Fixed points in conjunctive networks and maximal independent sets in graph contractions

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Abstract

For a graph G, let $\mathcal{C}$ be the set of conjunctive networks with interaction graph G, and let $\mathcal{H}$ be the set of graphs obtained from G by contracting some edges. Let $\mathrm{fix}(f)$ be the number of fixed points in a network $f \in \mathcal{C}$, and let $\mathrm{mis}(H)$ be the number of maximal independent sets in $H \in \mathcal{H}$. Our main result is

$$\mathrm{mis}(G) \;\leq\; \max_{H\in\mathcal{H}} \mathrm{mis}(H) \;\leq\; \max_{f\in\mathcal{C}} \mathrm{fix}(f) \;\leq\; \left(\frac{3}{2}\right)^{m(G)} \mathrm{mis}(G)$$

where $m(G)$ is the maximum size of a matching M of G such that every edge of M is contained in an induced copy of C_4 that contains no other edge of M. Thus if G has no induced C_4 then $\max_{H\in\mathcal{H}} \mathrm{mis}(H) = \max_{f\in\mathcal{C}} \mathrm{fix}(f) = \mathrm{mis}(G)$, and this contrasts with following complexity result: It is coNP-hard to decide if $\max_{f\in\mathcal{C}} \mathrm{fix}(G) = \mathrm{mis}(G)$ or if $\max_{H\in\mathcal{H}} \mathrm{mis}(H) = \mathrm{mis}(G)$, even if G has a unique induced copy of C_4.
Keywords: Boolean network, fixed point, maximal independent set, edge contraction.

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1 Introduction

This paper is at the frontier between Graph Theory and Boolean Network Theory. A *Boolean network* with n components is a discrete dynamical system usually defined to be a map

$$f:\{0,1\}^n\to\{0,1\}^n,\qquad x=(x_1,\dots,x_n)\mapsto f(x)=(f_1(x),\dots,f_n(x)).$$

The *interaction graph* of f is the digraph G on $[n]=\{1,\dots,n\}$ that contains an arc from j to i if f_i depends on x_j. Given a family $\mathcal{F}$ of Boolean networks and a family $\mathcal{G}$ of digraphs, we denote by $\mathrm{fix}(\mathcal{G},\mathcal{F})$ the maximum number of fixed points among all the networks in $\mathcal{F}$ with an interaction graph in $\mathcal{G}$. Here, we are mainly interested in *conjunctive networks*, that is, networks f such that each component f_i is a conjunction of positive or negative literals. The set of all Boolean networks is denoted $\mathbb{F}$, and the set of all conjunctive networks is denoted $\mathbb{C}$. The number of maximal independent set in G is denoted $\mathrm{mis}(G)$, and if $\mathcal{G}$ is a set of digraphs, then $\mathrm{mis}(\mathcal{G})=\max_{G\in\mathcal{G}}\mathrm{mis}(G)$.

Boolean networks have many applications. They are classical model for the dynamics of gene networks [14, 18, 19, 13], neural networks [15, 11, 6, 7], social interactions [16, 9] and more [20, 8]. From a theoretical point of view, the quantity $\mathrm{fix}(\mathcal{G},\mathcal{F})$ has deserved a lot of attention in different contexts, mostly in computational biology [3, 2, 1] and Information Theory [17, 5]. In computational biology, the motivation comes from the fact that fixed points have often a biological meaning and that the first reliable information on gene networks are often represented under the form of interaction graphs. Furthermore, gene networks controlling differentiation process are known to produce multiple stable states, so that upper bound on $\mathrm{fix}(\mathcal{G},\mathcal{F})$ are particularly relevant. In information theory, the study of $\mathrm{fix}(\mathcal{G},\mathcal{F})$ is strongly related to the network coding solvability problem [17, 5], and here again upper-bound are of special interest. It is then not so surprising that the following fundamental bound has been established independently in both contexts [3, 2, 17]:

$$\mathrm{fix}(G,\mathbb{F})\leq 2^{\tau(G)}.$$

Here $\tau(G)$ is the cycle transversal number of G, that is, the minimum number of vertices whose deletion leaves the graph acyclic. To go further, different tools have been used. For instance, Riis and Gadouleau proved the following approximation [5]: *If G is a digraph with n vertices and $\mathcal{G}$ is the set of subgraphs of G then*

$$\frac{2^n}{\sum_{k=0}^{n-\delta(G)}\binom{n}{k}}\leq A(n,n-\delta(G)+1)\leq \mathrm{fix}(\mathcal{G},\mathbb{F})\leq A(n,g(G))\leq \frac{2^n}{\sum_{k=0}^{\lfloor\frac{g(G)-1}{2}\rfloor}\binom{n}{k}}$$

