A fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity*

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Abstract

We propose and analyse an augmented mixed finite element method for the coupling of fluid flow with porous media flow. The flows are governed by a class of nonlinear Navier–Stokes and the linear Darcy equations, respectively, and the transmission conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. We apply dual-mixed formulations in both domains, and the nonlinearity involved in the Navier–Stokes region is handled by setting the strain and vorticity tensors as auxiliary unknowns. In turn, since the transmission conditions become essential, they are imposed weakly, which yields the introduction of the traces of the porous media pressure and the fluid velocity as the associated Lagrange multipliers. Furthermore, since the convective term in the fluid forces the velocity to live in a smaller space than usual, we augment the variational formulation with suitable Galerkin type terms arising from the constitutive and equilibrium equations of the Navier–Stokes equations, and the relation defining the strain and vorticity tensors. The resulting augmented scheme is then written equivalently as a fixed point equation, so that the well-known Schauder and Banach theorems, combined with classical results on bijective monotone operators, are applied to prove the unique solvability of the continuous and discrete systems. In particular, given an integer $k \geq 0$, piecewise polynomials of degree $\leq k$, Raviart–Thomas spaces of order k, continuous piecewise polynomials of degree $\leq k+1$, and piecewise polynomials of degree $\leq k$ are employed in the fluid for approximating the strain tensor, stress, velocity, and vorticity, respectively, whereas Raviart–Thomas spaces of order k and piecewise polynomials of degree $\leq k$ for the velocity and pressure, together with continuous piecewise polynomials of degree $\leq k+1$ for the traces, constitute feasible choices in the porous medium. Finally, several numerical results illustrating the good performance of the augmented mixed finite element method and confirming the theoretical rates of convergence are reported.

Key words: Navier–Stokes problem, Darcy problem, stress-velocity formulation, fixed point theory, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

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1 Introduction

The coupling of fluid flow, governed by the Navier–Stokes equations, and porous media flow, governed by the Darcy equations, has been intensively studied in recent decades (see, e.g., [3, 6, 9, 14, 17, 23, 24, 41, 47]) for the steady-state case and [15, 16] for the time dependent case. Applications include the interaction between surface and subsurface flows, modelling of blood flow, and others. In particular, in [41] it has been introduced and analyzed a DG discretization for the coupled problem considering the usual nonsymmetric interior penalty Galerkin (NIPG), symmetric interior penalty Galerkin (SIPG), and incomplete interior penalty Galerkin (IIPG) bilinear forms for the discretization of the Laplacian in both media and the upwind Lesaint-Raviart discretization of the convective term in the free fluid domain. In turn, in [3] the authors extend previous results on the Stokes–Darcy coupling (see [21] and [22]) and introduce an iterative subdomain method employing the velocity-pressure formulation for the Navier–Stokes equation and the primal one for the Darcy equation. More recently, a conforming mixed method for the coupled system has been introduced and analyzed in [23]. This work, which extends the previous results from [36], utilize the velocity-pressure formulation for the Navier–Stokes equation and the dual-mixed approach in the Darcy region, which yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier.

Now, in the context of incompressible Newtonian flows, the velocity-pressure formulation has been long time used in computations. However, when passing to non-Newtonian flows, the introduction of the stress as an additional unknown is very desirable, and thus the stress-velocity-pressure formulation has become frequently employed. A minor disadvantage, however, of such a formulation is the symmetry requirement for the stress tensor, for which several approaches have been derived. One is based on imposing the symmetry of the stress in a weak sense by introducing a Lagrange multiplier (see, e.g., [2]). Other one, and nowadays more popular approach, is based on the use of the so-called pseudostress instead of the stress in the Navier–Stokes equations. Indeed, in the context of mixed finite element methods, the Stokes and Navier–Stokes equations based on the pseudostress–velocity formulation are studied in [7] and [8], respectively, and the pseudostress-pressure-velocity formulation for the Stokes and Navier–Stokes equations has been introduced and analysed in [9]. While the latter formulation leads to a larger algebraic system, a hybridization technique can be used, however, to eliminate the pseudostress unknowns and, hence, to reduce its size. Furthermore, a new dual-mixed method for the Navier–Stokes equations, which introduces a nonlinear stress-like quantity that connects the stress and the convective term as a primary unknown together with the velocity and its gradient, has been proposed and analysed in [44]. The main advantage of this idea is that it allows for a unified analysis of Newtonian and non-Newtonian fluids. Moreover, the skew symmetry of the nonlinear terms is preserved, and therefore the classical theory for the mixed methods extends easily to that setting. On the other hand, it turns out that natural finite element candidates do not fulfill Babuška–Brezzi conditions for the associated discrete scheme and consequently construction of special finite elements is necessary.

The idea of a stress augmentation by the convective term has been later modified by connecting the pseudostress with the convective term in [12], where a new augmented mixed finite element method for the Navier–Stokes problem is proposed and analysed. The main unknowns are only comprised of the velocity and the aforementioned nonlinear pseudostress in this case. In order to guarantee the well-posedness of the resulting variational formulation, certain Galerkin least-square terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition are introduced. The idea of the augmented variational formulations goes back to [27] and later has been used for different kind of problems (see, e.g., [25, 28, 4, 26]). In order to prove well-posedness of both the continuous and discrete problems in [12], it suffices to apply the Lax–Milgram and Banach fixed point Theorems, which means, in particular, that no discrete inf-sup conditions are required there. As a consequence, arbitrary finite element subspaces of corresponding continuous spaces can be used. The results of [12] have been further extended in [10] to the Navier–Stokes equations with constant density and variable viscosity. In particular, the analysis in [10] focuses in developing a mixed finite element approach for those quasi-Newtonian fluids whose viscosity is a nonlinear function of the magnitude of the gradient of velocity. Another application of the idea of introducing the aforementioned nonlinear pseudostress has been done in [19] for the Boussinesq problem. In turn, in the context of stabilized methods, we can refer to [13], where two three-field (deviatoric stress-velocity-pressure) subgrid-scale type formulations of the Navier–Stokes problem with nonlinear viscosity has been studied. This approach allows to employ the same interpolation for all unknowns even in the convection-dominant case.

The purpose of this paper is to extend the results obtained in [10] and [32] to the coupled nonlinear Navier–Stokes and linear Darcy problem with constant density and variable viscosity in the fluid region. Unlike [10], in our model the viscosity depends nonlinearly only on the strain tensor, not on the whole gradient of the velocity. We define the pseudostress tensor as in [12] and subsequently eliminate the pressure unknown using the incompressibility condition. The transmission conditions consisting of mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law are imposed weakly, which results in additional Lagrange multipliers: the traces of the porous media pressure and the fluid velocity on the interface. We consider dual-mixed formulations in both domains. Similarly to [34, 32], in order to handle the nonlinearity in the fluid, the strain tensor and the vorticity are introduced as additional unknowns. Furthermore, the difficulty that the fluid velocity lives in H^1 instead of L^2 as usual, is resolved as in [10] by augmenting the variational formulation with residuals arising from the constitutive and equilibrium equations for the fluid flow, and the formulas for the strain and vorticity tensors. The resulting augmented variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness and uniqueness of both the continuous and discrete formulation is proved employing a generalized Babuška–Brezzi theory (see [30, 32]) and a fixed point argument. The rest of the paper is organized as follows. In Section 2 we introduce the continuous problem and identify the twofold saddle point structure of the corresponding variational system. The augmented fully-mixed variational formulation is then derived in Section 3, and, under the assumption that the data are sufficiently small, its well-posedness is proved there by combining fixed point theorems with the generalized Babuška–Brezzi theory. Next, hypotheses on the finite element spaces aiming to ensure the well-posedness of the corresponding Galerkin scheme are established in Section 4, and the discrete analogue of the theory applied to the continuous case is employed here for the respective proof. In turn, the associated a priori error estimate is derived in Section 5, whereas particular choices of discrete subspaces satisfying the hypotheses from Section 4 together with the rates of convergence of the Galerkin schemes, are specified in Section 6. Finally, we illustrate the accuracy of the augmented mixed finite element method with some numerical examples in Section 7.

We end this section by introducing some definitions and fixing some notations. Given the vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, with $n \in \{2, 3\}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor field $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} := (\zeta_{ij})_{i,j=1,n}$, we define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}\,(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathrm{R}^{n\times n}$. In addition, in what follows we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O}$ is a domain, Γ is an open or closed Lipschitz curve (respectively surface in R^3), and $s \in \mathrm{R}$, we define

$$\mathbf{H}^s(\mathcal{O}) := [H^s(\mathcal{O})]^n, \quad \mathbb{H}^s(\mathcal{O}) := [H^s(\mathcal{O})]^{n\times n}, \quad \text{and} \quad \mathbf{H}^s(\Gamma) := [H^s(\Gamma)]^n.$$

However, when $s = 0$ we usually write $\mathbf{L}^2(\mathcal{O}), \mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O}), \mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$ for $H^s(\mathcal{O})$, $\mathbf{H}^s(\mathcal{O})$ and $\mathbb{H}^s(\mathcal{O})$, and $\|\cdot\|_{s,\Gamma}$ for $H^s(\Gamma)$ and $\mathbf{H}^s(\Gamma)$. For $s \geq 0$, we write $|\cdot|_{s,\mathcal{O}}$ for the $\mathbf{H}^s$-seminorm. In addition, we recall that

$$\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O}) \right\},$$

is a standard Hilbert space in the realm of mixed problems (see, e.g. [5, 40]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div};\mathcal{O})$. The norms of $\mathbf{H}(\mathrm{div}\,;\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div}\,;\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. On the other hand, the following symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\lambda \quad \forall \xi, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Gamma := \int_\Gamma \boldsymbol{\xi} \cdot \boldsymbol{\lambda} \quad \forall \boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathrm{R}^n, P_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, we set $\mathbf{P}_k(S) := [P_k(S)]^n$ and $\mathbb{P}_k(S) := [P_k(S)]^{n\times n}$. Finally, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The continuos formulation

In this section we introduce the model problem and derive the corresponding weak formulation.

2.1 The model problem

In order to describe the geometry under consideration we let Ω_{S} and Ω_{D} be bounded and simply connected polyhedral domains in R^n, such that $\Omega_{\mathrm{S}} \cap \Omega_{\mathrm{D}} = \emptyset$ and $\partial\Omega_{\mathrm{S}} \cap \partial\Omega_{\mathrm{D}} = \Sigma \neq \emptyset$. Then, we let $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$ and Ω_{S} (and hence inward to Ω_{D} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Fig. 2.1 below) and by $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$. The problem we are interested in consists of the movement of an incompressible quasi-Newtonian viscous fluid that occupies Ω_{S} and that flows towards and from Ω_{D} through Σ, where Ω_{D} is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In Ω_{S}, the governing equations are those of the Navier–Stokes problem with constant density and variable viscosity, which are written in the following nonstandard stress–velocity–pressure formulation:

$$\boldsymbol{\sigma}_{\mathrm{S}} = \mu(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbb{I} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \operatorname{div} \mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}},$$
$$-\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}, \tag{2.1}$$

where $\boldsymbol{\sigma}_{\mathrm{S}}$ is the nonlinear stress tensor, $\mathbf{u}_{\mathrm{S}}$ is the velocity, p_{S} is the pressure, $\mu : \mathrm{R}^+ \to \mathrm{R}^+$ is the nonlinear kinematic viscosity, $\mathbf{e}(\mathbf{u}_{\mathrm{S}}) := \frac{1}{2}\{\nabla \mathbf{u}_{\mathrm{S}} + (\nabla \mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\}$ is the strain tensor (or symmetric part of the velocity gradient), $|\cdot|$ denotes the Euclidean norm in $\mathrm{R}^{n\times n}$, and $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$ is a known volume force.

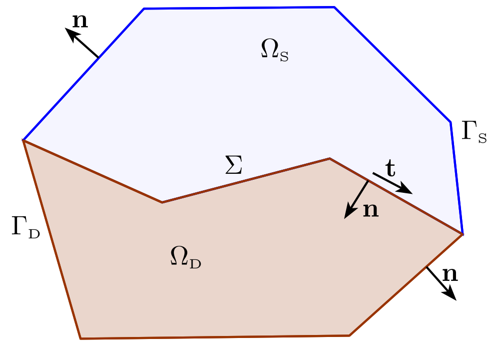

Figure 2.1: Sketch of a 2D geometry of our Navier–Stokes/Darcy model

Furthermore, we assume that μ is of class C^1, and that there exist constants $\mu_1, \mu_2 > 0$, such that

$$\mu_1 \le \mu(s) \le \mu_2 \quad \text{and} \quad \mu_1 \le \mu(s) + s\mu'(s) \le \mu_2 \qquad \forall s \ge 0, \tag{2.2}$$

which, according to the result provided in [39, Theorem 3.8], implies Lipschitz continuity and strong monotonicity of the nonlinear operator induced by μ. This fact will be used later on in Section 3. In addition, it is easy to see that the forthcoming analysis also applies to the slightly more general case of a viscosity function acting on $\Omega \times \mathrm{R}^+$, that is $\mu : \Omega \times \mathrm{R}^+ \to \mathrm{R}$. Some examples of nonlinear μ are the following:

$$\mu(s) := 2 + \frac{1}{1+s} \quad \text{and} \quad \mu(s) := \alpha_0 + \alpha_1(1+s^2)^{(\beta-2)/2}, \tag{2.3}$$

where $\alpha_0, \alpha_1 > 0$ and $\beta \in [1,2]$. The first example is basically academic but the second one corresponds to a particular case of the well-known Carreau law in fluid mechanics. It is easy to see that they both satisfy (2.2) with $(\mu_1, \mu_2) = (2,3)$ and $(\mu_1, \mu_2) = (\alpha_0, \alpha_0 + \alpha_1)$, respectively.

Now, in order to derive our fully-mixed formulation, we first observe, owing to the fact that $\operatorname{tr} \mathbf{e}(\mathbf{u}_{\mathrm{S}}) = \operatorname{div} \mathbf{u}_{\mathrm{S}}$, that the first two equations in (2.1) are equivalent to

$$\boldsymbol{\sigma}_{\mathrm{S}} = \mu(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbb{I} \quad \text{and} \quad p_{\mathrm{S}} = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}_{\mathrm{S}} + (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})) \quad \text{in} \quad \Omega_{\mathrm{S}}, \tag{2.4}$$

and hence, eliminating the pressure p_{S} (which anyway can be approximated later on by the postprocessed formula suggested by the second equation of (2.4)), the Navier–Stokes problem (2.1) can be rewritten as

$$\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}} = \mu(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - (\mathbf{u}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad -\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}. \tag{2.5}$$

Next, in order to handle the nonlinearity in $\boldsymbol{\sigma}_{\mathrm{S}}$ given by the term $\mu(|\mathbf{e}(\mathbf{u}_{\mathrm{S}})|)\mathbf{e}(\mathbf{u}_{\mathrm{S}})$, and employ the corresponding integration by parts formula, we adopt the approach from [34] (see also [35]) and introduce the additional unknowns

$$\mathbf{t}_{\mathrm{S}} := \mathbf{e}(\mathbf{u}_{\mathrm{S}}) \quad \text{and} \quad \boldsymbol{\rho}_{\mathrm{S}} := \frac{1}{2}\left\{\nabla \mathbf{u}_{\mathrm{S}} - (\nabla \mathbf{u}_{\mathrm{S}})^{\mathrm{t}}\right\} \quad \text{in} \quad \Omega_{\mathrm{S}}, \tag{2.6}$$

where $\boldsymbol{\rho}_{\mathrm{S}}$ is the vorticity (or skew-symmetric part of the velocity gradient). In this way, instead of (2.5), in the sequel we consider the set of equations with unknowns $\mathbf{t}_{\mathrm{S}}$, $\mathbf{u}_{\mathrm{S}}$, $\boldsymbol{\sigma}_{\mathrm{S}}$ and $\boldsymbol{\rho}_{\mathrm{S}}$, given by

$$\begin{gathered}\mathbf{t}_{\mathrm{S}} = \nabla\mathbf{u}_{\mathrm{S}} - \boldsymbol{\rho}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \boldsymbol{\sigma}_{\mathrm{S}}^{\mathrm{d}} = \mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}} - (\mathbf{u}_{\mathrm{S}}\otimes\mathbf{u}_{\mathrm{S}})^{\mathrm{d}} \quad \text{in} \quad \Omega_{\mathrm{S}},\\ -\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}}, \qquad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}},\end{gathered} \tag{2.7}$$

where both $\mathbf{t}_{\mathrm{S}}$ and $\boldsymbol{\sigma}_{\mathrm{S}}$ are symmetric tensors, and $\operatorname{tr}(\mathbf{t}_{\mathrm{S}}) = 0$ holds in Ω_{S}.

On the other hand, in Ω_{D} we consider the linearized Darcy model with homogeneous Neumann boundary condition on Γ_{D}:

$$\mathbf{u}_{\mathrm{D}} = -\mathbf{K}\nabla p_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}, \tag{2.8}$$

where $\mathbf{u}_{\mathrm{D}}$ and p_{D} denote the velocity and pressure, respectively, $f_{\mathrm{D}} \in L^2(\Omega_{\mathrm{D}})$ is a source term satisfying $\int_{\Omega_{\mathrm{D}}} f_{\mathrm{D}} = 0$, and $\mathbf{K} \in [L^\infty(\Omega_{\mathrm{D}})]^{n\times n}$ is a symmetric tensor describing the permeability of Ω_{D} divided by a constant approximation of the viscosity, satisfying with $C_{\mathbf{K}} > 0$

$$\mathbf{w}\cdot\mathbf{K}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}}\|\mathbf{w}\|^2, \tag{2.9}$$

for almost all $\mathbf{x} \in \Omega_{\mathrm{D}}$, and for all $\mathbf{w} \in \mathrm{R}^n$. Finally, the transmission conditions on Σ are given by

$$\begin{aligned}\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} &= \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma,\\ \boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n} + \sum_{l=1}^{n-1}\omega_l^{-1}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t}_l)\mathbf{t}_l &= -p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma,\end{aligned} \tag{2.10}$$

where $\{\omega_1, \ldots, \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.10) corresponds to mass conservation on Σ, whereas the second one establishes the balance of normal forces and a Beavers–Joseph–Saffman law.

2.2 The augmented fully-mixed variational formulation

In this section we proceed analogously to [32] (see also [37]) and derive a weak formulation of the coupled problem given by (2.7), (2.8), and (2.10). To this end, let us first introduce further notations and definitions. In what follows, given $\star \in \{\mathrm{S}, \mathrm{D}\}$, $u, v \in L^2(\Omega_\star)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$, we set

$$(u, v)_\star := \int_{\Omega_\star} uv, \quad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u}\cdot\mathbf{v}, \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

In addition, we let $\mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}})$ and $\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}})$ be the subspaces of symmetric and skew-symmetric tensors of $\mathbb{L}^2(\Omega_{\mathrm{S}})$, respectively, that is

$$\mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}}) := \{\mathbf{r}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \mathbf{r}_{\mathrm{S}}^{\mathrm{t}} = \mathbf{r}_{\mathrm{S}}\}$$

and

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) := \{\boldsymbol{\eta}_{\mathrm{S}} \in \mathbb{L}^2(\Omega_{\mathrm{S}}) : \boldsymbol{\eta}_{\mathrm{S}}^{\mathrm{t}} = -\boldsymbol{\eta}_{\mathrm{S}}\}.$$

Furthermore, we need to define the spaces

$$\begin{aligned}\mathbf{H}_0(\operatorname{div};\Omega_{\mathrm{D}}) &:= \{\mathbf{v}_{\mathrm{D}} \in \mathbf{H}(\operatorname{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}\},\\ \mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}}) &:= \{\mathbf{r}_{\mathrm{S}} \in \mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{S}}) : \operatorname{tr}\mathbf{r}_{\mathrm{S}} = 0\},\end{aligned}$$

and the space of traces $\mathbf{H}^{1/2}_{00}(\Sigma) := [H^{1/2}_{00}(\Sigma)]^n$, where

$$H^{1/2}_{00}(\Sigma) := \big\{ v|_\Sigma : \quad v \in H^1(\Omega_{\rm S}), \quad v = 0 \quad \text{on} \quad \Gamma_{\rm S} \big\} .$$

Observe that, if $E_{0,{\rm S}} : H^{1/2}(\Sigma) \to L^2(\partial\Omega_{\rm S})$ is the extension operator defined by

$$E_{0,{\rm S}}(\psi) := \begin{cases} \psi & \text{on } \Sigma \\ 0 & \text{on } \Gamma_{\rm S} \end{cases} \qquad \forall \psi \in H^{1/2}(\Sigma),$$

we have that

$$H^{1/2}_{00}(\Sigma) = \Big\{ \psi \in H^{1/2}(\Sigma) : \quad E_{0,{\rm S}}(\psi) \in H^{1/2}(\partial\Omega_{\rm S}) \Big\} ,$$

endowed with the norm $\|\psi\|_{1/2,00,\Sigma} := \|E_{0,{\rm S}}(\psi)\|_{1/2,\partial\Omega_{\rm S}}$. The dual space of $\mathbf{H}^{1/2}_{00}(\Sigma)$ is denoted by $\mathbf{H}^{-1/2}_{00}(\Sigma)$.

