A note on weak* convergence and compactness and their connection to the existence of the inverse-adjoint*

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Abstract

In this note we provide a systematic reasoning to arrive at the reflexivity of the underlying Banach space as a sufficient condition for guaranteeing that any compact operator transforms weak* convergence in strong convergence. Our starting point is an adaptation of the proof for the analogue result holding in the case of the weak convergence. Then, along the way, and as a by-product of the analysis, we characterize the existence of what we call the inverse-adjoint operator.

Key words: weak convergence, weak* convergence, compactness, inverse-adjoint

1 Introduction

The main motivation of this note arises from the need of finding out whether the well-known result that establishes that a compact operator transforms weak into strong convergence, does also hold true and under which conditions for the case of the weak* topology of the dual of a Banach space. In order to clarify this goal, we provide next some preliminary notations and definitions (see, e.g. [1], [2]). We begin by mentioning that, given X and Y Banach spaces over the same field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{L}(X,Y)$ denotes the space of linear and bounded operators from X into Y, whereas $\mathcal{K}(X,Y)$ stands for the closed subspace of $\mathcal{L}(X,Y)$ given by the set of compact operators. In addition, given $A \in \mathcal{L}(X,Y)$, its adjoint operator $A' : Y' \longrightarrow X'$ is defined by $A'(G) := G \circ A \;\; \forall\, G \in Y'$, which belongs to $\mathcal{L}(Y',X')$ and satisfies $\|A'\| = \|A\|$. Furthermore, we say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$ with respect to the weak topology $\sigma(X,X')$ of X, which is written $x_n \stackrel{w}{\longrightarrow} x$, if

$$\lim_{n\to+\infty} F(x_n) \,=\, F(x) \qquad \forall\, F \in X' .$$

In this way, the aforementioned result says that for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of X satisfying $x_n \stackrel{w}{\longrightarrow} x$ for some $x \in X$, there holds (see, e.g. [2, Theorem 5.12-4])

$$\|A(x_n) \,-\, A(x)\|_Y \stackrel{n\to+\infty}{\longrightarrow} 0 \qquad \forall\, A \in \mathcal{K}(X,Y) . \tag{1.1}$$

Also, we say that a sequence $\{G_n\}_{n\in\mathbb{N}} \subseteq Y'$ converges to $G \in Y'$ with respect to the weak* topology $\sigma(Y',Y)$ of Y', which is written $G_n \stackrel{w^*}{\longrightarrow} G$, if

$$\lim_{n\to+\infty} G_n(y) \,=\, G(y) \qquad \forall\, y \in Y .$$

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Notice that straightforward applications of the Uniform Boundedness Theorem allow to prove that every sequence converging either weakly or weakly* is bounded (see, e.g. [1, Propositions 3.5 and 3.13] or [2, Theorem 5.12-2 and Problem 5.12-4]). Now, given a sequence $\{G_n\}_{n\in\mathbb{N}}$ as above, and motivated by an eventual analogue of (1.1), we consider another Banach space Z over $\mathbb{K}$ and wonder under which assumptions there holds

$$\|B(G_n) - B(G)\|_Z \stackrel{n\to+\infty}{\longrightarrow} 0 \qquad \forall\, B \in \mathcal{K}(Y', Z)\,. \tag{1.2}$$

While the question raised by (1.2) seems very simple, and even the right answer might possibly be conjectured from what is already known for the weak convergence (cf. (1.1)), it turns out, up to the author's knowledge, that it has not been fully addressed yet in the classical textbooks on Functional Analysis (see, e.g. [1, Section 3.4] and [2, Section 5.12]), which somehow has motivated the present work. However, it is not difficult to see that a first straightforward answer to this inquiry would be given by Lemma 1.1 below in which the reflexivity of Y plays a crucial role. In this regard, we previously recall that given a generic Banach space X, we can define the linear operator $\mathcal{J}_X : X \longrightarrow X''$ mapping each $x \in X$ to the functional $\mathcal{J}_X(x)$ in the bi-dual X'', which is defined by $\mathcal{J}_X(x)(F) := F(x)$ $\forall\, F \in X'$. It is well-known that $\mathcal{J}_X$ is a linear isometry, and hence an injective operator, so that X is said to be reflexive when $\mathcal{J}_X$ is additionally surjective. The afore announced result is as follows.

