A POSTERIORI ERROR ESTIMATES FOR MAXWELL'S EIGENVALUE PROBLEM

DANIELE BOFFI, LUCIA GASTALDI, RODOLFO RODRÍGUEZ, AND IVANA ŠEBESTOVÁ

ABSTRACT. We introduce a residual error indicator for the edge finite element approximation of the eigenmodes of the Maxwell cavity problem. By using the known equivalence with a mixed problem we prove reliability and efficiency of the error indicator. Numerical results confirm the optimal behavior of an adaptive scheme based on the error indicator.

1. INTRODUCTION

The study of a posteriori analysis for the approximation of partial differential equations is nowadays recognized as a fundamental tool when dealing with problems with singular solutions [1, 30]. A posteriori analysis for eigenvalue problem arising from partial differential equations is a more recent research field which is now reaching its full maturity [29, 16, 17, 9, 11].

We consider the approximation of Maxwell's eigenvalue problem with edge finite elements. Our aim is to introduce a suitable a posteriori error indicator and to show that it is equivalent to the error of the eigenfunctions. Since the error for the eigenvalues can be bounded by the error of the eigenfunctions, our error indicator provides an upper bound for the eigenvalue error as well. For simplicity, we deal with simple eigenvalues; however, our analysis can be extended to multiple eigenvalues and to clusters of eigenvalues in the spirit of [8].

A posteriori analysis for Maxwell's equations is present in the literature mainly for what concerns the source problem [21, 4, 24, 23, 13, 26, 12, 31]. To the best of our knowledge, the eigenvalue problem has been considered only in [10] where the reliability analysis is performed up to higher order terms (asymptotically in the mesh size). Here, we improve the analysis by adding suitable (quite natural) terms to the error indicator, thus avoiding the introduction of the higher order terms in the error analysis. The reliability estimate shows that our error indicator yields asymptotically an upper bound for the error in the eigenfunctions.

The main tool for the analysis consists of a mixed formulation equivalent (both at continuous and discrete level) to the original problem. A superconvergence result, presented in Lemma 9, is crucial for the proof of our main reliability estimate.

The outline of the paper is as follows: after recalling the Maxwell's eigenvalue problem in Section 2, we introduce the indicator and perform the main reliability and efficiency analysis in Section 3; Section 4 contains the proofs of the main auxiliary results. Finally, Section 5 reports on a numerical test confirming the theoretical results and showing that the indicator can be successfully applied to drive an adaptive scheme.

2. Problem setting

Let $\Omega \subset \mathbb{R}^3$ be an open bounded polyhedral domain with Lipschitz boundary $\partial\Omega$ and let $\boldsymbol{n}$ be the outward normal unit vector. For the sake of simplicity we assume that Ω is simply connected and that its boundary is connected.

The eigenvalue problem for the Maxwell system consists in finding $\omega > 0$ and $\boldsymbol{E} : \Omega \to \mathbb{R}^3$ with $\boldsymbol{E} \neq \boldsymbol{0}$ such that

$$\begin{aligned}
&\mathbf{curl}\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\right) = \omega^2 \varepsilon \boldsymbol{E} && \text{in } \Omega,\\
&\operatorname{div}(\varepsilon \boldsymbol{E}) = 0 && \text{in } \Omega,\\
&\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} && \text{on } \partial\Omega.
\end{aligned}$$

Here $\boldsymbol{E}$ represents the electric field, while ε and μ are the electric permittivity and the magnetic permeability, respectively. Assuming that the medium is homogeneous and isotropic, ε and μ are positive constants; in such a case we can assume without loosing generality that $\varepsilon = \mu = 1$.

Before writing a variational formulation of the problem, we introduce the functional setting we will use. Boldface characters will indicate vector valued functions and the corresponding functional spaces. For a given domain D and $p \geq 1$, $\mathrm{L}^p(D)$ denotes the classical Lebesgue function space and $\mathbf{L}^p(D) := [\mathrm{L}^p(D)]^3$. For positive t, $\mathrm{H}^t(D)$ stands for the standard Sobolev space and $\mathbf{H}^t(D) := [\mathrm{H}^t(D)]^3$. Moreover, we introduce the following Hilbert spaces:

$$\begin{aligned}
\mathrm{H}_0^1(\Omega) &:= \left\{ v \in \mathrm{H}^1(\Omega) :\ v = 0 \text{ on } \partial\Omega \right\},\\
\mathbf{H}(\mathbf{curl};\Omega) &:= \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Omega) :\ \mathbf{curl}\,\boldsymbol{v} \in \mathbf{L}^2(\Omega) \right\},\\
\mathbf{H}_0(\mathbf{curl};\Omega) &:= \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega) :\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega \},\\
\mathbf{H}_0(\mathbf{curl}^0;\Omega) &:= \{ \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega) :\ \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0} \},\\
\mathbf{H}(\operatorname{div};\Omega) &:= \left\{ \boldsymbol{v} \in \mathbf{L}^2(\Omega) :\ \operatorname{div}\boldsymbol{v} \in \mathrm{L}^2(\Omega) \right\},\\
\mathbf{H}_0(\operatorname{div};\Omega) &:= \{ \boldsymbol{v} \in \mathbf{H}(\operatorname{div};\Omega) :\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \},\\
\mathbf{H}(\operatorname{div}^0;\Omega) &:= \{ \boldsymbol{v} \in \mathbf{H}(\operatorname{div};\Omega) :\ \operatorname{div}\boldsymbol{v} = 0 \},\\
\mathbf{H}_0(\operatorname{div}^0;\Omega) &:= \left\{ \boldsymbol{v} \in \mathbf{H}(\operatorname{div}^0;\Omega) :\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \right\}.
\end{aligned}$$

We will use the following norms:

$$\begin{aligned}
\|v\|_1^2 &:= \|v\|_0^2 + \|\nabla v\|_0^2\,,\\
\|\boldsymbol{v}\|_{\mathbf{curl}}^2 &:= \|\boldsymbol{v}\|_0^2 + \|\mathbf{curl}\,\boldsymbol{v}\|_0^2\,,\\
\|\boldsymbol{v}\|_{\operatorname{div}}^2 &:= \|\boldsymbol{v}\|_0^2 + \|\operatorname{div}\boldsymbol{v}\|_0^2\,,
\end{aligned}$$

where $\|\cdot\|_0$ denotes the norm of $[\mathrm{L}^2(\Omega)]^d$ for any integer $d \geq 1$; moreover, $(\cdot,\cdot)$ will denote the inner product of $[\mathrm{L}^2(\Omega)]^d$. Finally, for domains D different from Ω, we will denote by $\|\cdot\|_{0,D}$ and $(\cdot,\cdot)_D$ the corresponding norm and inner product.

For $\varepsilon = \mu = 1$, the variational form of the eigenvalue problem for the Maxwell system reads as follows: find $\omega > 0$ and $\boldsymbol{E} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\operatorname{div}^0;\Omega)$ with $\boldsymbol{E} \neq \boldsymbol{0}$ such that

$$(\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{F}) = \omega^2\,(\boldsymbol{E}, \boldsymbol{F}) \quad \forall \boldsymbol{F} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\operatorname{div}^0;\Omega). \tag{1}$$

From the computational point of view, it is difficult to enforce the divergence free constraint, hence usually the following form of the variational problem is preferred.

Problem 1. *Find* $\omega > 0$ *and* $\boldsymbol{E} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ *with* $\boldsymbol{E} \neq \mathbf{0}$ *such that*

$$(\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{F}) = \omega^2 (\boldsymbol{E}, \boldsymbol{F}) \quad \forall \boldsymbol{F} \in \mathbf{H}_0(\mathbf{curl};\Omega).$$

It is easy to check that the positive eigenvalues of Problem 1 are also eigenvalues of the original equation (1). In addition to them, Problem 1 admits the eigenvalue $\omega^2 = 0$ with eigenspace $\mathbf{H}_0(\mathbf{curl}^0;\Omega) = \nabla\left(\mathrm{H}_0^1(\Omega)\right)$.

Let $\boldsymbol{\mathcal{X}}_h$ be a finite dimensional subspace of $\mathbf{H}_0(\mathbf{curl};\Omega)$; then, an approximation of Problem 1 reads:

Problem 2. *Find* $\omega_h > 0$ *and* $\boldsymbol{E}_h \in \boldsymbol{\mathcal{X}}_h$ *with* $\boldsymbol{E}_h \neq \mathbf{0}$ *such that*

$$(\mathbf{curl}\,\boldsymbol{E}_h, \mathbf{curl}\,\boldsymbol{F}_h) = \omega_h^2 (\boldsymbol{E}_h, \boldsymbol{F}_h) \quad \forall \boldsymbol{F}_h \in \boldsymbol{\mathcal{X}}_h.$$

Sufficient conditions to guarantee that all the positive eigenvalues of Problem 2 are well separated from the vanishing one and that no spurious mode is generated by the numerical scheme are introduced in [7]. These conditions are based on the introduction of the following mixed formulation of (1) and on its finite element counterpart. For $\omega > 0$, let us denote $\lambda := \omega^2$, $\boldsymbol{u} := \omega \boldsymbol{E}$ and $\boldsymbol{\sigma} := -\,\mathbf{curl}\,\boldsymbol{E}/\omega$. Notice that, by definition, $\boldsymbol{\sigma} \in \mathbf{curl}\,(\mathbf{H}_0(\mathbf{curl};\Omega)) = \mathbf{H}_0(\mathrm{div}^0;\Omega)$ (cf. [2, Theor. 3.17]).

Problem 3. *Find* $\lambda \in \mathbb{R}$ *and* $(\boldsymbol{u}, \boldsymbol{\sigma}) \in \mathbf{H}_0(\mathbf{curl};\Omega) \times \mathbf{H}_0(\mathrm{div}^0;\Omega)$ *with* $(\boldsymbol{u}, \boldsymbol{\sigma}) \neq (\mathbf{0}, \mathbf{0})$ *such that*

$$\begin{aligned} &(\boldsymbol{u}, \boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{\sigma}) = 0 \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega), \\ &(\mathbf{curl}\,\boldsymbol{u}, \boldsymbol{\tau}) = -\lambda\,(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div}^0;\Omega). \end{aligned} \tag{2}$$

It is not difficult to check that the eigenvalues of Problem 3 are strictly positive and that they coincide with the non vanishing eigenvalues of Problem 1, so that Problem 1 and 3 are equivalent in the sense of [6, Prop. 11.2.1] (see also [7]). Moreover, since any solution of Problem 3 satisfies $\|\boldsymbol{u}\|_0^2 = \lambda\,\|\boldsymbol{\sigma}\|_0^2$, normalizing an eigenfunction of Problem 1 by imposing $\|\boldsymbol{E}\|_0 = 1$ corresponds to normalizing that of Problem 3 by $\|\boldsymbol{\sigma}\|_0 = 1$.