Now, we proceed with the derivation of our weak formulation. We begin by introducing two additional unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_{\rm S} \in \mathbf{H}^{1/2}_{00}(\Sigma) \quad \text{and} \quad \lambda := p_{\rm D} \in H^{1/2}(\Sigma).$$

Then, to derive the weak formulation of the coupled system (2.7)–(2.8)–(2.10) we proceed similarly to [32] (see also [11, 37]), that is, we test the first equations of (2.7) and (2.8) with arbitrary $\boldsymbol{\tau}_{\rm S} \in \mathbb{H}(\mathbf{div};\Omega_{\rm S})$ and $\mathbf{v}_{\rm D} \in \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})$, respectively, integrate by parts, utilize the identity $(\mathbf{t}_{\rm S},\boldsymbol{\tau}_{\rm S})_{\rm S} = (\mathbf{t}_{\rm S},\boldsymbol{\tau}^{\rm d}_{\rm S})_{\rm S}$ (which follows from the fact that $\mathbf{t}_{\rm S} : \mathbb{I} = \mathrm{tr}\,\mathbf{t}_{\rm S} = 0$), and impose the remaining equations weakly, as well as the symmetry of $\boldsymbol{\sigma}_{\rm S}$, to obtain the variational problem: Find $\mathbf{t}_{\rm S} \in \mathbb{L}^2_{\rm tr}(\Omega_{\rm S})$, $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}(\mathbf{div};\Omega_{\rm S})$, $\boldsymbol{\rho}_{\rm S} \in \mathbb{L}^2_{\rm skew}(\Omega_{\rm S})$, $\mathbf{u}_{\rm D} \in \mathbf{H}_0(\mathrm{div}\,;\Omega_{\rm D})$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, $\lambda \in H^{1/2}(\Sigma)$, $p_{\rm D} \in L^2(\Omega_{\rm D})$ and $\mathbf{u}_{\rm S}$ in a suitable space (to be specified below), such that

$$\begin{aligned}
(\mathbf{t}_{\rm S},\boldsymbol{\tau}^{\rm d}_{\rm S})_{\rm S} + (\mathbf{div}\boldsymbol{\tau}_{\rm S},\mathbf{u}_{\rm S})_{\rm S} + \langle \boldsymbol{\tau}_{\rm S}\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma + (\boldsymbol{\tau}_{\rm S},\boldsymbol{\rho}_{\rm S})_{\rm S} &= 0,\\
\big(\mathbf{K}^{-1}\mathbf{u}_{\rm D},\mathbf{v}_{\rm D}\big)_{\rm D} - (\mathrm{div}\,\mathbf{v}_{\rm D},p_{\rm D})_{\rm D} - \langle \mathbf{v}_{\rm D}\cdot\mathbf{n},\lambda\rangle_\Sigma &= 0,\\
(\mu(|\mathbf{t}_{\rm S}|)\mathbf{t}_{\rm S},\mathbf{r}_{\rm S})_{\rm S} - (\mathbf{r}_{\rm S},\boldsymbol{\sigma}^{\rm d}_{\rm S})_{\rm S} - ((\mathbf{u}_{\rm S}\otimes\mathbf{u}_{\rm S})^{\rm d},\mathbf{r}_{\rm S})_{\rm S} &= 0,\\
-(\mathbf{div}\boldsymbol{\sigma}_{\rm S},\mathbf{v}_{\rm S})_{\rm S} &= (\mathbf{f}_{\rm S},\mathbf{v}_{\rm S})_{\rm S},\\
(\mathrm{div}\,\mathbf{u}_{\rm D},q_{\rm D})_{\rm D} &= (f_{\rm D},q_{\rm D})_{\rm D},\\
(\boldsymbol{\sigma}_{\rm S},\boldsymbol{\eta}_{\rm S})_{\rm S} &= 0,\\
-\langle \boldsymbol{\varphi}\cdot\mathbf{n},\xi\rangle_\Sigma - \langle \mathbf{u}_{\rm D}\cdot\mathbf{n},\xi\rangle_\Sigma &= 0,\\
\langle \boldsymbol{\sigma}_{\rm S}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma &= 0,
\end{aligned} \tag{2.11}$$

for all $\mathbf{r}_{\rm S} \in \mathbb{L}^2_{\rm tr}(\Omega_{\rm S})$, $\boldsymbol{\tau}_{\rm S} \in \mathbb{H}(\mathbf{div};\Omega_{\rm S})$, $\boldsymbol{\eta}_{\rm S} \in \mathbb{L}^2_{\rm skew}(\Omega_{\rm S})$, $\mathbf{v}_{\rm D} \in \mathbf{H}_0(\mathrm{div}\,;\Omega_{\rm D})$, $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, $\xi \in H^{1/2}(\Sigma)$, $q_{\rm D} \in L^2(\Omega_{\rm D})$ and $\mathbf{v}_{\rm S} \in \mathbf{L}^2(\Omega_{\rm S})$, where

$$\langle \boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} := \sum_{l=1}^{n-1} \omega_l^{-1} \langle \boldsymbol{\varphi}\cdot\mathbf{t}_l,\boldsymbol{\psi}\cdot\mathbf{t}_l\rangle_\Sigma .$$

Notice that the third term in the third equation of the foregoing system requires $\mathbf{u}_{\rm S}$ to live in a smaller space than $\mathbf{L}^2(\Omega_{\rm S})$. In fact, by applying the Cauchy–Schwarz and Hölder inequalities and then the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega_S)$ into $\mathbf{L}^4(\Omega_S)$ (see e.g. [1, Theorem 6.3] or [46, Theorem 1.3.5]), we find that there holds

$$\left|((\mathbf{u}_S\otimes\mathbf{w}_S)^{\mathrm{d}},\mathbf{r}_S)_S\right| \le \|\mathbf{u}\|_{\mathbf{L}^4(\Omega_S)}\|\mathbf{w}_S\|_{\mathbf{L}^4(\Omega_S)}\|\mathbf{r}_S\|_{0,\Omega_S} \le \|\mathbf{i}_c\|^2\|\mathbf{u}_S\|_{1,\Omega_S}\|\mathbf{w}_S\|_{1,\Omega_S}\|\mathbf{r}_S\|_{0,\Omega_S}, \tag{2.12}$$

for all $\mathbf{u}_S,\mathbf{w}_S\in\mathbf{H}^1(\Omega_S)$ and $\mathbf{r}_S\in\mathbf{L}^2(\Omega_S)$. According to this, we propose to look for the unknown $\mathbf{u}_S$ in $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$ and to restrict the set of corresponding test functions $\mathbf{v}_S$ to the same space, where

$$\mathbf{H}^1_{\Gamma_S}(\Omega_S) := \{\mathbf{v}_S\in\mathbf{H}^1(\Omega_S):\mathbf{v}_S|_{\Gamma_S}=\mathbf{0}\}.$$

Next, analogously to [32], it is not difficult to see that the system (2.11) is not uniquely solvable since, given any solution $(\mathbf{t}_S,\boldsymbol{\sigma}_S,\boldsymbol{\rho}_S,\mathbf{u}_D,\boldsymbol{\varphi},\lambda,p_D,\mathbf{u}_S)$ in the indicated spaces, and given any constant $c\in\mathrm{R}$, the vector defined by $(\mathbf{t}_S,\boldsymbol{\sigma}_S-c\mathbb{I},\boldsymbol{\rho}_S,\mathbf{u}_D,\boldsymbol{\varphi},\lambda+c,p_D+c,\mathbf{u}_S)$ also becomes a solution. As a consequence of the above, from now on we require the Darcy pressure p_D to be in $L^2_0(\Omega_D)$, where

$$L^2_0(\Omega_D) := \left\{q\in L^2(\Omega_D):\ (q,1)_D=0\right\}.$$

In turn, due to the decomposition $L^2(\Omega_D)=L^2_0(\Omega_D)\oplus\mathrm{R}$, the boundary conditions $\mathbf{u}_D\cdot\mathbf{n}=0$ on Γ_D and $\mathbf{u}_S=\mathbf{0}$ on Γ_S, the first transmission condition in (2.10), and the fact that $\int_{\Omega_D} f_D=0$, guarantee that the fifth equation of (2.11) is equivalent to requiring it for all $q_D\in L^2_0(\Omega_D)$.

On the other hand, for convenience of the subsequent analysis, we consider the decomposition (see, for instance, [5],[29])

$$\mathbb{H}(\mathbf{div};\Omega_S)=\mathbb{H}_0(\mathbf{div};\Omega_S)\oplus\mathrm{R}\mathbb{I}, \tag{2.13}$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega_S) := \{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega_S):\quad (\operatorname{tr}\boldsymbol{\tau},1)_S=0\},$$

and redefine the stress tensor as $\boldsymbol{\sigma}_S:=\boldsymbol{\sigma}_S+l\mathbb{I}$, with the new unknowns $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div};\Omega_S)$ and $l\in\mathrm{R}$. In this way the first and last equations of (2.11) are rewritten, equivalently, as

$$\begin{aligned}
(\mathbf{t}_S,\boldsymbol{\tau}_S^{\mathrm{d}})_S+(\mathbf{div}\boldsymbol{\tau}_S,\mathbf{u}_S)_S+\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma+(\boldsymbol{\tau}_S,\boldsymbol{\rho}_S)_S &= 0 \quad \forall\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div};\Omega_S),\\
j\,\langle\boldsymbol{\varphi}\cdot\mathbf{n},1\rangle_\Sigma &= 0 \quad \forall j\in\mathrm{R},\\
\langle\boldsymbol{\sigma}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma-\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma}+\langle\boldsymbol{\psi}\cdot\mathbf{n},\lambda\rangle_\Sigma+l\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma &= 0 \quad \forall\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma).
\end{aligned} \tag{2.14}$$

Finally, consequently with the choice of the corresponding space for $\mathbf{u}_S$, and in order to be able to analyze the present variational formulation of (2.7), (2.8), and (2.10), we augment the resulting system through the incorporation of the following redundant Galerkin terms:

$$\begin{aligned}
\kappa_1\left(\boldsymbol{\sigma}_S^{\mathrm{d}}-\mu(|\mathbf{t}_S|)\mathbf{t}_S+(\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathrm{d}},\boldsymbol{\tau}_S^{\mathrm{d}}\right)_S=0 &\qquad \forall\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div};\Omega_S),\\
\kappa_2\left(\mathbf{div}\boldsymbol{\sigma}_S,\mathbf{div}\boldsymbol{\tau}_S\right)_S=-\kappa_2\left(\mathbf{f}_S,\mathbf{div}\boldsymbol{\tau}_S\right)_S &\qquad \forall\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div};\Omega_S),\\
\kappa_3\left(\mathbf{e}(\mathbf{u}_S)-\mathbf{t}_S,\mathbf{e}(\mathbf{v}_S)\right)_S=0 &\qquad \forall\mathbf{v}_S\in\mathbf{H}^1_{\Gamma_S}(\Omega_S),\\
\kappa_4\left(\boldsymbol{\rho}_S-\frac{1}{2}\left(\nabla\mathbf{u}_S-(\nabla\mathbf{u}_S)^{\mathrm{t}}\right),\boldsymbol{\eta}_S\right)_S=0 &\qquad \forall\boldsymbol{\eta}_S\in\mathbb{L}^2_{\mathrm{skew}}(\Omega_S),
\end{aligned} \tag{2.15}$$

where $\kappa_1,\kappa_2,\kappa_3$, and κ_4 are positive parameters to be specified later.

Now, it is clear that there are many different ways of ordering the augmented mixed variational formulation described above, but for the sake of the subsequent analysis we proceed as in [32] (see also [37, 11]), and adopt one leading to a doubly-mixed structure. To that end, we group the spaces, unknowns, and test functions as follows:

$$\mathbf{X} := \mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}}) \times \mathbb{H}_0(\mathbf{div};\Omega_{\mathrm{S}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_0(\mathrm{div}\,;\Omega_{\mathrm{D}}), \quad \mathbf{M} := \mathbf{H}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma),$$

$$\mathbb{X} := \mathbf{X} \times \mathbf{M}, \quad \text{and} \quad \mathbb{M} := L^2_0(\Omega_{\mathrm{D}}) \times \mathrm{R},$$

$$\underline{\mathbf{t}} := (\mathbf{t}_{\mathrm{S}}, \boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}}, \boldsymbol{\rho}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda) \in \mathbf{M}, \quad \underline{\mathbf{p}} := (p_{\mathrm{D}}, l) \in \mathbb{M},$$

$$\underline{\mathbf{r}} := (\mathbf{r}_{\mathrm{S}}, \boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{M}, \quad \underline{\mathbf{q}} := (q_{\mathrm{D}}, j) \in \mathbb{M},$$

where $\mathbf{X}$, $\mathbf{M}$, $\mathbb{X}$, and $\mathbb{M}$ are respectively endowed with the norms

$$\|\underline{\mathbf{r}}\|_{\mathbf{X}} := \|\mathbf{r}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div},\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\eta}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div}\,,\Omega_{\mathrm{D}}},$$

$$\|\underline{\boldsymbol{\psi}}\|_{\mathbf{M}} := \|\boldsymbol{\psi}\|_{1/2,00,\Sigma} + \|\xi\|_{1/2,\Sigma}, \quad \|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{X}} := \|\underline{\mathbf{r}}\|_{\mathbf{X}} + \|\underline{\boldsymbol{\psi}}\|_{\mathbf{M}} \quad \text{and} \quad \|\underline{\mathbf{q}}\|_{\mathbb{M}} := \|q_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + |j|.$$

Hence, the augmented fully-mixed variational formulation for the system (2.11) with the new equations (2.14) and (2.15) reads: Find $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_{\mathrm{S}})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X}, \\ [\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \qquad \forall \underline{\mathbf{q}} \in \mathbb{M}, \end{aligned} \tag{2.16}$$

where

$$[\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := [F, \underline{\mathbf{r}}] \quad \text{and} \quad [\mathbf{G}, \underline{\mathbf{q}}] := [G, q_{\mathrm{D}}], \tag{2.17}$$

with

$$[F, \underline{\mathbf{r}}] := -\kappa_2(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} \quad \text{and} \quad [G, q_{\mathrm{D}}] := -(f_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}}\,.$$

In addition, given $\mathbf{z}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, the operator $\mathbf{A}(\mathbf{z}_{\mathrm{S}}) : \mathbb{X} \to \mathbb{X}'$ is defined by

$$[\mathbf{A}(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := [a(\mathbf{z})(\underline{\mathbf{t}}), \underline{\mathbf{r}}] + [b(\underline{\mathbf{t}}), \underline{\boldsymbol{\psi}}] + [b(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}}] - [c(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}], \tag{2.18}$$

with

$$\begin{aligned}
[a(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}), \underline{\mathbf{r}}] \;&:=\; [a_1(\underline{\mathbf{t}}), \underline{\mathbf{r}}] + [a_2(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}), \underline{\mathbf{r}}], \\
[a_1(\underline{\mathbf{t}}), \underline{\mathbf{r}}] \;&:=\; (\mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}}, \mathbf{r}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{r}_{\mathrm{S}}, \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{t}_{\mathrm{S}}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} + \kappa_1(\boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{S}} - \mu(|\mathbf{t}_{\mathrm{S}}|)\mathbf{t}_{\mathrm{S}}, \boldsymbol{\tau}^{\mathrm{d}}_{\mathrm{S}})_{\mathrm{S}} \\
&\quad + \kappa_2(\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{div}\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{u}_{\mathrm{S}})_{\mathrm{S}} - (\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\boldsymbol{\tau}_{\mathrm{S}}, \boldsymbol{\rho}_{\mathrm{S}})_{\mathrm{S}} \\
&\quad - (\boldsymbol{\sigma}_{\mathrm{S}}, \boldsymbol{\eta}_{\mathrm{S}})_{\mathrm{S}} + \kappa_3(\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - \mathbf{t}_{\mathrm{S}}, \mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}} + \kappa_4\left(\boldsymbol{\rho}_{\mathrm{S}} - \frac{1}{2}(\nabla\mathbf{u}_{\mathrm{S}} - (\nabla\mathbf{u}_{\mathrm{S}})^{\mathrm{t}}), \boldsymbol{\eta}_{\mathrm{S}}\right)_{\mathrm{S}} \\
&\quad + (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\mathrm{D}}, \\
[a_2(\mathbf{z}_{\mathrm{S}})(\underline{\mathbf{t}}), \underline{\mathbf{r}}] \;&:=\; ((\mathbf{z}_{\mathrm{S}} \otimes \mathbf{u}_{\mathrm{S}})^{\mathrm{d}}, \kappa_1\boldsymbol{\tau}^{\mathrm{d}} - \mathbf{r}_{\mathrm{S}})_{\mathrm{S}}, \\
[b(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}] \;&:=\; \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\psi}\rangle_{\Sigma} - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \xi\rangle_{\Sigma}\,, \\
[c(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}] \;&:=\; \langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_{\Sigma} - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_{\Sigma} + \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma}\,,
\end{aligned} \tag{2.19}$$

whereas $\mathbf{B} : \mathbb{X} \to \mathbb{M}'$ is given by

$$[\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}] := [B_1(\underline{\mathbf{r}}), q_{\mathrm{D}}] + [B_2(\underline{\boldsymbol{\psi}}), j], \tag{2.20}$$

with

$$[B_1(\underline{\mathbf{r}}), q_{\mathrm{D}}] \;:=\; -(\mathrm{div}\,\mathbf{v}_{\mathrm{D}}, q_{\mathrm{D}})_{\mathrm{D}} \quad \text{and} \quad [B_2(\underline{\boldsymbol{\psi}}), j] \;:=\; j\,\langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_{\Sigma}\,.$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators.

3 Analysis of the continuous formulation

In this section we analyse the well-posedness of problem (2.16) by means of a fixed point argument and a result on the solvability of twofold saddle point problems. To that end we first collect some previous results and notations that will serve for the forthcoming analysis.

3.1 Preliminaries

We begin by recalling the following theorem to be employed next.

Theorem 3.1 *Let X_1, M_1, and M be Hilbert spaces, set $X := X_1 \times M_1$, and let X_1', M_1', M', and $X' := X_1' \times M_1'$, be their respective duals. Let $A_1 : X_1 \to X_1'$ be a nonlinear operator, and $S : M_1 \to M_1'$, $B_1 : X_1 \to M_1'$, and $B : X \to M'$ be linear bounded operators. We also let $B_1' : M_1 \to X_1'$ and $B' : M \to X'$ be the corresponding adjoints and define the nonlinear operator $A : X \to X'$, as:*

$$[A(\mathbf{s}, \boldsymbol{\phi}), (\mathbf{r}, \boldsymbol{\psi})] := [A_1(\mathbf{s}), \mathbf{r}] + [B_1'(\boldsymbol{\phi}), \mathbf{r}] + [B_1(\mathbf{s}), \boldsymbol{\psi}] - [S(\boldsymbol{\phi}), \boldsymbol{\psi}] \quad \forall\, (\mathbf{s}, \boldsymbol{\phi}), (\mathbf{r}, \boldsymbol{\psi}) \in X. \tag{3.1}$$

Finally, we let V be the kernel of B, that is

$$V := \{(\mathbf{r}, \boldsymbol{\psi}) \in X : \quad [B(\mathbf{r}, \boldsymbol{\psi}), q] = 0 \quad \forall q \in M\},$$

and let $\tilde{X}_1$ and $\tilde{M}_1$ be subspaces of X_1 and M_1, respectively, such that $V = \tilde{X}_1 \times \tilde{M}_1$.

