A posteriori error estimates for a Virtual Elements Method for the Steklov eigenvalue problem.

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Abstract

The paper deals with the a posteriori error analysis of a virtual element method for the Steklov eigenvalue problem. The virtual element method has the advantage of using general polygonal meshes, which allows implementing very efficiently mesh refinement strategies. We introduce a residual type a posteriori error estimator and prove its reliability and efficiency. We use the corresponding error estimator to drive an adaptive scheme. Finally, we report the results of a couple of numerical tests, that allow us to assess the performance of this approach.

Key words: virtual element method, a posteriori error estimates, Steklov eigenvalue problem, polygonal meshes
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1. Introduction

The *Virtual Element Method* (VEM), introduced in [7, 8], is a recent generalization of the Finite Element Method, which is characterized by the capability of dealing with very general polygonal/polyhedral meshes. The interest in numerical methods that can make use of general polytopal meshes has recently undergone a significant growth in the mathematical and engineering literature; among the large number of papers on this subject, we cite as a minimal sample [7, 9, 20, 26, 35, 36, 37]. Indeed, polytopal meshes can be very useful for a wide range of reasons including meshing of the domain, automatic use of hanging nodes, moving meshes and adaptivity. VEM has been applied successfully in a large range of problems; see for instance [1, 3, 6, 7, 8, 10, 12, 13, 16, 18, 29, 31, 32, 33].

The object of this paper is to introduce and analyze an a posteriori error estimator of residual type for the virtual element approximation of the Steklov eigenvalue problem. In fact, due to the large flexibility of the meshes to which the virtual element method is applied, mesh adaptivity becomes an appealing feature as mesh refinement strategies can be implemented very efficiently. For instance, hanging nodes can be introduced in the mesh to guarantee the mesh conformity without spreading the refined zones. In fact hanging nodes introduced by the refinement of a neighboring

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element are simply treated as new nodes since adjacent non matching element interfaces are perfectly acceptable. On the other hand, polygonal cells with very general shapes are admissible thus allowing us to adopt simple mesh coarsening algorithms.

The approximation of eigenvalue problems has been the object of great interest from both the practical and theoretical points of view, since they appear in many applications. We refer to [17] and the references therein for the state of art in this subject area. In particular, the Steklov eigenvalue problem, which involves the Laplace operator but is characterized by the presence of the eigenvalue in the boundary condition, appears in many applications; for example, the study of the vibration modes of a structure in contact with an incompressible fluid (see [14]) and the analysis of the stability of mechanical oscillators immersed in a viscous media (see [34]). One of its main applications arises from the dynamics of liquids in moving containers, i.e., sloshing problems (see [15, 19, 23, 24, 28, 39]).

On the other hand, adaptive mesh refinement strategies based on a posteriori error indicators play a relevant role in the numerical solution of partial differential equations in a general sense. For instance, they guarantee achieving errors below a tolerance with a reasonable computer cost in presence of singular solutions. Several approaches have been considered to construct error estimators based on the residual equations (see [2, 27, 38] and the references therein). In particular, for the Steklov eigenvalue problem we mention [4, 5, 25, 30, 40]. On the other hand, the design and analysis of a posteriori error bounds for the VEM is a challenging task. References [11, 21] are the only a posteriori error analyses for VEM currently available in the literature. In [11], a posteriori error bounds for the C^1-conforming VEM for the two-dimensional Poisson problem are proposed. In [21], a posteriori error bounds are introduced for the C^0-conforming VEM proposed in [22] for the discretization of second order linear elliptic reaction-convection-diffusion problems with non constant coefficients in two and three dimensions.

We have recently developed in [31] a virtual element method for the Steklov eigenvalue problem. Under standard assumptions on the computational domain, we have established that the resulting scheme provides a correct approximation of the spectrum and proved optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. In order to exploit the capability of VEM in the use of general polygonal meshes and its flexibility for the application of mesh adaptive strategies, we introduce and analyze an a posteriori error estimator for the virtual element approximation introduced in [31]. Since normal fluxes of the VEM solution are not computable, they will be replaced in the estimators by a proper projection. As a consequence of this replacement, new additional terms appear in the a posteriori error estimator, which represent the virtual inconsistency of VEM. Similar terms also appear in the other papers for a posteriori error estimates of VEM (see [11, 21]). We prove that the error estimator is equivalent to the error and use the corresponding indicator to drive an adaptive scheme.

The outline of this article is as follows: in Section 2 we present the continuous and discrete formulations of the Steklov eigenvalue problem together with the spectral characterization. Then, we recall the a priori error estimates for the virtual element approximation analyzed in [31]. In Section 3, we define the a posteriori error estimator and proved its reliability and efficiency. Finally, in Section 4, we report a set of numerical tests that allow us to assess the performance of an adaptive strategy driven by the estimator. We have also made a comparison between the proposed estimator and the standard edge-residual error estimator for a finite element method.

Throughout the article we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

2. The Steklov eigenvalue problem and its virtual element approximation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$. Let Γ_0 and Γ_1 be disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$ with $\Gamma_0 \neq \emptyset$. We denote by n the outward unit normal vector to $\partial\Omega$.

We consider the following eigenvalue problem:

Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = \begin{cases} \lambda w & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases}$$

By testing the first equation above with $v \in H^1(\Omega)$ and integrating by parts, we arrive at the following equivalent weak formulation:

Problem 1. *Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that*

$$\int_\Omega \nabla w \cdot \nabla v = \lambda \int_{\Gamma_0} wv \qquad \forall v \in H^1(\Omega).$$

According to [31, Theorem 2.1], we know that the solutions (λ, w) of the problem above are:

- $\lambda_0 = 0$, whose associated eigenspace is the space of constant functions in Ω;
- a sequence of positive finite-multiplicity eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \to \infty$.

The eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\Gamma_0)$. Therefore the eigenfunctions w^k corresponding to $\lambda_k > 0$ satisfy

$$\int_{\Gamma_0} w^k = 0. \tag{2.1}$$

We denote the bounded bilinear symmetric forms appearing in Problem 1 as follows:

$$a(w, v) := \int_\Omega \nabla w \cdot \nabla v, \qquad w, v \in H^1(\Omega),$$
$$b(w, v) := \int_{\Gamma_0} wv, \qquad w, v \in H^1(\Omega).$$

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. We assume that for every mesh $\mathcal{T}_h$, $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ are union of edges of elements $K \in \mathcal{T}_h$. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\mathcal{T}_h} h_K$.

For the analysis, we will make as in [7, 31] the following assumptions.

- **A1.** Every mesh $\mathcal{T}_h$ consists of a finite number of *simple* polygons (i.e., open simply connected sets with non self intersecting polygonal boundaries).
- **A2.** There exists $\gamma > 0$ such that, for all meshes $\mathcal{T}_h$, each polygon $K \in \mathcal{T}_h$ is star-shaped with respect to a ball of radius greater than or equal to γh_K.
- **A3.** There exists $\widehat{\gamma} > 0$ such that, for all meshes $\mathcal{T}_h$, for each polygon $K \in \mathcal{T}_h$, the distance between any two of its vertices is greater than or equal to $\widehat{\gamma} h_K$.

We consider now a simple polygon K and, for $k \in \mathbb{N}$, we define

$$\mathbb{B}_k(\partial K) := \left\{ v \in C^0(\partial K) : \; v|_\ell \in \mathbb{P}_k(\ell) \text{ for all edges } \ell \subset \partial K \right\}.$$

We then consider the finite-dimensional space defined as follows:

$$V_k^K := \left\{ v \in H^1(K) : \; v|_{\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v|_K \in \mathbb{P}_{k-2}(K) \right\}, \tag{2.2}$$

where, for $k = 1$, we have used the convention that $\mathbb{P}_{-1}(K) := \{0\}$. We choose in this space the degrees of freedom introduced in [7, Section 4.1]. Finally, for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and for a fixed $k \in \mathbb{N}$, we define

$$V_h := \left\{ v \in H^1(\Omega) : \; v|_K \in V_k^K \quad \forall K \in \mathcal{T}_h \right\}.$$

In what follows, we will use standard Sobolev spaces, norms and seminorms and also the broken H^1-seminorm

$$|v|_{1,h}^2 := \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{0,K}^2,$$

which is well defined for every $v \in L^2(\Omega)$ such that $v|_K \in H^1(K)$ for each polygon $K \in \mathcal{T}_h$.

We split the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v) = \sum_{K \in \mathcal{T}_h} a^K(u,v), \qquad u, v \in H^1(\Omega),$$

where

$$a^K(u,v) := \int_K \nabla u \cdot \nabla v, \qquad u, v \in H^1(K).$$

Due to the implicit space definition, we must have into account that we would not know how to compute $a^K(\cdot,\cdot)$ for $u_h, v_h \in V_h$. Nevertheless, the final output will be a local matrix on each element K whose associated bilinear form can be exactly computed whenever one of the two entries is a polynomial of degree k. This will allow us to retain the optimal approximation properties of the space V_h.

