A priori and a posteriori error analyses of a flux-based mixed-FEM for convection-diffusion-reaction problems *

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Abstract

In this paper we propose and analyze a new mixed-type finite element method for the numerical simulation of a diffusion-convection-reaction problem with non-homogeneous Dirichlet boundary condition. The method is based on a new formulation of the problem of interest consisting in a single variational equation posed in $H(\mathrm{div};\Omega)$, where the flux (or gradient of the primal variable u) is the main and only unknown. Consequently, we propose a conforming Raviart-Thomas approximation of order $k \geq 0$ for the flux, and the primal unknown u can be easily approximated through a simple post-processing procedure based on the equilibrium equation. We prove unique solvability of the resulting continuous and discrete problems by means of the generalized Lax-Milgram lemma. In particular, the well-posedness, stability and convergence of the Galerkin scheme can be achieved through a sufficiently small mesh-size assumption. Next, we derive a reliable and efficient residual-based a posteriori error estimator for the conforming method. The proof of reliability makes use of the global inf-sup condition, Helmholtz decomposition, and the local approximation properties of the Clément interpolant and Raviart-Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator. Finally, some numerical results confirming the good performance of the method and the theoretical properties of the a posteriori error estimator, and illustrating the capability of the corresponding adaptive algorithm to localize the singularities of the solution, are reported.

Key words: Convection-diffusion-reaction model; Mixed finite element

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1 Introduction

This paper is concerned with the numerical approximation of the convection-diffusion-reaction problem: Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nu\Delta\, u + \boldsymbol{\beta}\cdot\nabla u + \alpha u &= f \qquad \text{in} \quad \Omega\,,\\
u &= g \qquad \text{on} \quad \Gamma\,,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with Lipschitz boundary $\Gamma := \partial\Omega$, f and g are given data in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively, $\nu > 0$ is the diffusion coefficient, $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$ is the convective vector field satisfying $\mathrm{div}(\boldsymbol{\beta}) = 0$, and $\alpha > 0$ is the reaction coefficient.

For several years, the numerical analysis community has put great efforts in developing new efficient numerical methods to approximate the solution of (1.1), motivated basically by the diversity of applications where this model might be used, particularly, in engineering and industry. Roughly speaking, the equations described in (1.1) model, for instance, the behaviour of the concentration u of certain species in a physical medium occupying Ω (a fluid for example), which is moving with a given velocity $\boldsymbol{\beta}$. Over the last decades, several contributions have been made in order to obtain accurate approximations of (1.1), including stabilizations techniques, adaptive algorithms and combinations of both (see for instance [2, 3, 6, 18, 19]), most of them based on primal formulations and, up to the author's knowledge, few of them on mixed methods. It is worth mentioning that the main motivations, among others, to utilize mixed methods to solve numerically elliptic problems, and particularly (1.1), are: they give better approximations for the flux variable even for highly nonhomogeneous media with large jumps in the physical properties and they conserve mass locally. In this direction, the first work in developing a detailed analysis of a mixed finite element discretization of problem (1.1) is [15]. There, the authors provide the well-posedness and convergence analysis of a Raviart-Thomas approximation of (1.1). In particular, the solvability analysis in [15] requires a smallness assumption on the mesh size. Years later, in [33, 13] the work started in [15] was extended to quasilinear and nonlinear problems.

On the other hand, concerning adaptive algorithms, it was not until 2007 when Vohralík in [39] provided the first a posteriori error analysis for a mixed finite element discretization of (1.1). More precisely, in [39] the author introduced a reliable and efficient residual-based a posteriori error estimator for the lowest order Raviart-Thomas approximation of (1.1) on simplicial meshes, considering also the upwind-mixed scheme. Later on, the work developed in [39] was extended in [16, 17].

In this paper we attempt to contribute to the development of new numerical methods to approximate the solution of (1.1) by introducing a new flux-based mixed method for the model problem. More precisely, we adopt a technique recently applied to the Brinkman problem in [24] (see also [20] for a similar approach), which consists in introducing the flux $\boldsymbol{\sigma} := \nabla u$ as an auxiliary unknown, and in eliminating the unknown u through the equilibrium equation (assuming that $\alpha > 0$), and propose a new mixed-type formulation for (1.1) where $\boldsymbol{\sigma}$ is the only unknown of the resulting variational formulation posed in $H(\mathrm{div};\Omega)$. Consequently, a conforming Galerkin scheme defined by Raviart-Thomas elements of order $k \geq 0$ is introduced to approximate the flux. The analysis of the continuous and discrete schemes are carried out by means of the well-known generalized Lax-Milgram lemma. In particular, the well-posedness of the corresponding Galerkin scheme is attained under a smallness assumption of the mesh-size, which is in concordance with the results obtained in [15]. Moreover, similarly as in [24], the primal variable u can be approximated through a simple post-processing procedure, which finally leads us to an optimal convergent method for both variables $\boldsymbol{\sigma}$ and u.

We also derive a reliable and efficient residual-based a posteriori error estimator for this problem. Making use of the continuous global inf-sup condition, previously derived for the solvability analysis of the continuous problem, together with a suitable stable Helmholtz decomposition, and the approximation properties of the Raviart-Thomas and Clément interpolators, we prove the reliability of the estimator whereas inverse inequalities, the localization technique based on element-bubble and edge–bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator. We emphasize that all the estimates in our paper exhibit the explicit dependence of the corresponding constants on the parameters ν, α and $\boldsymbol{\beta}$. The rest of the paper is organized as follows. In Section 2 we define the convection-diffusion-reaction model, derive the flux-based formulation, and then show that it is well-posed. The associated mixed finite element method is introduced and analyzed in Section 3. Next, in Section 4 we derive a reliable and efficient residual-based a posteriori error estimator. Finally, some numerical results showing the good performance and robustness of the mixed finite element methods, confirming the reliability and efficiency of the estimator, and illustrating the behaviour of the associated adaptive algorithm are reported in Section 5.

We end this section by introducing some notations to be used below. In what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a domain, $\mathcal{S} \subset \mathbb{R}^d$ is a Lipschitz curve or surface, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^d \text{ and } \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^d .$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding norms and seminorms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ and $|\cdot|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$ and $\mathbf{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\mathcal{S}}$ and $|\cdot|_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In turn, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}(\mathbf{w}) \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [5]), which for the sake of simplicity will be denoted by $\mathbf{H}$. Moreover, suggested by the bilinear form A defined later on in (2.5), in this paper the Hilbert space $\mathbf{H}$ will be endowed with the norm

$$\|\mathbf{w}\|_{\mathbf{H}}^2 = \|\mathbf{w}\|_{0,\mathcal{O}}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\mathbf{w})\|_{0,\mathcal{O}}^2 .$$

The corresponding duality pairing with respect to the $L^2(\Gamma)$ inner product is denoted by $\langle \cdot\,,\cdot \rangle_\Gamma$. In addition, we will denote by $\|\cdot\|_{1/2,\Gamma}$ and $\|\cdot\|_{-1/2,\Gamma}$ the usual norms of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively. Furthermore, in the sequel, $\|\cdot\|_{\infty,\Omega}$ will denote either the norm of the Banach space $L^\infty(\mathcal{O})$, or the norm of the product space $[L^\infty(\mathcal{O})]^d$. In the latter, the norm is defined by

$$\|\mathbf{w}\|_{\infty,\Omega} = \max_{i=1,\ldots,d}\{\|w_i\|_{\infty,\Omega}\}.$$

Finally, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameter h and of the data α, ν and $\boldsymbol{\beta}$, which may take different values at different places.

2 Analysis of the continuous flux-based formulation

2.1 The continuous variational formulation

Since we are interested in using a flux-based formulation to approximate the solution of problem (1.1), to derive our variational formulation we first introduce the further unknown $\boldsymbol{\sigma} = \nabla u$, and rewrite (1.1) as

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla u && \text{in} \quad \Omega\,, \\ -\nu\,\mathrm{div}(\boldsymbol{\sigma}) + \boldsymbol{\beta}\cdot\boldsymbol{\sigma} + \alpha u &= f && \text{in} \quad \Omega\,, \\ u &= g && \text{on} \quad \Gamma\,. \end{aligned} \tag{2.1}$$

Then, as it is usual in the realm of mixed methods, we first test the first equation of (2.1) with $\boldsymbol{\tau} \in \mathbf{H}$, integrate by parts the expression $\displaystyle\int_\Omega \nabla u\cdot\boldsymbol{\tau}$, and utilize the Dirichlet boundary condition $u = g$ on Γ, to arrive at

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_\Omega u\,\mathrm{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\cdot\mathbf{n}\,, g\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau}\,\in\,\mathbf{H}\,. \tag{2.2}$$

However, instead of imposing the second equation of (2.1) weakly, here we proceed similarly to [24] and replace u in (2.2) by

$$u := \frac{1}{\alpha}\Big\{ f + \nu\,\mathrm{div}(\boldsymbol{\sigma}) - \boldsymbol{\beta}\cdot\boldsymbol{\sigma}\Big\} \quad \text{in} \quad \Omega\,, \tag{2.3}$$

to obtain the identity

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \frac{\nu}{\alpha}\int_\Omega \mathrm{div}(\boldsymbol{\sigma})\,\mathrm{div}(\boldsymbol{\tau}) - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\sigma})\,\mathrm{div}(\boldsymbol{\tau}) = -\frac{1}{\alpha}\int_\Omega f\,\mathrm{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau}\cdot\mathbf{n}\,, g\rangle_\Gamma, \tag{2.4}$$

for all $\boldsymbol{\tau} \in \mathbf{H}$.

In this way, defining the bilinear form $A : \mathbf{H}\times\mathbf{H} \to \mathbb{R}$ and the functional $F : \mathbf{H} \to \mathbb{R}$, respectively, as

$$A(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} + \frac{\nu}{\alpha}\int_\Omega \mathrm{div}(\boldsymbol{\zeta})\,\mathrm{div}(\boldsymbol{\tau}) - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\zeta})\,\mathrm{div}(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\zeta},\boldsymbol{\tau}\,\in\,\mathbf{H} \tag{2.5}$$

and

$$F(\boldsymbol{\tau}) := -\frac{1}{\alpha}\int_\Omega f\,\mathrm{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau}\cdot\mathbf{n}\,, g\rangle_\Gamma \quad \forall\,\boldsymbol{\tau}\,\in\,\mathbf{H}, \tag{2.6}$$

from (2.4) we obtain the variational problem: Find $\boldsymbol{\sigma} \in \mathbf{H}$ such that

$$A(\boldsymbol{\sigma},\boldsymbol{\tau}) = F(\boldsymbol{\tau})\,, \quad \text{for all } \boldsymbol{\tau}\in\mathbf{H}. \tag{2.7}$$

Remark 2.1 *Observe that in* (2.3) *we are using explicitly the fact that* $\alpha > 0$. *To study the case* $\alpha = 0$, *the second equation of* (2.1) *must be incorporated weakly, and consequently, the resulting formulation must be studied as a saddle-point problem. For more details we refer the reader to [15].*

Observe also that, although the variable u *is no longer an unknown, it can be easily recovered in terms of* $\boldsymbol{\sigma}$ *by using* (2.3).

2.2 Well-posedness of the continuous problem

In this section we study the solvability of problem (2.7) by means of the well-known Generalized Lax-Milgram Lemma, which for the sake of completeness, is provided next. For its proof we refer the reader to [22, Theorem 1.2].

