if} \quad K(a) \neq \emptyset; \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$K(a) \doteq \{x \in C : \; g(x) = a\}. \tag{9}$$

Notice that $K = K(0)$, and $K(a) \neq \emptyset$ if and only if $a \in g(C)$, that is,

$$\text{dom } \psi \doteq \{a \in \mathbb{R} : \; \psi(a) < +\infty\} = g(C).$$

The sets

$$\mathcal{F} \doteq F(C) + \mathbb{R}_+(0,1), \; \mathcal{E}_\rho \doteq \mathcal{F} - \rho(0,1) \quad (\rho \in \mathbb{R}). \tag{10}$$

will play an important role in our analysis.

Some topological and geometrical properties of ψ are shown in the following proposition, which is nothing else than Theorem 3.1 in [22], whose origin goes back to [23], where such a result was obtained for a class of integral functionals.

Proposition 3.1 *Let f, g, F be as above. The following assertions hold.*

(a) $(a, r) \in \text{epi } \psi \Longleftrightarrow (a, r + \frac{1}{k}) \in F(C) + \mathbb{R}_+(0,1), \quad \forall \; k \in \mathbb{N}.$
As a consequence, if $F(C) + \mathbb{R}_+(0,1)$ is convex then ψ is convex.

(b) $F(C) + \mathbb{R}_+(0,1) \subseteq \text{epi } \psi \subseteq \overline{F(C) + \mathbb{R}_+(0,1)}$.
Consequently,

$$\overline{\mathcal{E}_\mu} = \text{epi } \overline{\psi} - \mu(0,1); \; \overline{\text{co}} \; \mathcal{E}_\mu = \overline{\text{co}}(\text{epi } \psi) - \mu(0,1) = \text{epi}(\overline{\text{co}} \; \psi) - \mu(0,1).$$

Recall that (see for instance [24]) if $\mu = \psi(0) \in \mathbb{R}$ then

$$\nu = \psi^{**}(0) \tag{11}$$

Proposition 3.1 leads to the following characterization of lower semicontinuity of ψ at 0, which is a particular case of Theorem 3.1 in [36], under the additional assumption of existence of solution for problem (5). Notice that (12) reinforces (8). Part (b) may be also found in [36].

Proposition 3.2 *Assume that $\mu = \psi(0)$ is finite. Then,*

(*a*) $\overline{\psi}(0) = \psi(0)$ *if and only if*

$$\overline{\mathcal{E}_\mu} \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset. \tag{12}$$

(*b*) $\nu = \overline{\text{co}}\ \psi(0)$.

Proof It is Theorem 4.1 in [22]. □

Part (*a*) of the next theorem characterizes the zero duality gap for problem (5). It reduces to the lower semicontinuity of ψ at 0 under convexity of $\overline{\mathcal{E}}_\mu$, or equivalently under convexity of $\overline{\mathcal{F}}$, as a consequence of the previous proposition. A similar result was obtained in [32] for a semi-infinite optimization problem. Part (*b*) was established in [24].

Theorem 3.1 ([22, *Theorems* 4.1 *and* 4.2]) *Assume that $\mu = \psi(0)$ is finite. Then,*

(*a*) *zero duality holds for* (5)*, i.e., $\mu = \nu$ if, and only if*

$$\overline{\text{co}}\ \mathcal{E}_\mu \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset; \tag{13}$$

(*b*) *strong duality holds for* (5) *if, and only if*

$$\overline{\text{cone}}(\text{co}\ \mathcal{E}_\mu) \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset; \tag{14}$$

(*c*) *strong duality holds for* (5) *if, and only if*

$$\overline{\text{cone}}(\mathcal{E}_\mu) \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset \quad \textit{and}\ \overline{\text{cone}}(\mathcal{E}_\mu)\ \textit{is convex}. \tag{15}$$

Convexity of $\overline{\text{cone}}(\mathcal{E}_\mu)$ or equivalently of $\overline{\text{cone}}((g,f)(C) - \mu(0,1) + \mathbb{R}_+(0,1))$ may be obtained in several important instances. For example, if $C = \mathbb{R}^n$ and f, g are quadratic functions, such a convexity property is completely analyzed in [25]; whereas the convexity of $(g,f)(\mathbb{R}^n)$, with f being duadratic and g linear was discussed in [4], see also [33,43].

Remark 3.1 (The convex case) In case $(g,f)(C) + \mathbb{R}_+(0,1)$ is already convex and closed (this will occur for our problem (1), as we will show in Subsection 4.3), (*b*) of the preceding theorem, along with (7), ensures that strong duality holds for (5).

A (copositive) reformulation - dual representation of problem (1)

As we said in Section 1, problem (1) can be written, in a equivalent way, as

$$\mu_q = \inf \Big\{ \frac{1}{2} x^\top A x : \ x^\top e e^\top x - 1 = 0, \ x \in C \Big\}. \tag{16}$$

According to (6), its Lagrangian dual problem is

$$\nu_q \doteq \sup_{\lambda \in \mathbb{R}} \inf_{x \in C} \Big\{ \frac{1}{2} x^\top A x + \lambda (x^\top e e^\top x - 1) \Big\}. \tag{17}$$

It is not difficult to show that

$$\nu_q = \sup_{\lambda\in\mathbb{R}} \Big\{ -\lambda + \frac{1}{2}\inf_{x\in C} x^\top (A + 2\lambda ee^\top)x \Big\}.$$

Then,

$$\theta_0(\lambda) \doteq \inf_{x\in C} x^\top (A + 2\lambda ee^\top)x \in \{0, -\infty\},\ \forall\ \lambda \in \mathbb{R},$$

and

$$\begin{aligned}\theta_0(\lambda) &= \inf_{a\geq -1,\ y\in K} \Big\{(1+a)^2 y^\top Ay + 2(1+a)^2\lambda\Big\}\\ &= \inf_{a\geq -1} \Big\{ \inf_{y\in K} (1+a)^2 y^\top Ay + 2(1+a)^2\lambda\Big\}\\ &= \inf_{a\geq -1} 2[(1+a)^2\mu_q + (1+a)^2\lambda] = \inf_{a\geq -1} 2(\mu_q + \lambda)(1+a)^2.\end{aligned}$$

Thus,

$$A + 2\lambda ee^\top \text{ is copositive on } C \Longleftrightarrow \theta_0(\lambda) = 0 \Longleftrightarrow \lambda + \mu_q \geq 0. \qquad (18)$$

Hence, the dual problem (17) reduces to

$$\nu_q = \sup_{\lambda\in\mathbb{R}} \Big\{ -\lambda :\ A + 2\lambda ee^\top \text{ is copositive on } C\Big\} \leq \mu_q.$$

On the other hand, by the definition of μ_q, we get

$$x^\top Ax - 2\mu_q = x^\top (A - 2\mu_q ee^\top)x \geq 0,\ \ \forall\ x \in K,$$

which implies

$$x^\top (A - 2\mu_q ee^\top)x \geq 0,\ \ \forall\ x \in C,$$

that is, $A - 2\mu_q ee^\top$ is copositive on C. It yields $\nu_q \geq \mu_q$. Hence $\mu_q = \nu_q$, ensuring zero duality gap, and so strong duality between problem (1) and its dual (17) holds. Moreover, we also infer that the dual problem has $-\mu_q$ as the unique solution.

Thus, we found a dual problem to (1) for which strong duality holds. By (c) of Theorem 3.1,

$$\overline{\text{cone}}((g, f)(C) + \mathbb{R}_+(0, 1) - \mu_q(0, 1)) \ \text{ is convex},$$

where $g(x) = x^\top ee^\top x - 1$. Certainly, if $C = \mathbb{R}^n$ then $(g, f)(\mathbb{R}^n)$ is convex by Dines theorem (see [18]).