With this end, for any $K \in \mathcal{T}_h$ and for any sufficiently regular function φ, we define first

$$\overline{\varphi} := \frac{1}{N_K} \sum_{i=1}^{N_K} \varphi(P_i),$$

where P_i, $1 \le i \le N_K$, are the vertices of K. Then, we define the projector $\Pi_k^K : \; V_k^K \longrightarrow \mathbb{P}_k(K) \subseteq V_k^K$ for each $v_h \in V_k^K$ as the solution of

$$a^K\left(\Pi_k^K v_h, q\right) = a^K(v_h, q) \qquad \forall q \in \mathbb{P}_k(K),$$
$$\overline{\Pi_k^K v_h} = \overline{v_h}.$$

On the other hand, let $S^K(\cdot,\cdot)$ be any symmetric positive definite bilinear form to be chosen as to satisfy

$$c_0\, a^K(v_h, v_h) \le S^K(v_h, v_h) \le c_1\, a^K(v_h, v_h) \qquad \forall v_h \in V_k^K \text{ with } \Pi_k^K v_h = 0 \tag{2.4}$$

for some positive constants c_0 and c_1 independent of K. Then, set

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h), \qquad u_h, v_h \in V_h,$$

where $a_h^K(\cdot,\cdot)$ is the bilinear form defined on $V_k^K \times V_k^K$ by

$$a_h^K(u_h, v_h) := a^K\left(\Pi_k^K u_h, \Pi_k^K v_h\right) + S^K\left(u_h - \Pi_k^K u_h, v_h - \Pi_k^K v_h\right), \qquad u_h, v_h \in V_k^K.$$

Notice that the bilinear form $S^K(\cdot,\cdot)$ has to be actually computable for $u_h, v_h \in V_k^K$.

The following properties of $a_h^K(\cdot,\cdot)$ have been established in [7, Theorem 4.1].

- *k-Consistency*:
$$a_h^K(p, v_h) = a^K(p, v_h) \qquad \forall p \in \mathbb{P}_k(K), \quad \forall v_h \in V_k^K. \tag{2.5}$$
- *Stability*: There exist two positive constants α_* and α^*, independent of K, such that:
$$\alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h) \qquad \forall v_h \in V_k^K. \tag{2.6}$$

Now, we are in a position to write the virtual element discretization of Problem 1.

Problem 2. *Find $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, such that*

$$a_h(w_h, v_h) = \lambda_h b(w_h, v_h) \qquad \forall v_h \in V_h.$$

According to [31, Theorem 3.1] we know that the solutions (λ_h, w_h) of the problem above are:

- $\lambda_{h0} = 0$, whose associated eigenfunction are the constant functions in Ω.
- $\{\lambda_{hk}\}_{k=1}^{N_h}$, with $N_h := \dim\{v_h|_{\Gamma_0}, v_h \in V_h\} - 1$, which are positive eigenvalues repeated according to their respective multiplicities.

Moreover, the eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\Gamma_0)$. Therefore the eigenfunctions w_h^k corresponding to $\lambda_{hk} > 0$ satisfy

$$\int_{\Gamma_0} w_h^k = 0. \tag{2.7}$$

Let (λ, w) be a solution to Problem 1. We assume $\lambda > 0$ is a simple eigenvalue and we normalize w so that $\|w\|_{0,\Gamma_0} = 1$. Then, for each mesh $\mathcal{T}_h$, there exists a solution (λ_h, w_h) of Problem 2 such that $\lambda_h \to \lambda$, $\|w_h\|_{0,\Gamma_0} = 1$ and $\|w - w_h\|_{1,\Omega} \to 0$ as $h \to 0$. Moreover, according to (2.1) and (2.7), we have that w and w_h belong to the space

$$V := \left\{ v \in H^1(\Omega) : \int_{\Gamma_0} v = 0 \right\}.$$

Let us remark that the following generalized Poincaré inequality holds true in this space: there exists $C > 0$ such that

$$\|v\|_{1,\Omega} \le C|v|_{1,\Omega} \qquad \forall v \in V. \tag{2.8}$$

The following a priori error estimates have been proved in [31, Theorems 4.2–4.4]: there exists $C > 0$ such that for all $r \in [\frac{1}{2}, r_\Omega)$

$$\|w - w_h\|_{1,\Omega} \le C h^{\min\{r,k\}}, \tag{2.9}$$

$$|\lambda - \lambda_h| \le C h^{2\min\{r,k\}}, \tag{2.10}$$

$$\|w - w_h\|_{0,\Gamma_0} \le C h^{\min\{r,1\}/2 + \min\{r,k\}}, \tag{2.11}$$

where the constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. Let us remark that $r_\Omega > 1$, if Ω is convex, and $r_\Omega := \frac{\pi}{\omega}$ with ω being the largest re-entrant angle of Ω, otherwise.

3. A posteriori error analysis

The aim of this section is to introduce a suitable residual-based error estimator for the Steklov eigenvalue problem which be fully computable, in the sense that it depends only on quantities available from the VEM solution. Then, we will show its equivalence with the error. For this purpose, we introduce the following definitions and notations.

For any polygon $K \in \mathcal{T}_h$, we denote by $\mathcal{E}_K$ the set of edges of K and

$$\mathcal{E} := \bigcup_{K \in \mathcal{T}_h} \mathcal{E}_K.$$

We decompose $\mathcal{E} = \mathcal{E}_\Omega \cup \mathcal{E}_{\Gamma_0} \cup \mathcal{E}_{\Gamma_1}$, where $\mathcal{E}_{\Gamma_0} := \{\ell \in \mathcal{E} : \ell \subset \Gamma_0\}$, $\mathcal{E}_{\Gamma_1} := \{\ell \in \mathcal{E} : \ell \subset \Gamma_1\}$ and $\mathcal{E}_\Omega := \mathcal{E} \backslash (\mathcal{E}_{\Gamma_0} \cup \mathcal{E}_{\Gamma_1})$. For each inner edge $\ell \in \mathcal{E}_\Omega$ and for any sufficiently smooth function v, we define the jump of its normal derivative on ℓ by

$$\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right]_\ell := \nabla(v|_K) \cdot n_K + \nabla(v|_{K'}) \cdot n_{K'},$$

where K and K' are the two elements in $\mathcal{T}_h$ sharing the edge ℓ and n_K and $n_{K'}$ are the respective outer unit normal vectors.

As a consequence of the mesh regularity assumptions, we have that each polygon $K \in \mathcal{T}_h$ admits a sub-triangulation $\mathcal{T}_h^K$ obtained by joining each vertex of K with the midpoint of the ball with respect to which K is starred. Let $\widehat{\mathcal{T}}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{T}_h^K$. Since we are also assuming **A3**, $\{\widehat{\mathcal{T}}_h\}_h$ is a shape-regular family of triangulations of Ω.

We introduce bubble functions on polygons as follows (see [21]). An interior bubble function $\psi_K \in H_0^1(K)$ for a polygon K can be constructed piecewise as the sum of the cubic bubble functions for each triangle of the sub-triangulation $\mathcal{T}_h^K$ that attain the value 1 at the barycenter of each triangle. On the other hand, an edge bubble function ψ_ℓ for $\ell \in \partial K$ is a piecewise quadratic function attaining the value 1 at the barycenter of ℓ and vanishing on the triangles $T \in \widehat{\mathcal{T}}_h$ that do not contain ℓ on its boundary.

The following results which establish standard estimates for bubble functions will be useful in what follows (see [2, 38]).

Lemma 3.1 (Interior bubble functions). *For any $K \in \mathcal{T}_h$, let ψ_K be the corresponding interior bubble function. Then, there exists a constant $C > 0$ independent of h_K such that*

$$C^{-1}\|q\|_{0,K}^2 \leq \int_K \psi_K q^2 \leq \|q\|_{0,K}^2 \qquad \forall q \in \mathbb{P}_k(K),$$
$$C^{-1}\|q\|_{0,K} \leq \|\psi_K q\|_{0,K} + h_K\|\nabla(\psi_K q)\|_{0,K} \leq C\|q\|_{0,K} \qquad \forall q \in \mathbb{P}_k(K).$$

Lemma 3.2 (Edge bubble functions). *For any $K \in \mathcal{T}_h$ and $\ell \in \mathcal{E}_K$, let ψ_ℓ be the corresponding edge bubble function. Then, there exists a constant $C > 0$ independent of h_K such that*

$$C^{-1}\|q\|_{0,\ell}^2 \leq \int_\ell \psi_\ell q^2 \leq \|q\|_{0,\ell}^2 \qquad \forall q \in \mathbb{P}_k(\ell).$$

Moreover, for all $q \in \mathbb{P}_k(\ell)$, there exists an extension of $q \in \mathbb{P}_k(K)$ (again denoted by q) such that

$$h_K^{-1/2}\|\psi_\ell q\|_{0,K} + h_K^{1/2}\|\nabla(\psi_\ell q)\|_{0,K} \leq C\|q\|_{0,\ell}.$$

Remark 3.1. *A possible way of extending q from $\ell \in \mathcal{E}_K$ to K so that Lemma 3.2 holds is as follows: first we extend q to the straight line $L \supset \ell$ using the same polynomial function. Then, we extend it to the whole plain through a constant prolongation in the normal direction to L. Finally, we restrict the latter to K.*

The following lemma provides an error equation which will be the starting point of our error analysis. From now on, we will denote by $e := (w - w_h) \in V$ the eigenfunction error and by

$$J_\ell := \begin{cases} \frac{1}{2}\left[\!\left[\frac{\partial(\Pi_k^K w_h)}{\partial n}\right]\!\right]_\ell, & \ell \in \mathcal{E}_\Omega, \\ \lambda_h w_h - \frac{\partial(\Pi_k^K w_h)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_0}, \\ -\frac{\partial(\Pi_k^K w_h)}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_1}, \end{cases} \tag{3.1}$$

the edge residuals. Notice that J_ℓ are actually computable since they only involve values of w_h on Γ_0 (which are computable in terms of the boundary degrees of freedom) and $\Pi_k^K w_h \in \mathbb{P}_k(K)$ which is also computable.