Theorem 2.1 *Let H be a Hilbert space with induced norm $\|\cdot\|_H$ and let $B : H \times H \to \mathbb{R}$ be a bounded bilinear form. Assume that:*

(i) There exists $c > 0$, such that

$$\sup_{\phi \in H\setminus\{0\}} \frac{B(\psi,\phi)}{\|\phi\|_H} \geq c\,\|\psi\|_H \qquad \forall\, \psi \in H, \tag{2.8}$$

(ii)

$$\sup_{\psi \in H} B(\psi,\phi) > 0 \qquad \forall\, \phi \in H\setminus\{0\}. \tag{2.9}$$

Then, for each $F \in H'$, there exists a unique $\varphi \in H$, such that

$$B(\varphi,\psi) = F(\psi), \qquad \forall\, \psi \in H,$$

and

$$\|\varphi\|_H \leq \frac{1}{c}\|F\|_{H'}. \tag{2.10}$$

In what follow we focus on verifying that the bilinear form A (cf. (2.5)) satisfies the assumptions of Theorem 2.1. We begin with the inf-sup condition (2.8).

Lemma 2.1 *There exists $\gamma > 0$, depending only on α, $\|\boldsymbol{\beta}\|_{\infty,\Omega}$ and ν, such that*

$$\sup_{\boldsymbol{\zeta} \in \mathbf{H}\setminus\{\mathbf{0}\}} \frac{A(\boldsymbol{\tau},\boldsymbol{\zeta})}{\|\boldsymbol{\zeta}\|_{\mathbf{H}}} \geq \gamma\,\|\boldsymbol{\tau}\|_{\mathbf{H}} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}. \tag{2.11}$$

(The explicit expression of γ can be found in (2.26)*.)*

Proof. Let $\boldsymbol{\tau}$ be an arbitrary element in $\mathbf{H}$. First, from the definition of A (cf. (2.5)), we easily obtain that

$$A(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\tau})\mathrm{div}(\boldsymbol{\tau}). \tag{2.12}$$

For the last term of (2.12) we use Hölder inequality and $ab \leq \frac{1}{2}\left(a^2+b^2\right)$, to obtain

$$\frac{1}{\alpha}\left|\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\tau})\mathrm{div}(\boldsymbol{\tau})\right| \leq \frac{1}{\alpha}\|\boldsymbol{\beta}\cdot\boldsymbol{\tau}\|_{0,\Omega}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega} \leq \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{\nu}{2\alpha}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2, \tag{2.13}$$

which combined with (2.12), implies

$$A(\boldsymbol{\tau},\boldsymbol{\tau}) \geq \|\boldsymbol{\tau}\|_{0,\Omega}^2 - \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{\nu}{2\alpha}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2\,. \tag{2.14}$$

In turn, using again the definition of A, it readily follows that

$$A\left(\boldsymbol{\tau}, \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\boldsymbol{\tau}\right) = \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 - \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2\nu}\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\tau})\mathrm{div}(\boldsymbol{\tau})\,. \tag{2.15}$$

Now, let $z \in H_0^1(\Omega)$ be the unique weak solution of the boundary value problem

$$\begin{aligned} -\nu\Delta z - (\boldsymbol{\beta}\cdot\nabla z) + \alpha z &= \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\nu}\mathrm{div}(\boldsymbol{\tau}) \qquad \text{in} \quad \Omega\,, \\ z &= 0 \qquad\qquad\qquad\qquad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.16}$$

which owing to the assumption $\mathrm{div}(\boldsymbol{\beta}) = 0$ is clearly well posed. In addition, it is not difficult to see that its solution satisfies

$$\nu\|\nabla z\|_{0,\Omega}^2 + \alpha\|z\|_{0,\Omega}^2 = \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\nu}\int_\Omega z\,\mathrm{div}(\boldsymbol{\tau}),$$

which readily implies the estimates

$$\|z\|_{0,\Omega} \le \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega} \quad \text{and} \quad \|\nu\nabla z\|_{0,\Omega} \le \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\sqrt{\nu\alpha}}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}. \tag{2.17}$$

Then, we define

$$\tilde{\boldsymbol{\tau}} := \frac{\nu}{\alpha}\nabla z + \frac{1}{\alpha}z\boldsymbol{\beta}, \tag{2.18}$$

and observe from the first equation of (2.16), and the fact that $\mathrm{div}(\boldsymbol{\beta}) = 0$, that

$$\mathrm{div}(\tilde{\boldsymbol{\tau}}) = z - \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\mathrm{div}(\boldsymbol{\tau}), \tag{2.19}$$

to obtain

$$\begin{aligned} A(\boldsymbol{\tau},\tilde{\boldsymbol{\tau}}) &= \int_\Omega \boldsymbol{\tau}\cdot\tilde{\boldsymbol{\tau}} + \frac{\nu}{\alpha}\int_\Omega \mathrm{div}(\boldsymbol{\tau})\,\mathrm{div}(\tilde{\boldsymbol{\tau}}) - \frac{1}{\alpha}\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\tau})\,\mathrm{div}(\tilde{\boldsymbol{\tau}}) \\ &= \int_\Omega \boldsymbol{\tau}\cdot\left(\frac{\nu}{\alpha}\nabla z + \frac{1}{\alpha}z\boldsymbol{\beta}\right) + \frac{1}{\alpha}\int_\Omega(\nu\,\mathrm{div}(\boldsymbol{\tau}) - \boldsymbol{\beta}\cdot\boldsymbol{\tau})\left(z - \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\mathrm{div}(\boldsymbol{\tau})\right), \end{aligned}$$

which together to (2.15), and after integrating by parts and applying simple computations, implies

$$\begin{aligned} A(\boldsymbol{\tau},\tilde{\boldsymbol{\tau}}) &= -\frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2\nu}\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\tau})\mathrm{div}(\boldsymbol{\tau}) \\ &= -A\left(\boldsymbol{\tau}, \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\boldsymbol{\tau}\right) + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\boldsymbol{\tau}\|_{0,\Omega}^2. \end{aligned} \tag{2.20}$$

In this way, setting $\boldsymbol{\xi} := \boldsymbol{\tau} + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}$, from (2.14), (2.20), it follows straightforwardly that

$$A(\boldsymbol{\tau},\boldsymbol{\xi}) = A(\boldsymbol{\tau},\boldsymbol{\tau}) + A\left(\boldsymbol{\tau}, \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\boldsymbol{\tau}\right) + A(\boldsymbol{\tau},\tilde{\boldsymbol{\tau}}) \ge \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \frac{\nu}{2\alpha}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \ge \frac{1}{2}\|\boldsymbol{\tau}\|_{\mathbf{H}}^2. \tag{2.21}$$

To continue with the analysis we now need to estimate $\|\boldsymbol{\xi}\|_{\mathbf{H}}$ in terms of $\|\boldsymbol{\tau}\|_{\mathbf{H}}$. To that end, we first observe that

$$\|\boldsymbol{\xi}\|_{\mathbf{H}} = \left\|\boldsymbol{\tau} + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\boldsymbol{\tau} + \tilde{\boldsymbol{\tau}}\right\|_{\mathbf{H}} \leq \left(1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\right)\|\boldsymbol{\tau}\|_{\mathbf{H}} + \|\tilde{\boldsymbol{\tau}}\|_{\mathbf{H}}\,. \tag{2.22}$$

Now, to estimate $\|\tilde{\boldsymbol{\tau}}\|_{\mathbf{H}}$, we apply the first estimate in (2.17) and use the identity (2.19), to obtain

$$\|\mathrm{div}(\tilde{\boldsymbol{\tau}})\|_{0,\Omega} \leq \|z\|_{0,\Omega} + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega} \leq \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{\alpha\nu}\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}, \tag{2.23}$$

and from the definition of $\tilde{\boldsymbol{\tau}}$ in (2.18) and using again the estimates (2.17), we get

$$\|\tilde{\boldsymbol{\tau}}\|_{0,\Omega} \leq \frac{1}{\alpha}\|\nu\nabla z\|_{0,\Omega} + \frac{\sqrt{d}}{\alpha}\|\boldsymbol{\beta}\|_{\infty,\Omega}\|z\|_{0,\Omega} \leq \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{3/2}\nu^{1/2}}\left(1 + \frac{\sqrt{d}\|\boldsymbol{\beta}\|_{\infty,\Omega}}{\alpha^{1/2}\nu^{1/2}}\right)\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}, \tag{2.24}$$

which together with (2.23), yields

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathbf{H}} \leq \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{3/2}\nu^{1/2}}\left(3 + \frac{\sqrt{d}\|\boldsymbol{\beta}\|_{\infty,\Omega}}{\alpha^{1/2}\nu^{1/2}}\right)\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{2.25}$$

In this way, defining

$$C(\nu,\boldsymbol{\beta},\alpha) := 1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{\alpha\nu}\left(2 + \frac{\sqrt{d}\|\boldsymbol{\beta}\|_{\infty,\Omega}}{2\alpha^{1/2}\nu^{1/2}}\right),$$

from (2.23) and (2.25), it readily follows that

$$\|\boldsymbol{\xi}\|_{\mathbf{H}} \leq C(\nu,\boldsymbol{\beta},\alpha)\|\boldsymbol{\tau}\|_{\mathbf{H}},$$

which combined with (2.21), implies

$$\sup_{\substack{\boldsymbol{\zeta}\in\mathbf{H}\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{A(\boldsymbol{\tau},\boldsymbol{\zeta})}{\|\boldsymbol{\zeta}\|_{\mathbf{H}}} \geq \frac{A(\boldsymbol{\tau},\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_{\mathbf{H}}} \geq \frac{1}{2C(\nu,\boldsymbol{\beta},\alpha)}\|\boldsymbol{\tau}\|_{\mathbf{H}} \quad \forall\,\boldsymbol{\tau}\in\mathbf{H}\,,$$

which concludes the proof with

$$\gamma := \frac{1}{2C(\nu,\boldsymbol{\beta},\alpha)}. \tag{2.26}$$

□

We now turn to prove that A satisfies condition (ii) in Theorem 2.1.

