A mixed-primal finite element method for the Boussinesq problem with temperature-dependent viscosity*

Javier A. Almonacid† Gabriel N. Gatica‡ Ricardo Oyarzúa§

Abstract

In this paper we focus on the analysis of a mixed finite element method for a class of natural convection problems in two dimensions. More precisely, we consider a system based on the coupling of the steady-state equations of momentum (Navier-Stokes) and thermal energy by means of the Boussinesq approximation (coined the Boussinesq problem), where we also take into account a temperature dependence of the viscosity of the fluid. The construction of this finite element method begins with the introduction of the pseudostress and vorticity tensors, and a mixed formulation for the momentum equations, which is augmented with Galerkin-type terms, in order to deal with the non-linearity of these equations and the convective term in the energy equation, where a primal formulation is considered. The prescribed temperature on the boundary becomes an essential condition, which is weakly imposed, leading us to the definition of the normal heat flux through the boundary as a Lagrange multiplier. We show that this highly coupled problem can be uncoupled and analysed as a fixed-point problem, where Banach and Brouwer theorems will help us to provide sufficient conditions to ensure well-posedness of the problems arising from the continuous and discrete formulations, along with several applications of continuous injections guaranteed by the Rellich-Kondrachov theorem. Finally, we show some numerical results to illustrate the performance of this finite element method, as well as to prove the associated rates of convergence.

1 Introduction

Natural convection is a heat transfer process that is present is our everyday life: from the cooling of little electronic devices, to indoor climate systems, to environmental transport problems. Unlike what happens in forced convection (where the fluid flow is driven by external sources, e.g. a fan), buoyant forces arising from density variations constitute the main cause of movement. When these variations are small around an operating density (cf. [14]), and they depend solely on the temperature of the fluid, then the problem can be modelled using the equations of momentum (Navier-Stokes), mass and energy conservation, coupled by means of the Boussinesq approximation, what is commonly known as the Boussinesq equations, or simply, the Boussinesq problem. The devise of new finite element methods to approximate the solution of these equations has seen an increasing interest from the mathematical community. For instance, the problem with constant coefficients has been already considered in several works, in both primal and mixed-type formulations (see, e.g. [7, 13, 19], and [17, 15, 16, 22],

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†CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: jalmonacid@ci2ma.udec.cl.

‡CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

§GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl.

respectively, and the references therein). In particular, the authors in [15] propose an augmented mixed-primal formulation for the problem, where the sought quantities are the pseudostress, the velocity, the temperature and the normal heat flux through the boundary. Under sufficiently small data, they are able to prove that, when Raviart-Thomas elements are used to approximate the pseudostress, Lagrange elements for the velocity and temperature, and discontinuous piecewise polynomials for the normal heat flux, then the finite element method is optimally-convergent. Similarly in [17], the authors propose two formulations for this problem, each of them based on a dual-mixed formulation for the momentum equation, and a primal and mixed-primal one for the energy equation. Thus, when the velocity, trace-free gradient and normal heat flux are approximated by discontinuous piecewise polynomials, the stress by Raviart-Thomas elements and the temperature by Lagrange elements, the finite element methods are also optimally-convergent provided the data is sufficiently small.

On the other hand, there are several examples where an increase in the temperature of the fluid can produce a strong variation of its viscosity (even in isobaric conditions) such as the case of oils, lubricants, metal alloys and the magma beneath the surface of the earth, to name a few, meaning that the consideration of a temperature-dependent viscosity will provide a better quality model, at the cost of increasing the non-linearity of these equations. For instance, in a related context, the authors in [2] deal with a coupled flow-transport problem where the kinematic effective viscosity, the diffusion coefficient and the one-dimensional flux function describing hindered settling depend non-linearly on the concentration of species; a problem that under minor modifications, becomes a simplification of the Boussinesq equations, as the convective term in the Navier-Stokes equations is not present here. They propose a mixed-primal formulation, which turns out to be well-posed, and the corresponding finite element method is optimally convergent under smallness-of-data assumptions (the same approach is later applied to a more general case of this problem in [3] for a sedimentation-consolidation system, and to the a posteriori error analysis of it in [4]). In these works, the presence of variable parameters make the analysis more difficult, as the decoupling of the unknowns usually requires the usage of non-conventional embeddings and fixed-point strategies.

However, up to our knowledge, the full Boussinesq problem with temperature-dependent parameters is something that has not had great attention, until now (see, e.g. [29, 30, 31, 35, 36] and the references therein). Indeed, works such as [35] (and a stabilized version of it recently in [36]) deal with the unsteady problem, where backward euler discretization is used in time, and conforming finite elements in space, although the problem is linearized using information from the solution in the previous timestep. More recently, in [30] a conforming finite element method is developed for the problem with temperature-dependent parameters (viscosity and thermal conductivity) and Dirichlet boundary conditions. The finite element approximation is done using a pair of Stokes-stable elements for the velocity and pressure (Taylor-Hood and MINI-element), Lagrange elements for the temperature and discontinuous piecewise polynomials for the normal heat flux through the boundary, yielding an optimally convergent method, whose well-posedness is based on the assumption that the exact velocity and temperature live in $W^{1,\infty}(\Omega)$.

According to the above, we extend the results given by [15] to the case where the viscosity of the fluid depends on the temperature, considering in addition the original Cauchy stress tensor in the Navier-Stokes equations. To this end, we will introduce the pseudostress and vorticity tensors as new variables to construct a mixed formulation for the momentum equations, whereas for the energy equation we will consider a primal formulation, along with the introduction of the normal heat flux through the boundary as a Lagrange multiplier. Next, to achieve conformity and well-definiteness of the involved terms in the variational formulation, redundant Galerkin-type terms are included (similarly to what has been done in [2, 3, 9, 10, 11, 15] for coupled flow-transport, Boussinesq, Navier-Stokes, and related problems). Then, the well-posedness of the continuous and discrete problems will be proved using besides smallness-of-data assumptions, fixed-point arguments; a tool basically used in all the works referenced here so far. In particular, we use the fixed-point approach described in [15] that uncouples the problem into two formulations, one related to the mixed formulation of the momentum equations, and the other one to the primal formulation of the energy equation, which allows us to reuse the results for the latter problem. We then fulfill the hypotheses of the Banach and Brouwer fixed-point theorems for the continuous and discrete problems, respectively. In both cases, inspired by the techniques used in [2], the continuity of the operator is proved based on continuous injections guaranteed by the Rellich-Kondrachov and Sobolev embedding theorems. Finally, the finite element method is constructed with Raviart-Thomas elements of order k to approximate the pseudostress, Lagrange elements of order $k+1$ for the velocity and temperature, and discontinuous piecewise polynomials of degree $\leq k$ for the vorticity and normal heat flux through the boundary, which yields optimal a priori error estimates.

1.1 Outline

The rest of this work is organized as follows. First, we end this section by introducing some notation that will be used throughout the paper. Next, in Section 2, the Boussinesq problem is formally introduced, along with assumptions on the given data, to then rewrite the momentum equation in pseudostress-velocity-vorticity formulation. In Section 3, an augmented mixed-primal formulation is proposed, and the fixed-point approach that uncouples the problem is presented. Then, the well-posedness of the problem is proved by means of the Lax-Milgram theorem, the Babuška-Brezzi theory and the Banach fixed-point theorem. Next, in Section 4, an argument similar to the one applied in the previous section provides the well-posedness of the Galerkin scheme, but this time, thanks to the Brouwer fixed-point theorem. Then, after a specific choice of finite element subspaces, the corresponding a priori error estimates are derived in Section 5, to finally in Section 6 present some numerical examples that validate these results and illustrate the good performance of our augmented mixed-primal finite element method.

1.2 Preliminaries

Let us denote by $\Omega \subset R^2$ a given bounded domain with polyhedral boundary Γ, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{s,2}(\Omega) =: H^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $H^{1/2}(\Gamma)$ is the space of traces of functions in $H^1(\Omega)$ and $H^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,2}$ and $\mathbf{w} = (w_i)_{i=1,2}$, we set the gradient, divergence and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{2} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,2}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,2}$, we let $\mathbf{div}\,\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,2}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := R^{2\times 2}$. Furthermore, we recall that

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\right\},$$

equipped with the usual norm

$$\| \boldsymbol{\tau} \|_{\mathbf{div};\Omega}^2 := \| \boldsymbol{\tau} \|_{0,\Omega}^2 + \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2,$$

is a standard Hilbert space in the realm of mixed problems. Finally, in what follows, $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := R^2$. Also, we employ $\mathbf{0}$ to denote a generic null vector and use C, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2 The model

We begin by introducing formally the Boussinesq problem, along with assumptions on the data and the introduction of further notation used in this work.

2.1 The Boussinesq Equations

We are interested in obtaining the steady state of a non-isothermal, incompressible, Newtonian fluid flow in the region Ω. Hence, we consider the equations of momentum (Navier-Stokes), mass and thermal energy conservation, coupled by means of the Boussinesq approximation (cf. [14]). The problem (without dimensionless numbers for readability purposes) reads: Find a velocity field $\mathbf{u}$, a pressure field p and a temperature field φ such that

$$-\mathbf{div}\,(\mu(\varphi)\mathbf{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p - \varphi\mathbf{g} = 0 \quad \text{in } \Omega, \tag{2.1a}$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$-\mathrm{div}\,(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi = 0 \quad \text{in } \Omega, \tag{2.1c}$$

where $\mathbf{e}(\mathbf{u})$ is the strain rate tensor, which corresponds to the symmetric part of the velocity gradient tensor $\nabla\mathbf{u}$, that is, for any velocity $\mathbf{v}$,

$$\mathbf{e}(\mathbf{v}) := \frac{1}{2}\Big\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathtt{t}}\Big\},$$

$-\mathbf{g} \in \mathbf{L}^\infty(\Omega)$ is an external force per unit mass (e.g. gravity force, centrifugal force, coriolis force) , $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ is a uniformly positive definite tensor describing the thermal conductivity of the fluid (thus allowing the possibility of an anisotropy of the material, cf. [28]) and $\mu : R \to R^+$ is a temperature-dependent viscosity function, which is assumed to be bounded above and below by positive constants, that is, there exist $\mu_2 \geq \mu_1 > 0$ such that

$$\mu_1 \leq \mu(s) \leq \mu_2 \quad \forall\, s \in R. \tag{2.2}$$

We also assume that μ is a Lipschitz continuous function, that is, there exists $L_\mu > 0$ such that

$$|\mu(s) - \mu(t)| \leq L_\mu |s - t| \quad \forall\, s,t \in R. \tag{2.3}$$

Examples of temperature-dependent viscosity functions may include exponential and power-law correlations (see, e.g. [32])

$$\mu(s) = \exp\left(A + \frac{B}{s - s_0}\right), \quad \mu(s) = A(s - s_0)^B, \quad \forall\, s \in R,$$

where A, B are constants and s_0 is a reference temperature. It is worth noting that usually these functions are valid only in a predefined range of temperatures, something that may provide feasible bounds for (2.2).

In turn, concerning boundary conditions for the system (2.1), we consider Dirichlet conditions in both velocity and temperature:

$$\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma, \tag{2.4}$$

and

$$\varphi = \varphi_D \quad \text{on } \Gamma, \tag{2.5}$$

with $u_D \in \mathbf{H}^{1/2}(\Gamma)$ and $\varphi_D \in H^{1/2}(\Gamma)$. Here $\mathbf{u}_D$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} = 0, \tag{2.6}$$

which comes from an application of the divergence theorem when integrating over Ω the incompressibility condition (2.1b).

2.2 Introduction of the Pseudostress and Vorticity Tensors

Let $\boldsymbol{\sigma}$ be the pseudostress tensor defined as

$$\boldsymbol{\sigma} := \mu(\varphi)\mathbf{e}(\mathbf{u}) - \mathbf{u} \otimes \mathbf{u} - p\mathbb{I}. \tag{2.7}$$

Then, by taking trace in both sides of the previous equation, and using the incompressibility condition, it is possible to show that the pressure can be postprocessed as follows:

$$p = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u} \otimes \mathbf{u}). \tag{2.8}$$

Moreover, let $\boldsymbol{\omega}(\mathbf{v})$ be the skew-symmetric part of the tensor $\nabla\mathbf{v}$, that is,

$$\boldsymbol{\omega}(\mathbf{v}) = \frac{1}{2}\Big\{\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\Big\},$$

for any vector field $\mathbf{v}$, and let $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$ be the space of skew-symmetric tensors with components in $L^2(\Omega)$, i.e.,

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) := \{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\}.$$

Then, in what follows, we consider the vorticity tensor $\boldsymbol{\gamma}$ defined as

$$\boldsymbol{\gamma} := \boldsymbol{\omega}(\mathbf{u}) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega). \tag{2.9}$$

Thus, introducing this quantity in (2.7), and taking into account the new constitutive equation arising from the pseudostress definition when the pressure is taken as in (2.8), the associated boundary value problem becomes: Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi)$ such that

$$\nabla\mathbf{u} - \boldsymbol{\gamma} - \frac{1}{\mu(\varphi)}\,(\mathbf{u} \otimes \mathbf{u})^{\mathtt{d}} = \frac{1}{\mu(\varphi)}\,\boldsymbol{\sigma}^{\mathtt{d}} \quad \text{in } \Omega, \tag{2.10a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma} - \varphi\mathbf{g} = 0 \qquad \text{in } \Omega, \tag{2.10b}$$

$$-\mathrm{div}\,(\mathbb{K}\nabla\varphi) + \mathbf{u} \cdot \nabla\varphi = 0 \qquad \text{in } \Omega, \tag{2.10c}$$

$$\mathbf{u} = \mathbf{u}_D \qquad \text{on } \Gamma, \tag{2.10d}$$

$$\varphi = \varphi_D \qquad \text{on } \Gamma, \tag{2.10e}$$

$$\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u} \otimes \mathbf{u}) = 0. \tag{2.10f}$$

Notice here that the incompressibility condition is implicitly present in (2.10a). This can be shown by taking trace in both sides of this equation, having in mind that $\mathrm{tr}(\nabla\mathbf{u}) = \mathrm{div}\,\mathbf{u}$ and $\mathrm{tr}(\boldsymbol{\gamma}) = 0$. Also, uniqueness of a pressure solution of (2.1) is ensured with (2.10f) for it implies (according to (2.8)) that p lies in $L^2_0(\Omega) := \Big\{p \in L^2(\Omega) : \int_\Omega p = 0\Big\}$ (cf., e.g. [26]).

3 The Continuous Formulation

3.1 The Augmented Mixed-Primal Formulation

In this section, we derive a weak formulation of the problem (2.10). Multiplying the constitutive equation (2.10a) by a test function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, integrating by parts, and using the Dirichlet condition (2.10d), we obtain

$$\int_\Omega \frac{1}{\mu(\varphi)}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \frac{1}{\mu(\varphi)}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega). \tag{3.1}$$

In turn, the momentum equilibrium equation (2.10b) can be rewritten as

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega \varphi\mathbf{g}\cdot\mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega). \tag{3.2}$$

Next, for the energy equilibrium equation (2.10c), we consider an additional variable $\lambda := -\mathbb{K}\nabla\varphi\cdot\boldsymbol{\nu}$ on Γ, which is nothing but the normal heat flux through the boundary. Then, multiplying (2.10c) by a test function $\psi \in H^1(\Omega)$ and integrating by parts, it follows that

$$\int_\Omega \mathbb{K}\nabla\varphi\cdot\nabla\psi + \langle \lambda, \psi \rangle_\Gamma = -\int_\Omega \psi\mathbf{u}\cdot\nabla\varphi \quad \forall\, \psi \in H^1(\Omega), \tag{3.3}$$

where $\langle \cdot,\cdot \rangle_\Gamma$ stands for the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. On the other hand, we incorporate the Dirichlet condition (2.10e) as

$$\langle \xi, \varphi \rangle_\Gamma = \langle \xi, \varphi_D \rangle_\Gamma \quad \forall\, \xi \in H^{-1/2}(\Gamma), \tag{3.4}$$

whereas the symmetry of the pseudostress tensor is imposed by

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = 0 \quad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega). \tag{3.5}$$

Notice that, due to the tensor product in (3.1) and the term in the right-hand side of (3.3), $\mathbf{u}$ must live in a smaller space than $\mathbf{L}^2(\Omega)$. Indeed, by applying the Cauchy-Schwarz and Hölder inequalities, and then the continuous injection from $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ (cf. [1, Theorem 4.12], [33, Theorem 1.3.4]), we find that there exists positive constants $c_1(\Omega)$ and $c_2(\Omega)$ such that

$$\left|\int_\Omega (\mathbf{u}\otimes\mathbf{w})^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}\right| \le c_1(\Omega)\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\mathbf{w}\,\|_{1,\Omega}\|\,\boldsymbol{\tau}\,\|_{0,\Omega} \quad \forall\, \mathbf{u},\mathbf{w} \in \mathbf{H}^1(\Omega),\ \forall\, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega), \tag{3.6}$$

and

$$\left|\int_\Omega \psi\mathbf{u}\cdot\nabla\varphi\right| \le c_2(\Omega)\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\psi\,\|_{1,\Omega}|\varphi|_{1,\Omega} \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega) \quad \forall\, \varphi,\psi \in H^1(\Omega). \tag{3.7}$$

In this way, the variational formulation would be given, at first glance, by: Find $(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma},\varphi,\lambda) \in \mathbb{H}(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\times H^1(\Omega)\times H^{-1/2}(\Gamma)$ such that $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}+\mathbf{u}\otimes\mathbf{u}) = 0$, and

$$\int_\Omega \frac{1}{\mu(\varphi)}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \frac{1}{\mu(\varphi)}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma, \tag{3.8a}$$

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = \int_\Omega \varphi\mathbf{g}\cdot\mathbf{v}, \tag{3.8b}$$

$$\int_\Omega \mathbb{K}\nabla\varphi\cdot\nabla\psi + \langle \lambda, \psi \rangle_\Gamma = -\int_\Omega \psi\mathbf{u}\cdot\nabla\varphi, \tag{3.8c}$$

$$\langle \xi, \varphi \rangle_\Gamma = \langle \xi, \varphi_D \rangle_\Gamma, \tag{3.8d}$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}, \psi, \xi) \in \mathbb{H}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$. However, notice also that if $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ is a solution to (3.8), then, given any $d \in R$, $(\boldsymbol{\sigma} + d\mathbb{I}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$, is also a solution to this problem. To avoid this non-uniqueness issue, we consider the orthogonal decomposition (cf., e.g. [24, 33])

$$\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus R\mathbb{I}, \tag{3.9}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0 \right\}.$$

More precisely, for each $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega)$, it is known that there exists a unique $\boldsymbol{\zeta}_0 := \boldsymbol{\zeta} - \left(\frac{1}{2|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) \right) \mathbb{I}$ $\in \mathbb{H}_0(\mathbf{div};\Omega)$ and $c := \frac{1}{2|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) \in R$ such that

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_0 + c\mathbb{I}. \tag{3.10}$$

Then, the variational formulation (3.8) can be reformulated in terms of the $\mathbb{H}_0(\mathbf{div};\Omega)$-component of the pseudostress. The equivalence of these problems is addressed next.