Assume that

(i) $A_1|_{\tilde{X}_1} : \tilde{X}_1 \to \tilde{X}_1'$ is Lipschitz continuous and strongly monotone, that is, there exist constants $\gamma, \alpha > 0$ such that

$$\|A_1(\mathbf{s}) - A_1(\mathbf{r})\|_{\tilde{X}_1'} \leq \gamma \|\mathbf{s} - \mathbf{r}\|_{X_1} \quad \forall\, \mathbf{s}, \mathbf{r} \in \tilde{X}_1$$

and

$$[A_1(\mathbf{s}) - A_1(\mathbf{r}), \mathbf{s} - \mathbf{r}] \geq \alpha \|\mathbf{s} - \mathbf{r}\|_{X_1}^2 \quad \forall \mathbf{s}, \mathbf{r} \in \tilde{X}_1.$$

(ii) For each pair $(\mathbf{r}, \mathbf{r}^\perp) \in \tilde{X}_1 \times \tilde{X}_1^\perp$ there holds the pseudolinear property

$$A_1(\mathbf{r} + \mathbf{r}^\perp) = A_1(\mathbf{r}) + A_1(\mathbf{r}^\perp).$$

(iii) S is positive semi-definite on $\tilde{M}_1$, that is,

$$[S(\boldsymbol{\psi}), \boldsymbol{\psi}] \geq 0 \quad \forall \boldsymbol{\psi} \in \tilde{M}_1.$$

(iv) B_1 satisfies an inf-sup condition on $\tilde{X}_1 \times \tilde{M}_1$, that is, there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{r} \in \tilde{X}_1 \\ \mathbf{r} \neq \mathbf{0}}} \frac{[B_1(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_{X_1}} \geq \beta_1 \|\boldsymbol{\psi}\|_{M_1} \quad \forall \boldsymbol{\psi} \in \tilde{M}_1.$$

(v) B satisfies an inf-sup condition on $X \times M$, that is, there exists $\beta > 0$ such that

$$\sup_{\substack{(\mathbf{r}, \boldsymbol{\psi}) \in X \\ (\mathbf{r}, \boldsymbol{\psi}) \neq \mathbf{0}}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), q]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_X} \geq \beta \|q\|_M \quad \forall q \in M.$$

Then, there exists a unique $((\mathbf{t},\boldsymbol{\varphi}),p) \in X \times M$*, such that*

$$
\begin{aligned}
[A(\mathbf{t},\boldsymbol{\varphi}),(\mathbf{r},\boldsymbol{\psi})] + [B'(p),(\mathbf{r},\boldsymbol{\psi})] &= [F,(\mathbf{r},\boldsymbol{\psi})] \quad \forall (\mathbf{r},\boldsymbol{\psi}) \in X, \\
[B(\mathbf{t},\boldsymbol{\varphi}),q] &= [G,q] \qquad \forall q \in M.
\end{aligned}
\tag{3.2}
$$

Moreover, there exists $C>0$*, depending only on* $\alpha,\gamma,\beta_1,\beta,\|S\|$*, and* $\|B_1\|$ *such that*

$$
\|((\mathbf{t},\boldsymbol{\varphi}),p)\|_{X\times M} \le C\left\{\|F\|_{X'} + \|G\|_{M'}\right\}.
$$

Proof. See [32, Theorem 3.1]. □

Next, we recall that for each $\mathbf{r},\mathbf{s} \in \mathbb{L}^2(\Omega)$ (see [39, Theorem 3.8] for details) there holds

$$
\|\mu(|\mathbf{r}|)\mathbf{r} - \mu(|\mathbf{s}|)\mathbf{s}\|_{0,\Omega} \le L_\mu \|\mathbf{r}-\mathbf{s}\|_{0,\Omega}, \tag{3.3}
$$

$$
\int_\Omega \{\mu(|\mathbf{r}|)\mathbf{r} - \mu(|\mathbf{s}|)\mathbf{s}\} : (\mathbf{r}-\mathbf{s}) \ge \mu_1 \|\mathbf{r}-\mathbf{s}\|_{0,\Omega}^2. \tag{3.4}
$$

where $L_\mu := \max\{\mu_2, 2\mu_2 - \mu_1\}$, with μ_1 and μ_2 being the bounds of μ given in (2.2).

3.2 A fixed point approach

We begin the solvability analysis of (2.16) by defining the operator $\mathbf{T} : \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \to \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ by

$$
\mathbf{T}(\mathbf{z}_\mathrm{S}) := \mathbf{u}_\mathrm{S} \quad \forall\, \mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}), \tag{3.5}
$$

where $\mathbf{u}_\mathrm{S}$ is the third component of $\underline{\mathbf{t}} \in \mathbf{X}$, which in turn is the first component of the unique solution (to be confirmed below) of the nonlinear problem: Find $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}) \in \mathbb{X}\times\mathbb{M}$, such that

$$
\begin{aligned}
[\mathbf{A}(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{p}}] &= [\mathbf{F},(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{X}, \\
[\mathbf{B}(\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{q}}] &= [\mathbf{G},\underline{\mathbf{q}}] \qquad \forall \underline{\mathbf{q}} \in \mathbb{M},
\end{aligned}
\tag{3.6}
$$

It follows that $((\underline{\mathbf{t}},\underline{\boldsymbol{\varphi}}),\underline{\mathbf{p}}) \in \mathbb{X}\times\mathbb{M}$ is a solution of (2.16) if and only if $\mathbf{u}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ satisfies

$$
\mathbf{T}(\mathbf{u}_\mathrm{S}) = \mathbf{u}_\mathrm{S}. \tag{3.7}
$$

However, we remark in advance that the definition of $\mathbf{T}$ will make sense only in a closed ball of $\mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$. Now, it is clear that problem (3.6) has the same structure as the one in Theorem 3.1. Therefore, in what follows we apply this result to establish the well-posedness of (3.6), equivalently the well-definiteness of $\mathbf{T}$. To that end, we first observe that the kernel of the operator $\mathbf{B}$ (cf. (2.20)) can be written, equivalently, as

$$
\mathbb{V} := \left\{(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}) \in \mathbb{X} : \quad [\mathbf{B}(\underline{\mathbf{r}},\underline{\boldsymbol{\psi}}),\underline{\mathbf{q}}] = 0 \quad \forall\, \underline{\mathbf{q}} \in \mathbb{M}\right\} = \tilde{\mathbf{X}} \times \tilde{\mathbf{M}},
$$

where

$$
\tilde{\mathbf{X}} = \mathbb{L}^2_{\mathrm{tr}}(\Omega_\mathrm{S}) \times \mathbb{H}_0(\mathbf{div};\Omega_\mathrm{S}) \times \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) \times \tilde{\mathbf{H}}_0(\mathrm{div}\,;\Omega_\mathrm{D})
$$

and

$$
\tilde{\mathbf{M}} = \tilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \times H^{1/2}(\Sigma),
$$

with

$$
\tilde{\mathbf{H}}_0(\mathrm{div}\,;\Omega_\mathrm{D}) := \{\mathbf{v}_\mathrm{D} \in \mathbf{H}_0(\mathrm{div}\,;\Omega_\mathrm{D}) : \quad \mathrm{div}\,(\mathbf{v}_\mathrm{D}) \in P_0(\Omega_\mathrm{D})\}
$$

and

$$\tilde{\mathbf{H}}_{00}^{1/2}(\Sigma) := \left\{ \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \right\}.$$

At this point we recall, for later use, that the following inequalities hold (see, [5, Proposition 3.1, Chapter IV], [37, Lemma 3.2], and [5, 40], respectively, for details)

$$c_1(\Omega_\mathrm{S})\|\boldsymbol{\tau}_\mathrm{S}\|_{0,\Omega_\mathrm{S}}^2 \leq \|\boldsymbol{\tau}_\mathrm{S}^\mathrm{d}\|_{0,\Omega_\mathrm{S}}^2 + \|\mathbf{div}\boldsymbol{\tau}_\mathrm{S}\|_{0,\Omega_\mathrm{S}}^2 \qquad \forall\, \boldsymbol{\tau}_\mathrm{S} \in \mathbb{H}_0(\mathbf{div};\Omega_\mathrm{S}), \tag{3.8}$$

$$\|\mathbf{v}_\mathrm{D}\|_{0,\Omega_\mathrm{D}}^2 \geq C_{\mathrm{div}}\,\|\mathbf{v}_\mathrm{D}\|_{\mathbf{div},\Omega_\mathrm{D}}^2 \qquad \forall\,\mathbf{v}_\mathrm{D} \in \tilde{\mathbf{H}}_0(\mathbf{div};\Omega_\mathrm{D}), \tag{3.9}$$

$$\|\mathbf{e}(\mathbf{v}_\mathrm{S})\|_{0,\Omega_\mathrm{S}}^2 \geq C_{\mathrm{Ko}}\|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}^2 \qquad \forall\,\mathbf{v}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}). \tag{3.10}$$

In what follows, and through the verification of the hypotheses of Theorem 3.1, we provide sufficient conditions under which the operator $\mathbf{T}$ is well-defined. We begin with the Lipschitz-continuity and strong-monotonicity of $a(\mathbf{z}_\mathrm{S})(\cdot)$ for a given $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega)$.

Lemma 3.2 *Assume that*

$$\kappa_1 \in \left(0, \frac{2\delta_1\mu_1}{L_\mu}\right), \quad \kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1}\right)\right) \quad \textit{and} \quad \kappa_4 \in \left(0, 2\delta_3 C_{\mathrm{Ko}}\kappa_3\left(1 - \frac{\delta_2}{2}\right)\right),$$

with $\delta_1 \in \left(0, \dfrac{2}{L_\mu}\right)$, $\delta_2 \in (0,2)$, $\delta_3 \in (0,2)$, *and that* $\kappa_2 > 0$. *Then, there exists* $r_0 > 0$ *such that for each* $r \in (0, r_0)$, *the nonlinear operator* $a(\mathbf{z}_\mathrm{S})(\cdot)$ *is strongly-monotone on* $\tilde{\mathbf{X}}$ *and Lipschitz-continuous on* $\mathbf{X}$, *for each* $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ *such that* $\|\mathbf{z}_\mathrm{S}\| \leq r$, *with respective constants* $\alpha(\Omega) > 0$ *and* $\gamma(\Omega) > 0$, *independent of* $\mathbf{z}_\mathrm{S}$.

Proof. Let $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ such that $\|\mathbf{z}_\mathrm{S}\|_{1,\Omega} \leq r$, with $r \in (0, r_0)$ and r_0 to be defined below. We first observe that a_1 and $a_2(\mathbf{z}_\mathrm{S})$, and consequently $a(\mathbf{z}_\mathrm{S})$, are Lipschitz-continuous. In fact, using the Cauchy–Schwarz inequality, and the Lipschitz-continuity of the operator induced by μ (cf. (3.3)), we deduce from (2.19) that a_1 is Lipschitz continuous with a positive constant L_{a_1}, depending on L_μ, and the parameters $\kappa_i, i \in \{1, \ldots, 4\}$, that is

$$\|a_1(\underline{\mathbf{t}}) - a_1(\underline{\mathbf{r}})\|_{\mathbf{X}'} \leq L_{a_1}\|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\, \underline{\mathbf{t}}, \underline{\mathbf{r}} \in \mathbf{X}. \tag{3.11}$$

In addition, from (2.12) and (2.19) we easily obtain that

$$\begin{aligned} |[a_2(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}), \underline{\mathbf{r}}]| &\leq (\kappa_1^2+1)^{1/2}\|\mathbf{z}_\mathrm{S}\|_{\mathbf{L}^4(\Omega_\mathrm{S})}\|\mathbf{u}_\mathrm{S}\|_{\mathbf{L}^4(\Omega_\mathrm{S})}\|\underline{\mathbf{r}}\|_{\mathbf{X}} \\ &\leq c_2(\Omega_\mathrm{S})(\kappa_1^2+1)^{1/2}\|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\|\underline{\mathbf{t}}\|_{\mathbf{X}}\|\underline{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\, \underline{\mathbf{t}}, \underline{\mathbf{r}} \in \mathbf{X}, \end{aligned} \tag{3.12}$$

which, together with the linearity of $a_2(\mathbf{z}_\mathrm{S})$, and the Lipschitz-continuity of a_1, implies that

$$\begin{aligned} \|a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}) - a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{r}})\|_{\mathbf{X}'} &\leq (L_{a_1} + c_2(\Omega_\mathrm{S})(\kappa_1^2+1)^{1/2}\|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}})\|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|_{\mathbf{X}} \\ &\leq \gamma(\Omega)\|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\, \underline{\mathbf{t}}, \underline{\mathbf{r}} \in \mathbf{X}, \end{aligned} \tag{3.13}$$

with $\gamma(\Omega) := L_{a_1} + c_2(\Omega_\mathrm{S})(\kappa_1^2+1)^{1/2} r$. Now, for the strong monotonicity of $a(\mathbf{z}_\mathrm{S})$, we observe from the definition of a_1 (cf. (2.18)) that it readily follows that

$$
\begin{aligned}
[a_1(\underline{\mathbf{t}}) - a_1(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \;=\; & (\mu(|\mathbf{t}_{\rm S}|)\mathbf{t}_{\rm S} - \mu(|\mathbf{r}_{\rm S}|)\mathbf{r}_{\rm S}, \mathbf{t}_{\rm S} - \mathbf{r}_{\rm S})_{\rm S} + \kappa_1 \|(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})^{\rm d}\|^2_{0,\Omega_{\rm S}} \\
& - \kappa_1 (\mu(|\mathbf{t}_{\rm S}|)\mathbf{t}_{\rm S} - \mu(|\mathbf{r}_{\rm S}|)\mathbf{r}_{\rm S}, (\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})^{\rm d})_{\rm S} + \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})\|^2_{0,\Omega_{\rm S}} \\
& + \kappa_3 ||\mathbf{e}(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S})||^2_{0,\Omega_{\rm S}} - \kappa_3 (\mathbf{t}_{\rm S} - \mathbf{r}_{\rm S}, \mathbf{e}(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}))_{\rm S} + \kappa_4 \|\boldsymbol{\rho}_{\rm S} - \boldsymbol{\eta}_{\rm S}\|^2_{0,\Omega_{\rm S}} \\
& - \frac{1}{2}\kappa_4 (\nabla(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}) - (\nabla(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}))^{\rm t}, \boldsymbol{\rho}_{\rm S} - \boldsymbol{\eta}_{\rm S})_{\rm S} \\
& + (\mathbf{K}^{-1}(\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}), \mathbf{u}_{\rm D} - \mathbf{v}_{\rm D})_{\rm D}.
\end{aligned}
$$

Hence, we proceed similarly to the proof of [10, Lemma 3.4], utilize the Cauchy–Schwarz and Young inequalities, and apply (2.9), (3.3) and (3.4) to obtain that for any $\delta_1, \delta_2, \delta_3 > 0$, and for all $\underline{\mathbf{t}}, \underline{\mathbf{r}} \in \tilde{\mathbf{X}}$, there holds

$$
\begin{aligned}
[a_1(\underline{\mathbf{t}}) - a_1(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \;\geq\; & \mu_1 \|\mathbf{t}_{\rm S} - \mathbf{r}_{\rm S}\|^2_{0,\Omega_{\rm S}} + \kappa_1 \|(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})^{\rm d}\|^2_{0,\Omega_{\rm S}} \\
& - \frac{\kappa_1 L_\mu}{2} \left\{ \frac{1}{\delta_1} \|\mathbf{t}_{\rm S} - \mathbf{r}_{\rm S}\|^2_{0,\Omega_{\rm S}} + \delta_1 \|(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})^{\rm d}\|^2_{0,\Omega_{\rm S}} \right\} \\
& + \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})\|^2_{0,\Omega_{\rm S}} + \kappa_3 \|\mathbf{e}(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S})\|^2_{0,\Omega_{\rm S}} \\
& - \frac{\kappa_3}{2} \left\{ \frac{1}{\delta_2} \|\mathbf{t}_{\rm S} - \mathbf{r}_{\rm S}\|^2_{0,\Omega_{\rm S}} + \delta_2 \|\mathbf{e}\big(\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}\big)\|^2_{0,\Omega_{\rm S}} \right\} \\
& + \kappa_4 \|\boldsymbol{\rho}_{\rm S} - \boldsymbol{\eta}_{\rm S}\|^2_{0,\Omega_{\rm S}} - \frac{\kappa_4}{2} \left\{ \frac{1}{\delta_3} \|\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}\|^2_{1,\Omega_{\rm S}} + \delta_3 \|\boldsymbol{\rho}_{\rm S} - \boldsymbol{\eta}_{\rm S}\|^2_{0,\Omega_{\rm S}} \right\} \\
& + C_{\mathbf{K}} \|\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}\|^2_{0,\Omega_{\rm D}},
\end{aligned}
$$

which, together with (3.9) and the Korn's inequality (3.10), implies

$$
\begin{aligned}
[a_1(\underline{\mathbf{t}}) - a_1(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \;\geq\; & \left\{ \left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1} \right) - \frac{\kappa_3}{2\delta_2} \right\} \|\mathbf{t}_{\rm S} - \mathbf{r}_{\rm S}\|^2_{0,\Omega_{\rm S}} \\
& + \kappa_1 \left(1 - \frac{\delta_1 L_\mu}{2} \right) \|(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})^{\rm d}\|^2_{0,\Omega_{\rm S}} + \kappa_2 \|\mathbf{div}(\boldsymbol{\sigma}_{\rm S} - \boldsymbol{\tau}_{\rm S})\|^2_{0,\Omega_{\rm S}} \\
& + \left\{ C_{\rm Ko} \kappa_3 \left(1 - \frac{\delta_2}{2} \right) - \frac{\kappa_4}{2\delta_3} \right\} \|\mathbf{u}_{\rm S} - \mathbf{v}_{\rm S}\|^2_{1,\Omega_{\rm S}} \\
& + \kappa_4 \left(1 - \frac{\delta_3}{2} \right) \|\boldsymbol{\rho}_{\rm S} - \boldsymbol{\eta}_{\rm S}\|^2_{0,\Omega_{\rm S}} + C_{\mathbf{K}} C_{\rm div} \|\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}\|^2_{\mathrm{div},\Omega_{\rm D}}.
\end{aligned}
\tag{3.14}
$$

Then, assuming the stipulated hypotheses on $\delta_1, \kappa_1, \kappa_3, \delta_2, \delta_3, \kappa_4$, and κ_2, and applying the inequality (3.8), we can define the positive constants

$$
\begin{aligned}
\alpha_1(\Omega_{\rm S}) &:= \left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1} \right) - \frac{\kappa_3}{2\delta_2}, & \alpha_2(\Omega_{\rm S}) &:= \min\left\{ \kappa_1 \left(1 - \frac{\delta_1 L_\mu}{2} \right), \frac{\kappa_2}{2} \right\}, \\
\alpha_3(\Omega_{\rm S}) &:= \min\left\{ \alpha_1(\Omega_{\rm S}) c_1(\Omega_{\rm S}), \frac{\kappa_2}{2} \right\}, & \alpha_4(\Omega_{\rm S}) &:= C_{\rm Ko} \kappa_3 \left(1 - \frac{\delta_2}{2} \right) - \frac{\kappa_4}{2\delta_3}, \\
\alpha_5(\Omega_{\rm S}) &:= \kappa_4 \left(1 - \frac{\delta_3}{2} \right), & \alpha_6(\Omega_{\rm D}) &:= C_{\mathbf{K}} C_{\rm div},
\end{aligned}
$$

which allow us to deduce from (3.14) that

$$
[a_1(\underline{\mathbf{t}}) - a_1(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \;\geq\; \alpha_0(\Omega) \|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|^2_{\mathbf{X}} \qquad \forall\, \underline{\mathbf{t}},\, \underline{\mathbf{r}} \in \tilde{\mathbf{X}}, \tag{3.15}
$$

where

$$\alpha_0(\Omega) := \min_{i\in\{1,\ldots,5\}} \{\alpha_i(\Omega_\mathrm{S}), \alpha_6(\Omega_\mathrm{D})\} \tag{3.16}$$

is the strong monotonicity constant of a_1. Moreover, according to the definition of $a(\mathbf{z}_\mathrm{S})$ (cf. (2.19)), and combining (3.12) and (3.15), we obtain

$$[a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}) - a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \,\geq\, \left\{\alpha_0(\Omega) - c_2(\Omega_\mathrm{S})(\kappa_1^2+1)^{1/2}\|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\right\} \|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|^2_{\mathbf{X}}\,,$$

for all $\underline{\mathbf{t}}, \underline{\mathbf{r}} \in \tilde{\mathbf{X}}$. Consequently, by requiring $\|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq r_0$, with

$$r_0 := \frac{\alpha_0(\Omega)}{2c_2(\Omega_\mathrm{S})(\kappa_1^2+1)^{1/2}}, \tag{3.17}$$

the strong monotonicity of $a(\mathbf{z}_\mathrm{S})$ is ensured with a constant $\alpha(\Omega) := \dfrac{\alpha_0(\Omega)}{2}$ independent of $\mathbf{z}_\mathrm{S}$, that is

$$[a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}) - a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{r}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \geq \frac{\alpha_0(\Omega)}{2}\|\underline{\mathbf{t}} - \underline{\mathbf{r}}\|^2_{\mathbf{X}} \qquad \forall\, \underline{\mathbf{t}}, \underline{\mathbf{r}} \in \tilde{\mathbf{X}}. \tag{3.18}$$

$\square$

We continue with the pseudolinearity of $a(\mathbf{z}_\mathrm{S})(\cdot)$.