where $\delta(G)$ is the minimal in-degree of G, $g(G)$ is the directed girth of G (minimal length of a directed cycle), and $A(n,d)$ is the maximum cardinality of a binary code of length n with minimal Hamming distance d. This quantity $A(n,d)$ has been extensively studied, and the first and last inequalities are classic results in Information Theory, known as the Gilbert and sphere-packing bound. Beside, using graph transformation techniques, we recently prove in [1] the following equality, where the right hand side is a well known quantity in graph theory [4, 10]: *If $\mathcal{G}$ is the set of all loop-less connected digraphs with n vertices then*

$$\mathrm{fix}(\mathcal{G},\mathbb{C}) = \mathrm{mis}(\mathcal{G}). \tag{1}$$

In this paper, we study the quantity $\mathrm{fix}(G,\mathbb{C})$, pushing further the connexion with maximal independent sets. Our main results, the following, concerns (undirected) graphs, which are seen as loop-less symmetric digraphs. We denote by C_k the (undirected) cycle of length k.

Theorem 1. *For every graph G,*

$$\mathrm{mis}(G) \leq \mathrm{fix}(G,\mathbb{C}) \leq \left(\frac{3}{2}\right)^{m(G)} \mathrm{mis}(G)$$

where $m(G)$ is the maximum size of a matching M of G such that every edge of M is contained in an induced copy of C_4 that contains no other edge of M.

Before discuss briefly this theorem, let us note that it can be restated in purely graph theoretical terms. Let $\mathcal{H}(G)$ the set of graphs obtained from G by contracting some edges. For every connected subgraph C of G, let G/C be the graph obtained by contracting C into a single vertex c and by adding an edge between c and a new vertex c'. Let $\mathcal{H}'(G)$ be the set of graphs that can be obtained from G by repeating such an operation (by convention G is a member of $\mathcal{H}(G)$ and $\mathcal{H}'(G)$). We will prove the following equality.

Theorem 2. *For every graph G,* $\mathrm{mis}(\mathcal{H}'(G)) = \mathrm{fix}(G,\mathbb{C})$*.*

Furthermore, since every graph H in $\mathcal{H}(G)$ can be obtained from a graph H' in $\mathcal{H}'(G)$ by removing some pending vertices, and since H is then an induced subgraph of H', we have $\mathrm{mis}(H) \leq \mathrm{mis}(H')$, and as a consequence $\mathrm{mis}(\mathcal{H}(G)) \leq \mathrm{mis}(\mathcal{H}'(G))$. Putting things together, we then obtain the following graph theoretical version of our main result:

$$\mathrm{mis}(G) \leq \mathrm{mis}(\mathcal{H}(G)) \leq \mathrm{mis}(\mathcal{H}'(G)) \leq \left(\frac{3}{2}\right)^{m(G)} \mathrm{mis}(G).$$

The upper bound on $\mathrm{mis}(\mathcal{H}'(G))$, which is the non trivial part, is reached for the disjoint union of C_4. Suppose indeed that G is the disjoint union of k copies of C_4. Then $m(G) = k$ and $\mathrm{mis}(G) = \mathrm{mis}(C_4)^k = 2^k$ thus the upper bound is 3^k. Now, let H be the disjoint union of k copies of C_3. Then $H \in \mathcal{H}(G)$ and $\mathrm{mis}(H) = \mathrm{mis}(C_3)^k = 3^k$. Thus $\mathrm{mis}(\mathcal{H}(G))$ reaches the bound, and this forces $\mathrm{mis}(\mathcal{H}'(G))$ to reach the bound.

Obviously, if G has no induced C_4 then $m(G) = 0$ and we obtain the following corollary (which trivializes (1) for the class of graphs without induced copy of C_4).

Corollary 1. *For every graph G without induced copy of C_4,*

$$\text{mis}(\mathcal{H}(G)) = \text{fix}(G, \mathbb{C}) = \text{mis}(G).$$

This contrasts with the following complexity result.

Theorem 3. *Given an graph G, it is coNP-hard to decide if* $\text{fix}(G, \mathbb{C}) = \text{mis}(G)$ *or if* $\text{mis}(\mathcal{H}(G)) = \text{mis}(G)$*, even if G has a unique induced copy of C_4.*

The paper is organized as follows. In Section 2 we define with more precision the notions involved in the mentioned results. Then we prove Theorems 1, 2 and 3 in Sections 3, 4 and 5, respectively.