Lemma 3.3. *For any $v \in H^1(\Omega)$, we have the following identity:*

$$a(e,v) = \lambda b(w,v) - \lambda_h b(w_h,v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) + \sum_{K\in\mathcal{T}_h}\left[\int_K \Delta(\Pi_k^K w_h)v + \sum_{\ell\in\mathcal{E}_K}\int_\ell J_\ell v\right].$$

Proof. Using that (λ, w) is a solution of Problem 1, adding and subtracting $\Pi_k^K w_h$ and integrating by parts, we obtain

$$\begin{aligned} a(e,v) &= \lambda b(w,v) - a(w_h,v) \\ &= \lambda b(w,v) - \sum_{K\in\mathcal{T}_h}\left[a^K(w_h - \Pi_k^K w_h, v) + a^K(\Pi_k^K w_h, v)\right] \\ &= \lambda b(w,v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) - \sum_{K\in\mathcal{T}_h}\left[-\int_K \Delta(\Pi_k^K w_h)\,v + \int_{\partial K}\frac{\partial(\Pi_k^K w_h)}{\partial n}\,v\right] \\ &= \lambda b(w,v) - \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, v) \\ &\quad + \sum_{K\in\mathcal{T}_h}\left[\int_K \Delta(\Pi_k^K w_h)\,v - \sum_{\ell\in\mathcal{E}_K\cap(\mathcal{E}_{\Gamma_0}\cup\mathcal{E}_{\Gamma_1})}\int_\ell \frac{\partial(\Pi_k^K w_h)}{\partial n}\,v + \frac{1}{2}\sum_{\ell\in\mathcal{E}_K\cap\mathcal{E}_\Omega}\int_\ell \left[\!\left[\frac{\partial(\Pi_k^K w_h)}{\partial n}\right]\!\right]_\ell v\right]. \end{aligned}$$

Finally, the proof follows by adding and subtracting the term $\lambda_h b(w_h, v)$. □

For all $K \in \mathcal{T}_h$, we introduce the local terms θ_K and R_K and the local error indicator η_K by

$$\begin{aligned} \theta_K^2 &:= a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h), \\ R_K^2 &:= h_K^2\|\Delta(\Pi_k^K w_h)\|_{0,K}^2, \\ \eta_K^2 &:= \theta_K^2 + R_K^2 + \sum_{\ell\in\mathcal{E}_K} h_K\|J_\ell\|_{0,\ell}^2. \end{aligned}$$

We also introduce the global error estimator by

$$\eta^2 := \sum_{K\in\mathcal{T}_h}\eta_K^2.$$

Remark 3.2. *The indicators η_K include the terms θ_K which do not appear in standard finite element estimators. This term, which represent the virtual inconsistency of the method, has been introduced in [11, 21] for a posteriori error estimates of other VEM. Let us emphasize that it can be directly computed in terms of the bilinear form $S^K(\cdot,\cdot)$. In fact,*

$$\theta_K^2 = a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) = S^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h).$$

3.1. Reliability of the a posteriori error estimator

First, we provide an upper bound for the error.

Theorem 3.1. *There exists a constant $C > 0$ independent of h such that*

$$|w - w_h|_{1,\Omega} \leq C \left(\eta + \frac{\lambda + \lambda_h}{2} \|w - w_h\|_{0,\Gamma_0} \right).$$

Proof. Since $e = w - w_h \in V \subset H^1(\Omega)$, there exists $e_I \in V_h$ satisfying (see [31, Proposition 4.2])

$$\|e - e_I\|_{0,K} + h_K |e - e_I|_{1,K} \leq C h_K \|e\|_{1,K}. \tag{3.2}$$

Then, we have that

$$\begin{aligned}
|w - w_h|_{1,\Omega}^2 &= a(w - w_h, e) \\
&= a(w - w_h, e - e_I) + a(w, e_I) - a_h(w_h, e_I) + a_h(w_h, e_I) - a(w_h, e_I) \\
&= \underbrace{\lambda b(w, e) - \lambda_h b(w_h, e)}_{T_1} + \underbrace{\sum_{K \in \mathcal{T}_h} \left[\int_K \Delta(\Pi_k^K w_h)(e - e_I) + \sum_{\ell \in \mathcal{E}_K} \int_\ell J_\ell (e - e_I) \right]}_{T_2} \\
&\quad - \underbrace{\sum_{K \in \mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, e - e_I)}_{T_3} + \underbrace{a_h(w_h, e_I) - a(w_h, e_I)}_{T_4},
\end{aligned} \tag{3.3}$$

the last equality thanks to Lemma 3.3. Next, we bound each term T_i separately.

For T_1, we use the definition of $b(\cdot, \cdot)$, the fact that $\|w\|_{0,\Gamma_0} = \|w_h\|_{0,\Gamma_0} = 1$, a trace theorem and (2.8) to write

$$T_1 = \lambda + \lambda_h - (\lambda + \lambda_h) \int_{\Gamma_0} w w_h = \frac{\lambda + \lambda_h}{2} \|e\|_{0,\Gamma_0}^2 \leq C \frac{\lambda + \lambda_h}{2} \|e\|_{0,\Gamma_0} |e|_{1,\Omega}. \tag{3.4}$$

For T_2, first, we use a local trace inequality (see [13, Lemma 14]) and (3.2) to write

$$\|e - e_I\|_{0,\ell} \leq C \Big(h_K^{-1/2} \|e - e_I\|_{0,K} + h_K^{1/2} |e - e_I|_{1,K} \Big) \leq C h_K^{1/2} \|e\|_{1,K}.$$

Hence, using (3.2) again, we have

$$\begin{aligned}
T_2 &\leq C \sum_{K \in \mathcal{T}_h} \left[\|\Delta(\Pi_k^K w_h)\|_{0,K} \|e - e_I\|_{0,K} + \sum_{\ell \in \mathcal{E}_K} \|J_\ell\|_{0,\ell} \|e - e_I\|_{0,\ell} \right] \\
&\leq C \sum_{K \in \mathcal{T}_h} \left[h_K \|\Delta(\Pi_k^K w_h)\|_{0,K} \|e\|_{1,K} + \sum_{\ell \in \mathcal{E}_K} h_K^{1/2} \|J_\ell\|_{0,\ell} \|e\|_{1,K} \right] \\
&\leq C \left\{ \sum_{K \in \mathcal{T}_h} \left[h_K^2 \|\Delta(\Pi_k^K w_h)\|_{0,K}^2 + \sum_{\ell \in \mathcal{E}_K} h_K \|J_\ell\|_{0,\ell}^2 \right] \right\}^{1/2} |e|_{1,\Omega},
\end{aligned} \tag{3.5}$$

where for the last estimate we have used (2.8).

To bound T_3, we use the *stability* property (2.6) and (3.2) to write

$$T_3 \leq C \sum_{K \in \mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} \|e\|_{1,K} \leq C \left(\sum_{K \in \mathcal{T}_h} \theta_K^2 \right)^{1/2} |e|_{1,\Omega}, \tag{3.6}$$

where for the last estimate we have used Remark 3.2 and (2.8) again.

Finally, to bound T_4, we add and subtract $\Pi_k^K w_h$ on each $K \in \mathcal{T}_h$ and use the *k-consistency* property (2.5):

$$\begin{aligned} T_4 &= \sum_{K\in\mathcal{T}_h} \left[a_h^K(w_h - \Pi_k^K w_h, e_I) - a^K(w_h - \Pi_k^K w_h, e_I)\right] \\ &\le \sum_{K\in\mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} a_h^K(e_I, e_I)^{1/2} \\ &\quad + \sum_{K\in\mathcal{T}_h} a^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} a^K(e_I, e_I)^{1/2} \\ &\le C \sum_{K\in\mathcal{T}_h} a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)^{1/2} |e_I|_{1,K} \\ &\le C \left(\sum_{K\in\mathcal{T}_h} \theta_K^2\right)^{1/2} |e|_{1,\Omega}, \end{aligned} \tag{3.7}$$

where we have used the *stability* property (2.6), (3.2) and (2.8) for the last two inequalities.

Thus, the result follows from (3.3)–(3.7). □

Although the virtual approximate eigenfunction is w_h, this function is not known in practice. Instead of w_h, what can be used as an approximation of the eigenfunction is $\Pi_h w_h$, where Π_h is defined for $v_h \in V_h$ by

$$(\Pi_h v_h)|_K := \Pi_k^K v_h \quad \forall K \in \mathcal{T}_h.$$

Notice that $\Pi_h w_h$ is actually computable. The following result shows that an estimate similar to that of Theorem 3.1 holds true for $\Pi_h w_h$.