Let $\boldsymbol{\mathcal{M}}_h^0$ be a finite dimensional subspace of $\mathbf{H}_0(\mathrm{div}^0;\Omega)$. Then, the discretization of Problem 3 reads:

Problem 4. *Find* $\lambda_h \in \mathbb{R}$ *and* $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h) \in \boldsymbol{\mathcal{X}}_h \times \boldsymbol{\mathcal{M}}_h^0$ *with* $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h) \neq (\mathbf{0}, \mathbf{0})$ *such that*

$$\begin{aligned} &(\boldsymbol{u}_h, \boldsymbol{v}_h) + (\mathbf{curl}\,\boldsymbol{v}_h, \boldsymbol{\sigma}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{X}}_h, \\ &(\mathbf{curl}\,\boldsymbol{u}_h, \boldsymbol{\tau}_h) = -\lambda_h\,(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{M}}_h^0. \end{aligned} \tag{3}$$

The equivalence of Problems 2 and 4 has been proved in [7, Theor. 2.1] under the assumption that

$$\mathbf{curl}\,(\boldsymbol{\mathcal{X}}_h) \subseteq \boldsymbol{\mathcal{M}}_h^0. \tag{4}$$

More precisely, all the solutions with positive frequencies ω_h of Problem 2 correspond to solutions of Problem 4 with the identifications $\lambda_h = \omega_h^2$, $\boldsymbol{u}_h = \omega_h \boldsymbol{E}_h$ and $\boldsymbol{\sigma}_h = -\,\mathbf{curl}\,\boldsymbol{E}_h/\omega_h$. Moreover, as in the continuous case, normalizing the eigenfunctions of Problem 2 by $\|\boldsymbol{E}_h\|_0 = 1$ corresponds to normalizing those of Problem 4 by $\|\boldsymbol{\sigma}_h\|_0 = 1$.

The compatibility condition (4) is generally obtained by defining $\boldsymbol{\mathcal{M}}_h^0$ exactly equal to $\mathbf{curl}\,(\boldsymbol{\mathcal{X}}_h)$. Let us remark that Problems 3 and 4 are introduced only for the theoretical analysis, but not for the actual computations. In fact, Problem 2 is the one used in practice to compute the eigenvalues and eigenfunctions of Maxwell equations. Therefore, the fact that it would not be easy to find an algebraic basis of the space $\mathbf{curl}\,(\boldsymbol{\mathcal{X}}_h)$ does not matter in practice.

We recall the standard family of finite elements that we are going to use in this paper. Let $\{\mathcal{T}_h\}$ be a regular family of partitions of Ω into a finite number of tetrahedra. Let h_K be the diameter of the element $K \in \mathcal{T}_h$ and $h := \max_{K\in\mathcal{T}_h} h_K$. We denote by $\mathbb{P}_k(K)$ the space of polynomials of degree at most k on K and by $\widetilde{\mathbb{P}}_k(K)$ the subspace of homogeneous polynomials of degree k on K. For $k \geq 0$ we define

$$\begin{aligned} \boldsymbol{\mathcal{N}}_k(K) &:= [\mathbb{P}_k(K)]^3 \oplus \boldsymbol{x} \times [\widetilde{\mathbb{P}}_k(K)]^3, \\ \boldsymbol{\mathcal{RT}}_k(K) &:= [\mathbb{P}_k(K)]^3 \oplus \boldsymbol{x}\, \widetilde{\mathbb{P}}_k(K), \end{aligned}$$

where $\boldsymbol{x} = (x, y, z)$. Then, we set

$$\begin{aligned} \boldsymbol{\mathcal{X}}_h &:= \{\boldsymbol{v}_h \in \mathbf{H}_0(\mathbf{curl}; \Omega) :\ \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_k(K)\}\,, \\ \boldsymbol{\mathcal{M}}_h &:= \{\boldsymbol{\tau}_h \in \mathbf{H}_0(\mathrm{div}; \Omega) :\ \boldsymbol{\tau}_h|_K \in \boldsymbol{\mathcal{RT}}_k(K)\}\,, \\ \boldsymbol{\mathcal{M}}_h^0 &:= \mathbf{curl}\,(\boldsymbol{\mathcal{X}}_h) = \{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{M}}_h :\ \mathrm{div}\,\boldsymbol{\tau}_h = 0\}\,. \end{aligned}$$

Spaces $\boldsymbol{\mathcal{X}}_h$ and $\boldsymbol{\mathcal{M}}_h$ belong to the well-known families of *edge* and *face* spaces introduced by Nédélec in [22] and Raviart–Thomas in [25], respectively. Therefore, they are often called Nédélec and Raviart–Thomas spaces. We remark that the characterization of $\boldsymbol{\mathcal{M}}_h^0$ in the last line above is well known (see [22]) and it is a consequence of the commuting diagram property. Moreover, the compatibility condition (4) is satisfied by definition. These finite element spaces provide approximations of $\mathbf{H}_0(\mathbf{curl}; \Omega)$ and $\mathbf{H}_0(\mathrm{div}; \Omega)$. We recall here the properties of the corresponding interpolation operators which will be used in the following.

For $t > 1/2$ and $p > 2$,

$$\boldsymbol{I}_E : \left\{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}^t(\Omega) :\ \mathbf{curl}\,\boldsymbol{v} \in \mathbf{L}^p(\Omega)\right\} \to \boldsymbol{\mathcal{X}}_h$$

denotes the interpolant operator for the Nédélec spaces, which enjoys the following approximation properties (see, e.g. [6]): for any $\boldsymbol{v} \in \mathbf{H}^m(\Omega)$ with $1 < m \leq k+1$

$$\|\boldsymbol{v} - \boldsymbol{I}_E\boldsymbol{v}\|_0 \leq Ch^m\, |\boldsymbol{v}|_{\mathbf{H}^m(\Omega)}\,.$$

Moreover, for $1/2 < t \leq 1$ and $p > 2$,

$$\|\boldsymbol{v} - \boldsymbol{I}_E\boldsymbol{v}\|_0 \leq Ch^t \left[|\boldsymbol{v}|_{\mathbf{H}^t(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}\right].$$

Analogously, the interpolant operator for the Raviart–Thomas spaces is defined for $p > 2$ as

$$\boldsymbol{I}_F : \mathbf{H}(\mathrm{div}; \Omega) \cap \mathbf{L}^p(\Omega) \to \boldsymbol{\mathcal{M}}_h.$$

This interpolant is also well defined for any $\boldsymbol{\tau} \in \mathbf{H}^t(\Omega)$ with $1/2 < t \leq k+1$ and the following estimate holds true:

$$\|\boldsymbol{\tau} - \boldsymbol{I}_F\boldsymbol{\tau}\|_0 \leq Ch^t\, |\boldsymbol{\tau}|_{\mathbf{H}^t(\Omega)}\,.$$

It is well known that the interpolation operators defined above satisfy the following *commuting diagram* property for any $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}^t(\Omega)$ with $\mathbf{curl}\,\boldsymbol{v} \in \mathbf{L}^p(\Omega)$ ($t > 1/2$, $p > 2$):

$$\mathbf{curl}\,(\boldsymbol{I}_E\boldsymbol{u}) = \boldsymbol{I}_F(\mathbf{curl}\,\boldsymbol{u}).$$

The following a priori error estimates for Problems 3 and 4 follow from [5, Theor. 2 & 3] together with this lemma. From now on, for simplicity, we assume that λ is a simple eigenvalue; generalizations to multiple or clusters of eigenvalues can be performed following the analysis of [8].

Proposition 1. *Let λ be a simple eigenvalue of Problem 3 and $(\boldsymbol{u},\boldsymbol{\sigma})$ an associated eigenfunction with $\|\boldsymbol{\sigma}\|_0 = 1$. Then, there exists a solution $(\lambda_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ of Problem 4 with $\|\boldsymbol{\sigma}_h\|_0 = 1$, such that λ_h approximates λ as h goes to zero. Moreover, if the sign of $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ is chosen so that $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) > 0$, then $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ is also an approximation of $(\boldsymbol{u}, \boldsymbol{\sigma})$. In such a case, there exists a positive constant C independent of h such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le C \inf_{\substack{\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{M}}_h^0 \\ \boldsymbol{v}_h \in \boldsymbol{\mathcal{X}}_h}} \left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_0 + \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathbf{curl}} \right),$$

$$|\lambda - \lambda_h| \le C \left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 \right).$$

To end this section, we recall that it has been also proved in [5] that this finite element method does not introduce spurious eigenvalues. As a consequence of this, for h small enough, except for λ_h, all the other eigenvalues of Problem 3 are well separated from λ. More precisely, the following result holds true.

Proposition 2. *Let us enumerate the eigenvalues of Problems 3 and 4 in increasing order as follows: $0 < \lambda_1 \le \cdots \le \lambda_i \le \cdots$ and $0 < \lambda_{h,1} \le \cdots \le \lambda_{h,I_h}$, with $I_h := \dim(\boldsymbol{\mathcal{M}}_h^0)$. Let us assume that λ_J is a simple eigenvalue of Problem 3. Then, there exists $h_0 > 0$ such that*

$$|\lambda_J - \lambda_{h,i}| \ge \frac{1}{2} \min_{j \ne J} |\lambda_J - \lambda_j| \qquad \forall i \le I_h,\ i \ne J, \quad \forall h < h_0.$$

3. A posteriori error analysis

In this section, we introduce an error indicator valid for Problem 2 and show how it can be interpreted as an indicator for the mixed formulation of Problem 4.

