A note on stable Helmholtz decompositions in 3D*

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Abstract

The stability of Helmholtz decompositions in 3D is known to hold for convex polyhedral regions and for arbitrary (not necessarily convex) domains of class $C^{1,1}$. In this note we extend this result to non-convex polyhedral regions and to the case of homogeneous Neumann boundary conditions on a part of the boundary that is contained in the boundary of a convex extension of the original region. Some implications on the associated discrete Helmholtz decomposition and its application to the derivation of a posteriori error estimates, are also discussed.

Key words: Helmholtz decomposition, a posteriori error analysis

Mathematics subject classifications (2010): 65N15, 65N30, 35J20, 35J25, 35Q74, 65J10

1 Introduction

Given a domain $\mathcal{O}$ in R^3, we first introduce the well known Hilbert spaces

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \Big\{ \boldsymbol{v} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}(\boldsymbol{v}) \in \mathrm{L}^2(\mathcal{O}) \Big\}$$

and

$$\mathbb{H}(\mathbf{div};\mathcal{O}) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\mathcal{O}) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O}) \Big\},$$

where div denotes the distributional divergence operator acting on a vector field $\boldsymbol{v}$, **div** stands for the action of div along each row of a tensor field $\boldsymbol{\tau}$, and the spaces $\mathbf{L}^2(\mathcal{O})$ and $\mathbb{L}^2(\mathcal{O})$ correspond to the vector and tensorial versions of $\mathrm{L}^2(\mathcal{O})$, that is $\mathbf{L}^2(\mathcal{O}) := [\mathrm{L}^2(\mathcal{O})]^3$, and $\mathbb{L}^2(\mathcal{O}) := [\mathrm{L}^2(\mathcal{O})]^{3\times 3}$. Then, we say that $\mathbb{H}(\mathbf{div};\mathcal{O})$ admits a stable Helmholtz decomposition if there exist bounded linear operators $A : \mathbb{H}(\mathbf{div};\mathcal{O}) \to \mathbf{H}^2(\mathcal{O})$ and $B : \mathbb{H}(\mathbf{div};\mathcal{O}) \to \mathbb{H}^1(\mathcal{O})$, with $\mathbf{H}^m(\mathcal{O}) := [\mathrm{H}^m(\mathcal{O})]^3$ and $\mathbb{H}^m(\mathcal{O}) := [\mathrm{H}^m(\mathcal{O})]^{3\times 3}$ for each integer $m \geq 1$, such that there holds

$$\boldsymbol{\tau} = \nabla\big(A(\boldsymbol{\tau})\big) + \underline{\mathbf{curl}}\big(B(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O}), \tag{1.1}$$

where ∇ and $\underline{\mathbf{curl}}$ denote the distributional gradient and curl operators acting on each component of a vector and along each row of a tensor, respectively. The latter means that, given $\boldsymbol{v} := (v_j)_{j=1}^3 \in \mathbf{L}^2(\mathcal{O})$ and $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1}^3 \in \mathbb{L}^2(\mathcal{O})$, we define the distribution $\mathrm{curl}(\boldsymbol{v})$ as

$$\langle \mathrm{curl}(\boldsymbol{v}), \boldsymbol{\varphi} \rangle := \int_{\mathcal{O}} \boldsymbol{v} \cdot \mathrm{curl}(\boldsymbol{\varphi}) \qquad \forall\, \boldsymbol{\varphi} := (\varphi_j) \in \mathcal{D}(\mathcal{O})^3,$$

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with

$$\mathrm{curl}(\boldsymbol{\varphi}) \;=\; \nabla\times\boldsymbol{\varphi} \;:=\; \left(\frac{\partial\varphi_3}{\partial x_2}-\frac{\partial\varphi_2}{\partial x_3}\,,\;\frac{\partial\varphi_1}{\partial x_3}-\frac{\partial\varphi_3}{\partial x_1}\,,\;\frac{\partial\varphi_2}{\partial x_1}-\frac{\partial\varphi_1}{\partial x_2}\right),$$

and

$$\underline{\mathbf{curl}}(\boldsymbol{\tau}) \;:=\; \begin{pmatrix} \mathrm{curl}(\tau_{11},\tau_{12},\tau_{13}) \\ \mathrm{curl}(\tau_{21},\tau_{22},\tau_{23}) \\ \mathrm{curl}(\tau_{31},\tau_{32},\tau_{33}) \end{pmatrix}.$$

The above suggests to introduce the Hilbert spaces

$$\mathbf{H}(\mathrm{curl};\mathcal{O}) \;:=\; \Big\{\boldsymbol{v}\in\mathbf{L}^2(\mathcal{O}):\quad \mathrm{curl}(\boldsymbol{v})\in\mathbf{L}^2(\mathcal{O})\Big\}$$

and

$$\mathbb{H}(\underline{\mathbf{curl}};\mathcal{O}) \;:=\; \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\mathcal{O}):\quad \underline{\mathbf{curl}}(\boldsymbol{\tau})\in\mathbb{L}^2(\mathcal{O})\Big\}.$$

The norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$, $\mathbb{H}(\mathbf{div};\mathcal{O})$, $\mathbf{H}(\mathrm{curl};\mathcal{O})$, and $\mathbb{H}(\underline{\mathbf{curl}};\mathcal{O})$ are then denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$, $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, $\|\cdot\|_{\mathrm{curl};\mathcal{O}}$, and $\|\cdot\|_{\underline{\mathbf{curl}};\mathcal{O}}$, respectively. In addition, in what follows we use $\|\cdot\|_{m,\mathcal{O}}$ and $|\cdot|_{m,\mathcal{O}}$ to identify the norm and seminorm, respectively, of $\mathrm{H}^m(\mathcal{O})$ and its vector and tensorial versions given by $\mathbf{H}^m(\mathcal{O})$ and $\mathbb{H}^m(\mathcal{O})$, for any integer $m\geq 0$. Note, however, that when $m=0$ we usually write $\mathbf{L}^2(\mathcal{O})$ and $\mathbb{L}^2(\mathcal{O})$ instead of $\mathbf{H}^0(\mathcal{O})$ and $\mathbb{H}^0(\mathcal{O})$, respectively.