Lemma 3.3 *Given $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, for each pair $(\underline{\mathbf{t}}, \underline{\mathbf{t}}^\perp) \in \tilde{\mathbf{X}} \times \tilde{\mathbf{X}}^\perp$ there holds*

$$a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}} + \underline{\mathbf{t}}^\perp) = a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}) + a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}^\perp). \tag{3.19}$$

Proof. Let $\mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$. We first decompose $\mathbf{X}$ as $\mathbf{X} = \mathbf{X}^\mathrm{l} \times \mathbf{X}^\mathrm{r}$, with $\mathbf{X}^\mathrm{l} := \mathbb{L}^2_{\mathrm{tr}}(\Omega_\mathrm{S})$ and $\mathbf{X}^\mathrm{r} := \mathbb{H}_0(\mathbf{div};\Omega_\mathrm{S}) \times \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) \times \mathbf{H}_0(\mathrm{div}\,;\Omega_\mathrm{D})$. In addition, since $\mathbf{B}$ (cf. (2.20)) does not depend on the variable from $\mathbf{X}^\mathrm{l}$, we easily obtain that $\tilde{\mathbf{X}} = \mathbf{X}^\mathrm{l} \times \tilde{\mathbf{X}}^\mathrm{r}$, with

$$\tilde{\mathbf{X}}^\mathrm{r} := \mathbb{H}_0(\mathbf{div};\Omega_\mathrm{S}) \times \mathbf{H}^1\Gamma_\mathrm{S}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) \times \tilde{\mathbf{H}}_0(\mathrm{div}\,;\Omega_\mathrm{D}) \subseteq \mathbf{X}^\mathrm{r}\,,$$

which yields $\tilde{\mathbf{X}}^\perp = \{\mathbf{0}\} \times (\tilde{\mathbf{X}}^\mathrm{r})^\perp$. In turn, given $\underline{\mathbf{s}} = (\mathbf{0}, \underline{\mathbf{s}}^\mathrm{r})$, with $\underline{\mathbf{s}}^\mathrm{r} = (\boldsymbol{\sigma}_\mathrm{S}, \mathbf{u}_\mathrm{S}, \boldsymbol{\rho}_\mathrm{S}, \mathbf{u}_\mathrm{D}) \in \mathbf{X}^\mathrm{r}$ and $\underline{\mathbf{r}} = (\mathbf{r}_\mathrm{S}, \boldsymbol{\tau}_\mathrm{S}, \mathbf{v}_\mathrm{S}, \boldsymbol{\eta}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{X}$, there holds

$$\begin{aligned}
[a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{s}}), \underline{\mathbf{r}}] = \quad & -(\mathbf{r}_\mathrm{S}, \boldsymbol{\sigma}^\mathrm{d}_\mathrm{S})_\mathrm{S} + \kappa_1(\boldsymbol{\sigma}^\mathrm{d}_\mathrm{S}, \boldsymbol{\tau}^\mathrm{d}_\mathrm{S})_\mathrm{S} + \kappa_2(\mathbf{div}\boldsymbol{\sigma}_\mathrm{S}, \mathbf{div}\boldsymbol{\tau}_\mathrm{S})_\mathrm{S} + (\mathbf{div}\boldsymbol{\tau}_\mathrm{S}, \mathbf{u}_\mathrm{S})_\mathrm{S} - (\mathbf{div}\boldsymbol{\sigma}_\mathrm{S}, \mathbf{v}_\mathrm{S})_\mathrm{S} \\
& + (\boldsymbol{\tau}_\mathrm{S}, \boldsymbol{\rho}_\mathrm{S})_\mathrm{S} - (\boldsymbol{\sigma}_\mathrm{S}, \boldsymbol{\eta}_\mathrm{S})_\mathrm{S} + \kappa_3(\mathbf{e}(\mathbf{u}_\mathrm{S}), \mathbf{e}(\mathbf{v}_\mathrm{S}))_\mathrm{S} + \kappa_4\left(\boldsymbol{\rho}_\mathrm{S} - \frac{1}{2}(\nabla\mathbf{u}_\mathrm{S} - (\nabla\mathbf{u}_\mathrm{S})^\mathrm{t}), \boldsymbol{\eta}_\mathrm{S}\right)_\mathrm{S} \\
& + (\mathbf{K}^{-1}\mathbf{u}_\mathrm{D}, \mathbf{v}_\mathrm{D})_\mathrm{D} + ((\mathbf{z}\otimes\mathbf{u}_\mathrm{S})^\mathrm{d}, \kappa_1\boldsymbol{\tau}^\mathrm{d} - \mathbf{r}_\mathrm{S})_\mathrm{S},
\end{aligned}$$

which shows that $a(\mathbf{z}_\mathrm{S})$ is linear in $\{\mathbf{0}\} \times \mathbf{X}^\mathrm{r}$. Similarly, from the definition of $a(\mathbf{z}_\mathrm{S})$, we also find that for each $\underline{\mathbf{t}} := (\underline{\mathbf{t}}^\mathrm{l}, \underline{\mathbf{t}}^\mathrm{r}) \in \mathbf{X} = \mathbf{X}^\mathrm{l} \times \mathbf{X}^\mathrm{r}$ and for each $\underline{\mathbf{r}} \in \mathbf{X}$, there holds

$$[a(\mathbf{z}_\mathrm{S})(\mathbf{0}, \underline{\mathbf{t}}^\mathrm{r}) + a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}^\mathrm{l}, \mathbf{0}), \underline{\mathbf{r}}] = [a(\mathbf{z}_\mathrm{S})(\underline{\mathbf{t}}), \underline{\mathbf{r}}].$$

According to the previous analysis, it readily follows that $a(\mathbf{z}_\mathrm{S})$ satisfies (3.19). $\square$

Now, we establish the positive semi-definiteness of c.

Lemma 3.4 *There holds*

$$[c(\underline{\boldsymbol{\psi}}), \underline{\boldsymbol{\psi}}] \geq 0 \quad \forall\, \underline{\boldsymbol{\psi}} \in \mathbf{M}. \tag{3.20}$$

Proof. From the definition of operator c, it readily follows that

$$[c(\underline{\boldsymbol{\psi}}), \underline{\boldsymbol{\psi}}] := \sum_{l=1}^{n-1} \omega_l^{-1} \|\boldsymbol{\psi} \cdot \mathbf{t}_l\|_{0,\Sigma}^2 \geq 0 \quad \forall\, \underline{\boldsymbol{\psi}} \in \mathbf{M}, \tag{3.21}$$

which clearly confirms that c is positive semi-definite. □

We end the verification of the hypotheses of Theorem 3.1 with the corresponding inf-sup conditions for the bilinear forms b and $\mathbf{B}$.

Lemma 3.5 *There exist positive constants β_1 and β, such that*

$$\sup_{\substack{\underline{\mathbf{r}} \in \tilde{\mathbf{X}} \\ \underline{\mathbf{r}} \neq 0}} \frac{[b(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]}{\|\underline{\mathbf{r}}\|_{\mathbf{X}}} \geq \beta_1 \|\underline{\boldsymbol{\psi}}\|_{\mathbf{M}} \qquad \forall\, \underline{\boldsymbol{\psi}} \in \tilde{\mathbf{M}} \tag{3.22}$$

and

$$\sup_{\substack{(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X} \\ (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \neq \mathbf{0}}} \frac{[\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}]}{\|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{X}}} \geq \beta \|\underline{\mathbf{q}}\|_{\mathbb{M}} \qquad \forall\, \underline{\mathbf{q}} \in \mathbb{M}. \tag{3.23}$$

Proof. For the proof of (3.22) we refer the reader to [32, Lemma 4.3] whereas a slight modification of [33, Lemma 4.3] implies (3.23). We omit further details. □

We are now in position of establishing the well-posedness of (3.6) (equivalently the well-definiteness of $\mathbf{T}$).

Lemma 3.6 *Let $r \in (0, r_0)$, with r_0 given by (3.17). Assume that*

$$\kappa_1 \in \left(0, \frac{2\delta_1 \mu_1}{L_\mu}\right), \quad \kappa_3 \in \left(0, 2\delta_2 \left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1}\right)\right) \quad \text{and} \quad \kappa_4 \in \left(0, 2\delta_3 C_{\mathrm{Ko}} \kappa_3 \left(1 - \frac{\delta_2}{2}\right)\right),$$

with $\delta_1 \in \left(0, \dfrac{2}{L_\mu}\right)$, $\delta_2 \in (0,2)$, $\delta_3 \in (0,2)$, and that $\kappa_2 > 0$. Then, the problem (3.6) has a unique solution for each $\mathbf{z}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, such that $\|\mathbf{z}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq r$. Moreover, there exists a constant $c_{\mathbf{T}} > 0$, independent of $\mathbf{z}_{\mathrm{S}}$ and the data $\mathbf{f}_{\mathrm{S}}$ and f_{D}, such that there holds

$$\|\mathbf{T}(\mathbf{z}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} = \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{X} \times \mathbb{M}} \leq c_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\}. \tag{3.24}$$

Proof. Given $\mathbf{z}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, such that $\|\mathbf{z}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq r$, the well-posedness of (3.6) follows from Lemmas 3.2 – 3.5, and a straightforward application of Theorem 3.1. Now, concerning the estimate (3.24), we first deduce from the definitions of $\mathbf{F}$ and $\mathbf{G}$ (cf. (2.17)), and from the Cauchy–Schwarz and Young inequalities, that there exist constants $c_{\mathbf{F}} > 0$ and $c_{\mathbf{G}} > 0$, such that

$$\|\mathbf{F}\|_{\mathbb{X}'} \leq c_{\mathbf{F}} \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} \quad \text{and} \quad \|\mathbf{G}\|_{\mathbb{M}'} \leq c_{\mathbf{G}} \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}. \tag{3.25}$$

This fact and Theorem 3.1 imply the estimate

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{X} \times \mathbb{M}} \leq c_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\},$$

with $c_{\mathbf{T}}$ independent of $\mathbf{z}_{\mathrm{S}}$, which implies (3.24) and concludes the proof. □

We end this section by remarking that the constant $\alpha_0(\Omega)$ yielding the strong monotonicity of both a_1 and $a(\mathbf{z}_\mathrm{S})$ can be maximized by taking the parameters $\delta_1, \kappa_1, \delta_2, \kappa_3, \delta_3$, and κ_4 as the middle points of their feasible ranges, and by choosing κ_2 so that it maximizes the minimum defining $\alpha_2(\Omega_\mathrm{S})$. More precisely, we simply take

$$\begin{gathered}
\delta_1 = \frac{1}{L_\mu}, \quad \kappa_1 = \frac{\delta_1 \mu_1}{L_\mu} = \frac{\mu_1}{L_\mu^2}, \quad \delta_2 = 1, \quad \kappa_3 = \delta_2 \left(\mu_1 - \frac{\kappa_1 L_\mu}{2\delta_1} \right) = \frac{\mu_1}{2}, \quad \delta_3 = 1, \\
\kappa_4 = \delta_3 C_{\mathrm{Ko}} \kappa_3 \left(1 - \frac{\delta_2}{2} \right) = C_{\mathrm{Ko}} \frac{\mu_1}{4}, \quad \text{and} \quad \kappa_2 = 2\kappa_1 \left(1 - \frac{\delta_1 L_\mu}{2} \right) = \frac{\mu_1}{L_\mu^2},
\end{gathered} \tag{3.26}$$

which yields

$$\begin{gathered}
\alpha_1(\Omega_\mathrm{S}) = \frac{\mu_1}{4}, \quad \alpha_2(\Omega_\mathrm{S}) = \frac{\mu_1}{2L_\mu^2}, \quad \alpha_3(\Omega_\mathrm{S}) = \min\{c_1(\Omega_\mathrm{S}), 1\} \frac{\mu_1}{2L_\mu^2}, \\
\alpha_4(\Omega_\mathrm{S}) = C_{\mathrm{Ko}} \frac{\mu_1}{8}, \quad \alpha_5(\Omega_\mathrm{S}) = C_{\mathrm{Ko}} \frac{\mu_1}{8}, \quad \alpha_6(\Omega_\mathrm{D}) = C_{\mathbf{K}} C_{\mathrm{div}},
\end{gathered}$$

and hence

$$\alpha_0(\Omega) = \min \left\{ \min \left\{ C_{\mathrm{Ko}}, 1 \right\} \frac{\mu_1}{8}, \min \left\{ c_1(\Omega_\mathrm{S}), 1 \right\} \frac{\mu_1}{2L_\mu^2}, C_{\mathbf{K}} C_{\mathrm{div}} \right\}.$$

The explicit values of the stabilization parameters κ_i, $i \in \{1, \ldots, 4\}$, given in (3.26), will be employed in Section 7 for the corresponding numerical experiments.

3.3 Solvability analysis of the fixed point equation

In this section we proceed analogously to [10, Section 3.3] and establish existence of a fixed point of the operator $\mathbf{T}$ (cf. (3.5)) by means of the well known Schauder fixed point Theorem. The uniqueness can then be established by means of the Banach fixed point Theorem by utilizing the same estimates derived for the existence.

We begin by recalling the first of the aforementioned results (see, e.g. [18, Theorem 9.12-1(b)]).

Theorem 3.7 *Let W be a closed and convex subset of a Banach space X, and let $T : W \to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

The verification of the hypotheses of Theorem 3.7 is provided next.

Lemma 3.8 *Let $r \in (0, r_0)$, with r_0 given by (3.17), let W_r be the closed ball defined by $W_r := \left\{ \mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq r \right\}$, and assume that the data satisfy*

$$c_{\mathbf{T}} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\} \leq r, \tag{3.27}$$

with $c_{\mathbf{T}}$ the positive constant satisfying (3.31). Then there holds $\mathbf{T}(W_r) \subseteq W_r$.

Proof. It is a straightforward consequence of Lemma 3.6. □

We continue with the following result providing an estimate needed to derive next the required continuity and compactness properties of the operator $\mathbf{T}$.

Lemma 3.9 *Let $r \in (0, r_0)$, with r_0 given by (3.17), and let $W_r := \left\{ \mathbf{z}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \|\mathbf{z}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq r \right\}$. Then there exists a positive constant $C_{\mathbf{T}}$, depending on κ_1, $\|\mathbf{i}_c\|$, and $\alpha_0(\Omega)$, such that*

$$\|\mathbf{T}(\mathbf{z}_\mathrm{S}) - \mathbf{T}(\tilde{\mathbf{z}}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} \leq C_{\mathbf{T}} \|\mathbf{T}(\tilde{\mathbf{z}}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} \|\mathbf{z}_\mathrm{S} - \tilde{\mathbf{z}}_\mathrm{S}\|_{L^4(\Omega_\mathrm{S})} \qquad \forall\, \mathbf{z}_\mathrm{S}, \tilde{\mathbf{z}}_\mathrm{S} \in W_r. \tag{3.28}$$

Proof. Given r as indicated and $\mathbf{z}_{\rm S}, \tilde{\mathbf{z}}_{\rm S} \in W_r$, we let $\mathbf{u}_{\rm S} = \mathbf{T}(\mathbf{z}_{\rm S})$ and $\tilde{\mathbf{u}}_{\rm S} = \mathbf{T}(\tilde{\mathbf{z}}_{\rm S})$. According to the definition of $\mathbf{T}$, it follows that

$$\begin{aligned}
[\mathbf{A}(\mathbf{z}_{\rm S})(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X}, \\
[\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \quad \forall \underline{\mathbf{q}} \in \mathbb{M},
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{A}(\tilde{\mathbf{z}}_{\rm S})(\underline{\tilde{\mathbf{t}}}, \underline{\tilde{\boldsymbol{\varphi}}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\tilde{\mathbf{p}}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X}, \\
[\mathbf{B}(\underline{\tilde{\mathbf{t}}}, \underline{\tilde{\boldsymbol{\varphi}}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \quad \forall \underline{\mathbf{q}} \in \mathbb{M}.
\end{aligned}$$

Then, recalling the definition of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{G}$, in (2.18), (2.20) and (2.17), respectively, we subtract both problems to obtain

$$\begin{aligned}
[(a_1 + a_2(\mathbf{z}_{\rm S}))(\underline{\mathbf{t}}) - (a_1 + a_2(\tilde{\mathbf{z}}_{\rm S}))(\underline{\tilde{\mathbf{t}}}), \underline{\mathbf{r}}] + [b(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}] + [B_1(\underline{\mathbf{r}}), p_{\rm D} - \tilde{p}_{\rm D}] &= 0, \\
[b(\underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}), \underline{\boldsymbol{\psi}}] - [c(\underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}), \underline{\boldsymbol{\psi}}] + [B_2(\underline{\boldsymbol{\psi}}), l - \tilde{l}] &= 0, \\
[B_1(\underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}), q_{\rm D}] &= 0, \\
[B_2(\underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}), j] &= 0,
\end{aligned}$$

for all $(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}, q_{\rm D}, j) \in \mathbf{X} \times \mathbf{M} \times L^2_0(\Omega_{\rm D}) \times \mathrm{R}$. In particular, taking $\underline{\mathbf{r}} = \underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}$, $\underline{\boldsymbol{\psi}} = \underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}$, $q_{\rm D} = p_{\rm D} - \tilde{p}_{\rm D}$ and $j = l - \tilde{l}$ in the latter system, we get

$$[(a_1 + a_2(\mathbf{z}_{\rm S}))(\underline{\mathbf{t}}) - (a_1 + a_2(\tilde{\mathbf{z}}))(\underline{\tilde{\mathbf{t}}}), \underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}] = -[c(\underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}), \underline{\boldsymbol{\varphi}} - \underline{\tilde{\boldsymbol{\varphi}}}]. \tag{3.29}$$

Hence, adding and substracting $a_2(\mathbf{z}_{\rm S})(\underline{\tilde{\mathbf{t}}})$ in the second term on the left hand side of (3.29), and using the strong monotonicity of $a(\mathbf{z}_{\rm S}) = a_1 + a_2(\mathbf{z}_{\rm S})$ (cf. (3.18)), and the fact that c is positive semi-definite, it follows that

$$\frac{\alpha_0(\Omega)}{2} \|\underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}\|^2_{\mathbf{X}} \le [a_2(\tilde{\mathbf{z}}_{\rm S} - \mathbf{z}_{\rm S})(\underline{\tilde{\mathbf{t}}}), \underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}].$$

In this way, by applying the first inequality in (3.12) and then bounding $\|\tilde{\mathbf{u}}_{\rm S}\|_{\mathbf{L}^4(\Omega_{\rm S})}$ by $\|\mathbf{i}_c\| \|\tilde{\mathbf{u}}_{\rm S}\|_{1,\Omega_{\rm S}}$, we deduce that

$$\|\underline{\mathbf{t}} - \underline{\tilde{\mathbf{t}}}\|_{\mathbf{X}} \le \frac{2(\kappa_1^2 + 1)^{1/2} \|\mathbf{i}_c\|}{\alpha_0(\Omega)} \|\tilde{\mathbf{u}}_{\rm S}\|_{1,\Omega_{\rm S}} \|\mathbf{z}_{\rm S} - \tilde{\mathbf{z}}_{\rm S}\|_{\mathbf{L}^4(\Omega_{\rm S})},$$

which implies (3.28) with

$$C_{\mathbf{T}} := \frac{2(\kappa_1^2 + 1)^{1/2} \|\mathbf{i}_c\|}{\alpha_0(\Omega)}, \tag{3.30}$$

thus completing the proof. □

Owing to the above analysis, we establish now the announced properties of the operator $\mathbf{T}$.

Lemma 3.10 *Given $r \in (0, r_0)$, with r_0 defined by* (3.17)*, we let $W_r := \{\mathbf{z}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}) : \|\mathbf{z}_{\rm S}\|_{1,\Omega_{\rm S}} \le r\}$, and assume that the data $\mathbf{f}_{\rm S}$ and $f_{\rm D}$ satisfy* (3.27)*. Then, $\mathbf{T} : W_r \to W_r$ is continuous and $\overline{\mathbf{T}(W_r)}$ is compact.*

Proof. The required result follows straightforwardly from estimate (3.28) and the compactness of $\mathbf{i}_c : \mathbf{H}^1(\Omega_{\rm S}) \to \mathbf{L}^4(\Omega_{\rm S})$. We omit further details and refer to [10, Lemma 3.8]. □

We are now in position of establishing the main result of this section.

Theorem 3.11 *Suppose that the parameters κ_i, $i \in \{1, \ldots, 4\}$, satisfy the conditions required by Lemma 3.6. In addition, given $r \in (0, r_0)$, with r_0 defined by (3.17), we let $W_r := \{\mathbf{z}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \|\mathbf{z}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq r\}$, and assume that the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} satisfy (3.27). Then, the augmented fully-mixed formulation (2.16) has a unique solution $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ with $\mathbf{u}_{\mathrm{S}} \in W_r$, and there holds*

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{X}\times\mathbb{M}} \leq c_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\}. \tag{3.31}$$

Proof. The equivalence between (2.16) and the fixed point equation (3.7), together with Lemmas 3.8 and 3.10, confirm the existence of solution of (2.16) as a direct application of the Schauder fixed point Theorem 3.7. In addition, it is clear that the estimate (3.31) follows straightforwardly from (3.24). On the other hand, using the estimate (3.28), the continuity of the compact injection $\mathbf{i}_c$, and the definitions of $C_{\mathbf{T}}$ (cf. (3.30)) and r_0 (cf. (3.17)), we easily obtain

$$\|\mathbf{T}(\mathbf{z}_{\mathrm{S}}) - \mathbf{T}(\tilde{\mathbf{z}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \leq \frac{r}{r_0} \|\mathbf{z}_{\mathrm{S}} - \tilde{\mathbf{z}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}},$$

which, thanks to the Banach fixed point Theorem, implies that the solution is actually unique. □

4 The Galerkin scheme

In this section we introduce the Galerkin scheme of problem (2.16) and analyse its well-posedness by establishing suitable assumptions on the finite element subspaces involved.