Let us consider a solution $(\omega, \boldsymbol{E})$ of Problem 1 with $\omega > 0$ and $\|\boldsymbol{E}\|_0 = 1$. Let $(\omega_h, \boldsymbol{E}_h)$ be the solution of Problem 2 with $\|\boldsymbol{E}_h\|_0 = 1$ that approximates $(\omega, \boldsymbol{E})$. In particular, we assume that the sign of $\boldsymbol{E}_h$ has been chosen so that $(\boldsymbol{E}_h, \boldsymbol{E}) > 0$.

The local error indicators are defined for each $K \in \mathcal{T}_h$ as follows:

$$\begin{aligned} \mu_K^2 := {} & h_K^2 \left\| \boldsymbol{E}_h - \mathbf{curl}\left(\mathbf{curl}\, \boldsymbol{E}_h/\omega_h^2\right) \right\|_{0,K}^2 + h_K^2 \left\| \operatorname{div} \boldsymbol{E}_h \right\|_{0,K}^2 \\ & + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left[h_F \left\| [\![\left(\mathbf{curl}\, \boldsymbol{E}_h/\omega_h^2\right) \times \boldsymbol{n}]\!] \right\|_{0,F}^2 + h_F \left\| [\![\boldsymbol{E}_h \cdot \boldsymbol{n}]\!] \right\|_{0,F}^2 \right], \end{aligned} \tag{5}$$

where $\mathcal{F}_{\mathrm{I}}(K)$ denotes the set of the inner faces of the element K, h_F the diameter of F, $[\![\cdot]\!]$ the jump of a quantity across and inner face F and $\boldsymbol{n}$ a unit vector normal to F. Then, the global estimator is defined by

$$\mu^2 := \sum_{K \in \mathcal{T}_h} \mu_K^2.$$

Due to the equivalence of the original Maxwell eigenvalue problem with the mixed formulation presented in the previous section, the analysis of reliability and efficiency of this error indicator will rely on studying the same properties of an error indicator for Problem 4.

Let $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$ be a solution of Problem 3 with $\|\boldsymbol{\sigma}\|_0 = 1$ and hence $\|\boldsymbol{u}\|_0^2 = \lambda$ and let $(\lambda_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ be a solution of Problem 4 with $\|\boldsymbol{\sigma}_h\|_0 = 1$, $\|\boldsymbol{u}_h\|_0^2 = \lambda_h$ and

$(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) > 0$. Thanks to the compatibility condition (4), the second equation of (3) implies that $\mathbf{curl}\, \boldsymbol{u}_h = -\lambda_h \boldsymbol{\sigma}_h$; hence, from the first equation in (2), $(\boldsymbol{u}, \boldsymbol{u}_h) = -(\mathbf{curl}\, \boldsymbol{u}_h, \boldsymbol{\sigma}) = \lambda_h (\boldsymbol{\sigma}_h, \boldsymbol{\sigma}) > 0$, too.

For each $K \in \mathcal{T}_h$, we define the local error indicators for Problem 4 by

$$\begin{aligned}\eta_K^2 &:= h_K^2 \left\| \boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h \right\|_{0,K}^2 + h_K^2 \left\| \operatorname{div} \boldsymbol{u}_h \right\|_{0,K}^2 \\ &\quad + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left(h_F \left\| [\![\boldsymbol{\sigma}_h \times \boldsymbol{n}]\!] \right\|_{0,F}^2 + h_F \left\| [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!] \right\|_{0,F}^2 \right)\end{aligned}$$

and the corresponding global error estimator by

$$\eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2.$$

We observe that due to the equivalence between Problems 2 and 4, the following relations hold true:

$$\mu_K^2 = \frac{1}{\lambda_h} \eta_K^2 \quad \forall K \in \mathcal{T}_h \qquad \text{and} \qquad \mu^2 = \frac{1}{\lambda_h} \eta^2. \tag{6}$$

The first step of our a posteriori error analysis will be to show that the estimator η yields asymptotically an upper estimate for the error $\left\| \boldsymbol{u} - \boldsymbol{u}_h \right\|_0 + \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_0$.

Theorem 3. *Let $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$ and $(\lambda_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ be solutions of Problems 3 and 4, respectively, such that the latter approximates the former as h goes to zero. Then, there exist $\rho(h)$ tending to zero as $h \to 0$ and two positive constants C_1 and C_2 independent of the mesh size such that*

$$\left\| \boldsymbol{u} - \boldsymbol{u}_h \right\|_0 + \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_0 \le C_1 \eta + C_2 \rho(h) \left(\left\| \boldsymbol{u} - \boldsymbol{u}_h \right\|_0 + \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\|_0 \right).$$

Proof. In order to estimate the first term, we introduce the following Helmholtz decomposition of $\boldsymbol{u} - \boldsymbol{u}_h$ (see, e.g., [2, Theor. 3.12]): there exist $\alpha \in \mathrm{H}_0^1(\Omega)$ and $\boldsymbol{\beta} \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}_0(\operatorname{div}^0; \Omega)$ such that

$$\boldsymbol{u} - \boldsymbol{u}_h = \nabla \alpha + \mathbf{curl}\, \boldsymbol{\beta}. \tag{7}$$

Then, we have that $\alpha \in \mathrm{H}_0^1(\Omega)$ satisfies

$$(\nabla \alpha, \nabla \psi) = (\boldsymbol{u} - \boldsymbol{u}_h, \nabla \psi) \quad \forall \psi \in \mathrm{H}_0^1(\Omega),$$

whereas, thanks to [2, Cor. 3.16], $\left\| \boldsymbol{\beta} \right\|_0 \le C \left\| \mathbf{curl}\, \boldsymbol{\beta} \right\|_0$. Moreover, there exists $t > 1/2$ such that $\boldsymbol{\beta} \in \mathbf{H}^t(\Omega)$ and $\left\| \boldsymbol{\beta} \right\|_{\mathbf{H}^t(\Omega)} \le C \left\| \mathbf{curl}\, \boldsymbol{\beta} \right\|_0$, too (see [2, Prop. 3.7]).

Let α_h be the piecewise linear and continuous Scott-Zhang interpolant [27] of α on the mesh $\mathcal{T}_h$ that vanishes on $\partial \Omega$. Taking into account the first equations from (2) and (3) and the fact that $\nabla \alpha_h \in \boldsymbol{\mathcal{X}}_h$, we have that

$$\begin{aligned}\left\| \nabla \alpha \right\|_0^2 &= (\boldsymbol{u} - \boldsymbol{u}_h, \nabla \alpha) = -(\boldsymbol{u}_h, \nabla \alpha) = -(\boldsymbol{u}_h, \nabla(\alpha - \alpha_h)) \\ &= \sum_{K \in \mathcal{T}_h} \left[(\operatorname{div} \boldsymbol{u}_h, \alpha - \alpha_h)_K - (\boldsymbol{u}_h \cdot \boldsymbol{n}_K, \alpha - \alpha_h)_{\partial K} \right] \\ &\le \sum_{K \in \mathcal{T}_h} \left[\left\| \operatorname{div} \boldsymbol{u}_h \right\|_{0,K} \left\| \alpha - \alpha_h \right\|_{0,K} + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left\| [\![\boldsymbol{u}_h \cdot \boldsymbol{n}_K]\!] \right\|_{0,F} \left\| \alpha - \alpha_h \right\|_{0,F} \right],\end{aligned}$$

where $\boldsymbol{n}_K$ denotes the outer unit normal to ∂K. Hence, standard estimates for the Scott-Zhang interpolant lead to

$$\|\nabla\alpha\|_0^2 \le C \sum_{K\in\mathcal{T}_h}\left[h_K^2\,\|\operatorname{div}\boldsymbol{u}_h\|_{0,K}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathrm{I}}(K)} h_F\,\|[\![\boldsymbol{u}_h\cdot\boldsymbol{n}_K]\!]\|_{0,F}^2\right] \le C\eta^2. \tag{8}$$

Let us now consider the second term in the Helmholtz decomposition. Integrating by parts and using the second equation in (2) and the fact that $\operatorname{\mathbf{curl}}\boldsymbol{u}_h = -\lambda_h\boldsymbol{\sigma}_h$ (which in turn follows from the second equation in (3) and (4)), we obtain

$$\begin{aligned}\|\operatorname{\mathbf{curl}}\boldsymbol{\beta}\|_0^2 &= (\boldsymbol{u}-\boldsymbol{u}_h,\operatorname{\mathbf{curl}}\boldsymbol{\beta}) = (\operatorname{\mathbf{curl}}(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{\beta})\\ &= (\lambda-\lambda_h)(\boldsymbol{\sigma},\boldsymbol{\beta}) + \lambda_h(\boldsymbol{\sigma}-\boldsymbol{P}_h\boldsymbol{\sigma},\boldsymbol{\beta}) + \lambda_h(\boldsymbol{P}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\beta}).\end{aligned} \tag{9}$$

Here and thereafter, $\boldsymbol{P}_h:\ \mathbf{H}_0(\operatorname{div}^0;\Omega)\to\boldsymbol{\mathcal{M}}_h^0$ is the $\mathbf{L}^2(\Omega)$-orthogonal projection onto $\boldsymbol{\mathcal{M}}_h^0$ which satisfies the following approximation property for all $\boldsymbol{\tau}\in\mathbf{H}_0(\operatorname{div}^0;\Omega)\cap\mathbf{H}^t(\Omega)$ with $0<t\le 1$ [10, Lemma 3.2]:

$$\|\boldsymbol{\tau}-\boldsymbol{P}_h\boldsymbol{\tau}\|_0 \le Ch^t\,\|\boldsymbol{\tau}\|_{\mathbf{H}^t(\Omega)}. \tag{10}$$

For the first term on the right-hand side of (9), we use Proposition 1, the fact that $\|\boldsymbol{\sigma}\|_0 = 1$ and the bound for $\|\boldsymbol{\beta}\|_0$ to write

$$(\lambda-\lambda_h)(\boldsymbol{\sigma},\boldsymbol{\beta}) \le C\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0^2\right)\|\operatorname{\mathbf{curl}}\boldsymbol{\beta}\|_0.$$

For the second one, we proceed as follows:

$$\begin{aligned}\lambda_h(\boldsymbol{\sigma}-\boldsymbol{P}_h\boldsymbol{\sigma},\boldsymbol{\beta}) &= \lambda_h(\boldsymbol{\sigma}-\boldsymbol{P}_h\boldsymbol{\sigma},\boldsymbol{\beta}-\boldsymbol{P}_h\boldsymbol{\beta}) \le \lambda_h\|\boldsymbol{\sigma}-\boldsymbol{P}_h\boldsymbol{\sigma}\|_0\|\boldsymbol{\beta}-\boldsymbol{P}_h\boldsymbol{\beta}\|_0\\ &\le \lambda_h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\,Ch^t\,\|\boldsymbol{\beta}\|_{\mathbf{H}^t(\Omega)} \le Ch^t\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\,\|\operatorname{\mathbf{curl}}\boldsymbol{\beta}\|_0,\end{aligned}$$

where we have used that $\|\boldsymbol{\sigma}-\boldsymbol{P}_h\boldsymbol{\sigma}\|_0\le\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_0$ for all $\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{M}}_h^0$ and the bound for $\|\boldsymbol{\beta}\|_{\mathbf{H}^t(\Omega)}$.