2 Preliminaries

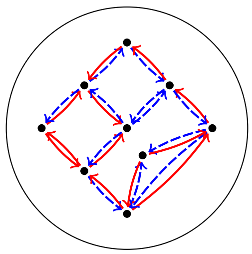
The vertex set of a digraph G is denoted $V = V(G)$ and its arc set is denoted $E = E(G)$. An arc from a vertex u to v is denoted uv. An arc vv is a *loop*. The in-neighbor of a vertex v is denoted $N_G(v)$, and if U is a set of vertices then $N_G(U) = \bigcup_{v \in U} N_G(v)$. We say that G is *symmetric* if $uv \in E$ for every $vu \in E$. The strongly connected components of G are seen as set of vertices, and a component C is *trivial* if it contains an unique vertex. We see *(undirected) graphs* as loop-less symmetric digraphs. The set of maximal independent sets of G is denoted $\text{Mis}(G)$ and $\text{mis}(G) = |\text{Mis}(G)|$. The subgraph of G induced by a set of vertices U is denoted $G[U]$, and $G - U$ denotes the subgraph induced by $V - U$.

A *signed digraph* G_σ is a digraph G together with an arc-labeling function σ that gives a positive or negative sign to each arc of G. Such a labeling is called a *repartition of signs* in G. We denote by G^+_σ (resp. G^-_σ) the spanning subgraph of G containing all the positive (resp. negative) arcs of G_σ. We denote by $G_{\sigma=+}$ (resp. $G_{\sigma=-}$) the signed digraph G_σ where $\sigma = \text{cst} = +$ (resp. $\sigma = \text{cst} = -$). This constant labeling is referred as the *full positive (resp. negative) repartition.* If $U \subseteq V$, then $G_\sigma[U] = G[U]_{\sigma_{|U}}$ and $G_\sigma - U = G_\sigma[V - U]$. We say that G_σ is a *simple signed graph* if G is a graph and σ is symmetrical, that is, $\sigma(uv) = \sigma(vu)$ for all $uv \in E$. Equivalently, G_σ is a simple signed graph if G, G^+ and G^- are graphs (see an illustration in Figure 1). In a simple signed graph, directions do not matter, and we thus speak about positive and negative *edges*, instead of positive and negative arcs.

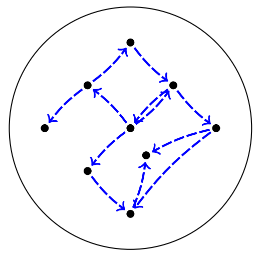
A *Boolean network* on a finite set V is a function

$$f : \{0,1\}^V \to \{0,1\}^V, \qquad x = (x_v)_{v \in V} \mapsto f(x) = (f_v(x))_{v \in V}$$

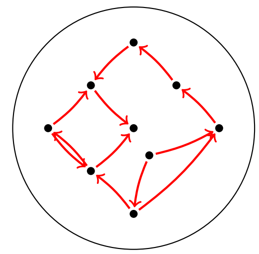
Thus each component f_v is a Boolean function from $\{0,1\}^V$ to $\{0,1\}$ (often called local update function). The *interaction graph* of f is the digraph G on V such that for all $u, v \in V$ there is an uv if and only if f_v depends on x_u, that is, there is $x, y \in \{0,1\}^V$ that only differs in $x_u \neq y_u$ such that $f_v(x) \neq f_v(y)$. We say that f is a *conjunctive network*



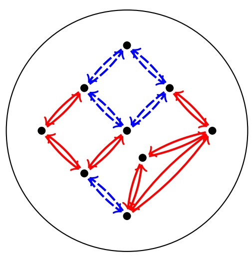
A signed graph G



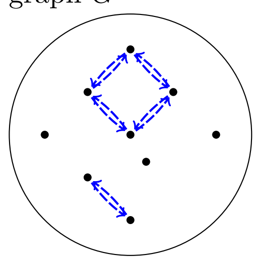
The corresponding graph G^+



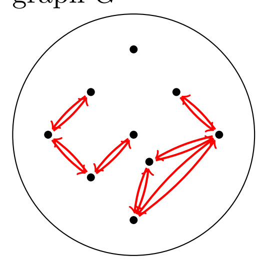
The corresponding graph G^-



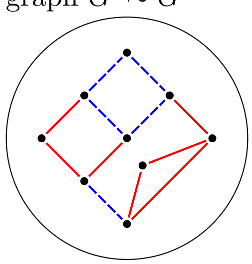
A *simple* signed graph $G' \sim G$



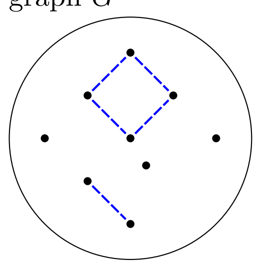
The corresponding graph G'^+



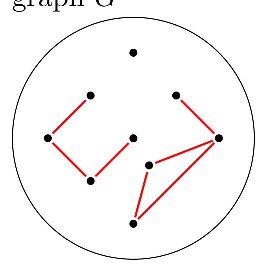
The corresponding graph G'^-



Another representation of G'