4 The generalized standard quadratic optimization problem: strong duality, hidden convexity and S-lemma

Let us go back to our problem

$$\mu_q \doteq \min\Big\{\frac{1}{2}x^\top Ax:\ e^\top x = 1,\ x\in C\Big\}, \tag{19}$$

where we recall that $C \subseteq \mathbb{R}^n$ is a pointed, closed, convex cone having non-empty interior. Obviously the feasible set to (19) is $K \doteq \{x \in C:\ e^\top x = 1\}$, which becomes a convex and compact base of C provided $e \in \text{int } C^*$ (it is non empty since C is pointed); A is a real symmetric matrix. We say that A is copositive on a cone P if $x^\top Ax \geq 0$ for all $x \in P$; it is strictly copositive on P if $x^\top Ax > 0$ for all $x \in P$, $x \neq 0$. We discuss the cases $\mu_q = 0$ and $\mu_q > 0$.

It is easy to check that:

- $\mu_q = 0 \Longleftrightarrow A$ is copositive but not strictly copositive on C;
- $\mu_q > 0 \Longleftrightarrow A$ is strictly copositive on C.

The specialization $C = \mathbb{R}^n_+$; $e = (1\ 1\ \dots\ 1)^\top \in \mathbb{R}^n$, discussed at the introduction, is termed the standard quadratic optimization problem and was studied in many papers as was mentioned in Section 1, and models quadratic allocation problems, portfolio optimization problems, the maximum weight clique problem, among others.

We will describe the three main dual problems associated to (19). To that end, we formulate problem (19) as a semi-infinite optimization problem:

$$\mu_q \doteq \min\{f(x):\ x\in X,\ g_j(x)\leq 0,\ \forall\ j \in J\}, \tag{20}$$

where $X \doteq \{x \in \mathbb{R}^n:\ e^\top x = 1\}$,

$$J \doteq -C^*,\ f(x) \doteq \frac{1}{2}x^\top Ax,\ g_j(x) \doteq j^\top x,\ j\in J, \tag{21}$$

and $g_0(x) \doteq e^\top x - 1$. Thus, we consider the following three dual problems:

$$\nu_0 \doteq \sup_{\lambda\in\mathbb{R}} \inf_{x\in C} \Big\{f(x) + \lambda g_0(x)\Big\}; \tag{22}$$

$$\nu_1 \doteq \sup_{\lambda\in\mathbb{R}^{(J)}_+} \inf_{x\in X} \Big\{f(x) + \sum_{j\in J}\lambda_j g_j(x)\Big\}; \tag{23}$$

$$\nu_2 \doteq \sup_{(\lambda_0,\lambda)\in\mathbb{R}\times\mathbb{R}^{(J)}_+} \inf_{x\in\mathbb{R}^n} \Big\{f(x) + \lambda_0 g_0(x) + \sum_{j\in J}\lambda_j g_j(x)\Big\}. \tag{24}$$

Here, $\mathbb{R}^{(J)}$ is the topological dual of $\mathbb{R}^J$ (it stands for the set of real-valued functions defined on J, endowed with the usual product topology), which is the space of generalized sequences $\lambda = (\lambda_j)_{j\in J}$ such that $\lambda_j \in \mathbb{R}$, for each

$j \in J$, and with only finitely many λ_j different from zero. The supporting set of λ is supp $\lambda \doteq \{j \in J : \lambda_j \neq 0\}$. Thus

$$\langle \lambda, z \rangle = \lambda(z) = \sum_{j \in J} \lambda_j z_j \doteq \sum_{j \in \text{supp } \lambda} \lambda_j z_j, \quad \forall\, z \in \mathbb{R}^J,\ \forall\, \lambda \in \mathbb{R}^{(J)}.$$

If $\lambda = 0$ then supp $\lambda = \emptyset$, and so we put $\sum_\emptyset = 0$. In addition, $\mathbb{R}^{(J)}_+$ denotes the non-negative cone in $\mathbb{R}^{(J)}$.

Before establishing the validity of strong duality for (19) with respect to (22), strong duality with respect to the duals (23) and (24) will be discussed first. Actually, such a property will be equivalent to copositivity of A on suitable cones. In what follows, some preliminaries are stated.

Set

$$M \doteq \text{cone co}\left(\bigcup_{j \in J} \text{epi } g_j^* \cup \text{epi } \delta_X^* \right).$$

Here, g_j^* (resp. δ_X^*), denotes the conjugate function of g_j (resp. δ_X). For general f, g_j, X, the problem formulated as in (20) with optimal value denoted by μ instead of μ_q, the following two constraint qualification conditions arise

$$M \ \text{ is closed;} \tag{25}$$

$$\text{epi } f^* + \overline{M} \ \text{ is closed.} \tag{26}$$

Notice that

$$M = \text{cone co}\left(\bigcup_{j \in J} \text{epi } g_j^* \right) + \text{epi } \delta_X^*.$$

Next theorem is taken from [19]

Theorem 4.1 (*[19, Theorem 2]*) *Given any proper lsc and convex functions f and g_j $(j \in J)$, X a non-empty convex closed set, under* (25) *and* (26) *and assuming μ finite, one gets*

$$\exists\ \lambda^* \in \mathbb{R}^{(J)}_+ : \quad f(x) + \sum_{j \in J} \lambda_j^* g_j(x) \geq \mu,\ \forall\ x \in X, \tag{27}$$

or, equivalently, there exists $\lambda^* \in \mathbb{R}^{(J)}_+$ such that

$$\mu = \sup_{x \in X} \{ f(x) + \sum_{j \in J} \lambda_j^* g_j(x) \}. \tag{28}$$

Remark 4.1 Theorem 1 in [19] asserts that if f is either linear or continuous at some point of the feasible set of (20), K, then, the fulfillment of (25) implies that (26) is also satisfied.

4.1 Characterizing strong duality with respect to (24)

We are ready to apply Theorem 4.1 to our model (19). The following result establishes that standard strong duality for problem (20) or equivalently, (19), with respect to (24) holds if, and only if A is positive semidefinite, i.e., copositivity on $\mathbb{R}^n$. So, this dual is not suitable when considering, for instance, the portfolio optimization problem since A is only copositive on a proper cone of $\mathbb{R}^n$.

Theorem 4.2 *Let us consider problem* (19) *or equivalently* (20) *with* C *being any pointed closed convex cone with nonempty interior, and* $e \in \operatorname{int} C^*$*. The following assertions are equivalent:*

(a) $A \succcurlyeq 0$;

(b) $A \succcurlyeq 0$ *and there exists* $\lambda^* \in \mathbb{R}_+^{(I)}$ *such that*

$$f(x) + \sum_{i\in I} \lambda_j^* g_j(x) \geq \mu_q, \ \forall \ x \in X; \tag{29}$$

(c) *there exists* $(\lambda_0^*, \lambda^*) \in \mathbb{R} \times \mathbb{R}_+^{(J)}$ *such that*

$$f(x) + \lambda_0^* g_0(x) + \sum_{j\in J} \lambda_j^* g_j(x) \geq \mu_q, \ \forall \ x \in \mathbb{R}^n, \tag{30}$$

or, equivalently,

$$\mu_q = \inf_{x\in\mathbb{R}^n} \Big\{ f(x) + \lambda_0^* g_0(x) + \sum_{j\in J} \lambda_j^* g_j(x) \Big\}.$$

Proof $(c) \Rightarrow (a)$: Assume to the contrary that there exists $\bar{x} \in \mathbb{R}^n$ such that $\bar{x}^\top A \bar{x} < 0$. Then, by setting $x = t\bar{x}$ with t going to $+\infty$ in (30), it yields a contradiction.
$(a) \Rightarrow (b)$: In view of Theorem 4.1 and Remark 4.1, we need to check only that condition (25) holds. Since

$$g_j^*(u) = \sup_{x\in\mathbb{R}^n} \{u^\top x - j^\top x\} = \delta_{\{j\}}(u),$$

one obtains epi $g_j^* = \{j\} \times \mathbb{R}_+$, and so

$$\text{co } \left(\bigcup_{j\in J} \text{epi } g_i^* \right) = \text{co}[(-C^*) \times \mathbb{R}_+] = (-C^*) \times \mathbb{R}_+.$$

