Study of stability and conservative numerical methods for a High order Nonlinear Schrödinger Equation.

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Abstract

In this work we present a Finite Difference scheme used to solve a High order Nonlinear Schrödinger Equation. These equations can model the propagation of solitons travelling in fiber optics ([3], [17]). The scheme is designed to preserve the numerical L^2 norm and the energy, for a suitable initial condition. We show numerical results displaying conservation properties of the schemes using solitons as initial conditions.

1 Introduction

We will study a numerical solution of a Higher order Non-Linear Schrödinger (HNLS) equation:

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + \gamma |u|^2 u + i\delta |u|^2 u_x + i\epsilon u|u|^2_x = 0, \qquad u(x,0) = u^0(x) \tag{1}$$

where $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ and $u = u(x,t)$, $x, t \in \mathbb{R}$ is a complex valued function. This equation plays an important rule in soliton theory. It has applications in the propagation of femtosecond optical pulses in a monomode optical fiber, accounting for additional effects such as third order dispersion, self-steeping of the pulse, and self-frequency shift [17]. We can also consider equation (1) as a generalization of the classical Nonlinear Schrödinger (NLS) equation

$$iu_t + \alpha u_{xx} + \gamma |u|^2 u = 0 \tag{2}$$

which can be obtained using $\beta = \delta = \epsilon = 0$ in (1). This equation describes the electric field envelope of a laser beam in a medium with Kerr nonlinearity [13]. It is also known in plasma physics, where it describes Langmuir waves in a plasma with non-homogeneous density [15]. If in (1) we also take $\alpha = \gamma = 0$, $\beta = 1$, $\epsilon = 0$ and $\delta = 6$, we can obtain the modified Korteweg-de Vries (KdV) equation which studies, for example, surface waves on conducting nonviscous incompressible liquid under the presence of a transverse electric field [25]. The KdV equation has also great importance in the study of surface water waves [18]. In this sense, numericallly solving (1) can also solve many subproblems derived from it.

Carvajal proved in [8] for $\beta\epsilon \neq 0$ the global well-posedness of the Cauchy Problem (1) in $H^s(\mathbb{R}), s > \frac{1}{4}$ when $\gamma = \alpha(\delta - 2\epsilon)/(3\beta)$. Meanwhile, Takaoka proved in [32], for $\beta = 1$, the local well-posedness for the Cauchy Problem (1) in $H^s(\mathbb{T})$, $s > \frac{1}{2}$, where $\mathbb{T}$ is a unidimensional torus. Similar conclusions were obtained also by Takaoka in [31] for $\beta = 0$, where the well-posedness is over $H^{\frac{1}{2}}(\mathbb{R})$. Regularity properties were studied by Alves et al. [2] when $\delta = \epsilon = 0$.

Exact solutions for (1) can be found using the Inverse Scattering Transform (IST) [1], proposed originally in Zakharov et al. [36]. Its integration depends on the values of β, δ and ϵ. In particular: for $\alpha = \frac{1}{2}$, $\gamma = 1$, and rewriting equation (1) as

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + i\varepsilon(\beta_1 u_{xxx} + \beta_2 |u|^2 u_x + \beta_3 |u|^2_x u) = 0 \tag{3}$$

where $\beta_1, \beta_2, \beta_3, \varepsilon$ are real constants, then exact solutions can be obtained via IST for the following cases:

- For the derivative NLS equation of type I: $\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1$ [3].
- For the derivative NLS equation of type II: $\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0$ [11].
- For the Hirota equation: $\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 0$ [14].
- For the Sasa-Satsuma equation: $\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$ [28].

Exact solutions are all of solitonic form. N-soliton solutions can also be obtained [14]. Potasek [27] shows some particular solutions that has been proven experimentally. But even when continuous solutions can be found for some specific initial conditions and some values for the real constans in (1), numerical solutions can prescinde from those requirements when computed. We can even use non-solitonic initial conditions in order to obtain a result. One way to compute numerical solutions is using the Finite Difference Method, whose computational implementation can be done in an fast and efficient way.

Other ways to obtain numerical solutions for (1) has been studied by different authors in the recents years. One of the first scheme were proposed by Delfour, Fortin and Payre [12], which solves the NLS equation (2) proposing a rule to discretize powers of the nonlinearity multiplying the γ term. Their method has a strong property: it preserves the discrete versions of both the L^2 norm and the energy of the numerical solution, where their continuous versions are given by:

$$||u||^2_{L^2(\Omega)}(t) = \int_\Omega |u(x,t)|^2 dx$$
$$E(t) := \frac{\alpha}{2}\int_\Omega |\nabla u(x,t)|^2 dx - \frac{\gamma}{4}\int_\Omega |u(x,t)|^4 dx$$

for $u = u(x,t) \in \Omega \subset \mathbb{R} \times \mathbb{R}^+ \longmapsto \mathbb{C}$ the exact solution of (1). The convergence of the numerical method is proved in Matsuo and Furihata [21]. Pazoto et al [24] proposed a finite difference scheme which solves the critical generalilzed Kortewetg-de Vries equation (GKdV-4) in a bounded domain. The higher-power term $u^4 u_x$ was rewritten as a linear combination of other derivatives in order to obtain specific conservation properties. Smadi and Bahloul [29] [30] combined a Compact Padé Finite Difference scheme [20] with a fourth order Runge-Kutta (RK4) scheme. They splitted the problem in two parts: a linear section which is solved using the finite difference scheme; and the nonlinear, which is solved using the RK4 scheme. The method was implemented with an interesting success, but no analysis of the error, convergence or conserved quantities was made.

The purpose of this work is to search for numerical solutions of the IVP (1) using a Finite Difference scheme which preserves the numerical L^2 norm and delimits the energy. The structure of this work is as follows: Section 2 illustrates the definitions, notation, and properties used along this paper. We will also define the proposed numerical scheme and show its properties. In Theorem 4 we demonstrate that the numerical scheme is of second order in both variables. Section 3 will present results for some experiments, and Section 4 will show our conclusions.

2 Notation, Numerical Scheme, and Properties

In this section we will introduce our numerical scheme and its properties, but first we shall introduce some notation.

For the space coordinate $x \in \mathbb{R}$, let us discretize it using a space-step Δx. We will do the same for the time coordinate $t \in \mathbb{R}.t > 0$, using a time-step Δt. For some $n \in \mathbb{N}$, we denote the solution vector in the n-th timestep as $u^n \in \mathbb{C}^{\mathbb{Z}}$, where u^n_j, $j \in \mathbb{Z}$ will be is its j-th element. We will use the usual inner product between two vectors $u, v \in \mathbb{C}^{\mathbb{Z}}$, which is denoted and defined as

$$(u,v) := u \cdot \overline{v} = \sum_{j\in\mathbb{Z}} u_j \overline{v_j} \Delta x \tag{4}$$

where $\overline{v_j}$ denotes the complex conjugate of the element v_j. This induces the discrete L^2 norm $||u||^2_2 := (u,u)$. From here, we define the difference between the maximum and minimum values of $||u||_2$ as

$$\Delta\ell_2 := \max_{n,m\in\mathbb{N}} \Big| ||u^{(n)}||_2 - ||u^{(m)}||_2 \Big|$$

We will now introduce the operators which shall be used to discretize the partial derivatives in (1). Let us define the operator $\boldsymbol{D}_0$, which represents a centered finite difference approximation of $\partial/\partial x$, such that it maps a vector $v \in \mathbb{C}^{\mathbb{Z}}$ to $\mathbb{C}^{\mathbb{Z}}$ where the j-th component of $\boldsymbol{D}_0 v$ is given by

$$[\boldsymbol{D}_0(v)]_j := D_0(v_j) = \frac{v_{j+1} - v_{j-1}}{2\Delta x}$$

The time derivative will be approximated using a forward finite difference quotient, whose operator $\boldsymbol{D}_t$ acting over a vector $v^n \in \mathbb{C}^{\mathbb{Z}}$ for some $n \in \mathbb{N}$ is given by

$$[\boldsymbol{D}_t(v^n)]_j := D_t(v_j^n) = \frac{v_j^{n+1} - v_j^n}{\Delta t}$$

In the same way we've defined the operator $\boldsymbol{D}_0$, we define the operator $\boldsymbol{D}_0^2$ which approximates the second space derivative using centered finite differences, where the j-th component of $\boldsymbol{D}_0^2 u$, for a $u \in \mathbb{C}^{\mathbb{Z}}$ is given by

$$[\boldsymbol{D}_0^2(u)]_j := D_0^2(u_j) = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

in the same sense, the operator $\boldsymbol{D}_0^3$ discretizes the third space derivative using centered finite differences, such that the j-th component is given by

$$[\boldsymbol{D}_0^3(u)]_j := D_0^3(u_j) = \frac{-\frac{1}{2}u_{j-2} + u_{j-1} - u_{j+1} + \frac{1}{2}u_{j+2}}{\Delta x^3}$$

We also define the forward and backwards finite difference operators for the first derivative, $\boldsymbol{D}_+$ and $\boldsymbol{D}_-$ respectively, as follows:

$$\begin{aligned}
[\boldsymbol{D}_+(u)]_j &:= D_+ u_j := \frac{u_{j+1} - u_j}{\Delta x} \\
[\boldsymbol{D}_-(u)]_j &:= D_- u_j := \frac{u_j - u_{j-1}}{\Delta x}
\end{aligned}$$

With these two operators, in combination with $\boldsymbol{D}_0$, it is easy to check that we can re-write $\boldsymbol{D}_0^2 u$ and $\boldsymbol{D}_0^3 u$ as

$$\begin{aligned}
\boldsymbol{D}_0^2 u &= \boldsymbol{D}^+ \boldsymbol{D}^- u \\
\boldsymbol{D}_0^3 u &= \boldsymbol{D}_0 \boldsymbol{D}^2 u = \boldsymbol{D}_0 \boldsymbol{D}^+ \boldsymbol{D}^-
\end{aligned} \tag{5}$$