Another representation of G'^+



Another representation of G'^-

Figure 1: Example of signed graph and simple signed graph. Dashed arcs are positive and solid arcs are negative.

if, for all $v \in V$, f_v is a conjunction of positive and negative literals. If f is a conjunctive network, then the *signed interaction graph* of f is the signed digraph G_σ where G is the interaction graph of f, and where for all arc uv we have $\sigma(uv) = +$ if x_u is a positive literal of f_v and $\sigma(uv) = -$ if $\overline{x_u}$ is a negative literal of f_v. Conversely, given any signed digraph G_σ, there is clearly a unique conjunctive network whose signed interaction graph is G_σ, namely the conjunctive network f^{G_σ} defined by

$$\forall v \in V, \quad \forall x \in \{0,1\}^V, \qquad f_v^{G_\sigma}(x) = \prod_{u \in N_{G_\sigma^+}(v)} x_u \prod_{u \in N_{G_\sigma^-}(v)} 1 - x_u.$$

We denote by $\mathrm{Fix}(G_\sigma)$ the set of fixed points of f^{G_σ} and by $\mathrm{fix}(G_\sigma)$ the number of fixed points in f^{G_σ}. Now, recall that the quantity we want to study is $\mathrm{fix}(G,\mathbb{C})$, defined in the introduction to be the maximum number of fixed points among all the conjunctive networks with interaction graph G. Since $G_\sigma \mapsto f^{G_\sigma}$ is a bijection between the signed versions of G and the conjunctive networks with interaction graph G, we have

$$\mathrm{fix}(G,\mathbb{C}) = \max_{\sigma : E \to \{+,-\}} \mathrm{fix}(G_\sigma).$$

We will often see a fixed point as the characteristic function of a subset of V. For that we set $\mathbf{1}(x) = \{v \in V \mid x_v = 1\}$, and we say that a subset $S \subseteq V$ is a *fixed set* of G_σ if $S = \mathbf{1}(x)$ for some fixed point $x \in \mathrm{Fix}(G_\sigma)$. By abuse of notation, the set of fixed sets of G_σ is also denoted $\mathrm{Fix}(G_\sigma)$. Thus, for all $S \subseteq V$, we have $S \in \mathrm{Fix}(G_\sigma)$ if and only if for all $v \in V$ we have

$$v \in S \iff N_{G_\sigma^-}(v) \cap S = \emptyset \text{ and } N_{G_\sigma^+}(v) \subseteq S. \tag{2}$$

For the full-negative repartition $\sigma = \mathrm{cst} = -$ we have $G^-_{\sigma=-} = G$ and $G^+_{\sigma=-} = \emptyset$ thus the equivalence becomes: $v \in S$ if and only if $N_G(v) \cap S = \emptyset$. If G is a graph, this is precisely the definition of a maximal independent set. Thus we have the following basic relation between fixed sets and maximal independent sets, already exhibited in [1].

Proposition 1. *For every graph G we have* $\mathrm{Fix}(G_{\sigma=-}) = \mathrm{Mis}(G)$.

We immediately obtain

$$\mathrm{mis}(G) \leq \mathrm{fix}(G,\mathbb{C}) \tag{3}$$

and thus only the upper bound in Theorem 1 has to be prove.

Another important observation is that for every signed digraph G_σ, if C is a strongly connected component of G_σ^+, then for all $S \in \mathrm{Fix}(G_\sigma)$ we have either $C \cap S = \emptyset$ or $C \subseteq S$. As a consequence, if G_σ^+ is strongly connected and $S \in \mathrm{Fix}(G_\sigma)$ then either $S = \emptyset$ or $S = V$. Thus G_σ has at most two fixed sets: $\emptyset$ and V. Clearly, V is a fixed set if and only if G_σ has no negative arcs, and $\emptyset$ is a fixed set in any case. We have thus the following proposition.

Proposition 2. *For every signed digraph G_σ such that G_σ^+ is strongly connected, we have* $\mathrm{Fix}(G_\sigma) = \{\emptyset, V\}$ *if* $\sigma = \mathrm{cst} = +$ *is the full positive labeling, and* $\mathrm{Fix}(G_\sigma) = \{\emptyset\}$ *otherwise.*

Remark about notations. In the following, we mainly consider signed digraphs. In order to simplify notations, given a signed digraph $G = H_\sigma$, we set $G^+ = H_\sigma^+$ and $G^- = H_\sigma^-$. Hence, if we consider a signed digraph G, the underlying repartition of sign is given by the two spanning subgraphs G^+ and G^-. Furthermore, all concepts that do not involve sign are applied on G or its underlying unsigned digraph H indifferently. For instance, we write $\mathrm{mis}(G)$ or $\mathrm{mis}(H)$ indifferently.