Hence

$$\text{cone co} \left(\bigcup_{i\in J} \text{epi } g_j^* \right) = (-C^*) \times \mathbb{R}_+.$$

On the other hand, by writing $X = e^\perp + \bar{x}$ with $\bar{x} \in X$, we get

$$\delta_X^*(u) = \sup_{x\in\mathbb{R}^n} \{u^\top x - \delta_X(x)\} = \sup_{x\in X} u^\top x = \sup_{v\in e^\perp} u^\top v + u^\top \bar{x} = \delta_{\mathbb{R}e}(u) + u^\top \bar{x}.$$

Thus,

$$\text{epi } \delta_X^* = \mathbb{R}(e,1) + \mathbb{R}_+(0,1).$$

Consequently,

$$\begin{aligned} M &= \text{cone co}\left(\bigcup_{j\in J} \text{epi } g_i^* \cup \text{epi } \delta_X^*\right) = \text{cone co}\left(\bigcup_{j\in J} \text{epi } g_j^*\right) + \text{epi } \delta_X^* \\ &= (-C^*)\times\mathbb{R}_+ + \mathbb{R}(e,1) + \mathbb{R}_+(0,1) = (-C^*)\times\mathbb{R}_+ + \mathbb{R}(e,1), \end{aligned}$$

which is closed, i.e., (25) is satisfied. Here we use the result: given two closed convex sets M and N, the set $M - N$ is closed provided $M^\infty \cap N^\infty = \{0\}$, see for instance [40]. Then, (27) holds, and so (28) holds as well, proving (b).
$(b) \Rightarrow (c)$: By setting $\varphi(x) = f(x) + \sum_{j\in J} \lambda_j^* g_j(x)$, it is not difficult to check that: $(g_0, \varphi)(\mathbb{R}^n) + \mathbb{R}_+(0,1)$ is convex (since φ is convex and g_0 is affine), and there exist $x_0, x_1 \in \mathbb{R}^n$ satisfying $g_0(x_0) < 0 < g_0(x_1)$. Thus, strong duality holds for problem (28) (a usual application of a convex separation theorem yields the conclusion), that is, there exists $\lambda_0^* \in \mathbb{R}$ such that

$$\varphi(x) + \lambda_0^* g_0(x) \geq \mu_q, \quad \forall\, x \in \mathbb{R}^n,$$

which is nothing else than (30). □

Remark 4.2 One can check easily that if (29) holds for some $\lambda^* \in \mathbb{R}_+^{(J)}$, then A is copositive on the hyperplane $e^\perp$. Such a notion arises naturally in the next subsection.

4.2 Characterizing strong duality with respect to (23)

In this subsection we deal with the second dual problem (23). To be more precise, next thorem shows that standard strong duality for problem (20) or equivalently, (19), with respect to (23) holds if, and only if A is copositive on $e^\perp$.

Theorem 4.3 *Let us consider problem* (19) *with C being any pointed closed convex cone with nonempty interior, and $e \in \text{int } C^*$. The following assertions are equivalent:*

(a) *A is copositive on $e^\perp$;*
(b) *there exists $\lambda^* \in \mathbb{R}_+^{(J)}$ such that*

$$f(x) + \sum_{j\in J} \lambda_j^* g_j(x) \geq \mu_q, \ \forall\, x \in X, \tag{31}$$

or, equivalently,

$$\mu_q = \inf_{x\in X} \Big\{ f(x) + \sum_{j\in J} \lambda_j^* g_j(x) \Big\}.$$

(c) *f is convex on X.*
(d) *f is convex on K.*

Hence, under any of the conditions (a)*,* (b)*,* (c) *or* (d)*, every local solution to problem* (19) *is global.*

Proof (b) $\Rightarrow$ (a): It is straightforward.
(a) $\Rightarrow$ (b): Take any $\bar{x} \in \underset{K}{\mathrm{argmin}} f$. By writing $X = \bar{x} + e^{\perp}$, one obtains for all $x \in X$,

$$x^{\top}Ax = (x - \bar{x})^{\top}A(x - \bar{x}) + 2\bar{x}^{\top}Ax - \bar{x}^{\top}A\bar{x} \tag{32}$$
$$\geq 2\bar{x}^{\top}Ax - \bar{x}^{\top}A\bar{x}, \tag{33}$$

where the inequality was obtained by copositivity of A on $e^{\perp}$. Thus,

$$\frac{1}{2}x^{\top}Ax \geq \bar{x}^{\top}Ax - \frac{1}{2}\bar{x}^{\top}A\bar{x},\ \forall\ x \in X. \tag{34}$$

Let us consider the convex problem:

$$\mu' \doteq \min\Big\{\bar{x}^{\top}Ax:\ x \in X,\ g_j(x) \leq 0,\ j \in J\Big\}. \tag{35}$$

It is easy to check that $\mu' = 2\mu_q$. Indeed, obviously $\mu' \leq \bar{x}^{\top}A\bar{x} = 2\mu_q$. On the other hand, by the first order optimality condition,

$$(A\bar{x})^{\top}(x - \bar{x}) \geq 0,\ \forall\ x \in K,$$

that is, $\mu' \geq 2\mu_q$, proving the claim.
We now check that strong duality holds for (35) with respect to the dual

$$\sup_{\lambda \in \mathbb{R}^{(J)}_{+}} \inf_{x \in X} \Big\{\bar{x}^{\top}Ax + \sum_{j \in J}\lambda_j g_j(x)\Big\}.$$

It will be a consequence, as before, of Theorem 4.1 with objective function given by $\bar{x}^{\top}Ax$. In fact, such a theorem is applicable since the assumptions are verified, see also Remark 4.1. Hence, there exists $\lambda^* = (\lambda^*_j) \in \mathbb{R}^{(J)}_{+}$ such that

$$\bar{x}^{\top}Ax + \sum_{j \in J}\lambda^*_j g_j(x) \geq 2\mu_q,\ \forall\ x \in X.$$

This along with inequality (34) yields that

$$f(x) + \sum_{j \in J}\lambda^*_j g_j(x) \geq (A\bar{x})^{\top}x + \sum_{j \in J}\lambda^*_j g_j(x) - \mu_q \geq 2\mu_q - \mu_q = \mu_q,\ \forall\ x \in X.$$

(a) $\Leftrightarrow$ (c): First observe that A is copositive on $e^{\perp}$ if, and only if

$$(x - y)^{\top}A(x - y) \geq 0 \quad \forall\, x, y \in X.$$

In other words, A is copositive on $e^{\perp}$ if, and only if A is copositive on $X - X$. On the other hand, given $t \in \ ]0,1[$, and $x, y \in X$, on combining the two identities:

$$f(x) = f(y) + \nabla f(y)^{\top}(x-y) + \frac{1}{2}(x-y)^{\top}A(x-y);$$

$$f(y+t(x-y)) = f(y) + t\nabla f(y)^{\top}(x-y) + \frac{t^2}{2}(x-y)^{\top}A(x-y),$$

one obtains

$$f(y+t(x-y)) = f(y) + t(f(x)-f(y)) - \frac{t}{2}(1-t)(x-y)^{\top}A(x-y),$$

from which the desired inequality follows.
$(c) \Leftrightarrow (d)$: One implication is obvious since $K \subseteq X$, and the other is obtained because of X is the affine hull of K ($\mathrm{aff}\, K$), which is the smallest affine set containing K, and the fact that the convexity of f on K implies the convexity of f on $\mathrm{aff}\, K$, see for instance [3, Exercise 3.20]. $\square$

Remark 4.3 In case $C = \mathbb{R}^n_+$ and $e = \mathbf{1}$, (d) is equivalent to ([12, Lemma 6])

$$P_{\bar{x}}^{\top} A P_{\bar{x}} \succcurlyeq 0 \ \text{ for any } \bar{x} \in X, \text{ with } P_{\bar{x}} \doteq I - \bar{x}\mathbf{1}^{\top}, \tag{36}$$

where I is the identity matrix of order n.