Corollary 3.1. *There exists a constant $C > 0$ independent of h such that*

$$|w - w_h|_{1,\Omega} + |w - \Pi_h w_h|_{1,h} \le C \left(\eta + \frac{\lambda + \lambda_h}{2} \|w - w_h\|_{0,\Gamma_0}\right).$$

Proof. For each polygon $K \in \mathcal{T}_h$, we have that

$$|w - \Pi_k^K w_h|_{1,K} \le |w - w_h|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}.$$

Then, summing over all polygons we obtain

$$|w - \Pi_h w_h|_{1,h} \le C \left(\sum_{K\in\mathcal{T}_h} |w - w_h|_{1,K}^2 + \sum_{K\in\mathcal{T}_h} |w_h - \Pi_k^K w_h|_{1,K}^2\right)^{1/2}.$$

Now, using (2.4) together with Remark 3.2, we have that

$$|w_h - \Pi_k^K w_h|_{1,K}^2 \le \frac{1}{c_0} S^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h) = \frac{1}{c_0}\theta_K^2 \le \frac{1}{c_0}\eta_K^2.$$

Thus, the result follows from Theorem 3.1. □

In what follows, we prove a convenient upper bound for the eigenvalue approximation.

Corollary 3.2. *There exists a constant $C > 0$ independent of h such that*

$$|\lambda - \lambda_h| \le C \left(\eta + \frac{\lambda + \lambda_h}{2} \|w - w_h\|_{0,\Gamma_0} \right)^2 .$$

Proof. From the symmetry of the bilinear forms together with the facts that $a(w, v) = \lambda b(w, v)$ for all $v \in H^1(\Omega)$, $a_h(w_h, v_h) = \lambda_h b(w_h, v_h)$ for all $v_h \in V_h$ and $b(w_h, w_h) = 1$, we have

$$\begin{aligned}
|\lambda - \lambda_h| &= \frac{|a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) + a_h(w_h, w_h) - a(w_h, w_h)|}{b(w_h, w_h)} \\
&\le C \left[|w - w_h|^2_{1,\Omega} + \|w - w_h\|^2_{0,\Gamma_0} + |a_h(w_h, w_h) - a(w_h, w_h)| \right] \\
&\le C \left[|w - w_h|^2_{1,\Omega} + |a_h(w_h, w_h) - a(w_h, w_h)| \right] , \qquad (3.8)
\end{aligned}$$

where we have also used a trace theorem and (2.8). We now bound the last term on the right-hand side above using the definition of $a_h(\cdot, \cdot)$ and (2.4):

$$\begin{aligned}
&|a_h(w_h, w_h) - a(w_h, w_h)| \\
&= \left| \sum_{K \in \mathcal{T}_h} \left[a^K(\Pi_k^K w_h, \Pi_k^K w_h) + S^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \right] - \sum_{K \in \mathcal{T}_h} a^K(w_h, w_h) \right| \\
&\le \left| \sum_{K \in \mathcal{T}_h} \left[a^K\left(\Pi_k^K w_h, \Pi_k^K w_h\right) - a^K(w_h, w_h) \right] \right| + \sum_{K \in \mathcal{T}_h} c_1\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \\
&= \sum_{K \in \mathcal{T}_h} (1 + c_1)\, a^K\left(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h\right) \\
&\le (1 + c_1) \sum_{K \in \mathcal{T}_h} \left(|w_h - w|^2_{1,K} + \left|w - \Pi_k^K w_h\right|^2_{1,K} \right) .
\end{aligned}$$

Finally, from the above estimate and (3.8) we obtain

$$|\lambda - \lambda_h| \le C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_h w_h|^2_{1,h} \right) . \qquad (3.9)$$

Hence, we conclude the proof thanks to Corollary 3.1. $\square$

According to (2.9) and (2.11), it seems reasonable to expect the term $\|w - w_h\|_{0,\Gamma_0}$ in the estimate of Theorem 3.1 to be of higher order than $|w - w_h|_{1,\Omega}$ and hence asymptotically negligible. However this cannot be rigorously derived from (2.9) and (2.11), which are only upper error bounds. In fact, the actual error $|w - w_h|_{1,\Omega}$ could be in principle of higher order than the estimate (2.9).

Our next goal is to prove that the term $\|w - w_h\|_{0,\Gamma_0}$ is actually asymptotically negligible in the estimates of Corollaries 3.1 and 3.2. With this aim, we will modify the estimate (2.11) and prove that

$$\|w - w_h\|_{0,\Gamma_0} \le C h^{\min\{r,1\}/2} \left(|w - w_h|_{1,\Omega} + |w - \Pi_h w_h|_{1,h} \right) . \qquad (3.10)$$

This proof is based on the arguments used in Section 4 from [31]. To avoid repeating them step by step, in what follows we will only report the changes that have to be made in order to prove (3.10).

We define in $H^1(\Omega)$ the bilinear form $\widehat{a}(\cdot, \cdot) := a(\cdot, \cdot) + b(\cdot, \cdot)$, which is elliptic [31, Lemma 2.1]. Let $u \in H^1(\Omega)$ be the solution of

$$\widehat{a}(u, v) = b(w, v) \qquad \forall v \in H^1(\Omega).$$

Since $a(w,v) = \lambda b(w,v)$ we have that $u = w/(\lambda+1)$. We also define in V_h the bilinear form $\widehat{a}_h(\cdot,\cdot) := a_h(\cdot,\cdot) + b(\cdot,\cdot)$, which is elliptic uniformly in h [31, Lemma 3.1]. Let $u_h \in V_h$ be the solution of

$$\widehat{a}_h(u_h, v_h) = b(w, v_h) \qquad \forall v_h \in V_h. \tag{3.11}$$

The arguments in the proof of Lemma 4.3 from [31] can be easily modified to prove that

$$\|u - u_h\|_{0,\Gamma_0} \le C h^{\min\{r,1\}/2} \left(|u - u_h|_{1,\Omega} + |u - \Pi_h u_h|_{1,h} \right).$$

Then, using this estimate in the proof of Theorem 4.4 from [31] yields

$$\|w - w_h\|_{0,\Gamma_0} \le C h^{\min\{r,1\}/2} \left(|u - u_h|_{1,\Omega} + |u - \Pi_h u_h|_{1,h} \right). \tag{3.12}$$

Now, since as stated above $u = w/(\lambda+1)$, we have that

$$|u - u_h|_{1,\Omega} \le \frac{|w - w_h|_{1,\Omega}}{|\lambda+1|} + \left| \frac{1}{\lambda+1} - \frac{1}{\lambda_h+1} \right| |w_h|_{1,\Omega} + \left| \frac{w_h}{\lambda_h+1} - u_h \right|_{1,\Omega}. \tag{3.13}$$

For the second term on the right hand side above, we use (3.9) to write

$$\left| \frac{1}{\lambda+1} - \frac{1}{\lambda_h+1} \right| = \frac{|\lambda - \lambda_h|}{|\lambda+1||\lambda_h+1|} \le C \left(|w - w_h|_{1,\Omega}^2 + |w - \Pi_h w_h|_{1,h}^2 \right). \tag{3.14}$$

To estimate the third term we recall first that

$$\widehat{a}_h(w_h, v_h) = (\lambda_h + 1) b(w_h, v_h) \qquad \forall v_h \in V_h.$$

Then, subtracting this equation divided by $\lambda_h + 1$ from (3.11) we have that

$$\widehat{a}_h\left(u_h - \frac{w_h}{\lambda_h+1}, v_h \right) = b(w - w_h, v_h) \qquad \forall v_h \in V_h.$$

Hence, from the uniform ellipticity of $\widehat{a}_h(\cdot,\cdot)$ in V_h, we obtain

$$\left\| u_h - \frac{w_h}{\lambda_h+1} \right\|_{1,\Omega}^2 \le C \|w - w_h\|_{0,\Gamma_0} \left\| u_h - \frac{w_h}{\lambda_h+1} \right\|_{0,\Gamma_0} \le C \|w - w_h\|_{0,\Gamma_0} \left\| u_h - \frac{w_h}{\lambda_h+1} \right\|_{1,\Omega}.$$

Therefore

$$\left\| u_h - \frac{w_h}{\lambda_h+1} \right\|_{1,\Omega} \le C \|w - w_h\|_{0,\Gamma_0} \le C \|w - w_h\|_{1,\Omega} \le C |w - w_h|_{1,\Omega}, \tag{3.15}$$

the last inequality because of Poincaré inequality (2.8). Then, substituting (3.14) and (3.15) into (3.13) we obtain

$$|u - u_h|_{1,\Omega} \le C \left(|w - w_h|_{1,\Omega} + |w - \Pi_h w_h|_{1,h} \right). \tag{3.16}$$

For the other term on the right hand side of (3.12) we have

$$|u - \Pi_h u_h|_{1,h} \le |u - u_h|_{1,\Omega} + |u_h - \Pi_h u_h|_{1,h}, \tag{3.17}$$

whereas

$$|u_h - \Pi_h u_h|_{1,h} \le \left| u_h - \frac{w_h}{\lambda_h+1} \right|_{1,\Omega} + \frac{|w_h - \Pi_h w_h|_{1,h}}{\lambda_h+1} + \left| \Pi_h \left(\frac{w_h}{\lambda_h+1} - u_h \right) \right|_{1,h}$$

$$\leq 2\left|u_h - \frac{w_h}{\lambda_h+1}\right|_{1,\Omega} + \frac{|w-w_h|_{1,\Omega}}{\lambda_h+1} + \frac{|w-\Pi_h w_h|_{1,h}}{\lambda_h+1}$$
$$\leq C\left(|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h}\right),$$

where we have used (3.15) for the last inequality. Substituting this and estimate (3.16) into (3.17) we obtain

$$|u-\Pi_h u_h|_{1,h} \leq C\left(|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h}\right).$$

Finally, substituting the above estimate and (3.16) into (3.12), we conclude the proof of the following result.