3 Proof of Theorem 1

As said above, we only have to prove the upper-bound. For a graph G, let $m'(G)$ be the maximum size of a matching M of G such that every edge uv of M is contained in an induced copy of C_4 in which u and v are the only vertices adjacent to an edge of M. Clearly, we have $m'(G) \leq m(G)$, and the upper bound that we will prove, the following, is thus slightly stronger than that of Theorem 1.

Theorem 4. *For every signed graph G,*

$$\mathrm{fix}(G) \leq \left(\frac{3}{2}\right)^{m'(G)} \mathrm{mis}(G).$$

The proof is based on the following three lemmas, which are proved in Sections 3.1, 3.2 and 3.3 respectively.

Lemma 1. *For every signed graph G there exists a simple signed graph G' obtained from G by changing the repartition of signs such that* $\mathrm{fix}(G) \leq \mathrm{fix}(G')$.

Lemma 2. *For every simple signed graph G there exists a simple signed graph G' obtained from G by changing the repartition of sign such that* $\mathrm{fix}(G) \leq \mathrm{fix}(G')$ *and such that the set of positive edges of G' is a matching.*

Lemma 3. *Let G be a simple signed graph whose set of positive edges is a matching. Let G' be the simple signed graph obtained from G by making negative a positive edge ab. If G has an induced copy of C_4 containing a and b and no other vertices adjacent to a positive edge, then* $\mathrm{fix}(G) \leq (3/2)\,\mathrm{fix}(G')$ *and otherwise* $\mathrm{fix}(G) \leq \mathrm{fix}(G')$.

Proof of Theorem 4, assuming Lemmas 1, 2 and 3. Let G be a signed graph. The following three previous lemmas show that we can change the repartition of sign in G in order to obtain a simple signed graph G_0 with the following properties: $\mathrm{fix}(G) \leq \mathrm{fix}(G_0)$; the set of positive edges of G_0 forms a matching, say $\{e_1, \ldots, e_m\}$; and each positive edge e_k belongs to an induced copy of C_4 in which the only vertices adjacent to a positive edge are the two vertices of e_k. Thus, by the definition of $m'(G)$, we have

$$m \le m'(G). \tag{4}$$

Now, for $0 \le k < m$, let G_{k+1} be the simple signed graph obtained from G_k by making negative the positive edge e_{k+1}. By Lemma 3, we have $\mathrm{fix}(G_k) \le (3/2)\,\mathrm{fix}(G_{k+1})$. Thus

$$\mathrm{fix}(G) \le \mathrm{fix}(G_0) \le (3/2)^m\,\mathrm{fix}(G_m). \tag{5}$$

Furthermore, since G_m has only negative edges, by Proposition 1, we have $\mathrm{fix}(G_m) = \mathrm{mis}(G_m) = \mathrm{mis}(G)$, and with (4) and (5) we obtain the theorem. □

3.1 Proof of Lemma 1 (symmetrization of signs)

Lemma 4. *Let G be a signed symmetric digraph, and let v be a vertex of G. Let $A(v)$ be the set of arcs uv such that uv and vu have different signs, and suppose that $A(v)$ contains at least one positive arc. Let G' be the signed graph obtained from G by changing the sign of each arc in $A(v)$. Then $\mathrm{fix}(G') \ge \mathrm{fix}(G)$.*

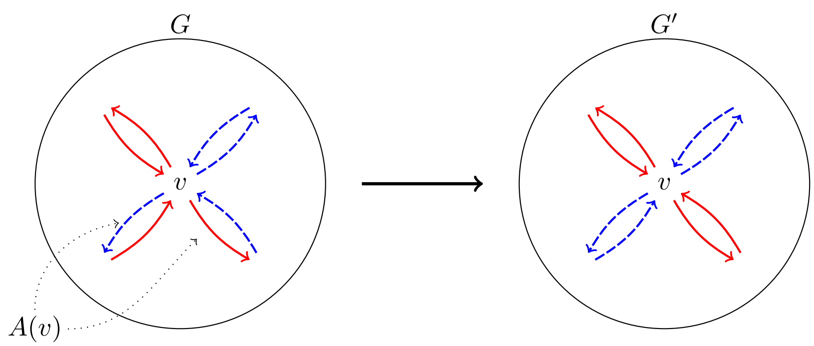


Figure 2: Example of transformation of the graph G as described in Lemma 4.