2.1 Numerical Scheme

In order to construct the numerical method, we will write a similar form of equation (1). This is because the terms multiplying δ and ϵ are rather complicated to deal with. The modification of equation (1) will then proceed by adding and substracting the same nonlinear term as follows:

$$iu_t + \alpha u_{xx} + \gamma |u|^2 u + i\beta u_{xxx} + i\delta\Big(|u|^2 u_x + (|u|^2 u)_x\Big) + i\epsilon u |u|_x^2 - i\delta(|u|^2 u)_x = 0 \tag{6}$$

For the time being, we will not discretize this expression directly. We will focus for a moment on the last term in (6). For a sufficiently differentiable function $u(x,t) : \mathbb{R}^2 \longrightarrow \mathbb{C}$, we can write $(|u|^2 u)_x$ as a convex combination of itself, $(|u|_x^2 u + |u|^2 u_x)$ and $(u^2 \overline{u_x} + 2|u|^2 u_x)$. Hence,

$$\frac{\partial}{\partial x}\left(|u|^2 u\right) = \alpha_0 \frac{\partial}{\partial x}\left(|u|^2 u\right) + \beta_0 \left(u^2 \frac{\partial \overline{u}}{\partial x} + 2|u|^2 \frac{\partial u}{\partial x}\right) + (1 - \alpha_0 - \beta_0)\left(u \frac{\partial}{\partial x}(|u|^2) + |u|^2 \frac{\partial u}{\partial x}\right) \tag{7}$$

where $\alpha_0, \beta_0 \in [0,1]$. Using (7), equation (6) can be written as

$$\begin{aligned}
&iu_t + \alpha u_{xx} + \gamma |u|^2 u + i\beta u_{xxx} + i\delta\Big(|u|^2 u_x + (|u|^2 u)_x\Big) + i\epsilon u |u|_x^2 \\
&-i\delta\Big[\alpha_0 \frac{\partial}{\partial x}\left(|u|^2 u\right) + \beta_0 \left(u^2 \tfrac{\partial \overline{u}}{\partial x} + 2|u|^2 \tfrac{\partial u}{\partial x}\right) + (1 - \alpha_0 - \beta_0)\left(u \tfrac{\partial}{\partial x}(|u|^2) + |u|^2 \tfrac{\partial u}{\partial x}\right)\Big] = 0
\end{aligned} \tag{8}$$

Which will be the expression to discretize using the finite differences, aiming to preserve the L^2 norm of the numerical solution.

Let us discretize the real axis as $x_j = j\Delta x$, $j \in \mathbb{Z}$, for a predefined spacestep Δx. We shall also discretize the time domain writing $t_n := t_0 + n\Delta t$, $n \in \mathbb{N}$, for a timestep Δt. For a given $n \in \mathbb{N}$, we define the vector $u^n \in \mathbb{C}^{\mathbb{Z}}$ to be the approximation of the function $u(x, t = t_n)$, solution of equation (1). With this, is straightforward to write $u_j^n \approx u(x_j, t_n)$. We also write $u_j^{n+\frac{1}{2}} := \frac{1}{2}\left(u_j^{n+1} + u_j^n\right)$. We shall now be on our way to propose the numerical scheme, based on a finite difference approximation of (8). The time derivative will be discretized using the forward finite difference quotient in time:

$$\frac{\partial u}{\partial t}(x_j, t_n) \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} = D_t(u_j^n)$$

The terms multiplied by α and β are discretized using a Crank-Nicolson method:

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) \approx \frac{1}{2}\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) = \frac{1}{2}\left(D_0^2(u_j^{n+1}) + D_0^2(u_j^n)\right) = D_0^2(u_j^{n+\frac{1}{2}})$$

$$\frac{\partial^3 u}{\partial x^3}(x_j, t_n) \approx \frac{1}{2}\left(\frac{-\frac{1}{2}u_{j-2}^{n+1} + u_{j-1}^{n+1} - u_{j+1}^{n+1} + \frac{1}{2}u_{i+2}^{n+1}}{\Delta x^3} + \frac{-\frac{1}{2}u_{j-2}^n + u_{j-1}^n - u_{j+1}^n + \frac{1}{2}u_{j+2}^n}{\Delta x^3}\right)$$
$$= \frac{1}{2}\left(D_0^3(u_j^{n+1}) + D_0^3(u_j^n)\right) = D_0^3(u_j^{n+\frac{1}{2}})$$

The discretization of the term multiplied by γ will be given by:

$$|u(x_j, t_n)|^2 u(x_j, t_n) \approx |u_j^{n+\frac{1}{2}}|^2 \left(u_j^{n+\frac{1}{2}}\right)$$

We will associate it an operator $F_\gamma : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ such that

$$u_j^{(p)} \longrightarrow [F_\gamma(u^{(p)})]_j := \left(\left|\frac{u_j^{(p)} + u_j^n}{2}\right|^2\right)\left(\frac{u_j^{(p)} + u_j^n}{2}\right)$$

For the terms multiplied by δ and $-\delta$, and using $\alpha_0 = \frac{1}{2}$, $\beta = 0$ in (8), we write:

$$|u(x_j, t_n)|^2 \left(u(x_j, t_n)\right)_x + \left(|u(x_j, t_n)|^2 u(x_j, t_n)\right)_x - \left(|u(x_j, t_n)|^2 u(x_j, t_n)\right)_x$$
$$\approx \frac{1}{2}\left(|u_j^{n+\frac{1}{2}}|^2 D_0(u_j^{n+\frac{1}{2}}) + D_0\left(|u_j^{n+\frac{1}{2}}|^2 u_j^{n+\frac{1}{2}}\right)\right) - \frac{1}{2}D_0(|u_j^{n+\frac{1}{2}}|^2)u_j^{n+\frac{1}{2}}$$

where we define the operator

$$F_\delta : \mathbb{C}^{\mathbb{Z}} \longrightarrow \mathbb{C}^{\mathbb{Z}}$$
$$u_j^{(p)} \longrightarrow [F_\delta(u^{(p)})]_j := \frac{1}{2}\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2 D_0\left(\frac{u_j^p + u_j^n}{2}\right) + D_0\left(\frac{u_j^p + u_j^n}{2}\,\frac{u_j^p + u_j^n}{2}\right)\right) - \frac{1}{2}D_0\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2\right)\left(\frac{u_j^p + u_j^n}{2}\right)$$

The ϵ term will be discretized directly:

$$u(x_j, t_n)\left|u(x_j, t_n)\right|_x^2 \approx u_j^{n+\frac{1}{2}} D_0\left(|u_j^{n+\frac{1}{2}}|^2\right)$$

where we define its representing function

$$F_\epsilon : \mathbb{C}^{\mathbb{Z}} \longrightarrow \mathbb{C}^{\mathbb{Z}}$$
$$u_j^{(p)} \longrightarrow [F_\epsilon(u^{(p)})]_j := \left(\frac{u_j^p + u_j^n}{2}\right) D_0\left(\left|\frac{u_j^p + u_j^n}{2}\right|^2\right)$$

Hence, $\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, and for a given $u^0 \in \mathbb{C}^N$ the numerical scheme will be given component-wise by

$$iD_t u_j^n + \alpha D_0^2(u_j^{n+\frac{1}{2}}) + \gamma[F_\gamma(u^{(n+1)})]_j + i\beta D_0^3(u_j^{n+\frac{1}{2}}) + i\delta[F_\delta(u^{(n+1)})]_j + i\epsilon[F_\epsilon(u^{(n+1)})]_j = 0 \tag{9}$$

2.2 Conservation of the L^2-norm

This numerical scheme was designed to preserve the L^2-norm of the numerical solution. In order to demonstrate that property, we will show the next lemma

Lemma 1 *$\forall \varphi \in \mathbb{C}^{\mathbb{Z}}$, we have*

$$Im(\boldsymbol{D}_0^2\varphi, \varphi) = 0 \tag{10}$$

$$Re(\boldsymbol{D}_0^3\varphi, \varphi) = 0 \tag{11}$$

$$Re(|\varphi|^2\boldsymbol{D}_0\varphi + \boldsymbol{D}_0\left(|\varphi|^2\varphi\right), \varphi) = 0 \tag{12}$$

$$(\varphi^2\boldsymbol{D}_0\overline{\varphi} + \boldsymbol{D}_0(|\varphi|^2\varphi), \varphi) = 0 \tag{13}$$

$$(\varphi\boldsymbol{D}_0(|\varphi|^2), \varphi) = 0 \tag{14}$$

Proof: To get (10), note that

$$[\boldsymbol{D}_0^2(\varphi)]_j = D_0^2(\varphi_j) = \frac{\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}}{\Delta x^2} = \frac{1}{\Delta x}\left(D_+\varphi_j - D_-\varphi_j\right)$$

Which can be extended to the other elements of $\boldsymbol{D}_0^2(\varphi)$. Hence, we have.

$$(\boldsymbol{D}_0^2(\varphi), \varphi) = \frac{1}{\Delta x}(\boldsymbol{D}_+\varphi - \boldsymbol{D}_-\varphi, \varphi) = \frac{1}{\Delta x}(-\overline{(\boldsymbol{D}_-\varphi, \varphi)} + \overline{(\boldsymbol{D}_+\varphi, \varphi)}) = \overline{(\boldsymbol{D}_0^2(\varphi), \varphi)}$$

from here, we get (10). In order to obtain (11), and as in the previous case, we must first note that

$$D_0^3\varphi_j = D_0(D_0^2\varphi_j)$$

Which can be extended to the other elements $\boldsymbol{D}_0\varphi$. Thus, equation (11) follows because $(\boldsymbol{D}_0 u, v) = -(u, \boldsymbol{D}_0 v)$. To obtain (12), we have

$$|\varphi|^2\boldsymbol{D}_0\varphi \cdot \overline{\varphi} = |\varphi|^2\overline{\varphi} \cdot \boldsymbol{D}_0\varphi = -\boldsymbol{D}_0\left(|\varphi|^2\overline{\varphi}\right) \cdot \varphi = -\overline{\boldsymbol{D}_0\left(|\varphi|^2\varphi\right) \cdot \overline{\varphi}}$$