For the third term, we use a superapproximation property for $\|\boldsymbol{P}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$ that we will prove in Section 4 (see Lemma 9) and the bound for $\|\boldsymbol{\beta}\|_0$ again:

$$\lambda_h(\boldsymbol{P}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\beta}) \le \rho(h)\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\right)\|\operatorname{\mathbf{curl}}\boldsymbol{\beta}\|_0.$$

Thus, inserting the last three inequalities in (9), we obtain

$$\|\operatorname{\mathbf{curl}}\boldsymbol{\beta}\|_0 \le C\rho(h)\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\right)$$

and, combining this with (7) and (8),

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \le C\eta + C\rho(h)\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\right). \tag{11}$$

Now, let us estimate $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$. By definition, $\operatorname{div}\boldsymbol{\sigma} = \operatorname{div}\boldsymbol{\sigma}_h = 0$ in Ω. Hence, applying [2, Theor. 3.17 and Cor. 3.19], there exists $\boldsymbol{w}\in\mathbf{H}_0(\operatorname{\mathbf{curl}};\Omega)\cap\mathbf{H}(\operatorname{div}^0;\Omega)$ such that

$$\boldsymbol{\sigma}-\boldsymbol{\sigma}_h = \operatorname{\mathbf{curl}}\boldsymbol{w}\quad\text{in }\Omega$$

with

$$\|\boldsymbol{w}\|_{\mathbf{curl}} \le C\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0. \tag{12}$$

Then, for any $\boldsymbol{v}_h \in \boldsymbol{\mathcal{X}}_h$, from the first equations in (2) and (3) we have

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0^2 &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,\boldsymbol{w}) \\ &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{v}_h)) + (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,\boldsymbol{v}_h) \\ &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{v}_h)) - (\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h)\,.\end{aligned} \tag{13}$$

Next, we construct $\boldsymbol{v}_h$ using the interpolant operator introduced in [26, Theor. 1]. In what follows we recall its definition and some of its properties. Let $\boldsymbol{\mathcal{X}}_h^0 := \{\boldsymbol{v}_h \in \mathbf{H}_0(\mathbf{curl};\Omega):\ \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_0(K)\}$ be the lowest-order Nédélec space. There exists an operator $\boldsymbol{\Pi}_h : \mathbf{H}_0(\mathbf{curl};\Omega) \to \boldsymbol{\mathcal{X}}_h^0$ such that for every $\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ there exist $\varphi \in \mathrm{H}_0^1(\Omega)$ and $\boldsymbol{z} \in \mathbf{H}_0^1(\Omega)$ which satisfy $\boldsymbol{w} - \boldsymbol{\Pi}_h\boldsymbol{w} = \nabla\varphi + \boldsymbol{z}$ and the following estimates for all $K \in \mathcal{T}_h$:

$$\begin{aligned} h_K^{-1}\|\varphi\|_{0,K} + \|\nabla\varphi\|_{0,K} &\le C\,\|\boldsymbol{w}\|_{0,\omega_K}\,, \\ h_K^{-1}\|\boldsymbol{z}\|_{0,K} + \|\nabla\boldsymbol{z}\|_{0,K} &\le C\,\|\mathbf{curl}\,\boldsymbol{w}\|_{0,\omega_K}\,. \end{aligned} \tag{14}$$

Here and thereafter ω_K is the union of the elements which share at least one vertex with K. Notice that as a consequence of (14), $\|\boldsymbol{\Pi}_h\boldsymbol{w}\|_0 \le C\,\|\boldsymbol{w}\|_{\mathbf{curl}}$.

Let us set $\boldsymbol{v}_h = \boldsymbol{\Pi}_h\boldsymbol{w}$ in (13) and bound separately the two terms in the last line. First, integration by parts and the first equation of (2) lead to

$$\begin{aligned}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{v}_h)) &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,\boldsymbol{z}) \\ &= -(\boldsymbol{u},\boldsymbol{z}) - \sum_{K\in\mathcal{T}_h}\left[(\mathbf{curl}\,\boldsymbol{\sigma}_h,\boldsymbol{z})_K + \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathrm{I}}(K)}([\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!],\boldsymbol{z})_F\right] \\ &= (\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{z}) - \sum_{K\in\mathcal{T}_h}\left[(\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h,\boldsymbol{z})_K + \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathrm{I}}(K)}([\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!],\boldsymbol{z})_F\right]\end{aligned}$$

and hence

$$\begin{aligned}|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{v}_h))| &\le \|\boldsymbol{u}-\boldsymbol{u}_h\|_0\,\|\boldsymbol{z}\|_0 \\ &+ \sum_{K\in\mathcal{T}_h}\left[\|\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h\|_{0,K}\,\|\boldsymbol{z}\|_{0,K} + \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathrm{I}}(K)}\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}\,\|\boldsymbol{z}\|_{0,F}\right].\end{aligned}$$

From (14), using standard trace estimates and (12), we infer

$$\begin{aligned}\|\boldsymbol{z}\|_{0,F}^2 &\le C\left(h_F^{-1}\|\boldsymbol{z}\|_{0,K}^2 + h_F\,\|\nabla\boldsymbol{z}\|_{0,K}^2\right) \\ &\le Ch_F\,\|\mathbf{curl}\,\boldsymbol{w}\|_{0,\omega_K}^2 = Ch_F\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_K}^2\,,\end{aligned}$$

whereas

$$\|\boldsymbol{z}\|_{0,K} \le Ch_K\,\|\mathbf{curl}\,\boldsymbol{w}\|_{0,\omega_K} \le Ch_K\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_K}\,.$$

Hence,

$$\begin{aligned}|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{w}-\boldsymbol{v}_h))| \le C\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\Bigg\{& h\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \\ &+ \sum_{K\in\mathcal{T}_h}\left[h_K^2\,\|\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h\|_{0,K}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathrm{I}}(K)}h_F\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}^2\right]^{1/2}\Bigg\}.\end{aligned} \tag{15}$$

To bound the second term in the last line of (13), we use the fact that $\boldsymbol{v}_h = \boldsymbol{\Pi}_h \boldsymbol{w} \in \boldsymbol{\mathcal{X}}_h^0 \subset \boldsymbol{\mathcal{X}}_h$ and (12) to write

$$\begin{aligned}|(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h)| &\le \|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \|\boldsymbol{\Pi}_h \boldsymbol{w}\|_0 \\ &\le C \|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \|\boldsymbol{w}\|_{\mathbf{curl}} \le C \|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 .\end{aligned}$$

Putting together the last inequality with (13), (15) and (11), we arrive at

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \le C\eta + C\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \le C\eta + C\rho(h)\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0\right)$$

which allows us to conclude the proof. □

In what follows, we prove the efficiency of the local indicators η_K.

Theorem 4. *There exists a positive constant C such that the following bounds hold true for any $K \in \mathcal{T}_h$:*

$$\begin{aligned} h_K \|\operatorname{div} \boldsymbol{u}_h\|_{0,K} &\le C \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} , \\ h_K \|\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h\|_{0,K} &\le C \left(h_K \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,K} \right) .\end{aligned} \tag{16}$$

Moreover, for any inner face F, let ω_F denote the union of the two tetrahedra sharing this face. Then, there exists another positive constant C such that

$$\begin{aligned} h_F^{1/2} \|[\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!]\|_{0,F} &\le C \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\omega_F} , \\ h_F^{1/2} \|[\![\boldsymbol{\sigma}_h \times \boldsymbol{n}]\!]\|_{0,F} &\le C \left(h_F \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\omega_F} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_F} \right) .\end{aligned} \tag{17}$$

Consequently, for each $K \in \mathcal{T}_h$,

$$\eta_K \le C \left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\widetilde{\omega}_K} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\widetilde{\omega}_K} \right) ,$$

where $\widetilde{\omega}_K$ is the union of the tetrahedra sharing a face with K.

Proof. The inequalities on the first lines of (16) and (17) have been proved in [10, Theor. 4.5]. Hence, we only have to prove the remaining estimates.

Let $b_K \in \mathrm{H}_0^1(\Omega)$ be the standard quartic bubble function on K which attains the value one at the barycenter of this element, extended by zero outside K. Let us set $\boldsymbol{\varphi} := b_K (\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h) \in \mathbf{H}_0^1(\Omega)$. Then, taking into account the first equation in (2), which implies that $\boldsymbol{u} + \mathbf{curl}\, \boldsymbol{\sigma} = \mathbf{0}$, we have

$$\begin{aligned} C \|\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h\|_{0,K}^2 &\le (\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h, \boldsymbol{\varphi})_K \\ &= -(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\varphi})_K - (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\, \boldsymbol{\varphi})_K \\ &\le \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} \|\boldsymbol{\varphi}\|_{0,K} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,K} \|\mathbf{curl}\, \boldsymbol{\varphi}\|_{0,K} .\end{aligned}$$

The application of an inverse inequality in the last term and the definition of $\boldsymbol{\varphi}$ yield

$$C \|\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h\|_{0,K}^2 \le C \left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} + \frac{1}{h_K} \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,K} \right) \|\boldsymbol{u}_h + \mathbf{curl}\, \boldsymbol{\sigma}_h\|_{0,K} ,$$

from which the desired estimate easily follows.