Proof. Let f and f' be the conjunctive networks associated with G and G' respectively (see example of transformation in Lemma 4). For each $x \in \mathrm{Fix}(G)$, let x' be defined by

$$x'_v = f'_v(x) \quad \text{and} \quad x'_u = x_u \text{ for all } u \neq v.$$

We will prove that $x \mapsto x'$ is an injection from $\mathrm{Fix}(G)$ to $\mathrm{Fix}(G')$. To prove that $x \mapsto x'$ is an injection, it is sufficient to prove that

$$\forall x \in \mathrm{Fix}(G), \qquad x_v = 0.$$

Let $x \in \mathrm{Fix}(G)$ and let uv be a positive arc of $A(v)$. If $x_v = 1 = f_v(x)$ then $x_u = 1$ and since vu is negative, we have $f_u(x) = 0 \neq x_u$, a contradiction.

Now, let us prove that $x' \in \mathrm{Fix}(G')$. Suppose, by contradiction, that $f'(x') \neq x'$. Since G' has no loop on v we have $x'_v = f'_v(x) = f'_v(x')$ thus there exists $u \neq v$ such that

$$f_u(x') = f'_u(x') \neq x'_u = x_u = f_u(x).$$

Thus $x \neq x'$. It means that $x_v = 0$ and $x'_v = 1 = f'_v(x)$, and that vu is an arc of G. Suppose first that vu is positive. Since $x_v = 0$ we have $f_u(x) = 0$ thus $x_u = 0$, and since $f'_v(x) = 1$ we deduce that uv is a negative arc of G'. So uv and vu have different signs in G', a contradiction. Suppose now that vu is negative. Since $x'_v = 1$ we have $f_u(x') = 0$ thus $x_u = 1$, and since $f'_v(x) = 1$ we deduce that uv is a positive arc of G'. Thus uv and vu have different signs in G', a contradiction. So $x' \in \mathrm{Fix}(G')$. □

Proof of Lemma 1. Suppose that G is a signed graph which is not simple. Then it contains at least one asymmetry of signs, that is, an arc uv such that uv and vu have not the same sign. If uv is positive then, by changing the sign of the arcs in $A(v)$, we obtained a signed graph G' with strictly less asymmetries of signs and, by the previous lemma, $\mathrm{fix}(G) \leq \mathrm{fix}(G')$. Otherwise, vu is positive and thus, by changing the sign of the arcs in $A(u)$, we also strictly decrease the number of asymmetries of signs without decreasing the number of fixed points. Obviously, we can apply this transformation until to have no asymmetry of signs. This proves Lemma 1. □

3.2 Proof of Lemma 2 (reduction of positive components)

Let G be a simple signed graph with vertex set V. In this section, we prove that we can change the repartition of signs, without decreasing the number of fixed sets, until to obtain a simple signed graph G' in which positive edges forms a matching.

For that we will introduced decomposition properties, also used to prove Lemma 3. Actually, if $U \subseteq V$ is such that G has no positive edge between U and $V - U$, then a lot of things on $\mathrm{Fix}(G)$ can be understand from $\mathrm{fix}(G[U])$ and the fixed set of the induced subgraphs of $G - U$. In particular, under the condition that there is no positive edge between U and $V - U$, we will prove that if $S \in \mathrm{Fix}(G[U])$ and $S' \in \mathrm{Fix}(G[U'])$, where U' is some subset of $V - U$ that only depends on U and S, then $S \cup S' \in \mathrm{Fix}(G)$. This very useful decomposition properties, essential for the rest of the proof, is based on the following definitions.

For all $U \subseteq V$, we denote by $\mathcal{N}_G(U)$ the union of $N_G(U)$ and the connected components C of G^+ such that $N_G(U) \cap C \neq \emptyset$. In other words, $\mathcal{N}_G(U)$ is the union of $N_G(U)$ and the set of vertices reachable from $N_G(U)$ with a path that contains only positive edges. Note that for all $U \subseteq V$, G has no positive edges between $\mathcal{N}_G(U)$ and $V - \mathcal{N}_G(U)$. We can see an illustration of $N_G(U)$ and $\mathcal{N}_G(U)$ in Figure 3.

