A divergence–conforming DG–mixed finite element method for the stationary Boussinesq problem *

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Abstract

In this work we propose and analyze a new fully divergence–conforming finite element method for the numerical simulation of the Boussinesq problem, describing the motion of a non-isothermal incompressible fluid subject to a heat source. We consider the standard velocity–pressure formulation for the fluid flow equation and the dual–mixed one for the heat equation. In this way, the unknowns of the resulting formulation are given by the velocity, the pressure, the temperature and the gradient of the latter. The corresponding Galerkin scheme makes use of the nonconforming exactly divergence–free approach to approximate the velocity and pressure, and employ standard $Hdiv$–conforming elements for the gradient of the temperature and discontinuous elements for the temperature. Since here we utilize a dual–mixed formulation for the heat equation, the temperature Dirichlet boundary condition becomes natural, thus there is no need of introducing a sufficiently small discrete lifting to prove well–posedness of the discrete problem. Moreover, the resulting numerical scheme yields exactly divergence–free velocity approximations; thus, it is probably energy-stable without the need to modify the underlying differential equations, and provide an optimal convergent approximation of the temperature gradient. The analysis of the continuous and discrete problems are carried out by means of a fixed–point strategy, under a sufficiently small data assumption. We derive optimal error estimates in the mesh size for smooth solutions and provide several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence.

Key words: stationary Boussinesq equations; divergence–conforming elements; Discontinuous Galerkin methods; Mixed-FEM.

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

Preliminares

Natural convection is a phenomenon present in different important applications in engineering and industry. Briefly, we can mention that electrical and electronic industries use it for the thermal regulation of components and devices of industrial equipments. Also, this phenomenon appears in geophysics and oceanography when studying climate predictions and oceanic flows. Roughly speaking, it refers to a fluid motion generated by density differences due to temperature gradients. Mathematically, it

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is modelled by the Navier–Stokes equations coupled to a convection-diffusion equation through the Boussinesq approximation (variations in density are neglected everywhere except in the buoyancy term), reason why it is often called the Boussinesq model.

In the last decades, the devising of suitable numerical methods for solving the Boussinesq equations and its generalizations, such as temperature-dependent coefficient problems, has become a very active research area (see, e.g. [1, 2, 8, 12, 13, 14, 18, 19, 23, 24, 31], and the references therein). In particular, in [8], which up to the authors's knowledge is one of the first works in analyzing a finite element discretization for the Boussinesq problem, it is introduced and analysed a primal formulation where the main unknowns of the respective system are the velocity, the pressure and the temperature of the fluid. There, suitable assumptions on the finite elements subspaces are introduced to ensuring that the associated Galerkin scheme is well posed and convergent. In particular, the use of any pair of stable Stokes elements for the fluid variables and Lagrange elements for the temperature leads to a convergent scheme.

Recently, in [24] it has been proposed and analyzed a new finite element method with exactly divergence–free velocities for the numerical simulation of a generalized Boussinesq problem where the viscosity and the thermal conductivity depend on the temperature of the fluid. The method proposed in [24], which is based on the works [10] and [11], makes use of divergence–conforming elements for the velocities, discontinuous elements for the pressure, and standard continuous elements for the temperature. Thus, the resulting method has the distinct property that it yields exactly divergence–free velocity approximations, which is an essential constraint of the governing equations, and is probably energy–stable without the need for symmetrization of the convective discretization. However, to ensure existence and stability of solution of the numerical method, since the temperature Dirichlet boundary condition becomes essential (due to the fact that the heat equation is discretized by using an H^1–conforming method), it is needed to construct a sufficiently small (in the L^3-norm) discrete lifting of the temperature boundary data. The latter is a delicate matter as the numerical construction of discrete liftings may be computationally expensive, which constitutes the main drawback of this approach.

Now, it is well-known that in several physical phenomena where the system exchanges energy with its surroundings through heat transfer, the heat flux $\boldsymbol{\phi} := -k\nabla\theta$ can be employed to calculate the energy balance, where θ is the temperature and κ the thermal conductivity. Hence, the gradient of the temperature is a must know variable and the more accurate its approximation, the better the approximation of the energy balance. Then, the employment of a mixed method for solving the corresponding heat equation seems to be the best option. In this direction, in [18] the authors introduce a new mixed formulation for the two-dimensional Boussinesq problem. There, it is introduced the gradient of the temperature as an additional unknown, which together to the velocity and its gradient, as well as the pressure and the temperature of the fluid, constitute the main unknowns of the resulting dual-mixed variational system. The associated Galerkin scheme makes use of the Raviart-Thomas element of lowest order for the gradient of the velocity and the temperature, and piecewise constants for the velocity, temperature and pressure. Existence of solution and convergence of the numerical scheme are proved near a nonsingular solution and only quasi–optimal error estimates are provided. In turn, in [13] it is introduced a new augmented fully–mixed finite element method for the stationary Boussinesq problem. The method is based on the introduction of a pseudostress tensor depending on the pressure, and the diffusive and convective terms of the Navier–Stokes equations for the fluid, and an auxiliary vector unknown involving the temperature, its gradient and the velocity for the heat equation. The resulting variational formulation is then augmented by using the constitutive and equilibrium equations of the system and the boundary conditions, and as a consequence, it is obtained an augmented fully-mixed formulation for the coupled problem, which allows the utilization of $Hdiv$–

conforming spaces for approximating the unknowns of both, the Navier-Stokes and convection-diffusion equations.

According to the discussion above, and with the purpose of contributing to the development of new numerical methods to approximate the solution of natural convection problems, allowing a direct approximation of the gradient of the temperature, in this work we propose and analyse a fully $Hdiv$–conforming finite element method for the numerical simulation of the Boussinesq problem. Here we consider the standard velocity–pressure formulation for the fluid flow equation and similarly to [18] we introduce the gradient of the temperature as a further unknown and employ a dual-mixed formulation for the heat equation. In this way, the unknowns of the resulting formulation are given by the velocity, the pressure, the temperature and its gradient. For the corresponding Galerkin scheme we employ the divergence–conforming approach utilized in [24] for the discretization of the fluid equation, and differently from [24], the heat equation is discretized by using standard $Hdiv$–conforming elements for the gradient of the temperature and discontinuous elements for the temperature. We emphasize that, since here we utilize a dual–mixed formulation for the heat equation, the temperature Dirichlet boundary condition becomes natural, thus there is no need of introducing a discrete lifting to ensure well–posedness of the problem. In turn, it allows to approximate directly the gradient of the temperature, thus avoiding numerical differentiation of the temperature field. Moreover, differently from [18] we prove that our method is optimal convergent, and this optimality can be achieved without incorporating any stabilization parameter as it is done in [13]. In addition, it exactly preserves the divergence–free velocity constraint. The analysis of the continuous and discrete problems are carried out by means of a sufficiently small data assumption and a fixed–point strategy. More precisely, similarly to the analysis in [13] (see also [12, 3]), we rewrite the variational problem as an equivalent fixed–point problem and apply the classical Schauder (Brouwer) and Banach fixed–point theorems to prove existence and uniqueness of solution of the continuous (discrete) problem. Finally, we derive optimal error estimates in the mesh size for smooth solutions.

The rest of the paper is organized as follows. In Section 2 we introduce the model problem, derive the corresponding weak formulation and analyze its existence and uniqueness of solution. Next in Section 3 we propose the $Hdiv$–conforming method and analyze the well–posedness of the corresponding Galerkin scheme by mimicking the analysis developed for the continuous problem. In Section 4 we prove that our numerical method is optimal convergent. Finally, in Section 5 we provide several numerical results illustrating the performance of the primal-mixed finite element method and conforming the theoretical rates of convergence.

We end this section by fixing some notations and well–known previous results. To that end, let us denote by $\Omega \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$, a given bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ.

In the sequel, standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{t,p}(\Omega)$ endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\Omega)}$, $\|\cdot\|_{\mathrm{W}^{t,p}(\Omega)}$ and the seminorm by $|\cdot|_{\mathrm{W}^{t,p}(\Omega)}$ respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$, and if $p = 2$, we write $\mathrm{H}^t(\Omega)$ in place of $\mathrm{W}^{t,2}(\Omega)$, and denote the norm by $\|\cdot\|_{t,\Omega}$ and the seminorm by $|\cdot|_{t,\Omega}$. The spaces of vector-valued functions are denoted in bold face. For example, $\mathbf{H}^t(\Omega) := \left[\mathrm{H}^t(\Omega)\right]^n$, $t \geq 0$ We denote $\mathrm{H}^{1/2}(\Gamma)$ as the space of traces of functions in $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. Along with the above we denote $\langle\cdot,\cdot\rangle$ as the duality pairing of $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ with respect to the $\mathrm{L}^2(\Gamma)$ inner product. By $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. We employ $\mathbf{0}$ to denote a generic null vector. In addition, in the sequel we will make use of the well-known Hölder, Poincaré inequalities, given respectively by

$$\int_\Omega |fg| \leq \|f\|_{\mathrm{L}^p(\Omega)}\, \|g\|_{\mathrm{L}^q(\Omega)}, \qquad \forall\, f \in \mathrm{L}^p(\Omega),\ \forall\, g \in \mathrm{L}^q(\Omega), \qquad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.1}$$

$$\|w\|_{1,\Omega} \leq C\,|w|_{1,\Omega}, \qquad \forall\, w \in \mathrm{H}_0^1(\Omega), \tag{1.2}$$

Finally, we recall that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^p(\Omega)$ for $p \geq 1$ if $n = 2$ or $p \in [1, 6]$ if $n = 3$. More precisely, we have the following inequality

$$\|w\|_{\mathrm{L}^p(\Omega)} \leq C_{Sob}(p)\,\|w\|_{1,\Omega}, \qquad \forall\, w \in \mathrm{H}^1(\Omega), \tag{1.3}$$

with $C_{Sob}(p) > 0$ depending only on $|\Omega|$, and p (see [26, Theorem 1.3.4]).

2 Continuous problem

In this section we introduce a model problem, cast it into weak form, discuss the stability properties of the forms involved, and review some theoretical properties regarding existence and uniqueness of solution. We start by introducing the model problem.

2.1 The model problem and its weak formulation

In this work we are interested in approximating the solution of the stationary Boussinesq problem consisting of a system of equations where the incompressible Navier–Stokes equation:

$$\begin{gathered} -\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{g}\,\theta = 0 \quad \text{in} \quad \Omega, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad \int_\Omega p = 0, \end{gathered} \tag{2.1}$$

is coupled with the convection-diffusion equation:

$$-\,\kappa\,\Delta\theta + \mathbf{u}\cdot\nabla\theta = 0 \quad \text{in} \quad \Omega, \qquad \theta = \theta_D \quad \text{on} \quad \Gamma, \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \{2, 3\}$, with polyhedral boundary Γ. Above, the unknowns are the velocity $\mathbf{u}$, the pressure p and the temperature θ of the fluid occupying the region Ω, and the given data are the fluid viscosity $\nu > 0$, the thermal conductivity $\kappa > 0$, the external force per unit mass $\mathbf{g} \in \mathbf{L}^2(\Omega)$, and the boundary temperature $\theta_D \in \mathrm{H}^{1/2}(\Gamma)$.

Now, since we want to derive a numerical scheme allowing a divergence–conforming approximation for the whole coupled system, differently from [24], here we introduce the flux

$$\boldsymbol{\sigma} := \kappa\,\nabla\theta \quad \text{in} \quad \Omega, \tag{2.3}$$

as a further unknown and realize that (2.2) can be rewritten as the following first–order set of equations,

$$\kappa^{-1}\,\boldsymbol{\sigma} - \nabla\theta = 0 \quad \text{in} \quad \Omega, \qquad -\operatorname{div}\boldsymbol{\sigma} + \kappa^{-1}\,\mathbf{u}\cdot\boldsymbol{\sigma} = 0 \quad \text{in} \quad \Omega, \qquad \theta = \theta_D \quad \text{on} \quad \Gamma. \tag{2.4}$$

As a consequence, in the sequel we derive our variational formulation based on the coupled system given by (2.1) and (2.4). To that end, as usual we first multiply the first equation of (2.1) by a test function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, integrate by parts to obtain

$$\nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v} - \int_\Omega p\operatorname{div}\mathbf{v} - \int_\Omega \theta(\mathbf{g}\cdot\mathbf{v}) = 0, \qquad \forall\,\mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.5}$$

and incorporate the second equation of (2.1) weakly through

$$\int_\Omega q\operatorname{div}\mathbf{u} = 0 \qquad \forall\, q \in \mathrm{L}_0^2(\Omega), \tag{2.6}$$

which suggests to look for the unknowns $\mathbf{u}$ and p in the Hilbert spaces $\mathbf{H}_0^1(\Omega)$ and $\mathrm{L}_0^2(\Omega)$, respectively.

Next, to write equations (2.4) in weak form we first recall that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^t(\Omega)$, with $t \geq 1$ if $n = 2$ and $1 \leq t \leq 6$ if $n = 3$ (see for instance [26, Theorem 1.3.4]), and observe that if particularly $t > 2$ and if $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{\sigma} \in \mathbf{L}^2(\Omega)$, then

$$\mathbf{u} \cdot \boldsymbol{\sigma} \in \mathrm{L}^r(\Omega), \tag{2.7}$$

with

$$r := \frac{2t}{t+2} \in \left(1, \frac{6-n}{2}\right), \qquad n \in \{2, 3\}. \tag{2.8}$$

Consequently, and according to the second equation of (2.4), we obtain that $\operatorname{div} \boldsymbol{\sigma} \in \mathrm{L}^r(\Omega)$ which suggests us to introduce the Banach space

$$H(\operatorname{div}_r; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \operatorname{div} \boldsymbol{\tau} \in \mathrm{L}^r(\Omega) \right\}, \quad \text{with} \quad r \in \left(1, \frac{6-n}{2}\right), \quad n \in \{2, 3\}, \tag{2.9}$$

equipped with the norm

$$\|\boldsymbol{\tau}\|^2_{H(\operatorname{div}_r;\Omega)} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{\tau}\|^2_{\mathrm{L}^r(\Omega)}.$$

Observe that $H(\operatorname{div}; \Omega) \subset H(\operatorname{div}_r; \Omega)$.

Then, multiplying the first equation of (2.4) by $\boldsymbol{\tau} \in H(\operatorname{div}_r; \Omega)$, integrating by parts, and using the boundary condition $\theta = \theta_D$ on Γ, we obtain

$$\kappa^{-1} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_\Omega \theta \operatorname{div} \boldsymbol{\tau} = \langle \boldsymbol{\tau} \cdot \mathbf{n}, \theta_D \rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in H(\operatorname{div}_r; \Omega). \tag{2.10}$$

Finally, since $-\operatorname{div} \boldsymbol{\sigma} + \kappa^{-1} \mathbf{u} \cdot \boldsymbol{\sigma} \in \mathrm{L}^r(\Omega)$, we impose this identity weakly as follows

$$\int_\Omega \psi \operatorname{div} \boldsymbol{\sigma} - \kappa^{-1} \int_\Omega \psi (\mathbf{u} \cdot \boldsymbol{\sigma}) = 0 \qquad \forall\, \psi \in \mathrm{L}^s(\Omega), \tag{2.11}$$

with

$$s := \frac{2t}{t-2}, \tag{2.12}$$

satisfying $\frac{1}{r} + \frac{1}{s} = 1$. Notice that $s > n$ for $n = 2, 3$ and that $\frac{1}{t} + \frac{1}{2} + \frac{1}{s} = 1$.