4.3 Characterizing strong duality with respect to (22)

We now analize the strong duality property in connection with (22), and it will be suitable for the portfolio optimization problem, since copositivity of A on C arises naturally in such a problem. Recall that $f(x) = \frac{1}{2}x^{\top}Ax$, $g_0(x) = e^{\top}x - 1$, and $F = (g_0, f)$. By following Section 3, denote, given $a \in \mathbb{R}$,

$$K(a) \doteq \{x \in C : \ g_0(x) = a\}.$$

Clearly $K(-1) = \{0\}$. The following proposition, whose proof is straightforward, collects some basic facts on the optimal value function $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\psi(a) \doteq \begin{cases} \min\{f(x) : \ x \in K(a)\} & \text{if } K(a) \neq \emptyset \\ +\infty \quad \text{otherwise.} \end{cases} \tag{37}$$

Proposition 4.1 *Let A be a real symmetric matrix; C be a pointed closed convex cone with nonempty interior, and $e \in \mathrm{int}\, C^*$. The following assertions hold.*

(a) *$K(a) \neq \emptyset$ if, and only if $a \geq -1$.*

(b) *Let $a > -1$. Then, $x \in K(a)$ if, and only if $\frac{1}{1+a}x \in K$. Hence*

$$\min_{x \in K(a)} f(x) = \mu_q(1+a)^2, \ \ \forall\, a \geq -1.$$

(*c*) *Let* $\mu_q > 0$. *Then,* $f(x) > \mu_q$ *for all* $x \in K(a)$ *and all* $a > 0$.

Now, denote the objective function of the dual problem (22) by

$$\theta(\lambda) \doteq \inf_{x \in C} L(\lambda, x),$$

where $L(\lambda, x) \doteq f(x) + \lambda g_0(x)$, and by $\mathcal{S}_D$ the solution set of the dual problem (22). We now describe the main properties shared by this problem, which appear here for the first time when C is non-polyhedral. In particular, They reveal a hidden convexity of the general standard quadratic optimization problem.

Notice that for the StQO problem (2), (*c*) below was obtained in [12, Theorem 4] by using the Frank-Wolfe theorem, which is not applicable here (see also [7]).

Theorem 4.4 *Let* A, C *and* e *be as in the preceding proposition. Then,*

(*a*) *the optimal value function* ψ *is given by*

$$\psi(a) = \begin{cases} \mu_q(1+a)^2 & \text{if} \quad a \geq -1; \\ +\infty & \text{if} \quad a < -1. \end{cases}$$

Thus, ψ *is convex if, and only if* $\mu_q \geq 0$.

(*b*) $F(C)$ *and* $F(C) + \mathbb{R}_+(0,1)$ *are closed, so*

$$\{(a, r) \in \mathbb{R}^2 : a \geq -1, \mu_q(1+a)^2 \leq r\} = \text{epi } \psi = F(C) + \mathbb{R}_+(0,1).$$

(*c*) *Let* $\mu_q > 0$. *Then, the objective function* θ *is given by*

$$\theta(\lambda) = \begin{cases} -\dfrac{\lambda^2}{4\mu_q} - \lambda & \text{if} \quad \lambda < 0; \\ -\lambda & \text{if} \quad \lambda \geq 0. \end{cases}$$

Hence $\mathcal{S}_D = \{-2\mu_q\}$ *and* $\mu_q = \nu_0$, *i.e., strong duality holds with respect to* (22).

(*d*) *Let* $\mu_q = 0$. *Then, the objective function* θ *is given by*

$$\theta(\lambda) = \begin{cases} -\infty & \text{if} \quad \lambda < 0; \\ -\lambda & \text{if} \quad \lambda \geq 0. \end{cases}$$

Hence $\mathcal{S}_D = \{0\}$ *and strong duality holds with respect to* (22).

Proof (*a*) is a consequence of the previous proposition.
(*b*):By virtue of (*b*) of Proposition 3.1, we need only to check the closedness of $\overline{F(C) + \mathbb{R}_+(0,1)}$. The same argument also shows that $F(C)$ is closed. Let $(a, r) \in \overline{F(C) + \mathbb{R}_+(0,1)}$. Then, there exist sequences $x_k \in C$, $q_k \geq 0$ satisfying $f(x_k) + q_k \to r$ and $g_0(x_k) \to a$. From the second relation, we deduce that $\|x_k\|$ is bounded. Thus, up to a subsequence, $x_k \to \bar{x} \in C$, implying that $q_k = f(x_k) + q_k - f(x_k) \to r - f(\bar{x})$. Setting $q \doteq r - f(\bar{x})$, we get $q \geq 0$, and

so $(a, r) = (g_0(\bar{x}), f(\bar{x}) + q) \in F(C) + \mathbb{R}_+(0, 1)$.
(c): From the inequality $(x^\top Ax + \lambda e^\top x)^2 \geq 0$ and taking into account that by assumption $x^\top Ax > 0$ for all $x \in C$, $x \neq 0$, one obtains

$$L(\lambda, x) \geq -\frac{\lambda^2 (e^\top x)^2}{2x^\top Ax} - \lambda, \ \forall\, x \in C, \ x \neq 0.$$

Since K is a base for C, we conclude

$$\frac{(e^\top x)^2}{2x^\top Ax} \leq \frac{1}{4\mu_q}, \ \forall\, x \in C, \ x \neq 0.$$

Hence

$$L(\lambda, x) \geq -\frac{\lambda^2}{4\mu_q} - \lambda, \ \forall\, x \in C. \tag{38}$$

In case $\lambda \geq 0$, it is easy to see that

$$\theta(\lambda) = \min_{x \in C} L(\lambda, x) = L(\lambda, 0) = -\lambda.$$

If $\lambda < 0$ and $\bar{x} \in \operatorname{argmin}_K f$, then by taking $x_0 = -\dfrac{\lambda}{2\mu_q}\bar{x} \in C$, we get

$$L(\lambda, x_0) = -\frac{\lambda^2}{4\mu_q} - \lambda.$$

Thus, from (38),

$$\theta(\lambda) = -\frac{\lambda^2}{4\mu_q} - \lambda.$$

(d): We consider only the case $\lambda < 0$ (if $\lambda \geq 0$ is exactly as in (c)), and check that

$$\inf \Big\{ L(\lambda, x) : \ x \in C, \ x^\top Ax = 0 \Big\} = -\infty.$$

Indeed, since there exists $x_0 \in C$, $x_0 \neq 0$, such that $x_0^\top Ax_0 = 0$ by assumption, we obtain

$$\begin{aligned} \inf \Big\{ L(\lambda, x) : \ x \in C, \ x^\top Ax = 0 \Big\} &= \inf \Big\{ \lambda(e^\top x - 1) : \ x \in C, \ x^\top Ax = 0 \Big\} \\ &\leq \inf_{t > 0} \lambda(te^\top x_0 - 1) = -\infty. \end{aligned}$$

Thus, $\theta(\lambda) = -\infty$ in case $\lambda < 0$. □

From the previous theorem, one characterizes the copositivity of A on C by means of the convexity of $F(C) + \mathbb{R}_+(0, 1)$. This result is new and corresponds to the nice challenge of proving convexity of joint-range for a pair of quadratic functions. When $C = \mathbb{R}^n$ a similar convexity result was analyzed in [43].