Hence, we have

$$Re\left[|\varphi|^2\boldsymbol{D}_0\varphi \cdot \overline{\varphi} + \boldsymbol{D}_0\left(|\varphi|^2\varphi\right) \cdot \overline{\varphi}\right] = 0$$

where we can conclude (12). To obtain (13), we have:

$$\varphi^2\boldsymbol{D}_0\overline{\varphi} \cdot \overline{\varphi} = \varphi^2\overline{\varphi} \cdot \boldsymbol{D}_0\overline{\varphi} = -\boldsymbol{D}_0(\varphi^2\overline{\varphi}) \cdot \overline{\varphi} = -\boldsymbol{D}_0(|\varphi|^2\varphi) \cdot \overline{\varphi}$$

Therefore, we can conclude (13). In the identity (14) we have

$$\varphi\boldsymbol{D}_0(|\varphi|^2) \cdot \overline{\varphi} = \boldsymbol{D}_0(|\varphi|^2) \cdot |\varphi|^2 = -|\varphi|^2 \cdot \boldsymbol{D}_0(|\varphi|^2) = -\boldsymbol{D}_0(|\varphi|^2)\varphi \cdot \overline{\varphi}$$

Then, we have (14), concluiding the proof of the lemma. ■

The previous lemmas will now give us reasons to write the numerical scheme (8). From the convex combination (7), we want to re-write it in function of the conserved quantities of the lemma 2. In other words, we want something like

$$\frac{\partial}{\partial x}(|\varphi|^2\varphi) = A\left(|\varphi|^2\varphi_x + (|\varphi|^2\varphi)_x\right) + B(|\varphi|_x^2\varphi) \tag{15}$$

where A and B are real constants whose values must be obtained. Comparing (15) with the convex combination (7), and solving the resulting linear system of equations, gives $A = B = \frac{1}{2}$ or $\alpha_0 = \frac{1}{2}$ and $\beta_0 = 0$. This leads to the proposed scheme in (8). Now we are in conditions to state the following theorem:

Theorem 1 *Let $u^0 \in \mathbb{C}^{\mathbb{Z}} : ||u^0||_2^2 < \infty$. Then, $\forall n \in \mathbb{N}$, and for $u^n \in \mathbb{C}^{\mathbb{Z}}$, the numerical method written in* (9) *preserves the numerical L^2 norm in time, i.e. for the numerical solution of* (1) *at the n-th timestep, we have*

$$\sum_{j\in\mathbb{Z}} |u_j^{n+1}|^2\Delta x = \sum_{j\in\mathbb{Z}} |u_j^n|^2\Delta x = ||u^0||_2^2 \tag{16}$$

Proof: we will multiply equation (9) by $\left(\overline{u}_j^{n+1}+\overline{u}_j^n\right)\Delta x$, sum all the terms of the array, and extract the complex part. We will study each resulting term separately, and neglecting by the moment the Δx term. For the time derivative term, we have

$$i\frac{u_j^{n+1}-u_j^n}{\Delta t}(\overline{u}_j^{n+1}+\overline{u}_j^n)=\frac{i}{\Delta t}\left(|u_j^{n+1}|^2-|u_j^n|^2+2iIm(u_j^{n+1}\overline{u}_j^n)\right) \tag{17}$$

Hence, when extracting the complex part, we have to demonstrate that the other terms of the scheme are reals or zeros. The complex part of the second term which multiplies the α constant, is zero due to (10) for $\varphi=u^{n+\frac{1}{2}}$. In the case of the γ term, the result obtained is real; thus, its complex part is equal to zero. The same conclusion is valid for the term multiplied by β, but using $\varphi=u^{n+\frac{1}{2}}$ in (11). Finally, the complex parts of the terms involving δ, ϵ are also zero as a consequence of (12), (13) and (14) in Lemma 1. Hence, adding all the previous results, and extracting the complex part, we have,

$$0=\sum_{j\in\mathbb{Z}}\left(|u_j^{n+1}|^2\Delta x-|u_j^n|^2\Delta x)\right), \quad n\in\mathbb{N}. \tag{18}$$

this allow us to conclude if we consider $||u^0||_2^2<\infty$. ■.

2.3 Conservation of the Energy.

Before presenting the result, let us define the discrete L^4 norm of u^n as

$$||u^n||_4^4:=\left((u^n)^2,(u^n)^2\right)=\sum_j|u_j^n|^4\Delta x$$

Let us also define the discrete energy $E^{(n)}$ of the numerical solution u^n, at a timestep n, as

$$E^{(n)}:=\frac{\alpha}{2}||D_+u^n||_2^2-\frac{\gamma}{4}||u^n||_4^4 \tag{19}$$

where

$$\Delta E:=\max_{n,m\in\mathbb{N}}\left|E^{(n)}-E^{(m)}\right|$$

Finally, we will define the space $H^2_{\Delta x}$ as the space endorsed with the inner product $(\cdot,\cdot)_{H^2_{\Delta x}}$ defined as

$$(u,v)_{H^2_{\Delta x}}:=(u,v)+(\boldsymbol{D}^+u,\boldsymbol{D}^+v)+(\boldsymbol{D}^2u,\boldsymbol{D}^2v)$$

where $(\cdot,\cdot)$ is the inner product defined in (4), and $u,v\in\mathbb{C}^{\mathbb{N}}:\{u,\boldsymbol{D}^+u,\boldsymbol{D}^2u\}\subset\ell_2,\{v,\boldsymbol{D}^+v,\boldsymbol{D}^2v\}\subset\ell_2$. For a $\varphi\in\mathbb{C}^{\mathbb{N}}$ then, this induces the following norm:

$$||\varphi||^2_{H^2_{\Delta x}}:=||\varphi||_2^2+||\boldsymbol{D}^+\varphi||_2^2+||\boldsymbol{D}^2\varphi||_2^2$$

The following result is obtained:

Theorem 2 *Let $u^n\in\mathbb{C}^N$ the numerical solution of* (1) *using scheme* (9)*, such that $u^n\in H^2_{\Delta x}$. If $3\beta\gamma=\alpha(\epsilon+2\delta)$, then the following property holds*

$$E^{(n+1)}=E^{(n)}+\mathcal{O}(\Delta t+\Delta x^2) \tag{20}$$

Before proving the theorem, we will state and prove the following lemma:

Lemma 2 *$\forall\varphi\in\mathbb{C}^{\mathbb{Z}}$, and for φ_+, $\varphi_-\in\mathbb{C}^{\mathbb{Z}}:(\varphi_+)_j=\varphi_{j+1}$, $(\varphi_-)_j=\varphi_{j-1}$, we have*

$$Re\Big(|\varphi|^2\boldsymbol{D}_0\varphi,\boldsymbol{D}^2\varphi\Big)=\frac{1}{2}|\varphi|^2\cdot\Big(\boldsymbol{D}^+\left(|\boldsymbol{D}^-\varphi|^2\right)\Big) \tag{21}$$

$$Re\Big(\boldsymbol{D}_0(|\varphi|^2)\varphi,\boldsymbol{D}^2\varphi\Big)=\frac{|\varphi_+|^2+2|\varphi|^2+|\varphi_-|^2}{4}\cdot\boldsymbol{D}^+\left(\boldsymbol{D}^-|\varphi|^2\right) \tag{22}$$

$$Re\Big(\boldsymbol{D}_0(|\varphi|^2\varphi),\boldsymbol{D}^2\varphi\Big)=\frac{|\varphi_+|^2+|\varphi|^2+|\varphi_-|^2}{2}\cdot\boldsymbol{D}^+\left(|\boldsymbol{D}^-\varphi|^2\right)+\frac{\Delta x^2}{2}\left(\boldsymbol{D}_0|\varphi|^2\right)\cdot\left|\boldsymbol{D}^2\varphi\right|^2 \tag{23}$$

$$Re\Big(\boldsymbol{D}^3\varphi,\boldsymbol{D}^2\varphi\Big)=0 \tag{24}$$

Proof: starting with (21), we have

$$|\varphi_j|^2 D_0\varphi_j\overline{D^2\varphi_j} = |\varphi_j|^2 D_0\varphi_j\overline{D^+D^-\varphi_j}$$
$$= \frac{1}{2}|\varphi_j|^2\Big(D^+\varphi_j\overline{D^-D^+\varphi_j}\Big) + \frac{1}{2}|\varphi_j|^2\Big(D^-\overline{D^+D^-\varphi_j}\Big) \tag{25}$$

At this point we will use the following identities for $a, b \in \mathbb{C}$:

$$Re\big(b(\overline{b}-\overline{a})\big) = \frac{1}{2}\big(|b|^2-|a|^2\big) + \frac{1}{2}|b-a|^2 \tag{26}$$
$$Re\big(a(\overline{b}-\overline{a})\big) = \frac{1}{2}\big(|b|^2-|a|^2\big) - \frac{1}{2}|b-a|^2 \tag{27}$$

using this over the real part in (25), we get

$$Re\Big(|\varphi_j|^2 D_0\varphi_j\overline{D^2\varphi_j}\Big) = \frac{1}{4}|\varphi_j|^2 D^+\big(|D\varphi_j|^2\big) - \frac{\Delta x^2}{4}|D^+D\varphi_j|^2|\varphi_j|^2$$
$$+ \frac{1}{4}|\varphi_j|^2 D^-\big(D^-(|D^+\varphi_j|^2)\big) + \frac{\Delta x^2}{4}|D^-D^+\varphi_j|^2|\varphi_j|^2$$
$$= \frac{1}{4}|\varphi_j|^2\Big(D^+\big(|D^-\varphi_j|^2\big) + D^-\big(|D^+\varphi_j|^2\big)\Big).$$