Now, with regards to an eventual stable Helmholtz decomposition for $\mathbb{H}(\mathbf{div};\mathcal{O})$, we first recall that once one finds a bounded operator A for which $\boldsymbol{\tau}-\nabla\big(A(\boldsymbol{\tau})\big)$ is divergence-free (usually through an auxiliary boundary value problem), the existence of an operator B completing the verification of (1.1) only is already well established in 2D and 3D (see, e.g. [9, Chapter I, Theorems 3.1 and 3.4]). In turn, the required boundedness of B is always guaranteed in 2D thanks to the fact that the corresponding operator $\underline{\mathbf{curl}}$ satisfies $\|\underline{\mathbf{curl}}\,\chi\|_{0,\mathcal{O}} = |\chi|_{1,\mathcal{O}}\;\;\forall\,\chi\in\mathrm{H}^1(\mathcal{O})$, whereas in the 3D case, in which the foregoing identity does not hold, additional geometric or regularity properties of the domain are needed to arrive at the same conclusion. In particular, the stability of a Helmholtz decomposition for $\mathbb{H}(\mathbf{div};\mathcal{O})$ has already been established for convex polyhedral regions in 3D (see, e.g. [10, Proposition 4.52]) by using [2, Theorems 2.17 and 3.12]. Moreover, the latter work by Amrouche et al. is actually a classical reference providing existence, uniqueness and regularity results concerning vector potentials associated with a divergence-free function in a bounded three-dimensional domain. Indeed, another consequence that follows also from [2] refers to the aforementioned stability for the case of arbitrary (not necessarily convex) domains of class $C^{1,1}$. In the present paper we make further use of some results from [2] to extend [10, Proposition 4.52] to non-convex polyhedral regions in 3D and to a special case of homogeneous Neumann boundary conditions on a part of the boundary. The rest of this work is organized as follows. In Section 2 we collect some preliminary results from [2]. Next, in Section 3 we provide our main contributions, and finally, the implications on the associated discrete Helmholtz decomposition and its application to a posteriori error analysis, are discussed in Section 4. Throughout the paper we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants that may take different values at different places.

2 Preliminary results

We begin with some additional notations and definitions. Throughout this section, $\mathcal{O}$ is a bounded and simply-connected polyhedral domain in $\mathbf{R}^3$ with boundary $\partial\mathcal{O}$, which implies, in particular, that there exists a unit exterior normal vector $\boldsymbol{\nu}$ almost everywhere on $\partial\mathcal{O}$. In this regard, we recall that there holds $\boldsymbol{\tau}\cdot\boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\partial\mathcal{O})$ (resp. $\boldsymbol{\tau}\boldsymbol{\nu} \in \mathbf{H}^{-1/2}(\partial\mathcal{O})$) $\quad \forall\,\boldsymbol{\tau} \in \mathbf{H}(\mathbf{div};\mathcal{O})$ (resp. $\forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$). Then, following closely [2], we introduce the Hilbert spaces:

$$\begin{aligned}
\mathbf{X}(\mathcal{O}) \,&:=\, \mathbf{H}(\mathrm{div};\mathcal{O}) \cap \mathbf{H}(\mathrm{curl};\mathcal{O})\,, \\
\mathbf{H}_0(\mathrm{curl};\mathcal{O}) \,&:=\, \Big\{ \boldsymbol{v} \in \mathbf{H}(\mathrm{curl};\mathcal{O}) : \quad \boldsymbol{v}\times\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O} \Big\}\,, \\
\mathbf{H}_0(\mathrm{div};\mathcal{O}) \,&:=\, \Big\{ \boldsymbol{v} \in \mathbf{H}(\mathrm{div};\mathcal{O}) : \quad \boldsymbol{v}\cdot\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O} \Big\}\,, \\
\mathbf{X}_{\mathbf{n}}(\mathcal{O}) \,&:=\, \mathbf{H}(\mathrm{div};\mathcal{O}) \cap \mathbf{H}_0(\mathrm{curl};\mathcal{O})\,, \\
\mathbf{X}_{\mathbf{t}}(\mathcal{O}) \,&:=\, \mathbf{H}_0(\mathrm{div};\mathcal{O}) \cap \mathbf{H}(\mathrm{curl};\mathcal{O})\,,
\end{aligned} \tag{2.1}$$

and its corresponding tensorial versions:

$$\begin{aligned}
\mathbb{X}(\mathcal{O}) \,&:=\, \mathbb{H}(\mathbf{div};\mathcal{O}) \cap \mathbb{H}(\underline{\mathbf{curl}};\mathcal{O})\,, \\
\mathbb{H}_0(\underline{\mathbf{curl}};\mathcal{O}) \,&:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\underline{\mathbf{curl}};\mathcal{O}) : \quad \big(\tau_{i1},\tau_{i2},\tau_{i3}\big)\times\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\,, \quad \forall\, i \in \{1,2,3\} \Big\}\,, \\
\mathbb{H}_0(\mathbf{div};\mathcal{O}) \,&:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O}) : \quad \boldsymbol{\tau}\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O} \Big\}\,, \\
\mathbb{X}_{\mathbf{n}}(\mathcal{O}) \,&:=\, \mathbb{H}(\mathbf{div};\mathcal{O}) \cap \mathbb{H}_0(\underline{\mathbf{curl}};\mathcal{O})\,, \\
\mathbb{X}_{\mathbf{t}}(\mathcal{O}) \,&:=\, \mathbb{H}_0(\mathbf{div};\mathcal{O}) \cap \mathbb{H}(\underline{\mathbf{curl}};\mathcal{O})\,.
\end{aligned} \tag{2.2}$$