There remains to prove the last bound in (17). For a fixed inner face F, we observe that $[\![\boldsymbol{\sigma}_h \times \boldsymbol{n}]\!] \in [\mathbb{P}_k(F)]^3$. Let $\boldsymbol{J}_F$ be the extension of $[\![\boldsymbol{\sigma}_h \times \boldsymbol{n}]\!]$ to ω_F such that $\boldsymbol{J}_F|_K \in [\mathbb{P}_k(K)]^3$ for $K \subset \omega_F$ and it is constant in the direction from the barycenter of F to the opposite vertex of K. Moreover, let $b_F \in \mathrm{H}_0^1(\omega_F)$ be the piecewise cubic function which attains the value one at the barycenter of F. Setting $\boldsymbol{\gamma} := \boldsymbol{J}_F b_F \in \mathbf{H}_0^1(\omega_F)$, we have from (2)

$$\begin{aligned}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \mathbf{curl}\,\boldsymbol{\gamma})_{\omega_F} &= -(\boldsymbol{u},\boldsymbol{\gamma})_{\omega_F} - \sum_{K\subset\omega_F}(\mathbf{curl}\,\boldsymbol{\sigma}_h,\boldsymbol{\gamma})_K + ([\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!],\boldsymbol{\gamma})_F \\ &= -(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\gamma})_{\omega_F} - \sum_{K\subset\omega_F}(\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h,\boldsymbol{\gamma})_K + ([\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!],\boldsymbol{\gamma})_F\,.\end{aligned}$$

Hence, from the definition of $\boldsymbol{\gamma}$,

$$\begin{aligned}C\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}^2 &\le ([\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!],\boldsymbol{\gamma})_F \\ &\le \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_F}\,\|\mathbf{curl}\,\boldsymbol{\gamma}\|_{0,\omega_F} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\omega_F}\,\|\boldsymbol{\gamma}\|_{0,\omega_F} \\ &\quad + \left(\sum_{K\subset\omega_F}\|\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h\|_{0,K}^2\right)^{1/2}\|\boldsymbol{\gamma}\|_{0,\omega_F}\,.\end{aligned} \tag{18}$$

Now, from standard computations, for each $K\subset\omega_F$ we have

$$\|\boldsymbol{J}_F\|_{0,K}^2 \le C h_F\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}^2\,,$$

which implies

$$\|\boldsymbol{\gamma}\|_{0,\omega_F} = \|\boldsymbol{J}_F b_F\|_{0,\omega_F} \le \|\boldsymbol{J}_F\|_{0,\omega_F} \le C h_F^{1/2}\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}\,.$$

On the other hand, using an inverse inequality again,

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{\gamma}\|_{0,\omega_F} &= \|\mathbf{curl}\,(\boldsymbol{J}_F b_F)\|_{0,\omega_F} \le \|(\mathbf{curl}\,\boldsymbol{J}_F)\,b_F\|_{0,\omega_F} + \|\nabla b_K\times\boldsymbol{J}_F\|_{0,\omega_F} \\ &\le \|\mathbf{curl}\,\boldsymbol{J}_F\|_{0,\omega_F} + \|\nabla b_K\|_{\mathrm{L}^\infty(\omega_F)}\,\|\boldsymbol{J}_F\|_{0,\omega_F} \le C h_F^{-1}\,\|\boldsymbol{J}_F\|_{0,\omega_F} \\ &\le C h_F^{-1/2}\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}\,.\end{aligned}$$

Inserting the last two inequalities into (18), we obtain

$$\begin{aligned}\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}^2 \le C\Bigg[& h_F^{-1/2}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\omega_F} + h_F^{1/2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\omega_F} \\ & + h_F^{-1/2}\left(\sum_{K\subset\omega_F} h_K^2\,\|\boldsymbol{u}_h+\mathbf{curl}\,\boldsymbol{\sigma}_h\|_{0,K}^2\right)^{1/2}\Bigg]\,\|[\![\boldsymbol{\sigma}_h\times\boldsymbol{n}]\!]\|_{0,F}\,,\end{aligned}$$

which together with the second inequality in (16) yield the claimed estimate. □

3.1. Reliability and efficiency of the original indicator μ.

In practice, what is actually solved is Problem 1 and the a posteriori error indicators μ_K are the ones that are used to drive an adaptive scheme to solve this problem. In what follows, we will derive reliability and efficiency estimates for these indicators. For the former, we have the following result.

Proposition 5 (Reliability)**.** *Let $(\omega,\boldsymbol{E})$ and $(\omega_h,\boldsymbol{E}_h)$ be solutions of Problems 1 and 2, respectively, such that the latter approximates the former as h goes to zero. Then, there exists a positive constant C such that, for h small enough,*

$$\begin{aligned}\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathbf{curl}} &\le C\mu \\ \left|\omega^2-\omega_h^2\right| &\le C\mu^2.\end{aligned}$$

Proof. By using the relations between the solutions of Problems 1 and 2 and those of Problems 3 and 4, respectively, and (6), some simple algebra yields

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\mathbf{curl}} \leq C\mu + C'\mu^2. \tag{19}$$

Now, since $\boldsymbol{u} = \omega\boldsymbol{E} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega) \subset \mathbf{H}^t(\Omega)$ for $t > 1/2$ and $\boldsymbol{\sigma} = \mathbf{curl}\,\boldsymbol{E}/\omega \in \mathbf{H}_0(\mathrm{div}^0;\Omega) \cap \mathbf{H}(\mathbf{curl};\Omega) \subset \mathbf{H}^t(\Omega)$ as well, by virtue of Proposition 1 we have that $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \to 0$ as h goes to zero. Then, according to Theorem 4, $\mu = \frac{1}{\omega_h}\eta \leq \frac{C}{\omega_h}(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0) \to 0$, too. Therefore, for h small enough, μ^2 is negligible in (19) and we conclude the first estimate of the proposition. The second one follows for h small enough from Theorem 3, Proposition 1 again and (6):

$$|\omega-\omega_h| = \frac{|\lambda-\lambda_h|}{\omega+\omega_h} \leq C\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_0^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0^2\right) \leq C\eta^2 = \frac{C}{\omega_h^2}\mu^2.$$

□

The consequences of Theorem 4 in terms of local bounds for the indicator μ are less evident than what has been proved for the reliability estimates. Although this is a typical situation when dealing with eigenvalue problems, we find it useful to detail the estimates that can be obtained in this particular case. The next proposition shows some useful bounds for the terms in the definition of μ.

Proposition 6. *There exists a positive constant C such that the following bounds hold true for any $K \in \mathcal{T}_h$:*

$$\begin{aligned} &h_K \|\mathrm{div}\,\boldsymbol{E}_h\|_{0,K} \leq C\|\boldsymbol{E}-\boldsymbol{E}_h\|_{0,K}, \\ &h_K \left\|\boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\,\boldsymbol{E}_h/\omega_h^2)\right\|_{0,K} \\ &\qquad\qquad \leq \frac{C}{\omega_h^2}\left[h_K\left\|\omega^2\boldsymbol{E}-\omega_h^2\boldsymbol{E}_h\right\|_{0,K} + \|\mathbf{curl}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{0,K}\right]. \end{aligned} \tag{20}$$

Moreover, there exists another positive constant C such that, for any inner face F,

$$\begin{aligned} &h_F^{1/2}\|[\![\boldsymbol{E}_h\cdot\boldsymbol{n}]\!]\|_{0,F} \leq C\|\boldsymbol{E}-\boldsymbol{E}_h\|_{0,\omega_F}, \\ &h_F^{1/2}\left\|[\![(\mathbf{curl}\,\boldsymbol{E}_h/\omega_h^2)\times\boldsymbol{n}]\!]\right\|_{0,F} \\ &\qquad\qquad \leq \frac{C}{\omega_h^2}\left[h_F\left\|\omega^2\boldsymbol{E}-\omega_h^2\boldsymbol{E}_h\right\|_{0,\omega_F} + \|\mathbf{curl}(\boldsymbol{E}-\boldsymbol{E}_h)\|_{0,\omega_F}\right], \end{aligned} \tag{21}$$

where ω_F is again the union of the two tetrahedra sharing the face F.

Proof. The first inequalities in (20) and (21) are obtained by using the same arguments as in the proof of Theorem 4.5 of [10]. Let us prove the estimate for the residual term in the second line of (20). Let $b_K \in \mathrm{H}_0^1(\Omega)$ be the standard quartic bubble function on K which attains the value one at the barycenter of this element, extended by zero outside K. Let us set $\boldsymbol{\varphi} := b_K\left[\omega_h^2\boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\,\boldsymbol{E}_h)\right] \in \mathbf{H}_0^1(\Omega)$.

Then, taking into account (1), which implies that $\omega^2 \boldsymbol{E} - \mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}) = \mathbf{0}$, we have

$$\begin{aligned}
\left\|\boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}_h/\omega_h^2)\right\|_{0,K}^2 &= \frac{1}{\omega_h^4}\left\|\omega_h^2 \boldsymbol{E}_h - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{E}_h\right\|_{0,K}^2 \\
&\leq \frac{C}{\omega_h^4}\left(\omega_h^2 \boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}_h), \boldsymbol{\varphi}\right)_K \\
&= -\frac{C}{\omega_h^4}\left[(\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h, \boldsymbol{\varphi})_K - (\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h), \mathbf{curl}\, \boldsymbol{\varphi})_K\right] \\
&\leq \frac{C}{\omega_h^4}\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_{0,K} \|\boldsymbol{\varphi}\|_{0,K} + \|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,K} \|\mathbf{curl}\, \boldsymbol{\varphi}\|_{0,K}.
\end{aligned}$$

Taking into account the definition of $\boldsymbol{\varphi}$ and using an inverse inequality we arrive at

$$\begin{aligned}
\left\|\boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}_h/\omega_h^2)\right\|_{0,K}^2 \leq \frac{C}{\omega_h^2}&\left\|\boldsymbol{E}_h - \mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}_h/\omega_h^2)\right\|_{0,K} \\
&\times\left[\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_{0,K} + h_K^{-1}\|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,K}\right],
\end{aligned}$$

which allows us to conclude the last inequality in (20). The proof of the last inequality in (21) can be obtained using similar arguments. □

Putting things together, we obtain from the above proposition that

$$\mu_K \leq C\left[\|\boldsymbol{E} - \boldsymbol{E}_h\|_{0,\widetilde{\omega}_K} + \|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\widetilde{\omega}_K} + h_K\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K}\right], \tag{22}$$

where $\widetilde{\omega}_K$ is, as above, the union of the tetrahedra sharing a face with K. It is clear that the presence of the term $\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K}$ in the right hand side of our estimate prevents the efficiency property from being *local*. Indeed, bounding this term involves dealing with the difference $|\omega^2 - \omega_h^2|$ which is not localized.