Lemma 3.4. *There exists $C>0$ independent of h such that*

$$\|w-w_h\|_{0,\Gamma_0} \leq Ch^{\min\{r,1\}/2}\left(|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h}\right).$$

Using this result, now it easy to prove that the term $\|w-w_h\|_{0,\Gamma_0}$ in Corollaries 3.1 and 3.2 is asymptotically negligible. In fact, we have the following result.

Theorem 3.2. *There exist positive constants C and h_0 such that, for all $h<h_0$, there holds*

$$|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h} \leq C\eta; \tag{3.18}$$
$$|\lambda-\lambda_h| \leq C\eta^2. \tag{3.19}$$

Proof. From Lemma 3.4 and Corollary 3.1 we have

$$|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h} \leq C\left(\eta + h^{\min\{r,1\}/2}\left(|w-w_h|_{1,\Omega} + |w-\Pi_h w_h|_{1,h}\right)\right).$$

Hence, it is straightforward to check that there exists $h_0>0$ such that for all $h<h_0$ (3.18) holds true.

On the other hand, from Lemma 3.4 and (3.18) we have that for all $h<h_0$

$$\|w-w_h\|_{0,\Gamma_0} \leq Ch^{\min\{r,1\}/2}\eta.$$

Then, for h small enough, (3.19) follows from Corollary 3.2 and the above estimate. □

3.2. Efficiency of the a posteriori error estimator

We will show in this section that the local error indicators η_K are efficient in the sense of pointing out which polygons should be effectively refined.

First, we prove an upper estimate of the volumetric residual term R_K.

Lemma 3.5. *There exists a constant $C>0$ independent of h_K such that*

$$R_K \leq C\left(|w-w_h|_{1,K} + \theta_K\right).$$

Proof. For any $K\in\mathcal{T}_h$, let ψ_K be the corresponding interior bubble function. We define $v := \psi_K\Delta(\Pi_k^K w_h)$. Since v vanishes on the boundary of K, it may be extended by zero to the whole domain Ω. This extension, again denoted by v, belongs to $H^1(\Omega)$ and from Lemma 3.3 we have

$$a^K(e,v) = -a^K\left(w_h - \Pi_k^K w_h, \psi_K\Delta(\Pi_k^K w_h)\right) + \int_K \Delta(\Pi_k^K w_h)\psi_K\Delta(\Pi_k^K w_h).$$

Since $\Delta(\Pi_k^K w_h) \in \mathbb{P}_{k-2}(K)$, using Lemma 3.1 and the above equality we obtain

$$\begin{aligned} C^{-1}\|\Delta(\Pi_k^K w_h)\|_{0,K}^2 &\le \int_K \psi_K \Delta(\Pi_k^K w_h)^2 \\ &= a^K\left(e, \psi_K \Delta(\Pi_k^K w_h)\right) + a^K\left(w_h - \Pi_k^K w_h, \psi_K \Delta(\Pi_k^K w_h)\right) \\ &\le C\left(|e|_{1,K} + \left|w_h - \Pi_k^K w_h\right|_{1,K}\right) \left|\psi_K \Delta(\Pi_k^K w_h)\right|_{1,K} \\ &\le C h_K^{-1}\left(|e|_{1,K} + \theta_K\right) \left\|\Delta(\Pi_k^K w_h)\right\|_{0,K}, \end{aligned} \tag{3.20}$$

where, for the last inequality, we have used again Lemma 3.1 and (2.4) together with Remark 3.2. Multiplying the above inequality by h_K allows us to conclude the proof. □

Next goal is to obtain an upper estimate for the local term θ_K.

Lemma 3.6. *There exists $C > 0$ independent of h_K such that*

$$\theta_K \le C\left(|w - w_h|_{1,K} + |w - \Pi_k^K w_h|_{1,K}\right).$$

Proof. From the definition of θ_K together with Remark 3.2 and estimate (2.4) we have

$$\theta_K \le C|w_h - \Pi_k^K w_h|_{1,K} \le C\left(|w_h - w|_{1,K} + |w - \Pi_k^K w_h|_{1,K}\right).$$

The proof is complete. □

The following lemma provides an upper estimate for the jump terms of the local error indicator.

Lemma 3.7. *There exists a constant $C > 0$ independent of h_K such that*

$$h_K^{1/2}\|J_\ell\|_{0,\ell} \le C\big(|w - w_h|_{1,K} + \theta_K\big) \qquad \forall \ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_1}, \tag{3.21}$$

$$h_K^{1/2}\|J_\ell\|_{0,\ell} \le C\big(|w - w_h|_{1,K} + \theta_K + h_K^{1/2}\|\lambda w - \lambda_h w_h\|_{0,\ell}\big) \qquad \forall \ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_0}, \tag{3.22}$$

$$h_K^{1/2}\|J_\ell\|_{0,\ell} \le C \sum_{K' \in \omega_\ell} \big(|w - w_h|_{1,K'} + \theta_{K'}\big) \qquad \forall \ell \in \mathcal{E}_K \cap \mathcal{E}_\Omega, \tag{3.23}$$

where $\omega_\ell := \{K' \in \mathcal{T}_h : \ell \in \mathcal{E}_{K'}\}$.

Proof. First, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_1}$, we extend $J_\ell \in \mathbb{P}_{k-1}(\ell)$ to the element K as in Remark 3.1. Let ψ_ℓ be the corresponding edge bubble function. We define $v := J_\ell \psi_\ell$. Then, v may be extended by zero to the whole domain Ω. This extension, again denoted by v, belongs to $H^1(\Omega)$ and from Lemma 3.3 we have that

$$a^K(e, v) = -a^K(w_h - \Pi_k^K w_h, J_\ell \psi_\ell) + \int_K \Delta\left(\Pi_k^K w_h\right) J_\ell \psi_\ell + \int_\ell J_\ell^2 \psi_\ell.$$

For $J_\ell \in \mathbb{P}_{k-1}(\ell)$, from Lemma 3.2 and the above equality we obtain

$$\begin{aligned} C^{-1}\|J_\ell\|_{0,\ell}^2 &\le \int_\ell J_\ell^2 \psi_\ell \le C\left[\left(|e|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}\right)|\psi_\ell J_\ell|_{1,K} + \left\|\Delta(\Pi_k^K w_h)\right\|_{0,K}\|J_\ell \psi_\ell\|_{0,K}\right] \\ &\le C\left[\left(|e|_{1,K} + |w_h - \Pi_k^K w_h|_{1,K}\right) h_K^{-1/2}\|J_\ell\|_{0,\ell} + h_K^{-1}\left(\theta_K + |e|_{1,K}\right) h_K^{1/2}\|J_\ell\|_{0,\ell}\right] \\ &\le C h_K^{-1/2}\|J_\ell\|_{0,\ell}\left(|e|_{1,K} + \theta_K\right), \end{aligned}$$

where we have used again Lemma 3.2 together with estimate (3.20). Multiplying by $h_K^{1/2}$ the above inequality allows us to conclude (3.21).

Secondly, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_0}$, we extend $v := J_\ell \psi_\ell$ to $H^1(\Omega)$ as in the previous case. Taking into account that in this case $J_\ell \in \mathbb{P}_k(\ell)$ and ψ_ℓ is a quadratic bubble function in K, from Lemma 3.3 we obtain

$$a^K(e,v) = \lambda \int_\ell w J_\ell \psi_\ell - \lambda_h \int_\ell w_h J_\ell \psi_\ell - a^K\left(w_h - \Pi_k^K w_h, J_\ell \psi_\ell\right) + \int_K \Delta\left(\Pi_k^K w_h\right) J_\ell \psi_\ell + \int_\ell J_\ell^2 \psi_\ell.$$

Then, repeating the previous arguments we obtain

$$\left|\int_\ell J_\ell^2 \psi_\ell\right| \leq C\left[\left|\lambda_h \int_\ell w_h J_\ell \psi_\ell - \lambda \int_\ell w J_\ell \psi_\ell\right| + h_K^{-1/2}\|J_\ell\|_{0,\ell}\left(\theta_K + |e|_{1,K}\right)\right].$$

Hence, using Lemma 3.2 and a local trace inequality we arrive at

$$\begin{aligned}\|J_\ell\|_{0,\ell}^2 &\leq C\left[\|\lambda w - \lambda_h w_h\|_{0,\ell}\|\psi_\ell J_\ell\|_{0,\ell} + h_K^{-1/2}\left(\theta_K + |e|_{1,K}\right)\|J_\ell\|_{0,\ell}\right] \\ &\leq C h_K^{-1/2}\|J_\ell\|_{0,\ell}\left(\theta_K + |e|_{1,K} + h_K^{1/2}\|\lambda w - \lambda_h w_h\|_{0,\ell}\right),\end{aligned}$$

where we have used Lemma 3.2 again. Multiplying by $h_K^{1/2}$ the above inequality yields (3.22).