The spaces $\mathbf{X}(\mathcal{O})$, $\mathbf{X}_{\mathbf{n}}(\mathcal{O})$, and $\mathbf{X}_{\mathbf{t}}(\mathcal{O})$ are all provided with the natural norm

$$\|\boldsymbol{v}\|_{\mathbf{X}(\mathcal{O})} \,:=\, \Big\{ \|\boldsymbol{v}\|_{0,\mathcal{O}}^2 \,+\, \|\mathrm{div}\,\boldsymbol{v}\|_{0,\mathcal{O}}^2 \,+\, \|\mathrm{curl}\,\boldsymbol{v}\|_{0,\mathcal{O}}^2 \Big\}^{1/2} \qquad \forall\,\boldsymbol{v} \in \mathbf{X}(\mathcal{O})\,,$$

whereas $\mathbb{X}(\mathcal{O})$, $\mathbb{X}_{\mathbf{n}}(\mathcal{O})$, and $\mathbb{X}_{\mathbf{t}}(\mathcal{O})$ are endowed analogously with

$$\|\boldsymbol{\tau}\|_{\mathbb{X}(\mathcal{O})} \,:=\, \Big\{ \|\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 \,+\, \|\underline{\mathbf{curl}}\,\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 \Big\}^{1/2} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{X}(\mathcal{O})\,. \tag{2.3}$$

In turn, the seminorm of $\mathbb{X}(\mathcal{O})$ is defined by

$$|\boldsymbol{\tau}|_{\mathbb{X}(\mathcal{O})} \,:=\, \Big\{ \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 \,+\, \|\underline{\mathbf{curl}}\,\boldsymbol{\tau}\|_{0,\mathcal{O}}^2 \Big\}^{1/2} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{X}(\mathcal{O})\,. \tag{2.4}$$

We now collect the tensorial versions of some results provided in [2].

Lemma 2.1 *Assume that $\mathcal{O}$ is convex. Then the spaces $\mathbb{X}_{\mathbf{n}}(\mathcal{O})$ and $\mathbb{X}_{\mathbf{t}}(\mathcal{O})$ are both continuously imbedded in $\mathbb{H}^1(\mathcal{O})$, that is, there exist positive constants $C_{\mathbf{n}}(\mathcal{O})$ and $C_{\mathbf{t}}(\mathcal{O})$, such that*

$$\|\boldsymbol{\tau}\|_{1,\mathcal{O}} \,\le\, C_{\mathbf{n}}(\mathcal{O})\,\|\boldsymbol{\tau}\|_{\mathbb{X}(\mathcal{O})} \qquad \forall\,\boldsymbol{\tau} \,\in\, \mathbb{X}_{\mathbf{n}}(\mathcal{O}) \tag{2.5}$$

and

$$\|\boldsymbol{\tau}\|_{1,\mathcal{O}} \,\le\, C_{\mathbf{t}}(\mathcal{O})\,\|\boldsymbol{\tau}\|_{\mathbb{X}(\mathcal{O})} \qquad \forall\,\boldsymbol{\tau} \,\in\, \mathbb{X}_{\mathbf{t}}(\mathcal{O})\,. \tag{2.6}$$

Proof. See [2, Theorem 2.17]. □

Lemma 2.2 *Let $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$. Then $\boldsymbol{\tau}$ is divergence-free in $\mathcal{O}$ if and only if there exists a unique $\boldsymbol{\psi} \in \mathbb{X}_{\mathbf{t}}(\mathcal{O})$ such that $\mathbf{div}\,\boldsymbol{\psi} = \mathbf{0}$ and $\boldsymbol{\tau} = \underline{\mathbf{curl}}\,\boldsymbol{\psi}$ in $\mathcal{O}$.*

Proof. See [2, Theorem 3.12]. □

Lemma 2.3 *The seminorm $|\cdot|_{\mathbb{X}(\mathcal{O})}$ (cf. (2.4)) is equivalent to $\|\cdot\|_{\mathbb{X}(\mathcal{O})}$ (cf. (2.3)) in $\mathbb{X}_{\mathbf{t}}(\mathcal{O})$, that is, there exists a positive constant $c_{\mathbf{t}}(\mathcal{O})$, such that*

$$\|\boldsymbol{\tau}\|_{\mathbb{X}(\mathcal{O})} \,\leq\, c_{\mathbf{t}}(\mathcal{O})\,|\boldsymbol{\tau}|_{\mathbb{X}(\mathcal{O})} \qquad \forall\,\boldsymbol{\tau}\,\in\,\mathbb{X}_{\mathbf{t}}(\mathcal{O})\,. \tag{2.7}$$

Proof. See [2, Corollary 3.16]. □

Although it was already mentioned in the Introduction, we emphasize here that in the case of a convex polyhedral region $\mathcal{O}$ in 3D, the stability of the Helmholtz decomposition for $\mathbb{H}(\mathbf{div};\mathcal{O})$ (cf. [10, Proposition 4.52]) follows straightforwardly from Lemmata 2.1 and 2.2.

3 Main results

We begin with the extension of [10, Proposition 4.52] to the case of a non-convex region. This result was already established and used for the first time in [7, Lemma 4.3] and [7, Section 4.2], respectively. However, we recall it next for sake of completeness and because similar arguments to those utilized in its proof will be employed for the second extension to be presented below. In what follows, Ω is a bounded and simply-connected polyhedral domain in $\mathbf{R}^3$ with boundary Γ.

Theorem 3.1 *There exist bounded linear operators $A:\mathbb{H}(\mathbf{div};\Omega)\to\mathbf{H}^2(\Omega)$ and $B:\mathbb{H}(\mathbf{div};\Omega)\to\mathbb{H}^1(\Omega)$, such that there holds*

$$\boldsymbol{\tau} \,=\, \nabla\big(A(\boldsymbol{\tau})\big) \,+\, \underline{\mathbf{curl}}\big(B(\boldsymbol{\tau})\big) \qquad \forall\,\boldsymbol{\tau}\,\in\,\mathbb{H}(\mathbf{div};\Omega)\,. \tag{3.1}$$

Equivalently, given $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$ there exist $\mathbf{z}:=A(\boldsymbol{\tau})\in\mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi}:=B(\boldsymbol{\tau})\in\mathbb{H}^1(\Omega)$ such that

$$\boldsymbol{\tau} \,=\, \nabla\mathbf{z} \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi}) \quad \text{in} \quad \Omega \quad \text{and} \quad \|\mathbf{z}\|_{2,\Omega}+\|\boldsymbol{\chi}\|_{1,\Omega} \,\leq\, c\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,, \tag{3.2}$$

where c is a positive constant independent of all the foregoing variables.