Theorem 4.5 *Let us consider problem* (19) *with* C *being any pointed closed convex cone with nonempty interior, and* $e \in \operatorname{int} C^*$. *The following assertions are equivalent:*

(a) *A is copositive on C (or, equivalently $\mu_q \geq 0$);*
(b) *Strong duality holds for* (19) *with respect to* (22)*, i.e., there exists $\lambda_0^* \in \mathbb{R}$ such that*

$$f(x) + \lambda_0^* g_0(x) \geq \mu_q, \ \forall\, x \in C, \tag{39}$$

or, equivalently,

$$\mu_q = \inf_{x \in C} \Big\{ f(x) + \lambda_0^* g_0(x) \Big\}.$$

(c) *$F(C) + \mathbb{R}_+(0,1)$ is convex.*

Proof Since, A is copositive on C if and only if $\mu_q \geq 0$, the equivalence between (a) and (c) is a consequence of (a) and (b) of Theorem 4.4. That (a) implies (b) follows from (c) and (d) of the same Theorem 4.4. For the implication (b) $\Rightarrow$ (a), assume to the contrary that there exists $\bar{x} \in C$, $\bar{x} \neq 0$, such that $\bar{x}^\top A \bar{x} < 0$. Then, by substituting x by $t\bar{x}$ with $t > 0$ in (39), and letting $t \to +\infty$, one reachess a contradiction. This proves (a). $\square$

In connection with the previous theorem some remarks are in order.

Remark 4.4 One can split Theorem 4.5 into two cases according to $\mu_q = 0$ or $\mu_q > 0$. Clearly, if $\mu_q = 0$, then (39) holds trivially by putting $\lambda_0^* = 0$. In case $\mu_q > 0$, by (c) of Theorem 4.4, (39) holds if, and only if $\lambda_0^* = -2\mu_q$. Hence one may write the equivalences:

- A is copositive but not strictly copositive on C $\iff$ $F(C) + \mathbb{R}_+(0,1)$ is (convex) polyhedral (see (b) of Theorem 4.4);
- A is strictly copositive on C $\iff$ $F(C) + \mathbb{R}_+(0,1)$ is convex and non-polyhedral (see (b) of Theorem 4.4)
  $$\iff \exists\, \lambda \neq 0 : f(x) + \lambda g_0(x) > 0, \ \forall\, x \in C.$$

Remark 4.5 The fact that convexity must appear in a natural way under strong duality was the object of many investigations by one of the authors. For the general problem (5), it was proved in [15] under a Slater-type condition, that the validity of strong duality is equivalent to the convexity of $\overline{\text{cone}}(F(C) + \mathbb{R}_+(0,1) - \mu(0,1))$. When $C = \mathbb{R}^n$ and f, g are quadratic functions, the authors in [25] established (under Slater condition): strong duality holds if, and only if $F(\mathbb{R}^n) + \mathbb{R}_+(0,1)$ is convex; such a convexity is also characterized in [25].

Remark 4.6 One could expect that under any of the conditions (a), (b) or (c) of Theorem 4.5, local solutions are global. Unfortunately, this is not true. The matrix A_2 of Section 4.5 shows this fact when $\mu_q > 0$.

4.4 The S-lemma

Let $P \subseteq \mathbb{R}$ be either $\{0\}$ or $\mathbb{R}_+$, $f, g : \mathbb{R}^n \to \mathbb{R}$ be quadratic functions not necessarily homogeneous, and C be any nonempty subset of $\mathbb{R}^n$. The S-lemma asks when (40) and (41) are equivalent:

$$x \in C, \ g(x) \in -P \Longrightarrow f(x) \geq 0; \tag{40}$$

$$\exists\ \lambda \in P^*,\ f(x) + \lambda g(x) \geq 0\ \forall\ x \in C. \tag{41}$$

This actually requires only the implication (40) $\Longrightarrow$ (41). Originally, the S-lemma (with $P = \mathbb{R}_+$ and $C = \mathbb{R}^n$) is due to Yakubovich [44,45], and the equivalence holds under the Slater condition: there exists $x_0 \in \mathbb{R}^n$ such that $g(x_0) < 0$. When $P = \{0\}$ and $C = \mathbb{R}^n$, a characterization of the validity of the S-lemma is given in the recent paper [43]; although an earlier result may be found in [42, Corollary 6] (see Corollary 5 in [42] in case $P = \mathbb{R}_+$). In the same paper [43], Theorem 9, respectively Corollary 2, provides a necessary and sufficient condition for the convexity of $(g, f)(\mathbb{R}^n)$, respectively $(g, f)(\mathbb{R}^n)+\mathbb{R}^2_+$, when g is affine. In the case when both functions f and g are quadratic, the convexity of $(g, f)(\mathbb{R}^n)$ or that of $(g, f)(\mathbb{R}^n) + \mathbb{R}_+ d$ $(0 \neq d \in \mathbb{R}^2)$ are fully analyzed in [25]. The extension to C equals an affine space with $P = \mathbb{R}_+$, was considered in [31]. A nice survey about S-lemma is presented in [39], and results on joint-range convexity of a finite number of quadratic forms are presented in [29].

Our (strict-version) S-lemma cannot be obtained from any result mentioned above. Indeed, all forms of S-lemma appearing in [39,43,42] require either $C = \mathbb{R}^n$ or C to be a linear subspace; and, on the other side, the proof of our version does not follow any argument employeed elsewhere.

We will see that a non strict version of S-lemma (with $P = \{0\}$) for our problem (19) always holds, and it is immediate. Thus, the strict case is the interesting part.

Lemma 4.1 *Let $C \subseteq \mathbb{R}^n$ be any pointed closed convex cone with nonempty interior, $e \in \operatorname{int} C^*$. Set $P = \{0\}$, $g(x) = e^\top x - 1$, $f(x) = \frac{1}{2}x^\top Ax$ with $A = A^\top$. Then,*

$$(40) \Longleftrightarrow (41).$$

Proof It needs only to check (40) $\Longrightarrow$ (41). From (40) it follows that $\mu_q \doteq \min\{f(x) :\ g(x) = 0,\ x \in C\} \geq 0$. This means A is copositive on C. Thus, (41) holds for $\lambda = 0$, and the proof is completed. □

However, a strict version of the S-lemma requires Theorem 4.5 or more precisely Remark 4.4. Such a version is established in what follows, and it is new.

Theorem 4.6 *Let $C \subseteq \mathbb{R}^n$ be any pointed closed convex cone with nonempty interior, $e \in \operatorname{int} C^*$. Set $P = \{0\}$, $g(x) = e^\top x - 1$, $f(x) = \frac{1}{2}x^\top Ax$ with $A = A^\top$. Then, (42) $\Longleftrightarrow$ (43), where*

$$x \in C,\ g(x) = 0 \Longrightarrow f(x) > 0; \tag{42}$$

$$\exists\ \lambda \neq 0,\ f(x) + \lambda g(x) > 0\ \forall\ x \in C. \tag{43}$$

Proof As usual, we will only check (42) $\Longrightarrow$ (43). From (42) it follows that $\mu_q \doteq \min\{f(x) :\ g(x) = 0,\ x \in C\} > 0$. This means A is strictly copositive on C. By the last part of Remark 4.4, there exists $\lambda \neq 0$ such that $f(x)+\lambda g(x) > 0$ for all $x \in C$, which is the desired result. □

4.5 Example: the bidimensional case

Just for illustration, let us consider $n = 2$, $\mathbf{1} = (1, 1)$ and $A = A^\top \in \mathbb{R}^{2\times 2}$ with

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The following proposition is easily obtained, see also [28,38], and from it, one infers that there is no relationship between copositivity on $\mathbf{1}^\perp$ and on $\mathbb{R}^2_+$.