Lemma 3.1. *Let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda) \in \mathbb{H}(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ be a solution to (3.8). Then, there exists $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$ defined as*

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \left(\frac{1}{2|\Omega|} \int_\Omega \mathrm{tr}(\mathbf{u} \otimes \mathbf{u}) \right) \mathbb{I} \tag{3.11}$$

such that $(\boldsymbol{\sigma}_0, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ satisfies

$$\int_\Omega \frac{1}{\mu(\varphi)} \boldsymbol{\sigma}_0^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \int_\Omega \mathbf{u} \cdot \mathbf{div}\, \boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \frac{1}{\mu(\varphi)} (\mathbf{u} \otimes \mathbf{u})^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma, \tag{3.12a}$$

$$- \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\sigma}_0 - \int_\Omega \boldsymbol{\sigma}_0 : \boldsymbol{\eta} = \int_\Omega \varphi \mathbf{g} \cdot \mathbf{v}, \tag{3.12b}$$

$$\int_\Omega \mathbb{K} \nabla \varphi \cdot \nabla \psi + \langle \lambda, \psi \rangle_\Gamma = - \int_\Omega \psi \mathbf{u} \cdot \nabla \varphi, \tag{3.12c}$$

$$\langle \xi, \varphi \rangle_\Gamma = \langle \xi, \varphi_D \rangle_\Gamma, \tag{3.12d}$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}, \psi, \xi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$. Conversely, if $(\boldsymbol{\sigma}_0, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ $\in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ is a solution to (3.12), then $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$, with $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$ satisfying (3.11), is also a solution of (3.8).

Proof. Let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ be a solution to (3.8). Then, since $\boldsymbol{\sigma}$ satisfies $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u} \otimes \mathbf{u}) = 0$, it is clear from (3.10) that $\boldsymbol{\sigma}_0$ defined as (3.11) is the $\mathbb{H}_0(\mathbf{div};\Omega)$-part of the orthogonal decomposition of $\boldsymbol{\sigma}$. Thus, it follows that $(\boldsymbol{\sigma}_0, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ indeed satisfies (3.12). Conversely, if $(\boldsymbol{\sigma}_0, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ satisfies (3.12), then using the fact that $\mathrm{tr}(\boldsymbol{\eta}) = 0$, $\forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, it readily follows that $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$, with $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 - \left(\frac{1}{2|\Omega|} \int_\Omega \mathrm{tr}(\mathbf{u} \otimes \mathbf{u}) \right) \mathbb{I}$ satisfies equations (3.12a)–(3.12d) and the identity $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u} \otimes \mathbf{u}) = 0$ holds. Hence, by taking the orthogonal decomposition of the test function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$ and applying the compatibility condition (2.6) as

$$0 = \int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} = \langle (d\mathbb{I})\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma \quad \forall\, d \in R,$$

we deduce that $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi, \lambda)$ satisfies (3.8), which concludes the proof. $\square$

Therefore, our analysis continues from the variational formulation (3.12), but re-denoting $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$. On the other hand, the fact that now $\mathbf{u} \in \mathbf{H}^1(\Omega)$ leads us to augment (3.12) with Galerkin terms that will allow us to effectively analyse the variational formulation:

$$\kappa_1 \int_\Omega \left\{ \mathbf{e}(\mathbf{u}) - \frac{1}{\mu(\varphi)}(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} - \frac{1}{\mu(\varphi)}\boldsymbol{\sigma}^{\mathtt{d}} \right\} : \mathbf{e}(\mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{3.13}$$

$$\kappa_2 \int_\Omega (\mathbf{div}\,\boldsymbol{\sigma} + \varphi\mathbf{g}) \cdot \mathbf{div}\,\boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega), \tag{3.14}$$

$$\kappa_3 \int_\Omega \left\{ \boldsymbol{\gamma} - \boldsymbol{\omega}(\mathbf{u}) \right\} : \boldsymbol{\eta} = 0 \qquad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega), \tag{3.15}$$

$$\kappa_4 \int_\Gamma \mathbf{u}\cdot\mathbf{v} = \kappa_4 \int_\Gamma \mathbf{u}_D \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{3.16}$$

where κ_1, κ_2, κ_3 and κ_4 are positive parameters to be specified later on. Notice that these terms arise from the constitutive equation (2.10a), the equilibrium equation (2.10b), the definition of the vorticity (2.9), and the boundary condition for $\mathbf{u}$ (2.10d).

Throughout the rest of the paper, we denote

$$\vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), \quad \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), \quad \vec{\boldsymbol{\varrho}} := (\boldsymbol{\varrho}, \mathbf{s}, \boldsymbol{\vartheta}) \tag{3.17}$$

as elements of $\mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$. In this way, we arrive at the following augmented mixed-primal formulation: Find $(\vec{\boldsymbol{\sigma}}, (\varphi,\lambda)) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\mathbf{A}_\varphi(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) + \mathbf{B}_{\mathbf{u},\varphi}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = F_\varphi(\vec{\boldsymbol{\tau}}) + F_D(\vec{\boldsymbol{\tau}}), \tag{3.18a}$$

$$\mathbf{a}(\varphi,\psi) + \mathbf{b}(\psi,\lambda) = F_{\mathbf{u},\varphi}(\psi), \tag{3.18b}$$

$$\mathbf{b}(\varphi,\xi) = G(\xi), \tag{3.18c}$$

for all $(\vec{\boldsymbol{\tau}}, (\psi,\xi)) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$, where, given an arbitrary $(\mathbf{w},\phi) \in \mathbf{H}^1(\Omega) \times H^1(\Omega)$, the forms $\mathbf{A}_\phi, \mathbf{B}_{\mathbf{w},\phi}$, $\mathbf{a}$, $\mathbf{b}$, and the functionals F_D, F_ϕ, $F_{\mathbf{w},\phi}$ and G are defined as

$$\begin{aligned}\mathbf{A}_\phi(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) := & \int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathtt{d}} : \left\{ \boldsymbol{\tau}^{\mathtt{d}} - \kappa_1 \mathbf{e}(\mathbf{v}) \right\} + \int_\Omega (\mathbf{u} + \kappa_2 \mathbf{div}\,\boldsymbol{\sigma}) \cdot \mathbf{div}\,\boldsymbol{\tau} + \kappa_1 \int_\Omega \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) \\ & + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} + \kappa_3 \int_\Omega \left\{ \boldsymbol{\gamma} - \boldsymbol{\omega}(\mathbf{u}) \right\} : \boldsymbol{\eta} + \kappa_4 \int_\Gamma \mathbf{u}\cdot\mathbf{v},\end{aligned} \tag{3.19}$$

$$\mathbf{B}_{\mathbf{w},\phi}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) := -\int_\Omega \frac{1}{\mu(\phi)}(\mathbf{u}\otimes\mathbf{w})^{\mathtt{d}} : \left\{ \kappa_1 \mathbf{e}(\mathbf{v}) - \boldsymbol{\tau}^{\mathtt{d}} \right\}, \tag{3.20}$$

for all $\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$;

$$\mathbf{a}(\varphi,\psi) := \int_\Omega \mathbb{K}\nabla\varphi\cdot\nabla\psi, \tag{3.21}$$

for all $\varphi, \psi \in H^1(\Omega)$;

$$\mathbf{b}(\psi,\xi) := \langle\, \xi, \psi \,\rangle_\Gamma, \tag{3.22}$$

for all $(\psi,\xi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$;

$$F_D(\vec{\boldsymbol{\tau}}) := \langle\, \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \,\rangle_\Gamma + \kappa_4 \int_\Gamma \mathbf{u}_D\cdot\mathbf{v}, \tag{3.23}$$

$$F_\phi(\vec{\boldsymbol{\tau}}) := \int_\Omega \phi \mathbf{g} \cdot (\mathbf{v} - \kappa_2 \mathbf{div}\, \boldsymbol{\tau}), \tag{3.24}$$

for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$;

$$F_{\mathbf{w},\phi}(\psi) = -\int_\Omega \psi \mathbf{w} \cdot \nabla \phi, \tag{3.25}$$

for all $\psi \in H^1(\Omega)$; and

$$G(\xi) = \langle\, \xi, \varphi_D \,\rangle_\Gamma, \tag{3.26}$$

for all $\xi \in H^{-1/2}(\Gamma)$.

Having defined the forms $\mathbf{A}_\phi$ and $\mathbf{B}_{\mathbf{w},\phi}$, the following properties can be proved by simple algebraical manipulations.

Lemma 3.2. *Let* $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}^1(\Omega)$*;* $\phi, \phi_1, \phi_2 \in H^1(\Omega)$ *and* $\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$*. Then, the following properties hold*

i) $(\mathbf{A}_{\phi_1} - \mathbf{A}_{\phi_2})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = \displaystyle\int_\Omega \frac{\mu(\phi_2) - \mu(\phi_1)}{\mu(\phi_1)\mu(\phi_2)} \boldsymbol{\sigma}^{\mathtt{d}} : \Big\{ \boldsymbol{\tau}^{\mathtt{d}} - \kappa_1 \mathbf{e}(\mathbf{v}) \Big\},$

ii) $(\mathbf{B}_{\mathbf{w},\phi_1} - \mathbf{B}_{\mathbf{w},\phi_2})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = \displaystyle\int_\Omega \frac{\mu(\phi_2) - \mu(\phi_1)}{\mu(\phi_1)\mu(\phi_2)} (\mathbf{u} \otimes \mathbf{w})^{\mathtt{d}} : \Big\{ \boldsymbol{\tau}^{\mathtt{d}} - \kappa_1 \mathbf{e}(\mathbf{v}) \Big\},$

iii) $(\mathbf{B}_{\mathbf{w}_1,\phi} - \mathbf{B}_{\mathbf{w}_2,\phi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = \displaystyle\int_\Omega \frac{1}{\mu(\phi)} \Big\{ \mathbf{u} \otimes (\mathbf{w}_1 - \mathbf{w}_2) \Big\}^{\mathtt{d}} : \Big\{ \boldsymbol{\tau}^{\mathtt{d}} - \kappa_1 \mathbf{e}(\mathbf{v}) \Big\}.$

3.2 A Fixed-Point Approach

Although (3.18) is a strongly coupled problem, it can be uncoupled using a fixed-point approach (see, e.g. [2, 3, 15, 16]) . Indeed, let $\mathbf{H} := \mathbf{H}^1(\Omega) \times H^1(\Omega)$ and consider the operator: $\mathbf{S} : \mathbf{H} \to \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ defined by

$$\mathbf{S}(\mathbf{w},\phi) = (\mathbf{S}_1(\mathbf{w},\phi), \mathbf{S}_2(\mathbf{w},\phi), \mathbf{S}_3(\mathbf{w},\phi)) := \vec{\boldsymbol{\sigma}}, \tag{3.27}$$

where $\vec{\boldsymbol{\sigma}}$ is the solution of the problem: Find $\vec{\boldsymbol{\sigma}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ such that

$$\mathbf{A}_\phi(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) + \mathbf{B}_{\mathbf{w},\phi}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = F_D(\vec{\boldsymbol{\tau}}) + F_\phi(\vec{\boldsymbol{\tau}}), \tag{3.28}$$

for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$. In addition, let $\widetilde{\mathbf{S}} : \mathbf{H} \to H^1(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\mathbf{w},\phi) := \varphi, \tag{3.29}$$

where $\varphi \in H^1(\Omega)$ is the first component of the solution of the problem: Find $(\varphi, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) = F_{\mathbf{w},\phi}(\psi) \qquad \forall\ \psi \in H^1(\Omega), \tag{3.30a}$$

$$\mathbf{b}(\varphi, \xi) = G(\xi) \qquad \forall\ \xi \in H^{-1/2}(\Gamma). \tag{3.30b}$$

In this way, by introducing the operator $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ as

$$\mathbf{T}(\mathbf{w},\phi) := (\mathbf{S}_2(\mathbf{w},\phi),\ \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{w},\phi),\phi)) \quad \forall\ (\mathbf{w},\phi) \in \mathbf{H}, \tag{3.31}$$

we realize that (3.18) can be rewritten as the fixed-point problem: Find $(\mathbf{u}, \varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\mathbf{u},\varphi) = (\mathbf{u},\varphi), \tag{3.32}$$

meaning that the subsequent analysis will focus on how to prove the existence and uniqueness of this fixed-point. In this regard, we remark that the primal formulation for the energy equation (2.1c) has been already considered in [15], and therefore, most of the related results to the operator $\widetilde{\mathbf{S}}$ will only be cited, unless some substantial difference appears.

3.3 Well-Posedness of the Uncoupled Problems

As usual, we consider

$$\| \vec{\boldsymbol{\tau}} \| := \left\{ \| \boldsymbol{\tau} \|_{\mathrm{div};\Omega}^2 + \| \mathbf{v} \|_{1,\Omega}^2 + \| \boldsymbol{\eta} \|_{0,\Omega}^2 \right\}^{1/2},$$

for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, and

$$\| (\psi, \xi) \| := \left\{ \| \psi \|_{1,\Omega}^2 + \| \xi \|_{-1/2,\Gamma}^2 \right\}^{1/2},$$

for all $(\psi, \xi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$. We begin by recalling the following lemmas which will be useful to prove below some ellipticity properties.

Lemma 3.3. *There exists $c_3(\Omega) > 0$ such that*

$$c_3(\Omega) \| \boldsymbol{\tau}_0 \|_{0,\Omega}^2 \leq \left\| \boldsymbol{\tau}^{\mathrm{d}} \right\|_{0,\Omega}^2 + \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I} \in \mathbb{H}(\mathbf{div};\Omega).$$

Proof. See [8, Proposition 3.1], [24, Lemma 2.3]. $\square$

Lemma 3.4. *There exists $\kappa_0(\Omega) > 0$ such that*

$$\kappa_0 \| \mathbf{v} \|_{1,\Omega}^2 \leq \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 + \| \mathbf{v} \|_{0,\Gamma}^2 \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Proof. See [23, Lemma 3.1]. $\square$

The following result establishes sufficient conditions for the operator $\mathbf{S}$ being well-defined, equivalently, (3.28) being well-posed.

Lemma 3.5. *Assume that for $\delta_1 \in (0, 2\mu_1)$, $\delta_2 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_4 > 0, \quad \textit{and} \quad \kappa_3 \in \left(0, 2\delta_2\kappa_0 \min\left\{\kappa_1\left(1 - \frac{\delta_1}{2\mu_1}\right), \kappa_4\right\}\right).$$

Then, there exists $r_0 > 0$ such that for each $r \in (0, r_0)$, the problem (3.28) has a unique solution $\vec{\boldsymbol{\sigma}} := \mathbf{S}(\mathbf{w}, \phi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ for each $(\mathbf{w}, \phi) \in \mathbf{H}$ such that $\| \mathbf{w} \|_{1,\Omega} \leq r$. Moreover, there exists a constant $C_{\mathbf{S}} > 0$, independent of $(\mathbf{w}, \phi)$, such that there holds

$$\| \mathbf{S}(\mathbf{w}, \phi) \| = \| \vec{\boldsymbol{\sigma}} \| \leq C_{\mathbf{S}} \left\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \right\}. \tag{3.33}$$

Proof. Let $(\mathbf{w}, \phi) \in \mathbf{H}$. It is clear from (3.19) and (3.20) that $\mathbf{A}_\phi$ and $\mathbf{B}_{\mathbf{w},\phi}$ are bilinear forms. For $\mathbf{A}_\phi$, thanks to the Cauchy-Schwarz inequality, the trace theorem with constant $c_0(\Omega)$, and the bounds for μ, we see that

$$\begin{aligned}
|\mathbf{A}_\phi(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}})| \leq{}& \frac{1}{\mu_1} \left\| \boldsymbol{\sigma}^{\mathrm{d}} \right\|_{0,\Omega} \left\| \boldsymbol{\tau}^{\mathrm{d}} \right\|_{0,\Omega} + \frac{\kappa_1}{\mu_1} \left\| \boldsymbol{\sigma}^{\mathrm{d}} \right\|_{0,\Omega} \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega} + \| \mathbf{u} \|_{0,\Omega} \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega} \\
&+ \kappa_2 \| \mathbf{div}\, \boldsymbol{\sigma} \|_{0,\Omega} \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega} + \kappa_1 \| \mathbf{e}(\mathbf{u}) \|_{0,\Omega} \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega} + \| \boldsymbol{\gamma} \|_{0,\Omega} \| \boldsymbol{\tau} \|_{0,\Omega} \\
&+ \| \mathbf{v} \|_{0,\Omega} \| \mathbf{div}\, \boldsymbol{\sigma} \|_{0,\Omega} + \| \boldsymbol{\sigma} \|_{0,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} + \kappa_3 \| \boldsymbol{\gamma} \|_{0,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} \\
&+ \kappa_3 \| \boldsymbol{\omega}(\mathbf{u}) \|_{0,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} + \kappa_4 c_0(\Omega)^2 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{v} \|_{1,\Omega}.
\end{aligned}$$