Proof. Since Ω is not assumed to be convex, we proceed as in the second part of the proof of [8, Lemma 3.3] by extending each tensor of $\mathbb{H}(\mathbf{div};\Omega)$ to a convex region containing the closure of Ω. More precisely, let Θ be a sufficiently large convex domain such that $\bar\Omega\subset\Theta$, and let $G:=\Theta\setminus\bar\Omega$ be the annular region with boundary $\partial G:=\Gamma\,\cup\,\partial\,\Theta$. Then, given $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$, we let $\boldsymbol{w}\in\mathbf{H}^1(G)$ be the unique solution (guaranteed by the Lax-Milgram Theorem) of the mixed boundary value problem:

$$\Delta\boldsymbol{w} \,=\, \mathbf{0} \quad \text{in} \quad G\,, \quad \nabla\boldsymbol{w}\,\boldsymbol{\nu} \,=\, \boldsymbol{\tau}\,\boldsymbol{\nu} \quad \text{on} \quad \Gamma\,, \quad \boldsymbol{w}=\mathbf{0} \quad \text{on} \quad \partial\,\Theta\,, \tag{3.3}$$

where $\boldsymbol{\nu}$ stands here for the inward (resp. outward) unit normal on Γ (resp. on $\partial\Theta$). It is clear that

$$\|\boldsymbol{w}\|_{1,G} \,\leq\, \widetilde{c}\,\|\boldsymbol{\tau}\,\boldsymbol{\nu}\|_{-1/2,\Gamma} \,\leq\, \widetilde{c}\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,, \tag{3.4}$$

with a constant $\widetilde{c}>0$, independent of $\boldsymbol{\tau}$. Next, we define $\widetilde{\boldsymbol{\tau}} \,:=\, \begin{cases} \boldsymbol{\tau} & \text{in} \quad \Omega \\ \nabla\boldsymbol{w} & \text{in} \quad G \end{cases}$, and observe, according to (3.3) and (3.4), that $\widetilde{\boldsymbol{\tau}}\in\mathbb{H}(\mathbf{div};\Theta)$ and

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div};\Theta} \,\leq\, \Big\{1+\widetilde{c}^{2}\Big\}^{1/2}\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,. \tag{3.5}$$

Hence, applying the Helmholtz decomposition provided by [10, Proposition 4.52] to Θ and $\widetilde{\boldsymbol{\tau}} \in \mathbb{H}(\mathbf{div};\Theta)$, we deduce that there exist $\widetilde{\mathbf{z}} \in \mathbf{H}^2(\Theta)$ and $\widetilde{\boldsymbol{\chi}} := \begin{pmatrix} \widetilde{\boldsymbol{\chi}}_1 \\ \widetilde{\boldsymbol{\chi}}_2 \\ \widetilde{\boldsymbol{\chi}}_3 \end{pmatrix} \in \mathbb{H}^1(\Theta)$, with $\widetilde{\boldsymbol{\chi}}_i := (\widetilde{\chi}_{i1}, \widetilde{\chi}_{i2}, \widetilde{\chi}_{i3})^{\mathtt{t}} \in \mathbf{H}^1(\Theta)$, $i \in \{1,2,3\}$, such that

$$\widetilde{\boldsymbol{\tau}} \,=\, \nabla\widetilde{\mathbf{z}} \,+\, \underline{\mathbf{curl}}(\widetilde{\boldsymbol{\chi}}) \quad \text{in} \quad \Theta \tag{3.6}$$

and

$$\|\widetilde{\mathbf{z}}\|_{2,\Theta} + \|\widetilde{\boldsymbol{\chi}}\|_{1,\Theta} \,\le\, C\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div};\Theta} \,\le\, \widetilde{C}\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,, \tag{3.7}$$

where the last inequality in (3.7) follows from (3.5), thus yielding $\widetilde{C} \,=\, C\left\{1+\widetilde{c}^2\right\}^{1/2}$. Then, defining $\mathbf{z} = A(\boldsymbol{\tau}) := \widetilde{\mathbf{z}}|_\Omega \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi} = B(\boldsymbol{\tau}) := \widetilde{\boldsymbol{\chi}}|_\Omega \in \mathbb{H}^1(\Omega)$, noting that certainly $\widetilde{\boldsymbol{\tau}}|_\Omega = \boldsymbol{\tau}$, and employing (3.6) - (3.7), we arrive at (3.2). Finally, since the extension $\widetilde{\boldsymbol{\tau}}$ and the mappings yielding $\widetilde{\mathbf{z}} \in \mathbf{H}^2(\Theta)$ and $\widetilde{\boldsymbol{\chi}} \in \mathbb{H}^1(\Theta)$ are all linear, we conclude the linearity of our implicit operators A and B, which ends the proof. □

We continue our analysis with the stability of the Helmholtz decomposition for $\mathbb{H}(\mathbf{div};\Omega)$ in 3D when a particular case of Neumann boundary conditions is considered. More precisely, we now let Γ_{N} be a part of Γ such that $\bar{\Gamma}_{\mathrm{N}} \subset \Gamma$ and $|\Gamma_{\mathrm{N}}| > 0$, and introduce the spaces

$$\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau}\boldsymbol{\nu} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{N}}\Big\}, \tag{3.8}$$

and

$$\mathbb{H}^1_{\mathrm{N}}(\Omega) \,:=\, \Big\{\boldsymbol{\chi} \in \mathbb{H}^1(\Omega): \quad \boldsymbol{\chi}|_{\Gamma_{\mathrm{N}}} \,\in\, \mathbb{P}_0(\Gamma_{\mathrm{N}})\Big\}, \tag{3.9}$$

where $\mathbb{P}_0(\Gamma_{\mathrm{N}})$ stands for the constant tensors on Γ_{N}.