Proposition 4.2 *Let A be as above. Then*

(*a*) *A is positive semidefinite if, and only if $a \geq 0,\ ac \geq b^2$.*
(*b*) *A is copositive on $\mathbf{1}^\perp$ if, and only if $a - 2b + c \geq 0$.*
(*c*) *A is copositive on $\mathbb{R}^2_+$ if, and only if $a \geq 0,\ c \geq 0,\ b \geq -\sqrt{ac}$.*
(*d*) *Every local solution to*

$$\min\{f(x_1, x_2) \doteq \frac{1}{2}x^\top Ax :\ x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0\}, \qquad (44)$$

is global if, and only if any of the following assertions holds:
(*d*1) $a + c \geq 2b$;
(*d*2) $a + c < 2b,\ a \geq b$;
(*d*3) $a + c < 2b,\ b \leq c$;
(*d*4) $a + c < 2b,\ a = c$.

Proof We have $x^\top Ax = ax_1^2 + 2bx_1x_2 + cx_2^2$.
(a): This is a consequence of the fact that A is positive semidefinite if, and only if all the principal minors are nonnegative.
(b): Let $x \in \mathbf{1}^\perp$. Then, $x^\top Ax = (a - 2b + c)x_1^2$, and so the result follows.
(c): The condition $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^2_+$, implies $a \geq 0$ and $c \geq 0$. Thus

$$x^\top Ax = (\sqrt{a}x_1 - \sqrt{c}x_2)^2 + 2(\sqrt{ac} + b)x_1x_2, \qquad (45)$$

which is nonnegative if $b \geq 0$. In case $b < 0$ and $a > 0,\ c > 0$, we require $b \geq -\sqrt{ac}$, since otherwise, by choosing $x_1 = \frac{\sqrt{c}}{\sqrt{a}}x_2$, $x_2 > 0$, in (45), one gets $x^\top Ax < 0$.
(d): It follows from the equivalent formulation to (44)

$$\min\left\{f(x_1, 1 - x_1) = \frac{1}{2}x_1^2(a - 2b + c) + x_1(b - c) + \frac{c}{2} :\ 0 \leq x_1 \leq 1\right\}, \qquad (46)$$

and by noticing that the function $\varphi(x_1) \doteq f(x_1, 1 - x_1)$ satisfies

$$\varphi'(0) = b - c,\ \varphi'(1) = a - b,\ \varphi(0) = \frac{c}{2},\ \varphi(1) = \frac{a}{2}.$$

□

Consider the matrices

$$A_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}; A_2 = \begin{pmatrix} 1/4 & 1 \\ 1 & 1/2 \end{pmatrix}.$$

One can check that A_1 is non-copositive on $\mathbb{R}^2_+$, and it is such that every local solution (to the associated minimization problem) is global; whereas A_2 is strictly copositive on $\mathbb{R}^2_+$ for which there is a local solution without being global.

5 Local optimality vs global

Very recently, second-order necessary and sufficient conditions for local (resp. global) optimality for a quadratic optimization problem on a polyhedron were established in Theorem 1.2 (resp. Theorem 2.3) of [9]; thus those results cannot be applied to our model. In this section, due to the special structure of the standard quadratic programming problem (19), we derive second-order sufficient and/or necessary conditions for local or global optimality. We refer to [5] for a method locating some particular local minima. Copositivity-based escape procedures for the StQO problem on the simplex are analyzed in [6].

Set, as before, $f(x) \doteq \frac{1}{2}x^\top Ax$ and $g_0(x) \doteq e^\top x - 1$.
Due to the assumptions on C and C^*, we can write

$$C^* \setminus \{0\} = \text{co}(\text{extrd } C^*),$$

where extrd C^* stands for the extremal directions of C^*. Recall that $d \in$ extrd C^* if and only if $d \in C^* \setminus \{0\}$ and for all $d_1, d_2 \in C^*$ such that $d = d_1 + d_2$, we have $d_1, d_2 \in \mathbb{R}_+ d$. Thus, for every $\lambda \in C^* \setminus \{0\}$, one has, for some $k \in \mathbb{N}$,

$$\lambda = \sum_{i=1}^{k} \lambda_i d_i, \quad \lambda_i > 0, \ d_i \in \text{extrd } C^*, \ i = 1, 2, \ldots, k. \tag{47}$$

In what follows, we need the notion of contingent cone. Given a set $M \subseteq \mathbb{R}^n$ and $x \in M$, the contingent cone of M at x, denoted by $T(M; x)$, is the set of vectors $v \in \mathbb{R}^n$ such that there exist $t_k > 0$, $x_k \in M$, $x_k \to x$, satisfying $t_k(x_k - x) \to v$. For a great account of its properties, we refer the book [2]. In particular, we recall that when M is closed and convex, then

$$T(M; x) = \overline{\bigcup_{t \geq 0} t(M - x)}, \ x \in M. \tag{48}$$

We now establish first and second-order necessary conditions for local optimality to problem (19).

Theorem 5.1 *Let A be any real symmetric matrix, and C, e be as before. Let $\bar{x}$ be any local solution to problem* (19) *with $\bar{\mu} \doteq f(\bar{x})$. Set $\lambda \doteq A\bar{x} - 2\bar{\mu}e$. The following hold.*

(*a*) $\lambda^\top \bar{x} = 0$ *and* $\lambda \in \text{bd}\, C^*$*; if additionally* $\bar{x}$ *is not a global solution, we have* $A\bar{x} - 2\mu_q e \in \text{int}\, C^*$.
(*b*) *If* $\bar{x} \in \text{ri}\, K$*, then* $\lambda = 0$.
(*c*) *If* $\lambda = 0$*, then* A *is copositive on* $T(K; \bar{x})$*, and so* $\bar{x}$ *is a global solution to* (19).

Proof (*a*): Let $\bar{x}$ be a local solution to problem (19), i. e., $\bar{x} \in \underset{K \cap U_0}{\text{argmin}}\, f$ for some open neighborhood, U_0, of $\bar{x}$. Clearly $\bar{x} \neq 0$ and $\bar{\mu} \geq \mu_q$. By the first order optimality condition, $\nabla f(\bar{x})^\top v \geq 0$ for all $v \in T(K \cap U_0; \bar{x}) = T(K; \bar{x})$. In other words,

$$\begin{aligned}\nabla f(\bar{x}) \in [T(X \cap C; \bar{x})]^* &= [T(X; \bar{x}) \cap T(C; \bar{x})]^* = \overline{[T(X; \bar{x})]^* + [T(C; \bar{x})]^*} \\ &= [T(X; \bar{x})]^* + [T(C; \bar{x})]^* \\ &= \mathbb{R}e + (\overline{C + \mathbb{R}\bar{x}})^* = \mathbb{R}e + (\bar{x}^\perp \cap C^*),\end{aligned}$$

where the first equality follows from Table 4.3 in [2] (since $0 \in \text{int}(X - C)$) and the second one is a consequence of Corollary 16.4.2 in [40] (since $\text{ri}(e^\perp) \cap \text{ri}(C + \mathbb{R}\bar{x}) \neq \emptyset$). Thus there exists $t \in \mathbb{R}$ satisfying $A\bar{x} - te \in \bar{x}^\perp \cap C^*$. It follows that $t = 2\bar{\mu}$ and so $\lambda = A\bar{x} - 2\bar{\mu}e \in C^*$, which gives $A\bar{x} - 2\bar{\mu}e \in \text{bd}\, C^*$. This implies, in case $\bar{x}$ is not a global solution ($\bar{\mu} > \mu_q$), that

$$A\bar{x} - 2\mu_q e = A\bar{x} - 2\bar{\mu}e + 2(\bar{\mu} - \mu_q)e \in C^* + \text{int}\, C^* = \text{int}\, C^*.$$