Summing over j, we get

$$Re\Big(|\varphi|^2\boldsymbol{D}_0\varphi, \boldsymbol{D}^2\varphi\Big) = \frac{1}{4}|\varphi|^2\cdot\Big(\boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)\Big) - \frac{1}{4}\Big(\big(\boldsymbol{D}^+|\varphi|^2\big)\cdot\big(|\boldsymbol{D}^+\varphi|^2\big)\Big)$$
$$= \frac{1}{2}|\varphi|^2\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)$$

hence, (21) is proved. To obtain (22), we will first require the following property for $a, b \in \mathbb{C}^{\mathbb{Z}}$:

$$D_0(a_jb_j) = a_{j+1}\frac{D^+b_j}{2} + a_{j-1}\frac{D^-b_j}{2} + b_jD_0a_j \tag{28}$$
$$D^-(a_jb_j) = b_{j-1}Da_j + a_jD^-b_j \tag{29}$$

Using this over all the components of φ in (22), we get

$$\Big(\boldsymbol{D}_0\varphi, \boldsymbol{D}^2\varphi\Big) = -\Big(\boldsymbol{D}^-\big(\boldsymbol{D}_0(|\varphi|^2)\varphi\big), \boldsymbol{D}^-\varphi\Big)$$
$$= -\Big(\boldsymbol{D}_0(|\varphi|^2)\boldsymbol{D}^-\varphi, \boldsymbol{D}^-\varphi\Big) - \Big(\varphi_-\boldsymbol{D}^-\boldsymbol{D}_0(|\varphi|^2)\boldsymbol{D}^-\varphi\Big)$$
$$= -\Big(\boldsymbol{D}_0(|\varphi|^2), |\boldsymbol{D}^-\varphi|^2\Big) - \Big(\varphi_-\boldsymbol{D}^-\boldsymbol{D}_0(|\varphi|^2), \boldsymbol{D}^-\varphi\Big)$$

extracting the real part,

$$Re\Big(\boldsymbol{D}_0\varphi, \boldsymbol{D}^2\varphi\Big) = \Big(|\varphi|^2, \boldsymbol{D}_0(|\varphi|^2)\Big) - \Big(\boldsymbol{D}^-\boldsymbol{D}_0(|\varphi|^2), (\frac{1}{2}\boldsymbol{D}^-|\varphi|^2 - \frac{\Delta x}{2}|\boldsymbol{D}^-\varphi|^2)\Big)$$
$$= \Big(|\varphi|^2, \boldsymbol{D}_0(|\varphi|^2)\Big) - \frac{1}{2}\Big(\boldsymbol{D}_0\big(\boldsymbol{D}^-|\varphi|^2\big), \boldsymbol{D}^-|\varphi|^2\Big) + \frac{\Delta x}{2}\Big(\boldsymbol{D}_0\boldsymbol{D}^-(|\varphi|^2)), |\boldsymbol{D}^-\varphi|^2\Big)$$
$$= \Big(|\varphi|^2, \boldsymbol{D}_0(|\varphi|^2)\Big) - \frac{\Delta x}{2}\Big(\boldsymbol{D}^-(|\varphi|^2), \boldsymbol{D}_0(|\boldsymbol{D}^-\varphi|^2)\Big)$$
$$= \Big(|\varphi|^2, \boldsymbol{D}_0(|\varphi|^2)\Big) - \left[\frac{1}{2}\Big(|\varphi|^2, \boldsymbol{D}_0\big(|\boldsymbol{D}_0\varphi|^2\big)\Big) - \Big(|\varphi|^2, \boldsymbol{D}_0\big(|\boldsymbol{D}^+\varphi|^2\big)\Big)\right]$$
$$= \Big(|\varphi|^2, \boldsymbol{D}_0\Big[|\boldsymbol{D}^-\varphi|^2 + |\boldsymbol{D}^+\varphi|^2\Big]\Big)$$
$$= -\Big(\boldsymbol{D}_0|\varphi|^2, \Big[|\boldsymbol{D}^-\varphi|^2 + |\boldsymbol{D}^+\varphi|^2\Big]\Big)$$
$$= -\frac{1}{4}\Big(\boldsymbol{D}^-|\varphi_+|^2 + 2\boldsymbol{D}^-|\varphi|^2 + \boldsymbol{D}^-|\varphi_-|^2, |\boldsymbol{D}^-\varphi|^2\Big)$$
$$= \Big(\frac{|\varphi_+|^2 + 2|\varphi|^2 + |\varphi_-|^2}{4}, \boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)\Big).$$

Hence, (22) is proved. To prove (23), and starting by using (28), we have

$$\Big(\boldsymbol{D}_0(|\varphi|^2\varphi),\boldsymbol{D}^2\varphi\Big)=\Big[\Big(\boldsymbol{D}_0(|\varphi|^2)\varphi,\boldsymbol{D}^2\varphi\Big)+\frac{1}{2}\Big(\varphi_+|^2\boldsymbol{D}^+\varphi+|\varphi_-|^2\boldsymbol{D}^-\varphi,\boldsymbol{D}^2\varphi\Big)\Big] \tag{30}$$

extracting the real part, and by (26) and (27) respectively, we have

$$Re\Big(|\varphi_+|^2\boldsymbol{D}^+\varphi,\boldsymbol{D}^2\varphi\Big)=\frac{1}{2}|\varphi_+|^2\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^+\varphi|^2\big)+\frac{\Delta x}{2}|\varphi_+|^2\cdot\big(|\boldsymbol{D}^2\varphi|^2\big) \tag{31}$$

$$Re\Big(|\varphi_-|^2\boldsymbol{D}^-\varphi,\boldsymbol{D}^2\varphi\Big)=\frac{1}{2}|\varphi_-|^2\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)-\frac{\Delta x}{2}|\varphi_-|^2\cdot\big(|\boldsymbol{D}^2\varphi|^2\big) \tag{32}$$

replacing (31) and (32) over the real part of (30), we get

$$\begin{aligned}Re\Big(\boldsymbol{D}_0(|\varphi|^2\varphi),\boldsymbol{D}^2\varphi\Big)&=Re\Big(\boldsymbol{D}_0(|\varphi|^2)\varphi,\boldsymbol{D}^2\varphi\Big)+\frac{1}{4}\big(|\varphi_+|^2+|\varphi_-|^2\big)\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)+\frac{\Delta x}{4}\big(|\varphi_+|^2-|\varphi_-|^2\big)\cdot\big(|\boldsymbol{D}^2\varphi|^2\big)\\&=Re\Big(\boldsymbol{D}_0(|\varphi|^2)\varphi,\boldsymbol{D}^2\varphi\Big)+\frac{1}{4}\big(|\varphi_+|^2+|\varphi_-|^2\big)\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-\varphi|^2\big)+\frac{\Delta x^2}{2}\big(\boldsymbol{D}_0|\varphi|^2\big)\cdot\big(|\boldsymbol{D}^2\varphi|^2\big)\end{aligned}$$

and recalling (22), the conclusion follows. Finally for (24),

$$\begin{aligned}\Big(\boldsymbol{D}^3\varphi,\boldsymbol{D}^2\varphi\Big)&=\Big(\boldsymbol{D}_0\boldsymbol{D}^2\varphi,\boldsymbol{D}^2\varphi\Big)\\&=-\Big(\boldsymbol{D}^2\varphi,\boldsymbol{D}_0\boldsymbol{D}^2\varphi\Big)\\&=-\overline{\Big(\boldsymbol{D}^3\varphi,\boldsymbol{D}^2\varphi\Big)}\end{aligned}$$

and hence, the proof of the Lemma is complete. ■

Proof of Theorem 2: we will multiply (9) with $\boldsymbol{D}_t\overline{u}_j^n$, sum over j, and extract the real part. This will lead us to

$$0=\frac{\alpha}{2\Delta t}(||\boldsymbol{D}^+u^{n+1}||^2-||\boldsymbol{D}^+u^n||^2)-\gamma\sum_{j\in\mathbb{Z}}Re\Big(F_\gamma(u^{n+1})_jD_t\overline{u}_j^n\Big) \tag{33}$$

$$+\beta\sum_{j\in\mathbb{Z}}Im\Big(D^3u_j^{n+\frac{1}{2}}D_t\overline{u_j}^n\Big)+\delta\sum_{j\in\mathbb{Z}}Im\Big(F_\delta(u^{n+1})_jD_t\overline{u}_j^n\Big)+\epsilon\sum_{j\in\mathbb{Z}}Im\Big(F_\epsilon(u^{n+1})_jD_t\overline{u}_j^n\Big) \tag{34}$$

and replacing $D_tu_j^n$ from the numerical scheme on the last three products,

$$\begin{aligned}0=&\frac{\alpha}{2\Delta t}(||\boldsymbol{D}^+u^{n+1}||^2-||\boldsymbol{D}^+u^n||^2)-\gamma\sum_{j\in\mathbb{Z}}Re\Big(F_\gamma(u^{n+1})D_t\overline{u}_j^n\Big)\\&+\sum_{j\in\mathbb{Z}}Im\Big[\Big(\beta D^3u_j^{n+\frac{1}{2}}+\delta F_\delta(u^{n+1})_j+\epsilon F_\epsilon(u^{n+1})_j\Big)\Big(\overline{i\alpha D^2u_j^{n+\frac{1}{2}}}-\overline{\beta D^3u_j^{n+\frac{1}{2}}}+\overline{i\gamma F_\gamma(u^{n+1})_j}+\overline{\delta F_\delta(u^{n+1})_j}+\overline{\epsilon F_\epsilon(u^{n+1})_j}\Big)\Big]\end{aligned}$$

$$\begin{aligned}0=&\frac{\alpha}{2\Delta t}\Big(||\boldsymbol{D}^+u^{n+1}||^2-||\boldsymbol{D}^+u^n||^2\Big)-\gamma\sum_{j\in\mathbb{Z}}Re\Big(F_\gamma(u^{n+1})D_t\overline{u}_j^n\Big)\\&+\alpha\beta\sum_{j\in\mathbb{Z}}Re\Big(D^3u_j^{n+\frac{1}{2}}\overline{D^2u_j^{n+\frac{1}{2}}}\Big)+\alpha\delta\sum_{j\in\mathbb{Z}}Re\Big(F_\delta(u^{n+1})_j\overline{D^2u_j^{n+1\frac{1}{2}}}\Big)+\alpha\epsilon\sum_{j\in\mathbb{Z}}Re\Big(F_\epsilon(u^{n+1})_j\overline{D^2u_j^{n+\frac{1}{2}}}\Big)\\&+\beta\gamma\sum_{j\in\mathbb{Z}}Re\Big(D^3u_j^{n+\frac{1}{2}}\overline{F_\gamma(u^{n+1})_j}\Big)+\gamma\delta\sum_{j\in\mathbb{Z}}Re\Big(F_\delta(u^{n+1})_j\overline{F_\gamma(u^{n+1})_j}\Big)+\gamma\epsilon\sum_{j\in\mathbb{Z}}Re\Big(F_\epsilon(u^{n+1})_j\overline{F_\gamma(u^{n+1})_j}\Big)\end{aligned} \tag{35}$$