Then, we establish next the stability of the Helmholtz decomposition for $\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ when Γ_{N} is contained in the boundary of a convex extension of Ω. We remark that the 2D version of this result, which, coherently with the comments in the Introduction, makes use of the identity $\|\underline{\mathbf{curl}}\,\chi\|_{0,\Omega} = |\chi|_{1,\Omega} \;\; \forall\,\chi \in \mathrm{H}^1(\Omega)$, was recently proved in [1, Lemma 3.9]. In turn, the proof of the following theorem, which provides the aforementioned stability in 3D, combines similar arguments to those from the proofs of Theorem 3.1 and [1, Lemma 3.9], and apply the preliminary results collected in Section 2.

Theorem 3.2 *Assume that there exists a convex domain Ξ such that $\bar{\Omega} \subseteq \Xi$ and $\Gamma_{\mathrm{N}} \subseteq \partial\Xi$. Then there exist bounded linear operators $A: \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \to \mathbf{H}^2(\Omega)$ and $B: \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \to \mathbb{H}^1_{\mathrm{N}}(\Omega)$, such that there holds*

$$\boldsymbol{\tau} \,=\, \nabla\big(A(\boldsymbol{\tau})\big) \,+\, \underline{\mathbf{curl}}\big(B(\boldsymbol{\tau})\big) \qquad \forall\,\boldsymbol{\tau} \,\in\, \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)\,. \tag{3.10}$$

Equivalently, given $\boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ there exist $\mathbf{z} := A(\boldsymbol{\tau}) \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi} := B(\boldsymbol{\tau}) \in \mathbb{H}^1_{\mathrm{N}}(\Omega)$ such that

$$\boldsymbol{\tau} \,=\, \nabla\mathbf{z} \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi}) \quad \text{in} \quad \Omega \quad \text{and} \quad \|\mathbf{z}\|_{2,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \,\le\, c\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,, \tag{3.11}$$

where c is a positive constant independent of all the foregoing variables.

Proof. We begin with a suitable extension of each tensor of $\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ to an intermediate convex domain Θ satisfying $\Omega \subseteq \Theta \subseteq \Xi$, $\Gamma_{\mathrm{N}} \subseteq \partial\Theta$, and $|\Xi\backslash\bar{\Theta}| > 0$ (see Figure 3.1 for a 2D illustration of the corresponding geometry). Note that the existence of such a Θ is guaranteed by the conditions required on Ω, Ξ and Γ_{N}. Then, given $\boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$, we proceed similarly as in the first part of the proof of Theorem 3.1 to find $\widetilde{\boldsymbol{\tau}} \in \mathbb{H}(\mathbf{div};\Theta)$ such that $\widetilde{\boldsymbol{\tau}}|_\Omega \,=\, \boldsymbol{\tau}$ and

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div};\Theta} \,\le\, \widetilde{C}\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,, \tag{3.12}$$

with a positive constant $\widetilde{C}$ independent of $\boldsymbol{\tau}$ and $\widetilde{\boldsymbol{\tau}}$. In fact, denoting $G := \Theta\backslash\bar{\Omega}$ and splitting $\partial G = \bar{\Sigma} \cup \bar{S}$, with $\Sigma := \partial G \cap \partial\Omega$ and $S := \partial G\backslash\bar{\Sigma}$, it suffices to set $\widetilde{\boldsymbol{\tau}} := \begin{cases} \boldsymbol{\tau} & \text{in} \quad \Omega \\ \nabla\widetilde{\boldsymbol{w}} & \text{in} \quad G \end{cases}$, where $\widetilde{\boldsymbol{w}} \in \mathbf{H}^1(G)$ is the unique solution of the mixed boundary value problem (see again Figure 3.1 for a 2D version of the geometry):

$$\Delta\widetilde{\boldsymbol{w}} = \mathbf{0} \quad \text{in} \quad G, \quad \nabla\widetilde{\boldsymbol{w}}\,\boldsymbol{\nu} = \boldsymbol{\tau}\,\boldsymbol{\nu} \quad \text{on} \quad \Sigma, \quad \boldsymbol{w} = \mathbf{0} \quad \text{on} \quad S\,. \tag{3.13}$$

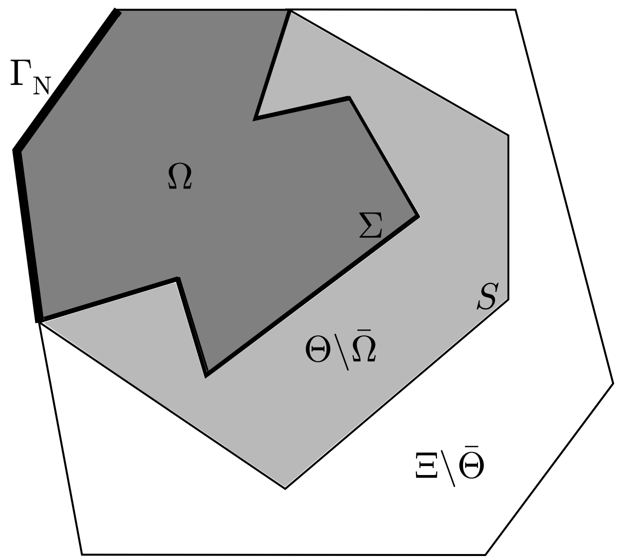

Figure 3.1: 2D illustration of the geometry for the proof of Theorem 3.2.