It follows that, there exists a constant $C_{\mathbf{A}} > 0$, depending only on μ_1, κ_1, κ_2, κ_3, κ_4 and $c_0(\Omega)$, such that

$$|\mathbf{A}_\phi(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}})| \leq C_{\mathbf{A}} \| \vec{\boldsymbol{\sigma}} \| \| \vec{\boldsymbol{\tau}} \|, \tag{3.34}$$

for all $\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$. On the other hand, for $\mathbf{B}_{\mathbf{w},\phi}$, using the estimation (3.6), we find that

$$|\mathbf{B}_{\mathbf{w},\phi}(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}})| \leq \frac{c_1(\Omega)(2+\kappa_1^2)^{1/2}}{\mu_1}\|\, \mathbf{w}\,\|_{1,\Omega}\|\, \vec{\boldsymbol{\sigma}}\,\|\|\, \vec{\boldsymbol{\tau}}\,\|, \tag{3.35}$$

for all $\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$. Hence, there exists a positive constant denoted by $\|\, \mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi}\,\|$, independent of $(\mathbf{w},\phi)$, such that

$$|(\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}})| \leq \|\, \mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi}\,\|\|\, \vec{\boldsymbol{\sigma}}\,\|\|\, \vec{\boldsymbol{\tau}}\,\|, \tag{3.36}$$

for all $\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$. On the other hand, by using the Cauchy-Schwarz and Young inequalities, we obtain that for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$ and for any $\delta_1,\delta_2>0$ there holds

$$\begin{aligned}
\mathbf{A}_\phi(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) &= \int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\tau}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} - \kappa_1 \int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\tau}^{\mathtt{d}} : \mathbf{e}(\mathbf{v}) + \kappa_2\|\, \mathbf{div}\,\boldsymbol{\tau}\,\|^2_{0,\Omega} + \kappa_1\|\, \mathbf{e}(\mathbf{v})\,\|^2_{0,\Omega}\\
&+ \kappa_3\|\, \boldsymbol{\eta}\,\|^2_{0,\Omega} - \kappa_3\int_\Omega \boldsymbol{\omega}(\mathbf{v}) : \boldsymbol{\eta} + \kappa_4\|\, \mathbf{v}\,\|^2_{0,\Gamma}\\
&\geq \frac{1}{\mu_2}\|\, \boldsymbol{\tau}^{\mathtt{d}}\,\|^2_{0,\Omega} - \frac{\kappa_1}{2\delta_1\mu_1}\|\, \boldsymbol{\tau}^{\mathtt{d}}\,\|^2_{0,\Omega} - \frac{\kappa_1\delta_1}{2\mu_1}\|\, \mathbf{e}(\mathbf{v})\,\|^2_{0,\Omega} + \kappa_2\|\, \mathbf{div}\,\boldsymbol{\tau}\,\|^2_{0,\Omega} + \kappa_1\|\, \mathbf{e}(\mathbf{v})\,\|^2_{0,\Omega}\\
&+ \kappa_3\|\, \boldsymbol{\eta}\,\|^2_{0,\Omega} - \frac{\kappa_3}{2\delta_2}\|\, \boldsymbol{\omega}(\mathbf{v})\,\|^2_{0,\Omega} - \frac{\kappa_3\delta_2}{2}\|\, \boldsymbol{\eta}\,\|^2_{0,\Omega} + \kappa_4\|\, \mathbf{v}\,\|_{0,\Gamma}\\
&= \left(\frac{1}{\mu_2} - \frac{\kappa_1}{2\mu_1\delta_1}\right)\|\, \boldsymbol{\tau}^{\mathtt{d}}\,\|^2_{0,\Omega} + \kappa_2\|\, \mathbf{div}\,\boldsymbol{\tau}\,\|^2_{0,\Omega} + \kappa_1\left(1 - \frac{\delta_1}{2\mu_1}\right)\|\, \mathbf{e}(\mathbf{v})\,\|^2_{0,\Omega} - \frac{\kappa_3}{2\delta_2}|\mathbf{v}|^2_{1,\Omega}\\
&+ \kappa_4\|\, \mathbf{v}\,\|^2_{0,\Gamma} + \kappa_3\left(1 - \frac{\delta_2}{2}\right)\|\, \boldsymbol{\eta}\,\|^2_{0,\Omega}.
\end{aligned}$$

Then, defining the following positive constants

$$\begin{gathered}
\alpha_1 := \min\left\{\frac{1}{\mu_2} - \frac{\kappa_1}{2\mu_1\delta_1}, \frac{\kappa_2}{2}\right\}, \quad \alpha_2 := \min\left\{\alpha_1 c_3(\Omega), \frac{\kappa_2}{2}\right\}, \quad \alpha_3 := \min\left\{\kappa_1\left(1 - \frac{\delta_1}{2\mu_1}\right), \kappa_4\right\},\\
\alpha_4 := \alpha_3\kappa_0 - \frac{\kappa_3}{2\delta_2}, \quad \alpha_5 := \kappa_3\left(1 - \frac{\delta_2}{2}\right),
\end{gathered} \tag{3.37}$$

and using Lemmas 3.3 and 3.4, it is possible to find a positive constant $\alpha(\Omega) := \min\{\alpha_2,\alpha_4,\alpha_5\}$, independent of $(\mathbf{w},\phi)$, such that

$$\mathbf{A}_\phi(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \geq \alpha(\Omega)\|\, \vec{\boldsymbol{\tau}}\,\|^2 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega),$$

which, together with the definition of $\mathbf{B}_{\mathbf{w},\phi}$ (cf. (3.20)) and the estimation (3.6), results in the fact that for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$ there holds

$$(\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi})(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \geq \left(\alpha(\Omega) - \frac{c_1(\Omega)(2+\kappa_1^2)^{1/2}}{\mu_1}\|\, \mathbf{w}\,\|_{1,\Omega}\right)\|\, \vec{\boldsymbol{\tau}}\,\|^2.$$

Therefore, we easily see that

$$(\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi})(\vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\tau}}) \geq \frac{\alpha(\Omega)}{2}\|\, \vec{\boldsymbol{\tau}}\,\|^2, \tag{3.38}$$

for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, provided that

$$\frac{\alpha(\Omega)}{2} \geq \frac{c_1(\Omega)(2+\kappa_1^2)^{1/2}}{\mu_1}\|\, \mathbf{w}\,\|_{1,\Omega},$$

that is,

$$\| \mathbf{w} \|_{1,\Omega} \leq \frac{\mu_1 \alpha(\Omega)}{2c_1(\Omega)(2+\kappa_1^2)^{1/2}} =: r_0, \tag{3.39}$$

thus proving ellipticity for $\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi}$ under the requirement (3.39). Concerning the functionals F_D and F_ϕ, it is clear from its definitions that they are linear, and by using the Cauchy-Schwarz inequality and the trace theorem, it is possible to show that

$$\| F_D \| \leq \| \mathbf{u}_D \|_{1/2,\Gamma} + \kappa_4 c_0(\Omega) \| \mathbf{u}_D \|_{0,\Gamma}, \tag{3.40}$$

and

$$\| F_\phi \| \leq (2+\kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega}. \tag{3.41}$$

In this way, denoting $M_{\mathbf{S}} := \max\{(2+\kappa_2^2)^{1/2}, \kappa_4 c_0(\Omega)\}$, we deduce from the previous inequalities that

$$\| F_\phi + F_D \| \leq M_{\mathbf{S}} \left\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega} + \| \mathbf{u}_D \|_{0,\Gamma} + \| \mathbf{u}_D \|_{1/2,\Gamma} \right\}. \tag{3.42}$$

Hence, by the Lax-Milgram theorem (see, e.g. [24, Theorem 1.1]), there is a unique solution $\vec{\boldsymbol{\sigma}} \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ of (3.28), and the corresponding continuous dependence result (3.33) is satisfied with $C_{\mathbf{S}} := \frac{2M_{\mathbf{S}}}{\alpha(\Omega)}$, which is clearly independent of $\mathbf{w}$ and ϕ. □

The foregoing lemma provides us with feasible ranges for the stabilization parameters κ_i, $i \in \{1,2,3,4\}$ such that the well-posedness of (3.28) is achieved. For computational purposes, we make a particular choice of these κ_i such that the ellipticity constant for $\mathbf{A}_\phi$, i.e., $\alpha(\Omega)$ is as large as possible. With this in mind, we first choose the middle points of the ranges for δ_1, δ_2 and κ_1, that is

$$\delta_1 = \mu_1, \quad \delta_2 = 1, \quad \kappa_1 = \frac{\mu_1 \delta_1}{\mu_2} = \frac{\mu_1^2}{\mu_2}. \tag{3.43}$$

Then, we aim to maximize α_1 and α_3 (cf. (3.37)) by taking

$$\kappa_2 = \frac{1}{\mu_2}, \quad \kappa_4 = \frac{\mu_1^2}{2\mu_2}, \tag{3.44}$$

and by choosing κ_3 as the middle point of its range:

$$\kappa_3 = \frac{\kappa_0 \mu_1^2}{2\mu_2}. \tag{3.45}$$

Notice that κ_0, the constant arising from the Korn-type inequality in Lemma 3.4, is still unknown. Nevertheless, [9] suggests that a heuristic choice for this parameter is enough for numerical computations.

In addition, throughout the rest of the article, and for purposes to be clarified below, further regularity will be assumed for the problem defining the operator $\mathbf{S}$. More precisely, we assume that $\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}(\Gamma)$, with $\varepsilon \in (0,1)$, and that for each $(\mathbf{z},\psi) \in \mathbf{H}$, with $\| \mathbf{z} \|_{1,\Omega} \leq r$, $r > 0$ given, there hold $(\boldsymbol{\zeta}, \mathbf{v}, \boldsymbol{\chi}) := \mathbf{S}(\mathbf{z},\psi) \in \mathbb{H}_0(\mathbf{div};\Omega) \cap \mathbb{H}^\varepsilon(\Omega) \times \mathbf{H}^{1+\varepsilon}(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{H}^\varepsilon(\Omega)$ and

$$\| \boldsymbol{\zeta} \|_{\varepsilon,\Omega} + \| \mathbf{v} \|_{1+\varepsilon,\Omega} + \| \boldsymbol{\chi} \|_{\varepsilon,\Omega} \leq \widetilde{C}_{\mathbf{S}}(r) \left\{ \| \mathbf{g} \|_{\infty,\Omega} \| \psi \|_{1,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \right\}, \tag{3.46}$$

with $\widetilde{C}_{\mathbf{S}}(r)$ being a positive constant independent of $\mathbf{z}$ but depending on the upper bound r of its $\mathbf{H}^1$-norm.

For $\widetilde{\mathbf{S}}$, a direct application of the Babuška-Brezzi theory provides the well-posedness of (3.30).

Lemma 3.6. *For each $(\mathbf{w}, \phi) \in \mathbf{H}$, there exists a unique pair $(\varphi, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ solution of the problem (3.30), and there holds*

$$\left\| \widetilde{\mathbf{S}}(\mathbf{w}, \phi) \right\| \leq \| (\varphi, \lambda) \| \leq C_{\widetilde{\mathbf{S}}} \left\{ \| \mathbf{w} \|_{1,\Omega} |\phi|_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \right\}. \tag{3.47}$$

Proof. See [15, Lemma 3.4]. □

3.4 Solvability Analysis of the Fixed-Point Equation

Having proved the well-posedness of the uncoupled problems (3.28) and (3.30), which ensures that operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$, and hence $\mathbf{T}$, are well-defined, we now aim to establish the existence of a unique fixed-point of the operator $\mathbf{T}$. To do so, we will verify the hypotheses of the Banach fixed-point theorem. We begin the analysis with the following result.

Lemma 3.7. *Let $r \in (0, r_0)$ with r_0 as given in (3.39) and let $W := \bar{B}(\mathbf{0}, r)$ be the closed ball in $\mathbf{H}$ with center at $\mathbf{0}$ and radius r, that is*

$$W := \big\{ (\mathbf{w}, \phi) \in \mathbf{H} : \| (\mathbf{w}, \phi) \| \leq r \big\}.$$

In addition, assume that the data satisfy

$$c(r) \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\} + C_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma} \leq r, \tag{3.48}$$

where

$$c(r) := (1 + C_{\widetilde{\mathbf{S}}} r) C_{\mathbf{S}} \max\{r, 1\}, \tag{3.49}$$

and $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$ are given in Lemmas 3.5 and 3.6, respectively. Then, there holds $\mathbf{T}(W) \subseteq W$.

Proof. The proof follows the scheme in [15, Lemma 3.5], but now based on the continuous dependance estimates (3.33) and (3.47). □

Next, we will establish some results that will help us to check under which conditions $\mathbf{T}$ becomes a continuous mapping.

Lemma 3.8. *Let $r \in (0, r_0)$ with r_0 as given in (3.39). Then, there exists a positive constant $\widehat{C}_{\mathbf{S}}(r)$ depending on r such that*

$$\begin{aligned} \| \mathbf{S}(\mathbf{w}, \phi) - \mathbf{S}(\mathbf{z}, \psi) \| \leq \widehat{C}_{\mathbf{S}}(r) \Big\{ & \| \mathbf{S}_1(\mathbf{w}, \phi) \|_{\varepsilon,\Omega} \| \phi - \psi \|_{L^{2/\varepsilon}(\Omega)} \\ & + \| \mathbf{S}_2(\mathbf{w}, \phi) \|_{1,\Omega} \Big(\| \mathbf{w} - \mathbf{z} \|_{1,\Omega} + \| \phi - \psi \|_{1,\Omega} \Big) + \| \mathbf{g} \|_{\infty,\Omega} \| \phi - \psi \|_{0,\Omega} \Big\} \end{aligned} \tag{3.50}$$

for all $(\mathbf{w}, \phi), (\mathbf{z}, \psi) \in \mathbf{H}$ such that $\| \mathbf{w} \|_{1,\Omega}, \| \mathbf{z} \|_{1,\Omega} \leq r$.

Proof. Let $(\mathbf{w}, \phi)$, $(\mathbf{z}, \psi) \in \mathbf{H}$ as indicated and let $\vec{\boldsymbol{\sigma}} := \mathbf{S}(\mathbf{w}, \phi)$ and $\vec{\boldsymbol{\varrho}} := \mathbf{S}(\mathbf{z}, \psi)$ be the corresponding solutions of (3.28). From this fact, by adding and subtracting the equality

$$(\mathbf{A}_\phi + \mathbf{B}_{\mathbf{w},\phi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) = (F_D + F_\phi)(\vec{\boldsymbol{\tau}}),$$

and the term $\mathbf{B}_{\mathbf{w},\psi}(\cdot,\cdot)$, it is possible to show that, for all $\vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\begin{aligned} &(\mathbf{A}_\psi + \mathbf{B}_{\mathbf{z},\psi})(\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}}, \vec{\boldsymbol{\tau}}) \\ &\qquad = (\mathbf{A}_\psi - \mathbf{A}_\phi)(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) + (\mathbf{B}_{\mathbf{z},\psi} - \mathbf{B}_{\mathbf{w},\psi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) + (\mathbf{B}_{\mathbf{w},\psi} - \mathbf{B}_{\mathbf{w},\phi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) + F_{\phi-\psi}(\vec{\boldsymbol{\tau}}). \end{aligned}$$

Hence, using the ellipticity of the bilinear form $\mathbf{A}_\psi + \mathbf{B}_{\mathbf{z},\psi}$ (cf. (3.38)), the foregoing expression and the properties of the bilinear forms (cf. Lemma 3.2), we obtain

$$
\begin{aligned}
\frac{\alpha(\Omega)}{2}\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|^2 \;\le\; & (\mathbf{A}_\psi + \mathbf{B}_{\mathbf{z},\psi})(\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}}, \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}}) \\
= & \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} \;\boldsymbol{\sigma}^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \\
+ & \int_\Omega \frac{1}{\mu(\psi)} [\mathbf{u} \otimes (\mathbf{w} - \mathbf{z})]^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \\
+ & \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} (\mathbf{u} \otimes \mathbf{w})^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \\
+ & \int_\Omega (\phi - \psi) \mathbf{g} \cdot [(\mathbf{u} - \mathbf{s}) - \kappa_2 \mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\varrho})]\,.
\end{aligned}
\tag{3.51}
$$

For the last term of (3.51), as it was done for proving the boundedness of F_ϕ in (3.41), we see that

$$
\begin{aligned}
& \left| \int_\Omega (\phi - \psi) \mathbf{g} \cdot [(\mathbf{u} - \mathbf{s}) - \kappa_2 \mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\varrho})] \right| \\
& \quad \le \; \| \mathbf{g} \|_{\infty,\Omega} \| \phi - \psi \|_{0,\Omega} \| (\mathbf{u} - \mathbf{s}) - \kappa_2 \mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{\varrho}) \|_{0,\Omega} \\
& \quad \le \; (2 + \kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} \| \phi - \psi \|_{0,\Omega} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|\,.
\end{aligned}
\tag{3.52}
$$

Then, for the second term of the right hand side of (3.51), using the estimation (3.6) and the lower bound of μ, we get

$$
\left| \int_\Omega \frac{1}{\mu(\psi)} [\mathbf{u} \otimes (\mathbf{w} - \mathbf{z})]^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \right| \le \widehat{C}_1 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{w} - \mathbf{z} \|_{1,\Omega} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|,
\tag{3.53}
$$

where $\widehat{C}_1 := \frac{c_1(\Omega)(2+\kappa_1^2)^{1/2}}{\mu_1}$. Now, for the third term, we use the Lipschitz continuity of μ, its lower bound, and the Hölder and Cauchy-Schwarz inequalities to show that

$$
\begin{aligned}
& \left| \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} (\mathbf{u} \otimes \mathbf{w})^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \right| \\
& \quad \le \; \widehat{C}_2 \| (\phi - \psi)(\mathbf{u} \otimes \mathbf{w}) \|_{0,\Omega} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \| \\
& \quad \le \; \widehat{C}_2 \| \phi - \psi \|_{L^4(\Omega)} \| \mathbf{u} \|_{\mathbf{L}^8(\Omega)} \| \mathbf{w} \|_{\mathbf{L}^8(\Omega)} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|\,,
\end{aligned}
\tag{3.54}
$$

where $\widehat{C}_2 := \frac{L_\mu (2+\kappa_1^2)^{1/2}}{\mu_1^2}$. At this point, we recall from the Rellich-Kondrachov Theorem (cf., e.g. [33, Theorem 1.3.5] that $H^1(\Omega)$ is compactly embedded (hence continuously) in $L^8(\Omega)$ when $\Omega \subset R^2$, meaning that the previous argument cannot be used in the three dimensional case, where the compact imbedding of $H^1(\Omega)$ into $L^r(\Omega)$ is valid only for $1 \le r \le 6$. That being said, there exists a constant C_i depending on the boundedness constants of the corresponding injections such that

$$
\left| \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} (\mathbf{u} \otimes \mathbf{w})^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \right| \le \widehat{C}_2 C_i r \| \mathbf{u} \|_{1,\Omega} \| \phi - \psi \|_{1,\Omega} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|.
\tag{3.55}
$$