In this way, from now on we fix $t > 2$, define r and s as in (2.8) and (2.12), respectively, and sum up properly equations (2.5)–(2.11), to arrive at the variational coupled problem: Find $(\mathbf{u}, p, \boldsymbol{\sigma}, \theta) \in \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega) \times H(\operatorname{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$ such that

$$\begin{aligned} A_{\mathrm{S}}(\mathbf{u}, \mathbf{v}) + O_{\mathrm{S}}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - B_{\mathrm{S}}(\mathbf{v}, p) - D(\theta, \mathbf{v}) &= 0, \\ B_{\mathrm{S}}(\mathbf{u}, q) &= 0, \\ A_{\mathrm{T}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau}, \theta) &= G(\boldsymbol{\tau}), \\ B_{\mathrm{T}}(\boldsymbol{\sigma}, \psi) + O_{\mathrm{T}}(\mathbf{u}; \boldsymbol{\sigma}, \psi) &= 0, \end{aligned} \tag{2.13}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}, \psi) \in \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega) \times H(\operatorname{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$, where the forms $A_{\mathrm{S}} : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, $A_{\mathrm{T}} : H(\operatorname{div}_r, \Omega) \times H(\operatorname{div}_r, \Omega) \to \mathbb{R}$, $B_{\mathrm{S}} : \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega) \to \mathbb{R}$, $B_{\mathrm{T}} : H(\operatorname{div}_r, \Omega) \times \mathrm{L}^s(\Omega) \to \mathbb{R}$, $O_{\mathrm{S}} : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, $O_{\mathrm{T}} : \mathbf{H}_0^1(\Omega) \times H(\operatorname{div}_r, \Omega) \times \mathrm{L}^s(\Omega) \to \mathbb{R}$, $D : \mathrm{L}^s(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, and the functional $G_{\rm T} : H(\mathrm{div}_r, \Omega) \to \mathbb{R}$ are defined, respectively, as

$$\begin{aligned} A_{\rm S}(\mathbf{u},\mathbf{v}) &= \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}, & A_{\rm T}(\boldsymbol{\sigma},\boldsymbol{\tau}) &= \kappa^{-1} \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}, \\ B_{\rm S}(\mathbf{v},q) &= \int_\Omega q \,\mathrm{div}\,\mathbf{v}, & B_{\rm T}(\boldsymbol{\tau},\psi) &= \int_\Omega \psi\,\mathrm{div}\,\boldsymbol{\tau}, \\ O_{\rm S}(\mathbf{w};\mathbf{u},\mathbf{v}) &= \int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}, & O_{\rm T}(\mathbf{w};\boldsymbol{\sigma},\psi) &= -\kappa^{-1}\int_\Omega (\mathbf{w}\cdot\boldsymbol{\sigma})\,\psi, \\ D(\theta,\mathbf{v}) &= \int_\Omega \theta\,\mathbf{g}\cdot\mathbf{v}, & G(\boldsymbol{\tau}) &= \langle \boldsymbol{\tau}\cdot\mathbf{n}, \theta_D\rangle_\Gamma\,. \end{aligned} \tag{2.14}$$

2.1.1 Stability properties

In what follows we establish the stability properties of the forms involved. We begin by observing that, after simple computations, the bilinear forms $A_{\rm S}$, $A_{\rm T}$, $B_{\rm S}$ and $B_{\rm T}$ are bounded:

$$|A_{\rm S}(\mathbf{u},\mathbf{v})| \le \nu\, \|\mathbf{u}\|_{1,\Omega}\,\|\mathbf{v}\|_{1,\Omega}, \qquad |A_{\rm T}(\boldsymbol{\sigma},\boldsymbol{\tau})| \le \kappa^{-1}\,\|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r,\Omega)}\,\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r,\Omega)}, \tag{2.15}$$

and

$$|B_{\rm S}(\mathbf{v},q)| \le \|\mathbf{v}\|_{1,\Omega}\|q\|_{0,\Omega}, \qquad |B_{\rm T}(\boldsymbol{\tau},\psi)| \le \|\boldsymbol{\tau}\|_{H(\mathrm{div}_r,\Omega)}\,\|\psi\|_{\mathrm{L}^s(\Omega)}. \tag{2.16}$$

In turn, owing to the Hölder's and Sobolev inequalities, (1.1) and (1.3), respectively, it is not difficult to see that $O_{\rm S}$ satisfies

$$|O_{\rm S}(\mathbf{w}_1-\mathbf{w}_2;\mathbf{u},\mathbf{v})| \le L_O^{\rm S}\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{L}^t(\Omega)}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \tag{2.17}$$

$$|O_{\rm S}(\mathbf{w}_1-\mathbf{w}_2;\mathbf{u},\mathbf{v})| \le C_O^{\rm S}\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}, \tag{2.18}$$

for all $\mathbf{w}_1,\mathbf{w}_2,\mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, with $L_O^{\rm S} := C_{Sob}(s)$ and $C_O^{\rm S} := C_{Sob}(s)C_{Sob}(t)$. Similarly, for $O_{\rm T}$ we have

$$|O_{\rm T}(\mathbf{w}_1-\mathbf{w}_2;\boldsymbol{\tau},\psi)| \le L_O^{\rm T}\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{L}^t(\Omega)}\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r;\Omega)}\|\psi\|_{\mathrm{L}^s(\Omega)}, \tag{2.19}$$

$$|O_{\rm T}(\mathbf{w}_1-\mathbf{w}_2;\boldsymbol{\tau},\psi)| \le C_O^{\rm T}\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega}\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r;\Omega)}\|\psi\|_{\mathrm{L}^s(\Omega)}, \tag{2.20}$$

for all $\mathbf{w}_1,\mathbf{w}_2 \in \mathbf{H}_0^1(\Omega), \boldsymbol{\tau}\in H(\mathrm{div}_r;\Omega)$, $\psi \in \mathrm{L}^s(\Omega)$, where $L_O^{\rm T} := \kappa^{-1}$ and $C_O^{\rm T} := \kappa^{-1}C_{Sob}(t)$.

Using again (1.1) and (1.3) we can readily see that

$$|D(\psi,\mathbf{v})| \le C_D\,\|\psi\|_{\mathrm{L}^s(\Omega)}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{1,\Omega} \quad \psi\in \mathrm{L}^s(\Omega),\ \mathbf{v}\in\mathbf{H}_0^1(\Omega) \tag{2.21}$$

and

$$|G(\boldsymbol{\tau})| \le C_G\,\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r,\Omega)}\,\|\theta_D\|_{1/2,\Gamma} \quad \boldsymbol{\tau}\in H(\mathrm{div}_r;\Omega), \tag{2.22}$$

with $C_D := C_{Sob}(t)$ and $C_G := C_{Sob}(s)$. For the latter we refer the reader to [9].

Now we let

$$\begin{aligned} \mathbf{K}_{\rm S} :=&\ \{\mathbf{v}\in\mathbf{H}_0^1(\Omega) : B_{\rm S}(\mathbf{v},q) = 0, \quad \forall\, q\in \mathrm{L}_0^2(\Omega)\} \\ =&\ \{\mathbf{v}\in\mathbf{H}_0^1(\Omega) : \mathrm{div}\,\mathbf{v} = 0, \text{ in } \Omega\}, \end{aligned} \tag{2.23}$$

and observe that for any $\mathbf{w}\in\mathbf{K}_{\rm S}$, there holds

$$O_{\rm S}(\mathbf{w},\mathbf{v},\mathbf{v}) = 0, \qquad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega). \tag{2.24}$$

In addition, owing to the well-known Poincaré inequality we have that A_{S} is elliptic, that is

$$A_{\mathrm{S}}(\mathbf{v},\mathbf{v}) \,\geq \alpha_{\mathrm{S}}\,\|\mathbf{v}\|_{1,\Omega}^2, \qquad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega), \tag{2.25}$$

which together with (2.24) implies that for each $\mathbf{w}\in\mathbf{K}_{\mathrm{S}}$, the bilinear form $A_{\mathrm{S}}(\cdot,\cdot)+O_{\mathrm{S}}(\mathbf{w};\cdot,\cdot)$: $\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0^1(\Omega)\to\mathbb{R}$ is $\mathbf{H}_0^1(\Omega)$–elliptic. Similarly, we let

$$\begin{aligned}\mathbf{K}_{\mathrm{T}} \;&:=\; \{\boldsymbol{\tau}\in H(\mathrm{div}_r,\Omega) : B_{\mathrm{T}}(\boldsymbol{\tau},\psi)=0, \forall\,\psi\in\mathrm{L}^s(\Omega)\}\\ &=\; \{\boldsymbol{\tau}\in H(\mathrm{div}_r,\Omega) : \mathrm{div}\,\boldsymbol{\tau}=0 \quad\text{in}\quad \Omega\}\,,\end{aligned} \tag{2.26}$$

and realize that A_{T} satisfies

$$A_{\mathrm{T}}(\boldsymbol{\tau},\boldsymbol{\tau}) \;\geq\; \kappa^{-1}\,\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r,\Omega)}^2 \qquad \forall\,\boldsymbol{\tau}\in\mathbf{K}_{\mathrm{T}}. \tag{2.27}$$

For the caracterization of $\mathbf{K}_{\mathrm{T}}$ and the proof of (2.27) we refer the reader to [9, Lemma 2.2].

Let us now recall that the bilinear forms B_{S} and B_{T} satisfy the following inf-sup conditions

$$\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\setminus\{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}} \geq \beta_{\mathrm{S}}\,\|q\|_{0,\Omega}, \qquad \forall\,q\in\mathrm{L}_0^2(\Omega), \tag{2.28}$$

$$\sup_{\boldsymbol{\tau}\in H(\mathrm{div}_r,\Omega)\setminus\{\mathbf{0}\}} \frac{B_{\mathrm{T}}(\boldsymbol{\tau},\psi)}{\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r,\Omega)}} \;\geq\; \beta_{\mathrm{T}}\,\|\psi\|_{\mathrm{L}^s(\Omega)}, \qquad \forall\,\psi\in\mathrm{L}^s(\Omega). \tag{2.29}$$

For the proof of these inequalities we refer the reader to [27, Section 2.2] and [9, Lemma 2.1] respectively.

Finally, let us consider the bilinear form $\mathbf{A} : H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)\to\mathbb{R}$, defined by

$$\mathbf{A}((\boldsymbol{\sigma},\theta),(\boldsymbol{\tau},\psi)) = A_{\mathrm{T}}(\boldsymbol{\sigma},\boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau},\theta) + B_{\mathrm{T}}(\boldsymbol{\sigma},\psi), \tag{2.30}$$

for all $(\boldsymbol{\sigma},\theta),(\boldsymbol{\tau},\psi)\in H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$. Owing to the properties (2.27) and (2.29) and applying [20, Proposition 2.36] it is not difficult to see that $\mathbf{A}$ satisfies the following inf-sup condition:

$$\sup_{(\boldsymbol{\tau},\psi)\in[H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)]\setminus\{\mathbf{0}\}} \frac{\mathbf{A}((\boldsymbol{\zeta},\phi),(\boldsymbol{\tau},\psi))}{\|(\boldsymbol{\tau},\psi)\|} \geq \gamma\|(\boldsymbol{\zeta},\phi)\| \qquad \forall\,(\boldsymbol{\zeta},\phi)\in H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega), \tag{2.31}$$

where $\gamma>0$ is the constant defined by

$$\gamma = \frac{\kappa\,\beta_{\mathrm{T}}^2}{\kappa^2\,\beta_{\mathrm{T}}^2 + 4\,\kappa\,\beta_{\mathrm{T}} + 2}. \tag{2.32}$$

2.2 Existence and uniqueness of solution

In what follows, similarly to [12] and [13] (see also [3]), we study the well–posedness of problem (2.13) by means of a fixed–point strategy and the classical Babuška–Brezzi theory. We begin by introducing the associated fixed-point operator.

2.2.1 The fixed–point operator

In view of the fixed-point strategy to be used in the proof of solvability of problem (2.13), let us introduce first the reduced problem: Find $(\mathbf{u},\boldsymbol{\sigma},\theta)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$ such that

$$\begin{aligned} A_{\mathrm{S}}(\mathbf{u},\mathbf{v})\,+\,O_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v})\,-D(\theta,\mathbf{v}) &= \;\; 0,\\ A_{\mathrm{T}}(\boldsymbol{\sigma},\boldsymbol{\tau})\,+B_{\mathrm{T}}(\boldsymbol{\tau},\theta) &= \;\; G(\boldsymbol{\tau}),\\ B_{\mathrm{T}}(\boldsymbol{\sigma},\psi)\,+O_{\mathrm{T}}(\mathbf{u};\boldsymbol{\sigma},\psi) &= \;\; 0,\end{aligned}\tag{2.33}$$

for all $(\mathbf{v},\boldsymbol{\tau},\psi)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$. It is not difficult to see that, according to the definition of $\mathbf{K}_{\mathrm{S}}$ and owing to the inf-sup condition (2.28), problem (2.33) is equivalent to (2.13). This result is established next. The proof of this result follows analogously to the proof of [25, Lemma 2.1], reason why it is omitted.

Lemma 2.1 *If $(\mathbf{u},p,\boldsymbol{\sigma},\theta)\in\mathbf{H}_0^1(\Omega)\times\mathrm{L}_0^2(\Omega)\times H(\mathrm{div}_r;\Omega)\times\mathrm{L}^s(\Omega)$ is a solution of (2.13), then $\mathbf{u}\in\mathbf{K}_{\mathrm{S}}$ and $(\mathbf{u},\boldsymbol{\sigma},\theta)$ is a solution of (2.33). Conversely, if $(\mathbf{u},\boldsymbol{\sigma},\theta)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r;\Omega)\times\mathrm{L}^s(\Omega)$ is a solution of (2.33), then there exists $p\in\mathrm{L}_0^2(\Omega)$ such that $(\mathbf{u},p,\boldsymbol{\sigma},\theta)$ is a solution of (2.13).*

Now, let us introduce the bounded and convex set

$$\mathbf{X}_1 := \left\{\mathbf{w}\in\mathbf{K}_{\mathrm{S}} : \|\mathbf{w}\|_{1,\Omega}\le\frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma}\right\}\tag{2.34}$$

and define the operator

$$\begin{aligned}\mathbf{T}:\;\; \mathbf{X}_1 &\;\to\;\; \mathbf{X}_1\\ \mathbf{w} &\;\to\;\; \mathbf{T}(\mathbf{w})=\mathbf{u},\end{aligned}\tag{2.35}$$

with $\mathbf{u}$ being the first component of the solution of the linearized version of problem (2.33): Find $(\mathbf{u},\boldsymbol{\sigma},\theta)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$ such that

$$\begin{aligned} A_{\mathrm{S}}(\mathbf{u},\mathbf{v})\,+\,O_{\mathrm{S}}(\mathbf{w};\mathbf{u},\mathbf{v}) &= \;\; D(\theta,\mathbf{v})\,,\\ A_{\mathrm{T}}(\boldsymbol{\sigma},\boldsymbol{\tau})\,+B_{\mathrm{T}}(\boldsymbol{\tau},\theta) &= \;\; G(\boldsymbol{\tau}),\\ B_{\mathrm{T}}(\boldsymbol{\sigma},\psi)\,+O_{\mathrm{T}}(\mathbf{w};\boldsymbol{\sigma},\psi) &= \;\; 0,\end{aligned}\tag{2.36}$$

for all $(\mathbf{v},\boldsymbol{\tau},\psi)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$.

It is clear that $(\mathbf{u},\boldsymbol{\sigma},\theta)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)$ is a solution of problem (2.33), if and only if, $\mathbf{T}(\mathbf{u})=\mathbf{u}$. This result, together with the equivalence between (2.13) and (2.33), imply the following relations:

$$\begin{aligned}\mathbf{T}(\mathbf{u})=\mathbf{u}\quad &\Leftrightarrow\quad (\mathbf{u},\boldsymbol{\sigma},\theta)\in\mathbf{K}_{\mathrm{S}}\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)\quad\text{satisfies (2.33)}\\ &\Leftrightarrow\quad (\mathbf{u},p,\boldsymbol{\sigma},\theta)\in\mathbf{H}_0^1(\Omega)\times\mathrm{L}_0^2(\Omega)\times H(\mathrm{div}_r,\Omega)\times\mathrm{L}^s(\Omega)\quad\text{satisfies (2.16).}\end{aligned}\tag{2.37}$$

In this way, in establishing the well-posedness of (2.13), or equivalently (2.33), it suffices to prove that $\mathbf{T}$ has a unique fixed-point in $\mathbf{X}_1$. Before proceeding with the solvability analysis, we first state the well-definiteness of the fixed-point operator $\mathbf{T}$.

2.2.2 Well-definiteness of the fixed-point operator

For the forthcoming analysis we define the sets

$$\mathbf{X} = \left\{ (\mathbf{w}, \phi) \in \mathbf{K}_{\mathrm{S}} \times \mathrm{L}^s(\Omega) : \|\mathbf{w}\|_{1,\Omega} \leq \frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma}, \quad \|\phi\|_{\mathrm{L}^s(\Omega)} \leq \frac{C_G}{\gamma} \|\theta_D\|_{1/2,\Gamma} \right\}, \tag{2.38}$$

and

$$\mathbf{X}_2 := \left\{ \phi \in L^s(\Omega) : \|\phi\|_{L^s(\Omega)} \leq \frac{C_G}{\gamma} \|\theta_D\|_{1/2,\Gamma} \right\}, \tag{2.39}$$

and the operators

$$\begin{aligned} \mathbf{R} : \quad \mathbf{X} \quad &\rightarrow \quad \mathbf{X}_1 \\ (\mathbf{w}, \phi) \quad &\rightarrow \quad \mathbf{R}(\mathbf{w}, \phi) = \mathbf{u}, \end{aligned} \tag{2.40}$$

with $\mathbf{u}$ being the unique solution of problem: Find $\mathbf{u} \in \mathbf{K}_{\mathrm{S}}$, such that

$$A_{\mathrm{S}}(\mathbf{u}, \mathbf{v}) + O_{\mathrm{S}}(\mathbf{w}; \mathbf{u}, \mathbf{v}) = D(\phi, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}_{\mathrm{S}}, \tag{2.41}$$

and

$$\begin{aligned} \mathbf{S} : \quad \mathbf{X}_1 \quad &\rightarrow \quad H(\mathrm{div}_r; \Omega) \times \mathbf{X}_2 \\ \mathbf{w} \quad &\rightarrow \quad \mathbf{S}(\mathbf{w}) = (\mathbf{S}_1(\mathbf{w}), \mathbf{S}_2(\mathbf{w})) = (\boldsymbol{\sigma}, \theta), \end{aligned} \tag{2.42}$$

where $(\boldsymbol{\sigma}, \theta) \in H(\mathrm{div}_r, \Omega) \times \mathbf{X}_2$ is the unique solution of problem: find $(\boldsymbol{\sigma}, \theta) \in H(\mathrm{div}_r; \Omega) \times \mathrm{L}^s(\Omega)$, such that

$$\begin{aligned} A_{\mathrm{T}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau}, \theta) \quad &= G(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}_r, \Omega), \\ B_{\mathrm{T}}(\boldsymbol{\sigma}, \psi) + O_{\mathrm{T}}(\mathbf{w}; \boldsymbol{\sigma}, \psi) \quad &= 0 \qquad \forall\, \psi \in \mathrm{L}^s(\Omega). \end{aligned} \tag{2.43}$$

Then, it readily follows that operator $\mathbf{T}$ can be rewritten in terms of $\mathbf{R}$ and $\mathbf{S}$ as follows

$$\mathbf{T}(\mathbf{w}) = \mathbf{R}(\mathbf{w}, \mathbf{S}_2(\mathbf{w})) \qquad \forall\, \mathbf{w} \in \mathbf{X}_1. \tag{2.44}$$

In addition, provided a fixed-point $\mathbf{u} = \mathbf{T}(\mathbf{u})$, the element $(\boldsymbol{\sigma}, \theta) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$ such that $(\mathbf{u}, \boldsymbol{\sigma}, \theta) \in \mathbf{K}_{\mathrm{S}} \times H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$ is a solution to (2.33) (see (2.37)), satisfies the identity

$$(\boldsymbol{\sigma}, \theta) = \mathbf{S}(\mathbf{u}). \tag{2.45}$$

According to the above, to proving that $\mathbf{T}$ is well–defined, it suffices to prove that operators $\mathbf{R}$ and $\mathbf{S}$ are both well–defined separately. We begin with the well–definiteness of $\mathbf{R}$.