Next, we define $\widetilde{\boldsymbol{f}} \in \mathbf{L}^2(\Xi)$ by

$$\widetilde{\boldsymbol{f}} := \begin{cases} \mathbf{div}\,\widetilde{\boldsymbol{\tau}} & \text{in} \quad \Theta\,, \\ \dfrac{-1}{|\,\Xi\backslash\bar{\Theta}\,|}\displaystyle\int_\Theta \mathbf{div}\,\widetilde{\boldsymbol{\tau}} & \text{in} \quad \Xi\backslash\bar{\Theta}\,, \end{cases}$$

observe that

$$\|\widetilde{\boldsymbol{f}}\|_{0,\Xi} \le \widetilde{C}(\Xi,\Theta)\,\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Theta}\,, \tag{3.14}$$

where $\widetilde{C}(\Xi,\Theta)$ is a positive constant depending on $|\Theta|$ and $|\Xi\backslash\bar{\Theta}|$, and let $\boldsymbol{w} \in \mathbf{H}^1(\Xi)$ be the unique weak solution of the Neumann boundary value problem:

$$\Delta\boldsymbol{w} = \widetilde{\boldsymbol{f}} \quad \text{in} \quad \Xi, \qquad \nabla\boldsymbol{w}\,\boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \partial\,\Xi, \qquad \int_\Xi \boldsymbol{w} = \mathbf{0}\,. \tag{3.15}$$

It follows, thanks to the elliptic regularity result for (3.15) and the estimate (3.14), that there actually hold $\boldsymbol{w} \in \mathbf{H}^2(\Xi)$ and

$$\|\boldsymbol{w}\|_{2,\Xi} \le c\,\|\widetilde{\boldsymbol{f}}\|_{0,\Xi} \le c\,\widetilde{C}(\Xi,\Theta)\,\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Theta}\,. \tag{3.16}$$

Then, introducing $\boldsymbol{\zeta} := \nabla\boldsymbol{w}$ in Ξ, we deduce from the boundary condition in (3.15) that $\boldsymbol{\zeta} \in \mathbb{X}_{\mathbf{t}}(\Xi)$ (see definition of $\mathbb{X}_{\mathbf{t}}(\Xi)$ in (2.2)), which, applying (2.6) (cf. Lemma 2.1) and (2.7) (cf. Lemma 2.3), recalling the definition of the seminorm $|\cdot|_{\mathbb{X}(\Xi)}$ (cf. (2.4)), and using that $\mathbf{div}\,\boldsymbol{\zeta} = \Delta \boldsymbol{w} = \widetilde{\boldsymbol{f}}$ in Ξ, yields

$$\begin{aligned} \|\boldsymbol{\zeta}\|_{1,\Xi} \,&\le\, C_{\mathbf{t}}(\Xi)\,\|\boldsymbol{\zeta}\|_{\mathbb{X}(\Xi)} \,\le\, C_{\mathbf{t}}(\Xi)\,c_{\mathbf{t}}(\Xi)\,|\boldsymbol{\zeta}|_{\mathbb{X}(\Xi)} \,=\, C_{\mathbf{t}}(\Xi)\,c_{\mathbf{t}}(\Xi)\,\|\Delta\boldsymbol{w}\|_{0,\Xi} \\ &=\, C_{\mathbf{t}}(\Xi)\,c_{\mathbf{t}}(\Xi)\,\|\widetilde{\boldsymbol{f}}\|_{0,\Xi} \,\le\, C_{\mathbf{t}}(\Xi)\,c_{\mathbf{t}}(\Xi)\,\widetilde{C}(\Xi,\Theta)\,\|\mathbf{div}\,\widetilde{\boldsymbol{\tau}}\|_{0,\Theta}\,. \end{aligned} \tag{3.17}$$

In turn, setting $\widetilde{\boldsymbol{\zeta}} := \boldsymbol{\zeta}|_{\Theta}$, we readily see that $\mathbf{div}\big(\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\zeta}}\big) = \mathbf{0}$ in Θ, which, according to Lemma 2.2, implies the existence of a unique $\widetilde{\boldsymbol{\chi}} \in \mathbb{X}_{\mathbf{t}}(\Theta)$, such that $\mathbf{div}\,\widetilde{\boldsymbol{\chi}} = \mathbf{0}$ and $\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\zeta}} = \underline{\mathbf{curl}}\,\widetilde{\boldsymbol{\chi}}$ in Θ. In this way, applying now (2.6) (cf Lemma 2.1) and (2.7) (cf. Lemma 2.3) to $\widetilde{\boldsymbol{\chi}}$ and the convex set Θ, recalling again the definition of the seminorm $|\cdot|_{\mathbb{X}(\Xi)}$ (cf. (2.4)), and employing the bound (3.17), we find that

$$\begin{aligned} \|\widetilde{\boldsymbol{\chi}}\|_{1,\Theta} \,&\le\, C_{\mathbf{t}}(\Theta)\,\|\widetilde{\boldsymbol{\chi}}\|_{\mathbb{X}(\Theta)} \,\le\, C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\,|\widetilde{\boldsymbol{\chi}}|_{\mathbb{X}(\Theta)} \,=\, C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\,\|\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\zeta}}\|_{0,\Theta} \\ &\le\, C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\Big\{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Theta} + \|\boldsymbol{\zeta}\|_{0,\Theta}\Big\} \,\le\, C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\Big\{\|\widetilde{\boldsymbol{\tau}}\|_{0,\Theta} + \|\boldsymbol{\zeta}\|_{1,\Xi}\Big\} \\ &\le\, C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\,C(\Xi,\Theta)\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div};\Theta}\,, \end{aligned} \tag{3.18}$$

where $C(\Xi,\Theta) := \Big\{1 + \Big(C_{\mathbf{t}}(\Xi)\,c_{\mathbf{t}}(\Xi)\,\widetilde{C}(\Xi,\Theta)\Big)^2\Big\}^{1/2}$. Furthermore, restricting to Ω the identity $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\zeta}} + \underline{\mathbf{curl}}\,\widetilde{\boldsymbol{\chi}}$ in Θ, we arrive at $\boldsymbol{\tau} = \nabla\mathbf{z} + \underline{\mathbf{curl}}\,\boldsymbol{\chi}$ in Ω, where $\mathbf{z} = A(\boldsymbol{\tau}) := \boldsymbol{w}|_{\Omega} \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi} = B(\boldsymbol{\tau}) := \widetilde{\boldsymbol{\chi}}|_{\Omega} \in \mathbb{H}^1(\Omega)$. Moreover, since $\underline{\mathbf{curl}}\,\boldsymbol{\chi}\,\boldsymbol{\nu} = \big(\boldsymbol{\tau} - \nabla\mathbf{z}\big)\,\boldsymbol{\nu} = \big(\boldsymbol{\tau} - \nabla\boldsymbol{w}\big)\,\boldsymbol{\nu} = \mathbf{0}$ on Γ_{N}, we deduce that $\boldsymbol{\chi}|_{\Gamma_{\mathrm{N}}} \in \mathbb{P}_0(\Gamma_{\mathrm{N}})$, and hence $\boldsymbol{\chi} \in \mathbb{H}^1_{\mathrm{N}}(\Omega)$. Finally, utilizing the estimates (3.16) and (3.18), we obtain