Lemma 1.1. *Let Y and Z be Banach spaces such that Y is reflexive, and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence in Y' converging weakly* to $G \in Y'$. Then (1.2) is satisfied*

Proof. It suffices to observe that when Y is reflexive, the weak* topology $\sigma(Y', Y)$ of Y' coincides with its weak topology $\sigma(Y', Y'')$, and hence $G_n \stackrel{w^*}{\longrightarrow} G$ is equivalent to saying $G_n \stackrel{w}{\longrightarrow} G$, whence an application of (1.1) to the context $X \longleftarrow Y'$ and $Y \longleftarrow Z$ completes the proof. □

Having established the above, and in order to enrich the knowledge on the connections between compactness and the weak and weak* convergences, in the following section we develop a systematic reasoning to arrive naturally at the reflexivity of Y as a sufficient condition for (1.2), starting precisely from one of the approaches to prove (1.1). Moreover, we remark in advance that proceeding in this way, and as interesting by-product of our forthcoming analysis, we will be able to characterize the eventual existence of what we call the inverse-adjoint of a given $B \in \mathcal{L}(Y', X')$, that is an operator $A \in \mathcal{L}(X, Y)$ such that $A' = B$.

2 The main results

We begin by trying to extend the analysis from the weak case to the weak* topology. Indeed, throughout this process we realize that similar arguments to those leading to (1.1) can be successfully employed to show (1.2) without assuming reflexivity of Y, but only for the particular case in which the given compact operator B admits an inverse-adjoint. More precisely, we have the following result.

Lemma 2.1. *Let X and Y be Banach spaces, and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence in Y' that converges weakly* to $G \in Y'$. In addition, let $B \in \mathcal{K}(Y', X')$ for which there exists $A \in \mathcal{L}(X, Y)$ such that $A' = B$. Then there holds*

$$\|B(G_n) - B(G)\|_{X'} \stackrel{n\to+\infty}{\longrightarrow} 0\,. \tag{2.1}$$

Proof. Given $A \in \mathcal{L}(X, Y)$ such that $A' = B$, we first observe, thanks to the weak* convergence of $\{G_n\}_{n\in\mathbb{N}}$, that for each $x \in X$ there holds

$$B(G_n)(x) = A'(G_n)(x) = G_n\big(A(x)\big) \stackrel{n\to+\infty}{\longrightarrow} G\big(A(x)\big) = A'(G)(x) = B(G)(x)\,,$$

which shows that $B(G_n) \stackrel{w^*}{\longrightarrow} B(G) \in X'$. Let us assume now, by contradiction, that $\{B(G_n)\}_{n\in\mathbb{N}}$ does not converge strongly to $B(G)$ in X'. It follows that there exists $\delta > 0$ and a subsequence $\{G_n^{(1)}\}_{n\in\mathbb{N}} \subseteq \{G_n\}_{n\in\mathbb{N}}$ such that

$$\|B(G_n^{(1)}) - B(G)\|_{X'} \geq \delta \qquad \forall\, n \in \mathbb{N}. \tag{2.2}$$

In turn, the boundedness of $\{G_n^{(1)}\}_{n\in\mathbb{N}}$ and the compactness of B imply the existence of a subsequence $\{G_n^{(2)}\}_{n\in\mathbb{N}} \subseteq \{G_n^{(1)}\}_{n\in\mathbb{N}}$ and $F \in X'$ such that

$$\|B(G_n^{(2)}) - F\|_{X'} \stackrel{n\to+\infty}{\longrightarrow} 0. \tag{2.3}$$