And, for the remaining term in (3.51), using the Lipschitz continuity of μ, along with Cauchy-Schwarz and Hölder inequalities, we see that with the constant $\widehat{C}_2$ introduced in (3.54)

$$
\begin{aligned}
\left| \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} \;\boldsymbol{\sigma}^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \right| \;\le\; & \widehat{C}_2 \| (\psi - \phi) \boldsymbol{\sigma}^{\mathrm{d}} \|_{0,\Omega} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \| \\
\le\; & \widehat{C}_2 \| \phi - \psi \|_{L^{2q}(\Omega)} \| \boldsymbol{\sigma} \|_{\mathbb{L}^{2p}(\Omega)} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|\,,
\end{aligned}
\tag{3.56}
$$

where $p, q \in [1, +\infty)$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Taking into consideration the further regularity assumed in (3.46), the Sobolev Embedding Theorem (cf. [1, Theorem 4.12], [33, Theorem 1.3.4]) establishes the continuous injection $H^\varepsilon(\Omega) \hookrightarrow L^{\varepsilon^*}(\Omega)$ with boundedness constant C_ε, where $\varepsilon^* = \frac{2}{1-\varepsilon}$. Thus, choosing p such that $2p = \varepsilon^*$, i.e., $p = \frac{1}{1-\varepsilon}$, there holds that effectively $\boldsymbol{\sigma} \in \mathbb{L}^{2p}(\Omega)$ and

$$\| \boldsymbol{\sigma} \|_{\mathbb{L}^{2p}(\Omega)} \le C_\varepsilon \| \boldsymbol{\sigma} \|_{\varepsilon,\Omega}.$$

With this choice of p, $2q$ becomes

$$2q = \frac{2p}{p-1} = \frac{2}{\varepsilon},$$

and (3.56) yields

$$\left| \int_\Omega \frac{\mu(\phi) - \mu(\psi)}{\mu(\psi)\mu(\phi)} \, \boldsymbol{\sigma}^{\mathrm{d}} : [(\boldsymbol{\sigma} - \boldsymbol{\varrho})^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u} - \mathbf{s})] \right| \le \widehat{C}_2 C_\varepsilon \| \boldsymbol{\sigma} \|_{\varepsilon,\Omega} \| \phi - \psi \|_{L^{2/\varepsilon}(\Omega)} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \|. \tag{3.57}$$

Therefore, putting (3.52), (3.53), (3.55) and (3.57) together into (3.51), it is possible to find a constant $\widehat{C}_{\mathbf{S}}(r) > 0$ depending on L_μ, μ_1, κ_1, κ_2, $c_1(\Omega)$, C_i, C_ε, and r such that

$$\begin{aligned} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\varrho}} \| \le \widehat{C}_{\mathbf{S}}(r) \Big\{ & \| \boldsymbol{\sigma} \|_{\varepsilon,\Omega} \| \phi - \psi \|_{L^{2/\varepsilon}(\Omega)} \\ & + \| \mathbf{u} \|_{1,\Omega} \Big(\| \mathbf{w} - \mathbf{z} \|_{1,\Omega} + \| \phi - \psi \|_{1,\Omega} \Big) + \| \mathbf{g} \|_{\infty,\Omega} \| \phi - \psi \|_{0,\Omega} \Big\}, \end{aligned} \tag{3.58}$$

and since $\boldsymbol{\sigma} = \mathbf{S}_1(\mathbf{w}, \phi)$ and $\mathbf{u} = \mathbf{S}_2(\mathbf{w}, \phi)$, the last inequality is exactly the required estimate (3.50). $\square$

Next, concerning the operator $\widetilde{\mathbf{S}}$, we recall the following result from [15].

Lemma 3.9. *There exists a positive constant $\widehat{C}_{\widetilde{\mathbf{S}}}$ such that*

$$\left\| \widetilde{\mathbf{S}}(\mathbf{w}, \phi) - \widetilde{\mathbf{S}}(\mathbf{z}, \psi) \right\| \le \widehat{C}_{\widetilde{\mathbf{S}}} \Big\{ \| \mathbf{w} \|_{1,\Omega} |\phi - \psi|_{1,\Omega} + \| \mathbf{w} - \mathbf{z} \|_{1,\Omega} |\psi|_{1,\Omega} \Big\}, \tag{3.59}$$

for all $(\mathbf{w}, \phi)$, $(\mathbf{z}, \psi) \in \mathbf{H}$.

Proof. See [15, Lemma 3.7]. $\square$

As a consequence of the previous lemmas, the following can be established for the operator $\mathbf{T}$.

Lemma 3.10. *Let $r \in (0, r_0)$ with r_0 as given in (3.39) and $W := \Big\{ (\mathbf{w}, \phi) \in \mathbf{H} : \| (\mathbf{w}, \phi) \| \le r \Big\}$. Then, there exists a constant $C_{\mathbf{T}} > 0$ such that*

$$\| \mathbf{T}(\mathbf{w}, \phi) - \mathbf{T}(\mathbf{z}, \psi) \| \le C_{\mathbf{T}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\} \| (\mathbf{w}, \phi) - (\mathbf{z}, \psi) \|, \tag{3.60}$$

for all $(\mathbf{w}, \phi), (\mathbf{z}, \psi) \in W$.

Proof. Since $\mathbf{T}(\mathbf{w}, \phi) = \Big(\mathbf{S}_2(\mathbf{w}, \phi), \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{w}, \phi), \phi) \Big)$ $\forall\, (\mathbf{w}, \phi) \in \mathbf{H}$, by applying the bounds obtained in Lemma 3.8 and Lemma 3.9 (cf. (3.50) and (3.59)), we find that

$$\begin{aligned} \| \mathbf{T}(\mathbf{w}, \phi) - \mathbf{T}(\mathbf{z}, \psi) \| &= \left\| \Big(\mathbf{S}_2(\mathbf{w}, \phi), \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{w}, \phi), \phi) \Big) - \Big(\mathbf{S}_2(\mathbf{z}, \psi), \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{z}, \psi), \psi) \Big) \right\| \\ &\le \| \mathbf{S}_2(\mathbf{w}, \phi) - \mathbf{S}_2(\mathbf{z}, \psi) \| + \left\| \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{w}, \phi), \phi) - \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{z}, \psi), \psi) \right\| \end{aligned}$$

and

$$\Big\| \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{w},\phi),\phi) - \widetilde{\mathbf{S}}(\mathbf{S}_2(\mathbf{z},\psi),\psi) \Big\| \\ \leq \; \widehat{C}_{\widetilde{\mathbf{S}}} \Big\{ \| \mathbf{S}_2(\mathbf{w},\phi) \|_{1,\Omega} | \phi - \psi |_{1,\Omega} + \| \mathbf{S}_2(\mathbf{w},\phi) - \mathbf{S}_2(\mathbf{z},\psi) \| r \Big\} \,,$$

which leads to

$$\begin{aligned} &\| \mathbf{T}(\mathbf{w},\phi) - \mathbf{T}(\mathbf{z},\psi) \| \\ &\leq \; (1 + \widehat{C}_{\widetilde{\mathbf{S}}} r) \| \mathbf{S}_2(\mathbf{w},\phi) - \mathbf{S}_2(\mathbf{z},\psi) \| + \widehat{C}_{\widetilde{\mathbf{S}}} \| \mathbf{S}_2(\mathbf{w},\phi) \|_{1,\Omega} | \phi - \psi |_{1,\Omega} \\ &\leq \; (1 + \widehat{C}_{\widetilde{\mathbf{S}}} r) \widehat{C}_{\mathbf{S}}(r) \Big\{ \| \mathbf{S}_1(\mathbf{w},\phi) \|_{\varepsilon,\Omega} \| \phi - \psi \|_{L^{2/\varepsilon}(\Omega)} \\ &+ \| \mathbf{S}_2(\mathbf{w},\phi) \|_{1,\Omega} \Big(\| \mathbf{w} - \mathbf{z} \|_{1,\Omega} + \| \phi - \psi \|_{1,\Omega} \Big) + \| \mathbf{g} \|_{\infty,\Omega} \| \phi - \psi \|_{0,\Omega} \Big\} \\ &+ \widehat{C}_{\widetilde{S}} \| \mathbf{S}_2(\mathbf{w},\phi) \|_{1,\Omega} | \phi - \psi |_{1,\Omega} \,. \end{aligned}$$

Next, considering the continuous injections $H^1(\Omega) \hookrightarrow L^{2/\varepsilon}(\Omega)$ and $H^{1+\varepsilon}(\Omega) \hookrightarrow H^1(\Omega)$ (guaranteed by the Sobolev embedding theorem, given that $\varepsilon \in (0,1)$) with boundedness constants $\widetilde{C}_\varepsilon$ and $\widetilde{C}_i$, respectively, and defining

$$C_1 := \widehat{C}_{\mathbf{S}}(r)(1 + \widehat{C}_{\widetilde{\mathbf{S}}} r), \quad C_2 := \max\Big\{ C_1 \widetilde{C}_\varepsilon, (C_1 + \widehat{C}_{\widetilde{\mathbf{S}}}) \widetilde{C}_i \Big\}, \quad C_3 := C_2 \widetilde{C}_{\mathbf{S}}(r) r + \widehat{C}_{\mathbf{S}} C_1, \quad C_4 = C_2 r,$$

where $\widehat{C}_{\mathbf{S}}(r)$ and $\widehat{C}_{\widetilde{\mathbf{S}}}$ are the constants defined in Lemma 3.8 and Lemma 3.9, respectively, it is possible to show from the previous estimate that (3.60) holds with $C_{\mathbf{T}} := \max\{C_3, C_4\}$. $\square$

We are now in a position to establish sufficient conditions for the existence and uniqueness of a fixed-point for our problem (3.32) (equivalently, the well-posedness of our variational problem (3.18)). Indeed, we have from Lemmas 3.5 and 3.6 that $\mathbf{T}$ is well-defined and maps the ball W of radius r (with $r \in (0, r_0)$, r_0 given by (3.39)) into the same ball; the latter thanks to Lemma 3.7. Furthermore, Lemma 3.10 guarantees that $\mathbf{T}$ is Lipschitz-continuous, and it becomes a contraction when the data is small enough. Therefore, thanks to the Banach fixed-point theorem, there exists a unique fixed-point $(\mathbf{u}, \varphi) \in \mathbf{H}$ for the problem (3.32). This fact provides us with the main result of this section.

Theorem 3.11. *Assume that for $\delta_1 \in (0, 2\mu_1)$, $\delta_2 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_4 > 0, \quad \text{and} \quad \kappa_3 \in \left(0, 2\delta_2\kappa_0 \min\left\{ \kappa_1 \left(1 - \frac{\delta_1}{2\mu_1}\right), \kappa_4 \right\}\right).$$

and let $W := \Big\{ (\mathbf{w},\phi) \in \mathbf{H} : \| (\mathbf{w},\phi) \| \leq r \Big\}$, with $r \in (0, r_0)$, r_0 as in (3.39). In addition, assume that the data satisfy

$$c(r) \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\} + C_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma} \leq r,$$

with $c(r)$ as in Lemma 3.7, and

$$C_{\mathbf{T}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\} < 1.$$

Then, the problem (3.18) has a unique solution $(\vec{\boldsymbol{\sigma}}, (\varphi, \lambda)) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$, with $(\mathbf{u},\varphi) \in W$. Moreover, there hold

$$\| \vec{\boldsymbol{\sigma}} \| \leq C_{\mathbf{S}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\}$$

and

$$\| (\varphi,\lambda) \| \leq C_{\widetilde{\mathbf{S}}} \Big\{ r \| \mathbf{u} \|_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \Big\},$$

with $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$ as in Lemma 3.5 and Lemma 3.6, respectively.

4 The Galerkin Scheme

In this section, we introduce and analyse the corresponding Galerkin scheme for the augmented mixed-primal formulation (3.18). The well-posedness of this scheme will be proved following basically the same techniques used throughout Section 3.4.

4.1 Preliminaries

Let us consider $\mathcal{T}_h$ a regular triangulation of Ω by triangles K of diameter h_K, and define the mesh size $h := \max\{h_K : K \in \mathcal{T}_h\}$. In addition, given an integer $k \geq 0$, for each $K \in \mathcal{T}_h$ we let $P_k(K)$ be the space of polynomial functions on K of degree $\leq k$. To begin with, we consider arbitrary finite-dimensional subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}} \subset \mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{H}_h^{\mathbf{u}} \subset \mathbf{H}^1(\Omega)$, $\mathbb{H}_h^{\gamma} \subset \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $H_h^{\varphi} \subset H^1(\Omega)$, $H_h^{\lambda} \subset H^{-1/2}(\Gamma)$ and denote

$$\vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), \quad \vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h), \quad \vec{\boldsymbol{\varrho}}_h := (\boldsymbol{\varrho}_h, \mathbf{s}_h, \boldsymbol{\vartheta}_h). \tag{4.1}$$

Hence, according to the continuous formulation (3.18), the corresponding Galerkin scheme reads: Find $(\vec{\boldsymbol{\sigma}}_h, (\varphi_h, \lambda_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma} \times H_h^{\varphi} \times H_h^{\lambda}$ such that

$$\mathbf{A}_{\varphi_h}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{B}_{\mathbf{u}_h,\varphi_h}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) = F_{\varphi_h}(\vec{\boldsymbol{\tau}}_h) + F_D(\vec{\boldsymbol{\tau}}_h), \tag{4.2a}$$

$$\mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) = F_{\mathbf{u}_h,\varphi_h}(\psi_h), \tag{4.2b}$$

$$\mathbf{b}(\varphi_h, \xi_h) = G(\xi_h), \tag{4.2c}$$

for all $(\vec{\boldsymbol{\tau}}_h, (\psi_h, \xi_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma} \times H_h^{\varphi} \times H_h^{\lambda}$, recalling that the forms $\mathbf{A}_{\varphi_h}$, $\mathbf{B}_{\mathbf{u}_h,\varphi_h}$, $\mathbf{a}$, and $\mathbf{b}$; and the functionals F_{φ_h}, F_D, $F_{\mathbf{u}_h,\varphi_h}$ and G are defined by (3.19)-(3.26). To prove the well-posedness of the foregoing problem, we proceed using a fixed-point approach as it was done in Section 3.3. Thus, we define $\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times H_h^{\varphi}$ and let $\mathbf{S}_h : \mathbf{H}_h \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$ be the operator defined as

$$\mathbf{S}_h(\mathbf{w}_h, \phi_h) = (\mathbf{S}_{1,h}(\mathbf{w}_h, \phi_h), \mathbf{S}_{2,h}(\mathbf{w}_h, \phi_h), \mathbf{S}_{3,h}(\mathbf{w}_h, \phi_h)) := \vec{\boldsymbol{\sigma}}_h \quad \forall\ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h, \tag{4.3}$$

where $\vec{\boldsymbol{\sigma}}_h$ is the solution to the problem: Find $\vec{\boldsymbol{\sigma}}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$ such that

$$\mathbf{A}_{\phi_h}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) + \mathbf{B}_{\mathbf{w}_h,\phi_h}(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) = F_D(\vec{\boldsymbol{\tau}}_h) + F_{\phi_h}(\vec{\boldsymbol{\tau}}_h), \tag{4.4}$$

for all $\vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$. In addition, let $\widetilde{\mathbf{S}}_h : \mathbf{H}_h \to H_h^{\varphi}$ be the operator defined by

$$\widetilde{\mathbf{S}}_h(\mathbf{w}_h, \phi_h) := \varphi_h \quad \forall\ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h, \tag{4.5}$$

where φ_h is the first component of the solution of the problem: Find $(\varphi_h, \lambda_h) \in H_h^{\varphi} \times H_h^{\lambda}$ such that

$$\mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) = F_{\mathbf{w}_h,\phi_h}(\psi_h) \quad \forall\ \psi_h \in H_h^{\varphi}, \tag{4.6a}$$

$$\mathbf{b}(\varphi_h, \xi_h) = G(\xi_h) \qquad \forall\ \xi_h \in H_h^{\lambda}. \tag{4.6b}$$

Therefore, by introducing the operator $\mathbf{T}_h : \mathbf{H}_h \to \mathbf{H}_h$ as

$$\mathbf{T}_h(\mathbf{w}_h, \phi_h) := (\mathbf{S}_{2,h}(\mathbf{w}_h, \phi_h), \widetilde{\mathbf{S}}_h(\mathbf{S}_{2,h}(\mathbf{w}_h, \phi_h), \phi_h)) \quad \forall\ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h, \tag{4.7}$$

problem (4.2) is now equivalent to the fixed-point problem: Find $(\mathbf{u}_h, \varphi_h) \in \mathbf{H}_h$ such that

$$\mathbf{T}_h(\mathbf{u}_h, \varphi_h) = (\mathbf{u}_h, \varphi_h). \tag{4.8}$$

4.2 Solvability Analysis

The proof of the well-posedness of the discrete problem (4.4) follows the same technique used in Lemma 3.5. In fact, it is clear that for every $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$, the bilinear form $\mathbf{A}_{\phi_h} + \mathbf{B}_{\mathbf{w}_h,\phi_h}$ is bounded in $(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}) \times (\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma})$ with boundedness constant depending only on μ_1, κ_1, κ_2, κ_3, κ_4, $c_0(\Omega)$, $c_1(\Omega)$ and $\| \mathbf{w}_h \|_{1,\Omega}$, and elliptic in this same space, provided that the stabilization parameters κ_i live in the same stipulated ranges, and $\| \mathbf{w}_h \|_{1,\Omega} \le r_0$, with r_0 as in (3.39). Also, F_D and F_{ϕ_h} are linear bounded functionals in $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$ as well. The foregoing discussion and the Lax-Milgram theorem allow us to conclude the following result.