4.1 Discrete setting

We first introduce a set of arbitrary discrete subspaces, namely

$$\begin{gathered} L^2_h(\Omega_\star) \subset L^2(\Omega_\star), \quad \mathbf{H}_h(\Omega_\star) \subset \mathbf{H}(\mathbf{div}; \Omega_\star), \quad \star \in \{\mathrm{S}, \mathrm{D}\}, \\ \mathbf{H}^1_h(\Omega_{\mathrm{S}}) \subset \mathbf{H}^1(\Omega_{\mathrm{S}}), \quad \mathbb{L}^2_{\mathrm{skew},h}(\Omega_{\mathrm{S}}) \subset \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\mathrm{S}}), \quad \Lambda^{\mathrm{S}}_h(\Sigma) \subset H^{1/2}_{00}(\Sigma), \quad \Lambda^{\mathrm{D}}_h(\Sigma) \subset H^{1/2}(\Sigma), \end{gathered} \tag{4.1}$$

and set

$$\begin{aligned} \mathbb{H}_h(\Omega_{\mathrm{S}}) &:= \{\boldsymbol{\tau}_{\mathrm{S}} \in \mathbb{H}(\mathbf{div}; \Omega_{\mathrm{S}}) : \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall\, \mathbf{c} \in \mathrm{R}^n\}, \\ \mathbb{H}_{h,0}(\Omega_{\mathrm{S}}) &:= \mathbb{H}_h(\Omega_{\mathrm{S}}) \cap \mathbb{H}_0(\mathbf{div}; \Omega_{\mathrm{S}}), \\ \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) &:= \mathbf{H}^1_h(\Omega_{\mathrm{S}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \\ \mathbf{H}_{h,0}(\Omega_{\mathrm{D}}) &:= \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}_0(\mathrm{div}\,; \Omega_{\mathrm{D}}), \\ \mathbb{L}^2_{\mathrm{tr}\,,h}(\Omega_{\mathrm{S}}) &:= [L^2_h(\Omega_{\mathrm{S}})]^{n\times n} \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega_{\mathrm{S}}), \\ L^2_{h,0}(\Omega_{\mathrm{D}}) &:= L^2_h(\Omega_{\mathrm{D}}) \cap L^2_0(\Omega_{\mathrm{D}}), \\ \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) &:= [\Lambda^{\mathrm{S}}_h(\Sigma)]^n. \end{aligned} \tag{4.2}$$

Then, defining the global spaces, unknowns, and test functions as follows

$$
\begin{aligned}
&\mathbf{X}_h := \mathbb{L}^2_{\mathrm{tr},h}(\Omega_\mathrm{S}) \times \mathbb{H}_{h,0}(\Omega_\mathrm{S}) \times \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew},h}(\Omega_\mathrm{S}) \times \mathbf{H}_{h,0}(\Omega_\mathrm{D})\,,\\
&\mathbf{M}_h := \boldsymbol{\Lambda}^\mathrm{S}_h(\Sigma) \times \Lambda^\mathrm{D}_h(\Sigma)\,, \quad \mathbb{X}_h := \mathbf{X}_h \times \mathbf{M}_h, \quad \mathbb{M}_h := L^2_{h,0}(\Omega_\mathrm{D}) \times \mathrm{R}\,,\\
&\underline{\mathbf{t}}_h := (\mathbf{t}_{\mathrm{S},h}, \boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{S},h}, \boldsymbol{\rho}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{M}_h\,,\\
&\underline{\mathbf{r}}_h := (\mathbf{r}_{\mathrm{S},h}, \boldsymbol{\tau}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \boldsymbol{\eta}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{M}_h\,,\\
&\underline{\mathbf{p}}_h := (p_{\mathrm{D},h}, l_h) \in \mathbb{M}_h\,, \quad \text{and} \quad \underline{\mathbf{q}}_h := (q_{\mathrm{D},h}, j_h) \in \mathbb{M}_h\,,
\end{aligned}
\tag{4.3}
$$

the Galerkin scheme associated with problem (2.16) reads: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$
\begin{aligned}
[\mathbf{A}(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h,\\
[\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \qquad\quad \forall \underline{\mathbf{q}}_h \in \mathbb{M}_h.
\end{aligned}
\tag{4.4}
$$

Now, we proceed similarly to [37] and [32] (see also [11]), an derive suitable hypotheses on the spaces (4.1) ensuring the well-posedness of problem (4.4). We begin by noticing that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_\mathrm{S})$ and $L^2_{h,0}(\Omega_\mathrm{D})$, we need to be able to eliminate multiples of the identity matrix and constant polynomials from $\mathbb{H}_h(\Omega_\mathrm{S})$ and $L^2_h(\Omega_\mathrm{D})$, respectively. This requirement is certainly satisfied if we assume:

(H.0) $[P_0(\Omega_\mathrm{S})]^n \subseteq \mathbf{H}_h(\Omega_\mathrm{S})$ and $P_0(\Omega_\mathrm{D}) \subseteq L^2_h(\Omega_\mathrm{D})$, where $P_0(\Omega_\mathrm{S})$ and $P_0(\Omega_\mathrm{D})$ are the spaces of constant polynomials on Ω_S and Ω_D, respectively. In particular, it follows that $\mathbb{I} \in \mathbb{H}_h(\Omega_\mathrm{S})$ for all h, and hence there holds the decomposition

$$
\mathbb{H}_h(\Omega_\mathrm{S}) = \mathbb{H}_{h,0}(\Omega_\mathrm{S}) \oplus P_0(\Omega_\mathrm{S})\mathbb{I}. \tag{4.5}
$$

Next, we look at the discrete kernel of $\mathbf{B}$, which is given by

$$
\mathbb{V}_h := \Big\{ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h : \quad [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h] = 0 \quad \forall \underline{\mathbf{q}}_h \in \mathbb{M}_h \Big\}.
$$

In order to have a more explicit definition of $\mathbb{V}_h$, we introduce the following assumption:

(H.1) $\mathrm{div}\, \mathbf{H}_h(\Omega_\mathrm{D}) \subseteq L^2_h(\Omega_\mathrm{D})$.

Then, owing to **(H.1)** and recalling the definition of $\mathbf{B}$ (cf. (2.20)), it follows that $\mathbb{V}_h = \tilde{\mathbf{X}}_h \times \tilde{\mathbf{M}}_h$, where

$$
\tilde{\mathbf{X}}_h = \mathbb{L}^2_{\mathrm{tr},h}(\Omega_\mathrm{S}) \times \mathbb{H}_{h,0}(\Omega_\mathrm{S}) \times \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \times \mathbb{L}^2_{\mathrm{skew},h}(\Omega_\mathrm{S}) \times \tilde{\mathbf{H}}_{h,0}(\Omega_\mathrm{D})
$$

and

$$
\tilde{\mathbf{M}}_h = \tilde{\boldsymbol{\Lambda}}^\mathrm{S}_h(\Sigma) \times \Lambda^\mathrm{D}_h(\Sigma)\,,
$$

with

$$
\tilde{\mathbf{H}}_{h,0}(\Omega_\mathrm{D}) := \Big\{ \mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{h,0}(\Omega_\mathrm{D}) : \quad \mathrm{div}\,(\mathbf{v}_{\mathrm{D},h}) \in P_0(\Omega_\mathrm{D}) \Big\}\,,
$$

and

$$
\tilde{\boldsymbol{\Lambda}}^\mathrm{S}_h(\Sigma) := \Big\{ \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}^\mathrm{S}_h(\Sigma) : \quad \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \Big\}. \tag{4.6}
$$

In particular, it readily follows that $\mathbb{V}_h \subseteq \mathbb{V}$.

On the other hand, for the subsequent analysis we need to ensure the discrete version of the inf-sup conditions (3.22) and (3.23) of b and $\mathbf{B}$, respectively, namely the existence of constants $\tilde{\beta}_1$, $\tilde{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\underline{\mathbf{r}}_h \in \tilde{\mathbf{X}}_h \\ \underline{\mathbf{r}}_h \neq 0}} \frac{[b(\underline{\mathbf{r}}_h), \underline{\boldsymbol{\psi}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbf{X}}} \geq \tilde{\beta}_1 \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{M}} \qquad \forall\, \underline{\boldsymbol{\psi}}_h \in \tilde{\mathbf{M}}_h\,, \tag{4.7}$$

and

$$\sup_{\substack{(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h \\ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \neq \mathbf{0}}} \frac{[\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h]}{\|(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)\|_{\mathbb{X}}} \geq \tilde{\beta} \|\underline{\mathbf{q}}_h\|_{\mathbb{M}} \qquad \forall\, \underline{\mathbf{q}}_h \in \mathbb{M}_h\,. \tag{4.8}$$

For (4.7) we apply the same diagonal argument utilized in [32, Section 5.2] (see also [37, Lemma 3.8]) to deduce that b satisfies the discrete inf-sup condition (4.7) if and only if the following hypothesis holds:

(H.2) There exist $\widehat{\beta}_{\mathrm{S}}, \widehat{\beta}_{\mathrm{D}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{\mathrm{S},h} \in \mathbb{H}_h(\Omega_{\mathrm{S}}) \\ \boldsymbol{\tau}_{\mathrm{S},h} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_{\mathrm{S}}}} \geq \widehat{\beta}_{\mathrm{S}} \|\boldsymbol{\psi}_h\|_{1/2,00,\Sigma} \qquad \forall \boldsymbol{\psi}_h \in \tilde{\boldsymbol{\Lambda}}^{\mathrm{S}}_h(\Sigma), \tag{4.9}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \tilde{\mathbf{H}}_{h,0}(\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{\mathrm{D},h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div}\,,\Omega_{\mathrm{D}}}} \geq \widehat{\beta}_{\mathrm{D}} \|\xi_h\|_{1/2,\Sigma} \qquad \forall \xi_h \in \Lambda^{\mathrm{D}}_h(\Sigma). \tag{4.10}$$

Similarly, employing the same arguments in [32, Section 5.2] we obtain that $\mathbf{B}$ satisfies the discrete inf-sup condition (4.8) provided that the following hypothesis holds

(H.3) There exist $\tilde{\beta}_{\mathrm{D}} > 0$, independent of h, and $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$, such that

$$\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}^{\mathrm{S}}_h(\Sigma) \quad \forall\, h \quad \text{and} \quad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1 \rangle_\Sigma \neq 0, \tag{4.11}$$

$$\sup_{\substack{\mathbf{v}_{\mathrm{D},h} \in \mathbf{H}_{h,0}(\Omega_{\mathrm{D}}) \\ \mathbf{v}_{\mathrm{D},h} \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_{\mathrm{D},h}, q_{\mathrm{D},h})_{\mathrm{D}}}{\|\mathbf{v}_{\mathrm{D},h}\|_{\mathrm{div}\,,\Omega_{\mathrm{D}}}} \geq \tilde{\beta}_{\mathrm{D}} \|q_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} \qquad \forall q_{\mathrm{D},h} \in L^2_{h,0}(\Omega_{\mathrm{D}}). \tag{4.12}$$

4.2 Well-posedness of the discrete problem

In what follows, we assume that hypotheses **(H.0)**, **(H.1)**, **(H.2)**, and **(H.3)** hold, and, analogously to the analysis of the continuous problem, apply a fixed point argument to prove the well-posedness of the Galerkin scheme (4.4). To that end, we let $\mathbf{T}_h : \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \to \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ be the discrete operator defined by

$$\mathbf{T}_h(\mathbf{z}_{\mathrm{S},h}) := \mathbf{u}_{\mathrm{S},h} \quad \forall\, \mathbf{z}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}), \tag{4.13}$$

where $\mathbf{u}_{\mathrm{S},h}$ is the third component of $\underline{\mathbf{t}}_h$, which in turn is the first component of the unique solution (to be confirmed below) of the discrete nonlinear problem: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{z}_h)(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{X}_h, \\ [\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \qquad \forall \underline{\mathbf{q}}_h \in \mathbb{M}_h. \end{aligned} \tag{4.14}$$

Then, similarly as for the continuous case, the Galerkin scheme (4.4) can be rewritten, equivalently, as the fixed point problem: Find $\mathbf{u}_{S,h} \in \mathbf{H}^1_{h,\Gamma_S}(\Omega_S)$ such that

$$\mathbf{T}_h(\mathbf{u}_{S,h}) = \mathbf{u}_{S,h}. \tag{4.15}$$

Now, in order to prove the well-posedness of problem (4.4), or equivalently the well-definiteness of operator $\mathbf{T}_h$ (cf. (4.13)), we will require the following discrete version of Theorem 3.1 (cf. [32, Theorem 3.3]).

Theorem 4.1 *In addition to the spaces and operators defined in Theorem* 3.1*, let* $X_{1,h}, M_{1,h}$*, and* M_h *be finite dimensional subspaces of* X_1, M_1*, and* M*, respectively, and let* $X_h := X_{1,h} \times M_{1,h} \subseteq X := X_1 \times M_1$*. In turn, let* V_h *be the discrete kernel of* B*, that is,*

$$V_h := \left\{ (\mathbf{r}_h, \boldsymbol{\psi}_h) \in X_h : \quad [B(\mathbf{r}_h, \boldsymbol{\psi}_h), q_h] = 0 \quad \forall q_h \in M_h \right\},$$

and let $\tilde{X}_{1,h}$ *and* $\tilde{M}_{1,h}$ *be subspaces of* $X_{1,h}$ *and* $M_{1,h}$ *respectively, such that* $V_h = \tilde{X}_{1,h} \times \tilde{M}_{1,h}$*. Assume that*

(i) $A_1|_{\tilde{X}_{1,h}} : \tilde{X}_{1,h} \to \tilde{X}'_{1,h}$ *is Lipschitz continuous and strongly monotone, that is, there exist constants* $\gamma_h, \alpha_h > 0$ *such that*

$$\|A_1(\mathbf{s}_h) - A_1(\mathbf{r}_h)\|_{\tilde{X}'_{1,h}} \le \gamma_h \|\mathbf{s}_h - \mathbf{r}_h\|_{X_1} \quad \forall\, \mathbf{s}_h,\, \mathbf{r}_h \in \tilde{X}_{1,h}$$

and

$$[A_1(\mathbf{s}_h) - A_1(\mathbf{r}_h), \mathbf{s}_h - \mathbf{r}_h] \ge \alpha_h \|\mathbf{s}_h - \mathbf{r}_h\|^2_{X_1} \quad \forall\, \mathbf{s}_h,\, \mathbf{r}_h \in \tilde{X}_{1,h}\,.$$

(ii) For each pair $(\mathbf{r}_h, \mathbf{r}_h^\perp) \in \tilde{X}_{1,h} \times \tilde{X}^\perp_{1,h}$ *there holds the pseudolinear property*

$$A_1(\mathbf{r}_h + \mathbf{r}_h^\perp) = A_1(\mathbf{r}_h) + A_1(\mathbf{r}_h^\perp).$$

(iii) S *is positive semi-definite on* $\tilde{M}_{1,h}$*, that is,*

$$[S(\boldsymbol{\psi}_h), \boldsymbol{\psi}_h] \ge 0 \quad \forall\, \boldsymbol{\psi}_h \in \tilde{M}_{1,h}\,.$$

(iv) B_1 *satisfies an inf-sup condition on* $\tilde{X}_{1,h} \times \tilde{M}_{1,h}$*, that is, there exists* $\beta_{1,h} > 0$ *such that*

$$\sup_{\substack{\mathbf{r}_h \in \tilde{X}_{1,h} \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[B_1(\mathbf{r}_h), \boldsymbol{\psi}_h]}{\|\mathbf{r}_h\|_{X_1}} \ge \beta_{1,h} \|\boldsymbol{\psi}_h\|_{M_{1,h}} \quad \forall\, \boldsymbol{\psi}_h \in \tilde{M}_{1,h}\,.$$

(v) B *satisfies an inf-sup condition on* $X_h \times M_h$*, that is, there exists* $\beta_h > 0$ *such that*

$$\sup_{\substack{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in X_h \\ (\mathbf{r}_h, \boldsymbol{\psi}_h) \neq \mathbf{0}}} \frac{[B(\mathbf{r}_h, \boldsymbol{\psi}_h), q_h]}{\|(\mathbf{r}_h, \boldsymbol{\psi}_h)\|_X} \ge \beta_h \|q_h\|_M \quad \forall\, q_h \in M_h\,.$$

Then, there exists a unique $((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X_h \times M_h$*, such that*

$$\begin{aligned}
[A(\mathbf{t}_h, \boldsymbol{\varphi}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] + [B'(p_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] &= [F, (\mathbf{r}_h, \boldsymbol{\psi}_h)] \quad \forall\, (\mathbf{r}_h, \boldsymbol{\psi}_h) \in X_h, \\
[B(\mathbf{t}_h, \boldsymbol{\varphi}_h), q_h] &= [G, q_h] \qquad\quad \forall\, q_h \in M_h\,.
\end{aligned} \tag{4.16}$$

Moreover, there exists $C_h > 0$*, depending only on* $\alpha_h, \gamma_h, \beta_{1,h}, \beta_h, \|S\|$*, and* $\|B_1\|$ *such that*

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h)\|_{X \times M} \le C_h \left\{ \|F|_{X_h}\|_{X'_h} + \|G|_{M_h}\|_{M'_h} \right\}.$$

The following lemma establishes the well-definiteness of operator $\mathbf{T}_h$.

Lemma 4.2 *Assume that hypotheses* **(H.0)**, **(H.1)**, **(H.2)**, *and* **(H.3)** *hold. Assume further that* κ_i, $i \in \{1, \ldots, 4\}$ *satisfy the conditions required by Lemma* 3.6. *Then, for each* $r \in (0, r_0)$, *with* r_0 *given by* (3.17), *the problem* (4.14) *has a unique solution* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h}) \in \mathbb{X} \times \mathbb{M}$ *for each* $\mathbf{z}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ *such that* $\|\mathbf{z}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \leq r$. *Moreover, there exists a constant* $\tilde{c}_\mathbf{T} > 0$, *independent of* $\mathbf{z}_{\mathrm{S},h}$ *and the data* $\mathbf{f}_\mathrm{S}$ *and* f_D, *such that there holds*

$$\|\mathbf{T}_h(\mathbf{z}_{\mathrm{S},h})\|_{1,\Omega_\mathrm{S}} \,\leq\, \|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h})\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \tilde{c}_\mathbf{T} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\}. \tag{4.17}$$

Proof. Let $\mathbf{z}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ such that $\|\mathbf{z}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \leq r$. Recalling that $\mathbb{X}_h \subseteq \mathbb{X}$, $\mathbb{M}_h \subseteq \mathbb{M}$ and $\mathbb{V}_h \subseteq \mathbb{V}$, a straightforward application of Lemmas 3.2, 3.3 and 3.4, implies, respectively, that hypotheses *(i)*, *(ii)* and *(iii)* in Theorem 4.1, hold. In turn, as already discussed in Section 4.1, the inf-sup conditions *(iv)* and *(v)* follow from hypotheses (**H.2**) and (**H.3**), respectively. Therefore, according to the above, a direct application of Theorem 4.1 allows us to conclude that there exists a unique $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h}) \in \mathbb{X}_h \times \mathbb{M}_h$ solution to (4.14), which satisfies

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h})\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \tilde{c}_\mathbf{T} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\},$$

with $\tilde{c}_\mathbf{T}$ independent of $\mathbf{z}_{\mathrm{S},h}$ and h. □

We are now in position of establishing the well-posedness of (4.4).

Theorem 4.3 *Assume that hypotheses* **(H.0)**, **(H.1)**, **(H.2)**, *and* **(H.3)** *hold. Assume further that* κ_i, $i \in \{1, \ldots, 4\}$ *satisfy the conditions required by Lemma* 3.6. *In addition, given* $r \in (0, r_0)$, *with* r_0 *defined by* (3.17), *let* $W^h_r := \left\{ \mathbf{z}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \|\mathbf{z}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \leq r \right\}$, *and assume that the data* $\mathbf{f}_\mathrm{S}$ *and* f_D *satisfy*

$$\tilde{c}_\mathbf{T} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\} \,\leq\, r\,, \tag{4.18}$$

with $\tilde{c}_\mathbf{T} > 0$ *the constant in* (4.17). *Then, there exists a unique* $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h}) \in \mathbb{X}_h \times \mathbb{M}_h$ *solution to* (4.4), *which satisfies* $\mathbf{u}_{\mathrm{S},h} \in W^h_r$ *and*

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}_h}), \underline{\mathbf{p}_h})\|_{\mathbb{X}\times\mathbb{M}} \,\leq\, \tilde{c}_\mathbf{T} \left\{ \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|f_\mathrm{D}\|_{0,\Omega_\mathrm{D}} \right\}. \tag{4.19}$$

Proof. We first observe, owing to (4.17), that the assumption (4.18) guarantees that $\mathbf{T}_h(W^h_r) \subseteq W^h_r$. Next, proceeding analogously to the proof of Lemma 3.9, that is, applying the strong monotonicity of $a(\mathbf{z}_{\mathrm{S},h}) : \mathbf{X}_h \to \mathbf{X}'_h$ for each $\mathbf{z}_{\mathrm{S},h} \in W^h_r$, and using again the boundedness of the compact injection $\mathbf{i}_c$, we find that

$$\|\mathbf{T}_h(\mathbf{z}_{\mathrm{S},h}) - \mathbf{T}_h(\tilde{\mathbf{z}}_{\mathrm{S},h})\|_{1,\Omega_\mathrm{S}} \leq C_\mathbf{T} \|\mathbf{i}_c\| \|\mathbf{T}_h(\tilde{\mathbf{z}}_h)\|_{1,\Omega_\mathrm{S}} \|\mathbf{z}_h - \tilde{\mathbf{z}}_h\|_{1,\Omega_\mathrm{S}} \quad \forall\, \mathbf{z}_{\mathrm{S},h}, \tilde{\mathbf{z}}_{\mathrm{S},h} \in W^h_r,$$

which, together with (3.30), (4.17), (4.18), and the definition of r_0 (cf. (3.17)), implies

$$\|\mathbf{T}_h(\mathbf{z}_{\mathrm{S},h}) - \mathbf{T}_h(\tilde{\mathbf{z}}_{\mathrm{S},h})\|_{1,\Omega_\mathrm{S}} \leq \frac{r}{r_0} \|\mathbf{z}_{\mathrm{S},h} - \tilde{\mathbf{z}}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \quad \forall\, \mathbf{z}_{\mathrm{S},h}, \tilde{\mathbf{z}}_{\mathrm{S},h} \in W^h_r\,,$$

thus confirming that $\mathbf{T}_h : W^h_r \to W^h_r$ is a contraction mapping. Then, the Banach fixed point Theorem and the equivalence between (4.4) and (4.15) imply the well-posedness of (4.4). In turn, the a priori estimate (4.19) follows directly from (4.17). □

5 A priori error estimate

In this section, we derive an a priori error estimate for the Galerkin scheme (4.4). To that end, we first establish some preliminary results that will be utilized in our subsequent analysis.