Nevertheless, the main interest of a posteriori error analysis is to build adaptive schemes and global efficiency is in general enough to guarantee their convergence (the reader is referred to [18, **?**] for elliptic eigenvalue problems and to [8] for mixed eigenvalue problems). In the present case, indeed, we have a global efficiency estimate. In fact, from (22) we have

$$\mu \leq C\left(\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl}} + h\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_0\right).$$

To bound the last term above, we write

$$\left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_0 \leq \left|\omega^2 - \omega_h^2\right| \|\boldsymbol{E}_h\|_0 + \omega^2 \|\boldsymbol{E} - \boldsymbol{E}_h\|_0.$$

Then, using the normalization constraint $\|\boldsymbol{E}_h\|_0 = 1$ and the well known identity (cf. [3, Lemma 9.1])

$$\omega^2 - \omega_h^2 = \|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)\|_0^2 - \omega^2\|\boldsymbol{E} - \boldsymbol{E}_h\|_0^2,$$

we obtain

$$\mu \leq C\left(\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl}} + h\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl}}^2\right).$$

Since $\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl}} \to 0$ as h goes to zero, the last term in the inequality above is asymptotically negligible. Thus, for h small enough, we derive the global efficiency estimate

$$\mu \leq C\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\mathbf{curl}}.$$

On the other hand, in the case of lowest order elements, it is possible to proceed as in [17] to obtain an estimate more local than (22). More precisely, in this case $\mathbf{curl}(\mathbf{curl}(\boldsymbol{E}_h|_K))$ vanishes, so that $h_K\|\boldsymbol{E}_h\|_{0,K} \leq \mu_K$. Moreover, as stated in the proof of Proposition 5, $\boldsymbol{u} = \omega \boldsymbol{E}$ and $\boldsymbol{\sigma} = \mathbf{curl}\, \boldsymbol{E}$ belong to $\mathbf{H}^t(\Omega)$ for some $t > 1/2$. Hence, according to Proposition 1, $\left|\omega^2 - \omega_h^2\right| \leq C h^{2\hat{t}}$ with $\hat{t} := \min\{t, 1\}$. Therefore, proceeding as above, we have

$$\begin{aligned} h_K \left\|\omega^2 \boldsymbol{E} - \omega_h^2 \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K} &\leq \left|\omega^2 - \omega_h^2\right| h_K \left\|\boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K} + \omega^2 h_K \left\|\boldsymbol{E} - \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K} \\ &\leq C h^{2\hat{t}} \mu_{\widetilde{\omega}_K} + \omega^2 h_K \left\|\boldsymbol{E} - \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K}, \end{aligned}$$

where $\mu_{\widetilde{\omega}_K} := \left(\sum_{K' \subset \widetilde{\omega}_K} \mu_{K'}^2\right)^{1/2}$. Then, substituting this estimate in (22) leads to

$$\mu_K \leq C \left[\left\|\boldsymbol{E} - \boldsymbol{E}_h\right\|_{0,\widetilde{\omega}_K} + \left\|\mathbf{curl}(\boldsymbol{E} - \boldsymbol{E}_h)\right\|_{0,\widetilde{\omega}_K} + h^{2\hat{t}} \mu_{\widetilde{\omega}_K}\right].$$

4. Auxiliary results

In what follows we will consider the following auxiliary source problem: given a solution $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$ of Problem 3, find $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{\sigma}}_h) \in \boldsymbol{\mathcal{X}}_h \times \boldsymbol{\mathcal{M}}_h^0$ such that

$$\begin{aligned} (\widehat{\boldsymbol{u}}_h, \boldsymbol{v}_h) + (\mathbf{curl}\, \boldsymbol{v}_h, \widehat{\boldsymbol{\sigma}}_h) &= 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{X}}_h, \\ (\mathbf{curl}\, \widehat{\boldsymbol{u}}_h, \boldsymbol{\tau}_h) &= -\lambda\, (\boldsymbol{\sigma}, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\mathcal{M}}_h^0. \end{aligned} \tag{23}$$

The existence and uniqueness of the solution to this problem is a consequence of the well-posedness of the source problem associated to (3) (see e.g. [7, Prop. 4.1]). It is clear that $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{\sigma}}_h)$ provides an approximation to $(\boldsymbol{u}, \boldsymbol{\sigma})$. Moreover, by simple computations using (3) and (23), we obtain

$$-\lambda_h\, (\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_h) = (\mathbf{curl}\, \boldsymbol{u}_h, \widehat{\boldsymbol{\sigma}}_h) = -\,(\widehat{\boldsymbol{u}}_h, \boldsymbol{u}_h) = (\mathbf{curl}\, \widehat{\boldsymbol{u}}_h, \boldsymbol{\sigma}_h) = -\lambda\, (\boldsymbol{\sigma}, \boldsymbol{\sigma}_h). \tag{24}$$

We recall that $\boldsymbol{P}_h : \mathbf{H}_0(\mathrm{div}^0; \Omega) \to \boldsymbol{\mathcal{M}}_h^0$ is the $\mathbf{L}^2(\Omega)$-orthogonal projection onto $\boldsymbol{\mathcal{M}}_h^0$. The following technical result shows that $\boldsymbol{P}_h \boldsymbol{\sigma}$ provides a higher-order approximation of $\widehat{\boldsymbol{\sigma}}_h$.

Lemma 7. *Let $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$ and $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{\sigma}}_h)$ be solutions of Problems 3 and (23), respectively. Then, there exists $\rho(h)$ tending to zero as $h \to 0$ such that*

$$\left\|\boldsymbol{P}_h \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\right\|_0 \leq \rho(h) \left\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_h\right\|_{\mathbf{curl}}.$$

Proof. Let $(\boldsymbol{w}, \boldsymbol{\psi}) \in \mathbf{H}_0(\mathbf{curl}; \Omega) \times \mathbf{H}_0(\mathrm{div}^0; \Omega)$ be the solution of the following problem:

$$\begin{aligned} (\boldsymbol{w}, \boldsymbol{v}) + (\mathbf{curl}\, \boldsymbol{v}, \boldsymbol{\psi}) &= 0 \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \\ (\mathbf{curl}\, \boldsymbol{w}, \boldsymbol{\tau}) &= -\,(\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h \boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div}^0; \Omega). \end{aligned} \tag{25}$$

By testing the first equation above with a smooth $\boldsymbol{v}$ with compact support in Ω, we derive that $\mathbf{curl}\, \boldsymbol{\psi} = -\boldsymbol{w}$. Hence, $\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \cap \mathbf{H}(\mathrm{div}^0; \Omega)$ and $\boldsymbol{\psi} \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}_0(\mathrm{div}^0; \Omega)$, so that both belong to $\mathbf{H}^t(\Omega)$ for some $t > 1/2$ (see [2, Prop. 3.7]) and

$$\|\boldsymbol{w}\|_{\mathbf{H}^t(\Omega)} + \|\boldsymbol{\psi}\|_{\mathbf{H}^t(\Omega)} \leq C \left(\|\boldsymbol{w}\|_{\mathbf{curl}} + \|\boldsymbol{\psi}\|_0\right) \leq C \left\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h \boldsymbol{\sigma}\right\|_0. \tag{26}$$

Let us denote by $\boldsymbol{\Pi}$ the Fortin operator introduced in [5]. In particular, we have

$$\begin{aligned} (\mathbf{curl}\, (\boldsymbol{w} - \boldsymbol{\Pi} \boldsymbol{w}), \boldsymbol{v}_h) &= 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{M}}_h^0, \\ \|\boldsymbol{w} - \boldsymbol{\Pi} \boldsymbol{w}\|_0 &\leq C h^t \|\boldsymbol{w}\|_{\mathbf{curl}}. \end{aligned} \tag{27}$$

Then,

$$
\begin{aligned}
\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}\|_0^2 &= -\left(\mathbf{curl}\,\boldsymbol{w}, \widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}\right) && \text{(25) second eq.}\\
&= -\left(\mathbf{curl}(\boldsymbol{\Pi}\boldsymbol{w}), \widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}\right) && \text{(27) first eq.}\\
&= -\left(\mathbf{curl}(\boldsymbol{\Pi}\boldsymbol{w}), \widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}\right) && \mathbf{curl}(\boldsymbol{\Pi}\boldsymbol{w}) \in \boldsymbol{\mathcal{M}}_h^0\\
&= \left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{\Pi}\boldsymbol{w}\right) && \text{(23), (2) first eqs.}\\
&= \left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}\right) + \left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{w}\right) && \\
&= \left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}\right) - \left(\mathbf{curl}\left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}\right), \boldsymbol{\psi}\right) && \text{(25) first eq.}\\
&= \left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{\Pi}\boldsymbol{w} - \boldsymbol{w}\right) - \left(\mathbf{curl}\left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}\right), \boldsymbol{\psi} - \boldsymbol{P}_h\boldsymbol{\psi}\right) && \text{(23) and (2) second eqs.}\\
&\le Ch^t\left[\|\widehat{\boldsymbol{u}}_h - \boldsymbol{u}\|_0 + \|\mathbf{curl}\left(\widehat{\boldsymbol{u}}_h - \boldsymbol{u}\right)\|_0\right]\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}\|_0 && \text{(27), (10) and (26)}
\end{aligned}
$$

which allows us to end the proof. □

The following auxiliary result shows that the term $\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_{\mathbf{curl}}$ that arises in the previous lemma is bounded by the actual error.