Lemma 4.1. *Assume that for $\delta_1 \in (0, 2\mu_1)$, $\delta_2 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_4 > 0, \quad \text{and} \quad \kappa_3 \in \left(0, 2\delta_2\kappa_0 \min\left\{\kappa_1\left(1 - \frac{\delta_1}{2\mu_1}\right), \kappa_4\right\}\right).$$

Then, for each $r \in (0, r_0)$, r_0 given by (3.39), and for each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$ such that $\| \mathbf{w}_h \|_{1,\Omega} \le r$, the problem (4.4) has a unique solution $\vec{\boldsymbol{\sigma}}_h := \mathbf{S}_h(\mathbf{w}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$. Moreover, with the same constant $C_{\mathbf{S}}$ from Lemma 3.5, which is independent of $(\mathbf{w}_h, \phi_h)$, there holds

$$\| \mathbf{S}_h(\mathbf{w}_h, \phi_h) \| = \| \vec{\boldsymbol{\sigma}}_h \| \le C_{\mathbf{S}} \left\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi_h \|_{0,\Omega} + \| \mathbf{u}_D \|_{0,\Gamma} + \| \mathbf{u}_D \|_{1/2,\Gamma} \right\}. \tag{4.9}$$

It is worthwhile to mention that, at this time, no further restrictions are added to either $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$ or $\mathbb{H}_h^{\gamma}$. Moreover, they can be chosen as any finite dimensional subspace of $\mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{H}^1(\Omega)$ and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively. On the other hand, let V_h be the discrete kernel of the operator induced by $\mathbf{b}$, that is

$$V_h := \left\{ \psi_h \in H_h^{\varphi} : \mathbf{b}(\psi_h, \xi_h) = 0 \quad \forall\, \xi_h \in H_h^{\lambda} \right\}. \tag{4.10}$$

which may not be necessarily contained in V, the continuous kernel. For this reason, ellipticity can not be assured (straightforwardly) for the bilinear form $\mathbf{a}$ in V_h, and so we must introduce further hypotheses on the discrete spaces H_h^{φ} and H_h^{λ}. Hence, we assume that the following discrete inf-sup conditions hold:

(H.1) There exists a constant $\widehat{\alpha} > 0$, independent of h such that

$$\sup_{\substack{\psi_h \in V_h \\ \psi_h \neq 0}} \frac{\mathbf{a}(\psi_h, \phi_h)}{\| \psi_h \|_{1,\Omega}} \ge \widehat{\alpha} \| \phi_h \|_{1,\Omega} \quad \forall\, \phi_h \in V_h, \tag{4.11}$$

(H.2) There exists a constant $\widehat{\beta} > 0$, independent of h such that

$$\sup_{\substack{\psi_h \in H_h^{\varphi} \\ \psi_h \neq 0}} \frac{\mathbf{b}(\psi_h, \xi_h)}{\| \psi_h \|_{1,\Omega}} \ge \widehat{\beta} \| \xi_h \|_{-1/2,\Gamma} \quad \forall\, \xi_h \in H_h^{\lambda}. \tag{4.12}$$

Having this in mind, we have the following result.

Lemma 4.2. *For each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h$, there exists a unique pair $(\varphi_h, \lambda_h) \in H_h^{\varphi} \times H_h^{\lambda}$ solution of problem (4.6), and there holds*

$$\left\| \widetilde{\mathbf{S}}_h(\mathbf{w}_h, \phi_h) \right\| \le \| (\varphi_h, \lambda_h) \| \le \widetilde{C}_{\widetilde{\mathbf{S}}} \left\{ \| \mathbf{w}_h \|_{1,\Omega} | \phi_h |_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \right\}, \tag{4.13}$$

where $\widetilde{C}_{\widetilde{\mathbf{S}}}$ is a positive constant depending on $\| \mathbf{a} \|$, $\widehat{\alpha}$, $\widehat{\beta}$ and $c_2(\Omega)$.

Proof. It comes as a direct application of the Babuška-Brezzi theory, since **(H.1)** and **(H.2)** are part of its main hypotheses (see [15, Lemma 4.2]). $\square$

The solvability of the fixed-point problem (4.8) is now proved by means of the Brouwer fixed-point theorem, which reads as follows (cf. [12, Theorem 9.9-2]).

Theorem 4.3 (Brouwer)**.** *Let W be a compact and convex subset of a finite-dimensional Banach space X, and let $T : W \to W$ be a continuous mapping. Then T has at least one fixed-point.*

The discrete version of Lemma 3.7 is given as follows.

Lemma 4.4. *Let $r \in (0, r_0)$ with r_0 as given in (3.39), and let W_h be the closed ball in $\mathbf{H}_h$ defined as*

$$W_h := \{(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h : \|(\mathbf{w}_h, \phi_h)\| \leq r\}.$$

Assume that the data satisfy

$$\widetilde{c}(r)\left\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{0,\Gamma} + \|\mathbf{u}_D\|_{1/2,\Gamma}\right\} + \widetilde{C}_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2,\Gamma} \leq r, \tag{4.14}$$

where

$$\widetilde{c}(r) := \max\{r, 1\}(1 + \widetilde{C}_{\widetilde{\mathbf{S}}} r) C_{\mathbf{S}},$$

with $C_{\mathbf{S}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}}}$ as in (4.9) and (4.13), respectively. Then, there holds $\mathbf{T}_h(W_h) \subset W_h$.

Proof. It follows the same ideas as in Lemma 3.7, but now using the estimates (4.9) and (4.13). $\square$

We now provide the discrete analogues of Lemmas 3.8 and 3.9, which will allow us to prove the continuity of $\mathbf{T}_h$.

Lemma 4.5. *Let $r \in (0, r_0)$ with r_0 as given in (3.39). Then, there exists a positive constant $\bar{C}_{\mathbf{S}}(r)$, depending on r, such that*

$$\begin{aligned}\|\mathbf{S}_h(\mathbf{w}_h, \phi_h) - \mathbf{S}_h(\mathbf{z}_h, \psi_h)\| \leq \bar{C}_{\mathbf{S}}(r)\Big\{&\|\mathbf{S}_{1,h}(\mathbf{w}_h, \phi_h)\|_{\mathbb{L}^4(\Omega)}\|\phi_h - \psi_h\|_{L^4(\Omega)} \\ &+ \|\mathbf{S}_{2,h}(\mathbf{w}_h, \phi_h)\|_{1,\Omega}\Big(\|\mathbf{w}_h - \mathbf{z}_h\|_{1,\Omega} + \|\phi_h - \psi_h\|_{1,\Omega}\Big) + \|\mathbf{g}\|_{\infty,\Omega}\|\phi_h - \psi_h\|_{0,\Omega}\Big\}.\end{aligned} \tag{4.15}$$

for all $(\mathbf{w}_h, \phi_h), (\mathbf{z}_h, \phi_h) \in \mathbf{H}_h$ such that $\|\mathbf{w}_h\|_{1,\Omega}, \|\mathbf{z}_h\|_{1,\Omega} \leq r$.

Proof. The procedure is almost *verbatim* to the one for Lemma 3.8, except that, instead of the regularity assumption (3.46), we only need to consider an L^4-L^4-L^2 argument, that is, to take $p = q = 2$ when applying the Hölder inequality in (3.56):

$$\left|\int_\Omega \frac{\mu(\phi_h) - \mu(\psi_h)}{\mu(\psi_h)\mu(\phi_h)}\, \boldsymbol{\sigma}_h^{\mathrm{d}} : [(\boldsymbol{\sigma}_h - \boldsymbol{\varrho}_h)^{\mathrm{d}} - \kappa_1 \mathbf{e}(\mathbf{u}_h - \mathbf{s}_h)]\right| \leq \widehat{C}_2\|\phi_h - \psi_h\|_{L^4(\Omega)}\|\boldsymbol{\sigma}_h\|_{\mathbb{L}^4(\Omega)}\|\vec{\boldsymbol{\sigma}}_h - \vec{\boldsymbol{\varrho}}_h\|,$$

with $\widehat{C}_2$ as in (3.54). The fact that $\|\boldsymbol{\sigma}_h\|_{\mathbb{L}^4(\Omega)} < +\infty$ and $\|\phi_h - \psi_h\|_{L^4(\Omega)} < +\infty$ is because $\boldsymbol{\sigma}_h$, ϕ_h and ψ_h will be chosen as piecewise polynomials functions. We omit further details. $\square$

Lemma 4.6. *There exists a positive constant $\bar{C}_{\widetilde{\mathbf{S}}}$ depending on $c_2(\Omega)$ (cf. (3.7)) and the discrete inf-sup constant $\widehat{\alpha}$ (cf. (4.11)) such that*

$$\left\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h, \phi_h) - \widetilde{\mathbf{S}}_h(\mathbf{z}_h, \psi_h)\right\| \leq \bar{C}_{\widetilde{\mathbf{S}}}\left\{\|\mathbf{w}_h\|_{1,\Omega}|\phi_h - \psi_h|_{1,\Omega} + \|\mathbf{w}_h - \mathbf{z}_h\|_{1,\Omega}|\psi_h|_{1,\Omega}\right\}, \tag{4.16}$$

for all $(\mathbf{w}_h, \phi_h), (\mathbf{z}_h, \psi_h) \in \mathbf{H}_h$.

Proof. It follows the same arguments as in Lemma 3.9 (cf. [15, Lemma 3.7]), but using the inf-sup condition (4.11) rather than the V-ellipticity of $\mathbf{a}$, which, of course, cannot be applied here. $\square$

As a result of the previous two lemmas, we have the following.

Lemma 4.7. *Let $r \in (0, r_0)$ with r_0 as given in (3.39) and $W_h := \Big\{ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h : \| (\mathbf{w}_h, \phi_h) \| \le r \Big\}$. Then, there exists a constant $C_{\mathbf{T}_h} > 0$ such that*

$$\begin{aligned} &\| \mathbf{T}_h(\mathbf{w}_h, \phi_h) - \mathbf{T}_h(\mathbf{z}_h, \psi_h) \| \\ &\qquad \le C_{\mathbf{T}_h} \Big\{ \| \mathbf{S}_{1,h}(\mathbf{w}_h, \phi_h) \|_{\mathbb{L}^4(\Omega)} + \| \mathbf{S}_2(\mathbf{w}_h, \phi_h) \|_{1,\Omega} + \| \mathbf{g} \|_{\infty,\Omega} \Big\} \| (\mathbf{w}_h, \phi_h) - (\mathbf{z}_h, \psi_h) \|, \end{aligned} \tag{4.17}$$

for all $(\mathbf{w}_h, \phi_h), (\mathbf{z}_h, \psi_h) \in W_h$.

Proof. It follows the same arguments of Lemma 3.10, but now using (4.15), (4.16), and the continuous injection $H^1(\Omega) \hookrightarrow L^4(\Omega)$ with boundedness constant $\bar{C}_i$. This results in a constant $C_{\mathbf{T}_h} := \max\{\bar{C}\bar{C}_i, \bar{C} + \bar{C}_{\widetilde{\mathbf{S}}}, \bar{C}\}$, where $\bar{C} := (1 + \bar{C}_{\widetilde{\mathbf{S}}} r)\bar{C}_{\mathbf{S}}(r)$. $\square$

Notice that the previous lemma provides the continuity required by the Brouwer fixed-point theorem, in the convex and compact set $W_h \subset \mathbf{H}_h$. Therefore, we have the following result.

Theorem 4.8. *Assume that for $\delta_1 \in (0, 2\mu_1)$, $\delta_2 \in (0, 2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_4 > 0, \quad \text{and} \quad \kappa_3 \in \left(0, 2\delta_2\kappa_0 \min\left\{\kappa_1\left(1 - \frac{\delta_1}{2\mu_1}\right), \kappa_4\right\}\right),$$

and let $W_h := \{(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h : \| (\mathbf{w}_h, \phi_h) \| \le r\}$, with $r \in (0, r_0)$, r_0 as in (3.39). In addition, suppose that the data satisfy

$$\widetilde{c}(r) \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\} + \widetilde{C}_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma} \le r,$$

with $\widetilde{c}(r)$ as in Lemma 4.4. Then, the problem (4.2) has at least one solution $(\vec{\boldsymbol{\sigma}}_h, (\varphi_h, \lambda_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\boldsymbol{\gamma}} \times H_h^{\varphi} \times H_h^{\lambda}$, with $(\mathbf{u}_h, \varphi_h) \in W_h$. Moreover, there hold

$$\| \vec{\boldsymbol{\sigma}}_h \| \le C_{\mathbf{S}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} + \| \mathbf{u}_D \|_{0,\Gamma} \Big\}$$

and

$$\| (\varphi_h, \lambda_h) \| \le \widetilde{C}_{\widetilde{\mathbf{S}}} \Big\{ r \| \mathbf{u}_h \|_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \Big\}.$$

4.3 Specific Finite Element Subspaces

Given an integer $k \ge 0$, for each $K \in \mathcal{T}_h$ we define the local Raviart-Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus P_k(K)\mathbf{x},$$

where according to the terminology described in Section 1, $\mathbf{P}_k(K) := [P_k(K)]^2$, and $\mathbf{x}$ is a generic vector in $\mathbf{R}$. Similarly, $\mathbf{C}(\bar{\Omega}) = [C(\bar{\Omega})]^2$. Thus, we consider the global Raviart-Thomas space of order k to approximate the pseudostress $\boldsymbol{\sigma}$, the Lagrange space given by continuous piecewise polynomial vectors of degree $\le k+1$, and piecewise skew-symmetric polynomial tensors of degree $\le k$ for the vorticity tensor $\boldsymbol{\gamma}$, respectively

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}; \Omega) : \mathbf{c}^{\mathtt{t}}\boldsymbol{\tau}_h\big|_K \in \mathbf{RT}_k(K), \quad \forall\, \mathbf{c} \in \mathbf{R},\ \forall\, K \in \mathcal{T}_h \right\}, \tag{4.18}$$

$$\mathbf{H}_h^{\mathbf{u}} := \left\{ \mathbf{v}_h \in \mathbf{C}(\bar{\Omega}) : \mathbf{v}_h\big|_K \in \mathbf{P}_{k+1}(K), \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.19}$$

$$\mathbb{H}_h^{\boldsymbol{\gamma}} := \left\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) : \boldsymbol{\eta}_h\big|_K \in \mathbb{P}_k(K), \quad \forall\, K \in \mathcal{T}_h \right\}. \tag{4.20}$$

To provide finite element subspaces for the approximation of the temperature φ and the normal component of the heat flux λ, we must have in mind the hypotheses **(H.1)** and **(H.2)** assumed for H_h^φ and H_h^λ (that is, the inf-sup conditions (4.11) and (4.12)).

For the temperature φ, we will consider continuous piecewise polynomials of degree $\le k+1$, that is

$$H_h^\varphi := \left\{ \psi_h \in C(\bar{\Omega}) : \psi_h\big|_K \in P_{k+1}(K), \quad \forall\ K \in \mathcal{T}_h \right\}, \tag{4.21}$$

and for the normal heat flux λ, we let $\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_m\}$ be an independent triangulation of Γ (made of straight segments), and define $\widetilde{h} := \max_{j\in\{1,\ldots,m\}} |\widetilde{\Gamma}_j|$. Then, with the same integer $k \ge 0$ used in definitions (4.18), (4.19), (4.20), we approximate λ by piecewise polynomials of degree $\le k$ over this new mesh, that is

$$H_{\widetilde{h}}^\lambda := \left\{ \xi_{\widetilde{h}} \in L^2(\Gamma) : \xi_{\widetilde{h}}\big|_{\widetilde{\Gamma}_j} \in P_k(\widetilde{\Gamma}_j) \quad \forall\ j \in \{1, \ldots, m\} \right\}. \tag{4.22}$$

It can be proved (cf. [15, Lemma 4.10], [24, Lemma 4.7]) that $H_{\widetilde{h}}^\lambda$ do satisfy **(H.2)**, provided that $h \le C_0 \widetilde{h}$, for some constant $C_0 > 0$ (for computational purposes, we consider $\tilde{h}$ as approximately $2h$). In turn, since $P_0(\Gamma) \subseteq H_{\widetilde{h}}^\lambda$, it is easy to see that $V_h \subseteq \widetilde{V}$, where

$$\widetilde{V} := \left\{ \psi \in H^1(\Omega) : \int_\Gamma \psi = 0 \right\},$$

and hence thanks to the generalized Poincaré inequality it follows that $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ are equivalent in $\widetilde{V}$. In this way, $\mathbf{a}$ becomes V_h-elliptic, which clearly yields **(H.1)**.