5.1 Preliminaries

We begin with the following Strang-type lemma,

Lemma 5.1 *Let X and M be Hilbert spaces, $\mathbf{F} \in (X \times M)' := X' \times M'$, and $P : X \times M \to X' \times M'$ a nonlinear operator. In addition, let $\{X_n\}_{n\in N}$ and $\{M_n\}_{n\in N}$ be sequences of finite dimensional subspaces of X and M, respectively, and for each $n \in N$ consider a nonlinear operator $P_n : X_n \times M_n \to (X_n \times M_n)' := X_n' \times M_n'$ and a functional $\mathbf{F}_n \in X_n' \times M_n'$. Assume that the family $\{P\} \cup \{P_n\}_{n\in N}$ is uniformly Lipschitz continuous with constant $C_{\mathrm{LC}} > 0$. Moreover, assume that P_n has a hemi-continuous first-order Gâteaux derivative $\mathcal{D}P_n(\vec{\mathbf{s}})(\cdot,\cdot)$, for all $\vec{\mathbf{s}} \in X \times M$, which satisfies the global inf-sup condition*

$$C_{\mathrm{G}}\,\|\vec{\mathbf{s}}_n\| \le \sup_{\substack{\vec{\mathbf{r}}_n\in X_n\times M_n\\ \vec{\mathbf{r}}_n\neq\mathbf{0}}} \frac{\mathcal{D}P_n(\vec{\mathbf{s}})(\vec{\mathbf{s}}_n,\vec{\mathbf{r}}_n)}{\|\vec{\mathbf{r}}_n\|} \quad \forall\,\vec{\mathbf{s}}_n \in X_n \times M_n\,, \tag{5.1}$$

with a constant $C_{\mathrm{G}} > 0$ independent of $\vec{\mathbf{s}}$. Furthermore, let $\vec{\mathbf{t}} := ((\mathbf{t},\boldsymbol{\varphi}),\mathbf{p}) \in X \times M$ and $\vec{\mathbf{t}}_n := ((\mathbf{t}_n,\boldsymbol{\varphi}_n),p_n) \in X_n \times M_n$ be such that

$$[P(\vec{\mathbf{t}}),\vec{\mathbf{r}}] = [\mathbf{F},\vec{\mathbf{r}}] \quad \forall\vec{\mathbf{r}} := ((\mathbf{r},\boldsymbol{\psi}),\mathbf{q}) \in X \times M \tag{5.2}$$

and

$$[P_n(\vec{\mathbf{t}}_n),\vec{\mathbf{r}}_n] = [\mathbf{F}_n,\vec{\mathbf{r}}_n] \quad \forall\,\vec{\mathbf{r}}_n := ((\mathbf{r}_n,\boldsymbol{\psi}_n),\mathbf{q}_n) \in X_n \times M_n\,. \tag{5.3}$$

Then for each $n \in N$, there holds

$$\begin{aligned} \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_n\| \;\le\; C_{\mathrm{ST}} \Bigg\{ & \sup_{\substack{\vec{\mathbf{r}}_n\in X_n\times M_n\\ \vec{\mathbf{r}}_n\neq\mathbf{0}}} \frac{|[\mathbf{F},\vec{\mathbf{r}}_n] - [\mathbf{F}_n,\vec{\mathbf{r}}_n]|}{\|\vec{\mathbf{r}}_n\|} \\ & + \inf_{\substack{\vec{\mathbf{s}}_n\in X_n\times M_n\\ \vec{\mathbf{s}}_n\neq\mathbf{0}}} \left(\|\vec{\mathbf{t}} - \vec{\mathbf{s}}_n\| + \sup_{\substack{\vec{\mathbf{r}}_n\in X_n\times M_n\\ \vec{\mathbf{r}}_n\neq\mathbf{0}}} \frac{|[P(\vec{\mathbf{s}}_n),\vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n),\vec{\mathbf{r}}_n]|}{\|\vec{\mathbf{r}}_n\|} \right) \Bigg\}, \end{aligned} \tag{5.4}$$

with $C_{\mathrm{ST}} := C_{\mathrm{G}}^{-1} \max\{1, C_{\mathrm{G}} + C_{\mathrm{LC}}\}$.

Proof. We proceed as in the proof of [30, Theorem 3.3] (see also [32, Theorem 3.5]). In fact, given $\vec{\mathbf{t}}_n$, $\vec{\mathbf{s}}_n \in X_n \times M_n$, we first observe that the hemi-continuity of $\mathcal{D}P_n$ implies that there exists $\mu_0 \in (0,1)$, such that

$$\begin{aligned} [P_n(\vec{\mathbf{t}}_n),\vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n),\vec{\mathbf{r}}_n] &= \int_0^1 \mathcal{D}P_n(\mu\vec{\mathbf{t}}_n + (1-\mu)\vec{\mathbf{s}}_n)(\vec{\mathbf{t}}_n - \vec{\mathbf{s}}_n,\vec{\mathbf{r}}_n)d\mu \\ &= \mathcal{D}P_n(\mu_0\vec{\mathbf{t}}_n + (1-\mu_0)\vec{\mathbf{s}}_n)(\vec{\mathbf{t}}_n - \vec{\mathbf{s}}_n,\vec{\mathbf{r}}_n)\,, \end{aligned}$$

and hence, by taking $\vec{\mathbf{s}} = \mu_0\vec{\mathbf{t}}_n + (1-\mu_0)\vec{\mathbf{s}}_n$ in (5.1), we find that

$$\|\vec{\mathbf{t}}_n - \vec{\mathbf{s}}_n\| \le C_{\mathrm{G}}^{-1} \sup_{\substack{\vec{\mathbf{r}}_n\in X_n\times M_n\\ \vec{\mathbf{r}}_n\neq\mathbf{0}}} \frac{[P_n(\vec{\mathbf{t}}_n),\vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n),\vec{\mathbf{r}}_n]}{\|\vec{\mathbf{r}}_n\|}. \tag{5.5}$$

In turn, using (5.2) and (5.3), and adding and subtracting appropriate terms, we easily obtain

$$[P_n(\vec{\mathbf{t}}_n), \vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n] = [\mathbf{F}_n, \vec{\mathbf{r}}_n] - [\mathbf{F}, \vec{\mathbf{r}}_n] + [P(\vec{\mathbf{t}}), \vec{\mathbf{r}}_n] - [P(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n] + [P(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n],$$

which, together with the Lipschitz continuity of P, implies

$$\Big|[P_n(\vec{\mathbf{t}}_n), \vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n]\Big| \le \Big|[\mathbf{F}, \vec{\mathbf{r}}_n] - [\mathbf{F}_n, \vec{\mathbf{r}}_n]\Big| + C_{\mathrm{LC}}\|\vec{\mathbf{t}} - \vec{\mathbf{s}}_n\|\|\vec{\mathbf{r}}_n\| + \Big|[P(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n] - [P_n(\vec{\mathbf{s}}_n), \vec{\mathbf{r}}_n]\Big|, \quad (5.6)$$

for all $\vec{\mathbf{s}}_n \in X_n \times M_n$. In this way, from (5.5), (5.6) and the triangle inequality we readily obtain (5.4), which concludes the proof. □

In addition, we will require the following linear version of Theorem 4.1.

Theorem 5.2 *Consider the notations and definitions given in Theorem* 4.1*. Assume that*

(i) $A_1|_{X_{1,h}} : X_{1,h} \to X'_{1,h}$ *is linear, bounded and* $\tilde{X}_{1,h}$*–elliptic, that is, there exist* γ_h, $\alpha_h > 0$, *such that*

$$\|A_1(\mathbf{r}_h)\|_{X'_{1,h}} \le \gamma_h \|\mathbf{r}_h\|_{X_1} \qquad \forall\, \mathbf{r}_h \in \tilde{X}_{1,h},$$

and

$$[A_1(\mathbf{r}_h), \mathbf{r}_h] \ge \alpha_h \|\mathbf{r}_h\|^2_{X_1} \qquad \forall\, \mathbf{r}_h \in \tilde{X}_{1,h}.$$

(ii) The conditions (iii)-(v) from Theorem (4.1) *are satisfied.*

Then, there exists a unique $(\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h) \in X \times M$ *solution of* (4.16)*. Moreover, there exists* $C_h > 0$, *depending only on* α_h, γ_h, $\beta_{1,h}$, β_h, $\|S\|$ *and* $\|B_1\|$, *such that*

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), p_h)\|_{X\times M} \le C_h\{\|F|_{X_h}\|_{X'_h} + \|G|_{M_h}\|_{M'_h}\}. \quad (5.7)$$

Proof. It reduces to verify the hypotheses of Theorem 4.1. We omit further details □

We observe here that (5.7) is equivalent to the global inf-sup condition

$$\|((\mathbf{s}_h, \boldsymbol{\psi}_h), \rho_h)\|_{X\times M} \;\le\; C_h \sup_{\big((\mathbf{r},\boldsymbol{\psi}),q\big)\in X_h\times M_h\backslash \mathbf{0}} \frac{[A(\mathbf{s}_h, \phi_h), (\mathbf{r}, \psi)] + [B'(\rho_h), (\mathbf{r}, \psi)] \;+\; [B(\mathbf{s}, \phi), q]}{\|((\mathbf{r}, \boldsymbol{\psi}), q)\|_{X\times M}}. \quad (5.8)$$

for all $((\mathbf{s}_h, \boldsymbol{\psi}_h), \rho_h) \in X_h \times M_h$.

5.2 The main result

In what follows, we establish the corresponding a priori error estimate of our Galerkin scheme (4.4). To that end, and for the sake of simplicity, hereafter we denote by $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ and $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ the solutions of problems (2.16) and (4.4), respectively. In turn, we let $\mathbf{P} : \mathbb{X} \times \mathbb{M} \to (\mathbb{X} \times \mathbb{M})' := \mathbb{X}' \times \mathbb{M}'$ and $\mathbf{P}_h : \mathbb{X}_h \times \mathbb{M}_h \to (\mathbb{X}_h \times \mathbb{M}_h)' := \mathbb{X}'_h \times \mathbb{M}'_h$, be the nonlinear operators obtained after adding on the left hand side of (2.16) and (4.4), respectively, that is

$$\begin{aligned}[\mathbf{P}(\vec{\mathbf{s}}), \vec{\mathbf{r}}] \;:=\; & [(a_1 + a_2(\mathbf{u}_{\mathrm{S}}))(\underline{\mathbf{s}}), \underline{\mathbf{r}}] + [b(\underline{\mathbf{s}}), \underline{\boldsymbol{\psi}}] + [b(\underline{\mathbf{r}}), \underline{\boldsymbol{\phi}}] - [c(\underline{\boldsymbol{\phi}}), \underline{\boldsymbol{\psi}}] \\ & + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{m}}] + [\mathbf{B}(\underline{\mathbf{s}}, \underline{\boldsymbol{\phi}}), \underline{\mathbf{q}}]\,,\end{aligned} \quad (5.9)$$

and

$$\begin{aligned}[\mathbf{P}_h(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] \;:=\; & [(a_1 + a_2(\mathbf{u}_{\mathrm{S},h}))(\underline{\mathbf{s}}_h), \underline{\mathbf{r}}_h] + [b(\underline{\mathbf{s}}_h), \underline{\boldsymbol{\psi}}_h] + [b(\underline{\mathbf{r}}_h), \underline{\boldsymbol{\phi}}_h] - [c(\underline{\boldsymbol{\phi}}_h), \underline{\boldsymbol{\psi}}_h] \\ & + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{m}}_h] \;+\; [\mathbf{B}(\underline{\mathbf{s}}_h, \underline{\boldsymbol{\phi}}_h), \underline{\mathbf{q}}_h]\,,\end{aligned} \quad (5.10)$$

for all $\vec{\mathbf{s}} = ((\underline{\mathbf{s}}, \underline{\boldsymbol{\phi}}), \underline{\mathbf{m}})$, $\vec{\mathbf{r}} = ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}) \in \mathbb{X}\times\mathbb{M}$ and $\vec{\mathbf{t}}_h = ((\underline{\mathbf{s}}_h, \underline{\boldsymbol{\phi}}_h), \underline{\mathbf{m}}_h)$, $\vec{\mathbf{r}}_h = ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h) \in \mathbb{X}_h\times\mathbb{M}_h$, respectively, where $\mathbf{u}_{\mathrm{S}} \in W_r$ and $\mathbf{u}_{\mathrm{S},h} \in W_r^h$ are the velocity solutions of (2.16) and (4.4), respectively.

According to the above, and denoting by $\mathcal{F} := (\mathbf{F}, \mathbf{G}) \in \mathbb{X}' \times \mathbb{M}'$, it follows that

$$[\mathbf{P}(\vec{\mathbf{t}}), \vec{\mathbf{r}}] = [\mathcal{F}, \vec{\mathbf{r}}] \qquad \forall\, \vec{\mathbf{r}} := ((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}) \in \mathbb{X}\times\mathbb{M} \tag{5.11}$$

and

$$[\mathbf{P}_h(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] = [\mathcal{F}, \vec{\mathbf{r}}_h] \qquad \forall\, \vec{\mathbf{r}}_h := ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h) \in \mathbb{X}_h\times\mathbb{M}_h. \tag{5.12}$$

Next, since the Lipschitz-continuity of a_1 (cf. (3.11)) holds at the continuous and discrete levels with the same constant, as well as the continuity of a_2, b, c and $\mathbf{B}$, we observe that the family $\{\mathbf{P}\}\cup\{\mathbf{P}_h\}_{h>0}$ is uniformly Lipschitz-continuous with constant denoted from now on by $C_{\mathrm{LC}} > 0$.

On the other hand, owing to the fact that μ is assumed to be of class C^1 (cf. (2.2)), it is not difficult to see that $a_1 : \mathbf{X} \to \mathbf{X}'$ has hemi-continuous first order Gâteaux derivative $\mathcal{D}a_1 : \mathbf{X} \to \mathcal{L}(\mathbf{X}, \mathbf{X}')$, which in particular implies that for any $\underline{\mathbf{s}}, \underline{\mathbf{r}} \in \mathbf{X}$, the mapping $\mathrm{R} \ni \mu \to \mathcal{D}a_1(\underline{\mathbf{s}} + \mu\underline{\mathbf{r}})(\underline{\mathbf{r}})(\cdot) \in \mathbf{X}'$ is continuous. Moreover, we have the following lemma.

Lemma 5.3 *For any $\underline{\mathbf{s}} \in \mathbf{X}$, the Gâteaux derivative $\mathcal{D}a_1(\underline{\mathbf{s}})$ constitutes a bounded bilinear form on $\mathbf{X}\times\mathbf{X}$ that becomes elliptic on $\tilde{\mathbf{X}}\times\tilde{\mathbf{X}}$, with boundedness and ellipticity constants L_{a_1} (cf. (3.11)) and $\alpha_0(\Omega)$ (cf. (3.16)), respectively.*

Proof. Given $\underline{\mathbf{s}} \in \mathbf{X}$, the Gâteaux derivative $\mathcal{D}a_1(\underline{\mathbf{s}})$ is the operator in $\mathcal{L}(\mathbf{X}, \mathbf{X}')$ (equivalently, the bilinear form on $\mathbf{X}\times\mathbf{X}$) defined by

$$\mathcal{D}a_1(\underline{\mathbf{s}})(\underline{\mathbf{r}}, \underline{\widehat{\mathbf{r}}}) := \lim_{\epsilon\to 0} \frac{[a_1(\underline{\mathbf{s}} + \epsilon\underline{\mathbf{r}}), \underline{\widehat{\mathbf{r}}}] - [a_1(\underline{\mathbf{s}}), \underline{\widehat{\mathbf{r}}}]}{\epsilon} \qquad \forall\, \underline{\mathbf{r}}, \underline{\widehat{\mathbf{r}}} \in \mathbf{X}\,.$$

The rest of the proof follows as in [30, Lemma 3.1] by employing the properties (3.11), (3.15) and the continuity of the mapping $\mathrm{R} \ni \mu \to \mathcal{D}a_1(\underline{\mathbf{s}} + \mu\underline{\mathbf{r}})(\underline{\mathbf{r}})(\cdot) \in \mathbf{X}'$. We omit further details. □

Now, due to the hemi-continuity of the first order Gâteaux derivative $\mathcal{D}a_1$, and since the operators defining $\mathbf{P}_h$ (besides a_1) are linear, we easily obtain that, given $\vec{\mathbf{s}} = ((\underline{\mathbf{s}}, \underline{\boldsymbol{\phi}}), \underline{\mathbf{m}}) \in \mathbb{X}\times\mathbb{M}$, the Gâteaux derivative of $\mathbf{P}_h$ at $\vec{\mathbf{s}}$ is obtained by replacing $[a_1(\underline{\mathbf{t}}_h), \underline{\mathbf{r}}_h]$ in (5.10) by $\mathcal{D}a_1(\underline{\mathbf{s}})(\underline{\mathbf{t}}_h, \underline{\mathbf{r}}_h)$, that is

$$\begin{aligned}\mathcal{D}\mathbf{P}_h(\vec{\mathbf{s}})(\vec{\mathbf{t}}_h, \vec{\mathbf{r}}_h) \;:=\; & \mathcal{D}a_1(\underline{\mathbf{s}})(\underline{\mathbf{t}}_h, \underline{\mathbf{r}}_h) + [a_2(\mathbf{u}_{\mathrm{S},h})(\underline{\mathbf{t}}_h), \underline{\mathbf{r}}_h] + [b(\underline{\mathbf{t}}_h), \underline{\boldsymbol{\psi}}_h] + [b(\underline{\mathbf{r}}_h), \underline{\boldsymbol{\varphi}}_h] \\ & - [c(\underline{\boldsymbol{\varphi}}_h), \underline{\boldsymbol{\psi}}_h] \,+\, [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] \,+\, [\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h]\,,\end{aligned} \tag{5.13}$$

for all $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h), \vec{\mathbf{r}}_h := ((\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$, which, according to Lemma 5.3, becomes a bounded bilinear form on $(\mathbb{X}_h \times \mathbb{M}_h) \times (\mathbb{X}_h \times \mathbb{M}_h)$. Moreover, since c is positive-semidefinite, and assuming for a moment that $(\mathbf{H.0}), (\mathbf{H.1}), (\mathbf{H.2})$ and $(\mathbf{H.3})$ hold, we obtain that the conditions *(iii)*–*(v)* in Theorem 4.1 are verified, and as result, having in mind Lemma 5.3, the bilinear form $\mathcal{D}\mathbf{P}_h(\vec{\mathbf{s}})(\cdot,\cdot)$ satisfies the hypotheses of Theorem 5.2. Moreover, in virtue of (5.8), there holds the global inf-sup condition

$$C_{\mathrm{G}}\,\|\vec{\mathbf{s}}_h\| \le \sup_{\substack{\vec{\mathbf{r}}_h \in \mathbb{X}_n \times \mathbb{M}_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{\mathcal{D}\mathbf{P}_h(\vec{\mathbf{s}})(\vec{\mathbf{s}}_h, \vec{\mathbf{r}}_h)}{\|\vec{\mathbf{r}}_h\|} \quad \forall\, \vec{\mathbf{s}}_h \in \mathbb{X}_h \times \mathbb{M}_h\,. \tag{5.14}$$

According to the foregoing analysis, it follows that the family of operators $\{\mathbf{P}\} \cup \{\mathbf{P}_h\}_{h>0}$ satisfy the hypotheses of Lemma 5.1, and consequently we can establish now the main result of this section.