Lemma 8. *Let $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$, $(\lambda_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ and $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{\sigma}}_h)$ be solutions of Problem 3, Problem 4 and (23), respectively, with $\|\boldsymbol{\sigma}\|_0 = \|\boldsymbol{\sigma}_h\|_0 = 1$. Then,*

$$\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_{\mathbf{curl}} \le C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0\right).$$

Proof. The triangle inequality yields

$$\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_h\|_{\mathbf{curl}} \le \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{curl}} + \|\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\|_{\mathbf{curl}}.$$

Thanks to (2), (3) and (4), we have that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{curl}}^2 = \|\boldsymbol{u} - \boldsymbol{u}_h\|_0^2 + \|\lambda\boldsymbol{\sigma} - \lambda_h\boldsymbol{\sigma}_h\|_0^2.$$

Subtracting (23) from (3), the stability of the resulting discrete problem (see [7]) leads to

$$\|\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\|_{\mathbf{curl}} \le C\|\lambda\boldsymbol{\sigma} - \lambda_h\boldsymbol{\sigma}_h\|_0.$$

Finally, from Proposition 1,

$$
\begin{aligned}
\|\lambda\boldsymbol{\sigma} - \lambda_h\boldsymbol{\sigma}_h\|_0 &\le \lambda\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |\lambda - \lambda_h|\,\|\boldsymbol{\sigma}_h\|_0\\
&\le \lambda\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2\right)\\
&\le C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0\right)
\end{aligned}
$$

and the lemma follows from the last four inequalities. □

Now we are in a position to prove the following superapproximation result which is an easy generalization of analogous properties known for the approximation of mixed Laplacian (see [15, 19] and [6, Ch. 7.4]).

Lemma 9. *Let $(\lambda, \boldsymbol{u}, \boldsymbol{\sigma})$ and $(\lambda_h, \boldsymbol{u}_h, \boldsymbol{\sigma}_h)$ be solutions of Problems 3 and 4, respectively, with $\|\boldsymbol{\sigma}\|_0 = \|\boldsymbol{\sigma}_h\|_0 = 1$ and such that the latter approximates the former as h goes to zero. Then, there exists $\rho(h)$ tending to zero as $h \to 0$ such that*

$$\|\boldsymbol{P}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le \rho(h)\left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_0\right).$$

Proof. Let $(\widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{\sigma}}_h)$ be the solution of (23). By the triangle inequality we have

$$\|\boldsymbol{P}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le \|\boldsymbol{P}_h\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_0 + \|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0 \,. \tag{28}$$

Hence, thanks to the last two lemmas, we only have to bound the second term on the right-hand side above.

Let us enumerate the eigenvalues of Problems 3 and 4 in increasing order as in Proposition 2. Let us assume that the eigenvalue λ we are considering is λ_J; then, for h small enough, $\lambda_h = \lambda_{h,J}$.

Let $\boldsymbol{\sigma}_{h,J} := \boldsymbol{\sigma}_h$ and, for each $i \neq J$ ($i \le I_h$), we choose an eigenfunction $(\boldsymbol{u}_{h,i}, \boldsymbol{\sigma}_{h,i})$ of Problem 4 corresponding to the eigenvalue $\lambda_{h,i}$, so that $\{\boldsymbol{\sigma}_{h,i}\}_{i=1}^{I_h}$ is an $\mathbf{L}^2(\Omega)$-orthogonal basis of $\boldsymbol{\mathcal{M}}_h^0$. Then, we can write

$$\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h = \sum_{i=1}^{I_h} \alpha_i \boldsymbol{\sigma}_{h,i} \qquad \text{with } \alpha_i := (\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_{h,i})$$

and

$$\|\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h\|_0^2 = \sum_{i=1}^{I_h} \alpha_i^2. \tag{29}$$

We bound separately α_J and α_i for $i \neq J$. For the former we have

$$\alpha_J = (\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) = (\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_h) - 1 = \frac{\lambda}{\lambda_h}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) - 1,$$

where we have used (24) for the last equality. Now,

$$\frac{\lambda}{\lambda_h}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) - 1 = \frac{\lambda}{\lambda_h} - 1 + \frac{\lambda}{\lambda_h}\left[(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) - 1\right] = \frac{\lambda - \lambda_h}{\lambda_h} - \frac{\lambda}{2\lambda_h}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2 \,.$$

Hence, from Proposition 1, it follows that

$$|\alpha_J| = \left|\frac{\lambda}{\lambda_h}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h) - 1\right| \le C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2\right). \tag{30}$$

On the other hand, for $i \neq J$, from (3) and (23) we have

$$\begin{aligned} -\lambda_{h,i}(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_{h,i}) &= (\mathbf{curl}\, \boldsymbol{u}_{h,i}, \widehat{\boldsymbol{\sigma}}_h) = -(\widehat{\boldsymbol{u}}_h, \boldsymbol{u}_{h,i}) \\ &= (\mathbf{curl}\, \widehat{\boldsymbol{u}}_h, \boldsymbol{\sigma}_{h,i}) = -\lambda(\boldsymbol{\sigma}, \boldsymbol{\sigma}_{h,i}) = -\lambda(\boldsymbol{P}_h\boldsymbol{\sigma}, \boldsymbol{\sigma}_{h,i}) \,. \end{aligned}$$

Adding $\lambda(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_{h,i})$ to both sides of this identity yields

$$(\lambda - \lambda_{h,i})(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_{h,i}) = \lambda(\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}, \boldsymbol{\sigma}_{h,i}),$$

which, provided $\lambda \neq \lambda_{h,i}$, leads to

$$(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\sigma}_{h,i}) = \frac{\lambda}{\lambda - \lambda_{h,i}}(\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{P}_h\boldsymbol{\sigma}, \boldsymbol{\sigma}_{h,i}) \,.$$

The definition of α_i, the orthogonality of $\{\boldsymbol{\sigma}_{h,i}\}_{i=1}^{I_h}$ and the last identity yield

$$\begin{aligned}\sum_{i\neq J}\alpha_i^2 &= \sum_{i\neq J}\alpha_i\left(\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h,\boldsymbol{\sigma}_{h,i}\right)=\sum_{i\neq J}\alpha_i\left(\widehat{\boldsymbol{\sigma}}_h,\boldsymbol{\sigma}_{h,i}\right)\\ &= \sum_{i\neq J}\alpha_i\frac{\lambda}{\lambda-\lambda_{h,i}}\left(\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{P}_h\boldsymbol{\sigma},\boldsymbol{\sigma}_{h,i}\right)\\ &\le \max_{i\neq J}\frac{\lambda}{|\lambda-\lambda_{h,i}|}\left(\sum_{i\neq J}\alpha_i^2\right)^{1/2}\left[\sum_{i\neq J}\left(\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{P}_h\boldsymbol{\sigma},\boldsymbol{\sigma}_{h,i}\right)^2\right]^{1/2}.\end{aligned}$$

Finally, Proposition 2 gives

$$\left(\sum_{i\neq J}\alpha_i^2\right)^{1/2}\le C\left\|\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{P}_h\boldsymbol{\sigma}\right\|_0$$

which, together with (30) and (29) yield

$$\left\|\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h\right\|_0^2\le C\left[\left\|\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{P}_h\boldsymbol{\sigma}\right\|_0^2+\left(\left\|\boldsymbol{u}-\boldsymbol{u}_h\right\|_0^2+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\|_0^2\right)^2\right].$$

This relation, inserted in (28), leads to

$$\left\|\boldsymbol{P}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\|_0\le C\left(\left\|\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{P}_h\boldsymbol{\sigma}\right\|_0+\left\|\boldsymbol{u}-\boldsymbol{u}_h\right\|_0^2+\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\|_0^2\right).$$

Thus, from the above inequality and Lemmas 7 and 8, we conclude the proof. □

5. Numerical results

In this section, we illustrate the behavior of the error indicators (5). Problem 1 has been discretized by using lowest-order edge elements ($k = 1$) on tetrahedral meshes that have been created with the mesh generator TetGen ([28]). The resulting algebraic eigenvalue problem has been solved using the Matlab routine `eigs`, that is based on the `ARPACK` package ([20]).

We have chosen a so called *Fichera domain*: $\Omega := (-1,1)^3\backslash[-1,0]^3$ (see Figure 1). Let us remark that this is the same example as in [10], where, for lowest-order elements, the error indicator was taken only as the term in (5) involving the jumps of the normal components of the computed eigenfunction.

We have computed the eigenpair corresponding to the smallest positive eigenvalue. The exact eigenpairs of this problem are not known. Because of this, first, we have computed them with highly refined structured 'uniform' meshes, which allowed us to obtain a very accurate approximation of the corresponding eigenvalue by means of a least squares extrapolation. These 'uniform' meshes have been obtained by subdividing the domain into equal hexahedra, each of them subdivided into six tetrahedra. By so doing, we have obtained $\omega^2 = 3.220$ as an approximate value of the smallest positive eigenvalue with four correct significant digits. This value for ω^2 was taken as the 'exact' eigenvalue. Let us remark that this agrees with what is reported in Monique Dauge's web page ([14]).