Lemma 2.2 *There holds*

$$\sup_{\boldsymbol{\tau}\in\mathbf{H}} A(\boldsymbol{\tau},\boldsymbol{\zeta}) > 0 \quad \forall\,\boldsymbol{\zeta}\in\mathbf{H}\backslash\{\mathbf{0}\}. \tag{2.27}$$

Proof. Given $\boldsymbol{\zeta} \in \mathbf{H}$, $\boldsymbol{\zeta} \neq 0$, by applying inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we first observe that

$$\begin{aligned} A(\boldsymbol{\zeta}, \boldsymbol{\zeta}) = & \ \|\boldsymbol{\zeta}\|_{0,\Omega}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\zeta})\mathrm{div}(\boldsymbol{\zeta}), \\ \geq & \ \|\boldsymbol{\zeta}\|_{0,\Omega}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 - \frac{1}{2}\|\boldsymbol{\zeta}\|_{0,\Omega}^2 - \frac{d}{2\alpha^2}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 , \\ \geq & \ \frac{1}{2}\|\boldsymbol{\zeta}\|_{0,\Omega}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 - \frac{d}{2\alpha^2}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 . \end{aligned} \tag{2.28}$$

On the other hand, let $\tilde{\boldsymbol{\sigma}} = \nabla z$, with $z \in H_0^1(\Omega)$ being the unique weak solution of the boundary value problem

$$\begin{aligned} -\Delta z + \frac{1}{\nu}(\boldsymbol{\beta} \cdot \nabla z) + \frac{\alpha}{\nu} z \ &= \ -\frac{d}{2\nu\alpha}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2 \mathrm{div}(\boldsymbol{\zeta}) \quad \text{in} \quad \Omega , \\ z \ &= \ 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

which, similarly as for (2.16), is well posed thanks to the fact that $\mathrm{div}(\boldsymbol{\beta}) = 0$. Then, noticing that

$$\mathrm{div}(\tilde{\boldsymbol{\sigma}}) = \frac{1}{\nu}(\boldsymbol{\beta} \cdot \tilde{\boldsymbol{\sigma}}) + \frac{\alpha}{\nu} z + \frac{d}{2\nu\alpha}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2 \mathrm{div}(\boldsymbol{\zeta}) \quad \text{in} \quad \Omega,$$

from the definition of A (cf. (2.5)), it readily follows that

$$\begin{aligned} A(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\zeta}) = & \ \int_\Omega \tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\zeta} + \frac{\nu}{\alpha}\int_\Omega \mathrm{div}(\tilde{\boldsymbol{\sigma}})\, \mathrm{div}(\boldsymbol{\zeta}) - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta} \cdot \tilde{\boldsymbol{\sigma}})\, \mathrm{div}(\boldsymbol{\zeta}) \\ = & \ \int_\Omega \nabla z \cdot \boldsymbol{\zeta} + \frac{\nu}{\alpha}\int_\Omega \left\{ \frac{1}{\nu}(\boldsymbol{\beta} \cdot \tilde{\boldsymbol{\sigma}}) + \frac{\alpha}{\nu} z + \frac{d}{2\nu\alpha}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2 \mathrm{div}(\boldsymbol{\zeta}) \right\} \mathrm{div}(\boldsymbol{\zeta}) \\ & -\frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta} \cdot \tilde{\boldsymbol{\sigma}})\, \mathrm{div}(\boldsymbol{\zeta}) \\ = & \ \frac{d}{2\alpha^2}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 . \end{aligned} \tag{2.29}$$

Therefore, taking $\tilde{\boldsymbol{\tau}} := \boldsymbol{\zeta} + \tilde{\boldsymbol{\sigma}}$, from (2.28) and (2.29), we obtain

$$\sup_{\boldsymbol{\tau} \in \mathbf{H}} A(\boldsymbol{\tau}, \boldsymbol{\zeta}) \geq A(\tilde{\boldsymbol{\tau}}, \boldsymbol{\zeta}) = A(\boldsymbol{\zeta}, \boldsymbol{\zeta}) + A(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\zeta}) = \frac{1}{2}\|\boldsymbol{\zeta}\|_{0,\Omega}^2 + \frac{\nu}{\alpha}\|\mathrm{div}(\boldsymbol{\zeta})\|_{0,\Omega}^2 > 0,$$

which clearly implies (2.27) and concludes the proof. □

We conclude our analysis by establishing the continuity of the bilinear form A and the functional F.

Lemma 2.3 *The following estimates hold:*

$$|A(\boldsymbol{\zeta}, \boldsymbol{\tau})| \leq C_A \|\boldsymbol{\zeta}\|_{\mathbf{H}} \|\boldsymbol{\tau}\|_{\mathbf{H}} \qquad \forall\, \boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbf{H}, \tag{2.30}$$

and

$$|F(\boldsymbol{\tau})| \leq C_F \|\boldsymbol{\tau}\|_{\mathbf{H}} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}, \tag{2.31}$$

with $C_A = \sqrt{2}\left\{1 + \frac{d}{\alpha\nu}\|\boldsymbol{\beta}\|_{\infty,\Omega}^2\right\}^{1/2}$ *and* $C_F = \left\{\frac{2}{\alpha\nu}\|f\|_{0,\Omega}^2 + (1 + \frac{2\alpha}{\nu})\|g\|_{1/2,\Gamma}^2\right\}^{1/2}$.

Proof. For (2.30) we simply apply Hölder inequality, whereas estimate (2.31) follows from Hölder inequality and the continuity of the normal trace (see for instance [22, Theorem 1.7]). We omit further details. □

Having verified the hypotheses of Theorem 2.1, we now proceed to establish the well-posedness of problem (2.7).

Theorem 2.2 *Let $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$, $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$, $\alpha, \nu > 0$, and assume that* $\mathrm{div}(\boldsymbol{\beta}) = 0$*. Then, there exists a unique solution $\boldsymbol{\sigma} \in \mathbf{H}$ to* (2.7)*. In addition, the following estimate holds*

$$\|\boldsymbol{\sigma}\|_{\mathbf{H}} \leq \frac{1}{\gamma} \left\{ \frac{2}{\alpha\nu} \|f\|_{0,\Omega}^2 + \left(1 + \frac{2\alpha}{\nu}\right) \|g\|_{1/2,\Gamma}^2 \right\}^{1/2}, \tag{2.32}$$

with γ defined in (2.26)*.*

Proof. Thanks to Lemmas 2.1, 2.2 and 2.3, the unique solvability of (2.7) and estimate (2.32) follow from a straightforward application of Theorem 2.1. □

We end this section with the converse of the derivation of (2.7). More precisely, the following theorem establishes that the unique solution of (2.7) together with u given by (2.3) solves the original convection-diffusion-reaction problem (2.1). This result will be employed later on in Section 4 to prove the efficiency of the a posteriori error estimator.

Theorem 2.2 *Let $\boldsymbol{\sigma} \in \mathbf{H}$ be the unique solution of* (2.7) *and let $u \in L^2(\Omega)$ be defined by* (2.3)*. Then $\nabla u = \boldsymbol{\sigma}$ in Ω, $u \in H^1(\Omega)$, and $u = g$ on Γ.*

Proof. Let $\boldsymbol{\sigma}$ be the unique element in $\mathbf{H}$ satisfying (2.7), or equivalently (2.4). Reordering the variational equation we easily arrive at

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_\Omega \frac{1}{\alpha} \Big\{ f + \nu\, \mathrm{div}(\boldsymbol{\sigma}) - \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \Big\} \mathrm{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau} \cdot \mathbf{n}\,, g \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbf{H},$$

which together to the definition of u (cf. (2.3)), implies

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_\Omega u\, \mathrm{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau} \cdot \mathbf{n}\,, g \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}.$$

Then, using the density of $[C_0^\infty(\Omega)]^d$ in $\mathbf{H}$, and integrating by parts backwardly, from the latter we can readily deduce that $\nabla u = \boldsymbol{\sigma}$ in Ω, $u \in H^1(\Omega)$ and $u = g$ on Γ. □

3 Discrete problem

In this section we introduce the Galerkin scheme associated to problem (2.7), we prove its solvability, and derive the corresponding a priori error estimate and rate of convergence. We begin by introducing the finite element scheme.

3.1 Galerkin scheme

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega} \subseteq \mathbb{R}^d$ by triangles T (in $\mathbb{R}^2$) or tetrahedrons T (in $\mathbb{R}^3$) of diameter h_T such that $\bar{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$, and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. In addition, given an integer $l \geq 0$ and a subset S of $\mathbb{R}^d$, we denote by $P_l(S)$ the space of polynomials defined in S of total degree at most l, and let $\mathbf{P}_l(S) := [P_l(S)]^d$. Then, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas space of order k on T as (see, e.g. [5])

$$\mathbf{RT}_k(T) = \mathbf{P}_k(T) \, + \, P_k(T)\, \mathbf{x}\,,$$

where, $\mathbf{x} := (x_1, \ldots, x_d)^t$ is a generic vector of $\mathbb{R}^d$. In this way, defining the following finite element subspace for the unknown $\boldsymbol{\sigma} \in \mathbf{H}$:

$$\mathbf{H}_h := \{\boldsymbol{\tau}_h \in \mathbf{H} : \quad \boldsymbol{\tau}_h|_T \, \in \, \mathbf{RT}_k(T) \quad \forall\, T \, \in \, \mathcal{T}_h\}\,,$$

the Galerkin scheme associated with (2.7) reads: Find $\boldsymbol{\sigma}_h \in \mathbf{H}_h$, such that

$$A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = F(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \, \in \mathbf{H}_h, \tag{3.1}$$

where A and F are defined in (2.5) and (2.6), respectively.

We conclude the description of our Galerkin scheme by recalling the approximation properties of the finite dimensional subspace $\mathbf{H}_h$. To that end, we need to introduce the Raviart-Thomas interpolation operator (see [5], [35]), $\Pi_h^k : \mathbf{H}^1(\Omega) \to \mathbf{H}_h$, which, given $\boldsymbol{\tau} \in \mathbf{H}^1(\Omega)$, and denoting by $\mathbf{n}$ the outward unit normal on each face/edge of the triangulation, is characterized by the following identities:

$$\int_e p\, \Pi_h^k(\boldsymbol{\tau}) \cdot \mathbf{n} = \int_e p\, \boldsymbol{\tau} \cdot \mathbf{n} \qquad \forall \text{ face/edge } e \, \in \, \mathcal{T}_h\,, \quad \forall\, p \, \in \, P_k(e)\,, \quad \text{when } k \geq 0\,, \tag{3.2}$$

and

$$\int_T \Pi_h^k(\boldsymbol{\tau}) \cdot \mathbf{p} = \int_T \boldsymbol{\tau} \cdot \mathbf{p} \qquad \forall\, T \in \, \mathcal{T}_h\,, \quad \forall\, \mathbf{p} \, \in \, \mathbf{P}_{k-1}(T)\,, \quad \text{when} \quad k \geq 1\,. \tag{3.3}$$

It is easy to show, using (3.2) and (3.3) that (cf. [34, Section 3.4.2, eq (3.4.23)])

$$\operatorname{div}(\Pi_h^k(\boldsymbol{\tau})) = \mathcal{P}_h^k(\operatorname{div}(\boldsymbol{\tau})), \tag{3.4}$$

where $\mathcal{P}_h^k \colon L^2(\Omega) \to Y_h^k$ is the $L^2(\Omega)$–orthogonal projector on Y_h^k, with

$$Y_h^k := \{v_h \in L^2(\Omega) : \; v_h|_T \in P_k(T), \quad \forall\, T \in \mathcal{T}_h\}, \tag{3.5}$$

which satisfies the approximation property:

$$\|v - \mathcal{P}_h^k(v)\|_{0,T} \leq C\, h_T^m |v|_{m,T} \quad \forall\, T \in \mathcal{T}_h, \tag{3.6}$$

for each $v \in H^m(\Omega)$, with $0 \leq m \leq k+1$ (see, e.g [10]).