$$\|\mathbf{z}\|_{2,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \,\le\, \|\boldsymbol{w}\|_{2,\Xi} + \|\widetilde{\boldsymbol{\chi}}\|_{1,\Theta} \,\le\, \Big\{c\,\widetilde{C}(\Xi,\Theta) + C_{\mathbf{t}}(\Theta)\,c_{\mathbf{t}}(\Theta)\,C(\Xi,\Theta)\Big\}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div};\Theta}\,,$$

which, thanks to (3.12), gives the stability estimate in (3.11) and completes the proof. □

We find it important to remark here that the introduction of the auxiliary intermediate convex region Θ is crucial for the foregoing proof. Otherwise, instead of $\|\widetilde{\boldsymbol{\chi}}\|_{1,\Theta}$ and the use of the fact that $\mathbb{X}_{\mathbf{t}}(\Theta)$ is continuously imbedded in $\mathbb{H}^1(\Theta)$ (cf. Lemma 2.1) to derive (3.18), we would have had to deal with $\|\widetilde{\boldsymbol{\chi}}\|_{1,\Omega}$, for which, due to the lack of convexity of Ω, such an imbedding result is unfortunately not available for this domain. This is exactly the reason why, previously to the introduction of the key boundary value problem (3.15) and the derivation of its consequences, we need to extend the given $\boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ to a tensor $\widetilde{\boldsymbol{\tau}} \in \mathbb{H}(\mathbf{div};\Theta)$. In other words, our proof of Theorem 3.2 can be summarized as a two-steps approach through the auxiliary boundary value problems (3.13) and (3.15).

4 Application to a posteriori error analysis

Along the derivation of residual-based a posteriori error estimates for mixed finite element methods in continuum mechanics and related areas, one is usually faced to the problem of estimating the norms of functionals $F \in H'$, where H is either the whole space $\mathbb{H}(\mathbf{div};\Omega)$ or a subspace of it. More precisely, these functional arise from the mixed variational formulation of the underlying boundary value problem and the application of the global continuous inf-sup condition to the total error, whereas the definition of H depends on the particular boundary conditions involved. In addition, thanks to the Galerkin "*orthogonality condition*", there usually holds $F(\boldsymbol{\tau}_h) = 0 \quad \forall\,\boldsymbol{\tau}_h \in H_h$, where H_h is the finite element subspace of H, and therefore, given any $\boldsymbol{\tau}_h \in H_h$, one can write

$$\|F\|_{H'} \,=\, \sup_{\boldsymbol{\tau}\in H\backslash\mathbf{0}} \frac{|F(\boldsymbol{\tau} - \boldsymbol{\tau}_h)|}{\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}}\,. \tag{4.1}$$

As a consequence, the aforementioned estimation problem reduces first to choose a suitable $\boldsymbol{\tau}_h \in H_h$, and then to obtain a residual expression $\boldsymbol{\eta}_F$, splitted into explicitly computable local quantities, such that

$$|F(\boldsymbol{\tau} - \boldsymbol{\tau}_h)| \;\le\; C\,\boldsymbol{\eta}_F\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau} \in H\,, \tag{4.2}$$

thus yielding

$$\|F\|_{H'} \;\le\; C\,\boldsymbol{\eta}_F\,.$$

Moreover, for sake of efficiency of the resulting global a posteriori error estimator, it is also very desirable that each one of the local residual quantities forming part of $\boldsymbol{\eta}_F$ be bounded by local or quasi-local expressions of the true error. In this regard, we remark that actually the suitability of $\boldsymbol{\tau}_h$ depends on whether one is able or not of obtaining such a $\boldsymbol{\eta}_F$ satisfying (4.2) and the foregoing efficiency issue, and this is exactly the place where the Helmholtz decomposition and its discrete version (to be introduced below) enter into play.

Indeed, let us first suppose that $H = \mathbb{H}(\mathbf{div};\Omega)$ (which occurs, for instance, when the Poisson problem with Dirichlet boundary conditions is under consideration) and that H_h is the usual tensorial Raviart-Thomas space of order $k \ge 0$. Then, given $\boldsymbol{\tau} \in H$ and its stable Helmholtz decomposition provided by Theorem 3.1, that is

$$\boldsymbol{\tau} \;=\; \nabla\mathbf{z} \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi}) \quad \text{in} \quad \Omega\,, \tag{4.3}$$

with $\mathbf{z} \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi} \in \mathbb{H}^1(\Omega)$, we introduce what we call its discrete Helmholtz decomposition as

$$\boldsymbol{\tau}_h \;:=\; \Pi_h^k(\nabla\mathbf{z}) \,+\, \underline{\mathbf{curl}}\big(\mathbf{I}_h(\boldsymbol{\chi})\big)\,, \tag{4.4}$$

where $\Pi_h^k : \mathbb{H}^1(\Omega) \to H_h$ is the corresponding Raviart-Thomas interpolation operator (see [3], [6]), and $\mathbf{I}_h$ is the tensorial version of the well-known Clément interpolation operator mapping $\mathrm{H}^1(\Omega)$ into the classical Lagrange finite element subspace of degree 1 (see [5]). Certainly, all the above assuming that a regular family of triangulations $\Big\{\mathcal{T}_h\Big\}_{h>0}$ of Ω made of tetrahedra T with diameter h_T has been introduced. Note in this case that the fact that $\underline{\mathbf{curl}}\big(\mathbf{I}_h(\boldsymbol{\chi})\big)$ is a divergence-free piecewise constant tensor implies that it belongs to the Raviart-Thomas space of order 0, and therefore to H_h, whence $\boldsymbol{\tau}_h$ does belong to H_h, thus confirming the applicability of the identity (4.1). In this way, it follows from (4.3) and (4.4) that