Theorem 5.4 *Assume that the hypotheses* $(\mathbf{H.0}), (\mathbf{H.1}), (\mathbf{H.2})$ *and* $(\mathbf{H.3})$*, as well as the conditions on* $\kappa_i, i \in \{1, \ldots, 4\}$ *required by Lemma* 3.6*, hold. Let* $r \in (0, r_0)$*, with* r_0 *defined by* (3.17) *and assume further that the data* $\mathbf{f}_{\mathrm{S}}$ *and* f_{D} *satisfy*

$$c_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\} \le \frac{r}{\alpha_0(\Omega) C_{\mathrm{ST}}}, \tag{5.15}$$

with $c_{\mathbf{T}}$ *and* $\alpha_0(\Omega)$ *being the positive constants satisfying* (3.31) *and* (3.16)*, respectively. In addition, let* $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ *with* $\mathbf{u}_{\mathrm{S}} \in W_r$*, and* $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ *with* $\mathbf{u}_{\mathrm{S},h} \in W_r^h$ *be the unique solutions of problems* (2.16) *and* (4.4)*, respectively. Then there exists a positive constant* $C > 0$*, depending only on* $\alpha_0(\Omega)$ *and* C_{ST}*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \le C \mathrm{dist}\,(\vec{\mathbf{t}}, \mathbb{X}_h \times \mathbb{M}_h). \tag{5.16}$$

Proof. From the strang-type estimate (5.4) and from (5.11) and (5.12), we first obtain

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \le C_{\mathrm{ST}} \inf_{\vec{\mathbf{s}}_h \in \mathbb{X}_h\times\mathbb{M}_h} \left\{ \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbb{X}\times\mathbb{M}} + \sup_{\substack{\vec{\mathbf{r}}_h \in \mathbb{X}_h\times\mathbb{M}_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{\Big| [\mathbf{P}(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] - [\mathbf{P}_h(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] \Big|}{\|\vec{\mathbf{r}}_h\|_{\mathbb{X}\times\mathbb{M}}} \right\}. \tag{5.17}$$

In turn, utilizing the definition of $\mathbf{P}$ and $\mathbf{P}_h$ (resp. (5.9) and (5.10)), applying the estimate (3.12), adding and subtracting $\vec{\mathbf{t}}$, and bounding both $\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}$ and $\|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}$ by $r_0 = \frac{\alpha_0(\Omega)}{2 c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2}}$, we find that

$$\begin{aligned}
\Big| [\mathbf{P}(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] - [\mathbf{P}_h(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] \Big| &= \Big| [a_2(\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h})(\vec{\mathbf{s}}_h), \vec{\mathbf{r}}_h] \Big| \\
&\le c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \left\{ \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbb{X}\times\mathbb{M}} + \|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \right\} \|\vec{\mathbf{r}}_h\|_{\mathbb{X}\times\mathbb{M}} \\
&\le \left\{ 2 c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} r_0 \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbb{X}\times\mathbb{M}} + c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} \|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \right\} \|\vec{\mathbf{r}}_h\|_{\mathbb{X}\times\mathbb{M}} \\
&= \left\{ \alpha_0(\Omega) \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbb{X}\times\mathbb{M}} + c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} \|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \right\} \|\vec{\mathbf{r}}_h\|_{\mathbb{X}\times\mathbb{M}},
\end{aligned}$$

which, replaced back into (5.17), taking infimun, and using that $\|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \le \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}}$, yields

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \le C_{\mathrm{ST}}\{1 + \alpha_0(\Omega)\} \mathrm{dist}\,(\vec{\mathbf{t}}, \mathbb{X}_h \times \mathbb{M}_h) + C_{\mathrm{ST}} c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} \|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}}. \tag{5.18}$$

Finally, recalling from (3.31) that $\|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \le c_{\mathbf{T}} \left\{ \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|f_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} \right\}$, and employing assumption (5.15), we obtain that

$$C_{\mathrm{ST}} c_2(\Omega_{\mathrm{S}})(\kappa_1^2+1)^{1/2} \|\vec{\mathbf{t}}\|_{\mathbb{X}\times\mathbb{M}} \le \frac{1}{2}, \tag{5.19}$$

which, together with (5.18), implies (5.16) with $C = 2 C_{\mathrm{ST}}\{1 + \alpha_0(\Omega)\}$, thus ending the proof. □

6 Particular choices of discrete spaces

We now introduce specific discrete spaces satisfying hypotheses $(\mathbf{H.0})$, $(\mathbf{H.1})$, $(\mathbf{H.2})$, and $(\mathbf{H.3})$ in 2D and 3D. To this end, we let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D}, which are formed by shape-regular triangles (in R^2) or tetrahedra (in R^3) of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$). Then, for each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we set the local Raviart–Thomas space of order k as

$$\mathrm{RT}_k(T) := \mathbf{P}_k(T) \oplus P_k(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_n)^{\mathrm{t}}$ is a generic vector of R^n.

6.1 Raviart–Thomas elements in 2D

We define the discrete subspaces in (4.1) as follows:

$$
\begin{aligned}
L^2_h(\Omega_\star) &:= \left\{ q_h \in L^2(\Omega_\star) : \quad q_h|_T \in P_k(T) \quad \forall T \in \mathcal{T}_h^\star \right\}, \quad \star \in \{\mathrm{S},\mathrm{D}\}, \\
\mathbf{H}_h(\Omega_\star) &:= \left\{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star) : \quad \tau_h|_T \in \mathrm{RT}_k(T) \quad \forall T \in \mathcal{T}_h^\star \right\}, \quad \star \in \{\mathrm{S},\mathrm{D}\}, \\
\mathbf{H}^1_h(\Omega_\mathrm{S}) &:= \left\{ \mathbf{v}_h \in [\mathcal{C}(\overline{\Omega}_\mathrm{S})]^2 : \quad \mathbf{v}_h|_T \in \mathbf{P}_{k+1}(T) \quad \forall T \in \mathcal{T}_h^\mathrm{S} \right\}, \\
\mathbb{L}^2_{\mathrm{tr},h}(\Omega_\mathrm{S}) &:= \left\{ \mathbf{r}_h|_T \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_\mathrm{S}) : \quad \mathbf{r}_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^\mathrm{S} \right\}, \\
\mathbb{L}^2_{\mathrm{skew},h}(\Omega_\mathrm{S}) &:= \left\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_\mathrm{S}) : \quad \boldsymbol{\eta}_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^\mathrm{S} \right\}.
\end{aligned} \tag{6.1}
$$

In addition, in order to introduce the particular subspaces $\Lambda_h^\mathrm{S}(\Sigma)$ and $\Lambda_h^\mathrm{D}(\Sigma)$, we follow the simplest approach suggested in [37] and [45]. To this end, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that, since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we set

$$
\begin{aligned}
\Lambda_h^\mathrm{S}(\Sigma) &:= \left\{ \psi_h \in \mathcal{C}(\Sigma) : \quad \psi_h|_e \in P_{k+1}(e) \quad \forall \text{ edge } e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \right\}, \\
\Lambda_h^\mathrm{D}(\Sigma) &:= \left\{ \xi_h \in \mathcal{C}(\Sigma) : \quad \xi_h|_e \in P_{k+1}(e) \quad \forall \text{ edge } e \in \Sigma_{2h} \right\}.
\end{aligned} \tag{6.2}
$$

Then, we define the global spaces $\mathbb{X}_h$ and $\mathbb{M}_h$ by combining (4.2), (4.3), (6.1) and (6.2). Now, concerning hypotheses **(H.0)**–**(H.3)**, we start mentioning that **(H.0)** and **(H.1)** are straightforward from the definitions in (6.1). In turn, it is well known that the discrete inf-sup condition (4.12) in **(H.3)** holds (see for instance [5, Chapter IV] or [29, Section 4.2]). In addition, the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}^{1/2}_{00}(\Sigma)$ satisfying (4.11) follows as explained in [37, Section 2.5] or [38, Section 3.2]. Finally, the inf-sup conditions (4.9) and (4.10) in **(H.2)** can be derived by combining the results in [37, Section 2.5] and [45, Theorem A.1]. We omit further details and refer the reader to [32, Section 5.3.1] for the verification of these inf-sup conditions.

According to the above, we conclude that the Galerkin scheme (4.4) defined with the spaces in (6.1) is well posed. Moreover, by employing the approximations properties of the finite element subspaces involved (see, e.g. [5, 29, 40, 43]), and the a priori estimate (5.16), we can easily obtain the following result.

Theorem 6.1 *Assume that the hypotheses of Theorem* 5.4 *hold. Let* $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ *with* $\mathbf{u}_\mathrm{S} \in W_r$ *and* $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ *with* $\mathbf{u}_{\mathrm{S},h} \in W_r^h$ *be the unique solutions of the problems* (2.16) *and* (4.4)*, respectively. Assume further that there exists* $\delta > 0$*, such that* $\mathbf{t}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$*,* $\boldsymbol{\sigma}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$*,* $\mathbf{div}\boldsymbol{\sigma}_\mathrm{S} \in \mathbf{H}^\delta(\Omega_\mathrm{S})$*,* $\mathbf{u}_\mathrm{S} \in \mathbf{H}^{1+\delta}(\Omega_\mathrm{S})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\boldsymbol{\rho}_\mathrm{S} \in \mathbb{H}^\delta(\Omega_\mathrm{S})$*,* $\mathbf{u}_\mathrm{D} \in \mathbf{H}^\delta(\Omega_\mathrm{D})$*, and* $\mathrm{div}\,\mathbf{u}_\mathrm{D} \in H^\delta(\Omega_\mathrm{D})$*. Then,* $p_\mathrm{D} \in H^{1+\delta}(\Omega_\mathrm{D})$*,* $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h*, such that*

$$
\begin{aligned}
\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \,\le\, C\, h^{\min\{\delta,k+1\}} \Big\{ & \|\mathbf{t}\|_{\delta,\Omega_\mathrm{S}} + \|\boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{div}\boldsymbol{\sigma}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{S}\|_{1+\delta,\Omega_\mathrm{S}} \\
& + \|\boldsymbol{\rho}_\mathrm{S}\|_{\delta,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{\delta,\Omega_\mathrm{D}} + \|p_\mathrm{D}\|_{1+\delta,\Omega_\mathrm{D}} \Big\}.
\end{aligned}
$$

Proof. From the second equation of (2.11), we readily obtain that $\nabla p_{\rm D} = -\mathbf{K}^{-1}\mathbf{u}_{\rm D}$, which implies that $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$, whence $\lambda = p_{\rm D}|_\Sigma \in H^{1/2+\delta}(\Sigma)$. The rest of the proof follows from the a priori estimate (5.16), the approximation properties of the discrete spaces involved and the fact that, owing to the trace theorems in $\Omega_{\rm S}$ and $\Omega_{\rm D}$, respectively, there holds

$$\|\boldsymbol{\varphi}\|_{1/2+\delta,\Sigma} \le c\,\|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}} \quad \text{and} \quad \|\lambda\|_{1/2+\delta,\Sigma} \le c\,\|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}}\,.$$

□

6.2 Raviart–Thomas elements in 3D

Let us now consider the discrete spaces:

$$\begin{aligned}
L^2_h(\Omega_\star) &:= \big\{ q_h \in L^2(\Omega_\star): \quad q_h|_T \in P_k(T) \quad \forall\, T \in \mathcal{T}^\star_h \big\}\,, \quad \star \in \{\mathrm{S},\mathrm{D}\},\\
\mathbf{H}_h(\Omega_\star) &:= \{ \tau_h \in \mathbf{H}(\mathrm{div}\,;\Omega_\star): \quad \tau_h|_T \in \mathrm{RT}_k(T) \quad \forall\, T \in \mathcal{T}^\star_h\}\,, \quad \star \in \{\mathrm{S},\mathrm{D}\},\\
\mathbf{H}^1_h(\Omega_{\rm S}) &:= \big\{ \mathbf{v}_h \in [\mathcal{C}(\overline{\Omega}_{\rm S})]^3: \quad \mathbf{v}_h|_T \in \mathbf{P}_{k+1}(T) \quad \forall\, T \in \mathcal{T}^{\rm S}_h \big\}\,,\\
\mathbb{L}^2_{\mathrm{tr},h}(\Omega_{\rm S}) &:= \big\{ \mathbf{r}_h|_T \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_{\rm S}): \quad \mathbf{r}_h|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}^{\rm S}_h \big\}\,,\\
\mathbb{L}^2_{\mathrm{skew},h}(\Omega_{\rm S}) &:= \big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega_{\rm S}): \quad \boldsymbol{\eta}_h|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}^{\rm S}_h \big\}\,.
\end{aligned} \tag{6.3}$$

Now, in order to define the discrete spaces for the unknowns on the interface Σ, we introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by triangles K of diameter $\widehat{h}$, and define $\widehat{h}_\Sigma := \max\{\widehat{h}_K : K \in \Sigma_{\widehat{h}}\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$\begin{aligned}
\Lambda^{\rm S}_h(\Sigma) &:= \big\{ \psi_h \in \mathcal{C}(\Sigma): \quad \psi_h|_K \in P_{k+1}(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \psi_h = 0 \quad \text{on} \quad \partial K \big\}\,,\\
\Lambda^{\rm D}_h(\Sigma) &:= \big\{ \xi_h \in \mathcal{C}(\Sigma): \quad \xi_h|_K \in P_{k+1}(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \big\}\,.
\end{aligned} \tag{6.4}$$

In this way, we define the global spaces $\mathbb{X}_h$ and $\mathbb{M}_h$ by combining (4.2), (4.3), (6.3), and (6.4).

Now, for the verification of hypotheses **(H.0)**–**(H.3)** we first observe that applying the same arguments as for the 2D case, it follows that **(H.0)**, **(H.1)** and **(H.3)** hold. However, for the inf-sup conditions in **(H.2)** we need to proceed differently and apply [31, Lemma 7.5]. More precisely, utilizing [31, Lemma 7.5] we conclude that there exists $C_0 \in (0,1)$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \le C_0 \widehat{h}_\Sigma$, the inf-sup conditions (4.9) and (4.10) hold.

Having verified hypotheses **(H.0)**–**(H.3)** we conclude that the Galerkin scheme (4.4) defined with the spaces in (6.3) is well posed. In addition, owing again to the approximations properties of the finite element subspaces involved (see, e.g. [5, 29, 40, 43]), and the a priori estimate (5.16), the following result holds.

Theorem 6.2 *Assume that the hypotheses of Theorem* 5.4 *hold. Let* $\vec{\mathbf{t}} := ((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{X} \times \mathbb{M}$ *with* $\mathbf{u}_{\rm S} \in W_r$ *and* $\vec{\mathbf{t}}_h := ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ *with* $\mathbf{u}_{\mathrm{S},h} \in W^h_r$ *be the unique solutions of the problems* (2.16) *and* (4.4)*, respectively. Assume further that there exists* $\delta > 0$*, such that* $\mathbf{t}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$*,* $\boldsymbol{\sigma}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$*,* $\mathbf{div}\boldsymbol{\sigma}_{\rm S} \in \mathbf{H}^\delta(\Omega_{\rm S})$*,* $\mathbf{u}_{\rm S} \in \mathbf{H}^{1+\delta}(\Omega_{\rm S})$*,* $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$*,* $\boldsymbol{\rho}_{\rm S} \in \mathbb{H}^\delta(\Omega_{\rm S})$*,* $\mathbf{u}_{\rm D} \in \mathbf{H}^\delta(\Omega_{\rm D})$*, and* $\mathrm{div}\,\mathbf{u}_{\rm D} \in H^\delta(\Omega_{\rm D})$*. Then,* $p_{\rm D} \in H^{1+\delta}(\Omega_{\rm D})$*,* $\lambda \in H^{1/2+\delta}(\Sigma)$*, and there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned}
\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbb{X}\times\mathbb{M}} \le C\, h^{\min\{\delta,k+1\}} \Big\{ & \|\mathbf{t}\|_{\delta,\Omega_{\rm S}} + \|\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|\mathbf{div}\boldsymbol{\sigma}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|\mathbf{u}_{\rm S}\|_{1+\delta,\Omega_{\rm S}}\\
& + \|\boldsymbol{\rho}_{\rm S}\|_{\delta,\Omega_{\rm S}} + \|\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|\mathrm{div}\,\mathbf{u}_{\rm D}\|_{\delta,\Omega_{\rm D}} + \|p_{\rm D}\|_{1+\delta,\Omega_{\rm D}} \Big\}\,.
\end{aligned}$$

7 Numerical results

In this section we present three examples illustrating the performance of our augmented mixed finite element scheme (4.4), and confirming the rates of convergence provided by Theorem 6.1. Our implementation is based on a *FreeFem++* code (see [42]), in conjunction with the direct linear solver UMFPACK (see [20]). Regarding the implementation of the Newton iterative method, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in R^N, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{L}^2_{\mathrm{tr},h}(\Omega_\mathrm{S}), \mathbb{H}_{h,0}(\Omega_\mathrm{S}), \mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}), \mathbb{L}^2_{\mathrm{skew},h}(\Omega_\mathrm{S}), \mathbf{H}_{h,0}(\Omega_\mathrm{D}), \mathbf{\Lambda}^\mathrm{S}_h(\Sigma)$, $\Lambda^\mathrm{D}_h(\Sigma)$, and $L^2_{h,0}(\Omega_\mathrm{D})$, and *tol* is a fixed tolerance to be specified for each example. As usual, the individual errors are denoted by:

$$\mathrm{e}(\mathbf{t}_\mathrm{S}) := \|\mathbf{t}_\mathrm{S} - \mathbf{t}_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}}, \quad \mathrm{e}(\boldsymbol{\sigma}_\mathrm{S}) := \|\boldsymbol{\sigma}_\mathrm{S} - \boldsymbol{\sigma}_{\mathrm{S},h}\|_{\mathbf{div},\Omega_\mathrm{S}}, \quad \mathrm{e}(\mathbf{u}_\mathrm{S}) := \|\mathbf{u}_\mathrm{S} - \mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}},$$

$$\mathrm{e}(\boldsymbol{\rho}_\mathrm{S}) := \|\boldsymbol{\rho}_\mathrm{S} - \boldsymbol{\rho}_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}}, \quad \mathrm{e}(p_\mathrm{S}) := \|p_\mathrm{S} - p_{\mathrm{S},h}\|_{0,\Omega_\mathrm{S}}, \quad \mathrm{e}(\mathbf{u}_\mathrm{D}) := \|\mathbf{u}_\mathrm{D} - \mathbf{u}_{\mathrm{D},h}\|_{\mathrm{div},\Omega_\mathrm{D}},$$

$$\mathrm{e}(p_\mathrm{D}) := \|p_\mathrm{D} - p_{\mathrm{D},h}\|_{0,\Omega_\mathrm{D}}, \quad \mathrm{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,00,\Sigma}, \quad \mathrm{e}(\lambda) := \|\lambda - \lambda_h\|_{1/2,\Sigma}.$$

where $p_{\mathrm{S},h}$ is the postprocessed pressure given by

$$p_{\mathrm{S},h} := -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}_{\mathrm{S},h} + (\mathbf{u}_{\mathrm{S},h} \otimes \mathbf{u}_{\mathrm{S},h})) - l_h \quad \text{in} \quad \Omega_\mathrm{S}.$$

In addition, we define the experimental rates of convergence

$$r(\%) := \frac{\log(\mathrm{e}(\%)/\widehat{\mathrm{e}}(\%))}{\log(h/\widehat{h})} \qquad \text{for each } \% \in \{\mathbf{t}_\mathrm{S}, \boldsymbol{\sigma}_\mathrm{S}, \mathbf{u}_\mathrm{S}, \boldsymbol{\rho}_\mathrm{S}, p_\mathrm{S}, \mathbf{u}_\mathrm{D}, p_\mathrm{D}, \boldsymbol{\varphi}, \lambda\},$$

where e and $\widehat{\mathrm{e}}$ denote errors computed on two consecutive meshes of sizes h and $\widehat{h}$, respectively.

The examples to be considered in this section are described next. In all of them we choose $\mathbf{K} = \mathbb{I}$, $\omega = 1$, and according to (3.26), the stabilization parameters are taken as $\kappa_1 = \mu_1/L_\mu^2$, with $L_\mu := \max\{\mu_2, 2\mu_2 - \mu_1\}$, $\kappa_2 = \kappa_1$, $\kappa_3 = \mu_1/2$, and $\kappa_4 = C_{\mathrm{Ko}}\mu_1/4$, with $C_{\mathrm{Ko}} = 0.5$. In addition, the null mean value of $\mathrm{tr}\,\boldsymbol{\sigma}_{\mathrm{S},h}$ and $p_{\mathrm{D},h}$ over Ω_S and Ω_D, respectively, are fixed via a penalization strategy.

In our first example we consider a porous unit square, coupled with a semi-disk-shaped fluid domain, i.e., $\Omega_\mathrm{D} := (-0.5, 0.5)^2$ and $\Omega_\mathrm{S} := \{(x_1, x_2) : x_1^2 + (x_2 - 0.5)^2 < 0.25,\, x_2 > 0.5\}$ (see bottom left panel of Figure 7.2). In this case, we set the nonlinear viscosity to

$$\mu(s) := 2 + \frac{1}{1+s} \quad \text{for } s \geq 0.$$

The data $\mathbf{f}_\mathrm{S}$ and f_D are chosen so that the exact solution in the tombstone-shaped domain Ω is given by the smooth functions

$$p_\mathrm{S}(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2), \quad \mathbf{u}_\mathrm{S}(\mathbf{x}) = -\mathbf{curl}\,(\cos(\pi x_1)\cos(\pi x_2)),$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{S}$, and

$$p_\mathrm{D}(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2) \qquad \forall\, \mathbf{x} := (x_1, x_2) \in \Omega_\mathrm{D},$$

where $\mathbf{curl}\,(v) := \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)^{\mathrm{t}}$ for any sufficiently smooth function v. Notice that this solution satisfies $\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ and the boundary condition $\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0$ on Γ_{D}. However, the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_{S} is non-homogeneous. Then, we need to modify accordingly the functional $\mathbf{F}$ (cf. (2.17)), as follows

$$[\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := -\kappa_2(\mathbf{f}_{\mathrm{S}}, \mathbf{div}\boldsymbol{\tau}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \mathbf{g}\rangle_{\Gamma_{\mathrm{S}}} \quad \forall\, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{X},$$

where $\mathbf{g} := \mathbf{u}_{\mathrm{S}}|_{\Gamma_{\mathrm{S}}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{S}})$.