We will study each term in (35) by components if possible. For the second term in (35), let us note that

$$|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}-\frac{|u_j^{n+1}|^2+|u_j^n|^2}{2}u_j^{n+\frac{1}{2}}=-\frac{\Delta t^2}{8}u_j^{n+\frac{1}{2}}|D_tu_j^n|^2.$$

Hence,

$$\begin{aligned}Re\Big(F_\gamma(u^{n+1})_j\overline{D_tu_j^n}\Big)&=Re\Big(|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}\overline{D_tu_j^n}\Big)\\&=Re\Big(\frac{|u_j^{n+1}|^2+|u_j^n|^2}{2}u_j^{n+\frac{1}{2}}\overline{D_tu_j^n}-\frac{\Delta t^2}{8}u_j^{n+\frac{1}{2}}|D_tu_j^n|^2\overline{D_tu_j^n}\Big)\\&=\frac{1}{4\Delta t}\Big(|u_j^{n+1}|^4-|u_j^n|^4\Big)-\frac{\Delta t^2}{8}Re\Big(u_j^{n+\frac{1}{2}}|D_tu_j^n|^2\overline{D_tu_j^n}\Big)\end{aligned} \tag{36}$$

Meanwhile, and thanks to (24) in Lemma 2, we can get rid of the third term in (35). For the fourth term, and by (22) in Lemma 2:

$$\begin{aligned} Re\Big(F_\delta(u^{n+1})\overline{\boldsymbol{D}^2u^{n+1\frac{1}{2}}}\Big) &= \boldsymbol{D}_0(|u^{n+\frac{1}{2}}|^2)u^{n+\frac{1}{2}}\cdot\overline{\boldsymbol{D}^2u^{n+\frac{1}{2}}} \\ &= \frac{|u_+^{n+\frac{1}{2}}|^2+2|u^{n+\frac{1}{2}}|^2+|u_-^{n+\frac{1}{2}}|^2}{8}\cdot\boldsymbol{D}^+\Big(\boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big) \end{aligned} \tag{37}$$

where $u_+\in\mathbb{C}^{\mathbb{Z}}:(u_+)_j=u_{j+1}$, and $u_-\in\mathbb{C}^{\mathbb{Z}}:(u_-)_j=u_{j-1}$. For the fifth term in (35), and using (21), (22) and (23) in Lemma 2:

$$\begin{aligned} Re\Big(F_\epsilon(u^{n+1})\overline{\boldsymbol{D}^2u^{n+\frac{1}{2}}}\Big) &= \frac{1}{2}\Big(|u^{n+\frac{1}{2}}|^2\boldsymbol{D}_0u^{n+\frac{1}{2}}+\boldsymbol{D}_0\Big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}\Big)-\boldsymbol{D}_0(|u^{n+\frac{1}{2}}|^2)u^{n+\frac{1}{2}}\Big)\cdot\overline{\boldsymbol{D}^2u} \\ &= \frac{1}{4}|u^{n+\frac{1}{2}}|^2\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-u^{n+\frac{1}{2}}|^2\big)+\frac{|u_+^{n+\frac{1}{2}}|^2+|u^{n+\frac{1}{2}}|^2+|u_-^{n+\frac{1}{2}}|^2}{4}\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-u^{n+\frac{1}{2}}|^2\big) \\ &\quad+\frac{\Delta x^2}{4}\boldsymbol{D}_0\big(|u^{n+\frac{1}{2}}|^2\big)\cdot\big|\boldsymbol{D}^2u^{n+\frac{1}{2}}\big|^2-\frac{|u_+^{n+\frac{1}{2}}|^2+2|u^{n+\frac{1}{2}}|^2+|u_-^{n+\frac{1}{2}}|^2}{8}\cdot\boldsymbol{D}^+\big(\boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\big) \\ &= \frac{|u_+^{n+\frac{1}{2}}|^2+2|u^{n+\frac{1}{2}}|^2+|u_-^{n+\frac{1}{2}}|^2}{8}\cdot\boldsymbol{D}^+\big(|\boldsymbol{D}^-u^{n+\frac{1}{2}}|^2\big)+\frac{\Delta x^2}{4}\boldsymbol{D}_0\big(|u^{n+\frac{1}{2}}|^2\big)\cdot\big|\boldsymbol{D}^2u^{n+\frac{1}{2}}\big|^2. \end{aligned} \tag{38}$$

Meanwhile, the sixth term can be worked thanks to (23) in Lemma 2:

$$\begin{aligned} Re\Big(D^3u_j^{n+\frac{1}{2}}\overline{F_\gamma(u^{n+1})_j}\Big) &= Re\Big(D^3u_j^{n+\frac{1}{2}}|u_j^{n+\frac{1}{2}}|^2\overline{u_j^{n+\frac{1}{2}}}\Big) \\ &= -Re\Big(D_0\big(|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}\big)\overline{D^2u_j^{n+\frac{1}{2}}}\Big) \\ &= -\frac{|\varphi_{j+1}|^2+|\varphi_j|^2+|\varphi_{j-1}|^2}{2}D^+\big(|D^-\varphi_j|^2\big)+\frac{\Delta x^2}{2}D_0\big(|\varphi_j|^2\big)\big|D^2\varphi_j\big|^2. \end{aligned} \tag{39}$$

The last two terms in (35) require more effort. For the last one, and because $Re(\boldsymbol{D}u,u)=0$, $\forall u\in\mathbb{C}^{\mathbb{Z}}$, we have

$$\begin{aligned} Re\Big(F_\epsilon(u^{n+1})_j\overline{F_\gamma(u^{n+1})_j}\Big) &= Re\Bigg(\frac{1}{2}|u_j^{n+\frac{1}{2}}|^2D_0u_j^{n+\frac{1}{2}}\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}}+\frac{1}{2}D_0\big(|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}\big)\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}} \\ &\qquad-\frac{1}{2}D_0|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}\overline{|u_j^{n+\frac{1}{2}}|^2u_j^{n+\frac{1}{2}}}\Bigg) \\ &= \frac{1}{2}Re\Bigg(|u_j^{n+\frac{1}{2}}|^4\overline{u_j^{n+\frac{1}{2}}}D_0u_j^{n+\frac{1}{2}}-|u_j^{n+\frac{1}{2}}|^4D_0\big(|u_j^{n+\frac{1}{2}}|^2\big)\Bigg) \end{aligned} \tag{40}$$

using the following identities for $a,b\in\mathbb{R}$,

$$\begin{aligned} b^2(b-a) &= \frac{1}{3}(b^3-a^3)-\frac{1}{3}(b-a)^3+b(b-a)^2 \\ a^2(b-a) &= \frac{1}{3}(b^3-a^3)-\frac{1}{3}(b-a)^3-a(b-a)^2 \end{aligned}$$

we can write

$$|u_j^{n+\frac{1}{2}}|^4D_0\big(|u_j^{n+\frac{1}{2}}|^2\big)=\frac{\Delta x^2}{6}\Big(D^+|u_j^{n+\frac{1}{2}}|^2\Big)^3 \tag{41}$$

on the other hand, and by $a(b-a)=\frac{1}{2}(a^2-b^2)+\frac{1}{2}(a-b)^2$, we can write

$$|u_j^{n+\frac{1}{2}}|^4\overline{u_j^{n+\frac{1}{2}}}D_0u_j^{n+\frac{1}{2}}=\frac{\Delta x^2}{12}(D^+|u_j^{n+\frac{1}{2}}|^2)^3+\frac{\Delta x^2}{4}D^+\big(|u_j^{n+\frac{1}{2}}|^4\big)|u_j^{n+\frac{1}{2}}|^2 \tag{42}$$

then, replacing (41) and (42) in (40), we will get

$$Re\Big(F_\epsilon(u^{n+1})_j\overline{F_\gamma(u^{n+1})_j}\Big)=-\frac{\Delta x^2}{24}\big(D^+|u_j^{n+\frac{1}{2}}|^2\big)^3+\frac{\Delta x^2}{8}D^+\big(|u_j^{n+\frac{1}{2}}|^4\big)|D^+u_j^{n+\frac{1}{2}}|^2. \tag{43}$$

Using the same technique, we can write

$$Re\Big(F_\delta(u^{n+1})_j\overline{F_\gamma(u^{n+1})_j}\Big) = \frac{\Delta x^2}{6}\Big(D^+|u_j^{n+\frac{1}{2}}|^2\Big)^3. \tag{44}$$