Finally, for $\ell \in \mathcal{E}_K \cap \mathcal{E}_\Omega$, we extend $v := J_\ell \psi_\ell$ to $H^1(\Omega)$ as above again. Taking into account that $J_\ell \in \mathbb{P}_{k-1}(\ell)$ and ψ_ℓ is a quadratic bubble function in K, from Lemma 3.3 we obtain

$$a(e,v) = -\sum_{K' \in \omega_\ell} a^{K'}(w_h - \Pi_k^{K'} w_h, J_\ell \psi_\ell) + \sum_{K' \in \omega_\ell} \int_{K'} \Delta\left(\Pi_k^{K'} w_h\right) J_\ell \psi_\ell + \sum_{K' \in \omega_\ell} \int_\ell J_\ell^2 \psi_\ell.$$

Then, proceeding analogously to the previous case we obtain

$$\|J_\ell\|_{0,\ell}^2 \leq C h_K^{-1/2}\|J_\ell\|_{0,\ell}\left[\sum_{K' \in \omega_\ell}\left(|e|_{1,K'} + \theta_{K'}\right)\right].$$

Thus, the proof is complete. □

Now, we are in a position to prove an upper bound for the local error indicators η_K.

Theorem 3.3. *There exists $C > 0$ such that*

$$\eta_K^2 \leq C\left[\sum_{K' \in \omega_K}\left(|w - \Pi_k^{K'} w_h|_{1,K'}^2 + |w - w_h|_{1,K'}^2 + \sum_{\ell \in \mathcal{E}_K \cap \mathcal{E}_{\Gamma_0}} h_K \|\lambda w - \lambda_h w_h\|_{0,\ell}^2\right)\right],$$

where $\omega_K := \{K' \in \mathcal{T}_h : K' \text{ and } K \text{ share an edge}\}$.

Proof. It follows immediately from Lemmas 3.5–3.7. □

According to the above theorem, the error indicators η_K^2 provide lower bounds of the error terms $\sum_{K' \in \omega_K}\left(|w - \Pi_k^{K'} w_h|_{1,K'}^2 + |w - w_h|_{1,K'}^2\right)$ in the neighborhood ω_K of K. For those elements K with an edge on Γ_0, the term $h_K\|\lambda w - \lambda_h w_h\|_{0,\ell}^2$ also appears in the estimate. Let us remark that it is reasonable to expect this terms to be asymptotically negligible. In fact, this is the case at least for the global estimator $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$ as is shown in the following result.

Corollary 3.3. *There exists a constant $C > 0$ such that*

$$\eta^2 \leq C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_h w_h|^2_{1,h} \right).$$

Proof. From Theorem 3.3 we have that

$$\eta^2 \leq C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_h w_h|^2_{1,h} + h\|\lambda w - \lambda_h w_h\|^2_{0,\Gamma_0} \right).$$

The last term on the right hand side above is bounded as follows:

$$\|\lambda w - \lambda_h w_h\|^2_{0,\Gamma_0} \leq 2\lambda^2 \|w - w_h\|^2_{0,\Gamma_0} + 2|\lambda - \lambda_h|^2,$$

where we have used that $\|w_h\|_{0,\Gamma_0} = 1$. Now, by using a trace inequality and Poincaré inequality (2.8) we have

$$\|w - w_h\|_{0,\Gamma_0} \leq C|w - w_h|_{1,\Omega}.$$

On the other hand, using the estimate (3.9), we have

$$|\lambda - \lambda_h|^2 \leq (|\lambda| + |\lambda_h|)|\lambda - \lambda_h| \leq C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_h w_h|^2_{1,h} \right).$$

Therefore,

$$\eta^2 \leq C \left(|w - w_h|^2_{1,\Omega} + |w - \Pi_h w_h|^2_{1,h} \right)$$

and we conclude the proof. □

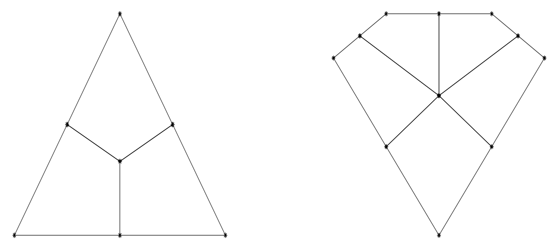
4. Numerical results

In this section, we will investigate the behavior of an adaptive scheme driven by the error indicator in two numerical tests that differ in the shape of the computational domain Ω and, hence, in the regularity of the exact solution. With this aim, we have implemented in a MATLAB code a lowest-order VEM ($k = 1$) on arbitrary polygonal meshes following the ideas proposed in [8].

To complete the choice of the VEM, we had to choose the bilinear forms $S^K(\cdot, \cdot)$ satisfying (2.4). In this respect, we proceeded as in [7, Section 4.6]: for each polygon K with vertices $P_1, \ldots, P_{N_K}$, we used

$$S^K(u, v) := \sum_{r=1}^{N_K} u(P_r)v(P_r), \qquad u, v \in V_1^K.$$

In all our tests we have initiated the adaptive process with a coarse triangular mesh. In order to compare the performance of VEM with that of a finite element method (FEM), we have used two different algorithms to refine the meshes. The first one is based on a classical FEM strategy for which all the subsequent meshes consist of triangles. In such a case, for $k = 1$, VEM reduces to FEM. The other procedure to refine the meshes is described in [11]. It consists of splitting each element into n quadrilaterals (n being the number of edges of the polygon) by connecting the barycenter of the element with the midpoint of each edge as shown in Figure 1 (see [11] for more details). Notice that although this process is initiated with a mesh of triangles, the successively created meshes will contain other kind of convex polygons, as can be seen in Figures 3 and 7.



(a) Triangle K refined into 3 quadrilaterals.

(b) Pentagon K refined into 5 quadrilaterals.

Figure 1: Example of refined elements for VEM strategy.

Since we have chosen $k = 1$, according to the definition of the local virtual element space V_1^K (cf. (2.2)), the term $R_K^2 := h_K^2\|\Delta w_h\|_{0,K}^2$ vanishes. Thus, the error indicators reduce in this case to

$$\eta_K^2 = \theta_K^2 + \sum_{\ell\in\mathcal{E}_K} h_K\|J_\ell\|_{0,\ell}^2 \qquad \forall K \in \mathcal{T}_h.$$

Let us remark that in the case of triangular meshes, the term $\theta_K^2 := a_h^K(w_h - \Pi_k^K w_h, w_h - \Pi_k^K w_h)$ vanishes too, since $V_1^K = \mathbb{P}_1(K)$ and hence Π_k^K is the identity. By the same reason, the projection Π_k^K also disappears in the definition (3.1) of J_ℓ. Therefore, for triangular meshes, not only VEM reduces to FEM, but also the error indicator becomes the classical well-known edge-residual error estimator (see [5]):

$$\eta_K^2 := \sum_{\ell\in\mathcal{E}_K} h_K\|J_\ell\|_{0,\ell}^2 \qquad \text{with} \qquad J_\ell := \begin{cases} \dfrac{1}{2}\left[\!\left[\dfrac{\partial w_h}{\partial n}\right]\!\right]_\ell, & \ell \in \mathcal{E}_\Omega, \\ \lambda_h w_h - \dfrac{\partial w_h}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_0}, \\ -\dfrac{\partial w_h}{\partial n}, & \ell \in \mathcal{E}_{\Gamma_1}. \end{cases}$$

In what follows, we report the results of a couple of tests. In both cases, we will restrict our attention to the approximation of the eigenvalues. Let us recall that according to Corollary 3.2, the global error estimator η^2 provides an upper bound of the error of the computed eigenvalue.