Lemma 2.2 *For each* $(\mathbf{w}, \phi) \in \mathbf{X}$*, there exists a unique* $\mathbf{u} \in \mathbf{X}_1$*, such that* $\mathbf{R}(\mathbf{w}, \phi) = \mathbf{u}$.

Proof. Let $(\mathbf{w}, \phi) \in \mathbf{X}$. Owing to the $\mathbf{H}_0^1(\Omega)$–ellipticity of the bilinear form $A_{\mathrm{S}}(\cdot,\cdot) + O_{\mathrm{S}}(\mathbf{w};\cdot,\cdot)$, the well–posedness of (2.41) is a direct consequence of the Lax-Milgram lemma, which is clearly equivalent to the existence of a unique $\mathbf{u} \in \mathbf{X}_1$, such that $\mathbf{R}(\mathbf{w}, \phi) = \mathbf{u}$. Moreover, from (2.41) with $\mathbf{v} = \mathbf{u}$, from (2.21) and from the definition of $\mathbf{X}$ (cf. (2.38)), it readily follows that

$$\|\mathbf{u}\|_{1,\Omega} \leq \frac{C_D}{\alpha_{\mathrm{S}}} \|\mathbf{g}\|_{0,\Omega} \|\phi\|_{\mathrm{L}^s(\Omega)} \leq \frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma}. \tag{2.46}$$

which implies that $\mathbf{u} \in \mathbf{X}_1$. □

Now, to prove that the mapping $\mathbf{S}$ is well defined, for a fixed $\mathbf{w} \in \mathbf{X}_1$, let us define the bilinear form $\mathcal{A}_{\mathbf{w}} : H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega) \to \mathbb{R}$, given by

$$\mathcal{A}_{\mathbf{w}}((\boldsymbol{\sigma}, \theta), (\boldsymbol{\tau}, \psi)) := \mathbf{A}((\boldsymbol{\sigma}, \theta), (\boldsymbol{\tau}, \psi)) + O_{\mathrm{T}}(\mathbf{w}; \boldsymbol{\sigma}, \psi) \quad \forall\, (\boldsymbol{\sigma}, \theta), (\boldsymbol{\tau}, \psi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega), \quad (2.47)$$

where $\mathbf{A}$ is the bilinear form defined in (2.30). Using the estimate (2.31), it is not difficult to see that $\mathcal{A}_{\mathbf{w}}$ satisfies the conditions of the Banach-Nečas-Babuška theorem (cf. [20, Theorem 2.6]). This result is established next.

Lemma 2.3 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that*

$$\frac{C_O^{\mathrm{T}} C_D C_G}{\alpha_{\mathrm{S}} \gamma^2} \|\theta_D\|_{1/2,\Gamma} \|\mathbf{g}\|_{0,\Omega} \leq \frac{1}{2}. \quad (2.48)$$

Then, for each $\mathbf{w} \in \mathbf{X}_1$, the bilinear form $\mathcal{A}_{\mathbf{w}}$ satisfies the following estimates

$$S_1 := \sup_{(\boldsymbol{\tau},\psi) \in [H(\mathrm{div}_r,\Omega) \times \mathrm{L}^s(\Omega)] \setminus \{\mathbf{0}\}} \frac{\mathcal{A}_{\mathbf{w}}((\boldsymbol{\zeta}, \phi), (\boldsymbol{\tau}, \psi))}{\|(\boldsymbol{\tau}, \psi)\|} \geq \frac{\gamma}{2} \|\boldsymbol{\zeta}\|_{H(\mathrm{div}_r;\Omega)} + \gamma \|\phi\|_{\mathrm{L}^s(\Omega)}, \quad (2.49)$$

$\forall\, (\boldsymbol{\zeta}, \phi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$, and

$$S_2 := \sup_{(\boldsymbol{\tau},\psi) \in H(\mathrm{div}_r,\Omega) \times \mathrm{L}^s(\Omega)} \mathcal{A}_{\mathbf{w}}((\boldsymbol{\tau}, \psi), (\boldsymbol{\zeta}, \phi)) > 0 \quad \forall\, (\boldsymbol{\zeta}, \phi) \in [H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)] \setminus \{\mathbf{0}\}, \quad (2.50)$$

where $\gamma > 0$ is the constant defined in (2.32).

Proof. In what follows we proceed analogously to the proof of [9, Theorem 4.1].

First, using (2.31) and the continuity of O_{T} in (2.18), we easily obtain that for all $(\boldsymbol{\zeta}, \phi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$,

$$S_1 \geq \gamma \|(\boldsymbol{\zeta}, \phi)\| - C_O^{\mathrm{T}} \|\mathbf{w}\|_{1,\Omega} \|\boldsymbol{\zeta}\|_{H(\mathrm{div}_r,\Omega)}.$$

Hence, utilizing the fact that $\mathbf{w} \in \mathbf{X}_1$ and assumption (2.48), from the latter inequality we easily obtain (2.49). In turn, since the bilinear form $\mathbf{A}$ (cf. (2.30)) is symmetric, it is clear that (2.31) yields

$$\sup_{(\boldsymbol{\tau},\psi) \in [H(\mathrm{div}_r,\Omega) \times \mathrm{L}^s(\Omega)] \setminus \{\mathbf{0}\}} \frac{\mathbf{A}((\boldsymbol{\tau}, \psi), (\boldsymbol{\zeta}, \phi))}{\|(\boldsymbol{\tau}, \psi)\|} \geq \gamma \|(\boldsymbol{\zeta}, \phi)\| \quad \forall\, (\boldsymbol{\zeta}, \phi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega),$$

which together to the continuity of O_{T}, the fact that $\mathbf{w} \in \mathbf{X}_1$ and estimate (2.48), implies

$$\begin{aligned} \widehat{S}_2 := \sup_{(\boldsymbol{\tau},\psi) \in [H(\mathrm{div}_r,\Omega) \times \mathrm{L}^s(\Omega)] \setminus \{\mathbf{0}\}} \frac{\mathcal{A}_{\mathbf{w}}((\boldsymbol{\tau}, \psi), (\boldsymbol{\zeta}, \phi))}{\|(\boldsymbol{\tau}, \psi)\|} &\geq \gamma \|(\boldsymbol{\zeta}, \phi)\| - C_O^{\mathrm{T}} \|\mathbf{w}\|_{1,\Omega} \|\phi\|_{\mathrm{L}^s(\Omega)} \\ &\geq \gamma \|\boldsymbol{\zeta}\|_{H(\mathrm{div}_r;\Omega)} + \frac{\gamma}{2} \|\phi\|_{\mathrm{L}^s(\Omega)}, \end{aligned}$$

for all $(\boldsymbol{\zeta}, \phi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$. Then, since $S_2 \geq \widehat{S}_2$, from the latter inequality it is not difficult to see that $\mathcal{A}_{\mathbf{w}}$ satisfies (2.50), which concludes the proof. □

The following lemma establishes the well-definiteness of $\mathbf{S}$.

Lemma 2.4 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that* (2.48) *holds. Then, for each $\mathbf{w} \in \mathbf{X}_1$, there exists a unique $(\boldsymbol{\sigma}, \theta) \in H(\mathrm{div}_r; \Omega) \times \mathbf{X}_2$, such that $\mathbf{S}(\mathbf{w}) = (\boldsymbol{\sigma}, \theta)$.*

Proof. The existence of a unique $(\boldsymbol{\sigma},\theta) \in H(\mathrm{div}_r;\Omega) \times \mathrm{L}^s(\Omega)$ such that $\mathbf{S}(\mathbf{w}) = (\boldsymbol{\sigma},\theta)$ is a direct consequence of Lemma 2.3 and the Banach-Nečas-Babuška theorem. In turn, since $(\boldsymbol{\sigma},\theta)$ satisfies equations (2.43), using (2.22) we readily obtain

$$|\mathcal{A}_{\mathbf{w}}((\boldsymbol{\sigma},\theta),(\boldsymbol{\tau},\psi))| = |G(\boldsymbol{\tau})| \leq C_G \|\theta_D\|_{1/2,\Gamma} \|(\boldsymbol{\tau},\psi)\|, \tag{2.51}$$

which together to (2.49) implies that $\theta \in \mathbf{X}_2$ and concludes the proof. □

Remark 2.1 *Observe that from* (2.51) *we actually deduce that* $\mathbf{S}(\mathbf{w}) \in \mathbf{Z}$, *with*

$$\mathbf{Z} := \left\{ (\boldsymbol{\tau},\psi) \in H(\mathrm{div}_r,\Omega) \times \mathrm{L}^s(\Omega) : \frac{\gamma}{2}\|\boldsymbol{\tau}\|_{H(\mathrm{div}_r;\Omega)} \;+\; \gamma\|\psi\|_{\mathrm{L}^s(\Omega)} \leq C_G \|\theta_D\|_{1/2,\Gamma} \right\}. \tag{2.52}$$

The latter will be employed later to conclude the corresponding a priori estimate.

We conclude this section by establishing the well–definiteness of operator $\mathbf{T}$.

Theorem 2.5 *Assume that* (2.48) *holds. Then, for each* $\mathbf{w} \in \mathbf{X}_1$, *there exists a unique* $\mathbf{u} \in \mathbf{X}_1$ *such that* $\mathbf{u} = \mathbf{T}(\mathbf{w})$.

Proof. According to the identity (2.44), the well–definiteness of $\mathbf{T}$ follows straightforwardly from Lemmas 2.2 and 2.4. □

2.2.3 The main result

Now we prove the main result of this section, namely, existence and uniqueness of solution of problem (2.13). First, we address the existence of solution by employing the classical Schauder's fixed point theorem written in the following form:

> *Let W be a closed and convex subset of a Banach space X and let $T : W \to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

Next, for the uniqueness of solution we apply the classical Banach's fixed–point theorem, which, for the sake of completeness, is established now:

> *Let X be a Banach space and let $T : X \to X$ be a contraction mapping, that is, there exists $0 < \rho < 1$, such that*
> $$\|T(x) - T(y)\|_X \leq \rho\|x - y\|_X, \forall\, x, y \in X.$$
> *Then there exists a unique fixed–point of T in X.*

We begin with the existence of solution. To that end we first establish the following preliminary results.

Lemma 2.6 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $s = \frac{2t}{t-2}$. Then there exist positive constants $L_{\mathbf{R}_1}$ and $L_{\mathbf{R}_2}$, and $C_{\mathbf{R}}$ such that*

$$\|\mathbf{R}(\mathbf{w}_1,\phi_1) - \mathbf{R}(\mathbf{w}_2,\phi_2)\|_{1,\Omega} \leq L_{\mathbf{R}_1}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^t(\Omega)} + L_{\mathbf{R}_2}\|\phi_1 - \phi_2\|_{\mathrm{L}^s(\Omega)} \tag{2.53}$$

and

$$\|\mathbf{R}(\mathbf{w}_1,\phi_1) - \mathbf{R}(\mathbf{w}_2,\phi_2)\|_{1,\Omega} \leq C_{\mathbf{R}}\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} + L_{\mathbf{R}_2}\|\phi_1 - \phi_2\|_{\mathrm{L}^s(\Omega)}, \tag{2.54}$$

for all $(\mathbf{w}_1,\phi_1)$, $(\mathbf{w}_2,\phi_2) \in \mathbf{X}$.

Proof. Let $(\mathbf{w}_1, \phi_1)$, $(\mathbf{w}_2, \phi_2) \in \mathbf{X}$, and $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbf{X}_1$ such that $\mathbf{R}(\mathbf{w}_1, \phi_1) = \mathbf{u}_1$ and $\mathbf{R}(\mathbf{w}_2, \phi_2) = \mathbf{u}_2$. Then, from the definition of $\mathbf{R}$ (cf. (2.41)) it readily follows that

$$A_{\mathrm{S}}(\mathbf{u}_i, \mathbf{v}) + O_{\mathrm{S}}(\mathbf{w}_i; \mathbf{u}_i, \mathbf{v}) = D(\phi_i, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K}_{\mathrm{S}}, \quad \forall\, i \in \{1, 2\}. \tag{2.55}$$

Then, from (2.55), by subtracting both equations, choosing the test function $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, using the identity (2.24), and adding and subtracting suitable terms, we obtain

$$A_{\mathrm{S}}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = -\,O_{\mathrm{S}}(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) + D(\phi_1 - \phi_2, \mathbf{u}_1 - \mathbf{u}_2). \tag{2.56}$$

In this way, estimate (2.53) can be obtained after a straightforward application of the ellipticity of A_{S} (cf. (2.25)), the continuity of D (cf. (2.21)), the fact that $\mathbf{u}_1 \in \mathbf{X}_1$, and estimate (2.17), with constants $L_{\mathbf{R}_1}$ and $L_{\mathbf{R}_2}$, given by

$$L_{\mathbf{R}_1} = \frac{L_O^{\mathrm{S}} C_D C_G}{\gamma \alpha_{\mathrm{S}}^2} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \quad \text{and} \quad L_{\mathbf{R}_2} = \frac{C_D}{\alpha_{\mathrm{S}}} \|\mathbf{g}\|_{0,\Omega}. \tag{2.57}$$

Analogously, estimate (2.54) can be obtained from (2.56) by employing (2.25), (2.21), the fact that $\mathbf{u}_1 \in \mathbf{X}_1$, and estimate (2.18), with constant $C_{\mathbf{R}}$ given by

$$C_{\mathbf{R}} = \frac{C_O^{\mathrm{S}} C_D C_G}{\gamma \alpha_{\mathrm{S}}^2} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma}. \tag{2.58}$$

$\square$

Lemma 2.7 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that hypothesis* (2.48) *holds. Then, there exist positive constants $L_{\mathbf{S}}$ and $C_{\mathbf{S}}$, such that*

$$\frac{1}{2} \|\mathbf{S}_1(\mathbf{w}_1) - \mathbf{S}_1(\mathbf{w}_2)\|_{H(\mathrm{div}_r;\Omega)} + \|\mathbf{S}_2(\mathbf{w}_1) - \mathbf{S}_2(\mathbf{w}_2)\|_{\mathrm{L}^s(\Omega)} \le L_{\mathbf{S}} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^t(\Omega)} \tag{2.59}$$

and

$$\frac{1}{2} \|\mathbf{S}_1(\mathbf{w}_1) - \mathbf{S}_1(\mathbf{w}_2)\|_{H(\mathrm{div}_r;\Omega)} + \|\mathbf{S}_2(\mathbf{w}_1) - \mathbf{S}_2(\mathbf{w}_2)\|_{\mathrm{L}^s(\Omega)} \le C_{\mathbf{S}} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} \tag{2.60}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}_1$.