The latter certainly implies that $B(G_n^{(2)}) \stackrel{w^*}{\longrightarrow} F \in X'$, which, thanks to the uniqueness of the weak* limit, yields $F = B(G)$. In this way, (2.3) becomes $\|B(G_n^{(2)}) - B(G)\|_{X'} \stackrel{n\to+\infty}{\longrightarrow} 0$, which contradicts (2.2) and finishes the proof. □

The foregoing result suggests to find out now in what cases each $B \in \mathcal{L}(Y', X')$ posseses an inverse-adjoint operator $A \in \mathcal{L}(X,Y)$. In other words, letting $T : \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(Y',X')$ be the linear operator mapping each $A \in \mathcal{L}(X,Y)$ to its adjoint $A' \in \mathcal{L}(Y',X')$, which is certainly an injective operator, we focus in what follows to determining under which hypotheses T becomes surjective. The answer to it, given by the next lemma, is far from being unexpected.

Lemma 2.2. *T is bijective if and only if Y is reflexive. Moreover, in the latter case there holds $T^{-1}(B) \in \mathcal{K}(X,Y)$ for each $B \in \mathcal{K}(Y',X')$.*

Proof. Let us first suppose that T is bijective, and let $\mathcal{G} \in Y''$. Then, given any non-null functional $F \in X'$, we define the operator $B \in \mathcal{L}(Y',X')$ by

$$B(G) := \mathcal{G}(G)\, F \qquad \forall\, G \in Y'. \tag{2.4}$$

Note that the linearity and boundedness of B follow from the same properties of $\mathcal{G}$. Thus, letting $A := T^{-1}(B) \in \mathcal{L}(X,Y)$, we obviously have $A' = B$, and hence (2.4) yields

$$\mathcal{G}(G)\, F(x) = A'(G)(x) = G(A(x)) \qquad \forall\, G \in Y', \quad \forall\, x \in X. \tag{2.5}$$

Next, taking any $\widetilde{x} \in X$ such that $F(\widetilde{x}) \neq 0$, which is possible thanks to our choice of F, and defining $\widetilde{y} := \dfrac{A(\widetilde{x})}{F(\widetilde{x})}$, we deduce from (2.5) that $\mathcal{G}(G) = G(\widetilde{y}) \quad \forall\, G \in Y'$. This identity shows that $\mathcal{G} = \mathcal{J}_Y(\widetilde{y})$, and therefore Y is reflexive. Conversely, we now assume that Y is reflexive and show in what follows that, given any $B \in \mathcal{L}(Y',X')$, there exists $A \in \mathcal{L}(X,Y)$ such that $A' = B$. Indeed, since $B' \in \mathcal{L}(X'',Y'')$ and $\mathcal{J}_Y$ is bijective, we consider the following diagram

$$\begin{array}{ccc} X'' & \stackrel{B'}{\longrightarrow} & Y'' \\ \mathcal{J}_X \Big\uparrow & & \Big\downarrow \mathcal{J}_Y^{-1} \\ X & \stackrel{A}{\longrightarrow} & Y \end{array}$$

in which we define $A := \mathcal{J}_Y^{-1} \circ B' \circ \mathcal{J}_X \in \mathcal{L}(X,Y)$. In order to prove that $A' = B$, we first notice that there holds $G\big(\mathcal{J}_Y^{-1}(\mathcal{G})\big) = \mathcal{G}(G) \quad \forall\, G \in Y', \quad \forall\, \mathcal{G} \in Y''$. In this way, we find that

$$\begin{aligned} A'(G)(x) &= G(A(x)) = G\big(\mathcal{J}_Y^{-1}(B'(\mathcal{J}_X(x)))\big) \\ &= B'(\mathcal{J}_X(x))(G) = J_X(x)\big(B(G)\big) = B(G)(x) \qquad \forall\, G \in Y', \quad \forall x \in X, \end{aligned}$$

from which it follows that $A'(G) = B(G) \ \ \forall\, G \in Y'$, thus proving that $A' = B$. Finally, whenever $B \in \mathcal{K}(Y', X')$ there certainly holds $B' \in \mathcal{K}(X'', Y'')$, and hence $T^{-1}(B) := \mathcal{J}_Y^{-1} \circ B' \circ \mathcal{J}_X$ is compact as well. □