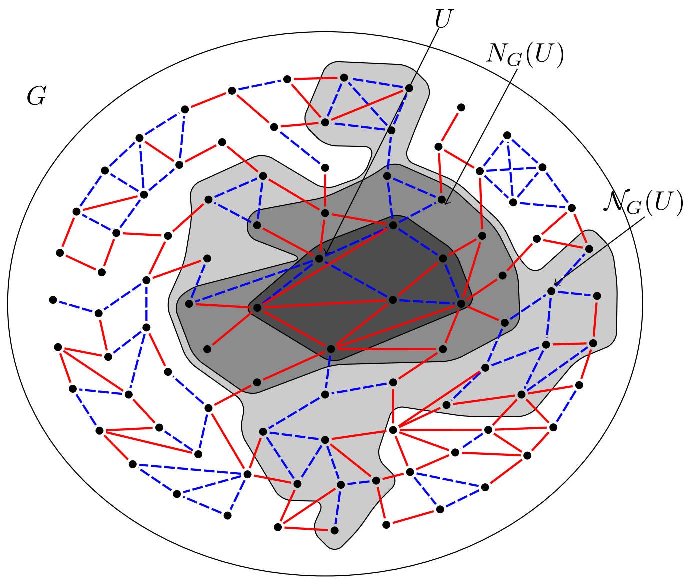


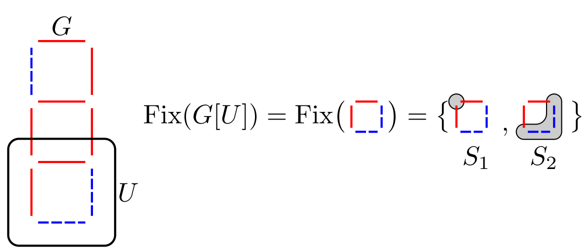
Figure 3: Example of the sets $N_G(U)$ and $\mathcal{N}_G(U)$.

We are now in position to introduce our decomposition tool. For all $U \subseteq V$ we set

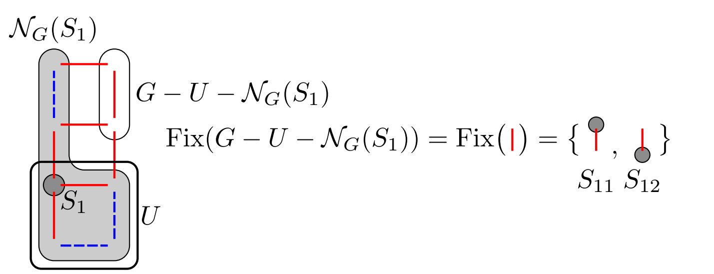
$$\mathrm{Fix}(G, U) = \{S \cup S' \mid S \in \mathrm{Fix}(G[U]),\ S' \in \mathrm{Fix}(G - U - \mathcal{N}_G(S))\}$$

and $\mathrm{fix}(G, U) = |\mathrm{Fix}(G, U)|$.

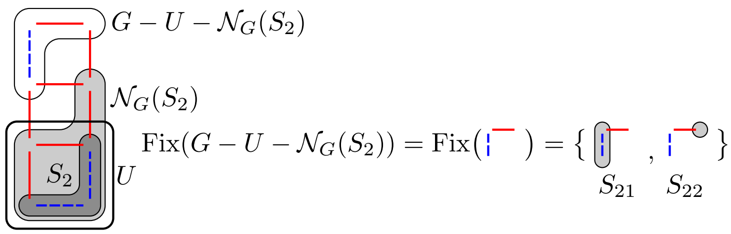
Example 1. *Here is an example where $G[U]$ contains 2 fixed sets, S_1 and S_2:*



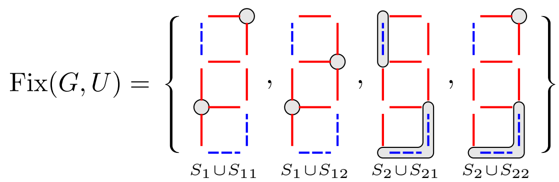
The induced subgraph $G - U - \mathcal{N}_G(S_1)$ has two fixed sets, S_{11} and S_{12}:



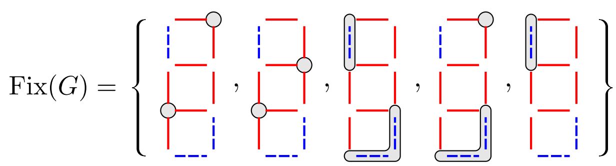
The induced subgraph $G - U - \mathcal{N}_G(S_2)$ has also two fixed sets, S_{21} and S_{22}:



Thus $\mathrm{Fix}(G, U)$ *contains the following 4 sets:*



These 4 sets are all fixed sets, since



Lemma 5. *If $U \subseteq V$ and G has no positive edge between U and $V - U$ then*

$$\mathrm{Fix}(G, U) \subseteq \mathrm{Fix}(G).$$

Proof. Let $S \in \mathrm{Fix}(G[U])$. Let $U' = V(G) - U - \mathcal{N}_G(S)$ and let $S' \in \mathrm{Fix}(G[U'])$. We want to prove that $S \cup S' \in \mathrm{Fix}(G)$. Let f, f^U and $f^{U'}$ be the conjunctive networks of G, $G[U]$ and $G[U']$, respectively. Let $x \in \{0,1\}^V$ be such that $\mathbf{1}(x) = S \cup S'$ and let us prove that x is a fixed point of f. Note that $x_{|U}$ is a fixed point of f^U and $x_{|U'}$ is a fixed point of $f^{U'}$.