Proof. Given $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}_1$, let $(\boldsymbol{\sigma}_1, \theta_1)$, $(\boldsymbol{\sigma}_2, \theta_2) \in H(\mathrm{div}_r, \Omega) \times \mathbf{X}_2$, such that $\mathbf{S}(\mathbf{w}_1) = (\boldsymbol{\sigma}_1, \theta_1)$ and $\mathbf{S}(\mathbf{w}_2) = (\boldsymbol{\sigma}_2, \theta_2)$. From the definition of $\mathbf{S}$ (cf. (2.42) and (2.43)), there holds

$$A_{\mathrm{T}}(\boldsymbol{\sigma}_i, \boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau}, \theta_i) + B_{\mathrm{T}}(\boldsymbol{\sigma}_i, \psi) + O_{\mathrm{T}}(\mathbf{w}_i; \boldsymbol{\sigma}_i, \psi) = G(\boldsymbol{\tau}),$$

for $i \in \{1, 2\}$ and for all $(\boldsymbol{\tau}, \psi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$, which after simple computations, implies

$$A_{\mathrm{T}}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau}, \theta_1 - \theta_2) + B_{\mathrm{T}}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \psi) + O_{\mathrm{T}}(\mathbf{w}_1; \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \psi) = -O_{\mathrm{T}}(\mathbf{w}_1 - \mathbf{w}_2; \boldsymbol{\sigma}_2, \psi),$$

for all $(\boldsymbol{\tau}, \psi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$, or equivalently

$$\mathcal{A}_{\mathbf{w}_1}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \theta_1 - \theta_2), (\boldsymbol{\tau}, \psi)) = -\,O_{\mathrm{T}}(\mathbf{w}_1 - \mathbf{w}_2; \boldsymbol{\sigma}_2, \psi), \tag{2.61}$$

for all $(\boldsymbol{\tau}, \psi) \in H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$, where $\mathcal{A}_{\mathbf{w}_1}$ is the bilinear form defined in (2.47).

In turn, let us observe that for $i \in \{1, 2\}$, $\mathbf{S}(\mathbf{w}_i) = (\boldsymbol{\sigma}_i, \theta_i) \in \mathbf{Z}$ (cf. (2.52)), which in particular implies that $\|\boldsymbol{\sigma}_2\|_{H(\mathrm{div}_r;\Omega)} \le \frac{2C_G}{\gamma} \|\theta_D\|_{1/2,\Gamma}$. Then, using this inequality, the fact that $\mathbf{w}_1 \in \mathbf{X}_1$, and assumption (2.48), it is not difficult to see that estimate (2.59) follows from (2.49), (2.19) and (2.61), with

$$L_{\mathbf{S}} = \frac{2L_O^{\mathrm{T}} C_G}{\gamma^2} \|\theta_D\|_{1/2,\Gamma}. \tag{2.62}$$

Analogously, estimate (2.60) follows from (2.49), (2.20) and (2.61), with $C_{\mathbf{S}}$ given by

$$C_{\mathbf{S}} = \frac{2C_O^{\mathrm{T}} C_G}{\gamma^2} \|\theta_D\|_{1/2,\Gamma}. \tag{2.63}$$

□

Now, we are in position of establishing the existence of solution of (2.13).

Theorem 2.8 *Given $t > 2$ if $n = 2$ and $t \in (2, 6)$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that hypothesis* (2.48) *holds. Then there exists $(\mathbf{u}, p, \boldsymbol{\sigma}, \theta) \in \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega) \times H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$ solution to* (2.13)*. Moreover, there exists $C > 0$, independent of the solution, such that*

$$\|\mathbf{u}\|_{1,\Omega} + \|p\|_{0,\Omega} + \|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r,\Omega)} + \|\theta\|_{\mathrm{L}^s(\Omega)} \leq C\|\theta_D\|_{1/2,\Gamma}. \tag{2.64}$$

Proof. We begin the proof by noticing that here, for the 3D case, the interval $(2, 6)$ must be open in both sides since we shall employ the fact that $\mathrm{H}^1(\Omega)$ is compactly embedded into $\mathrm{L}^t(\Omega)$ (see [26, Theorem 1.3.5]).

As mentioned before, and according to (2.37), to proving the existence of solution of (2.13) it suffices to proving that $\mathbf{T}$ has at least one fixed-point by means of the Shauder's fixed-point theorem. To that end we first recall that (cf. (2.44))

$$\mathbf{T}(\mathbf{w}) = \mathbf{R}(\mathbf{w}, \mathbf{S}_2(\mathbf{w})) \qquad \forall\, \mathbf{w} \in \mathbf{X}_1,$$

and observe that from (2.53) and (2.59), there holds

$$\begin{aligned} \|\mathbf{T}(\mathbf{w}_1) - \mathbf{T}(\mathbf{w}_2)\|_{1,\Omega} \leq\ & L_{\mathbf{R}_1}\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^t(\Omega)} + L_{\mathbf{R}_2}\|\mathbf{S}_2(\mathbf{w}_1) - \mathbf{S}_2(\mathbf{w}_2)\|_{\mathrm{L}^s(\Omega)} \\ \leq\ & (L_{\mathbf{R}_1} + L_{\mathbf{R}_2} L_{\mathbf{S}})\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^t(\Omega)}. \end{aligned} \tag{2.65}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}_1$.

From (2.65) and (1.3), it can be readily seen that $\mathbf{T}$ is a continuous operator. Moreover, using the fact that $\mathrm{L}^t(\Omega)$ is compactly embedded into $\mathrm{H}^1(\Omega)$ it is not difficult to prove that $\overline{\mathbf{T}(\mathbf{X}_1)}$ is compact. In fact, let $\{\mathbf{z}_k\}_{k\in\mathbb{N}}$ be a sequence of $\mathbf{X}_1$, which is clearly bounded. It follows that there exists a subsequence $\{\tilde{\mathbf{z}}_k\}_{k\in\mathbb{N}} \subseteq \{\mathbf{z}_k\}_{k\in\mathbb{N}}$ and $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ such that $\tilde{\mathbf{z}}_k \overset{w}{\longrightarrow} \mathbf{z}$. Then since $\mathrm{H}^1(\Omega)$ is compactly embedded into $\mathrm{L}^t(\Omega)$, we deduce that $\tilde{\mathbf{z}}_k \longrightarrow \mathbf{z}$ in $\mathbf{L}^t(\Omega)$, which owing to (2.65) implies that $\mathbf{T}(\tilde{\mathbf{z}}_k) \longrightarrow \mathbf{T}(\mathbf{z})$ in $\mathbf{H}_0^1(\Omega)$. Hence, $\overline{\mathbf{T}(\mathbf{X}_1)}$ is compact. Therefore, by applying the Shauder's fixed-point theorem we obtain that $\mathbf{T}$ has at least one fixed–point, or equivalently, problem (2.13) has at least one solution.

Now, to deduce (2.64) we first observe that $\mathbf{u} \in \mathbf{X}_1$ (cf. (2.34)) which implies that

$$\|\mathbf{u}\|_{1,\Omega} \leq \frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma}. \tag{2.66}$$

In turn, from Remark 2.1 and according to (2.45), it follows that $(\boldsymbol{\sigma}, \theta) = \mathbf{S}(\mathbf{u}) \in \mathbf{Z}$ (cf. (2.52)), which implies

$$\frac{\gamma}{2}\|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r;\Omega)} + \gamma\|\theta\|_{\mathrm{L}^s(\Omega)} \leq C_G\|\theta_D\|_{1/2,\Gamma}. \tag{2.67}$$

Finally, for the estimate of the pressure p, we make use of the inf-sup condition of B_{S} (2.28), the continuity of A_{S}, O_{S} and D (cf. (2.15), (2.21)), and the first equation of (2.13), to arrive at

$$\begin{aligned} \beta_{\mathrm{S}}\,\|p\|_{0,\Omega} &\le \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\setminus\{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v},p)}{\|\mathbf{v}\|_{1,\Omega}} \\ &= \sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)\setminus\{\mathbf{0}\}} \frac{A_{\mathrm{S}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v}) - D(\theta,\mathbf{v})}{\|\mathbf{v}\|_{1,\Omega}} \\ &\le \left(\nu\,\|\mathbf{u}\|_{1,\Omega} + C_O^{\mathrm{S}}\,\|\mathbf{u}\|_{1,\Omega}^2 + C_D\,\|\theta\|_{\mathrm{L}^s(\Omega)}\,\|\mathbf{g}\|_{0,\Omega}\right). \end{aligned} \tag{2.68}$$

From (2.66)–(2.68) we easily deduce (2.64), which concludes the proof. □

We end the analysis of the continuous problem (2.13) by establishing the corresponding uniqueness result.

Theorem 2.9 *Given $t > 2$ if $n = 2$ and $t \in (2, 6)$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that*

$$\frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma}\left(\frac{C_O^{\mathrm{S}}}{\alpha_{\mathrm{S}}} + \frac{2C_O^{\mathrm{T}}}{\gamma}\right)\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} < 1. \tag{2.69}$$

Then, the solution of problem (2.13) *is unique.*

Proof. First, we observe that if we assume (2.69), then hypothesis (2.48) holds, which implies that problem (2.13) admits at least one solution.

Now, proceeding analogously to the proof of Theorem 2.8, that is, recalling that (cf. (2.44))

$$\mathbf{T}(\mathbf{w}) = \mathbf{R}(\mathbf{w}, \mathbf{S}_2(\mathbf{w})) \qquad \forall\,\mathbf{w}\in\mathbf{X}_1,$$

and using estimates (2.54) and (2.60), we easily deduce that

$$\|\mathbf{T}(\mathbf{w}_1) - \mathbf{T}(\mathbf{w}_2)\|_{1,\Omega} \le (C_{\mathbf{R}} + L_{\mathbf{R}_2}C_{\mathbf{S}})\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega}, \tag{2.70}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}$. Then, observing that

$$C_{\mathbf{R}} + L_{\mathbf{R}_2}C_{\mathbf{S}} = \frac{C_D C_G}{\alpha_{\mathrm{S}}\gamma}\left(\frac{C_O^{\mathrm{S}}}{\alpha_{\mathrm{S}}} + \frac{2C_O^{\mathrm{T}}}{\gamma}\right)\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma},$$

from (2.70) and hypothesis (2.69) we readily obtain that operator $\mathbf{T}$ is a contraction mapping. Therefore, employing the classical Banach's fixed–point theorem we conclude that $\mathbf{T}$ possesses a unique fixed–point $\mathbf{u}\in\mathbf{X}_1$, or equivalently, according to (2.37), problem (2.13) has a unique solution. □

3 A nonconforming finite element discretization

In this section we introduce our nonconforming finite element method for approximating the solution of problem (2.13). As we shall see next in the forthcoming sections, the analysis of the corresponding discrete problem follows straightforwardly by adapting the fixed-point strategy introduced and analyzed in Section 2.2. We emphasize that the latter is feasible since, similarly as in [24], we consider here the exactly divergence–free finite element method proposed in [11] to approximate the velocity of the fluid. More precisely, using the incompressibility property of the fluid, which it is exactly satisfied here, we can introduce the discrete version of problem (2.33) and the corresponding discrete fixed–point operator, satisfying the same properties as the ones described in Sections 2.2.1–2.2.3.

We start by introducing our Galerkin scheme and reviewing the discrete stability properties of the forms involved.

3.1 The discrete problem

Let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles K (resp. tetrahedra K) of diameter h_K and define the mesh size $h = \max\{h_K : K \in \mathcal{T}_h\}$. In addition given an integer $l \geq 0$, for each $K \in \mathcal{T}_h$ we let $P_l(K)$ be the space of polynomials functions on K of degree $\leq l$, and define the corresponding local Raviart–Thomas space of order l as

$$\mathbf{RT}_l(K) := \mathbf{P}_l(K) \oplus \widetilde{P}_l(K)\mathbf{x},$$

where $\mathbf{P}_l(K) = [P_l(K)]^n$, $\mathbf{x}$ is the generic vector in $\mathbb{R}^n$, and $\widetilde{P}_l(K) \subset P_l(K)$ denotes the space of polynomials of total degree equal to l. Employing this definitions we introduce the finite–dimensional spaces

$$\begin{aligned} \mathbf{H}_h^l :=& \quad \{\mathbf{z} \in H(\mathrm{div}\,;\Omega) : \mathbf{z}|_K \in \mathbf{RT}_l(K), \quad \forall\, K \in \mathcal{T}_h\} \\ Y_h^l :=& \quad \left\{z \in \mathrm{L}^2(\Omega) : z|_K \in P_l(K), \quad \forall\, K \in \mathcal{T}_h\right\}, \end{aligned} \tag{3.1}$$

and then, for a given $k \geq 1$, we define the finite element subspaces to approximate the unknowns $\mathbf{u}$, and p, $\boldsymbol{\sigma}$, and θ, respectively by

$$\mathbf{V}_h^{\mathbf{u}} := \mathbf{H}_h^k \cap H_0(\mathrm{div}\,;\Omega), \quad Q_h^p := Y_h^k, \quad \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} := \mathbf{H}_h^{k-1} \quad \text{and} \quad \Psi_h^\theta := Y_h^{k-1}, \tag{3.2}$$

where

$$H_0(\mathrm{div}\,;\Omega) := \{\mathbf{v} \in H(\mathrm{div}\,;\Omega) : \mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \Gamma\}\,.$$

Observe that $\boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \subset H(\mathrm{div}\,;\Omega) \subset H(\mathrm{div}_r;\Omega)$, whereas $\mathbf{V}_h^{\mathbf{u}}$ is not a subspace of $\mathbf{H}_0^1(\Omega)$, thus the method is nonconforming. Then in order to deal with the nonconformity of our approach we must introduce discrete versions of the forms A_{S} and O_{S}. To do that we first introduce some additional notations and definitions.

For each K we denote by $\mathbf{n}_K$ the unit outward normal vector on the boundary ∂K. In addition, we denote by $\mathcal{E}_I(\mathcal{T}_h)$ the set of all interior edges (faces) of $\mathcal{T}_h$, by $\mathcal{E}_B(\mathcal{T}_h)$ the set of all boundary edges (faces), and define $\mathcal{E}_h(\mathcal{T}_h) = \mathcal{E}_I(\mathcal{T}_h) \cup \mathcal{E}_B(\mathcal{T}_h)$. The $(n-1)$-dimensional diameter of an edge (face) e is denoted by h_e.

We will use standard average and jump operators. To define them, let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$, and $e = \partial K^+ \cap \partial K^- \in \mathcal{E}_I(\mathcal{T}_h)$. Let $\mathbf{u}$ and $\mathbf{M}$ be a piecewise smooth vector-valued, respectively matrix-valued function, and let us denote by $\mathbf{u}^\pm$, $\mathbf{M}^\pm$ the traces of $\mathbf{u}$, $\mathbf{M}$ on e, taken from within the interior of $K^\pm$. Then, we define the jump of $\mathbf{u}$, respectively the mean value of $\mathbf{M}$ at $\mathbf{x} \in e$ by

$$[\![\mathbf{u}]\!] = \mathbf{u}^+ \otimes \mathbf{n}_{K^+} + \mathbf{u}^- \otimes \mathbf{n}_{K^-}, \qquad \{\!\{\mathbf{M}\}\!\} = \frac{1}{2}(\mathbf{M}^+ + \mathbf{M}^-), \tag{3.3}$$

where we denote by $\mathbf{u} \otimes \mathbf{n}$ the tensor product matrix $[\mathbf{u} \otimes \mathbf{n}]_{i,j} = u_i n_j$, $1 \leq i,j \leq n$. For a boundary edge (face) $e = \partial K^+ \cap \Gamma$, we set $[\![\mathbf{u}]\!] = \mathbf{u}^+ \otimes \mathbf{n}$, with $\mathbf{n}$ denoting the unit outward normal vector on Γ, and $\{\!\{\mathbf{M}\}\!\} = \mathbf{M}^+$.

For the discrete version of A_{S} we take the symmetric interior penalty (SIP) form (see [4]) given by

$$\begin{aligned} A_{\mathrm{S}}^h(\mathbf{u},\mathbf{v}) = \nu \int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \nu \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e \{\!\{\nabla_h \mathbf{u}\}\!\} : [\![\mathbf{v}]\!] \\ - \nu \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \int_e \{\!\{\nabla_h \mathbf{v}\}\!\} : [\![\mathbf{u}]\!] + \sum_{e \in \mathcal{E}_h(\mathcal{T}_h)} \frac{\nu a_0}{h_e} \int_e [\![\mathbf{u}]\!] : [\![\mathbf{v}]\!]. \end{aligned} \tag{3.4}$$

Here, $a_0 > 0$ is the interior penalty parameter, and we denote by ∇_h the broken gradient operator. As discussed in [11], other choices for A_{S}^h are equally feasible (such as LDG or BR methods), provided that the stability properties in Section 3.1.1 below hold. For the convection term, we take the standard upwind form [22] defined by

$$O_{\mathrm{S}}^h(\mathbf{w};\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla_h)\mathbf{u}\cdot\mathbf{v} + \sum_{K\in\mathcal{T}_h}\int_{\partial K\setminus\Gamma}\frac{1}{2}(\mathbf{w}\cdot\mathbf{n}_K - |\mathbf{w}\cdot\mathbf{n}_K|)(\mathbf{u}^e-\mathbf{u})\cdot\mathbf{v}, \tag{3.5}$$

where $\mathbf{u}^e$ is the trace of $\mathbf{u}$ taken from within the exterior of K. We note that convective forms with no upwinding can also be chosen in our setting, such as the trilinear form in [17, Section 6].