4.1. Test 1: Sloshing in a square domain.

We have chosen for this test a problem with known analytical solution. It corresponds to the computation of the sloshing modes of a two-dimensional fluid contained in the domain $\Omega := (0,1)^2$ with a horizontal free surface Γ_0 as shown in Figure 2. The solutions of this problem are

$$\lambda_n = n\pi\tanh(n\pi), \qquad w_n(x,y) = \cos(n\pi x)\sinh(n\pi y), \qquad n \in \mathbb{N}.$$

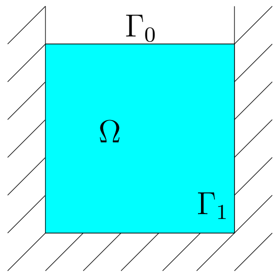
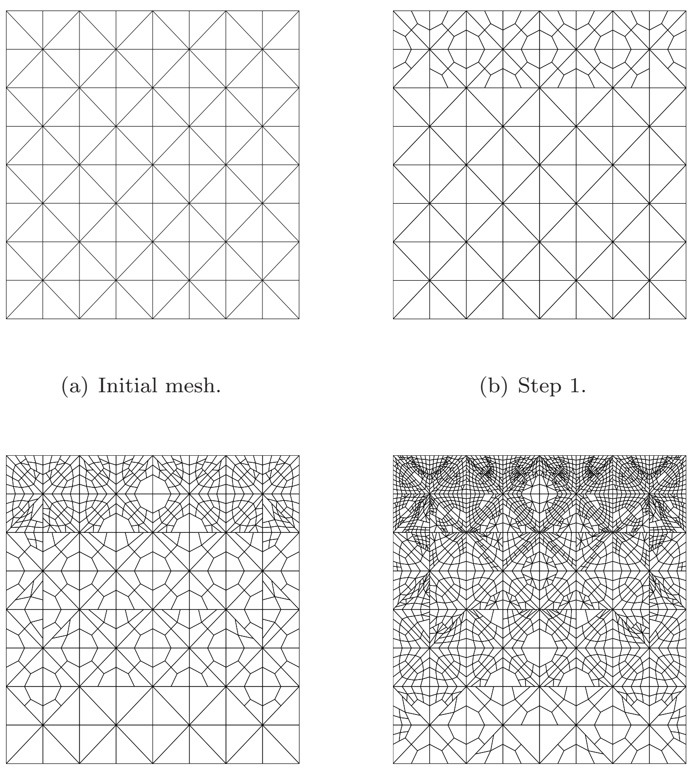


Figure 2: Test 1. Sloshing in a square domain.

We have used the two refinement procedures (VEM and FEM) described above. Both schemes are based on the strategy of refining those elements K which satisfy

$$\eta_K \geq 0.5 \max_{K' \in \mathcal{T}_h} \{\eta_{K'}\}.$$

Figures 3 and 4 show the adaptively refined meshes obtained with VEM and FEM procedures, respectively.



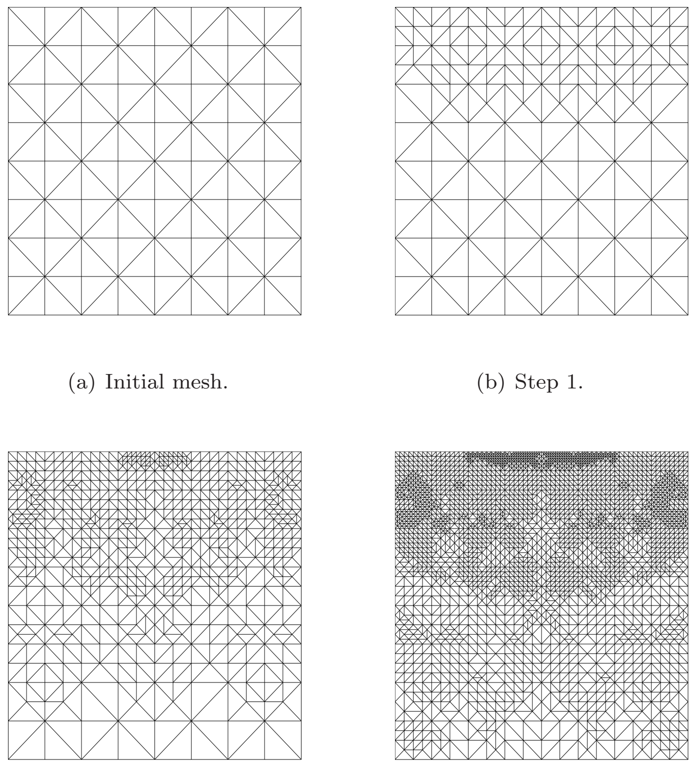
(a) Initial mesh.

(b) Step 1.

(c) Step 3.

(d) Step 6.

Figure 3: Test 1. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1, 3 and 6.



(a) Initial mesh. (b) Step 1.

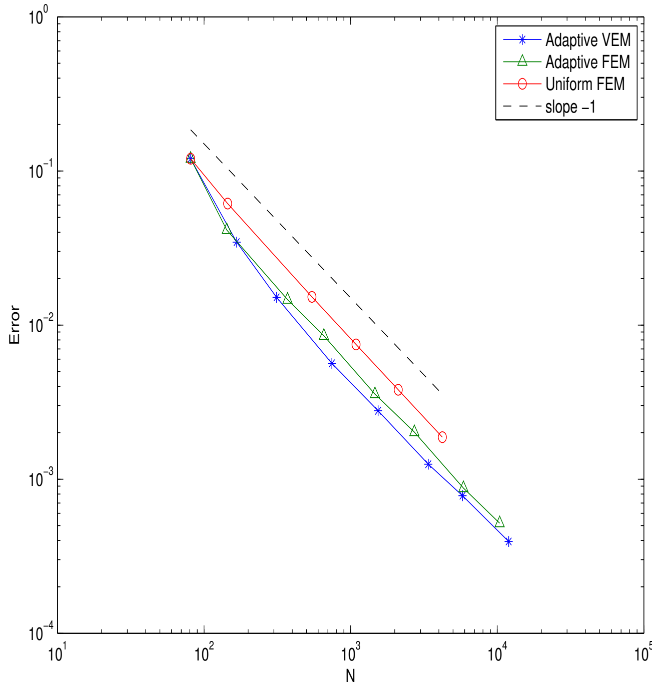
(c) Step 3. (d) Step 6.

Figure 4: Test 1. Adaptively refined meshes obtained with FEM scheme at refinement steps 0, 1, 3 and 6.

Since the eigenfunctions of this problem are smooth, according to (2.9) we have that $|\lambda - \lambda_h| = \mathcal{O}(h^2)$. Therefore, in case of uniformly refined meshes, $|\lambda - \lambda_h| = \mathcal{O}\left(N^{-1}\right)$, where N denotes the number of degrees of freedom which is the optimal convergence rate that can be attained.

Figure 5 shows the error curves for the computed lowest eigenvalue on uniformly refined meshes and adaptively refined meshes with FEM and VEM schemes. The plot also includes a line of slope -1, which correspond to the optimal convergence rate of the method $\mathcal{O}\left(N^{-1}\right)$.

Figure 5: Test 1. Error curves of $|\lambda_1 - \lambda_{h1}|$ for uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").



It can be seen from Figure 5 that the three refinement schemes lead to the correct convergence rate. Moreover, the performance of adaptive VEM is slightly better than that of adaptive FEM, while this is also better than uniform FEM.

We report in Table 1, the errors $|\lambda_1 - \lambda_{h1}|$ and the estimators η^2 at each step of the adaptive VEM scheme. We include in the table the terms $\theta^2 := \sum_{K\in\mathcal{T}_h} \theta_K^2$ which arise from the inconsistency of VEM and $J^2 := \sum_{K\in\mathcal{T}_h} \left(\sum_{\ell\in\mathcal{E}_K} h_K \|J_\ell\|_{0,\ell}^2\right)$ which arise from the edge residuals. We also report in the table the effectivity indexes $|\lambda_1 - \lambda_{h1}|/\eta^2$.

Table 1: Test 1. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

N	λ_{h1}	$\lvert\lambda_1 - \lambda_{h1}\rvert$	θ^2	J^2	η^2	$\frac{\lvert\lambda_1 - \lambda_{h1}\rvert}{\eta^2}$
38	3.2499	0.1200	0	0.8245	0.8245	0.1456
167	3.1644	0.0345	0.0111	0.2469	0.2580	0.1339
313	3.1450	0.0151	0.0117	0.1108	0.1225	0.1234
745	3.1355	0.0056	0.0054	0.0427	0.0481	0.1171
1540	3.1327	0.0028	0.0033	0.0216	0.0249	0.1113
3392	3.1311	0.0013	0.0015	0.0102	0.0117	0.1069
5806	3.1307	0.0008	0.0009	0.0064	0.0073	0.1069
11973	3.1303	0.0004	0.0005	0.0032	0.0037	0.1075

It can be seen from Table 1 that the effectivity indexes are bounded above and below far from zero and that the inconsistency and edge residual terms are roughly speaking of the same order, none of them being asymptotically negligible.

4.2. Test 2:

The aim of this test is to assess the performance of the adaptive scheme when solving a problem with a singular solution. In this test Ω consists of a unit square from which it is subtracted an equilateral triangle as shown in Figure 6. In this case Ω has a reentrant angle $\omega = \frac{5\pi}{3}$. Therefore, the Sobolev exponent is $r_\Omega := \frac{\pi}{\omega} = 3/5$, so that the eigenfunctions will belong to $H^{1+r}(\Omega)$ for all $r < 3/5$, but in general not to $H^{1+3/5}(\Omega)$. Therefore, according to (2.9), using quasi-uniform meshes, the convergence rate for the eigenvalues should be $|\lambda - \lambda_h| \approx \mathcal{O}\left(h^{6/5}\right) \approx \mathcal{O}\left(N^{-3/5}\right)$. An efficient adaptive scheme should lead to refine the meshes in such a way that the optimal order $|\lambda - \lambda_h| = \mathcal{O}\left(N^{-1}\right)$ could be recovered.