According to [8, 24], the approximation properties of the specific finite element subspaces introduced here are

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) There exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div};\Omega)$ with $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \le C h^s \left\{ \|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} \right\}, \tag{4.23}$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \le C h^s \|\mathbf{u}\|_{s+1,\Omega}, \tag{4.24}$$

($\mathbf{AP}_h^{\boldsymbol{\gamma}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\operatorname{dist}\left(\boldsymbol{\gamma}, \mathbb{H}_h^{\boldsymbol{\gamma}}\right) \le C h^s \|\boldsymbol{\gamma}\|_{s,\Omega}, \tag{4.25}$$

($\mathbf{AP}_h^{\varphi}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\varphi \in H^{s+1}(\Omega)$, there holds

$$\operatorname{dist}\left(\varphi, H_h^\varphi\right) \le C h^s \|\varphi\|_{s+1,\Omega}, \tag{4.26}$$

($\mathbf{AP}_{\widetilde{h}}^{\lambda}$) there exists $C > 0$, independent of $\widetilde{h}$, such that for each $s \in (0, k+1]$, and for each $\lambda \in H^{-1/2+s}(\Gamma)$, there holds

$$\operatorname{dist}\left(\lambda, H_{\widetilde{h}}^\lambda\right) \le C \widetilde{h}^s \|\lambda\|_{-1/2+s,\Gamma}. \tag{4.27}$$

5 A Priori Error Analysis

Consider in addition to the notation introduced in (3.17) and (4.1) $\vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_h, \mathbf{w}_h, \boldsymbol{\chi}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}$. Then, let $(\vec{\boldsymbol{\sigma}}, (\varphi, \lambda)) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times H^1(\Omega) \times H^{-1/2}(\Gamma)$, with $(\mathbf{u}, \varphi) \in W$ be the solution of the continuous problem (3.18), and $(\vec{\boldsymbol{\sigma}}_h, (\varphi_h, \lambda_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma} \times H_h^{\varphi} \times H_h^{\lambda}$, with $(\mathbf{u}_h, \varphi_h) \in W_h$ be a solution of the discrete problem (4.2), that is,

$$
\begin{aligned}
(\mathbf{A}_{\varphi} + \mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\sigma}}, \vec{\boldsymbol{\tau}}) &= (F_{\varphi} + F_D)(\vec{\boldsymbol{\tau}}) && \forall\, \vec{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega), \\
(\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\tau}}_h) &= (F_{\varphi_h} + F_D)(\vec{\boldsymbol{\tau}}_h) && \forall\, \vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma},
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
\mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) &= F_{\mathbf{u},\varphi}(\psi) && \forall\, \psi \in H^1(\Omega), \\
\mathbf{b}(\varphi, \xi) &= G(\xi) && \forall\, \xi \in H^{-1/2}(\Gamma); \\
\mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) &= F_{\mathbf{u}_h,\varphi_h}(\psi_h) && \forall\, \psi_h \in H_h^{\varphi}, \\
\mathbf{b}(\varphi_h, \xi_h) &= G(\xi_h) && \forall\, \xi_h \in H_h^{\lambda}.
\end{aligned}
\tag{5.2}
$$

In order to derive an upper bound for $\| (\vec{\boldsymbol{\sigma}}, (\varphi, \lambda)) - (\vec{\boldsymbol{\sigma}}_h, (\varphi_h, \lambda_h)) \|$, we will apply the standard Strang Lemma for elliptic variational problems to the pair (5.1), whereas for the pair (5.2), a Strang-type estimate for saddle point problems will be applied, as we only have a difference between the functionals involved at continuous and discrete levels. We refer to [34, Theorems 11.1 and 11.12] to further information on these results, which we recall next.

Lemma 5.1. *Let V be a Hilbert space, $F \in V'$, and $A : V \times V \to R$ be a bounded and V-elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite-dimensional subspaces of V, and for each $h > 0$, consider a bounded bilinear form $A_h : V_h \times V_h \to R$ and a functional $F_h \in V_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic in V_h, that is, there exists a constant $\widetilde{\alpha} > 0$, independent of h, such that*

$$
A_h(v_h, v_h) \geq \widetilde{\alpha} \| v_h \|_V^2 \quad \forall v_h \in V_h, \ \forall h > 0.
$$

In turn, let $u \in V$ and $u_h \in V_h$ such that

$$
A(u, v) = F(v) \quad \forall\, v \in V \qquad \text{and} \qquad A_h(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h.
$$

Then, for each $h > 0$, there holds

$$
\begin{aligned}
\| u - u_h \|_V \leq C_{ST} \Bigg\{ & \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|F(w_h) - F_h(w_h)|}{\| w_h \|_V} \\
& + \inf_{\substack{v_h \in V_h \\ v_h \neq 0}} \left(\| u - v_h \|_V + \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{|A(v_h, w_h) - A_h(v_h, w_h)|}{\| w_h \|_V} \right) \Bigg\},
\end{aligned}
\tag{5.3}
$$

where $C_{ST} := \widetilde{\alpha}^{-1} \max\{1, \| A \|\}$.

Lemma 5.2. *Let H and Q be Hilbert spaces, $F \in H'$, $G \in Q'$, and let $a : H \times H \to R$ and $b : H \times Q \to R$ be bounded bilinear forms satisfying the hypotheses of the Babuška-Brezzi theory. Furthermore, let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be sequences of finite-dimensional subspaces of H and Q, respectively, and for each $h > 0$, consider functionals $F_h \in H_h'$, $G_h \in Q_h'$. In addition, assume that a and b satisfy the hypotheses of the discrete Babuška-Brezzi theory uniformly on H_h and Q_h, that is,* *there exists positive constants $\bar{\alpha}$ and $\bar{\beta}$, both independent of h, such that, denoting by V_h the discrete kernel of the operator induced by b, there holds*

$$\sup_{\substack{\psi_H \in V_h \\ \psi_h \neq 0}} \frac{a(\psi_h, \phi_h)}{\| \psi_h \|_H} \geq \bar{\alpha} \| \phi_h \|_H \quad \forall\, \phi_h \in V_h \qquad \text{and} \qquad \sup_{\substack{\psi_h \in H_h \\ \psi_h \neq 0}} \frac{b(\psi_h, \xi_h)}{\| \psi_h \|_H} \geq \bar{\beta} \| \xi_h \|_Q \quad \forall\, \xi_h \in Q_h. \tag{5.4}$$

In turn, let $(\varphi, \lambda) \in H \times Q$ and $(\varphi_h, \lambda_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a(\varphi, \psi) + b(\psi, \lambda) &= F(\psi) \quad \forall\, \psi \in H, \\ b(\varphi, \xi) &= G(\xi) \quad \forall\, \xi \in Q; \end{aligned}$$

and

$$\begin{aligned} a(\varphi_h, \psi_h) + b(\psi_h, \lambda_h) &= F_h(\psi_h) \quad \forall\, \psi_h \in H_h, \\ b(\varphi_h, \xi_h) &= G_h(\xi_h) \quad \forall\, \xi_h \in Q_h. \end{aligned}$$

Then, for each $h > 0$, there holds

$$\begin{aligned} \| \varphi - \varphi_h \|_H + \| \lambda - \lambda_h \|_Q \leq \bar{C}_{ST} \Bigg\{ & \inf_{\substack{\psi_h \in H_h \\ \psi_h \neq 0}} \| \varphi - \psi_h \|_H + \inf_{\substack{\xi_h \in Q_h \\ \xi_h \neq 0}} \| \lambda - \xi_h \|_Q \\ & + \sup_{\substack{\phi_h \in H_h \\ \phi_h \neq 0}} \frac{|F(\phi_h) - F_h(\phi_h)|}{\| \phi_h \|_H} + \sup_{\substack{\eta_h \in Q_h \\ \eta_h \neq 0}} \frac{|G(\eta_h) - G_h(\eta_h)|}{\| \eta_h \|_Q} \Bigg\}, \end{aligned} \tag{5.5}$$

where $\bar{C}_{ST}$ is a positive constant depending only on $\| a \|$, $\| b \|$, $\bar{\alpha}$ and $\bar{\beta}$.

5.1 Céa's Estimate

In what follows, we denote as usual

$$\operatorname{dist}\left(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\right) := \inf_{\vec{\boldsymbol{\tau}}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\tau}}_h \|$$

and

$$\operatorname{dist}\Big((\varphi, \lambda), H_h^{\varphi} \times H_h^{\lambda}\Big) := \inf_{(\psi_h, \xi_h) \in H_h^{\varphi} \times H_h^{\lambda}} \| (\varphi, \lambda) - (\psi_h, \xi_h) \|.$$

Then, we have the following lemma establishing a preliminary estimate for $\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \|$.

Lemma 5.3. *Let $C_{ST} := \frac{2}{\alpha(\Omega)} \max\{1, \| \mathbf{A}_\varphi + \mathbf{B}_{\mathbf{u},\varphi} \|\}$, where $\frac{\alpha(\Omega)}{2}$ is the ellipticity constant of $\mathbf{A}_\varphi + \mathbf{B}_{\mathbf{u},\varphi}$ (cf. (3.38)). Then, there holds*

$$\begin{aligned} \| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \| \leq C_{ST} \Bigg\{ & \bigg(1 + 2C_{\mathbf{A}} + \widehat{C}_1 \Big(\| \mathbf{u} \|_{1,\Omega} + \| \mathbf{u}_h \|_{1,\Omega} \Big) \bigg) \operatorname{dist}\left(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\right) \\ & + \Big\{ (2 + \kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} + \widehat{C}_2 C_\varepsilon \widetilde{C}_\varepsilon \| \boldsymbol{\sigma} \|_{\varepsilon,\Omega} + \widehat{C}_2 C_i \| \mathbf{u} \|_{1,\Omega}^2 \Big\} \| \varphi - \varphi_h \|_{1,\Omega} \\ & + \widehat{C}_1 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \Bigg\}. \end{aligned} \tag{5.6}$$

Proof. From Lemma 3.5, we see that $\mathbf{A}_\varphi + \mathbf{B}_{\mathbf{u},\varphi}$ and $\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h,\varphi_h}$ are bilinear, bounded and uniformly elliptic forms with ellipticity constant $\dfrac{\alpha(\Omega)}{2}$. Also, $F_\varphi + F_D$ and $F_{\varphi_h} + F_D$ are linear bounded functionals in $\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$ and $\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\gamma}_h$, respectively. Hence , a straightforward application of Lemma 5.1 to the pair (5.1) yields

$$\|\,\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\sigma}}_h\,\|\le C_{ST}\left\{\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\gamma}_h\\ \vec{\boldsymbol{\tau}}_h\neq\vec{\mathbf{0}}}}\frac{|F_\varphi(\vec{\boldsymbol{\tau}}_h)-F_{\varphi_h}(\vec{\boldsymbol{\tau}}_h)|}{\|\,\vec{\boldsymbol{\tau}}_h\,\|}+\inf_{\substack{\vec{\boldsymbol{\zeta}}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\gamma}_h\\ \vec{\boldsymbol{\zeta}}_h\neq\vec{\mathbf{0}}}}\left(\left\|\,\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\,\right\|\right.\right.$$
$$\left.\left.+\sup_{\substack{\vec{\boldsymbol{\tau}}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\times\mathbf{H}^{\mathbf{u}}_h\times\mathbb{H}^{\gamma}_h\\ \vec{\boldsymbol{\tau}}_h\neq\vec{\mathbf{0}}}}\frac{|(\mathbf{A}_\varphi+\mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)-(\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|}{\|\,\vec{\boldsymbol{\tau}}_h\,\|}\right)\right\},\tag{5.7}$$

where $C_{ST}:=\dfrac{2}{\alpha(\Omega)}\max\{1,\|\,\mathbf{A}_\varphi+\mathbf{B}_{\mathbf{u},\varphi}\,\|\}$. First, we notice that

$$|F_\varphi(\vec{\boldsymbol{\tau}}_h)-F_{\varphi_h}(\vec{\boldsymbol{\tau}}_h)|=|F_{\varphi-\varphi_h}(\vec{\boldsymbol{\tau}}_h)|\le\|\,\mathbf{g}\,\|_{\infty,\Omega}\|\,\varphi-\varphi_h\,\|_{0,\Omega}(2+\kappa_2^2)^{1/2}\|\,\vec{\boldsymbol{\tau}}_h\,\|.\tag{5.8}$$

Then, in order to estimate the last supremum in (5.7), we add and subtract suitable terms to write

$$\begin{aligned}&(\mathbf{A}_\varphi+\mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)-(\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)\\ &\quad=\ (\mathbf{A}_\varphi+\mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\zeta}}_h-\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)+(\mathbf{A}_\varphi-\mathbf{A}_{\varphi_h})(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)+(\mathbf{B}_{\mathbf{u},\varphi}-\mathbf{B}_{\mathbf{u},\varphi_h})(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)\\ &\quad+\ (\mathbf{B}_{\mathbf{u},\varphi_h}-\mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\sigma}},\vec{\boldsymbol{\tau}}_h)+(\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)\end{aligned}$$

and so, using boundedness of the bilinear forms $\mathbf{A}_\varphi$, $\mathbf{B}_{\mathbf{u},\varphi}$, $\mathbf{A}_{\varphi_h}$, $\mathbf{B}_{\mathbf{u}_h,\varphi_h}$ (cf. (3.34), (3.35)) and the properties of these forms stated in Lemma 3.2, we get

$$\begin{aligned}&|(\mathbf{A}_\varphi+\mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)-(\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\tau}}_h)|\\ &\le\ \left\{C_{\mathbf{A}}+\widehat{C}_1\|\,\mathbf{u}\,\|_{1,\Omega}\right\}\left\|\,\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\,\right\|\|\,\vec{\boldsymbol{\tau}}_h\,\|\\ &+\left|\int_\Omega\frac{\mu(\varphi_h)-\mu(\varphi)}{\mu(\varphi_h)\mu(\varphi)}\boldsymbol{\sigma}^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\\ &+\left|\int_\Omega\frac{\mu(\varphi_h)-\mu(\varphi)}{\mu(\varphi_h)\mu(\varphi)}(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\\ &+\left|\int_\Omega\frac{1}{\mu(\varphi_h)}[\mathbf{u}\otimes(\mathbf{u}-\mathbf{u}_h)]^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\\ &+\left\{C_{\mathbf{A}}+\widehat{C}_1\|\,\mathbf{u}_h\,\|_{1,\Omega}\right\}\left\|\,\vec{\boldsymbol{\sigma}}-\vec{\boldsymbol{\zeta}}_h\,\right\|\|\,\vec{\boldsymbol{\tau}}_h\,\|\,,\end{aligned}\tag{5.9}$$

with $\widehat{C}_1$ defined as in (3.53). A similar procedure to the one realized in the proof of Lemma 3.8 will lead us to suitable bounds for the second, third and fourth terms of the foregoing inequality, respectively

$$\left|\int_\Omega\frac{\mu(\varphi_h)-\mu(\varphi)}{\mu(\varphi_h)\mu(\varphi)}\boldsymbol{\sigma}^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\le\widehat{C}_2C_\varepsilon\widetilde{C}_\varepsilon\|\,\boldsymbol{\sigma}\,\|_{\varepsilon,\Omega}\|\,\varphi-\varphi_h\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\tau}}_h\,\|,$$

$$\left|\int_\Omega\frac{\mu(\varphi_h)-\mu(\varphi)}{\mu(\varphi_h)\mu(\varphi)}(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\le\widehat{C}_2C_i\|\,\mathbf{u}\,\|_{1,\Omega}^2\|\,\varphi-\varphi_h\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\tau}}_h\,\|,$$

and

$$\left|\int_\Omega\frac{1}{\mu(\varphi_h)}[\mathbf{u}\otimes(\mathbf{u}-\mathbf{u}_h)]^{\mathtt{d}}:[\boldsymbol{\tau}_h^{\mathtt{d}}-\kappa_1\mathbf{e}(\mathbf{v}_h)]\right|\le\widehat{C}_1\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\mathbf{u}-\mathbf{u}_h\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\tau}}_h\,\|,$$

with $\widehat{C}_2$ defined as in (3.54). Putting the last three inequalities back into (5.9) results in

$$
\begin{aligned}
&|(\mathbf{A}_\varphi + \mathbf{B}_{\mathbf{u},\varphi})(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h,\varphi_h})(\vec{\boldsymbol{\zeta}}_h, \vec{\boldsymbol{\tau}}_h)| \\
&\quad \leq \left\{2C_{\mathbf{A}} + \widehat{C}_1\Big(\|\,\mathbf{u}\,\|_{1,\Omega} + \|\,\mathbf{u}_h\,\|_{1,\Omega}\Big)\right\}\Big\|\,\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\zeta}}_h\,\Big\|\|\,\vec{\boldsymbol{\tau}}_h\,\| \\
&\quad + \widehat{C}_2\Big\{C_\varepsilon \widetilde{C}_\varepsilon \|\,\boldsymbol{\sigma}\,\|_{\varepsilon,\Omega} + C_i\|\,\mathbf{u}\,\|_{1,\Omega}^2\Big\}\|\,\varphi - \varphi_h\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\tau}}_h\,\| \\
&\quad + \widehat{C}_1\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\mathbf{u} - \mathbf{u}_h\,\|_{1,\Omega}\|\,\vec{\boldsymbol{\tau}}_h\,\|\,.
\end{aligned}
$$

This expression, together with (5.8), and back into (5.7), results in (5.6), concluding this way the proof. $\square$

Then, for $\|\,(\varphi,\lambda) - (\varphi_h,\lambda_h)\,\|$, we recall the following result from [15].