(*b*): We already know that $\nabla f(\bar{x})^\top (x - \bar{x}) \geq 0$ for all $x \in K$. If $\bar{x} \in \text{ri}\, K$, then $\nabla f(\bar{x}) \in (\text{aff}\, K)^*$ (the affine hull of K); and so $A\bar{x} = 2\bar{\mu}e$, i.e., $\lambda = 0$.
(*c*): Let $v \in T(K; \bar{x})$. Then, there exist $t_k > 0$, $x_k \in K$, $x_k \to \bar{x}$ such that $t_k(x_k - \bar{x}) \to v$. Thus for all k sufficiently large,

$$\begin{aligned}0 &\leq f(x_k) - 2\bar{\mu}e^\top x_k - f(\bar{x}) + 2\bar{\mu}e^\top \bar{x} \\ &= (\nabla f(\bar{x}) - 2\bar{\mu}e)^\top (x_k - \bar{x}) + \frac{1}{2}(x_k - \bar{x})^\top A(x_k - \bar{x}) = \frac{1}{2}(x_k - \bar{x})^\top A(x_k - \bar{x}).\end{aligned}$$

Hence $v^\top A v \geq 0$, proving the copositivity on $T(K; \bar{x})$. Therefore, given any $x \in K$, the equality

$$\begin{aligned}f(x) - f(\bar{x}) &= f(x) - 2\bar{\mu}g_0(x) - f(\bar{x}) + 2\bar{\mu}g_0(\bar{x}) \\ &= (A\bar{x} - 2\bar{\mu}e)^\top (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^\top A(x - \bar{x}),\end{aligned}$$

yields the desired result, since $K - \bar{x} \subseteq T(K; \bar{x})$ by (48).

We now provide a characterization for a feasible point to be optimal under copositivity of A. Recall that $L(\lambda, x) \doteq f(x) + \lambda g_0(x)$.

Theorem 5.2 *Let* A *be any real symmetric matrix, and* C, e *be as before. The following assertions hold:*

(*a*) *Assume that* $\bar{x} \in K$. *Then,*

$$\left.\begin{array}{c}\mu_q \geq 0\\ \bar{x} \in \underset{K}{\operatorname{argmin}}\ f\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\exists\ \lambda_0 \in \mathbb{R}\\ \bar{x} \in \underset{C}{\operatorname{argmin}}\ L(\lambda_0, \cdot).\end{array}\right.$$

We already now that $\lambda_0 = -2\mu_q$.

(*b*) *Assume that* $\mu_q > 0$ (*and so* $-2\mu_q$ *is a solution to the dual problem* (22)). *Then,*

$$x_0 \in \underset{C}{\operatorname{argmin}}\ L(-2\mu_q, \cdot) \Longrightarrow x_0 \in K.$$

Proof (*a*): We know that $\mu_q \geq 0$ is equivalent to the validity of strong duality with respect to (22), which means

$$\mu_q = L(\lambda_0, \bar{x}) = f(\bar{x}) = \inf_{x \in C}\ L(\lambda_0, \cdot) \tag{49}$$

For the other implication, we have

$$\mu_q \leq f(\bar{x}) = L(\lambda_0, \bar{x}) = \inf_{x \in C} L(\lambda_0, x) \leq \inf_{x \in K} L(\lambda_0, x) = \inf_{x \in K} f(x) = \mu_q,$$

and so strong duality holds, i.e., $\mu_q \geq 0$. This completes the proof of (*a*).
(*b*): Set $x_0 = ty_0$ for some $1 \neq t > 0$ and $y_0 \in K$. By (49), we can write

$$\begin{aligned}\mu_q &= f(x_0) - 2\mu_q(e^\top x_0 - 1) = t^2 f(y_0) - 2\mu_q(t-1)\\ &\geq t^2\mu_q - 2\mu_q(t-1).\end{aligned}$$

This implies $0 \geq \mu_q(t-1)^2$, which is impossible if $\mu_q > 0$.

Next, we derive a sufficient condition for global optimality. In the particular case $C = \mathbb{R}^n_+$, this condition is a consequence of Theorem 2.1 in [9]. For a general convex cone C, the result is new.

Proposition 5.1 *Let* C *be as above. If* $\bar{x}$ *is feasible for* (19), A *is copositive on* $K - \bar{x}$ *or equivalently on* $T(K; \bar{x})$, *and* $A\bar{x} - 2\bar{\mu}e \in C^*$ *holds with* $\bar{\mu} = f(\bar{x})$, *then* $\bar{x} \in \underset{K}{\operatorname{argmin}}\ f$.

Proof For all $x \in K$, one obtains

$$\begin{aligned}f(x) - f(\bar{x}) &= f(x) - 2\bar{\mu}g_0(x) - f(\bar{x}) + 2\bar{\mu}g_0(\bar{x})\\ &= (A\bar{x} - 2\bar{\mu}e)^\top(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^\top A(x - \bar{x}),\end{aligned}$$

from which the result follows. □

We now deal with the standard quadratic optimization problem, that is, when $C = \mathbb{R}^n_+$ and $e = \mathbf{1}$. Denote, given $\bar{x} \in K$,

$$I = I(\bar{x}) \doteq \{i:\ \bar{x}_i = 0\},\ I_+ \doteq \{i \in I:\ \lambda_i > 0\}.$$

$$Z(\bar{x}) \doteq \Big\{v \in \mathbb{R}^n:\ v_i = 0,\ i \in I_+;\ v_i \geq 0,\ i \in I \setminus I_+;\ \sum_{i \in I \setminus I_+} v_i + \sum_{i \notin I} v_i = 0\Big\}.$$

It is not difficult to check that

$$T(K;\bar{x}) = \Big\{ v \in \mathbb{R}^n :\ v_i \geq 0,\ i \in I;\ \sum_{i=1}^{n} v_i = 0 \Big\}.$$

Hence, if $I_+ = \emptyset$, then $Z(\bar{x}) = T(K;\bar{x})$. The following result is a consequence of Theorem 4.4.3 in [3] and the above remark.

Theorem 5.3 *Let A, C, e be as just mentioned. Let $\bar{x}$ be any local solution to problem (2) with $\bar{\mu} \doteq f(\bar{x})$. Then*

(*a*) *$\bar{x}$ is a KKT point:*

$$A\bar{x} - 2\bar{\mu}\mathbf{1} - \lambda = 0,\ \lambda \geq 0,\ \lambda_i \bar{x}_i = 0,\ i = 1, \ldots, n,$$

and A is copositive on $Z(\bar{x})$.

(*b*) *If additionally $I_+ = \emptyset$, then $\bar{x}$ is a global solution.*

Example 5.1 Take the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By computing, one obtains ($x = (x_1, x_2, x_3)$),

$$x^\top Ax = x_1^2 + 2x_1x_2 + 2x_2x_3,\ \ f(x_1, x_2, 1 - x_1 - x_2) = \frac{1}{2}x^\top Ax = \frac{1}{2}(x_1^2 - 2x_2^2 + 2x_2).$$

Then: $\mu_q = 0$, that is, A is copositive on $\mathbb{R}^3_+$ but it is not strictly copositive on $\mathbb{R}^3_+$; A is not copositive on $\mathbf{1}^\perp$ since $f(-1, 1, 0) = -\frac{1}{2}$. Moreover, the associated StQO problem has two solutions, namely $\mathrm{argmin}_K f = \{\bar{x}^1 \doteq (0, 0, 1), \bar{x}^2 \doteq (0, 1, 0)\}$. One can also check that no local-nonglobal solution exists. Moreover, $\bar{\lambda}^1 = A\bar{x}^1 = (0, 1, 0)$, $\bar{\lambda}^2 = A\bar{x}^2 = (1, 0, 1)$, so (*b*) of Theorem 5.3 is not satisfied at $\bar{x}^1$ or $\bar{x}^2$. Notice that

$$Z(\bar{x}^1) = \{t(1, 0, -1) :\ t \geq 0\},\ T(K;\bar{x}^1) = \{t(1, 0, -1) + s(0, 1, -1) : t \geq 0, s \geq 0\},$$

and

$$Z(\bar{x}^2) = \{(0, 0, 0)\},\ T(K;\bar{x}^2) = \{t(1, -1, 0) + s(0, -1, 1) :\ t \geq 0,\ s \geq 0\}.$$