In our second example we consider the regions $\Omega_{\mathrm{S}} := (0,1)^2$ and $\Omega_{\mathrm{D}} := (0,1)\times(-1,0)$. The viscosity follows a Carreau law (cf. (2.3)) with $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta = 1$, that is

$$\mu(s) := 0.5 + 0.5(1 + s^2)^{-1/2} \quad \text{for } s \geq 0,$$

and the data $\mathbf{f}_{\mathrm{S}}$ and f_{D} are chosen so that the exact solution is given by

$$p_{\mathrm{S}}(\mathbf{x}) = x_1^2 - x_2^2, \quad \mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \mathbf{curl}\,(x_1(x_1-1)(x_2-1)\sin(\pi x_1)\sin(\pi x_2))$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}}$, and

$$p_{\mathrm{D}}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2) \qquad \forall \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{D}}.$$

Finally, in Example 3 we consider $\Omega_{\mathrm{S}} := (0,1)^2$ and let Ω_{D} be the L-shaped domain given by $(-1,1)^2 \setminus \overline{\Omega}_{\mathrm{S}}$. The viscosity follows a Carreau law with $\alpha_0 = 0.5$, $\alpha_1 = 0.5$, and $\beta = 1.5$, that is

$$\mu(s) := 0.5 + 0.5(1 + s^2)^{-1/4} \quad \text{for } s \geq 0.$$

The data $\mathbf{f}_{\mathrm{S}}$ and f_{D} are chosen so that the exact solution is given by

$$p_{\mathrm{S}}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2), \quad \mathbf{u}_{\mathrm{S}}(\mathbf{x}) = \mathbf{curl}\,(\sin(\pi x_1)\sin(\pi x_2))$$

for all $\mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{S}}$, and

$$p_{\mathrm{D}}(\mathbf{x}) = \cos(\pi x_1)\cos(\pi x_2) \qquad \forall \mathbf{x} := (x_1, x_2) \in \Omega_{\mathrm{D}}.$$

In Tables 7.1, 7.2 and 7.3 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the finite element spaces introduced in Section 6.1 with $k = 0$, and solving the nonlinear problem with a tolerance $tol = 1E{-}6$, which required around five Newton iterations. We observe that the rate of convergence $O(h^{k+1})$ predicted by Theorem 6.1 (when $\delta = k + 1$) is attained in all the variables (with $k = 0$). In addition, some components of the approximate solutions for the three examples are displayed in Figures 7.1, 7.2 and 7.3. All the figures were obtained with 785349, 1470527 and 2190236 degrees of freedom for the Examples 1, 2, and 3, respectively.

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dof	e($\mathbf{t}_S$)	$r(\mathbf{t}_S)$	e($\boldsymbol{\sigma}_S$)	$r(\boldsymbol{\sigma}_S)$	e($\mathbf{u}_S$)	$r(\mathbf{u}_S)$	e($\boldsymbol{\rho}_S$)	$r(\boldsymbol{\rho}_S)$	e(p_S)	$r(p_S)$
859	0.4148	–	4.6754	–	0.9306	–	1.2060	–	0.6239	–
3205	0.2057	1.0541	2.4721	0.9576	0.4707	0.9459	0.7050	0.8068	0.3409	0.9084
12542	0.1039	0.9941	1.2795	0.9638	0.2386	1.0366	0.3537	1.0091	0.1529	1.1735
50281	0.0481	1.1004	0.6393	0.9914	0.1144	1.0582	0.1681	1.0634	0.0687	1.1424
198922	0.0249	0.9703	0.3494	0.8894	0.0580	1.0209	0.0883	0.9468	0.0360	0.9352
785349	0.0127	0.9790	0.1743	1.0089	0.0294	1.0023	0.0452	0.9728	0.0184	0.9865

dof	e($\mathbf{u}_D$)	$r(\mathbf{u}_D)$	e(p_D)	$r(p_D)$	e($\boldsymbol{\varphi}$)	$r(\boldsymbol{\varphi})$	e(λ)	$r(\lambda)$	iter
859	1.1862	–	0.0586	–	1.0668	–	0.2034	–	5
3205	0.6054	1.0330	0.0299	1.0342	0.5573	0.9368	0.0978	1.0564	5
12542	0.3049	1.0072	0.0151	1.0067	0.2703	1.0436	0.0478	1.0317	5
50281	0.1516	1.0146	0.0075	1.0148	0.1344	1.0080	0.0243	0.9778	5
198922	0.0755	1.0024	0.0037	1.0025	0.0673	0.9973	0.0119	1.0254	5
785349	0.0379	1.0074	0.0019	1.0065	0.0336	1.0030	0.0060	1.0025	5

Table 7.1: Example 1, Degrees of freedom, errors, convergence history and Newton iteration count for the augmented finite element formulation.

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dof	e($\mathbf{t}_\mathrm{S}$)	$r(\mathbf{t}_\mathrm{S})$	e($\boldsymbol{\sigma}_\mathrm{S}$)	$r(\boldsymbol{\sigma}_\mathrm{S})$	e($\mathbf{u}_\mathrm{S}$)	$r(\mathbf{u}_\mathrm{S})$	e($\boldsymbol{\rho}_\mathrm{S}$)	$r(\boldsymbol{\rho}_\mathrm{S})$	e(p_S)	$r(p_\mathrm{S})$
1595	0.2408	–	1.0549	–	0.5103	–	0.6426	–	0.1225	–
5975	0.1160	1.1009	0.5503	0.9814	0.2513	1.0682	0.3750	0.8119	0.0597	1.0834
23486	0.0557	1.0713	0.2689	1.0455	0.1202	1.0763	0.1841	1.0383	0.0295	1.0275
93248	0.0281	0.9887	0.1368	0.9783	0.0607	0.9880	0.1005	0.8763	0.0149	1.0015
373093	0.0138	1.0256	0.0673	1.0247	0.0303	1.0026	0.0503	1.0003	0.0071	1.0666
1470527	0.0067	1.0641	0.0334	1.0218	0.0148	1.0468	0.0256	0.9829	0.0033	1.1104

dof	e($\mathbf{u}_\mathrm{D}$)	$r(\mathbf{u}_\mathrm{D})$	e(p_D)	$r(p_\mathrm{D})$	e($\boldsymbol{\varphi}$)	$r(\boldsymbol{\varphi})$	e(λ)	$r(\lambda)$	iter
1595	1.2205	–	0.0606	–	0.5111	–	0.2195	–	5
5975	0.6001	1.0874	0.0296	1.0968	0.2138	1.2577	0.1001	1.1326	5
23486	0.3026	1.0030	0.0149	1.0018	0.1030	1.0531	0.0494	1.0179	5
93248	0.1518	1.0069	0.0075	1.0073	0.0551	0.9032	0.0246	1.0065	5
373093	0.0757	1.0012	0.0037	1.0014	0.0266	1.0529	0.0122	1.0066	5
1470527	0.0381	1.0043	0.0019	1.0034	0.0139	0.9353	0.0062	0.9796	5

Table 7.2: Example 2, Degrees of freedom, errors, convergence history and Newton iteration count for the augmented finite element formulation.

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dof	$\mathrm{e}(\mathbf{t}_{\mathrm{S}})$	$r(\mathbf{t}_{\mathrm{S}})$	$\mathrm{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(\boldsymbol{\rho}_{\mathrm{S}})$	$r(\boldsymbol{\rho}_{\mathrm{S}})$	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$
2317	0.8558	–	3.7298	–	1.5556	–	2.9909	–	0.6252	–
8754	0.4521	0.9623	1.9581	0.9717	0.7868	1.0278	1.8055	0.7611	0.3568	0.8457
34744	0.2089	1.1268	0.9615	1.0382	0.3881	1.0317	0.9612	0.9202	0.1501	1.2637
138089	0.1061	0.9810	0.4818	1.0003	0.1946	0.9993	0.4855	0.9887	0.0761	0.9844
552690	0.0524	1.0196	0.2409	1.0005	0.0964	1.0135	0.2488	0.9650	0.0372	1.0309
2190236	0.0241	1.1328	0.1187	1.0315	0.0469	1.0514	0.1216	1.0435	0.0177	1.0826

dof	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathrm{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathrm{e}(\lambda)$	$r(\lambda)$	iter
2317	2.1509	–	0.1229	–	1.4795	–	0.5225	–	5
8754	1.0877	1.0236	0.0548	1.2126	0.7885	0.9080	0.2181	1.2604	5
34744	0.5389	1.0127	0.0267	1.0355	0.3985	0.9847	0.0948	1.2024	5
138089	0.2706	0.9998	0.0134	1.0018	0.1882	1.0822	0.0525	0.8526	5
552690	0.1353	0.9991	0.0067	1.0003	0.0932	1.0135	0.0267	0.9773	5
2190236	0.0677	1.0029	0.0033	1.0003	0.0468	0.9930	0.0136	0.9747	5

Table 7.3: EXAMPLE 3, Degrees of freedom, errors, convergence history and Newton iteration count for the augmented finite element formulation.

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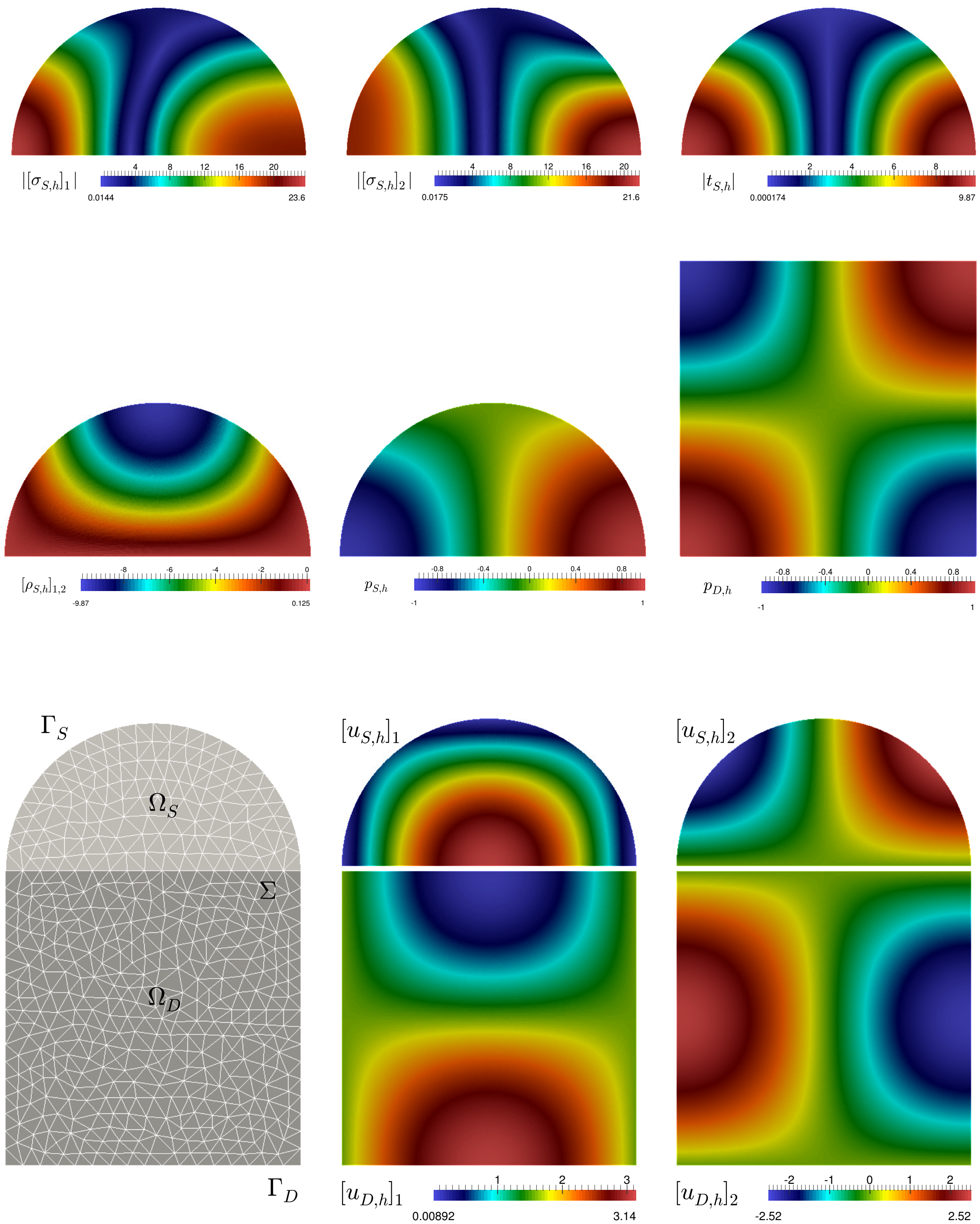

Figure 7.1: Example 1: Approximated spectral norm of the stress tensor components and the strain tensor (top panels), skew-symmetric part of the Navier–Stokes velocity gradient, Navier–Stokes pressure field, and Darcy pressure field (centre panels), and geometry configuration and velocity components on the whole domain (bottom row).

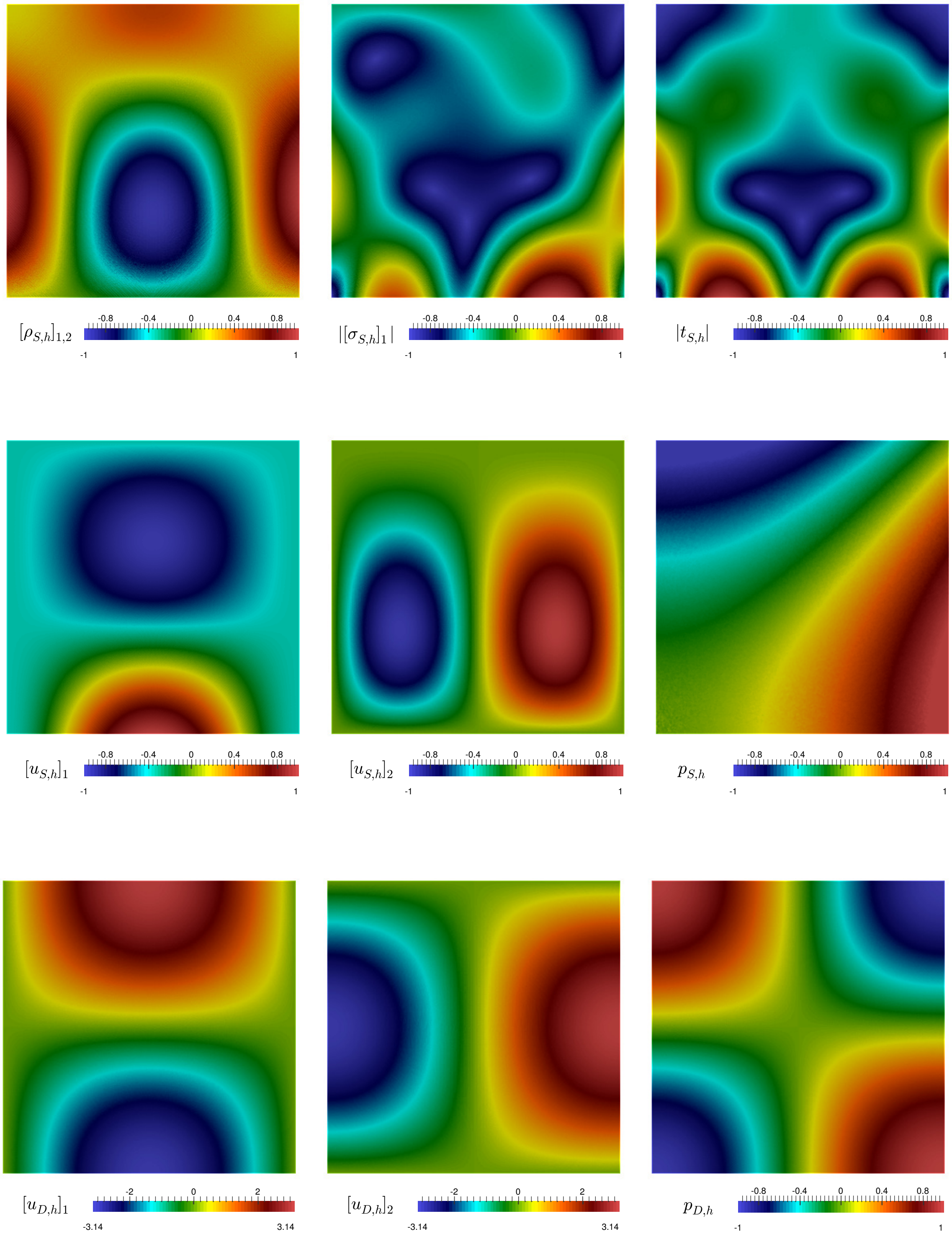

Figure 7.2: Example 2: Skew-symmetric part of the Navier–Stokes velocity gradient, approximated spectral norm of one of the components of the stress tensor and the strain tensor (top panels), Navier–Stokes velocity components and Navier–Stokes pressure field (centre panels), and Darcy velocity components and Darcy pressure field (bottom row).

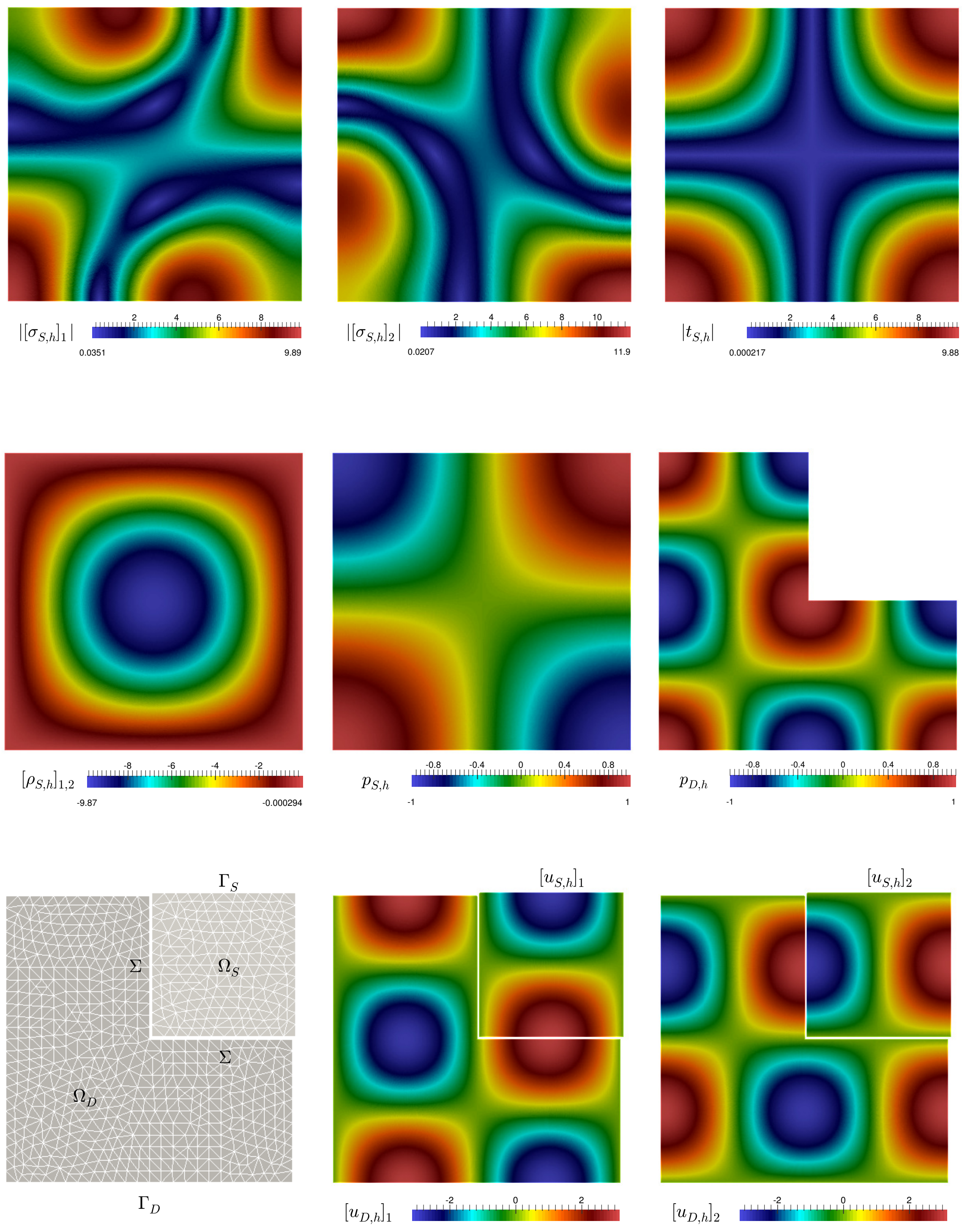

Figure 7.3: Example 3: Approximated spectral norm of the stress tensor components and the strain tensor (top panels), skew-symmetric part of the Navier–Stokes velocity gradient, Navier–Stokes pressure field, and Darcy pressure field (centre panels), and geometry configuration and velocity components on the whole domain (bottom row).

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