Lemma 5.4. *There exists a constant $\widehat{C}_{ST}$, depending only on $\|\,\mathbf{a}\,\|$, $\|\,\mathbf{b}\,\|$, $\widehat{\alpha}$ and $\widehat{\beta}$ (cf. (4.11), (4.12)), such that*

$$
\begin{aligned}
&\|\,(\varphi,\lambda) - (\varphi_h,\lambda_h)\,\| \\
&\leq \widehat{C}_{ST}\Big\{c_2(\Omega)|\,\varphi\,|_{1,\Omega}\|\,\mathbf{u} - \mathbf{u}_h\,\|_{1,\Omega} + c_2(\Omega)\|\,\mathbf{u}_h\,\|_{1,\Omega}|\,\varphi - \varphi_h\,|_{1,\Omega} + \mathrm{dist}\,\Big((\varphi,\lambda), H_h^\varphi \times H_h^\lambda\Big)\Big\}.
\end{aligned} \tag{5.10}
$$

Proof. See [15, Lemma 5.4]. $\square$

Having established bounds for $\|\,\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\,\|$ and $\|\,(\varphi,\lambda) - (\varphi_h,\lambda_h)\,\|$, we are now able to derive the Céa estimate for the global error. Indeed, by adding the estimates (5.6) and (5.10), we have

$$
\begin{aligned}
&\|\,\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\,\| + \|\,(\varphi,\lambda) - (\varphi_h,\lambda_h)\,\| \\
&\quad \leq C_{ST}\Big\{1 + 2C_{\mathbf{A}} + \widehat{C}_1\Big(\|\,\mathbf{u}\,\|_{1,\Omega} + \|\,\mathbf{u}_h\,\|_{1,\Omega}\Big)\Big\}\ \mathrm{dist}\,\big(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\big) \\
&\quad + \widehat{C}_{ST}\ \mathrm{dist}\,\Big((\varphi,\lambda), H_h^\varphi \times H_h^\lambda\Big) + \Big\{C_{ST}\Big((2+\kappa_2^2)^{1/2}\|\,\mathbf{g}\,\|_{\infty,\Omega} \\
&\quad + \widehat{C}_2 C_\varepsilon \widetilde{C}_\varepsilon\|\,\boldsymbol{\sigma}\,\|_{\varepsilon,\Omega} + \widehat{C}_2 C_i\|\,\mathbf{u}\,\|_{1,\Omega}^2\Big) + \widehat{C}_{ST}c_2(\Omega)\|\,\mathbf{u}_h\,\|_{1,\Omega}\Big\}\ \|\,\varphi - \varphi_h\,\|_{1,\Omega} \\
&\quad + \Big\{C_{ST}\widehat{C}_1\|\,\mathbf{u}\,\|_{1,\Omega} + \widehat{C}_{ST}c_2(\Omega)\|\,\varphi\,\|_{1,\Omega}\Big\}\ \|\,\mathbf{u} - \mathbf{u}_h\,\|_{1,\Omega}
\end{aligned} \tag{5.11}
$$

where we recall that C_i, C_ε, and $\widetilde{C}_\varepsilon$ are boundedness constants coming from the injections $H^1(\Omega) \hookrightarrow L^8(\Omega)$, $H^1(\Omega) \hookrightarrow L^{2/(1-\varepsilon)}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{2/\varepsilon}(\Omega)$, respectively. In turn, notice that the terms $\|\,\mathbf{u}\,\|_{1,\Omega}$, $\|\,\varphi\,\|_{1,\Omega}$, $\|\,\mathbf{u}_h\,\|_{1,\Omega}$ and $\|\,\varphi_h\,\|_{1,\Omega}$ can be bounded by data using the estimates (3.33), (3.47), (4.9), and (4.13), respectively; and $\|\,\boldsymbol{\sigma}\,\|_{\varepsilon,\Omega}$ as well by using the further regularity assumption (3.46). Therefore, after some algebraic work, and introducing the constants:

$$
C_1 := \widehat{C}_1, \quad C_2 := C_{ST}\widehat{C}_2 C_\varepsilon \widetilde{C}_\varepsilon, \quad C_3 := C_{ST}\widehat{C}_2 C_i, \quad C_4 := \widehat{C}_{ST}c_2(\Omega)C_{\widetilde{\mathbf{S}}},
$$

$$
\mathbf{C}_0(\mathbf{g},\mathbf{u}_D) := C_{\mathbf{S}}\left\{r\|\,\mathbf{g}\,\|_{\infty,\Omega} + \|\,\mathbf{u}_D\,\|_{1/2,\Gamma} + \|\,\mathbf{u}_D\,\|_{0,\Gamma}\right\},
$$

$$
\mathbf{C}_{0,\varepsilon}(\mathbf{g},\mathbf{u}_D) := \widetilde{C}_{\mathbf{S}}(r)\left\{r\|\,\mathbf{g}\,\|_{\infty,\Omega} + \|\,\mathbf{u}_D\,\|_{1/2+\varepsilon,\Gamma} + \|\,\mathbf{u}_D\,\|_{0,\Gamma}\right\},
$$

$$
\mathbf{C}_1(\mathbf{g},\mathbf{u}_D,\varphi_D) := C_{ST}(2+\kappa_2^2)^{1/2}\|\,\mathbf{g}\,\|_{\infty,\Omega} + C_2\mathbf{C}_{0,\varepsilon}(\mathbf{g},\mathbf{u}_D) + C_3\mathbf{C}_0(\mathbf{g},\mathbf{u}_D)^2 + \widehat{C}_{ST}c_2(\Omega)\mathbf{C}_0(\mathbf{g},\mathbf{u}_D),
$$

$$\mathbf{C}_2(\mathbf{g}, \mathbf{u}_D, \varphi_D) := (C_{ST}C_1 + C_4 r)\mathbf{C}_0(\mathbf{g}, \mathbf{u}_D) + C_4 \| \varphi_D \|_{1/2,\Omega},$$

and

$$\mathbf{C}(\mathbf{g}, \mathbf{u}_D, \varphi_D) := \max\Big\{\mathbf{C}_1(\mathbf{g}, \mathbf{u}_D, \varphi_D), \mathbf{C}_2(\mathbf{g}, \mathbf{u}_D, \varphi_D)\Big\},$$

it can be shown that

$$\begin{aligned}\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \| + \| (\varphi, \lambda) - (\varphi_h, \lambda_h) \| &\leq \Big\{1 + 2C_A + 2C_1\mathbf{C}_0(\mathbf{g}, \mathbf{u}_D)\Big\} \operatorname{dist}\left(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\right) \\ &+ \widehat{C}_{ST} \operatorname{dist}\Big((\varphi, \lambda), H_h^{\varphi} \times H_h^{\lambda}\Big) + \mathbf{C}(\mathbf{g}, \mathbf{u}_D, \varphi_D)\Big\{\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \| + \| (\varphi, \lambda) - (\varphi_h, \lambda_h) \|\Big\}, \end{aligned} \tag{5.12}$$

which leads us to the main result of this section.

Theorem 5.5. *Assume the data $\mathbf{g}$, $\mathbf{u}_D$ and φ_D satisfy*

$$\mathbf{C}_i(\mathbf{g}, \mathbf{u}_D, \varphi_D) \leq \frac{1}{2} \quad \forall\, i \in \{1, 2\}. \tag{5.13}$$

Then, there exists a positive constant C depending only on parameters, data and other constants, all of them independent of h, such that

$$\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \| + \| (\varphi, \lambda) - (\varphi_h, \lambda_h) \| \leq C\Big\{\operatorname{dist}\left(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\right) + \operatorname{dist}\Big((\varphi, \lambda), H_h^{\varphi} \times H_h^{\lambda}\Big)\Big\} \tag{5.14}$$

Proof. The hypotheses (5.13) assures us that $\mathbf{C}(\mathbf{g}, \mathbf{u}_D, \varphi_D) \leq \frac{1}{2}$, and hence,

$$\begin{aligned}&\| \vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h \| + \| (\varphi, \lambda) - (\varphi_h, \lambda_h) \| \\ &\qquad \leq 2\Big\{1 + 2C_A + 2C_1\mathbf{C}_0(\mathbf{g}, \mathbf{u}_D)\Big\} \operatorname{dist}\left(\vec{\boldsymbol{\sigma}}, \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\right) + 2\widehat{C}_{ST} \operatorname{dist}\Big((\varphi, \lambda), H_h^{\varphi} \times H_h^{\lambda}\Big),\end{aligned}$$

thus proving the Céa estimate (5.14) with $C := 2 \cdot \max\{1 + 2C_A + 2C_1\mathbf{C}_0(\mathbf{g}, \mathbf{u}_D), \widehat{C}_{ST}\}$. □

We end this section with the corresponding rates of convergence of the Galerkin Scheme (4.2) when the finite element subspaces (4.18)-(4.22) are used.

Theorem 5.6. *In addition to the hypotheses of Theorems 3.11, 4.8 and 5.5, assume that there exists $s > 0$ such that $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega)$, $\varphi \in H^{s+1}(\Omega)$ and $\lambda \in H^{-1/2+s}(\Gamma)$. Then, there exists $\widehat{C} > 0$, independent of h and $\widetilde{h}$ such that for all $h \leq C_0\widetilde{h}$ there holds*

$$\begin{aligned}\Big\| (\vec{\boldsymbol{\sigma}}, (\varphi, \lambda)) - (\vec{\boldsymbol{\sigma}}_h, (\varphi_h, \lambda_{\widetilde{h}})) \Big\| &\leq \widehat{C}\widetilde{h}^{\min\{s,k+1\}} \| \lambda \|_{-1/2+s,\Gamma} \\ &+ \widehat{C}h^{\min\{s,k+1\}}\Big\{\| \boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{div}\,\boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{u} \|_{s+1,\Omega} + \| \boldsymbol{\gamma} \|_{s,\Omega} + \| \varphi \|_{s+1,\Omega}\Big\}.\end{aligned} \tag{5.15}$$

Proof. It follows from the Céa's estimate (5.14) and the approximation properties ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{\gamma}$), ($\mathbf{AP}_h^{\varphi}$), and ($\mathbf{AP}_{\widetilde{h}}^{\lambda}$) described in Section 4.3. □

5.2 Postprocessing of the Pressure

Equation (2.8) and the orthogonal decomposition for the pseudostress tensor provided in Lemma 3.1 (recall that $\boldsymbol{\sigma}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \subset \mathbb{H}_0(\mathbf{div}; \Omega)$) suggests that the discrete pressure should take the form

$$p_h = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h + c_h\mathbb{I} + \mathbf{u}_h \otimes \mathbf{u}_h), \quad \text{with } c_h := -\frac{1}{2|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h). \tag{5.16}$$

On the other hand, since $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$, the modified equation for the continuous pressure becomes

$$p = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma} + c\mathbb{I} + \mathbf{u}\otimes\mathbf{u}), \quad \text{with } c := -\frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u}). \tag{5.17}$$

Then, it is easy to prove that there exists a constant $\widehat{C}$ independent of h and $\widetilde{h}$ such that

$$\| p - p_h \|_{0,\Omega} \leq \widehat{C} \left\{ \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{\mathbf{div};\Omega} + \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \right\}, \tag{5.18}$$

meaning that the rate of convergence of p_h corresponds to the same one provided for the rest of the variables, according to (5.15).

6 Numerical Results

We present in this section two examples that will illustrate the performance of our augmented mixed-primal finite element method on a set of quasi-uniform triangulations. The computational implementation is based on a `FreeFem++` code (cf. [27]) and the use of the direct linear solvers UMFPACK (cf. [20]) for the first example, and the Multifrontal Massively Parallel Solver MUMPS (cf. [5]) for the second one. Here, the iterative method comes straightforward from the uncoupling strategy presented in Section 4.1. Then, as a stopping criteria, we finish the algorithm when the relative error between two consecutive iterations of the complete coefficient vector measured in the discrete ℓ^2 norm is sufficiently small, this is,

$$\frac{\| \mathbf{coeff}^{\,m+1} - \mathbf{coeff}^{\,m} \|_{\ell^2}}{\| \mathbf{coeff}^{\,m+1} \|_{\ell^2}} < \mathtt{tol},$$

where `tol` is a specified tolerance.

Let us first define the error per variable

$$\begin{aligned} &e(\boldsymbol{\sigma}) := \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{\mathbf{div};\Omega}, \quad e(\mathbf{u}) := \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega}, \quad e(p) := \| p - p_h \|_{0,\Omega}, \\ &e(\boldsymbol{\gamma}) := \| \boldsymbol{\gamma} - \boldsymbol{\gamma}_h \|_{0,\Omega}, \quad e(\varphi) := \| \varphi - \varphi_h \|_{1,\Omega}, \quad e(\lambda) := \| \lambda - \lambda_{\widetilde{h}} \|_{0,\Gamma}, \end{aligned}$$

as well as their corresponding rates of convergence

$$\begin{aligned} &r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad r(\mathbf{u}) := \frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')}, \quad r(p) := \frac{\log(e(p)/e'(p))}{\log(h/h')}, \\ &r(\boldsymbol{\gamma}) := \frac{\log(e(\boldsymbol{\gamma})/e'(\boldsymbol{\gamma}))}{\log(h/h')}, \quad r(\varphi) := \frac{\log(e(\varphi)/e'(\varphi))}{\log(h/h')}, \quad r(\lambda) := \frac{\log(e(\lambda)/e'(\lambda)),}{\log(\widetilde{h}/\widetilde{h}')}, \end{aligned}$$

where h and h' (respectively $\widetilde{h}$ and $\widetilde{h}'$) denote two consecutive mesh sizes with errors e and e'.

6.1 Example 1: Smooth Exact Solution

In our first example, we consider $\Omega := [0,1]^2$, viscosity, thermal conductivity and body force given by,

$$\mu(\varphi) = \exp(-\varphi), \quad \mathbb{K} = \exp(x+y)\mathbb{I}, \quad \mathbf{g} = (0,-1)^{\mathtt{t}},$$

and boundary conditions such that the exact solution is given by $\mathbf{u}(x,y) = (u_1(x,y), u_2(x,y))^{\mathtt{t}}$ with

$$u_1(x,y) = 4y(x^2-1)^2(y^2-1), \quad u_2(x,y) = -4x(y^2-1)^2(x^2-1),$$

Finite Element: $\mathbb{RT}_0$ - $\mathbf{P}_1$ - P_0 - P_1 - P_0							
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$
960	0.1901	3.6546e-01	-	6.7123e-01	-	7.5087e-02	-
3536	0.0950	1.7831e-01	1.0353	2.9451e-01	1.1885	3.1834e-02	1.2380
13682	0.0490	8.7436e-02	1.0763	1.4031e-01	1.1199	1.4561e-02	1.1814
53895	0.0244	4.3350e-02	1.0076	6.8960e-02	1.0201	6.9382e-03	1.0646
216315	0.0140	2.1638e-02	1.2426	3.3814e-02	1.2745	3.4183e-03	1.2660
$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(\varphi)$	$r(\varphi)$	$\widetilde{h}$	$e(\lambda)$	$r(\lambda)$	Iterations
4.8085e-01	-	3.9769e-02	-	0.2500	8.7301e-01	-	12
1.9790e-01	1.2808	1.8860e-02	1.0763	0.1250	4.2801e-01	1.0284	11
9.1585e-02	1.1638	8.9611e-03	1.1240	0.0625	2.0754e-01	1.0443	10
4.4504e-02	1.0364	4.6255e-03	0.9497	0.0312	1.0216e-01	1.0226	10
2.1647e-02	1.2889	2.2669e-03	1.2754	0.0156	5.0642e-02	1.0124	10

Table 6.1: Convergence history for Example 1, with a uniform mesh refinement and a first order approximation.

and

$$p(x,y) = (x-0.5)(y-0.5), \quad \varphi(x,y) = \cos(xy) + 1.$$

Notice that in this case, nonzero source terms appear in the momentum and energy equations. Nevertheless, the well-posedness of the corresponding problems is still ensured, since the smoothness of the exact solution provides right-hand sides with terms in $L^2(\Omega)$, thus only requiring a minor modification of the variational formulation in its right-hand side. Concerning the stabilization parameters, these are taken as pointed out in Section 3.3, that is

$$\kappa_1 = \frac{\mu_1^2}{\mu_2}, \quad \kappa_2 = \frac{1}{\mu_2}, \quad \kappa_3 = \frac{\kappa_0 \mu_1^2}{2\mu_2}, \quad \kappa_4 = \frac{\mu_1^2}{2\mu_2},$$

where the bounds for the viscosity function are estimated in

$$\mu_1 := \exp(5), \quad \mu_2 := \exp(-5),$$

and for κ_0, we simply take $\kappa_0 = 1$. Finally, we set the fixed-point algorithm such that it starts with $(\mathbf{u}, \varphi) = (\mathbf{0}, 0)$ and stops when error between consecutive iterations reaches $\texttt{tol} = 1\mathrm{e}-08$.

In Figure 6.1, we compare the approximations to the velocity, pressure (postprocessed according to (5.16)) and temperature fields, respectively, with their exact counterparts when using 216,315 DOF and a first order approximation, thus showing the good quality of our numerical results. On the other hand, we show in Tables 6.1 and 6.2 the convergence history for a sequence of uniform mesh refinements when the finite element spaces described in Section 4.3 are used with $k = 0$ and $k = 1$, respectively. It can be observed that the rates of convergence are the ones expected from Theorem 5.6 (with $s = k + 1$), that is $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively.

6.2 Example 2: Natural Convection in a Square Cavity

In a second example we consider the natural convection of a fluid in a square cavity with differentially heated walls. This phenomenon has been widely studied with different types of boundary conditions (see, e.g. [6, 18, 21]). For instance, we first recall from [18] the problem with dimensionless numbers:

Finite Element: $\mathbb{RT}_1$ - $\mathbf{P}_2$ - P_1 - P_2 - P_1							
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$
2780	0.1901	2.7406e-02	-	4.7383e-02	-	1.6020e-02	-
10412	0.1025	6.8657e-03	2.2428	9.5429e-03	2.5965	5.0303e-03	1.8769
40658	0.0490	1.6687e-03	1.9165	2.1032e-03	2.0491	1.1836e-03	1.9604
160913	0.0256	4.2746e-04	2.0974	5.3457e-04	2.1094	2.9724e-04	2.1279
$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(\varphi)$	$r(\varphi)$	$\widetilde{h}$	$e(\lambda)$	$r(\lambda)$	Iterations
3.2956e-02	-	2.4371e-03	-	0.2500	5.9381e-02	-	10
6.0862e-03	2.7369	4.7855e-04	2.6375	0.1250	1.4765e-02	2.0078	10
1.3314e-03	2.0592	9.9904e-05	2.1225	0.0625	3.6813e-03	2.0039	10
3.4391e-04	2.0845	2.2527e-05	2.2938	0.0312	9.1906e-04	2.0020	10

Table 6.2: Convergence history for Example 1, with a uniform mesh refinement and a second order approximation.