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References

1. Ahmed, F.; Dür, M.; Still, G., Copositive programming via semi-infinite optimization, *J. Optim. Theory Appl.*, **159** (2013), 322–340.
2. Aubin, J. P.; Frankowska, H., Set-Valued Analysis, Birkhäuser, Boston, Basel, Berlin, 1990.
3. Bazaraa, M.S.; Sherali, H. D.; Shetty, C. M., Nonlinear programming, theory and algorithms, Wiley-Interscience, A John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
4. Beck, A., On the convexity of a class of quadratic mappings and its application to the problem of finding the smallest ball enclosing a given intersection of balls, *J. Global Optim.*, **39** (2007), 113–126.
5. Best, M. J.; Ding, B., Global and local quadratic minimization, *J. Global Optim.*, **10** (1997), 77–90.
6. Bomze, I. M., Global escape strategies for maximizing quadratic forms over a simplex, *J. Global Optim.*, **11** (1997), 325–338.
7. Bomze, I. M., On standard quadratic optimization problems, *J. Global Optim.*, **13** (1998), 369–387.
8. Bomze, I. M., Copositivity relaxation beats Lagrangian dual bounds in quadratically and linearly constrained quadratic optimization problems, *SIAM, J. Optim.*, **25** (2015), 1249–1275.
9. Bomze, I. M., Copositivity for second-order optimality conditions in general smooth optimization problems, *Optimization*, **65** (2016), 779–795.
10. Bomze, I. M.; de Klerk, E., Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, *J. Global Optim.*, **24** (2002), 163–185.
11. Bomze, I. M.; Dür M.; de Klerk E.; Roos, C.; Quist, A. J.; Terlaky, T., On copositive programming and standard quadratic optimization problems, *J. Global Optim.*, **18** (2000), 301–320.
12. Bomze, I. M.; Locatelli, M.; Tardella, F., New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability, *Math. Program., A* **115** (2008), 31–64.
13. Borwein, J. M.; Lewis, A. S., Partially finite convex programming, part I: Quasi relative interiors and duality theory, *Math. Program., A*, **57** (1992), 15–48.
14. Burer, S., A gentle-geometric introduction to copositive optimization, *Math. Program., B*, **151** (2015), 89–116.
15. Cárcamo, G.; Flores-Bazán, F., Strong duality and KKT conditions in nonconvex optimization with a single equality constraint and geometric constraint, *Math. Program., B*, **168** (2018), 369-400.
16. de Klerk E.; Pasechnik, D. V., A linear programming reformulation of the standard quadratic optimization problem, *J. Global Optim.*, **24** (2007), 75–84.
17. Dickinson, Peter J. C., An improved characterisation of the interior of the completely positive cone, *Electron. J. Linear Algebra*, **20** (2010), 723–729.
18. Dines, L. L., On the mapping of quadratic forms, *Bull. Am. Math. Soc.*, **47** (1941), 494–498.
19. Dinh, N,; Goberna, M. A.; Lopez, M. A.; Son, T. Q., New Farkas-type constraint qualifications in convex infinite programming, *ESAIM: Control, Optim. and Calculus of Variations*, **13** (2007), 580–597.
20. Dür, M.; Still G., Interior points of the completely positive cone, *Electron. J. Linear Algebra*, **17** (2008), 48–53.
21. Eichfelder G.; Jahn, J., Set-semidefinite optimization, *J. Convex Anal.*, **15** (2008), 767–801.
22. Flores-Bazán, F.; Echegaray, W.; Flores-Bazán, Fernando; Ocaña, E., Primal or dual strong-duality in non convex optimization and a class of quasiconvex problems having zero duality gap, *J. Global Optim.*, **69**(2017), 823–845.
23. Flores-Bazán, F.; Jourani, A.; Mastroeni, G. On the convexity of the value function for a class of nonconvex variational problems: existence and optimality conditions, *SIAM J. Control Optim.*, **52** (2014), 3673–3693.
24. Flores-Bazán, F.; Mastroeni, G., Strong duality in cone constrained nonconvex optimization, *SIAM J. Optim*, **23** No 1 (2013), 153–169.

25. Flores-Bazán, F.; Opazo, F., Characterizing the convexity of joint-range for a pair of inhomogeneous quadratic functions and strong duality, *Minimax Theory Appl.*, **1** (2016), 257–290.
26. Gaddum, J. W., Linear inequalities and quadratic forms, *Pacific Journal of Mathematics*, **8** (1958), 411–414.
27. Gibbons, L. E.; Hearn, D. W.; Pardalos, P. M.; Ramana, M. V., Continuous characterizations of the maximum clique problem, *Math. Oper. Res.*, **22** (1997), 754–768.
28. Hadeler, K. P., On copositive matrices, *Linear Alg. Appl.*, **49** (1983), 79–89.
29. Hiriart-Urruty, J. B.; Torki, M., Permanently going back and forth between "quadratic world" and the "convexity world" in optimization, *Appl. Math. Optim.*, **45** (2002), 169–184.
30. Ibaraki, T.; Katoh, N., Resource allocations problems: algorithm approaches. MIT Press, Cambridge, 1988.
31. Jeyakumar, V.; Lee, G. M.; Li, G. Y., Alternative theorems for quadratic inequality systems and global quadratic optimization, *SIAM, J. Optim.*, **20** (2009), 983–1001.
32. Jeyakumar, V.; Wolkowicz, H., Zero duality gaps in infinite-dimensional programming, *J. Optim. Theory Appl.*, **67** (1990), 87–108.
33. Jeyakumar, V.; Li, G. Y., Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimzation, *Math. Program. A*, **147** (2014), 171–206.
34. Markowitz, H. M., Portfolio selection, *J. Finance*, **7** (1952), 77–91.
35. Markowitz, H. M., The general mean-variance portfolio selection problem, in Howison, S. D. et al (eds) Mathematical models in finance; **93-99**. Chapman & Hall, London (1995).
36. Mastroeni, G., Some applications of the image space analysis to the duality theory for constrained extremum problems, *J. Global Optim.*, **46** (2010), 603–614.
37. Motzkin, T. S.; Strauss, E. G., Maxima for graphs and a new proof of a theorem of Turán, *Can. J. Math.*, **17** (1965), 533–540.
38. Ping, L.; Yu, F. Y., Criteria for copositive matrices of order four, *Linear Alg. Appl.*, **194** (1993), 109–124.
39. Pólik, I.; Terlaky, T., A survey of the S-Lemma, *SIAM, Review*, **49** (2007), 371–418.
40. Rockafellar, R. T., *Convex Analysis*, Princeton University Press, Princeton, (1970).
41. Rockafellar, R. T., Conjugate Duality and Optimization, SIAM, (1974).
42. Sturm, J. F.; Zhang, S., On cones of nonnegative quadratic functions, *Math. Oper. Res.*, **28** (2003), 246–267.
43. Xia, Y.; Wang, S.; Sheu, R. L., S-Lemma with equality and its applications, *Math. Prog.*, Ser A, **156** (2016), 513–547.
44. Yakubovich, V. A., S-procedure in nonlinear control theory, Vestnik Leningrad. Univ., **1** (1971), 62–77 (in Russian).
45. Yakubovich, V. A., S-procedure in nonlinear control theory, Vestnik Leningrad. Univ., **4** (1977), 73–93 (English translation).