Let us now summarize the approximation properties of Π_h^k. We begin with the local estimates: For each $\boldsymbol{\tau} \in \mathbf{H}^m(\Omega)$, such that $\operatorname{div}(\boldsymbol{\tau}) \in H^m(\Omega)$, with $1 \leq m \leq k+1$, there holds (see, e.g. [5], [22])

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,T} \leq C\, h_T^m \, |\boldsymbol{\tau}|_{m,T} \quad \forall\, T \, \in \, \mathcal{T}_h, \tag{3.7}$$

and

$$\|\mathrm{div}(\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau}))\|_{0,T} \le C\, h_T^m\, |\mathrm{div}(\boldsymbol{\tau})|_{m,T} \quad \forall\, T \,\in\, \mathcal{T}_h\,. \tag{3.8}$$

In addition, given $s \in (0,1)$, it can be proved that Π_h^k can also be defined in the larger space $\mathbf{H}^s(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$, and that the following estimate holds: Given $k \ge 0$, there holds (see [1, Theorem 3.4]):

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \le c(\Omega)\, h^s \|\boldsymbol{\tau}\|_{s,\Omega} \qquad \forall\, \boldsymbol{\tau} \,\in\, \mathbf{H}^s(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)\,. \tag{3.9}$$

In particular, for $k = 0$ we have the more precise local estimate (see [22, Lemma 3.19]):

$$\|\boldsymbol{\tau} - \Pi_h^0(\boldsymbol{\tau})\|_{0,T} \le C\, h_T^s \left\{ \|\boldsymbol{\tau}\|_{s,T} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,T} \right\} \qquad \forall T \in \mathcal{T}_h,\, \forall\, \boldsymbol{\tau} \,\in\, \mathbf{H}^s(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega). \tag{3.10}$$

As a consequence of the estimates above, we find that $\mathbf{H}_h$ satisfies the following approximation property (see also [32]): For each $m \,\in\, (0, k+1]$ and for each $\boldsymbol{\tau} \,\in\, \mathbf{H}^m(\Omega) \cap \mathbf{H}$ with $\mathrm{div}(\boldsymbol{\tau}) \in H^m(\Omega)$, there exists $\boldsymbol{\tau}_h \,\in\, \mathbf{H}_h$ such that

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{H}} \le C_{app}\, h^m \left\{ \|\boldsymbol{\tau}\|_{m,\Omega} + \sqrt{\frac{\nu}{\alpha}} \|\mathrm{div}(\boldsymbol{\tau})\|_{m,\Omega} \right\}. \tag{3.11}$$

3.2 Well-posedness of the discrete problem

In the sequel, we proceed similarly as in Section 2.2 and prove the solvability of (3.1) by means of Theorem 2.1. However, since problem (3.1) is posed in a finite dimensional subspace, and since the problem is linear, it is well-known that (i) and (ii) in Theorem 2.1 are equivalent, and consequently, it suffices to prove just one of the two conditions. In particular, in what follows we concentrate in proving condition (ii). This result is established next.

Lemma 3.1 *There exists $h_0 > 0$ such that for each $h \le h_0$, there holds*

$$\sup_{\boldsymbol{\sigma}_h \in \mathbf{H}_h} A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) > 0 \qquad \forall\, \boldsymbol{\tau}_h \,\in\, \mathbf{H}_h \backslash \{\mathbf{0}\}. \tag{3.12}$$

(The explicit expression of h_0 can be found in (3.21)*.)*

Proof. Let $\boldsymbol{\tau}_h \in \mathbf{H}_h$, such that $\boldsymbol{\tau}_h \neq \mathbf{0}$. We first observe that

$$A(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \ge \frac{1}{2} \|\boldsymbol{\tau}_h\|_{\mathbf{H}}^2 - \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2} \|\mathrm{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}^2. \tag{3.13}$$

In turn, let $z \in H_0^1(\Omega)$ be the unique weak solution of the boundary value problem

$$\begin{aligned} -\Delta z + \frac{1}{\nu}(\boldsymbol{\beta} \cdot \nabla z) + \frac{\alpha}{\nu} z \;&=\; -\frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu}\, \mathrm{div}(\boldsymbol{\tau}_h) \qquad \text{in} \quad \Omega\,, \\ z \;&=\; 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \Gamma. \end{aligned} \tag{3.14}$$

Similarly as for Lemma 2.1 we can deduce that z satisfies the estimates:

$$\|z\|_{0,\Omega} \le \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2} \|\mathrm{div}(\boldsymbol{\tau}_h)\|_{0,\Omega} \quad \text{and} \quad |z|_{1,\Omega} \le \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{3/2}\nu^{1/2}} \|\mathrm{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}, \tag{3.15}$$

which combined with the first equation of (3.14), imply

$$\|\Delta z\|_{0,\Omega} \leq \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{1/2}\nu^{3/2}} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \|\boldsymbol{\tau}_h\|_{\mathbf{H}}. \tag{3.16}$$

Moreover, since $z \in H_0^1(\Omega)$ and $\Delta z \in L^2(\Omega)$, a well known regularity result on polygonal domains (see [28, 29, 30]) implies that there exists $\delta \in (0,1]$, such that $z \in H^{1+\delta}(\Omega)$, and $\|z\|_{1+\delta,\Omega} \leq \widehat{c}\|\Delta z\|_{0,\Omega}$, with $\widehat{c}$ independent of h, ν, $\boldsymbol{\beta}$ and α. It is clear then that $\nabla z \in H^\delta(\Omega)$, which implies that $\Pi_h^k(\nabla z)$ is well defined (see Section 3.1). Then, we define

$$\tilde{\boldsymbol{\sigma}} := \nabla z \in [H^\delta(\Omega)]^d \quad \text{and} \quad \tilde{\boldsymbol{\sigma}}_h := \Pi_h^k(\tilde{\boldsymbol{\sigma}}) \in \mathbf{H}_h, \tag{3.17}$$

and observe from (3.16), (3.4) and from the first equation of (3.14), respectively, that

$$\|\tilde{\boldsymbol{\sigma}}\|_{\delta,\Omega} \leq \|z\|_{1+\delta,\Omega} \leq \widehat{c}\frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{1/2}\nu^{3/2}} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \|\boldsymbol{\tau}_h\|_{\mathbf{H}}, \tag{3.18}$$

$$\int_\Omega \operatorname{div}(\tilde{\boldsymbol{\sigma}}_h) \operatorname{div}(\boldsymbol{\tau}_h) = \int_\Omega \operatorname{div}(\tilde{\boldsymbol{\sigma}}) \operatorname{div}(\boldsymbol{\tau}_h) \quad \text{and} \quad \operatorname{div}(\tilde{\boldsymbol{\sigma}}) = \frac{1}{\nu}(\boldsymbol{\beta}\cdot\tilde{\boldsymbol{\sigma}}) + \frac{\alpha}{\nu} z + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu} \operatorname{div}(\boldsymbol{\tau}_h).$$

From the above, it follows that

$$\begin{aligned} A(\tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) = \; & \int_\Omega \tilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + \frac{\nu}{\alpha}\int_\Omega \operatorname{div}(\tilde{\boldsymbol{\sigma}}_h)\operatorname{div}(\boldsymbol{\tau}_h) - \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot\tilde{\boldsymbol{\sigma}}_h)\operatorname{div}(\boldsymbol{\tau}_h) \\ = \; & \int_\Omega \tilde{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h))\operatorname{div}(\boldsymbol{\tau}_h) + \int_\Omega z\operatorname{div}(\boldsymbol{\tau}_h) \\ & + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\operatorname{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}^2. \end{aligned}$$

Furthermore, we integrate by parts the term $\int_\Omega z\operatorname{div}(\boldsymbol{\tau}_h)$, and apply Hölder inequality to derive from the latter that

$$\begin{aligned} A(\tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) = \; & \int_\Omega (\tilde{\boldsymbol{\sigma}}_h - \tilde{\boldsymbol{\sigma}}) \cdot \boldsymbol{\tau}_h + \frac{1}{\alpha}\int_\Omega (\boldsymbol{\beta}\cdot(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h))\operatorname{div}(\boldsymbol{\tau}_h) + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\operatorname{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}^2 \\ \geq \; & -\left(\|\boldsymbol{\tau}_h\|_{0,\Omega} + \frac{\sqrt{d}}{\alpha}\|\boldsymbol{\beta}\|_{\infty,\Omega}\|\operatorname{div}(\boldsymbol{\tau}_h)\|_{0,\Omega} \right) \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\operatorname{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}^2 \\ \geq \; & -\left(1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{\alpha\nu} \right)^{1/2} \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\|\boldsymbol{\tau}_h\|_{\mathbf{H}} + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^2}\|\operatorname{div}(\boldsymbol{\tau}_h)\|_{0,\Omega}^2. \end{aligned} \tag{3.19}$$

In this way, we combine the estimates (3.13) and (3.19), apply the estimate (3.9) to the term $\|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}$, and utilize the bound (3.18), to get

$$\begin{aligned} A(\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) \geq \; & \left\{ \frac{1}{2}\|\boldsymbol{\tau}_h\|_{\mathbf{H}} - \left(1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}}{\alpha\nu} \right)^{1/2} \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \right\} \|\boldsymbol{\tau}_h\|_{\mathbf{H}} \\ \geq \; & \left\{ \frac{1}{2} - Kh^\delta \right\} \|\boldsymbol{\tau}_h\|_{\mathbf{H}}^2 \end{aligned} \tag{3.20}$$

with

$$K = \widehat{c}\, c(\Omega) \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{1/2}\nu^{3/2}} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \left(1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{\alpha\nu} \right)^{1/2}$$

Therefore, defining

$$h_0 := \left\{ \frac{1}{4K} \right\}^{1/\delta}, \tag{3.21}$$

and assuming that $h \leq h_0$, from (3.20) we deduce that

$$A(\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) \geq \frac{1}{4} \|\boldsymbol{\tau}_h\|_{\mathbf{H}}^2, \tag{3.22}$$

which yields

$$\sup_{\boldsymbol{\sigma}_h \in \mathbf{H}_h} A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \geq A(\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) > 0,$$

and finishes the proof. □

Let us observe that, thanks to the estimates (3.9), (3.18) and (3.16), $\tilde{\boldsymbol{\sigma}}_h \in \mathbf{H}_h$ defined in (3.17) satisfies

$$\begin{aligned} \|\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \leq & \ \|\tilde{\boldsymbol{\sigma}}_h - \tilde{\boldsymbol{\sigma}}\|_{0,\Omega} + \|\tilde{\boldsymbol{\sigma}}\|_{0,\Omega} \leq c(\Omega) h^\delta \|\tilde{\boldsymbol{\sigma}}\|_{\delta,\Omega} + \|\tilde{\boldsymbol{\sigma}}\|_{0,\Omega} \leq \bar{c} \|\tilde{\boldsymbol{\sigma}}\|_{\delta,\Omega} \\ \leq & \ \bar{c}\, \widehat{c} \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha^{1/2}\nu^{3/2}} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \|\boldsymbol{\tau}_h\|_{\mathbf{H}} \end{aligned}$$

and

$$\begin{aligned} \sqrt{\frac{\nu}{\alpha}} \|\mathrm{div}(\tilde{\boldsymbol{\sigma}}_h)\|_{0,\Omega} = & \ \sqrt{\frac{\nu}{\alpha}} \|\mathcal{P}_h^k(\mathrm{div}(\tilde{\boldsymbol{\sigma}}))\|_{0,\Omega} \leq \sqrt{\frac{\nu}{\alpha}} \|\mathrm{div}(\tilde{\boldsymbol{\sigma}})\|_{0,\Omega} \\ \leq & \ \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \|\boldsymbol{\tau}_h\|_{\mathbf{H}}, \end{aligned}$$

from which

$$\|\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h\|_{\mathbf{H}} \leq \widehat{C}(\nu, \boldsymbol{\beta}, \alpha) \|\boldsymbol{\tau}_h\|_{\mathbf{H}}, \tag{3.23}$$

with

$$\widehat{C}(\nu, \boldsymbol{\beta}, \alpha) := 1 + \frac{d\|\boldsymbol{\beta}\|_{\infty,\Omega}^2}{2\alpha\nu^{3/2}} \left(\sqrt{\frac{d}{\alpha\nu}} \|\boldsymbol{\beta}\|_{\infty,\Omega} + 2 \right) \left(\bar{c}\, \widehat{c}\, \alpha^{1/2} + \nu^{1/2} \right).$$

Hence, assuming that $h \leq h_0$, with h_0 the positive constant defined in (3.21), from (3.22) and (3.23), it readily follows that

$$\sup_{\boldsymbol{\sigma}_h \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\sigma}_h\|_{\mathbf{H}}} \geq \frac{A(\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h + \tilde{\boldsymbol{\sigma}}_h\|_{\mathbf{H}}} \geq \widehat{\gamma} \|\boldsymbol{\tau}_h\|_{\mathbf{H}}, \tag{3.24}$$

with

$$\widehat{\gamma} := \frac{1}{4\widehat{C}(\nu, \boldsymbol{\beta}, \alpha)}. \tag{3.25}$$

From this estimate and the fact that $\mathbf{H}_h$ is a finite dimensional space we can easily deduce that A satisfies condition (i). This result is established now.