Having introduced the additional notations described above, we now introduce our discrete problem: Find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$ such that

$$\begin{aligned} A_{\mathrm{S}}^h(\mathbf{u}_h,\mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) - B_{\mathrm{S}}(\mathbf{v},p_h) - D(\theta_h,\mathbf{v}) &= 0,\\ B_{\mathrm{S}}(\mathbf{u}_h,q) &= 0,\\ A_{\mathrm{T}}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau},\theta_h) &= G(\boldsymbol{\tau}),\\ B_{\mathrm{T}}(\boldsymbol{\sigma}_h,\psi) + O_{\mathrm{T}}(\mathbf{u}_h;\boldsymbol{\sigma}_h,\psi) &= 0, \end{aligned} \tag{3.6}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}, \psi) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$, where B_{S}, D, A_{T}, B_{T}, O_{T} and G are the forms and functional defined in Section 2.1.

3.1.1 Discrete stability properties

Here we discuss the stability properties of the forms involved restricted to the corresponding discrete spaces. We begin by observing that the form A_{T} and B_{T} as well as the functional G_{T}, are continuous with the exact same constants described in Section 2.1.1 (see (2.15)–(2.22)). To establishing the continuity of the remaining forms we need to introduce first the broken norms

$$\|\mathbf{v}\|_{1,\mathcal{T}_h}^2 = \sum_{K\in\mathcal{T}_h}\|\nabla_h\mathbf{v}\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h} a_0\, h_e^{-1}\,\|[\![\mathbf{v}]\!]\|_{0,e}^2 \qquad \mathbf{v}\in\mathbf{H}^1(\mathcal{T}_h), \tag{3.7}$$

and

$$\|\mathbf{v}\|_{2,\mathcal{T}_h}^2 = \|\mathbf{v}\|_{1,\mathcal{T}_h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\,|\mathbf{v}|_{2,K}^2 \qquad \mathbf{v}\in\mathbf{H}^2(\mathcal{T}_h), \tag{3.8}$$

where

$$\mathbf{H}^l(\mathcal{T}_h) := \left\{\mathbf{v}\in\mathbf{L}^2(\Omega) : \mathbf{v}|_K\in\mathbf{H}^r(K) \quad \forall\, K\in\mathcal{T}_h\right\}, \quad l\geq 1.$$

By using the inverse estimate $|p|_{2,K} \leq C h_K^{-1}\,|p|_{1,K}$, for all $K\in\mathcal{T}_h$, $p\in P_k(K)$, it can be readily seen that

$$\|\mathbf{v}\|_{2,\mathcal{T}_h} \leq C\,\|\mathbf{v}\|_{1,\mathcal{T}_h} \qquad \mathbf{v}\in\mathbf{V}_h^{\mathbf{u}}. \tag{3.9}$$

In turn, the following Sobolev inequality holds: for each $t\in I(n)\subset\mathbb{R}$, with $I(2)=[1,\infty)$ and $I(3)=[1,6]$, there exists a constant $C_{emb}(t)>0$, such that

$$\|\mathbf{v}\|_{\mathbf{L}^t(\Omega)} \leq C_{emb}(t)\|\mathbf{v}\|_{1,\mathcal{T}_h} \qquad \mathbf{v}\in\mathbf{H}^1(\mathcal{T}_h). \tag{3.10}$$

For $n=2$, this has been proved in [28, Lemma 6.2]. In the case $n=3$, the proof follows along the lines of [32, Lemma 5.15, Theorem 5.16].

To proving the continuity of the form A_{S}^h we can proceed analogously to [5] and utilize the trace inequalities

$$\|v\|_{0,\partial K} \leq C\,(h_K^{-1/2}\,\|v\|_{0,K} + h_K^{1/2}\,|v|_{1,K}), \qquad v\in\mathrm{H}^1(K) \tag{3.11}$$

and

$$\|p\|_{0,\partial K} \leq C\, h_K^{-1/2}\, \|p\|_{0,K}, \qquad p \in P_k(K), \tag{3.12}$$

to obtain

$$|A_{\mathrm{S}}^h(\mathbf{u},\mathbf{v})| \leq C_A^{\mathrm{S}}\, \|\mathbf{u}\|_{1,\mathcal{T}_h}\, \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{u},\mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}, \tag{3.13}$$

and

$$|A_{\mathrm{S}}^h(\mathbf{u},\mathbf{v})| \leq \tilde{C}_A^{\mathrm{S}}\, \|\mathbf{u}\|_{2,\mathcal{T}_h}\, \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{u} \in \mathbf{H}^2(\mathcal{T}_h),\, \mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}. \tag{3.14}$$

Now, owing to the Sobolev embedding (3.10) with $t = 4$ and the trace inequalities (3.11) and (3.12) (see for instance [24, Lemma 3.4]), we have

$$|O_{\mathrm{S}}^h(\mathbf{w}_1 - \mathbf{w}_2;\mathbf{u},\mathbf{v})| \;\leq\; \widehat{C}_O^{\mathrm{S}}\, \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h}\, \|\mathbf{u}\|_{1,\mathcal{T}_h}\, \|\mathbf{v}\|_{1,\mathcal{T}_h}, \tag{3.15}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2, \mathbf{u} \in \mathbf{H}^2(\mathcal{T}_h)$ and for all $\mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}$, with $\widehat{C}_O^{\mathrm{S}} > 0$ independent of h. Similarly, for O_{T} (cf. (2.14)), we have that

$$|O_{\mathrm{T}}(\mathbf{w}_1 - \mathbf{w}_2;\boldsymbol{\tau},\psi)| \;\leq\; \widehat{C}_O^{\mathrm{T}}\, \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h}\, \|\boldsymbol{\tau}\|_{H(\mathrm{div}_r;\Omega)}\, \|\psi\|_{\mathrm{L}^s(\Omega)}, \tag{3.16}$$

with $\widehat{C}_O^{\mathrm{T}} := \kappa^{-1} C_{emb}(t)$, and for D we easily get that for $t > 2$ if $n = 2$ and $t \in (2,6]$ if $n = 3$, and for $s = \frac{2t}{t-2}$, there holds

$$|D(\psi,\mathbf{v})| \;\leq\; \widehat{C}_D\, \|\mathbf{g}\|_{0,\Omega}\, \|\psi\|_{\mathrm{L}^s(\Omega)}\, \|\mathbf{v}\|_{1,\mathcal{T}_h}, \qquad \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h),\, \psi \in \Psi_h^{\theta}, \tag{3.17}$$

with $\widehat{C}_D := C_{emb}(t)$. In addition, by using the Hölder's inequality (1.1) we readily obtain

$$|B_{\mathrm{S}}(\mathbf{v},q)| \;\leq\; \|\mathbf{v}\|_{1,\mathcal{T}_h}\, \|q\|_{0,\Omega}, \qquad \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h),\, q \in Q_h^p. \tag{3.18}$$

Next, we establish the ellipticity of the forms A_{S}^h and A_{T}. First, for A_{S}^h we recall that, provided that $a_0 > 0$ is a sufficiently large constant (see [17, Lemma 4.12]), there holds

$$A_{\mathrm{S}}^h(\mathbf{v},\mathbf{v}) \geq \widehat{\alpha}_{\mathrm{S}}\, \|\mathbf{v}\|_{1,\mathcal{T}_h}^2, \qquad \forall\, \mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}. \tag{3.19}$$

Let us now define the sets

$$\mathbf{K}_{\mathrm{S},h} := \left\{\mathbf{v} \in \mathbf{V}_h^{\mathbf{u}} : B_{\mathrm{S}}(\mathbf{v},q) = 0,\, \forall\, q \in Q_h^p\right\} \quad \text{and} \quad \mathbf{K}_{\mathrm{T},h} := \left\{\boldsymbol{\tau} \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} : B_{\mathrm{T}}(\boldsymbol{\tau},\psi) = 0, \forall\, \psi \in \Psi_h^{\theta}\right\}.$$

Since the pairs $(\mathbf{V}_h^{\mathbf{u}}, Q_h^p)$ and $(\boldsymbol{\Xi}_h^{\boldsymbol{\sigma}}, \Psi_h^{\theta})$ satisfy

$$\mathrm{div}\, \mathbf{V}_h^{\mathbf{u}} \subseteq Q_h^p \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \subseteq \Psi_h^{\theta},$$

it readily follows that

$$\mathbf{K}_{\mathrm{S},h} := \{\mathbf{v} \in \mathbf{V}_h^{\mathbf{u}} : \mathrm{div}\, \mathbf{v} = 0 \quad \text{in} \quad \Omega\} \tag{3.20}$$

and

$$\mathbf{K}_{\mathrm{T},h} = \{\boldsymbol{\tau} \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} : \mathrm{div}\, \boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega\}\,. \tag{3.21}$$

In particular, on $\mathbf{K}_{\mathrm{T},h}$ we have that A_{T} is elliptic:

$$A_{\mathrm{T}}(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \kappa^{-1}\, \|\boldsymbol{\tau}\|_{H(\mathrm{div}_r;\Omega)}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbf{K}_{\mathrm{T},h}. \tag{3.22}$$

Moreover, on $\mathbf{K}_{\mathrm{S},h}$ it is well-known (see, e.g., [22, 10]) that O_{S}^h satisfies

$$O_{\mathrm{S}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2} \sum_{e\in\mathcal{E}_I(\mathcal{T}_h)} \int_e |\mathbf{w}\cdot\mathbf{n}_e| |[\![\mathbf{v}]\!]|^2\, ds \geq 0 \qquad \mathbf{w} \in \mathbf{K}_{\mathrm{S},h},\;\; \mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}, \tag{3.23}$$

which together to (3.19) implies that for each $\mathbf{w} \in \mathbf{K}_{\mathrm{S},h}$ the bilinear form $A_{\mathrm{S}}^h(\cdot,\cdot)+O_{\mathrm{S}}^h(\mathbf{w};\cdot,\cdot)$ is elliptic. In (3.23), in the integrals over edges (faces) e, the vector $\mathbf{n}_e$ denotes any unit vector normal to e.

Now, let us recall that the bilinear forms B_{S} and B_{T} satisfy the following inf-sup conditions:

$$\sup_{\mathbf{V}_h^{\mathbf{u}} \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} \geq \widehat{\beta}_{\mathrm{S}} \|q_h\|_{0,\Omega}, \qquad \forall\, q_h \in Q_h^p, \tag{3.24}$$

and

$$\sup_{\boldsymbol{\tau}_h \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{T}}(\boldsymbol{\tau}_h, \psi_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div}_r,\Omega)}} \geq \widehat{\beta}_{\mathrm{T}} \|\psi_h\|_{L^s(\Omega)}, \qquad \forall\, \psi_h \in \Psi_h^{\theta}, \tag{3.25}$$

where $\widehat{\beta}_{\mathrm{S}}$ and $\widehat{\beta}_{\mathrm{T}}$ are positive constants independent of the mesh size. The proof of (3.24) follows along the lines of [29] from the surjectivity of div $: \mathbf{H}_0^1(\Omega) \to L_0^2(\Omega)$ and the properties of the Raviart–Thomas interpolator, whereas the proof of (3.25) can be found in [9, Lemma 3.3].

We end this section by observing that, owing to the properties (3.22) and (3.25) and applying [20, Proposition 2.36], the bilinear form $\mathbf{A}$ defined in (2.30) satisfies the following discrete inf-sup condition:

$$\sup_{(\boldsymbol{\tau}_h,\psi_h) \in [\boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}] \setminus \{\mathbf{0}\}} \frac{\mathbf{A}((\boldsymbol{\zeta}_h, \phi_h), (\boldsymbol{\tau}_h, \psi_h))}{\|(\boldsymbol{\tau}_h, \psi_h)\|} \geq \widehat{\gamma} \|(\boldsymbol{\zeta}_h, \phi_h)\| \qquad \forall\, (\boldsymbol{\zeta}_h, \phi_h) \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}, \tag{3.26}$$

where $\widehat{\gamma} > 0$ is the constant defined by

$$\widehat{\gamma} = \frac{\kappa\, \widehat{\beta}_{\mathrm{T}}^2}{\kappa^2\, \widehat{\beta}_{\mathrm{T}}^2 + 4\,\kappa\, \widehat{\beta}_{\mathrm{T}} + 2}. \tag{3.27}$$

3.2 Existence and uniqueness of solution of the discrete problem

In this section we address the well-posedness of problem (3.6). Here we proceed analogously to the continuous case and prove the existence and uniqueness of solution of (3.6) by means of a fixed–point strategy. For the uniqueness of solution we use again the Banach's fixed–point theorem, whereas for the existence result we employ the well–known Brower's fixed–point theorem in the following form:

Let W be a compact and convex subset of a finite dimensional Banach space X and let $T : W \to W$ be a continuous mapping. Then T has at least one fixed point.

Similarly to the analysis of the continuous problem, we begin by introducing the fixed–point operator.

3.2.1 The discrete fixed–point operator

Analogously to the continuous case we start by defining the following reduced version of problem (3.6): find $(\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{K}_{\mathrm{S},h} \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$ such that

$$\begin{aligned} A_{\mathrm{S}}^h(\mathbf{u}_h, \mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - D(\theta_h, \mathbf{v}) &= 0, \\ A_{\mathrm{T}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + B_{\mathrm{T}}(\boldsymbol{\tau}, \theta_h) &= G(\boldsymbol{\tau}), \\ B_{\mathrm{T}}(\boldsymbol{\sigma}_h, \psi) + O_{\mathrm{T}}(\mathbf{u}_h; \boldsymbol{\sigma}_h, \psi) &= 0, \end{aligned} \tag{3.28}$$

for all $(\mathbf{v}, \boldsymbol{\tau}, \psi) \in \mathbf{K}_{\mathrm{S},h} \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$, with $\mathbf{K}_{\mathrm{S},h}$ being the set defined in (3.20). Owing to the inf-sup condition (3.24), here we also obtain that both (3.6) and (3.28) are equivalent.

Lemma 3.1 *If* $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$ *is a solution of* (3.6)*, then* $\mathbf{u}_h \in \mathbf{K}_{\mathrm{S},h}$ *and* $(\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h)$ *is a solution of* (3.28)*. Conversely, if* $(\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h)$ *is a solution of* (3.28)*, then there exists* $p_h \in Q_h^p$*, such that* $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h)$ *is a solution of* (3.6)*.*

We continue by defining the bounded and convex set

$$\mathbf{X}_{1,h} := \left\{ \mathbf{w}_h \in \mathbf{K}_{\mathrm{S},h} : \|\mathbf{w}_h\|_{1,\mathcal{T}_h} \le \frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}} \widehat{\gamma}} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \right\} \tag{3.29}$$

and the discrete version of the operator $\mathbf{T}$ (cf. (2.35)):

$$\begin{aligned} \mathbf{T}_h : \quad \mathbf{X}_{1,h} \quad &\to \quad \mathbf{X}_{1,h} \\ \mathbf{w}_h \quad &\to \quad \mathbf{T}_h(\mathbf{w}_h) = \mathbf{u}_h, \end{aligned} \tag{3.30}$$

with $\mathbf{u}_h$ being the first component of the solution of the linearized version of problem (3.28): Find $(\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{K}_{\mathrm{S},h} \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$ such that

$$\begin{aligned} A_{\mathrm{S}}^h(\mathbf{u}_h, \mathbf{v}_h) + O_{\mathrm{S}}^h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) &= D(\theta_h, \mathbf{v}_h), \\ A_{\mathrm{T}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + B_{\mathrm{T}}(\boldsymbol{\tau}_h, \theta_h) &= G(\boldsymbol{\tau}_h), \\ B_{\mathrm{T}}(\boldsymbol{\sigma}_h, \psi_h) + O_{\mathrm{T}}(\mathbf{w}_h; \boldsymbol{\sigma}_h, \psi_h) &= 0, \end{aligned} \tag{3.31}$$

for all $(\mathbf{v}_h, \boldsymbol{\tau}_h, \psi_h) \in \mathbf{K}_{\mathrm{S},h} \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$.

As for the continuous case, we have the following equivalences

$$\begin{aligned} \mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h \quad &\Leftrightarrow \quad (\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{K}_{\mathrm{S},h} \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta \quad \text{satisfies (3.28)} \\ &\Leftrightarrow \quad (\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta \quad \text{satisfies (3.6)}. \end{aligned} \tag{3.32}$$

In consequence, to proving that problem (3.6) is well posed, in what follows we prove equivalently that $\mathbf{T}_h$ possesses a unique fixed–point. Before doing that, we first verify that our discrete fixed–point operator is well–defined. This is addressed in the following section.