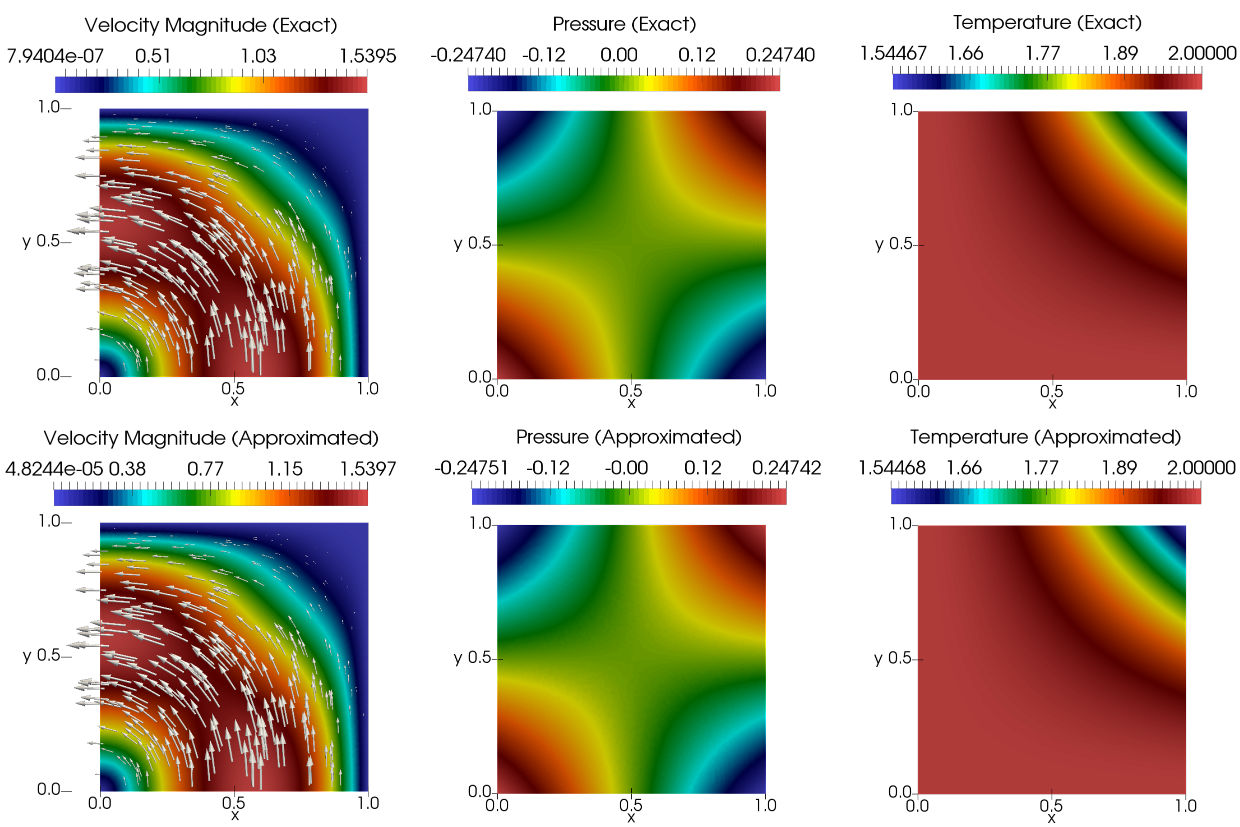

Figure 6.1: Graphical comparison of the exact solution $(\mathbf{u}, p, \varphi)$ (upper row) and its numerical approximation $(\mathbf{u}_h, p_h, \varphi_h)$ (lower row) for the data given in Example 1. Results calculated with 216,315 DOF and a first-order approximation $(\mathbb{RT}_0 - \mathbf{P}_1 - P_0 - P_1 - P_0)$.

Find $(\mathbf{u}, p, \varphi)$ such that

$$\begin{aligned} -\mathrm{Pr}\,\mathbf{div}\,(2\mu(\varphi)\mathbf{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p - \mathrm{Ra}\,\varphi\mathbf{g} &= 0 && \text{in } \Omega, \\ \mathrm{div}\,\mathbf{u} &= 0 && \text{in } \Omega, \\ -\mathrm{div}\,(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma, \\ \varphi &= \varphi_D && \text{on } \Gamma. \end{aligned}$$

where Pr and Ra are the Prandtl and Rayleigh numbers, defined respectively as the ratio of momentum diffusivity to thermal diffusivity, and the ratio of buoyancy forces to viscosity forces times the Prandtl number. Hence, we model the cavity as $\Omega = [0,1]^2$ and we consider Prandtl and Rayleigh numbers as

$$\mathrm{Pr} = 0.5, \quad \mathrm{Ra} = 2000.$$

In addition, the viscosity, thermal conductivity and body force will be given by

$$\mu(\varphi) = \exp(-\varphi), \quad \mathbb{K} = \mathbb{I}, \quad \mathbf{g} = (0,1)^{\mathbf{t}};$$

and the boundary conditions will be taken as in [18], that is

$$\mathbf{u}_D = \mathbf{0} \quad \text{and} \quad \varphi_D(x,y) = \frac{1}{2}\Big(1 - \cos(2\pi x)\Big)\Big(1 - y\Big),$$

both on Γ. The last condition results in the left, top and right walls with zero-temperature, and the described sinusoidal profile in the bottom wall, with a peak of temperature $\varphi = 1$ at $x = 0.5$. In this case, there are not source terms present, and the analytical solution is unknown. Therefore, to construct the convergence history for this example, we consider a solution calculated with 3,493,345 DOF as the exact solution. Concerning the stabilization parameters, it can be seen by redoing the analysis in Lemma 3.5 that these become

$$\kappa_1 = \frac{2\mathrm{Pr}\mu_1^2}{\mu_2}, \quad \kappa_2 = \frac{1}{\mu_2}, \quad \kappa_3 = \frac{\kappa_0\mu_1^2\mathrm{Pr}}{\mu_2}, \quad \kappa_4 = \frac{\mathrm{Pr}\mu_1^2}{\mu_2},$$

and we consider $\kappa_0 = 1$. In this regard, the viscosity bounds are estimated according to the maximum and minimum values of the temperature on the boundary, that is,

$$\mu_1 = \exp(-1), \quad \mu_2 = \exp(0) = 1.$$

Here, the fixed-point algorithm starts with $(\mathbf{u}, \varphi) = (\mathbf{10^{-3}}, 0.5)$ and stops when error between consecutive iterations reaches $\mathtt{tol} = 1\mathrm{e}-08$.

Some contours of the pressure, temperature, velocity and vorticity fields are available in Figure 6.2, where it is possible to see the expected physical behaviour from [18], that is, convection currents form inside the cavity in a symmetric configuration and, due to the relatively low Rayleigh number, the heat transfer throughout the fluid is mainly due to conduction. On the other hand, since the solution is smooth, it makes sense to expect convergence of $\mathcal{O}(h)$ when the approximation is made using the finite element subspaces from Section 4.3 with $k = 0$; a fact that can be verified from the results in Table 6.3.

Finite Element: $\mathbb{RT}_0$ - $\mathbf{P}_1$ - P_0 - P_1 - P_0							
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$
960	0.1901	103.8610	-	41.2596	-	42.9775	-
3536	0.1026	30.9625	1.9610	8.9754	2.4716	10.0549	2.3537
13682	0.0490	10.1080	1.5167	2.7914	1.5825	2.6802	1.7914
53895	0.0256	4.4485	1.2640	1.1727	1.3355	1.0559	1.4345
216315	0.0140	2.1506	1.1992	0.5559	1.2315	0.4864	1.2789
855293	0.0078	1.0832	1.1697	0.2720	1.2194	0.2433	1.1812
$e(\boldsymbol{\gamma})$	$r(\boldsymbol{\gamma})$	$e(\varphi)$	$r(\varphi)$	$\widetilde{h}$	$e(\lambda)$	$r(\lambda)$	Iterations
62.2405	-	0.5055	-	0.2500	0.9127	-	194
15.9318	2.2079	0.1840	1.6372	0.1250	0.5288	0.7875	20
7.1721	1.0814	0.0739	1.2371	0.0625	0.2660	0.9911	17
3.8341	0.9644	0.0346	1.1677	0.0312	0.1372	0.9558	14
1.8853	1.1711	0.0173	1.1429	0.0156	0.0688	0.9949	14
1.0067	1.0701	0.0087	1.1705	0.0078	0.0324	1.0885	14

Table 6.3: Convergence history for Example 2, with a uniform mesh refinement and a first order approximation.

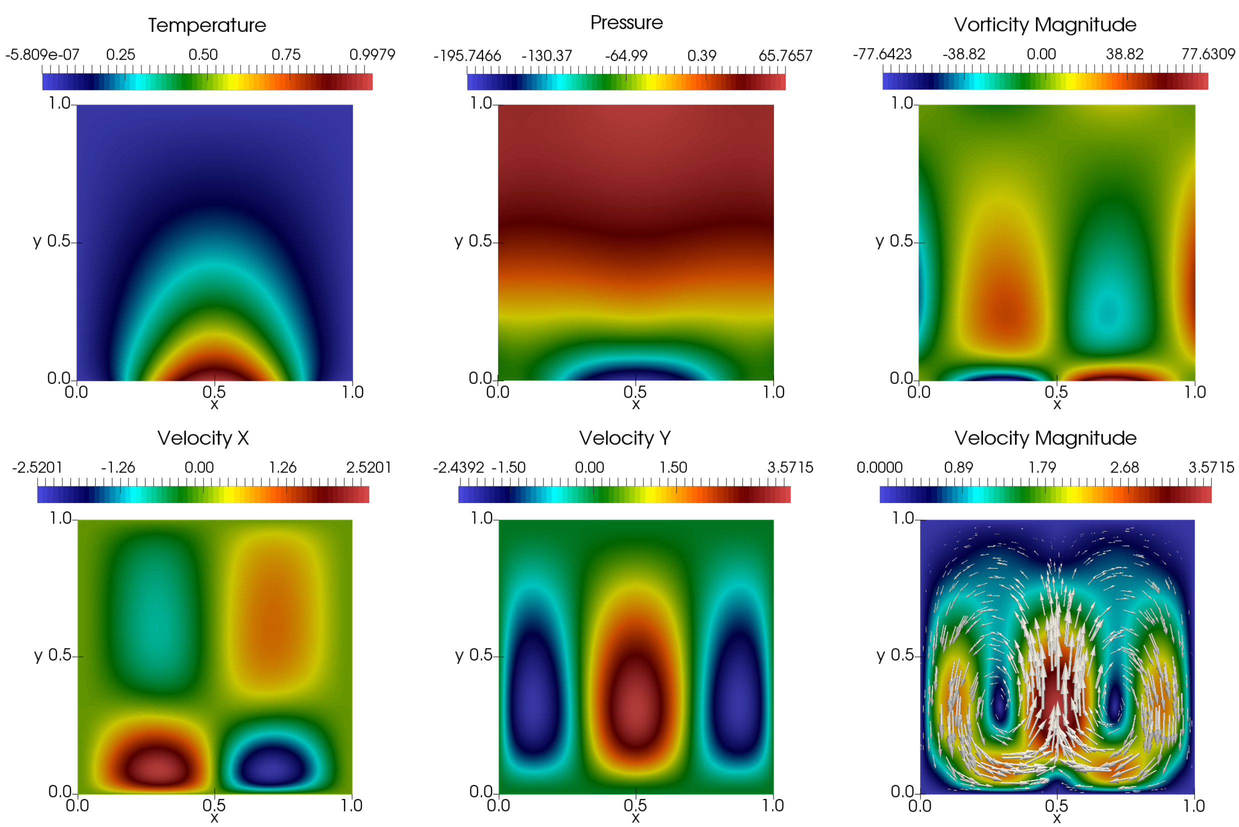

Figure 6.2: Contours of temperature, pressure and vorticity magnitude (taken as $2\gamma_{21}$ to coincide with the usual definition of vorticity) in the upper row, and velocity in the lower row for the data given in Example 2. Results calculated with 3,493,345 DOF and a first order approximation ($\mathbb{RT}_0 - \mathbf{P}_1 - P_0 - P_1 - P_0$).

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces*, Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] M. Alvarez, G.N. Gatica and R. Ruiz-Baier, *An augmented mixed-primal finite element method for a coupled flow-transport problem.* ESAIM: Math. Model. Numer. Anal. 49 (2015), 1399-1427.

[3] M. Alvarez, G.N. Gatica and R. Ruiz-Baier, *A mixed-primal finite element approximation of a sedimentation-consolidation system.* Math. Models Methods Appl. Sci. 26 (2016), no. 5, 867-900.

[4] M. Alvarez, G.N. Gatica and R. Ruiz-Baier, *A posteriori error analysis for a viscous flow-transport problem.* ESAIM: Math. Model. Numer. Anal. 50 (2016), no. 6, 1789-1816.

[5] P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent, *Multifrontal parallel distributed symmetric and unsymmetric solvers.* Comput. Methods Appl. Mech. Engrg. 184 (2000), 501-520.

[6] G. Barakos, E. Mitsoulis and D. Assimacopoulos, *Natural convection flow in a square cavity revisited: laminar and turbulent models with wall functions.* Internat. J. Numer. Methods Fluids 18 (1994), 695-719.

[7] C. Bernardi, B. Métivet and B. Pernaud-Thomas, *Couplage des équations de Navier-Stokes et de la chaleur: le modèle et son approximation par éléments finis.* (French) [Coupling of Navier-Stokes and heat equations: the model and its finite-element approximation] RAIRO Modél. Math. Anal. Numér. 29 (1995), no. 7, 871-921.

[8] F. Brezzi, and M. Fortin, *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, New York, 1991.

[9] J. Camaño, G.N. Gatica, R. Oyarzúa and R. Ruiz-Baier, *An augmented stress-based mixed finite element method for the steady state Navier-Stokes equations with nonlinear viscosity.* Numer. Methods Partial Differential Equations 33 (2017), no. 5, 1692-1725.

[10] J. Camaño, G.N. Gatica, R. Oyarzúa and G. Tierra, *An augmented mixed finite element method for the Navier-Stokes equations with variable viscosity.* SIAM J. Numer. Anal. 54 (2016), no. 2, 1069-1092.

[11] J. Camaño, R. Oyarzúa and G. Tierra, *Analysis of an augmented mixed-FEM for the Navier-Stokes problem.* Math. Comp. 86 (2017), no. 304, 589-615.

[12] P. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013

[13] A. Çibik and S. Kaya, *A projection-based stabilized finite element method for steady-state natural convection problem.* J. Math. Anal. Appl. 381 (2011), no. 2, 469-484.

[14] I.M. Cohen and P.K. Kundu, *Fluid Mechanics*, Third edition. Academic Press, Elsevier, Amsterdam, 2004.

[15] E. Colmenares, G.N. Gatica and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem.* Numer. Methods Partial Differential Equations 32 (2016), no. 2, 445-478.

[16] E. Colmenares, G.N. Gatica and R. Oyarzúa, *An augmented fully-mixed finite element method for the stationary Boussinesq problem.* Calcolo 54 (2017), no. 1, 167-205.

[17] E. Colmenares and M. Neilan, *Dual-mixed finite element methods for the stationary Boussinesq problem.* Comput. Math. Appl. 72 (2016), no. 7, 1828-1850.

[18] A. Dalal and M.K. Das, *Natural convection in a rectangular cavity heated from below and uniformly cooled from the top and both sides.* Numer. Heat Tr. A-Appl 49 (2006), no. 3, 301-322.

[19] H. Dallmann and D. Arndt, *Stabilized finite element methods for the Oberbeck-Boussinesq model.* J. Sci. Comput. 69 (2016), no. 1, 244-273.

[20] T. Davis, *Algorithm 832: UMFPACK V4.3 - an unsymmetric-pattern multifrontal method*. ACM Trans. Math. Software 30 (2004), no. 2, 196-199.

[21] G. De Vahl Davis, *Natural convection of air in a square cavity: A bench mark numerical solution*. Internat. J. Numer. Methods Fluids 3 (1983), 249-264.

[22] M. Farhoul, S. Nicaise and L. Paquet, *A mixed formulation of Boussinesq equations: analysis of nonsingular solutions*. Math. Comp. 69 (2000), no. 231, 965-986.

[23] G.N. Gatica, *An augmented mixed finite element method for linear elasticity with non-homogeneous Dirichlet conditions*. Electron. Trans. Numer. Anal. 26 (2007), 421-438.

[24] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method: Theory and Applications*. Springer Briefs in Mathematics, Springer, Cham, 2014.

[25] G.N. Gatica, A. Márquez and M.A. Sánchez, *Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations*. Comput. Methods Appl. Mech. Engrg. 199 (2010), no. 17-20, 1064-1079.

[26] V. Girault and P-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, 1986.

[27] F. Hecht, *New Development in FreeFem++*. J. Numer. Math. 20 (2012), no. 3-4, 251-265.

[28] M. Ishii and T. Hibiki, *Thermo-Fluid Dynamics of Two-Phase Flow*, Second edition. SpringerLink : Bücher, Springer, New York, 2010.

[29] R. Oyarzúa, T. Qin and D. Schötzau, *An exactly divergence-free finite element method for a generalized Boussinesq problem*. IMA J. Numer. Anal. 34 (2014), no. 3, 1104-1135.

[30] R. Oyarzúa and P. Zúñiga, *Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters*. J. Comput. Appl. Math. 323 (2017), 71-94.

[31] C.E. Pérez, J-M. Thomas, S. Blancher and R. Creff, *The steady Navier-Stokes/energy system with temperature-dependent viscosity - Part 2: The discrete problem and numerical experiments*. Internat. J. Numer. Methods Fluids 56 (2008), no. 1, 91-114.

[32] B.E. Poling, J.M. Prausnitz, J.P. O'Connell , *The Properties of Gases and Liquids*, Fifth edition. McGraw Hill, New York, 2001.

[33] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*. Vol. 23. Springer-Verlag, Berlin, 1994.

[34] J.E. Roberts and J.M. Thomas, *Mixed and Hybrid Methods*, P.G. Ciarlet and J.L. Lions, editors. Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1). North-Holland, Amsterdam, 1991.

[35] M. Tabata and D. Tagami, *Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients*. Numer. Math. 100 (2005), no. 2, 351-372.

[36] T. Zhang and H. Liang *Decoupled stabilized finite element methods for the Boussinesq equations with temperature-dependent coefficients*. Internat. J. Heat Mass Tr. 110 (2017), 151-165.