Lemma 3.2 *Let $h_0 > 0$ defined as in (3.21) and assume that $h \leq h_0$. Then, there holds*

$$\sup_{\boldsymbol{\tau}_h \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{A(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \geq \widehat{\gamma} \|\boldsymbol{\zeta}_h\|_{\mathbf{H}} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbf{H}_h, \tag{3.26}$$

with $\widehat{\gamma} > 0$ defined in (3.25).

Proof. Since conditions (i) and (ii) are equivalent and since (3.24) holds, the results is a direct consequence of [22, Lemma 1.2]. □

We are now in position of establishing the main result of this section.

Theorem 3.1 *Let $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$, $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$, $\alpha, \nu > 0$, and assume that $\operatorname{div}(\boldsymbol{\beta}) = 0$. In addition, let $h_0 > 0$ be the positive constant defined in (3.21). Then, for each $h \leq h_0$, there exists a unique $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ solution to (3.1), which satisfies the estimates*

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{H}} \leq \frac{1}{\widehat{\gamma}} \left\{ \frac{2}{\alpha\nu} \|f\|_{0,\Omega}^2 + \left(1 + \frac{2\alpha}{\nu}\right) \|g\|_{1/2,\Gamma}^2 \right\}^{1/2} \tag{3.27}$$

and

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} \leq \left(1 + \frac{C_A}{\widehat{\gamma}}\right) \inf_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{H}} \tag{3.28}$$

where $\boldsymbol{\sigma} \in \mathbf{H}$ is the unique solution of (2.7) and $\widehat{\gamma}$ and C_A are the positive constants defined in (3.25) and (2.30), respectively.

Proof. The existence and uniqueness of solution follow straightforwardly from Theorem 2.1 and Lemma 3.1 (or equivalently Lemma 3.2). In turn, estimate (3.27) is a direct consequence of Lemma 3.2 whereas the Céa estimate (3.28) follows from Lemmas 2.3 and 3.2 and the triangle inequality. □

Remark 3.1 *It is important to notice from the definition of h_0 in (3.21) that for fixed α and $\boldsymbol{\beta}$, there holds*

$$\lim_{\nu \to 0} h_0 = 0,$$

that explains theoretically the lack of accuracy of our approach for small values of ν, which can be visualized in the numerical experiments provided next in Section 5 (Example 3). However, this drawback can be substantially improved with the a posteriori error estimator proposed in Section 4, which is corroborated numerically in Section 5.

The following theorem provides the theoretical rate of convergence of the Galerkin scheme (3.1), under suitable regularity assumptions on the exact solution.

Theorem 3.2 *Let $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ be the unique solutions of the continuous and discrete formulations (2.7) and (3.1), respectively. Assume that $\boldsymbol{\sigma} \in \mathbf{H}^m(\Omega)$ and $\operatorname{div}(\boldsymbol{\sigma}) \in H^m(\Omega)$ for some $m \in (0, k+1]$. Then, there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} \leq C_{app} \left(1 + \frac{C_A}{\widehat{\gamma}}\right) h^m \left\{ \|\boldsymbol{\sigma}\|_{m,\Omega} + \sqrt{\frac{\nu}{\alpha}} \|\operatorname{div}(\boldsymbol{\sigma})\|_{m,\Omega} \right\}, \tag{3.29}$$

where $\widehat{\gamma}$ and C_A are the positive constants defined in (3.25) and (2.30), respectively.

Proof. This result is a straightforward consequence of the Céa estimate (3.28) and the approximation property (3.11). □

We end this section by introducing a suitable approximation for $u \in L^2(\Omega)$ (cf. (2.3)). To that end let us assume, for the sake of simplicity, that $f|_K \in C(\bar{T})$ and $\boldsymbol{\beta}|_T \in [C(\bar{T})]^d$,for all $T \in \mathcal{T}_h$. Then, if $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ is the solution of (3.1), resembling (2.3) we propose the following piecewise approximation for u:

$$u_h := \frac{1}{\alpha}\Big\{ f + \nu\,\mathrm{div}(\boldsymbol{\sigma}_h) - \boldsymbol{\beta}\cdot\boldsymbol{\sigma}_h \Big\} \quad \text{in} \quad \Omega\,. \tag{3.30}$$

Observe that this piecewise function is not necessarily polynomial on each $T \in \mathcal{T}_h$.

The following result establishes the rate of convergence for this post-processed approximation.

Corollary 3.3 *Let $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ be the unique solutions of the continuous and discrete problems* (2.7) *and* (3.1)*, respectively, and let u, $u_h \in L^2(\Omega)$ be the functions defined in* (2.3) *and* (3.30)*, respectively. Assume that $\boldsymbol{\sigma} \in \mathbf{H}^m(\Omega)$ and $\mathrm{div}(\boldsymbol{\sigma}) \in H^m(\Omega)$ for some $m \in (0, k+1]$. Then, there holds*

$$\|u - u_h\|_{0,\Omega} \,\le\, C_{app}\left(\frac{\nu}{\alpha} \,+\, \frac{d}{\alpha^2}\|\boldsymbol{\beta}\|^2_{\infty,\Omega}\right)^{1/2}\left(1 + \frac{C_A}{\widehat{\gamma}}\right) h^m \Big\{ \|\boldsymbol{\sigma}\|_{m,\Omega} \,+\, \sqrt{\frac{\nu}{\alpha}}\|\mathrm{div}(\boldsymbol{\sigma})\|_{m,\Omega} \Big\}, \tag{3.31}$$

where $\widehat{\gamma}$ and C_A are the positive constants defined in (3.25) *and* (2.30)*, respectively.*

Proof. This result follows straightforwardly from Theorem 3.2 and the the fact that

$$\|u - u_h\|_{0,\Omega} \le \left(\frac{\nu}{\alpha} \,+\, \frac{d}{\alpha^2}\|\boldsymbol{\beta}\|^2_{\infty,\Omega}\right)^{1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}}. \tag{3.32}$$

□

4 A residual-based a posteriori error estimator

In this section we derive a reliable and efficient residual-based a posteriori error estimate for our Galerkin method (3.1). For the sake of simplicity we restrict ourselves to the two-dimensional case since minor modifications allow to extend our approach to $\mathbb{R}^3$. We begin with some notations and definitions to be utilized below.

For each $T \in \mathcal{T}_h$ we let $\mathcal{E}(T)$ be the set of edges of T and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$, subdivided as follows:

$$\mathcal{E}_h \,=\, \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)\,,$$

where $\mathcal{E}_h(\Omega) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Omega \,\}$ and $\mathcal{E}_h(\Gamma) := \{\, e \in \mathcal{E}_h \,:\, e \subseteq \Gamma \,\}$. In what follows, h_e stands for the length of a given edge $e \in \mathcal{E}_h$. Now, let $v \in L^2(\Omega)$ such that $v|_T \in C(T)$ for each $T \in \mathcal{T}_h$. Then, given $e \in \mathcal{E}_h(\Omega)$, we denote by $[\![v]\!]$ the jump of v across e, that is $[\![v]\!] := (v|_{T'})|_e - (v|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathtt{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2, n_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $e \in \mathcal{E}_h(\Omega)$ and $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\boldsymbol{\tau}|_T \in [C(T)]^2$ for each $T \in \mathcal{T}_h$, we let $[\![\boldsymbol{\tau}\cdot\mathbf{t}_e]\!] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\cdot\mathbf{t}_e$, where T'

and T'' are the triangles of $\mathcal{T}_h$ having e as an edge. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for a sufficiently smooth vector field $\boldsymbol{\tau} := (\tau_1, \tau_2)^{\mathtt{t}}$, we let

$$\operatorname{rot} \boldsymbol{\tau} := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}.$$

Now, let $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ be the unique solutions of (2.7) and (3.1), respectively. Then, we define the global a posteriori error estimator:

$$\Theta := \left\{ \sum_{T \in \mathcal{T}_h} \Theta_T^2 \right\}^{1/2}, \tag{4.1}$$

where for each $T \in \mathcal{T}_h$:

$$\begin{aligned}\Theta_T^2 := \;& h_T^2 \, \|\nabla u_h - \boldsymbol{\sigma}_h\|_{0,T}^2 + h_T^2 \, \|\operatorname{rot}(\boldsymbol{\sigma}_h)\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega)} \left\{ h_e \, \|[\![u_h]\!]\|_{0,e}^2 + h_e \|[\![\boldsymbol{\sigma}_h \cdot \mathbf{t}]\!]\|_{0,e}^2 \right\} \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \, \|g - u_h\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma)} h_e \left\| \boldsymbol{\sigma}_h \cdot \mathbf{t} - \frac{dg}{d\mathbf{t}} \right\|_{0,e}^2,\end{aligned}$$

where $u_h \in L^2(\Omega)$ is the approximation of u defined in (3.30). Note that the above requires that $\frac{dg}{d\mathbf{t}} \in L^2(e)$ for each $e \in \mathcal{E}_h(\Gamma)$. This is fixed below by assuming that $g \in H^1(\Gamma)$.

In what follows we prove that Θ is reliable and efficient. We start with the reliability of the estimator.

4.1 Reliability

The main result of this section is stated as follows.

Theorem 4.1 *Let $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ be the respective unique solutions of* (2.7) *and* (3.1) *and assume that $g \in H^1(\Gamma)$. Then there exists $C_{\mathtt{rel}} > 0$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} \le \frac{C_{\mathtt{rel}}}{\gamma} \Theta \quad \text{and} \quad \|u - u_h\|_{0,\Omega} \le \frac{1}{\gamma} \left(\frac{\nu^2}{\alpha^2} + \|\boldsymbol{\beta}\|_{\infty,\Omega}^2 \right) C_{\mathtt{rel}} \Theta, \tag{4.2}$$

with $\gamma > 0$ defined in (2.26).