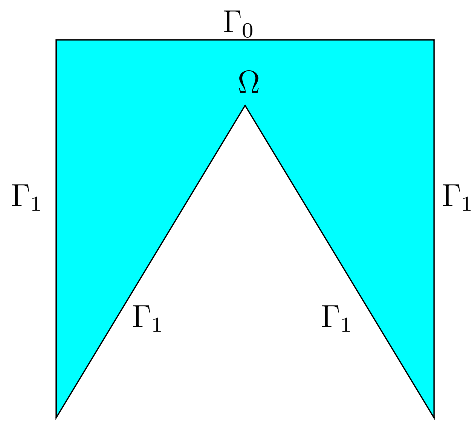
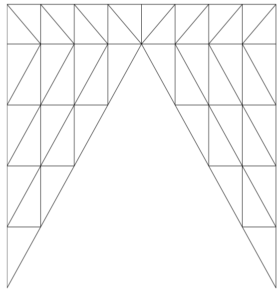
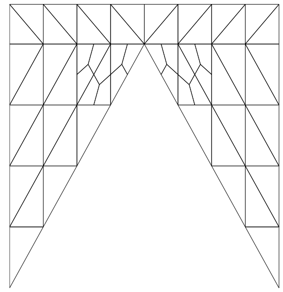


Figure 6: Test 2. Domain Ω.

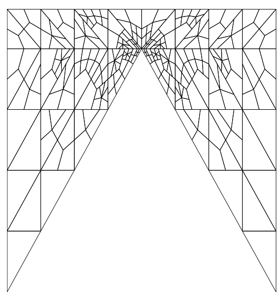
Figures 7 and 8 show the adaptively refined meshes obtained with the VEM and FEM adaptive schemes, respectively.



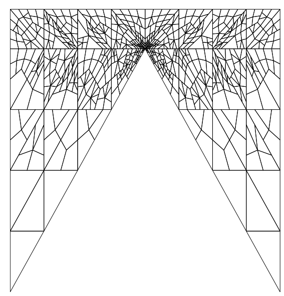
(a) Initial mesh.



(b) Step 1.

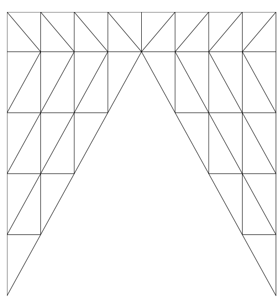


(c) Step 4.

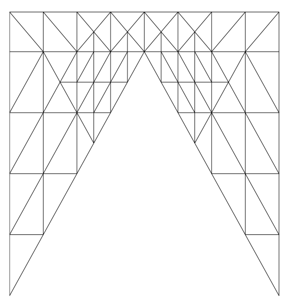


(d) Step 6.

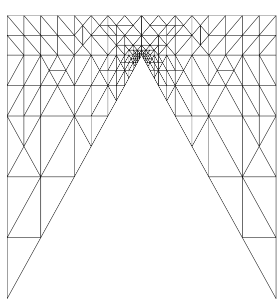
Figure 7: Test 2. Adaptively refined meshes obtained with VEM scheme at refinement steps 0, 1, 4 and 6.



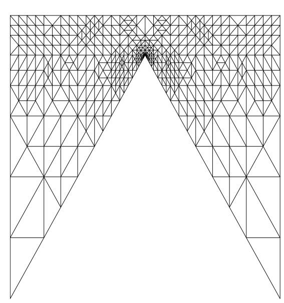
(a) Initial mesh.



(b) Step 1.



(c) Step 4.



(d) Step 6.

Figure 8: Test 2. Adaptively refined meshes obtained with FEM scheme at refinement steps 0, 1, 4 and 6.

In order to compute the errors $|\lambda_1 - \lambda_{h1}|$, due to the lack of an exact eigenvalue, we have used an approximation based on a least squares fitting of the computed values obtained with extremely refined meshes. Thus, we have obtained the value $\lambda_1 = 1.9288$, which has at least four correct significant digits.

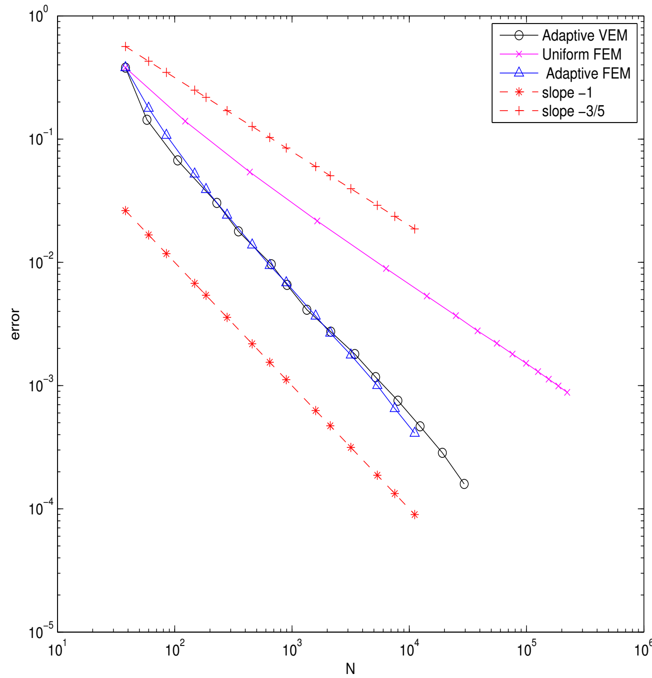
We report in Table 2 the lowest eigenvalue λ_{h1} computed with each of the three schemes. Each table includes the estimated convergence rate.

Table 2: Test 2. Eigenvalue λ_{h1} computed with different schemes: uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").

Uniform FEM		Adaptive VEM		Adaptive FEM	
N	λ_{h1}	N	λ_{h1}	N	λ_{h1}
38	2.3083	38	2.3083	38	2.3083
123	2.0686	58	2.0721	60	2.1067
437	1.9828	106	1.9960	85	2.0362
1641	1.9505	229	1.9592	148	1.9810
6353	1.9377	350	1.9467	185	1.9678
14137	1.9341	666	1.9384	280	1.9530
24993	1.9325	909	1.9354	458	1.9427
38291	1.9316	1340	1.9329	646	1.9382
55921	1.9310	2141	1.9315	895	1.9356
75993	1.9306	3438	1.9306	1593	1.9325
99137	1.9303	5172	1.9300	2122	1.9315
125353	1.9301	8014	1.9296	3178	1.9306
154641	1.9299	12365	1.9293	5341	1.9298
187001	1.9298	19153	1.9291	7522	1.9295
222433	1.9297	29403	1.9290	11124	1.9292
Order	$\mathcal{O}\left(N^{-0.68}\right)$	Order	$\mathcal{O}\left(N^{-1.10}\right)$	Order	$\mathcal{O}\left(N^{-1.16}\right)$
λ_1	1.9288	λ_1	1.9288	λ_1	1.9288

It can be seen from Table 2, that the uniform refinement leads to a convergence rate close to that predicted by the theory $\mathcal{O}\left(N^{-3/5}\right)$. Instead, Tables 2 show that the adaptive VEM and FEM schemes allow us to recover the optimal order of convergence $\mathcal{O}\left(N^{-1}\right)$. This can be clearly seen from Figure 9, where the three error curves are reported. The plot also includes lines of slopes -1 and $-3/5$, which correspond to the convergence rates of each scheme.

Figure 9: Test 2. Error curves of $|\lambda_1 - \lambda_{h1}|$ for uniformly refined meshes ("Uniform FEM"), adaptively refined meshes with FEM ("Adaptive FEM") and adaptively refined meshes with VEM ("Adaptive VEM").



Finally, we report in Table 3 the same information as in Table 1 for this test. Similar conclusions as in the previous test follow from this table.

Table 3: Test 2. Components of the error estimator and effectivity indexes on the adaptively refined meshes with VEM.

N	λ_{h1}	$\lvert\lambda_1 - \lambda_{h1}\rvert$	θ^2	J^2	η^2	$\frac{\lvert\lambda_1 - \lambda_{h1}\rvert}{\eta^2}$
38	2.3083	0.3795	0	2.3181	2.3181	0.1637
58	2.0721	0.1433	0.0379	0.8231	0.8609	0.1664
106	1.9960	0.0672	0.0368	0.4188	0.4556	0.1475
229	1.9592	0.0304	0.0216	0.1942	0.2158	0.1408
350	1.9467	0.0179	0.0164	0.1359	0.1522	0.1173
666	1.9384	0.0096	0.0094	0.0749	0.0844	0.1143
909	1.9354	0.0066	0.0068	0.0556	0.0624	0.1052
1340	1.9329	0.0041	0.0047	0.0408	0.0454	0.0907
2141	1.9315	0.0027	0.0032	0.0275	0.0308	0.0891
3438	1.9306	0.0018	0.0022	0.0178	0.0199	0.0904

Conclusions

We have derived an a posteriori error indicator for the VEM solution of the Steklov eigenvalue problem. We have proved that it is efficient and reliable. For lowest order elements on triangular meshes, VEM coincides with FEM and the a posteriori error indicators also coincide with the classical ones. However VEM allows using general polygonal meshes including hanging nodes, which is particularly interesting when designing an adaptive scheme. We have implemented such a scheme driven by the proposed error indicators. We have assessed its performance by means of a couple of tests which allow us to confirm that the adaptive scheme yields optimal order of convergence for regular as well as singular solutions.

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