In this way, as a straightforward consequence of Lemmas 2.1 and 2.2, we obtain the following particular version of Lemma 1.1.

Lemma 2.3. *Let X and Y be Banach spaces such that Y is reflexive, and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence in Y' converging weakly* to $G \in Y'$. Then there holds*

$$\|B(G_n) \,-\, B(G)\|_{X'} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \forall\, B \,\in\, \mathcal{K}(Y', X')\,. \tag{2.6}$$

Moreover, we are now able to extend Lemma 2.3 to the full version of Lemma 1.1, thus providing an alternative proof of this result.

Lemma 2.4. *Let Y and Z be Banach spaces such that Y is reflexive, and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence in Y' converging weakly* to $G \in Y'$. Then there holds*

$$\|B(G_n) \,-\, B(G)\|_{Z} \overset{n\to+\infty}{\longrightarrow} 0 \qquad \forall\, B \,\in\, \mathcal{K}(Y', Z)\,. \tag{2.7}$$

Proof. Given $B \in \mathcal{K}(Y', Z)$, we consider the following diagram

$$\begin{array}{ccc} Y' & \overset{B}{\longrightarrow} & Z \\ & \overset{\widetilde{B}}{\searrow} & \Big\downarrow \mathcal{J}_Z \\ & & Z'' \end{array} \tag{2.8}$$

in which we define $\widetilde{B} := \mathcal{J}_Z \circ B \in \mathcal{K}(Y', Z'') = \mathcal{K}\big(Y', (Z')'\big)$. Then, recalling that $\mathcal{J}_Z$ is an isometry, and applying Lemma 2.3 to the context $X \longleftarrow Z'$, $Y \longleftarrow Y$, and $B \longleftarrow \widetilde{B}$, we find that for $\{G_n\}_{n\in\mathbb{N}}$ and G as indicated, there holds

$$\|B(G_n) \,-\, B(G)\|_Z \,=\, \|\mathcal{J}_Z\big(B(G_n) - B(G)\big)\|_{Z''} \,=\, \|\widetilde{B}(G_n) - \widetilde{B}(G)\|_{(Z')'} \overset{n\to+\infty}{\longrightarrow} 0\,, \tag{2.9}$$

which proves (2.7) and finishes the proof. □

It would remain to see whether the reflexivity of Y is also a necessary condition or not for (2.7). Meanwhile we leave this issue as an open question.

Finally, we realize that the same diagram (2.8) suggests the following extension of Lemma 2.1 to the case of $B \in \mathcal{K}(Y', Z)$.

Lemma 2.5. *Let Y and Z be Banach spaces, and let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence in Y' that converges weakly* to $G \in Y'$. In addition, let $B \in \mathcal{K}(Y', Z)$ for which there exists $A \in \mathcal{L}(Z', Y)$ such that $A' = \widetilde{B} := \mathcal{J}_Z \circ B \in \mathcal{K}(Y', Z'') = \mathcal{K}\big(Y', (Z')'\big)$. Then there holds*

$$\|B(G_n) \,-\, B(G)\|_Z \overset{n\to+\infty}{\longrightarrow} 0\,. \tag{2.10}$$

Proof. Similarly to the proof of Lemma 2.4, it suffices to use the identity (2.9) and then apply Lemma 2.1 to the context $X \longleftarrow Z'$, $Y \longleftarrow Y$, and $B \longleftarrow \widetilde{B}$. □

References

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