Proof. To derive (4.2) we start by observing that, owing to the definition of A, F and u_h (cf. (2.5), (2.6) and (3.30), respectively), the following identity holds

$$\begin{aligned} A(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \;& -\frac{1}{\alpha} \int_\Omega (f + \nu \operatorname{div}(\boldsymbol{\sigma}_h) - (\boldsymbol{\beta} \cdot \boldsymbol{\sigma}_h)) \operatorname{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau} \cdot \mathbf{n}, g \rangle_\Gamma - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \\ = \;& -\int_\Omega u_h \operatorname{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau} \cdot \mathbf{n}, g \rangle_\Gamma - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}. \end{aligned} \tag{4.3}$$

Then, defining the residual operator $R : \mathbf{H} \to \mathbb{R}$, by

$$R(\boldsymbol{\tau}) = -\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} - \int_\Omega u_h \operatorname{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau} \cdot \mathbf{n}, g \rangle_\Gamma, \tag{4.4}$$

from (4.3) and the inf-sup condition (2.11) with $\boldsymbol{\tau} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \in \mathbf{H}$, we easily obtain the estimate

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} \;\leq\; \frac{1}{\gamma} \sup_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{R(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}}. \tag{4.5}$$

In this way, noticing that R has the same structure of the functional R_1 defined in [26, Section 3.1], we proceed analogously to [26, Lemma 3.8], that is, we apply the stable Helmhotlz decomposition provided by [26, Lemma 3.3] and apply integration by parts and the approximation properties of the Raviart-Thomas and Clément interpolator operators (see [11] for more details on the Clément operator), to deduce that

$$\sup_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{R(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \leq C_{\mathtt{rel}}\Theta,$$

with $C_{\mathtt{rel}} > 0$ independent of h, ν, $\boldsymbol{\beta}$ and α, which implies the first estimate of (4.2). We end the proof by observing that the second estimate in (4.2) is a direct consequence of (3.32) and the above. □

4.2 Efficiency

In this section we apply suitable well-known results available in the literature, and make frequent use of the identities provided by Theorem 2.2, to derive the efficiency of the estimator Θ established in the following theorem.

Theorem 4.2 *There exists $C_{\mathtt{eff}} > 0$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

$$C_{\mathtt{eff}}\Theta \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} + \|u - u_h\|_{0,\Omega} + \text{h.o.t.}, \tag{4.6}$$

where h.o.t. *stands for high order terms.*

In what follows, we bound each term defining the local indicator Θ_T to derive (4.6). To that end, since u_h is not necessarily a piecewise polynomial function, from now on we assume that f and $\boldsymbol{\beta}$ belong at least to $H^{k+4}(T)$ and $[H^{k+4}(T)]^2$, respectively. By doing that we can proceed similarly as in [9, Section 6.2] and apply the approximation properties of the orthogonal projection $\mathcal{P}_h^k$, inverse inequalities, and the localization technique introduced in [37] based on triangle-bubble and edge-bubble functions, as well as known results mainly from [7], [8] and [21], to obtain the desired estimate. To do that, let us first introduce further notations and preliminary results.

Given $T \in \mathcal{T}_h$, we let ψ_T be the usual triangle-bubble function which satisfy: $\psi_T \in P_3(T)$, supp(ψ_T)$\subseteq T$, $\psi_T = 0$ on ∂T, and $0 \leq \psi_T \leq 1$ in T. In addition, the following result holds.

Lemma 4.1 *Given $k \in \mathbb{N} \cup \{0\}$, there exists a positive constant c_1, depending only on k and the shape regularity of the triangulations (minimum angle condition), such that for each triangle T there holds*

$$\|\psi_T\, q\|_{0,T}^2 \leq \|q\|_{0,T}^2 \leq c_1 \|\psi_T^{1/2}\, q\|_{0,T}^2 \qquad \forall\, q \in P_k(T). \tag{4.7}$$

Proof. See [36, Lemma 4.1]. □

The following inverse estimate will be also needed.

Lemma 4.2 *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leq m$. Then, there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each triangle T there holds*

$$|q|_{m,T} \leq c h_T^{l-m} |q|_{l,T}, \quad \forall\, q \in P_k(T). \tag{4.8}$$

Proof. See [10, Theorem 3.2.6]. □

In turn, we will need the following inequality (see, for instance, [12])

$$\|v\|_{0,e}^2 \leq C(h_e^{-1} \|v\|_{0,T}^2 + h_e |v|_{1,T}^2) \quad \forall\, v \in H^1(T), \tag{4.9}$$

where T is a generic triangle having e as an edge, and C is a positive constant depending only on the minimum angle of T.

Now, we estimate each one of the terms defining Θ_T.

Lemma 4.3 *There exists $C > 0$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

$$h_T \|\nabla u_h - \boldsymbol{\sigma}_h\|_{0,T} \leq C \left(\|u - u_h\|_{0,T} + h_T \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} + \text{h.o.t.} \right) \tag{4.10}$$

for all $T \in \mathcal{T}_h$.

Proof. First observe that, according to the further assumptions on f and $\boldsymbol{\beta}$, we have that $\nabla u_h \in [H^{k+3}(K)]^2$. In turn, applying the triangle inequality, we easily get

$$\|\nabla u_h - \boldsymbol{\sigma}_h\|_{0,T} \leq \|\nabla u_h - \mathcal{P}_h^{k+2}(\nabla u_h)\|_{0,T} + \|\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h\|_{0,T}. \tag{4.11}$$

Then, proceeding analogously to [9, Lemma 6.11], that is, using (4.7) and adding and subtracting suitable terms, we obtain

$$\begin{aligned} \|\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h\|_{0,T}^2 \leq \; & c_1 \|\psi_T^{1/2} \,(\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h)\|_{0,T}^2 \\ = \; & c_1 \int_T \psi_T \,(\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h) \left((\mathcal{P}_h^{k+2}(\nabla u_h) - \nabla u_h) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \right) \\ & + c_1 \int_T \psi_T \,(\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h)\,(\nabla u_h - \nabla u)\,, \end{aligned} \tag{4.12}$$

and integrating by parts the last term in (4.12), we find that

$$\int_T \psi_T \,(\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h)(\nabla u_h - \nabla u) = - \int_T \operatorname{div} \left(\psi_T \,(\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h) \right) (u_h - u). \tag{4.13}$$

Then, from (4.12), (4.13), the triangle and Cauchy-Schwarz inequalities, and estimates (4.7) and (4.8), we easily find

$$\|\mathcal{P}_h^{k+2}(\nabla u_h) - \boldsymbol{\sigma}_h\|_{0,T} \leq c \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} + \|\mathcal{P}_h^{k+2}(\nabla u_h) - \nabla u_h\|_{0,T} + h_T^{-1} \|u - u_h\|_{0,T} \right)$$

which together to (3.6) and (4.11), imply

$$h_T \|\nabla u_h - \boldsymbol{\sigma}_h\|_{0,T} \leq C \left(\|u - u_h\|_{0,T} + h_T \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} + h_T^{k+4} |\nabla u_h|_{k+3,T} \right)$$

which concludes the proof. □

Lemma 4.4 *There exists $C > 0$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

$$h_e^{1/2} \| [\![u_h]\!] \|_{0,e} \leq C \left(\sum_{T \subseteq \omega_e} \| u - u_h \|_{0,T} + h_e \sum_{T \subseteq \omega_e} \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T} + \text{h.o.t.} \right), \quad \forall\, e \in \mathcal{E}_h(\Omega). \quad (4.14)$$

Proof. Let $e \in \mathcal{E}_h(\Omega)$. First we use the fact that $u \in H^1(\Omega)$ and utilize estimate (4.9), to obtain

$$\| [\![u_h]\!] \|_{0,e} = \| [\![u_h - u]\!] \|_{0,e} \leq C \left(h_e^{-1/2} \sum_{T \subseteq \omega_e} \| u - u_h \|_{0,T} + h_e^{1/2} \sum_{T \subseteq \omega_e} | u - u_h |_{1,T} \right). \quad (4.15)$$

On the other hand, using the identity $\boldsymbol{\sigma} = \nabla u$ in Ω and the triangle inequality, we easily get

$$| u - u_h |_{1,T} \leq \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T} + \| \boldsymbol{\sigma}_h - \nabla u_h \|_{0,T}. \quad (4.16)$$

Then, from (4.15), (4.16) and (4.10) and the fact that $h_e \leq h_T$, we find

$$h_e^{1/2} \| [\![u_h]\!] \|_{0,e} \leq C \left(\sum_{T \subseteq \omega_e} \| u - u_h \|_{0,T} + h_e \sum_{T \subseteq \omega_e} \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T} + h_e^{k+4} \sum_{T \subseteq \omega_e} | \nabla u_h |_{k+3,T} \right)$$

which concludes the proof. □

Lemma 4.5 *There exist positive constants C_i, $i \in \{1, 2, 3\}$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

a) $h_T \, \| \text{rot}\, (\boldsymbol{\sigma}_h) \|_{0,T} \leq C_1 \, \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T} \qquad \forall\, T \in \mathcal{T}_h$,

b) $h_e^{1/2} \, \| [\![\boldsymbol{\sigma}_h \cdot \mathbf{t}]\!] \|_{0,e} \leq C_2 \, \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,\omega_e} \qquad \forall\, e \in \mathcal{E}_h(\Omega)$,

c) $h_e^{1/2} \, \| g - u_h \|_{0,e} \leq C_3 \, (\| u - u_h \|_{0,T_e} + h_{T_e} \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T_e} + h.o.t.)$, $\forall\, e \in \mathcal{E}_h(\Gamma)$, *where T_e is the triangle having e as an edge.*

Proof. For a) we refer to [38, Lemma 6.1] or apply the technical result given by [4, Lemma 4.3] . Similarly, for b) we refer to [38, Lemma 6.2]. Alternatively, b) follows from a slight modification of the proof of [4, Lemma 4.4] (see also [27, Lemma 4.10]). Finally, the proof of c) follows from slight modifications of the proof of [27, Lemma 4.14] and Lemma 4.3. □

We end the efficiency analysis with the following result. Here, for simplicity we assume that g is piecewise polynomial. If g were not piecewise polynomial, but sufficiently smooth, then similarly as in lemmas 4.3 and 4.4, higher order terms would appear in (4.17).

Lemma 4.6 *Assume that g is piecewise polynomial. Then there exists $C > 0$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

$$h_e \left\| \boldsymbol{\sigma}_h \cdot \mathbf{t} - \frac{\partial g}{\partial \mathbf{t}} \right\|_{0,e}^2 \leq C \| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \|_{0,T_e}^2, \quad (4.17)$$

where T_e is the triangle of $\mathcal{T}_h$ having e as an edge.

Proof. For the proof we refer to [27, Lemma 4.15]. □

We end this section by observing that the efficiency estimate (4.6) follows from Lemmas 4.3, 4.4, 4.5 and 4.6.

4.3 Three dimensional case

In what follows we briefly discuss about the a posteriori error estimator in the three dimensional case. We start by introducing some notations.

Given $\boldsymbol{\tau}$ a sufficiently smooth vector field, we let

$$\mathbf{curl}\,\boldsymbol{\tau} := \nabla \times \boldsymbol{\tau}\,.$$

On the other hand, given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its faces, and let $\mathcal{E}_h$ be the set of all the faces of the triangulation $\mathcal{T}_h$. Then, we write $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$, where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h \,:\, e \subset \Omega\}$ and $\mathcal{E}_h(\Gamma) := \{e \in \mathcal{E}_h \,:\, e \subset \Gamma\}$. The faces of the tetrahedrons of $\mathcal{T}_h$ are denoted by e and their corresponding diameters by h_e. Also for each face $e \in \mathcal{E}_h$ we fix a unit normal $\mathbf{n}_e$ to e. In addition, if $\boldsymbol{\tau}$ is a sufficiently smooth vector field, and $e \in \mathcal{E}_h(\Omega)$, we let $[\![\boldsymbol{\tau} \times \mathbf{n}_e]\!] := (\boldsymbol{\tau}|_{T'} - \boldsymbol{\tau}|_{T''})\,|_e \times \mathbf{n}_e$, where T' and T'' are the elements of $\mathcal{T}_h$ having e as a common face. As in the previous section, from now on, when no confusion arises, we simple write $\mathbf{n}$ instead of $\mathbf{n}_e$.