3.2.2 Well-definiteness of the discrete fixed-point operator

Let us first define the sets

$$\mathbf{X}_h = \left\{ (\mathbf{w}_h, \phi_h) \in \mathbf{K}_{S,h} \times \Psi_h^\theta : \|\mathbf{w}_h\|_{1,\mathcal{T}_h} \le \frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}} \widehat{\gamma}} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma}, \quad \|\phi_h\|_{\mathrm{L}^s(\Omega)} \le \frac{C_G}{\widehat{\gamma}} \|\theta_D\|_{1/2,\Gamma} \right\}, \tag{3.33}$$

and

$$\mathbf{X}_{2,h} := \left\{ \phi_h \in \Psi_h^\theta : \|\phi_h\|_{\mathrm{L}^s(\Omega)} \le \frac{C_G}{\widehat{\gamma}} \|\theta_D\|_{1/2,\Gamma} \right\}, \tag{3.34}$$

and the discrete operators

$$\begin{aligned} \mathbf{R}_h : \quad \mathbf{X}_h \quad &\to \quad \mathbf{X}_{1,h} \\ (\mathbf{w}_h, \phi_h) \quad &\to \quad \mathbf{R}(\mathbf{w}_h, \phi_h) = \mathbf{u}_h, \end{aligned} \tag{3.35}$$

with $\mathbf{u}_h$ being the unique solution of problem: Find $\mathbf{u}_h \in \mathbf{K}_{\mathrm{S},h}$, such that

$$A_{\mathrm{S}}^h(\mathbf{u}_h, \mathbf{v}_h) + O_{\mathrm{S}}^h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) = D(\phi_h, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{K}_{\mathrm{S},h}, \tag{3.36}$$

and

$$\begin{aligned} \mathbf{S}_h : \quad \mathbf{X}_{1,h} &\rightarrow \quad \Xi_h^{\boldsymbol{\sigma}} \times \mathbf{X}_{2,h} \\ \mathbf{w}_h \quad &\rightarrow \quad \mathbf{S}_h(\mathbf{w}_h) = (\mathbf{S}_{1,h}(\mathbf{w}_h), \mathbf{S}_{2,h}(\mathbf{w}_h)) = (\boldsymbol{\sigma}_h, \theta_h), \end{aligned} \tag{3.37}$$

where $(\boldsymbol{\sigma}_h, \theta_h) \in \Xi_h^{\boldsymbol{\sigma}} \times \mathbf{X}_{2,h}$ is the unique solution of problem: find $(\boldsymbol{\sigma}_h, \theta_h) \in \Xi_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$, such that

$$\begin{aligned} A_{\mathrm{T}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \, + B_{\mathrm{T}}(\boldsymbol{\tau}_h, \theta_h) \quad &= \; G(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \Xi_h^{\boldsymbol{\sigma}}, \\ B_{\mathrm{T}}(\boldsymbol{\sigma}_h, \psi_h) \, + O_{\mathrm{T}}(\mathbf{w}_h; \boldsymbol{\sigma}_h, \psi_h) \quad &= 0 \qquad\quad \forall\, \psi_h \in \Psi_h^{\theta}. \end{aligned} \tag{3.38}$$

Analogously to the continuous case, from the definition of operators $\mathbf{T}_h$, $\mathbf{R}_h$ and $\mathbf{S}_h$ it can be readily seen that the following discrete version of (2.44) holds:

$$\mathbf{T}_h(\mathbf{w}_h) = \mathbf{R}_h(\mathbf{w}_h, \mathbf{S}_{2,h}(\mathbf{w}_h)) \qquad \forall\, \mathbf{w}_h \in \mathbf{X}_{1,h}. \tag{3.39}$$

In addition, provided a fixed–point $\mathbf{u}_h = \mathbf{T}_h(\mathbf{u}_h)$, we have that the pair $(\boldsymbol{\sigma}_h, \theta_h) \in \Xi_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$, such that $(\mathbf{u}_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{K}_{\mathrm{S},h} \times \Xi_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$ is a solution to (3.28), satisfies the identity

$$(\boldsymbol{\sigma}_h, \theta_h) = \mathbf{S}_h(\mathbf{u}_h). \tag{3.40}$$

The following lemma establishes that operator $\mathbf{R}_h$ is well-defined.

Lemma 3.2 *For each $(\mathbf{w}_h, \phi_h) \in \mathbf{X}_h$, there exists a unique $\mathbf{u}_h \in \mathbf{X}_{1,h}$, such that $\mathbf{R}_h(\mathbf{w}_h, \phi_h) = \mathbf{u}_h$.*

Proof. Since for any $\mathbf{w}_h \in \mathbf{K}_{\mathrm{S},h}$ the bilinear form $A_{\mathrm{S}}^h(\cdot,\cdot) + O_{\mathrm{S}}^h(\mathbf{w}_h; \cdot, \cdot)$ is $\mathbf{V}_h^{\mathbf{u}}$–elliptic, the proof follows analogously to the proof of Lemma 2.2 by means of the Lax-Milgram lemma. We omit further details. □

Next, we prove that operator $\mathbf{S}_h$ is well-defined. To this end we first establish the discrete version of Lemma 2.3.

Lemma 3.3 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that*

$$\frac{\widehat{C}_O^{\mathrm{T}} \widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}} \widehat{\gamma}^2} \|\theta_D\|_{1/2,\Gamma} \|\mathbf{g}\|_{0,\Omega} \leq \frac{1}{2}. \tag{3.41}$$

Then, for each $\mathbf{w}_h \in \mathbf{X}_{1,h}$, the bilinear form $\mathcal{A}_{\mathbf{w}_h}$ satisfies the following estimate

$$\sup_{(\boldsymbol{\tau}_h, \psi_h) \in [\Xi_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}] \setminus \{\mathbf{0}\}} \frac{\mathcal{A}_{\mathbf{w}_h}((\boldsymbol{\zeta}_h, \phi_h), (\boldsymbol{\tau}_h, \psi_h))}{\|(\boldsymbol{\tau}_h, \psi_h)\|} \geq \frac{\widehat{\gamma}}{2} \|\boldsymbol{\zeta}_h\|_{H(\mathrm{div}_r;\Omega)} \, + \, \widehat{\gamma} \|\phi_h\|_{\mathrm{L}^s(\Omega)}, \tag{3.42}$$

$\forall\, (\boldsymbol{\zeta}_h, \phi_h) \in \Xi_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$, where $\widehat{\gamma} > 0$ is the constant defined in (3.27).

Proof. By applying the estimates (3.16) and (3.26), this result follows analogously to the proof of estimate (2.49). We omit further details. □

Employing this lemma we can easily obtain that operator $\mathbf{S}_h$ is well-defined. This result is established next.

Lemma 3.4 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that* (3.41) *holds. Then, for each $\mathbf{w}_h \in \mathbf{X}_{1,h}$, there exists a unique $(\boldsymbol{\sigma}_h, \theta_h) \in \Xi_h^{\boldsymbol{\sigma}} \times \mathbf{X}_{2,h}$, such that $\mathbf{S}_h(\mathbf{w}_h) = (\boldsymbol{\sigma}_h, \theta_h)$.*

Proof. Similarly to the proof of Lemma 2.4, by applying Lemma 3.3 and the discrete version of the Banach-Nečas-Babuška theorem (cf. [20, Theorem 2.6]) it can be readily seen that problem (3.38) admits a unique solution $(\boldsymbol{\sigma}_h, \theta_h) \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta}$. In addition, using (3.42) we can easily obtain that $\theta_h \in \mathbf{X}_{2,h}$. We emphasize here that differently from the continuous case, the discrete version of (2.50) is not needed since for finite dimensional linear problems, injectivity is equivalent to surjectivity. We omit further details. □

Remark 3.1 *Similarly to the continuous case, employing* (3.26) *one can actually deduce that* $\mathbf{S}_h(\mathbf{w}_h) = (\boldsymbol{\sigma}_h, \theta_h) \in \mathbf{Z}_h$*, for all* $\mathbf{w}_h \in \mathbf{X}_{1,h}$*, where*

$$\mathbf{Z}_h := \left\{ (\boldsymbol{\tau}_h, \psi_h) \in \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^{\theta} : \frac{\widehat{\gamma}}{2} \|\boldsymbol{\tau}_h\|_{H(\mathrm{div}_r;\Omega)} + \widehat{\gamma} \|\psi_h\|_{L^s(\Omega)} \leq C_G \|\theta_D\|_{1/2,\Gamma} \right\}. \tag{3.43}$$

Now we are in position of establishing the well-definiteness of operator $\mathbf{T}_h$.

Theorem 3.5 *Assume that* (3.41) *holds. Then, for each* $\mathbf{w}_h \in \mathbf{X}_{1,h}$*, there exists a unique* $\mathbf{u}_h \in \mathbf{X}_{1,h}$ *such that* $\mathbf{u}_h = \mathbf{T}_h(\mathbf{w}_h)$*.*

Proof. This result is a direct consequence of the identity (3.39) and Lemmas 3.2 and 3.4. □

3.2.3 Main result

In this section we provide the well–posedness of problem (3.6). As already announced first we employ the Brower's fixed point theorem to prove existence of solution whereas for the uniqueness result we make use of the Banach's fixed point theorem. We begin by establishing the discrete version of Lemmas 2.6 and 2.7.

Lemma 3.6 *Given* $t > 2$ *if* $n = 2$ *and* $t \in (2, 6]$ *if* $n = 3$*, let* $s = \frac{2t}{t-2}$*. Then there exist positive constants* $\widehat{C}_{\mathbf{R}_1}$ *and* $\widehat{C}_{\mathbf{R}_2}$ *such that*

$$\|\mathbf{R}_h(\mathbf{w}_1, \phi_1) - \mathbf{R}_h(\mathbf{w}_2, \phi_2)\|_{1,\mathcal{T}_h} \leq \widehat{C}_{\mathbf{R}_1} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h} + \widehat{C}_{\mathbf{R}_2} \|\phi_1 - \phi_2\|_{\mathrm{L}^s(\Omega)}, \tag{3.44}$$

for all $(\mathbf{w}_1, \phi_1)$, $(\mathbf{w}_2, \phi_2) \in \mathbf{X}_h$*.*

Proof. Using the estimates (3.19), (3.15) and (3.17), the result can be obtained analogously to the proof of Lemma 2.6 with constants $C_{\mathbf{R}_1}$ and $C_{\mathbf{R}_2}$ given by

$$\widehat{C}_{\mathbf{R}_1} = \frac{\widehat{C}_O^{\mathrm{S}} \widehat{C}_D C_G}{\widehat{\gamma} \widehat{\alpha}_{\mathrm{S}}^2} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma} \quad \text{and} \quad \widehat{C}_{\mathbf{R}_2} = \frac{\widehat{C}_D}{\widehat{\alpha}_{\mathrm{S}}} \|\mathbf{g}\|_{0,\Omega}. \tag{3.45}$$

We omit further details. □

Lemma 3.7 *Given* $t > 2$ *if* $n = 2$ *and* $t \in (2, 6]$ *if* $n = 3$*, let* $r = \frac{2t}{t+2}$ *and* $s = \frac{2t}{t-2}$*. Assume that hypothesis* (3.41) *holds. Then, there exists a positive constant* $\widehat{C}_{\mathbf{S}}$*, such that*

$$\frac{1}{2} \|\mathbf{S}_1(\mathbf{w}_1) - \mathbf{S}_1(\mathbf{w}_2)\|_{H(\mathrm{div}_r;\Omega)} + \|\mathbf{S}_2(\mathbf{w}_1) - \mathbf{S}_2(\mathbf{w}_2)\|_{L^s(\Omega)} \leq \widehat{C}_{\mathbf{S}} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h} \tag{3.46}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}_{1,h}$*.*

Proof. Utilizing exactly the same arguments employed in the proof of Lemma 2.7 we can obtain (3.46) with

$$\widehat{C}_{\mathbf{S}} = \frac{2\widehat{C}_O^{\mathrm{T}} C_G}{\widehat{\gamma}^2} \|\theta_D\|_{1/2,\Gamma}. \tag{3.47}$$

We omit further details. □

In the following result we establish that problem (3.6) admits a solution.

Theorem 3.8 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that hypothesis (3.41) holds. Then there exists $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$ solution to (3.6). Moreover, there exists $\widehat{C} > 0$, independent of the solution, such that*

$$\|\mathbf{u}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_h\|_{H(\mathrm{div}_r,\Omega)} + \|\theta_h\|_{\mathrm{L}^s(\Omega)} \le \widehat{C}\|\theta_D\|_{1/2,\Gamma}. \tag{3.48}$$

Proof. We start by noticing that from (3.39), (3.44) and (3.46), analogously to the proof of Theorem 2.8, we obtain

$$\begin{aligned} \|\mathbf{T}_h(\mathbf{w}_1) - \mathbf{T}_h(\mathbf{w}_2)\|_{1,\mathcal{T}_h} \le &\ \widehat{C}_{\mathbf{R}_1}\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h} + \widehat{C}_{\mathbf{R}_2}\|\mathbf{S}_{2,h}(\mathbf{w}_1) - \mathbf{S}_{2,h}(\mathbf{w}_2)\|_{\mathrm{L}^s(\Omega)} \\ \le &\ (\widehat{C}_{\mathbf{R}_1} + \widehat{C}_{\mathbf{R}_2}\widehat{C}_{\mathbf{S}})\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h}, \end{aligned} \tag{3.49}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in \mathbf{X}_{1,h}$, which implies that $\mathbf{T}_h$ is continuous on $\mathbf{X}_{1,h}$. Then, as previously announced, by utilizing the classical Brower's fixed–point theorem, we conclude that $\mathbf{T}_h$ admits a fixed–point $\mathbf{u}_h \in \mathbf{X}_{1,h}$, or equivalently, owing to (3.32), that problem (3.6) admits at least one solution $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$. Moreover, since $\mathbf{u}_h \in \mathbf{X}_{1,h}$ and $(\boldsymbol{\sigma}_h, \theta_h) = \mathbf{S}_h(\mathbf{u}_h) \in \mathbf{Z}_h$ (see (3.40) and Remark 3.1), there holds

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \le \frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}}\widehat{\gamma}}\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} \quad \text{and} \quad \frac{\widehat{\gamma}}{2}\|\boldsymbol{\sigma}_h\|_{H(\mathrm{div}_r;\Omega)} + \widehat{\gamma}\|\theta_h\|_{\mathrm{L}^s(\Omega)} \le C_G\|\theta_D\|_{1/2,\Gamma} \tag{3.50}$$

In turn, from the inf–sup condition (3.24), and the continuity of A_h^{S} O_{S}^h and D (see (3.13), (3.15) and (3.17), respectively), analogously to (2.68) we can deduce that

$$\widehat{\beta}_{\mathrm{S}}\,\|p_h\|_{0,\Omega} \le \left(C_A^{\mathrm{S}}\,\|\mathbf{u}_h\|_{1,\mathcal{T}_h} + \widehat{C}_O^{\mathrm{S}}\,\|\mathbf{u}_h\|_{1,\mathcal{T}_h}^2 + \widehat{C}_D\,\|\theta_h\|_{\mathrm{L}^s(\Omega)}\,\|\mathbf{g}\|_{0,\Omega}\right). \tag{3.51}$$

Then, from (3.50) and (3.51) we readily obtain (3.48), which concludes the proof. □

We conclude this section by stating the uniqueness of solution of problem (3.6)

Theorem 3.9 *Given $t > 2$ if $n = 2$ and $t \in (2, 6]$ if $n = 3$, let $r = \frac{2t}{t+2}$ and $s = \frac{2t}{t-2}$. Assume that*

$$\frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}}\widehat{\gamma}}\left(\frac{\widehat{C}_O^{\mathrm{S}}}{\alpha_{\mathrm{S}}} + \frac{2\widehat{C}_O^{\mathrm{T}}}{\widehat{\gamma}}\right)\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} < 1. \tag{3.52}$$

Then, the solution of problem (3.6) is unique.

Proof. From (3.45) and (3.47) it can be readily seen that

$$(\widehat{C}_{\mathbf{R}_1} + \widehat{C}_{\mathbf{R}_2}\widehat{C}_{\mathbf{S}}) = \frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}}\widehat{\gamma}}\left(\frac{\widehat{C}_O^{\mathrm{S}}}{\alpha_{\mathrm{S}}} + \frac{2\widehat{C}_O^{\mathrm{T}}}{\widehat{\gamma}}\right)\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma},$$

which together to (3.49) and (3.52) implies that $\mathbf{T}_h$ is a contraction mapping. Therefore, the uniqueness of solution of problem (3.6) is a direct consequence of (3.32) and the Banach's fixed–point theorem.
□

4 Error analysis

In what follows we carry out the error analysis of the finite element approximation presented in Section 3 under an extra regularity assumption on the solution and, similarly to the above, by assuming that the data θ_D is sufficiently small in the $\mathrm{H}^{1/2}$–norm. We start by establishing some previous results and definitions.