$$F(\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h) \;=\; F\Big(\nabla\mathbf{z} \,-\, \Pi_h^k(\nabla\mathbf{z})\Big) \,+\, F\Big(\underline{\mathbf{curl}}\big(\boldsymbol{\chi} - \mathbf{I}_h(\boldsymbol{\chi})\big)\Big) \;=:\; F_1(\mathbf{z}) \,+\, F_2(\boldsymbol{\chi})\,, \tag{4.5}$$

which transforms the deduction of (4.2) into the seeking of suitable upper bounds for the component functionals F_1 and F_2. In fact, by using the characterization and approximation properties of Π_h^k (see, e.g. [6, Section 3.4]), one arrives at the estimate

$$|F_1(\mathbf{z})| \;=\; \Big|F\Big(\nabla\mathbf{z} \,-\, \Pi_h^k(\nabla\mathbf{z})\Big)\Big| \;\le\; C\,\boldsymbol{\eta}_{F_1}\,\|\mathbf{z}\|_{2,\Omega}\,, \tag{4.6}$$

whereas integrating by parts (so that the $\underline{\mathbf{curl}}$ operator is taken away from the Clément interpolation error $\boldsymbol{\chi} - \mathbf{I}_h(\boldsymbol{\chi})$), and then applying the approximation properties of $\mathbf{I}_h$ (cf. [5]), one finds that

$$|F_2(\boldsymbol{\chi})| \;=\; \Big|F\Big(\underline{\mathbf{curl}}\big(\boldsymbol{\chi} - \mathbf{I}_h(\boldsymbol{\chi})\big)\Big)\Big| \;\le\; C\,\boldsymbol{\eta}_{F_2}\,\|\boldsymbol{\chi}\|_{1,\Omega}\,, \tag{4.7}$$

where $\boldsymbol{\eta}_{F_1}$ and $\boldsymbol{\eta}_{F_2}$ are residual expressions giving rise to $\boldsymbol{\eta}_F$. Finally, combining (4.5), (4.6), and (4.7) with the stability estimate given in (3.2), the required inequality (4.2) is attained. In turn, the efficiency issue concerning $\boldsymbol{\eta}_F$ is handled in the standard way by applying inverse inequalities (see [4]) and the localization technique based on tetrahedron-bubble and face-bubble functions (see [10]).

On the other hand, in what follows we take $\Gamma_{\rm N}$ as indicated in Section 3 and assume the hypotheses on Ω and $\Gamma_{\rm N}$ specified in Theorem 3.2. Then, we suppose that $H = \mathbb{H}_{\rm N}(\mathbf{div};\Omega)$ (cf. (3.8)) (which arises, in particular, when one considers the Poisson problem with homogeneous Neumann boundary condition on $\Gamma_{\rm N}$ and Dirichlet boundary condition on $\Gamma_{\rm D} := \Gamma\backslash\bar{\Gamma}_{\rm N}$), and let H_h be the usual tensorial Raviart-Thomas space of order $k \geq 0$ with vanishing normal components on $\Gamma_{\rm N}$. In this way, given $\boldsymbol{\tau} \in H$, its stable Helmholtz decomposition from Theorem 3.2 reads

$$\boldsymbol{\tau} \,=\, \nabla\mathbf{z} \,+\, \underline{\mathbf{curl}}(\boldsymbol{\chi}) \quad \text{in} \quad \Omega\,, \tag{4.8}$$

with $\mathbf{z} \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\chi} \in \mathbb{H}^1_{\rm N}(\Omega)$ (cf. (3.9)). Since $\boldsymbol{\chi}|_{\Gamma_{\rm N}} \in \mathbb{P}_0(\Gamma_{\rm N})$, it is clear that $\underline{\mathbf{curl}}(\boldsymbol{\chi})\,\boldsymbol{\nu} = \mathbf{0}$ on $\Gamma_{\rm N}$, which together with (4.8) and the fact that $\boldsymbol{\tau}\,\boldsymbol{\nu} = \mathbf{0}$ on $\Gamma_{\rm N}$, imply that $\nabla\mathbf{z}\,\boldsymbol{\nu}$ vanishes on $\Gamma_{\rm N}$ as well. Then, proceeding as in (4.4), we introduce the discrete Helmholtz decomposition

$$\boldsymbol{\tau}_h \,:=\, \Pi_h^k(\nabla\mathbf{z}) \,+\, \underline{\mathbf{curl}}\big(\mathbf{I}_h(\boldsymbol{\chi})\big)\,, \tag{4.9}$$

for which it only remains to verify that it belongs to H_h. In fact, since one of the characterization properties of Π_h^k guarantees that this operator preserves normal components given by piecewise polynomials of degree $\leq k$, we easily find that $\Pi_h^k(\nabla\mathbf{z})\,\boldsymbol{\nu} \,=\, \nabla\mathbf{z}\,\boldsymbol{\nu} \,=\, \mathbf{0}$ on $\Gamma_{\rm N}$. In turn, since the Clément interpolant preserves a constant value on any part of the boundary, we deduce that $\mathbf{I}_h(\boldsymbol{\chi})$ is also a constant tensor on $\Gamma_{\rm N}$, which yields $\underline{\mathbf{curl}}\big(\mathbf{I}_h(\boldsymbol{\chi})\big)\,\boldsymbol{\nu} \,=\, \mathbf{0}$ on $\Gamma_{\rm N}$. In this way we conclude from (4.9) that $\boldsymbol{\tau}_h\,\boldsymbol{\nu} \,=\, 0$ on $\Gamma_{\rm N}$, which confirms that $\boldsymbol{\tau}_h \in H_h$. The rest of the analysis is exactly as for the previous case, and therefore further details are omitted.

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