Let us emphasize that such an extrapolation procedure to compute a more accurate approximation of an unknown quantity can only be used for scalar unknowns like the eigenvalues, but not for functional unknowns as is the case of the eigenfunctions. This is the reason why, in what follows, we will focus on the computation of

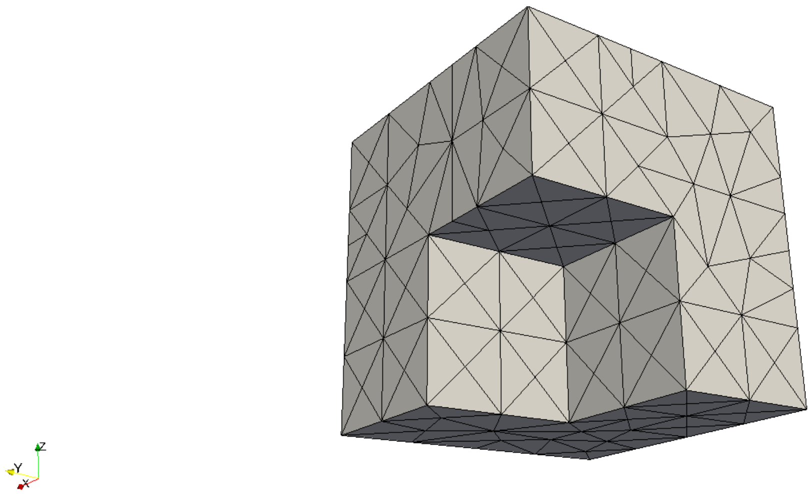

FIGURE 1. Domain with the initial mesh.

the eigenvalues. Moreover, let us remark that according to Proposition 5, μ^2 provides an asymptotic upper estimate for the error of the eigenvalue approximation; namely, for h small enough there holds

$$\left|\omega^2 - \omega_h^2\right| \leq C\mu^2.$$

We have applied an adaptive scheme driven by the error indicators μ_K. Notice that, for the lowest-order edge elements, $\operatorname{div} \boldsymbol{E}_h = 0$ and $\mathbf{curl}(\mathbf{curl}\, \boldsymbol{E}_h) = \mathbf{0}$ on each element K. Therefore, the local error indicators reduce to

$$\mu_K^2 = h_K^2 \left\|\boldsymbol{E}_h\right\|_{0,K}^2 + \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathrm{I}}(K)} \left(\frac{h_F}{\omega_h^4} \left\| [\![\mathbf{curl}\, \boldsymbol{E}_h \times \boldsymbol{n}]\!] \right\|_{0,F}^2 + h_F \left\| [\![\boldsymbol{E}_h \cdot \boldsymbol{n}]\!] \right\|_{0,F}^2 \right).$$

In this test, we started the computations with the unstructured mesh consisting of 581 elements, which is shown in Figure 1. Then, we proceeded with the adaptive refinement process. Figures 2 and 3 display the fifth and the last adaptively refined mesh, respectively.

Figure 4 displays a log-log plot of the errors between the computed approximations of the smallest positive eigenvalue and the 'exact' one, versus the number N of elements of the meshes. The figure shows the results obtained with 'uniform' meshes and with adaptively refined meshes. The very accurate agreement between the eigenvalues computed with 'uniform' meshes and the line obtained by a least square fitting of them is a clear indication of the reliability of the value taken as 'exact'.

The slope $-2/3$ of the optimal order of convergence for the used lowest-order edge elements is very close to the slope 0.660 of the line obtained by a least squares fitting of the values computed with the adaptive scheme. The results are very similar to those reported in [10], although, for instance, in the latter, the slope was a bit steeper: -0.75.

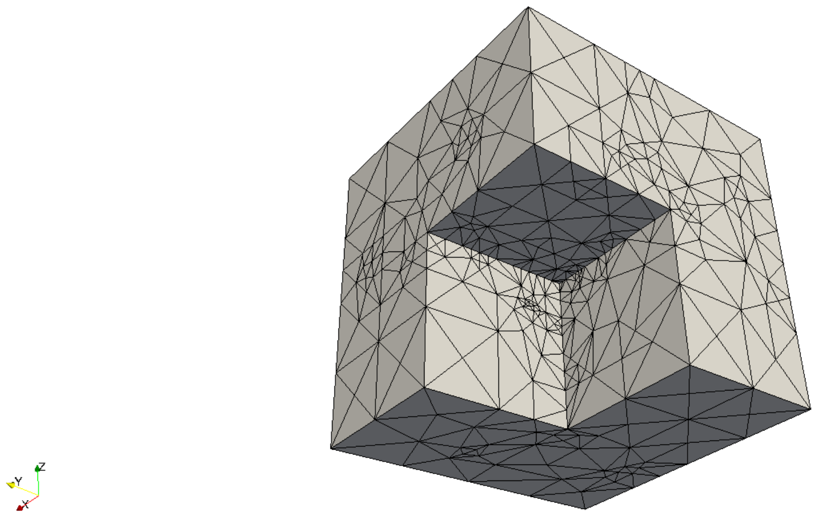

FIGURE 2. Adaptively refined mesh at the fifth step.

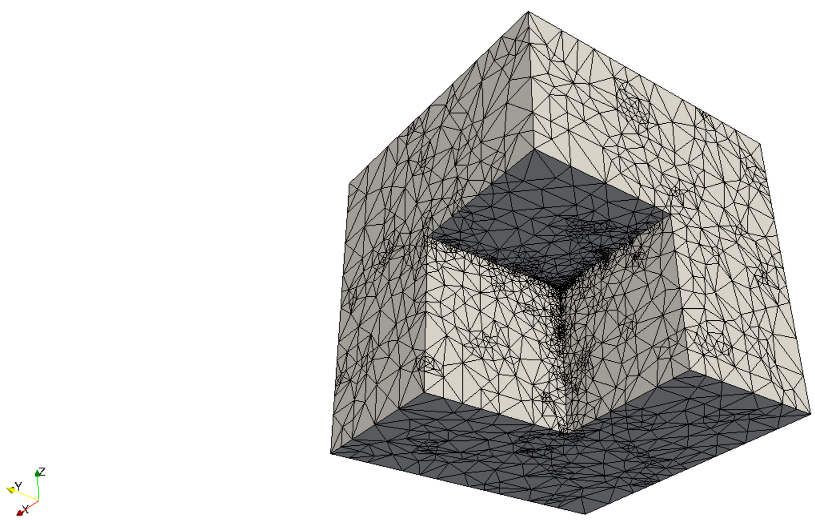

FIGURE 3. Adaptively refined mesh at the last refinement step.

We report in Table 1 the squared estimator for the error of the computed smallest positive eigenvalue and its three components:

$$\mu_1^2 := \sum_{K\in\mathcal{T}_h} h_K^2 \, \|\boldsymbol{E}_h\|_{0,K}^2 \,,$$

$$\mu_2^2 := \sum_{K\in\mathcal{T}_h} \frac{1}{2} \sum_{F\in\mathcal{F}_{\mathrm{I}}(K)} \frac{h_F}{\omega_h^4} \, \|[\![\mathbf{curl}\,\boldsymbol{E}_h \times \boldsymbol{n}]\!]\|_{0,F}^2 \,,$$

$$\mu_3^2 := \sum_{K\in\mathcal{T}_h} \frac{1}{2} \sum_{F\in\mathcal{F}_{\mathrm{I}}(K)} h_F \, \|[\![\boldsymbol{E}_h \cdot \boldsymbol{n}]\!]\|_{0,F}^2 \,.$$

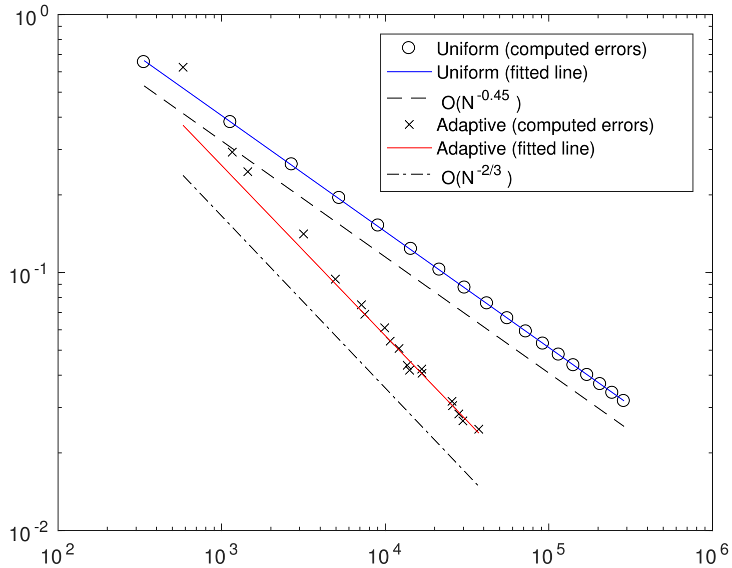

FIGURE 4. Error curves for the smallest positive eigenvalue of the Maxwell's equations on the Fichera domain computed with 'uniform' and adaptively refined meshes: log-log plots of the respective errors versus the number of elements.

The table also shows the error and the ratio of the estimator to this error, which plays the role of an *effectivity index*.

We observe that μ_1^2 is around 8.5% of μ^2, μ_2^2 around 11.5%, and μ_3^2 around 80%. Consequently, the first two components are significantly smaller than the third one (which was the only one considered in [10]), but not asymptotically negligible. On the other hand, the effectivity indexes are bounded above and below away from zero and, except for the coarsest meshes, they take values in a very narrow range (between 5.18 and 6.25).

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N	μ_1^2	μ_2^2	μ_3^2	μ^2	$\omega^2-\omega_h^2$	$\mu^2/(\omega^2-\omega_h^2)$
581	0.213	0.247	1.723	2.183	0.625	3.49
1161	0.104	0.133	1.111	1.348	0.293	4.60
1445	0.089	0.119	0.907	1.115	0.246	4.53
3167	0.057	0.070	0.540	0.667	0.141	4.72
4955	0.043	0.053	0.391	0.488	0.094	5.18
7153	0.036	0.046	0.327	0.409	0.075	5.45
7498	0.034	0.045	0.305	0.384	0.069	5.57
9922	0.030	0.040	0.271	0.341	0.061	5.59
10718	0.026	0.037	0.245	0.309	0.054	5.69
12140	0.026	0.036	0.230	0.292	0.051	5.76
13622	0.023	0.034	0.217	0.274	0.044	6.27
14090	0.022	0.033	0.205	0.261	0.042	6.25
16712	0.020	0.028	0.190	0.237	0.042	5.62
16779	0.020	0.027	0.182	0.230	0.041	5.64
25591	0.015	0.021	0.144	0.181	0.032	5.70
25800	0.015	0.021	0.139	0.175	0.030	5.75
28159	0.014	0.020	0.127	0.161	0.028	5.68
29828	0.013	0.019	0.120	0.152	0.027	5.71
37295	0.011	0.016	0.109	0.136	0.025	5.49

TABLE 1. Error estimators, components of these estimators, eigenvalue error and effectivity indexes.

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Dipartimento di Matematica "F. Casorati", Università di Pavia, Italy
E-mail address: daniele.boffi@unipv.it
URL: http://www-dimat.unipv.it/boffi/

DICATAM Sez. di Matematica, Università di Brescia, Italy
E-mail address: lucia.gastaldi@unibs.it
URL: http://lucia-gastaldi.unibs.it

CI²MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: rodolfo@ing-mat.udec.cl
URL: http://www.ing-mat.udec.cl/~rodolfo/

CI²MA, Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: isebestova@udec.cl