In the sequel we make use of the Raviart-Thomas operator $\mathbf{\Pi}_h^l : \mathbf{H}^1(\Omega) \to \mathbf{H}_h^l$ (cf. (3.1)) which, given $l \geq 0$, is characterized by the following identities:

$$\int_e \mathbf{\Pi}_h^l(\mathbf{z}) \cdot \mathbf{n}\, \xi \;=\; \int_e \mathbf{z} \cdot \mathbf{n}\, \xi \qquad \forall\, \text{edge/face } e \in \mathcal{T}_h, \quad \forall\, \xi \in P_l(e) \qquad \text{when} \quad l \geq 0\,, \tag{4.1}$$

and

$$\int_K \mathbf{\Pi}_h^l(\mathbf{z}) \cdot \mathbf{q} \;=\; \int_K \mathbf{z} \cdot \mathbf{q} \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \mathbf{q} \in \mathbf{P}_{l-1}(K) \qquad \text{when} \quad l \geq 1\,. \tag{4.2}$$

We shall also employ the L^2-projection $\mathcal{P}_h^l : \mathrm{L}^2(\Omega) \to Y_h^l$, which is defined through the identity

$$\int_\Omega \lambda(z - \mathcal{P}_h^l(z)) = 0 \quad \forall\, \lambda \in Y_h^l.$$

Next, to simplify the subsequent analysis, we denote the corresponding errors by

$$\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h, \quad e_p = p - p_h, \quad \mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \quad e_\theta = \theta - \theta_h$$

and, given $k \geq 1$, we decompose these errors into

$$\mathbf{e}_{\mathbf{u}} = \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, \quad e_p = \xi_p + \chi_p, \quad e_\theta = \xi_\theta + \chi_\theta, \quad \mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}}. \tag{4.3}$$

with

$$\begin{aligned} \boldsymbol{\xi}_{\mathbf{u}} &= \mathbf{u} - \mathbf{\Pi}_h^k(\mathbf{u}), & \boldsymbol{\chi}_{\mathbf{u}} &= \mathbf{\Pi}_h^k(\mathbf{u}) - \mathbf{u}_h, \\ \xi_p &= p - \mathcal{P}_h^k(p), & \chi_p &= \mathcal{P}_h^k(p) - p_h, \\ \boldsymbol{\xi}_{\boldsymbol{\sigma}} &= \boldsymbol{\sigma} - \mathbf{\Pi}_h^{k-1}(\boldsymbol{\sigma}), & \boldsymbol{\chi}_{\boldsymbol{\sigma}} &= \mathbf{\Pi}_h^{k-1}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h, \\ \xi_\theta &= \theta - \mathcal{P}_h^{k-1}(\theta), & \chi_\theta &= \mathcal{P}_h^{k-1}(\theta) - \theta_h. \end{aligned} \tag{4.4}$$

Owing to the approximation properties of operators $\mathbf{\Pi}_h^l$ and $\mathcal{P}_h^l$ (see for instance [7, 21]), it can be readily seen that

$$\begin{aligned} &\|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h} \leq Ch^k \|\mathbf{u}\|_{k+1,\Omega}, & &\|\xi_p\|_{0,\Omega} \leq Ch^{k+1}\|p\|_{k+1,\Omega} \\ &\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} \leq Ch^k\{\|\boldsymbol{\sigma}\|_{k,\Omega} + \|\mathrm{div}\,\boldsymbol{\sigma}\|_{\mathrm{W}^{k,r}(\Omega)}\}, & &\|\xi_\theta\|_{\mathrm{L}^s(\Omega)} \leq Ch^k\|p\|_{\mathrm{W}^{k,s}(\Omega)}. \end{aligned} \tag{4.5}$$

In particular, for the last two estimates we refer the reader to [20, Section 1.6.3] and [9, Section 3.1].

Having introduced the main tools to be employed in the forthcoming analysis, now we establish the the theoretical rate of convergence of our method.

Theorem 4.1 *Assume that estimates* (2.69) *and* (3.52) *hold and let* $(\mathbf{u}, p, \boldsymbol{\sigma}, \theta) \in \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega) \times H(\mathrm{div}_r, \Omega) \times \mathrm{L}^s(\Omega)$ *and* $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \theta_h) \in \mathbf{V}_h^{\mathbf{u}} \times Q_h^p \times \mathbf{\Xi}_h^{\boldsymbol{\sigma}} \times \Psi_h^\theta$ *be the unique solutions of* (2.13) *and* (3.6)*, respectively. Assume further that*

$$\frac{C_G}{\widehat{\alpha}_{\mathrm{S}}\gamma}\left(\frac{\widehat{C}_O^{\mathrm{S}} C_D}{\alpha_{\mathrm{S}}} + \frac{2\widehat{C}_O^{\mathrm{T}}\widehat{C}_D}{\widehat{\gamma}}\right)\|\mathbf{g}\|_{0,\Omega}\|\theta_D\|_{1/2,\Gamma} \leq \frac{1}{2}, \tag{4.6}$$

and for a given $k \geq 1$, $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $p \in \mathrm{H}^{k+1}(\Omega)$, $\boldsymbol{\sigma} \in \mathbf{H}^{k}(\Omega)$, $\operatorname{div}\boldsymbol{\sigma} \in \mathrm{W}^{k,r}(\Omega)$ *and* $\theta \in \mathrm{W}^{k,s}(\Omega)$. *Then, there exists* $C > 0$, *independent of* h, *such that*

$$\begin{aligned}\|\mathbf{e_u}\|_{1,\mathcal{T}_h} + \|e_p\|_{0,\Omega} + \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{H(\operatorname{div}_r;\Omega)} + \|e_\theta\|_{\mathrm{L}^s(\Omega)} \;\leq\; \\ Ch^k \Big\{ \|\mathbf{u}\|_{k+1,\Omega} + \|p\|_{k+1,\Omega} + \|\boldsymbol{\sigma}\|_{k,\Omega} + \|\operatorname{div}\boldsymbol{\sigma}\|_{\mathrm{W}^{k,r}(\Omega)} + \|\theta\|_{\mathrm{W}^{k,s}(\Omega)} \Big\}.\end{aligned} \tag{4.7}$$

Proof. First, since we are assuming that $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, it can be readily seen by integration by parts that

$$A^h_{\mathrm{S}}(\mathbf{u},\mathbf{v}) \,+\, O^h_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v}) \,-\, B_{\mathrm{S}}(\mathbf{v},p) - D(\theta,\mathbf{v}) \,=\, 0 \quad \forall\, \mathbf{v} \in \mathbf{V}^{\mathbf{u}}_h,$$

from which

$$A^h_{\mathrm{S}}(\mathbf{u},\mathbf{v}) \,+\, O^h_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v}) - D(\theta,\mathbf{v}) \,=\, 0 \quad \forall\, \mathbf{v} \in \mathbf{K}_{\mathrm{S},h}.$$

Thus, combining the latter identity with the first equation of (3.28) we obtain the orthogonality property

$$A^h_{\mathrm{S}}(\mathbf{e_u},\mathbf{v}) \,+\, O^h_{\mathrm{S}}(\mathbf{u};\mathbf{u},\mathbf{v}) - O^h_{\mathrm{S}}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}) - D(e_\theta,\mathbf{v}) \,=\, 0 \quad \forall\, \mathbf{v} \in \mathbf{K}_{\mathrm{S},h}, \tag{4.8}$$

which after simple algebraic manipulations can be rewritten as

$$\begin{aligned} A^h_{\mathrm{S}}(\boldsymbol{\chi}_{\mathbf{u}},\mathbf{v}) \,+\, O^h_{\mathrm{S}}(\mathbf{u}_h;\boldsymbol{\chi}_{\mathbf{u}},\mathbf{v}) = \;& -O^h_{\mathrm{S}}(\boldsymbol{\chi}_{\mathbf{u}};\mathbf{u},\mathbf{v}) + D(\chi_\theta,\mathbf{v}) \\ & -O^h_{\mathrm{S}}(\boldsymbol{\xi}_{\mathbf{u}};\mathbf{u},\mathbf{v}) + D(\xi_\theta,\mathbf{v}) - O^h_{\mathrm{S}}(\mathbf{u}_h;\boldsymbol{\xi}_{\mathbf{u}},\mathbf{v}) - A^h_{\mathrm{S}}(\boldsymbol{\xi}_{\mathbf{u}},\mathbf{v}), \end{aligned}$$

for all $\mathbf{v} \in \mathbf{K}_{\mathrm{S},h}$. In particular, for $\mathbf{v} = \boldsymbol{\chi}_{\mathbf{u}} \in \mathbf{K}_{\mathrm{S},h}$, employing the ellipticity of the bilinear form $A^h_{\mathrm{S}}(\cdot,\cdot) + O^h_{\mathrm{S}}(\mathbf{u}_h;\cdot,\cdot)$, and the continuity of O^h_{S} and D (see (3.15) and (3.17), respectively), we obtain

$$\widehat{\alpha}_{\mathrm{S}}\|\boldsymbol{\chi}_{\mathbf{u}}\|^2_{1,\mathcal{T}_h} \leq \widehat{C}^{\mathrm{S}}_O\|\mathbf{u}\|_{1,\mathcal{T}_h}\|\boldsymbol{\chi}_{\mathbf{u}}\|^2_{1,\mathcal{T}_h} + \widehat{C}_D\|\mathbf{g}\|_{0,\Omega}\|\chi_\theta\|_{\mathrm{L}^s(\Omega)}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + L_1(\boldsymbol{\xi}_{\mathbf{u}},\xi_\theta)\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}, \tag{4.9}$$

with

$$L_1(\boldsymbol{\xi}_{\mathbf{u}},\xi_\theta) := (\widehat{C}^{\mathrm{S}}_O\|\mathbf{u}\|_{1,\mathcal{T}_h} + \widehat{C}^{\mathrm{S}}_O\|\mathbf{u}_h\|_{1,\mathcal{T}_h} + C^{\mathrm{S}}_A)\|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \widehat{C}_D\|\mathbf{g}\|_{0,\Omega}\|\xi_\theta\|_{\mathrm{L}^s(\Omega)}.$$

On the other hand, since $\boldsymbol{\Xi}^{\boldsymbol{\sigma}}_h \subset H(\operatorname{div}_r;\Omega)$ and $\Psi^\theta_h \subseteq \mathrm{L}^s(\Omega)$, from the third and fourth equations of (2.33) and (3.28) it follows that the following orthogonality property holds

$$\begin{aligned} A_{\mathrm{T}}(\mathbf{e}_{\boldsymbol{\sigma}},\boldsymbol{\tau}) \,+ B_{\mathrm{T}}(\boldsymbol{\tau},e_\theta) \,=\;& 0, \\ B_{\mathrm{T}}(\mathbf{e}_{\boldsymbol{\sigma}},\psi) \,+\, [\,O_{\mathrm{T}}(\mathbf{u};\boldsymbol{\sigma},\psi) - O_{\mathrm{T}}(\mathbf{u}_h;\boldsymbol{\sigma}_h,\psi)\,] \,=\;& 0, \end{aligned}$$

for all $(\boldsymbol{\tau},\psi) \in \boldsymbol{\Xi}^{\boldsymbol{\sigma}}_h \times \Psi^\theta_h$. From this property, the definition of the bilinear form $\mathcal{A}_{\mathbf{w}}$ (cf. (2.47)), the decomposition (4.3), and simple computations it can be obtained the identity

$$\mathcal{A}_{\mathbf{u}_h}((\boldsymbol{\chi}_{\boldsymbol{\sigma}},\chi_\theta),(\boldsymbol{\tau},\psi_h)) = -O_T(\boldsymbol{\chi}_{\mathbf{u}};\boldsymbol{\sigma},\psi) - \mathcal{A}_{\mathbf{u}_h}((\boldsymbol{\xi}_{\boldsymbol{\sigma}},\xi_\theta),(\boldsymbol{\tau},\psi)) - O_T(\boldsymbol{\xi}_{\mathbf{u}};\boldsymbol{\sigma},\psi). \tag{4.10}$$

Then, utilizing the inf-sup condition (3.42), and the continuity of A_{T}, B_{T} and O_{T} in (2.15), (2.16) and (3.16), respectively, from (4.10) we obtain

$$\frac{\widehat{\gamma}}{2}\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{H(\operatorname{div}_r;\Omega)} + \widehat{\gamma}\|\chi_\theta\|_{\mathrm{L}^s(\Omega)} \leq \widehat{C}^{\mathrm{T}}_O\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}\|\boldsymbol{\sigma}\|_{H(\operatorname{div}_r;\Omega)} + L_2(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\xi_\theta,\boldsymbol{\xi}_{\mathbf{u}}), \tag{4.11}$$

with

$$L_2(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\xi_\theta,\boldsymbol{\xi}_{\mathbf{u}}) := (1 + \kappa^{-1} + \widehat{C}^{\mathrm{T}}_O\|\mathbf{u}_h\|_{1,\mathcal{T}_h})\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\operatorname{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \widehat{C}^{\mathrm{T}}_O\|\boldsymbol{\sigma}\|_{H(\operatorname{div}_r;\Omega)}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}.$$

In particular, from (4.11) we have that

$$\|\chi_\theta\|_{\mathrm{L}^s(\Omega)} \le \frac{\widehat{C}_O^{\mathrm{T}}}{\widehat{\gamma}} \|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r;\Omega)} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \frac{1}{\widehat{\gamma}} L_2(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \xi_\theta, \boldsymbol{\xi}_{\mathbf{u}}),$$

which combined with (4.9), implies

$$\left(\widehat{\alpha}_{\mathrm{S}} - \widehat{C}_O^{\mathrm{S}} \|\mathbf{u}\|_{1,\mathcal{T}_h} - \frac{\widehat{C}_D \widehat{C}_O^{\mathrm{T}}}{\widehat{\gamma}} \|\mathbf{g}\|_{0,\Omega} \|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r;\Omega)}\right) \|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \le L_1(\boldsymbol{\xi}_{\mathbf{u}}, \xi_\theta) + \frac{\widehat{C}_D}{\widehat{\gamma}} \|\mathbf{g}\|_{0,\Omega} L_2(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \xi_\theta, \boldsymbol{\xi}_{\mathbf{u}}). \tag{4.12}$$

Hence, recalling that (see (2.66), (2.67) and (3.50))

$$\|\boldsymbol{\sigma}\|_{H(\mathrm{div}_r;\Omega)} \le \frac{2C_G}{\gamma} \|\theta_D\|_{1/2,\Gamma}, \qquad \|\mathbf{u}\|_{1,\mathcal{T}_h} \le \|\mathbf{u}\|_{1,\Omega} \le \frac{C_D C_G}{\alpha_{\mathrm{S}} \gamma} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma}$$

and

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h} \le \frac{\widehat{C}_D C_G}{\widehat{\alpha}_{\mathrm{S}} \widehat{\gamma}} \|\mathbf{g}\|_{0,\Omega} \|\theta_D\|_{1/2,\Gamma},$$

we have

$$L_1(\boldsymbol{\xi}_{\mathbf{u}}, \xi_\theta) \le C_1(\|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)}) \quad \text{and} \quad L_2(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \xi_\theta, \boldsymbol{\xi}_{\mathbf{u}}) \le C_2(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}),$$

which together to (4.12), and estimate (4.6), yields

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \le C(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}). \tag{4.13}$$

Using this estimate, from (4.11) we also obtain

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\chi_\theta\|_{\mathrm{L}^s(\Omega)} \le C(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}). \tag{4.14}$$

In this way, from (4.13) and (4.14) and utilizing (4.3) and the triangle inequality, we readily obtain

$$\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|e_\theta\|_{\mathrm{L}^s(\Omega)} \le C(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h}). \tag{4.15}$$

Now, to estimate e_p we first observe that, owing to the discrete inf-sup condition (3.24), there holds

$$\begin{aligned} \widehat{\beta}_{\mathrm{S}} \|\chi_p\|_{0,\Omega} &\le \sup_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{u}} \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v}_h, \chi_p)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} \\ &\le \sup_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{u}} \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v}_h, e_p)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} + \sup_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{u}} \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v}_h, -\xi_p)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} \\ &\le \sup_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{u}} \setminus \{\mathbf{0}\}} \frac{B_{\mathrm{S}}(\mathbf{v}_h, e_p)}{\|\mathbf{v}_h\|_{1,\mathcal{T}_h}} + \|\xi_p\|_{0,\Omega}. \end{aligned} \tag{4.16}$$

In turn, from the first equations of (2.13) and (3.6), and adding and subtracting suitable terms, analogously to (4.8) we can obtain

$$\begin{aligned} B_{\mathrm{S}}(\mathbf{v}, e_p) &= A_{\mathrm{S}}^h(\mathbf{e}_{\mathbf{u}}, \mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{u}; \mathbf{u}, \mathbf{v}) - O_{\mathrm{S}}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - D(e_\theta, \mathbf{v}) \\ &= A_{\mathrm{S}}^h(\mathbf{e}_{\mathbf{u}}, \mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{e}_{\mathbf{u}}; \mathbf{u}, \mathbf{v}) + O_{\mathrm{S}}^h(\mathbf{u}_h; \mathbf{e}_{\mathbf{u}}, \mathbf{v}) - D(e_\theta, \mathbf{v}), \end{aligned} \tag{4.17}$$

for all $\mathbf{v} \in \mathbf{V}_h^{\mathbf{u}}$. Then, utilizing (4.15), (4.17), (2.66) and (3.50), from (4.16) we obtain

$$\|\chi_p\|_{0,\Omega} \le C(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\xi_p\|_{0,\Omega}),$$

which together to the triangle inequality and (4.3), implies

$$\|e_p\|_{0,\Omega} \le C(\|\boldsymbol{\xi_\sigma}\|_{H(\mathrm{div}_r;\Omega)} + \|\xi_\theta\|_{\mathrm{L}^s(\Omega)} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} + \|\xi_p\|_{0,\Omega}).$$

This latter estimate, together to (4.15), (4.5) and the fact that $\|\boldsymbol{\xi}_{\mathbf{u}}\|_{1,\mathcal{T}_h} \le \|\boldsymbol{\xi}_{\mathbf{u}}\|_{2,\mathcal{T}_h}$, readily implies (4.7), which concludes the proof. □