Replacing (36), (37), (38), (39), and the sums of (43) and (44), all in (35); using the fact that $3\beta\gamma = \alpha(\epsilon + 2\delta)$, multiplying by Δt, and recalling definition (19), we get

$$\begin{aligned}
E^{(n+1)} &= E^{(n)} - \gamma\frac{\Delta t^3}{8}Re\Big(u^{n+\frac{1}{2}}|\boldsymbol{D}_t u^n|^2 \cdot \overline{\boldsymbol{D}_t u^n}\Big) \\
&\quad + \Delta t\beta\gamma\bigg(\frac{\Delta x^2}{8}\boldsymbol{D}^2\Big(|u^{n+\frac{1}{2}}|^2\Big)\cdot\boldsymbol{D}^+\Big(|\boldsymbol{D}^- u^{n+\frac{1}{2}}|^2\Big) + \frac{\Delta x^2}{4}\Big(\frac{3\epsilon}{\epsilon+2\delta} - 2\Big)\boldsymbol{D}_0\Big(|u^{n+\frac{1}{2}}|^2\Big)\cdot\Big(|\boldsymbol{D}^2 u^{n+\frac{1}{2}}|^2\Big)\bigg) \\
&\quad + \Delta t\epsilon\gamma\bigg(\Big(-\frac{\Delta x^2}{24}\big(\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\big)^2\cdot\big(\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\big) + \frac{\Delta x^2}{8}\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^4\cdot|\boldsymbol{D}^+u^{n+\frac{1}{2}}|^2\Big)\bigg) \\
&\quad + \delta\gamma\frac{\Delta x^2}{6}\Big(\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)^2\cdot\Big(\boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big) \\
&= E^{(n)} + \mathcal{O}(\Delta t + \Delta x^2)
\end{aligned}$$

hence, the theorem is proved. ■

2.4 Estimation of the trunctation error

We will define the truncation error τ_k^n, for an exact solution φ_k^n of (1), as

$$\begin{aligned}
\tau_k^n := \quad & iD_t\varphi_k^n + \alpha D_2\varphi_k^{n+\frac{1}{2}} + \gamma|\varphi_k^{n+\frac{1}{2}}|^2\varphi_k^{n+\frac{1}{2}} + i\beta D_3\varphi_k^{n+\frac{1}{2}} \\
& + i\frac{\delta}{2}\bigg[|\varphi_k^{n+\frac{1}{2}}|^2 D_0\varphi_k^{n+\frac{1}{2}} + D_0\Big(|\varphi_k^{n+\frac{1}{2}}|^2\varphi_k^{n+\frac{1}{2}}\Big) - \varphi_k^{n+\frac{1}{2}}D_0\Big(|\varphi_k^{n+\frac{1}{2}}|^2\Big)\bigg] + i\epsilon\varphi_k^{n+\frac{1}{2}}D_0\Big(|\varphi_k^{n+\frac{1}{2}}|^2\Big)
\end{aligned} \tag{45}$$

The following estimate holds:

Theorem 3 *Assuming that $\varphi^n = \varphi(x, t_n)$, solution of (1) at the time $t = t_n = n\Delta t$, is in $C^2\big[[-T,T], C^5\big]$, then there exist a real constant $C_0 > 0$ such that*

$$\Delta t\sum_{n=0}^{N}||\tau^{(n)}||^2 \le C_0^2 T(\Delta t^4 + \Delta x^4) \tag{46}$$

where $T = N\Delta t$.

Proof: for the exact solution φ^n, we will calculate its Taylor expansion over $(x_k, t_{\overline{n}}) := (x_k, t_n + \frac{\Delta t}{2})$ for each term in (45). It is known that the linear terms are approximations of order 2 in both space and time variables; hence, we will only study the nonlinear terms. We will also use the following notation: we shall use $\varphi_{x,k}^n$ to denote the partial derivative $\frac{\partial\varphi}{\partial x}$ evaluated in (x_k, t_n), $\varphi_{xx,k}^n$ will denote the derivative $\frac{\partial\varphi^2}{\partial x^2}$ over (x_k, t_n), and $\varphi_{xt,k}^n$ will denote the derivative $\frac{\partial\varphi^2}{\partial t\partial x}$. Similar forms will be used for further derivatives.

We will study each term separately:

- γ *term:* let us recall the following:

$$\begin{aligned}
\varphi_k^{n+1} &= \varphi_k^{\overline{n}} + \frac{\Delta t}{2}\varphi_{t,k}^{\overline{n}} + \frac{\Delta t^2}{4}\varphi_{tt,k}^{\overline{n}} + \mathcal{O}(\Delta t^3) \\
\varphi_k^{n} &= \varphi_k^{\overline{n}} - \frac{\Delta t}{2}\varphi_{t,k}^{\overline{n}} + \frac{\Delta t^2}{4}\varphi_{tt,k}^{\overline{n}} + \mathcal{O}(\Delta t^3)
\end{aligned}$$

hence,

$$\frac{\varphi_k^{n+1}+\varphi_k^n}{2} = \varphi_k^{\overline{n}} + \mathcal{O}(\Delta t^2) \tag{47}$$

then,

$$|\varphi_k^{n+\frac{1}{2}}|^2\varphi_k^{n+\frac{1}{2}} = |\varphi_k^{\overline{n}}|^2\varphi_k^{\overline{n}} + \mathcal{O}(\Delta t^2)$$

- *δ and ϵ terms.* We need to study the last three terms in (45). For the one multiplying $\frac{\delta}{2}$, due to (47), and because $\boldsymbol{D}_0$ is a second order aproximation of the first derivative, we can write

$$|\varphi_k^{n+\frac{1}{2}}|^2 D_0\varphi_k^{n+\frac{1}{2}} = |\varphi_k^{\overline{n}}|^2\varphi_{x,k}^{\overline{n}} + \mathcal{O}(\Delta t^2 + \Delta x^2).$$

Using similar arguments over the term multiplying $\frac{\delta}{2}$,

$$D_0\left(|\varphi_k^{n+\frac{1}{2}}|^2\right)\varphi_k^{n+\frac{1}{2}} = |\varphi_k^{\overline{n}}|_x^2\varphi_k^{\overline{n}} + \mathcal{O}(\Delta t^2 + \Delta x^2).$$

For the last term in (45),

$$\varphi_k^{n+\frac{1}{2}} D_0\left(|\varphi_k^{n+\frac{1}{2}}|^2\right) = \varphi_k^{\overline{n}}|\varphi_k^{\overline{n}}|_x^2 + \mathcal{O}(\Delta x^2 + \Delta t^2)$$

we can then conclude after combining all the previous results, extracting the square at both sides, summing from $n = 0$ to $n = N \in \mathbb{N}$, and multiplying by Δt for a constant $C_0 \in \mathbb{R}$ depending on the continuity conditions of φ. ■

3 Numerical Results

In this section we will show some numerical experiments with results supporting the theorems demonstrated in the previous sections. In particular, we will test the scheme with some known examples whose exact solutions are previously known. Finally, we will test the code for an initial condition representing two colliding solitons.

3.1 Computing Strategy

Let us consider the space domain $[-L, L] \subset \mathbb{R}$. The calculation of the numerical solution will consider periodic boundary conditions, i.e. $u_{-N}^n = u_N^n$, where $u_N^n = u(N\Delta x = L, t_n)$, $N \in \mathbb{N}$. Some comments on this regard must be made:

- A bounded domain in $\mathbb{R}$ is considered because of computational limitations. As such, the space domain will be discretized using $2N$ equally spaced grids. Also, the numerical solution u^n will be considered as a complex-valued vector with $2N$ elements for each timestep. Theorems proved in the previous sections will still hold.
- The finite difference operators $\boldsymbol{D}_0$, $\boldsymbol{D}_0^2$ and $\boldsymbol{D}_0^3$ can be represented as matrices in $\mathbb{R}^{2N\times 2N}$ operating over complex-valued vectors. Furthermore: because these operators will act only over the numerical solution u^n, and because

of our choice on the boundary conditions, their matrix representations can be written as:

$$\boldsymbol{D}_0 = \frac{1}{\Delta x}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

$$\boldsymbol{D}_0^2 = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

$$\boldsymbol{D}_0^3 = \frac{1}{\Delta x^3}\begin{bmatrix} 0 & -1 & \frac{1}{2} & & & -\frac{1}{2} & 1 \\ 1 & 0 & -1 & \frac{1}{2} & & & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ \frac{1}{2} & & & -\frac{1}{2} & 1 & 0 & -1 \\ -1 & \frac{1}{2} & & & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

- Therefore, the numerical scheme can be rewritten using matrix notation: for a given $u^0 \in \mathbb{C}^N$, the numerical scheme (9) allows us to compute $u^n \in \mathbb{C}^N$, the numerical approximation of the solution $u(\cdot, t_n) \in L^2(\Omega)$, as follows:

$$\left[-\frac{1}{\Delta t}\boldsymbol{I} + i\frac{\alpha}{2}\boldsymbol{D}_0^2 - \frac{\beta}{2}\boldsymbol{D}_0^3\right]u^{n+1} + \left[\frac{1}{\Delta t}\boldsymbol{I} + i\frac{\alpha}{2}\boldsymbol{D}_0^2 - \frac{\beta}{2}\boldsymbol{D}_0^3\right]u^n = \delta F_\delta(u^{n+1}) + \epsilon F_\epsilon(u^{n+1}) - i\gamma F_\gamma(u^{n+1}) \quad (48)$$

To compute the numerical solution, we will use a fixed-point method in order to solve equation (48) for each time-step. As in Delfour, Fortin and Payre [12], for a $u^{p=1} = u^n \in \mathbb{C}^{\mathbb{Z}}$ given, we compute a sequence of complex vectors $\{u^p\}, p = 2, 3, 4, \ldots,$ until a stopping criteria is verified. The sequence is given by

$$u^p = \left[-\frac{1}{\Delta t}\boldsymbol{I} + i\frac{\alpha}{2}\boldsymbol{D}_0^2 - \frac{\beta}{2}\boldsymbol{D}_0^3\right]^{-1} \quad (49)$$

$$\left[\delta F_\delta(u^{p-1}) + \epsilon F_\epsilon(u^{p-1}) - i\gamma F_\gamma(u^{p-1}) - \delta F_{-\delta}(u^{p-1}) - \left(\frac{1}{\Delta t}\boldsymbol{I} + i\frac{\alpha}{2}\boldsymbol{D}_0^2 - \frac{\beta}{2}\boldsymbol{D}_0^3\right)u^n\right] \quad (50)$$

In other words, we have to solve a linear system of equations many times per timestep until a stopping criterion is fulfilled, where the matrix to invert has a pentadiagonal structure. Because of the pentadiagonal structure of the matrix, and because of the periodic boundary conditions, we've used the method proposed by Navon [22]. The resulting pentadiagonal system (with no periodic boundary conditions) is solved using the algorithm PTRANS-II proposed by Askar and Karawia [4]. As a stopping criterion, we've considered two consecutive terms in the sequence $\{u^p\}, p = 1, 2, \ldots$. The fixed point scheme stops if we have

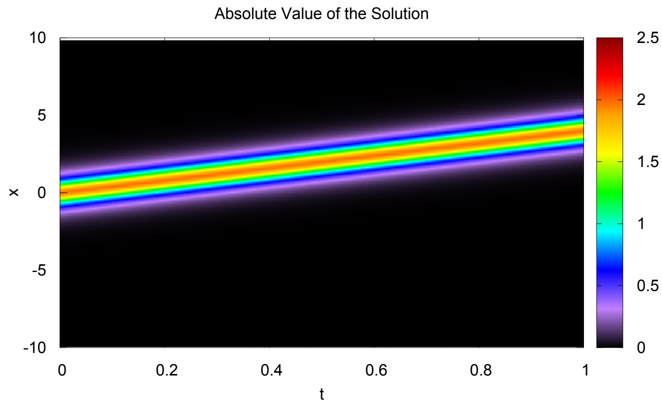
$$||u^p - u^{p-1}||_{L^2(\Omega)} < \hat{\delta}$$

where $\hat{\delta}$ is given. In that case, we do $u^{n+1} = u^p$ in order to continue with the next timestep. The scheme has linear convergence for Δt enoughly small.