Now, let $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$ be the respective unique solutions of (2.7) and (3.1) and let $u,\, u_h \in L^2(\Omega)$ be the functions defined in (2.3) and (3.30), respectively. Then we define the global a posteriori error estimator

$$\widehat{\Theta} := \left\{ \sum_{T\in\mathcal{T}_h} \widehat{\Theta}_T^2 \right\}^{1/2},$$

where for each $T \in \mathcal{T}_h$:

$$\begin{aligned}
\widehat{\Theta}_T^2 \;:=\;& h_T^2\, \|\nabla u_h - \boldsymbol{\sigma}_h)\|_{0,T}^2 \,+\, h_T^2\, \|\mathbf{curl}(\boldsymbol{\sigma}_h)\|_{0,T}^2 \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega)} \Big\{ h_e\, \|[\![u_h]\!]\|_{0,e}^2 \,+\, h_e\, \|[\![\boldsymbol{\sigma}_h \times \mathbf{n}]\!]\|_{0,e}^2 \Big\} \\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\, \|g - u_h\|_{0,e}^2 \,+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma)} h_e\, \|\nabla g \times \mathbf{n} - \boldsymbol{\sigma}_h \times \mathbf{n}\|_{0,e}^2\,.
\end{aligned}$$

The reliability of this estimator can be proved essentially by using the same arguments employed for the 2D case. In particular, analogously to the 2D case, here it is needed a stable Helmholtz decomposition for $H(\mathrm{div};\Omega)$. This result is established in the following lemma. For its proof we refer the reader to [23, Theorem 3.1].

Lemma 4.7 *For each $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ there exist $z \in H^2(\Omega)$ and $\boldsymbol{\chi} \in [H^1(\Omega)]^3$, such that there hold $\boldsymbol{\tau} = \nabla z + \mathbf{curl}\boldsymbol{\chi}$ in Ω, and*

$$\|z\|_{2,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \le C\|\boldsymbol{\tau}\|_{\mathbf{H}}\,,$$

where C is a positive constant independent of $\boldsymbol{\tau}$.

Finally, to prove the efficiency of the 3D estimator it suffices to control the new terms since the analysis of the rest of the terms is straightforward. The following lemma provides these desired estimates.

Lemma 4.8 *There exist positive constants C_i, $i \in \{1,2,3\}$, independent of h, ν, $\boldsymbol{\beta}$ and α, such that*

a) $h_T \, \|\mathbf{curl}(\boldsymbol{\sigma}_h)\|_{0,T} \,\leq\, C_1 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T} \qquad \forall\, T \in \mathcal{T}_h,$

b) $h_e^{1/2} \, \|[\![\boldsymbol{\sigma}_h \times \mathbf{n}]\!]\|_{0,e} \,\leq\, C_2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\omega_e} \qquad \forall\, e \in \mathcal{E}_h(\Omega),$

c) *Assume that g is piecewise polynomial. Then there exists $C_3 > 0$, independent of h, such that $h_e \, \|\nabla g \times \mathbf{n} - \boldsymbol{\sigma}_h \times \mathbf{n}\|_{0,e}^2 \leq C_3 \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,T_e}^2$, where T_e is the triangle of $\mathcal{T}_h$ having e as an edge.*

Proof. For a) we refer to [25, Lemma 4.9]. Similarly, for b) we refer to [25, Lemma 4.10]. Finally, the proof of c) follows from a slight modification of the proof of [25, Lemma 4.13]. □

5 Numerical results

In this section we present three numerical examples, illustrating the performance of the mixed finite element scheme (3.1), confirming the reliability and efficiency of the a posteriori error estimator Θ derived in Section 4, and showing the behaviour of the associated adaptive algorithm. Our implementation is based on a *FreeFem++* code (see [31]), in conjunction with the direct linear solver UMFPACK (see [14]).

In what follows, N stands for the total number of degrees of freedom defining $\mathbf{H}_h$. Denoting by $\boldsymbol{\sigma} \in \mathbf{H}$ and $\boldsymbol{\sigma}_h \in \mathbf{H}_h$, the respective solutions of (2.7) and (3.1), and by u and u_h the post-processed functions defined in (2.3) and (3.30), respectively, the individual errors are defined by

$$\mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}}, \quad \mathrm{e}(u) := \|u - u_h\|_{0,\Omega},$$

and

$$\mathrm{e}(\boldsymbol{\sigma}, u) := \left\{ (\mathrm{e}(\boldsymbol{\sigma}))^2 + (\mathrm{e}(u))^2 \right\}^{1/2}.$$

The effectivity index with respect to Θ is given by

$$\mathrm{eff}(\Theta) := \mathrm{e}(\boldsymbol{\sigma}, u)/\Theta.$$

Furthermore, we define the experimental rates of convergence

$$r(\boldsymbol{\sigma}) := \frac{\log(\mathrm{e}(\boldsymbol{\sigma})/\mathrm{e}'(\boldsymbol{\sigma}))}{\log(h/h')}, \quad r(u) := \frac{\log(\mathrm{e}(u)/\mathrm{e}'(u))}{\log(h/h')} \quad \text{and} \quad r(\boldsymbol{\sigma}, u) := \frac{\log(\mathrm{e}(\boldsymbol{\sigma}, u)/\mathrm{e}'(\boldsymbol{\sigma}, u))}{\log(h/h')},$$

where h and h' are two consecutive meshsizes with errors e and e$'$. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. Example 1 is used to illustrate the performance of the three dimensional mixed finite element scheme under a quasi-uniform refinement, whereas Examples 2 and 3 are utilized to illustrate the behaviour of the adaptive algorithm associated to the a posteriori error estimator Θ defined in (4.1). For the last two examples we apply the following adaptive procedure from [37]:

1) Start with a coarse mesh $\mathcal{T}_h$.

2) Solve the discrete problem (3.1) for the current mesh $\mathcal{T}_h$.

3) Compute $\Theta_T := \Theta$ for each triangle $T \in \mathcal{T}_h$.

4) Check the stopping criterion and decide whether to finish or go to next step.

5) Use *blue-green* refinement on those $T' \in \mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies

$$\Theta_{T'} \geq \frac{1}{2} \max_{T \in \mathcal{T}_h} \{\Theta_T : T \in \mathcal{T}_h\} .$$

6) Define resulting meshes as current meshes $\mathcal{T}_h$, and go to step 2.

In Example 1 we choose the domain $\Omega := (0,1)^3$, the vector field $\boldsymbol{\beta} = (x_1, x_2, -2x_3)^t$, the parameters $\alpha = \nu = 1$, and take f and g so that the exact solution is given by the smooth function

$$u(x_1, x_2, x_3) = e^{x_3 + x_1 x_2} + x_1 x_2 x_3.$$

In Table 5.1 below we summarize the convergence history obtained for this example, considering a sequence of quasi-uniform triangulations and a RT_0 approximation. We observe there that the rate of convergence $O(h)$ predicted by Theorem 3.2 and Corollary 3.3 is attained in all the cases. Computed solutions are shown in Figure 1

Our second test focuses on the case where, under uniform mesh refinement, the convergence rates are affected by the loss of regularity of the exact solutions. The problem setting is as follows: we consider $\nu = 1$, $\boldsymbol{\beta} = (1,1)^t$, $\alpha = 1$, the domain is taken as the non-convex pacman–shaped domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \backslash [0,1] \times [-1, 0]$, and the exact solution to (1.1) is given by

$$u(r, \theta) = r^{2/3} \sin(2\theta/3),$$

where (r, θ) are the polar coordinates. A simple calculation shows that $\mathrm{div}(\boldsymbol{\sigma}) = \Delta u = 0$. In addition, because of the power of r, the partial derivatives of this solution are singular at the origin, and hence Theorem 3.2 and Corollary 3.3 only yield a rate of convergence of $O(h^{2/3})$. In Table 5.2 we present the convergence history of the method (in its lowest-order configuration), considering firstly a quasi-uniform refinement (table at the top) and secondly an adaptive refinement (table at the bottom). In the first table we clearly observe that the $O(h^{2/3})$ predicted by the theory (Theorem 3.2 and Corollary 3.3) is attained in the three cases. In turn, in the second table we observe that the experimental rates of convergence obtained with the adaptive algorithm recover the order h, thus improving the rate around $2/3$ obtained with the quasi-uniform refinement. In addition, in both tables we observe that the effectivity index remains bounded. The behaviour of the convergence history described in Table 5.2 is illustrated next in Figure 3. Finally, in Figure 2 we display intermediate meshes obtained with the refinement procedure described above. As noticed there, the adaptive algorithm is able to recognize a neighborhood of the singular point $(0,0)$.

In our last example we assess, on the one hand, the performance of our Galerkin scheme for critical values of ν, and on the other hand, the capability of our adaptive algorithm to capturing

N	h	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(u)$	$r(u)$	$\mathsf{e}(\boldsymbol{\sigma}, u)$	$r(\boldsymbol{\sigma}, u)$
120	0.7071	1.7448	–	1.3280	–	2.1927	–
864	0.3536	0.9059	0.9457	0.6893	0.9460	1.1383	0.9458
6528	0.1768	0.4577	0.9850	0.3479	0.9865	0.5749	0.9856
50688	0.0884	0.2295	0.9960	0.1744	0.9966	0.2882	0.9962
399360	0.0442	0.1148	0.9989	0.0872	0.9992	0.1442	0.9990

Table 5.1: EXAMPLE 1: convergence history for the RT_0 approximation of the three dimensional version of the convection-diffusion-reaction problem (2.7) under a quasi-uniform refinement.

the presence of boundary layers. Here, we consider the domain $\Omega = (0,1)^2$, the vector field $\boldsymbol{\beta} = (0,1)$, the parameters $\alpha = 1$ and $\nu = 0.001$, and the exact solution to (1.1) given by

$$u(x_1, x_2) = 0.5\tanh((x_1 - 0.5)/p) + 0.5,$$

with $p = 0.02$. Notice that this solution presents a boundary layer at $x_1 = 0.5$. In Table 5.3 we summarize the convergence history obtained for this example, considering a RT_0 approximation. There we can see that, due to the presence of the aforementioned boundary layer, the method provides bad results for the first meshes, but thanks to the adaptive algorithm, the convergence of the method rapidly attains the desired $O(h)$ and the effectivity index is stabilized. The good performance of our adaptive algorithm can be also seen in Figure 4 where we display (on the left) the approximation of u computed by the post-processing formula (3.30) with 2563846 degrees of freedom, and the adapted mesh with 432048 degrees of freedom (on the right). In the first one we can see that, in spite of the presence of the boundary layer, the post-processing formula provides an accurate approximation of u, and in the second one we observe that the adaptive algorithm successfully captures the high gradients of the solution at $x = 0.5$.