Remark 4.1 *Although the theory above has been developed considering the Raviart-Thomas element for the unknowns* $\mathbf{u}$ *and* $\boldsymbol{\sigma}$*, the analysis can be easily adapted for other choices of finite element spaces, such as the well–known Brezzi–Douglas–Marini element. In fact, given an integer* $k \ge 1$*, one can consider the space*

$$\mathbf{H}_h^k := \{\mathbf{z} \in H(\mathrm{div}\,;\Omega) : \mathbf{z}|_K \in \mathbf{P}_k(K), \quad \forall\, K \in \mathcal{T}_h\},$$

and redefine problem (3.6) *with the spaces*

$$\mathbf{V}_h^{\mathbf{u}} := \mathbf{H}_h^k \cap H_0(\mathrm{div}\,;\Omega), \quad Q_h^p := Y_h^{k-1} \cap \mathrm{L}_0^2(\Omega) \quad \boldsymbol{\Xi}_h^{\boldsymbol{\sigma}} := \mathbf{H}_h^k \quad and \quad \Psi_h^\theta := Y_h^{k-1}, \tag{4.18}$$

where Y_h^k *is the space defined in* (3.1)*. Analogously to the analysis developed above it is not difficult to prove that problem* (3.6) *is well posed, and employing the approximation properties of the corresponding discrete spaces (see [7, Chapter 2, Section 2.5]) one can easily obtain the following theoretical rate of convergence*

$$\begin{aligned}\|\mathbf{e_u}\|_{1,\mathcal{T}_h} + \quad & \|e_p\|_{0,\Omega} + \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{H(\mathrm{div}_r;\Omega)} + \|\theta\|_{\mathrm{L}^s(\Omega)} \le \\ & Ch^k \Big\{ \|\mathbf{u}\|_{k+1,\Omega} + \|p\|_{k,\Omega} + \|\boldsymbol{\sigma}\|_{k,\Omega} + \|\mathrm{div}\,\boldsymbol{\sigma}\|_{\mathrm{W}^{k,r}(\Omega)} + \|\theta\|_{\mathrm{W}^{k,s}(\Omega)} \Big\}.\end{aligned} \tag{4.19}$$

We end this remark by mentioning that clearly both, Raviart-Thomas and BDM elements, can also be combined for discretazing the coupled problem, that is, one can choose $\mathbf{RT}_k - \mathrm{P}_k$ *to approximate the pair* $(\mathbf{u}, p)$ *and* $\mathbf{BDM}_k - \mathrm{P}_{k-1}$ *for* $(\boldsymbol{\sigma}, \theta)$*, or vice versa.*

5 Numerical results

In this section we present some numerical results illustrating the performance of our mixed finite element scheme (3.6) on a set of quasi-uniform triangulations of the corresponding domain and considering the finite element spaces introduces in Section 3 and in Remark 4.1. Our implementation is based on a *FreeFem++* code, in conjunction with the direct linear *UMFPACK*. For all the examples below we define the penalization term as $a_0 = 5$, choose $r = 4/3$, $s = 4$, and utilize a Picard-type algorithm with a fixed tolerance $tol = 1e - 6$. The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \le tol,$$

where $\|\cdot\|_{l^2}$ stands for the usual euclidean norm in $\mathbb{R}^{\mathrm{dof}}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{V}_h^{\mathbf{u}}$, Q_h^p, $\boldsymbol{\Xi}_h^{\boldsymbol{\sigma}}$ and Ψ_h^θ. Now, we introduce some additional notations. The individual errors are denoted by $e(\mathbf{u})$, $e(p)$, $e(\boldsymbol{\sigma})$ and $e(\theta)$. Also, we let $r(\mathbf{u})$, $r(p)$, $r(\boldsymbol{\sigma})$ and $r(\theta)$ be the experimental rates of convergence given by

$$r(\mathbf{u}) := \frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')}, \quad r(p) := \frac{\log(e(p)/e'(p))}{\log(h/h')},$$
$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad r(\theta) := \frac{\log(e(\theta)/e'(\theta))}{\log(h/h')},$$

where h and h' denote two consecutive mesh sizes with their respective errors e and e'.

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega = (-1/2, 3/2) \times (0, 2)$. We consider the thermal conductivity $\kappa = 1$, the external force $\mathbf{g} = (0, -1)^t$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\mathbf{u}(x,y) := \begin{pmatrix} 1 - e^{\lambda x}\cos(2\pi y) \\ \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y) \end{pmatrix}, \qquad p(x,y) := \frac{-1}{2} e^{2\lambda x} + \bar{p}, \qquad \theta(x,y) := x^2 y^2 + 1,$$

where

$$\lambda := \frac{-8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}},$$

with $\nu > 0$ being the viscosity of the fluid and the constant $\bar{p}$ is chosen in such a way that $\int_\Omega p = 0$. We observe that $(\mathbf{u}, p)$ is the well-known analytic solution for the Navier-Stokes problem obtained by Kovasznay in [30].

In Table 1 we summarize the convergence history, on a sequence of quasi-uniform triangulations, of two finite element families corresponding to $\mathbf{RT}_1 - P_1 - \mathbf{RT}_0 - P_0$ and $\mathbf{BDM}_1 - \mathrm{P}_0 - \mathbf{BDM}_1 - \mathrm{P}_0$ and considering the viscosity $\nu = 1$. We observe there that the rate of convergence $O(h)$ predicted by Theorem 4.1 and Remark 4.1 is attained in all the cases. In addition, the last columns of Table 1 illustrates that the velocity is practically divergence free for all refinement steps, and for both $\mathbf{RT}_1 - P_1 - \mathbf{RT}_0 - P_0$ and $\mathbf{BDM}_1 - \mathrm{P}_0 - \mathbf{BDM}_1 - \mathrm{P}_0$. We remark that in both cases the algorithm stopped after around 10 iterations.

ERRORS AND RATES OF CONVERGENCE FOR THE DG-MIXED
$\mathbf{RT}_1 - P_1 - \mathbf{RT}_0 - P_0$ APPROXIMATION.

h	DOF	$\mathbf{e}(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	$\mathbf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\theta)$	$r(\theta)$	$\Vert\mathrm{div}\,\mathbf{u}_h\Vert_{l^\infty}$
0.745	402	69.249	–	67.956	–	3.835	–	1.033	–	1.21e-13
0.380	1665	44.241	0.6655	21.272	1.7252	1.719	1.1917	0.634	0.7241	1.03e-13
0.190	6417	23.284	0.9261	12.898	0.7218	0.861	0.9983	0.309	1.0379	1.42e-14
0.095	25560	11.346	1.0377	6.863	0.9107	0.422	1.0301	0.147	1.0677	1.51e-13
0.053	101688	5.592	1.2100	3.452	1.1751	0.210	1.1912	0.081	1.0315	1.55e-14
0.027	410331	2.700	1.0556	1.698	1.0289	0.104	1.0265	0.039	1.0623	1.58e-13

ERRORS AND RATES OF CONVERGENCE FOR THE DG-MIXED
$\mathbf{BDM}_1 - \mathrm{P}_0 - \mathbf{BDM}_1 - \mathrm{P}_0$ APPROXIMATION.

h	DOF	$\mathbf{e}(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	$\mathbf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\theta)$	$r(\theta)$	$\Vert\mathrm{div}\,\mathbf{u}_h\Vert_{l^\infty}$
0.745	402	69.249	–	68.724	–	2.692	–	1.033	–	1.21e-13
0.380	1665	44.241	0.6655	27.826	1.3430	1.053	1 .3947	0.635	0.7230	1.03e-13
0.190	6417	23.284	0.9261	17.840	0.6413	0.402	1.3880	0.309	1.0393	1.43e-14
0.095	25560	11.346	1.0377	9.733	0.8747	0.178	1.1769	0.147	1.0679	1.51e-13
0.053	101688	5.592	1.2100	4.705	1.2430	0.087	1.2284	0.081	1.0315	1.55e-14
0.027	410331	2.700	1.0556	2.324	1.0226	0.042	1.0501	0.039	1.0623	1.57e-13

Table 1: EXAMPLE 1: Degrees of freedom, meshsizes, errors, rates of convergence and ... for the mixed $\mathbf{RT}_1 - P_1 - \mathbf{RT}_0 - P_0$ and $\mathbf{BDM}_1 - \mathrm{P}_0 - \mathbf{BDM}_1 - \mathrm{P}_0$ approximations of the Boussinesq equations.

In our second example we study the behaviour of a fluid in a square cavity $\Omega = (0,1)^2$ with differentially heated walls. To that end we first recall from [15] the problem with dimensionless numbers:

$$\begin{aligned} -\mathrm{Ra}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathrm{Pr}\;\mathrm{Ra}\,\mathbf{g}\,\theta &= 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \Gamma, \\ -\kappa\,\Delta\theta + \mathbf{u}\cdot\nabla\theta &= 0 \quad \text{in} \quad \Omega, \\ \theta &= \theta_D \quad \text{on} \quad \Gamma, \end{aligned} \tag{5.1}$$

where Pr and Ra are the Prandtl and Rayleigh numbers. Here we fix the Prandtl and Rayleigh numbers as

$$\mathrm{Pr} = 0.5 \quad \text{and} \quad \mathrm{Ra} = 2000,$$

the thermal conductivity $\kappa = 1$, and analogously to [15] we choose the boundary condition $\theta_D(x, y) = 0.5(1 - \cos(2\pi x))(1 - y)$ on Γ. Notice that $\theta_D = 0$ on the left, top and right walls whereas on the bottom wall θ_D has a sinusoidal profile with a peak of temperature $\theta_D = 1$ at $x = 0.5$. For the natural convection problem in a cavity with other boundary conditions we refer the reader to [6, 16]. In Figures 1 and 2 we display the approximate solutions obtained with a $\mathbf{BDM}_1 - \mathrm{P}_0 - \mathbf{RT}_0 - P_0$ discretization with 254294 degrees of freedom. In Figure 1 we show the components of the velocity, the velocity vector field and the pressure. There, it is possible to see the expected physical behaviour from [15], that is, convection currents form inside the cavity in a symmetric configuration. In addition, in Figure 2 it can be seen the components of the temperature gradient, the heat-flux vector field, and the temperature. There, as expected, we observe that the heat-flux moves from higher temperature regions to lower temperature regions.

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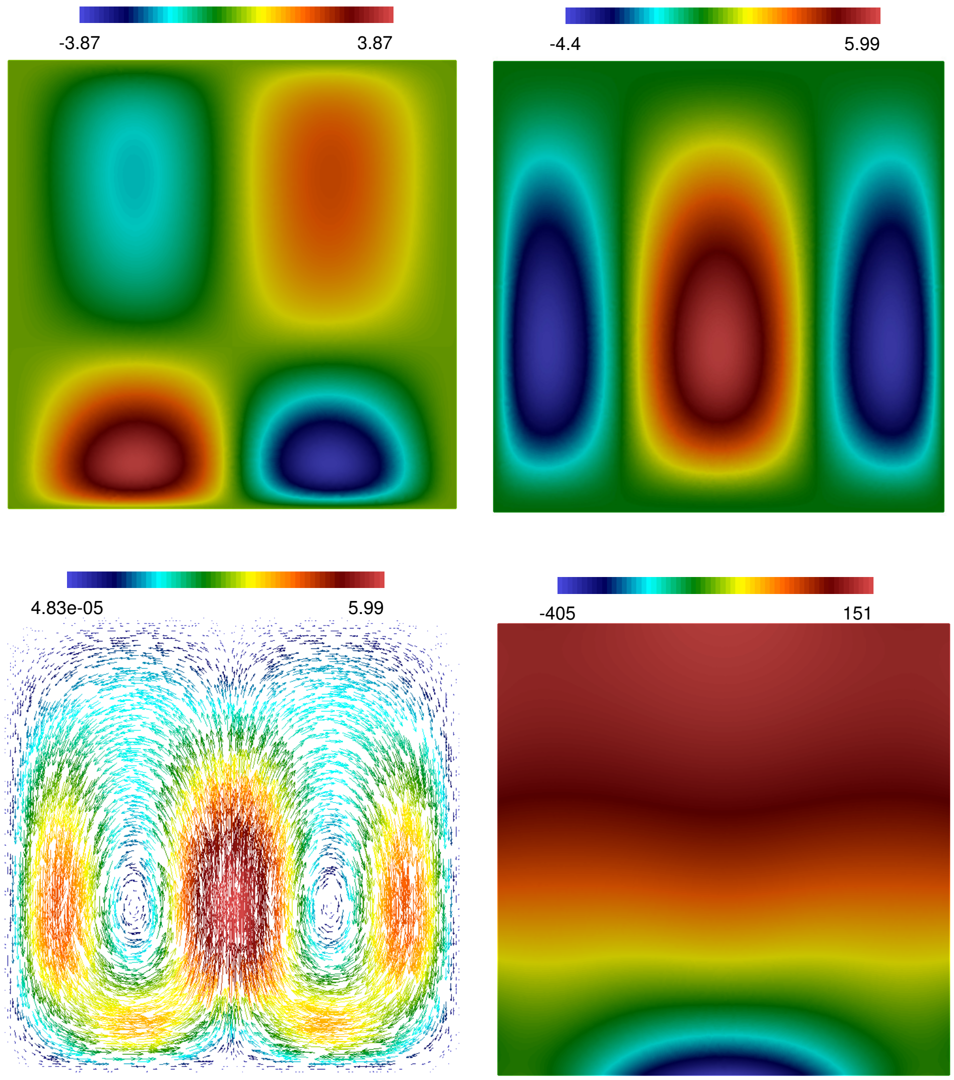

Figure 1: EXAMPLE 2: First (top-left) and second (top-right) components of $\mathbf{u}_h$, velocity vector field (bottom-left) and pressure (bottom-right).

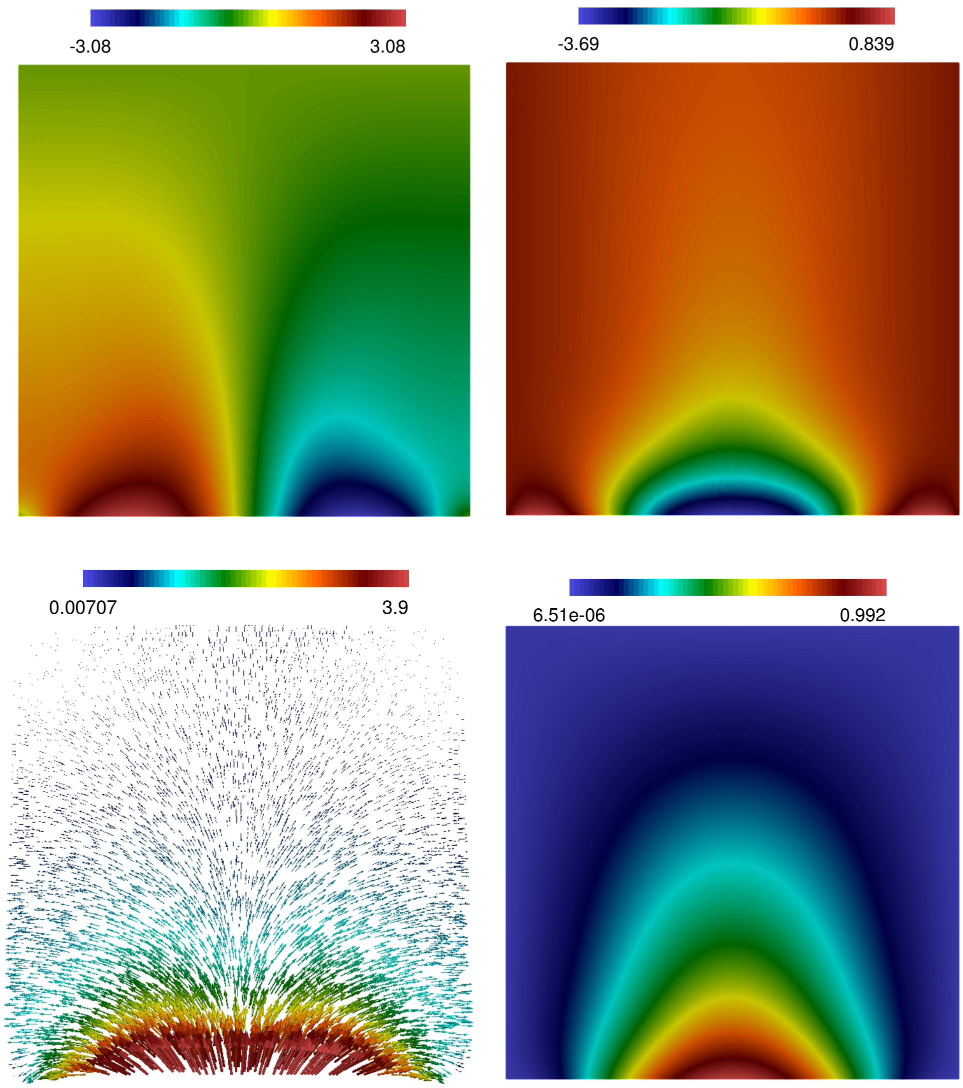

Figure 2: Example 2: First (top-left) and second (top-right) components of σ_h, heat-flux (bottom-left) and temperature (bottom-right).

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