3.2 Case 1: modified KdV equation for a 1-soliton

In (1), using $\alpha = \gamma = \epsilon = 0$, $\beta = 1$, $\delta = 6$, we can derive the modified KdV equation:

$$u_t + u_{xxx} + 6u^2 u_x = 0 \quad (51)$$



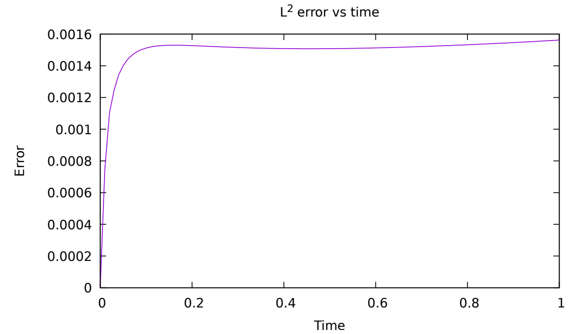


Figure 1: Case 1. Left: absolute value of the numerical solution for (51). Right: numerical error of the solution, in $\ell 2$ norm.

where an exact solution is given by

$$u(x,t) = 2sech(2x - 8t) \tag{52}$$

We have computed the numerical solution for this problem using the numerical scheme (48) with $u(x, 0)$ in (52) as an initial condition. The space domain is $\Omega = [-120\pi, 120\pi]$, and $t \in [0, 1]$, while $\Delta t = 10^{-2}$ is fixed. In the space variable, we've used $\Delta x = \frac{120\pi}{2^{14}} \approx 0.023$. Figure 1 left shows the time evolution of the soliton, and Figure1 right shows the numerical error of the numerical solution. The numerical L^2 norm is conserved with a difference of $\Delta \ell_2 = 3.597083 \cdot 10^{-15}$.

3.3 Case 2: solution for a HNLS equation.

The next example will consider a solution for (1) from Potasek and Tabor [27]:

$$u(x,t) = u_0 e^{i(nt+rx)} sech(kx + lt) \tag{53}$$

in (1), the equation parameters are $\alpha = \beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 0$, $\epsilon = 1$. Inside the fixed-point iteration, we have used $\hat{\delta} = 10^{-15}$. For $k = 1$, this yields $n \approx 0.733796$, $r = -\frac{1}{6}$, $l = -\frac{7}{24}$ and $|u_0|^2 = \frac{3}{2}$. Figure 2 presents the results of a simulation using $\Delta t = 0.01$ and $\Delta x = \frac{28\pi}{2^{15}} \approx 0.002684$. Figure 2 right shows the behavior of the error, while figure 2 left shows the absolute value of the numerical solution. There, the numerical L^2 norm is completely conserved; that is, $\Delta \ell_2 = 0$, while $\Delta E = 1.344651 \cdot 10^{-6}$. Both of those quantities are shown in figure 3.

3.4 Case 3: solution for another HNLS equation.

Let us now consider the solution given in Kumar and Chand [19]. In particular: we will consider the following solution:

$$u(x,t) = Ae^{i(-kt+\omega x+\theta)} sech(B(t - vx)) \tag{54}$$

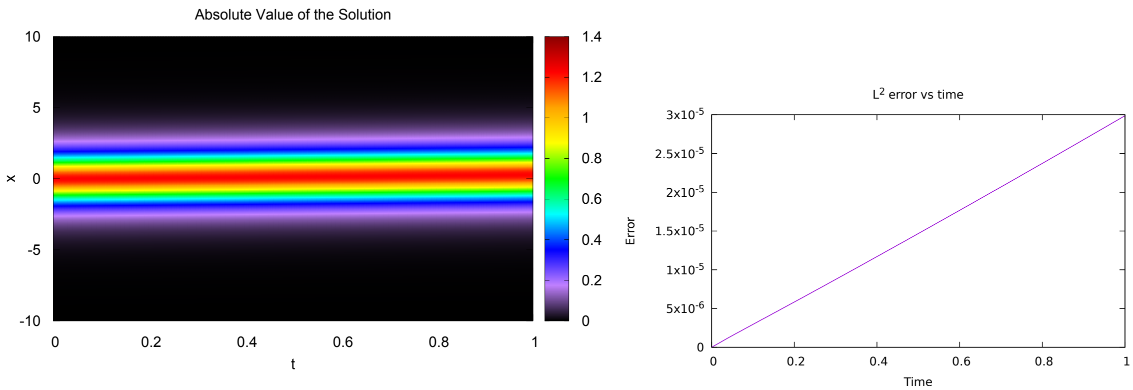


Figure 2: Case 2. Left: color plot of the numerical solution (in absolute value) of equation (53). Right : time behavior of the error.

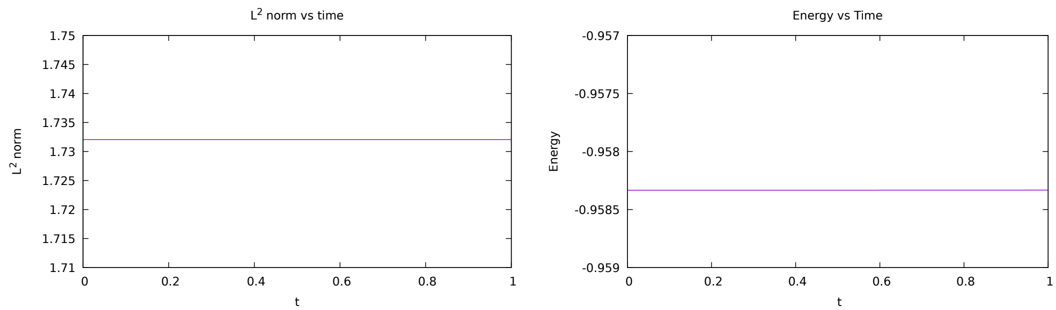


Figure 3: Case 2. Time evolution of the conserved quantities for the numerical solution.

where, besides the energy conservation condition $3\beta\gamma = \alpha(\epsilon + 2\delta)$, another condition is imposed in order to prove the existence of solitonic solutions: $\delta = -2\epsilon$. Furthermore,

$$\omega = \frac{\alpha v \pm \sqrt{\alpha^2 v^2 + 3\beta^2 v^3 B - 3\beta v}}{3\beta v}$$

$$B = \pm\frac{1}{v}\sqrt{\frac{k - \alpha\omega^2 + \beta\omega^3}{3\beta\omega - \alpha}}$$

$$A = \pm\sqrt{\frac{2(k - \alpha\omega^2 + \beta\omega^3)}{\delta\omega - \gamma}}$$

the parameters k and v can be chosen, and in our case $k = 0.0001$ and $v = 10$. With this parameters, a non-linear system is solved for ω in order to get the values of A and B. Figure 4 shows the time evolution and the numerical error of the solution. Figure 5 shows the evolution of the numerical L^2 norm and the energy, where $\Delta\ell_2 = 0$ and $\Delta E = 5.71114 \cdot 10^{-6}$

3.5 Case 4: colliding solitons for the KdV equation

Using again equation (**??**), let us consider the following function as initial condition describing two colliding solitons:

$$u(x, 0) = \sqrt{5}sech(\sqrt{5}(x + 5)) + sech(x - 5)$$

in this computation, $\Delta x = \frac{120\pi}{2^{14}}$, $\Delta t = 0.01$, $t \in [0, 1]$. Figure 6 shows the behavior of the solitons and the evolution of the numerical L^2 norm.

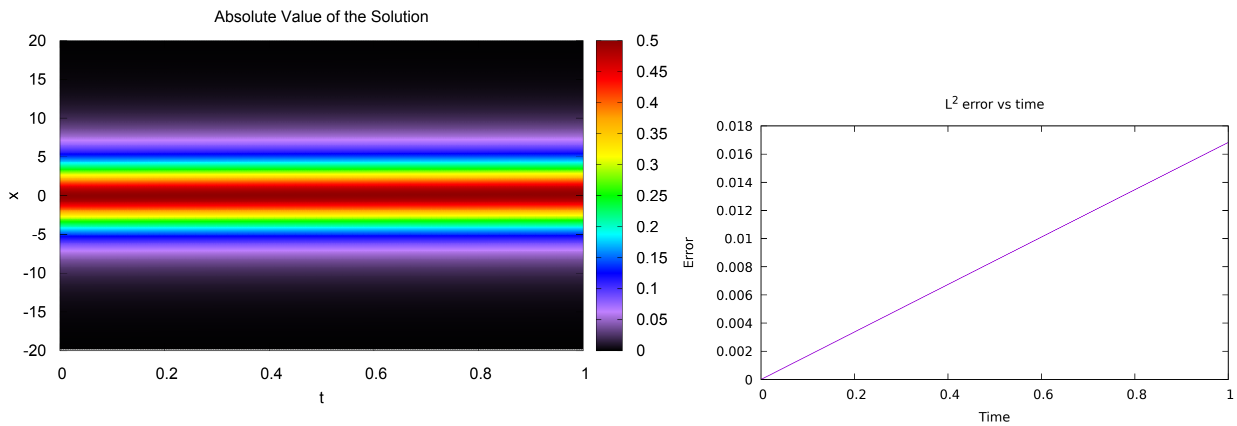


Figure 4: Case 3. Left: color plot of the numerical solution (in absolute value) of equation (53). Right : time behavior of the error.