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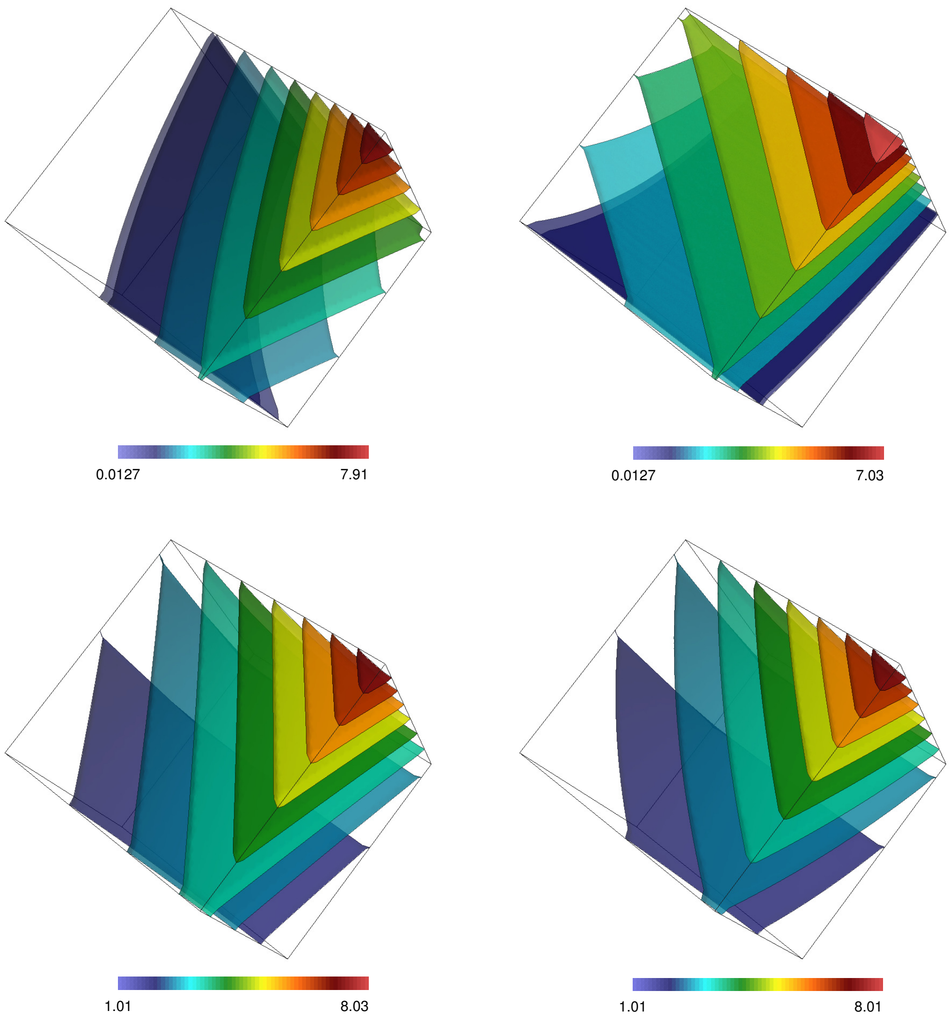


Figure 1: Example 1: Iso-surfaces of the approximated components of $\boldsymbol{\sigma}$ (1st component, top left; 2nd component, top right; 3rd component, bottom left), and postprocessed u_h (bottom right).

RT$_0$ SCHEME WITH QUASI-UNIFORM REFINEMENT

N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(u)$	$r(u)$	$\mathsf{e}(\boldsymbol{\sigma},u)$	$r(\boldsymbol{\sigma},u)$	Θ	eff
224	0.1682	–	0.0868	–	0.1893	–	1.0690	0.1771
856	0.1122	0.6036	0.0527	0.7449	0.1240	0.6312	0.6538	0.1896
3359	0.0698	0.6942	0.0359	0.5610	0.0785	0.6683	0.4388	0.1789
13288	0.0457	0.6161	0.0216	0.7408	0.0505	0.6405	0.2681	0.1885
52881	0.0282	0.7013	0.0145	0.5734	0.0317	0.6763	0.1756	0.1805
210894	0.0178	0.6669	0.0092	0.6638	0.0200	0.6663	0.1142	0.1750

RT$_0$ SCHEME WITH ADAPTIVE REFINEMENT

N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(u)$	$r(u)$	$\mathsf{e}(\boldsymbol{\sigma},u)$	$r(\boldsymbol{\sigma},u)$	Θ	eff
224	0.1682	–	0.0868	–	0.1893	–	1.0690	0.1771
413	0.0993	1.7208	0.0521	1.6721	0.1122	1.7104	0.5977	0.1877
756	0.0635	1.4829	0.0350	1.3089	0.0725	1.4439	0.3803	0.1906
1447	0.0436	1.1572	0.0244	1.1170	0.0499	1.1477	0.2601	0.1920
2939	0.0301	1.0462	0.0171	1.0083	0.0346	1.0371	0.1788	0.1934
6073	0.0206	1.0479	0.0117	1.0361	0.0237	1.0450	0.1235	0.1918
12447	0.0144	0.9895	0.0082	1.0002	0.0166	0.9921	0.0857	0.1934
25446	0.0100	1.0347	0.0057	0.9983	0.0115	1.0258	0.0600	0.1916
51788	0.0070	0.9844	0.0040	1.0078	0.0081	0.9902	0.0419	0.1930

Table 5.2: EXAMPLE 2: convergence history and effectivity index for the RT$_0$ approximation of the two dimensional version of the convection-diffusion-reaction problem (2.7) under quasi-uniform and adaptive refinements.

N	$\mathsf{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathsf{e}(u)$	$r(u)$	$\mathsf{e}(\boldsymbol{\sigma},u)$	$r(\boldsymbol{\sigma},u)$	Θ	eff
344	12.9896	–	3.8016	–	13.5345	–	40.9555	0.3305
937	130.5260	–	18.7232	–	131.8621	–	197.1978	0.6687
1796	35.3616	4.0097	1.9558	6.9438	35.4157	4.0410	18.4105	1.9237
4222	7.2989	3.6862	0.5177	3.1100	7.3172	3.6898	5.7747	1.2671
11559	3.4620	1.4795	0.1416	2.5751	3.4649	1.4845	1.7219	2.0123
31951	1.8537	1.2275	0.0678	1.4489	1.8549	1.2291	0.8331	2.2264
75216	1.2569	0.9063	0.0421	1.1103	1.2576	0.9078	0.5275	2.3842
182381	0.7989	1.0220	0.0263	1.0654	0.7993	1.0233	0.3290	2.4293
432048	0.5250	0.9723	0.0168	1.0319	0.5253	0.9735	0.2122	2.4752
1032102	0.3416	0.9862	0.0110	0.9861	0.3417	0.9874	0.1376	2.4842
2563846	0.2183	0.9825	0.0070	0.9962	0.2184	0.9836	0.0879	2.4841

Table 5.3: EXAMPLE 3: convergence history and effectivity index for the RT$_0$ approximation of the two dimensional version of the convection-diffusion-reaction problem (2.7) under adaptive refinement.

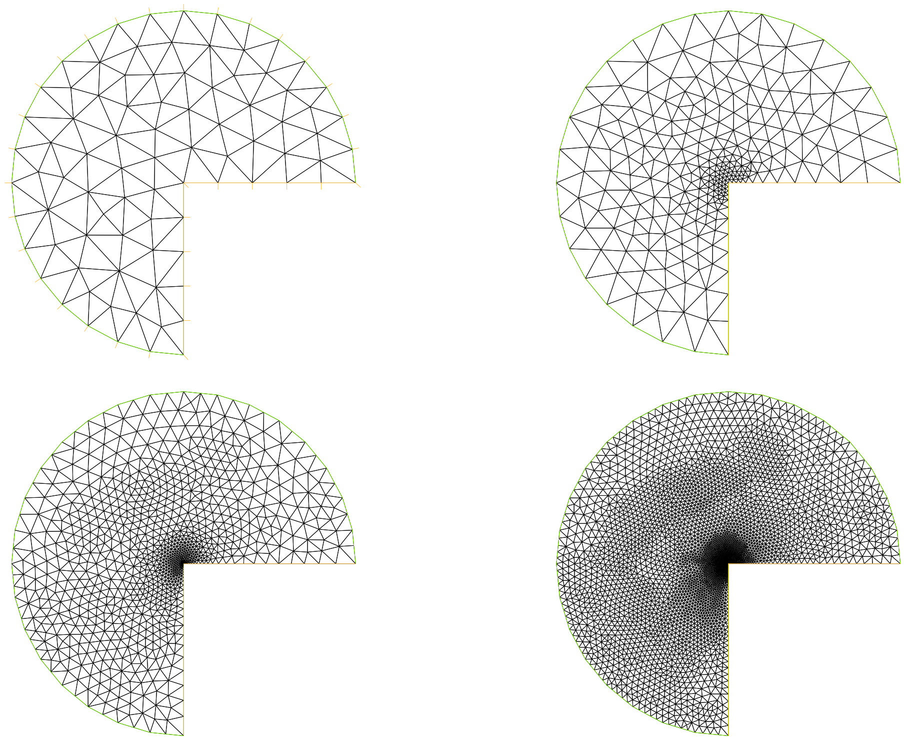


Figure 2: Example 2: four snapshots of successively refined meshes according to the indicator Θ.

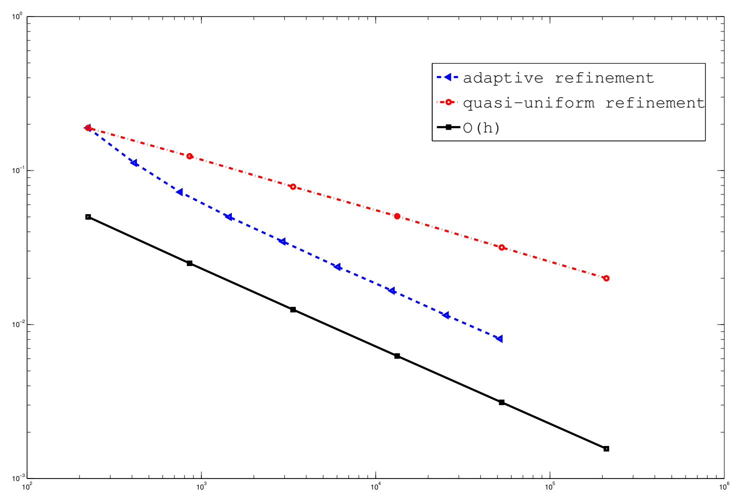
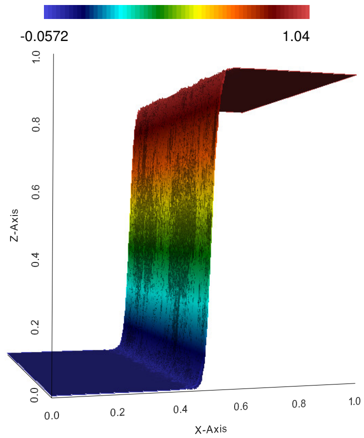


Figure 3: Example 2: log-log plot of the total errors vs. degrees of freedom associated to uniform and adaptive mesh refinements.



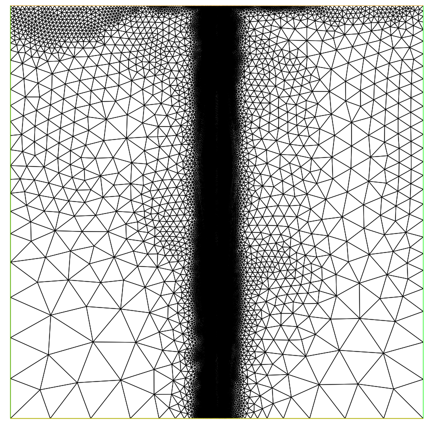


Figure 4: Example 3: postprocessed u_h (left) and refined mesh (right) according to the indicator Θ .

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