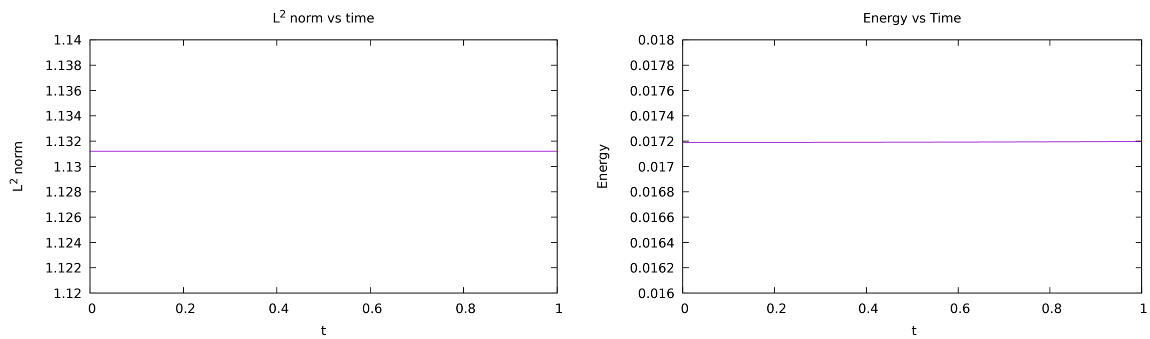


Figure 5: Case 3. Time evolution of the conserved quantities for the numerical solution.

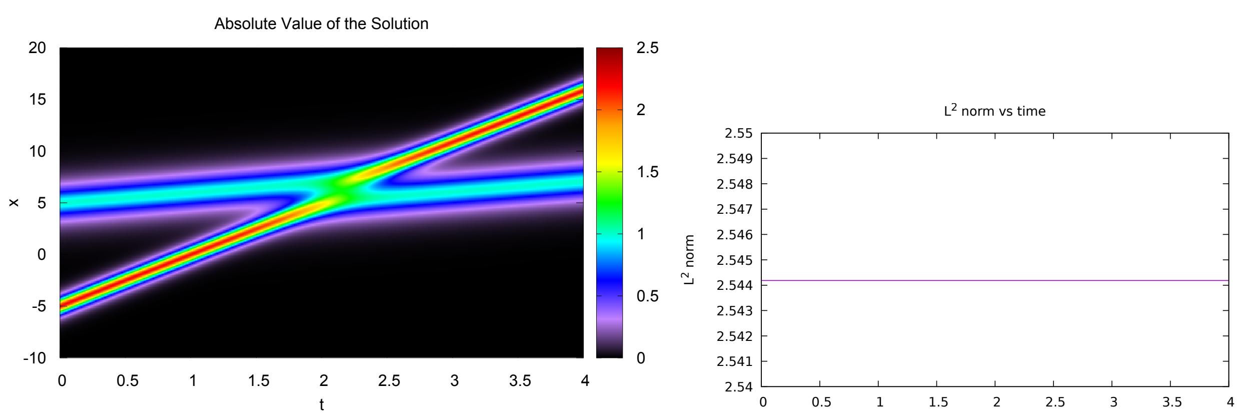


Figure 6: Case 3. Left: color plot of the numerical solution (in absolute value). Right : time behavior of the numerical L^2 norm.

3.6 Case 5: colliding solitons for a HNLS equation.

This experiment consist in use the following expression as initial condition [14]

$$u_0(x) := \frac{G(x)}{F(x)}.$$

Both functions are defined as follows:

$$F(x) = 1 + a(1,1^*)e^{\eta_1+\eta_1^*} + a(1,2^*)e^{\eta_1+\eta_2^*} + a(2,1^*)e^{\eta_2+\eta_1^*} + a(2,2^*)e^{\eta_2+\eta_2^*} + a(1,2,1^*,2^*)e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}$$
$$G(x) = e^{\eta_1} + e^{\eta_2} + a(1,2,1^*)e^{\eta_1+\eta_2+\eta_1^*} + a(1,2,2^*)e^{\eta_1+\eta_2+\eta_2^*}$$

where for $\omega := \frac{\alpha}{2\gamma}$ and $P_1, P_2 \in \mathbb{C}$, $i, j = 1, 2$,

$$\begin{aligned}
\eta_i &= P_i x - \eta_i^0 \\
\Omega_i &= -i\beta P_i^2 + \gamma P_i^3 \\
a(i,j^*) &= \frac{\omega}{(P_i + P_j^*)^2} \\
a(i,j) &= \frac{(P_i - P_j)^2}{\omega} \\
a(i^*,j^*) &= \frac{(P_i - P_j^*)^2}{\omega} \\
a(i,j,k^*) &= a(i,j)a(i,k^*)a(j,k^*) \\
a(i,j,k^*,l^*) &= a(i,j)a(i,k^*)a(i,l^*)a(j,k^*)a(j,l^*)a(k^*,l^*)
\end{aligned}$$

The constants P_1, P_2, η_1^0 and η_2^0 can be chosen in order to obtain colliding solitons. For our experiment, we have done $P_1 = -1$, $P_2 = \frac{1}{2}$, $\eta_1^0 = -1$, and $\eta_2^0 = 1$. In (1), $\alpha = 0.5$, $\beta = -1$, $\gamma = 1$, $\delta = -6$ and $\epsilon = 0$. Also, $\Delta t = 0.01$ and $\Delta x = \frac{120}{2^{14}}$.

Figure 7 shows the behavior of the solitons. Figure 8 shows the behavior of the discrete L^2 norm and the energy. The energy is not preserved in this example. This is maybe due to the nature of the iniial condition.

4 Conclusion

We have proposed a new way to solve equation (1) using a finite difference scheme. The procedure involved the re-writing of a particular nonlinearity as a convex combination in order to get the conservation of the numerical L^2 norm. The numerical energy can also be conserved for an appropiate initial condition. The algorithm proposed in this paper can be programmed with ease in any computer with a linear algebra library available. This work can be an inspiration for solving other problems involving higher order derivatives and nonlinearities, and can be easely adapted to work on other problems involving full damping terms like in [23], [33], and localized weak damping like [6] and [10], as well as when the damping coefficient $\lambda(x,t)$ may vanish at infinity [9].

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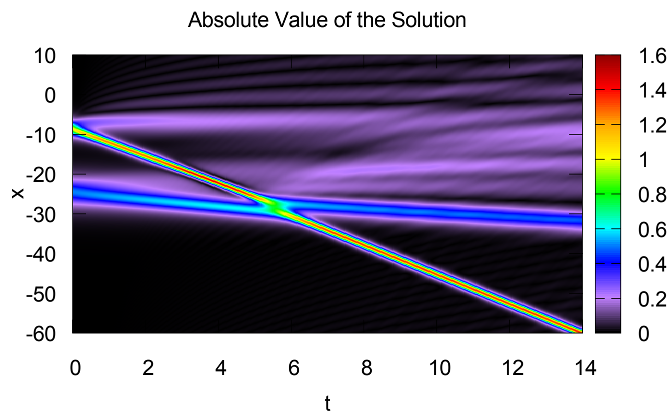
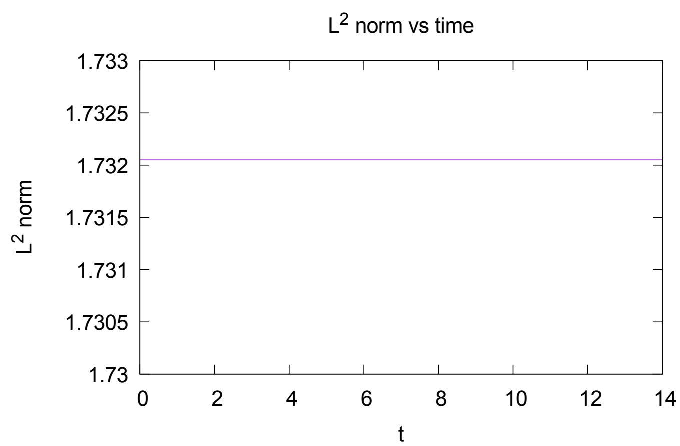


Figure 7: Case 5. Color plot of the numerical solution (in absolute value).



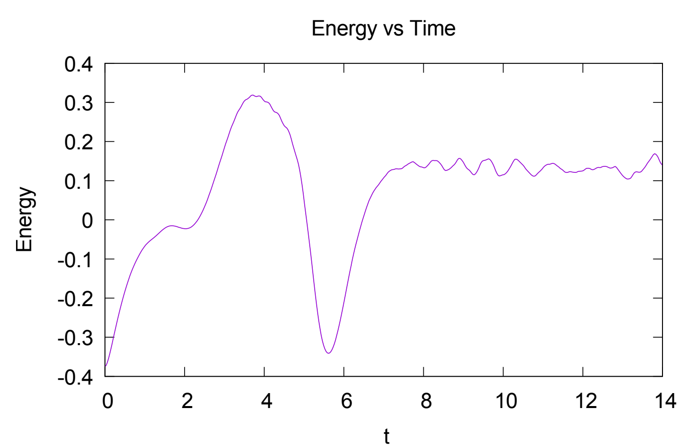


Figure 8: Case 5. Time evolution of the numerical L^2 norm and the energy for the numerical solution.

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