Suppose first that $x_v = 0$. There are three possibilities. First, if $v \in U$ then $f_v(x) \leq f^U_v(x_{|U}) = x_v = 0$ thus $f_v(x) = 0$. Second, if $v \in U'$ then $f_v(x) \leq f^{U'}_v(x_{|U'}) = x_v = 0$ thus $f_v(x) = 0$. Finally, suppose that $v \in \mathcal{N}_G(S) - U$. If there is an edge uv with $u \in S$ then $x_u = 1$ and since there is no positive edge between U and $V - U$, this edge is negative thus $f_v(x) = 0$. Otherwise, by definition of $\mathcal{N}_G(S)$, v belongs to a non-trivial component C of G^+ such that $C \subseteq \mathcal{N}_G(S)$. Since G has no positive edge between U and $V - U$ we have $C \cap U = \emptyset$, thus $C \subseteq \mathcal{N}_G(S) - U$. Thus G has positive edge uv with $u \in \mathcal{N}_G(S) - U$. So $x_u = 0$ and we deduce that $f_v(x) = 0$. Hence, in every case, $f_v(x) = 0 = x_v$.

Suppose now that $x_v = 1$ and $v \in S$. If $f_v(x) = 0$ then G has a positive edge uv with $x_u = 0$ or a negative edge uv with $x_u = 1$. If $u \in U$ then $f^U_v(x_{|U}) = 0 \neq x_v$, a contradiction. Thus $u \notin U$ and we deduce from the condition of the statement that uv is a negative edge with $x_u = 1$. Thus $u \in S' \subseteq U'$ and we have a contradiction with the fact that $u \in N_G(v) \subseteq \mathcal{N}_G(S)$. Therefore $f_v(x) = 1 = x_v$.

Suppose finally that $x_v = 1$ and $v \in S'$. If $f_v(x) = 0$ then G has a positive edge uv with $x_u = 0$ or a negative edge uv with $x_u = 1$. If $u \in U'$ then $f_v^{U'}(x_{|U'}) = 0 \neq x_v$, a contradiction. Thus $u \notin U'$. Since there is no positive edge between U and $V - U$, and no positive edge between $\mathcal{N}_G(S)$ and $V - \mathcal{N}_G(S)$, we deduce that there is no positive edge between U' and $V - U'$. Thus uv is a negative edge with $x_u = 1$. Hence, $u \in S \subseteq U$ and we deduce that $v \in N_G(u) \subseteq N_G(S)$, a contradiction. Thus $f_v(x) = 1 = x_v$. □

The following is an immediate consequence.

Lemma 6. *If $U \subseteq V$ and G has no positive edge between U and $V - U$ then*

$$\mathrm{fix}(G[U]) \leq \mathrm{fix}(G, U) \leq \mathrm{fix}(G)$$

Remark 1. *We deduce that* $\mathrm{fix}(G) \geq 1$ *for every simple signed graph G. Indeed, if G has no positive edge then* $\mathrm{fix}(G) = \mathrm{mis}(G) \geq 1$. *Otherwise, G^+ has a connected component C, and by Lemma 6 and Proposition 2 we have* $\mathrm{fix}(G) \geq \mathrm{fix}(G[U]) \geq 1$.

We now prove a lemma with that gives a stronger conclusion under stronger conditions.

Lemma 7. *If $U \subseteq V$, if G has no positive edge between U and $V - U$, and if every vertex in U is adjacent to a positive edge, then*

$$\mathrm{Fix}(G, U) = \mathrm{Fix}(G).$$

Proof. By Lemma 5, it is sufficient to prove that $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(G, U)$. Let f be the conjunctive network of G, and let x be a fixed point of f. Let

$$S = \mathbf{1}(x) \cap U, \qquad U' = V - U - \mathcal{N}_G(S), \qquad S' = \mathbf{1}(x) \cap U'.$$

Let f^U and $f^{U'}$ be the conjunctive networks of $G[U]$ and $G[U']$.

Let $v \in U$. Since v is adjacent to a positive edge and since G has no positive edge between U and $V - U$, we deduce that $G[U]$ has a positive edge uv. If $x_v = 0$ then $x_u = 0$ and we deduce that $f_v^U(x_{|U}) = 0 = x_v$. If $x_v = 1$ then $f_v(x) = 1 \leq f_v^U(x_{|U})$ thus $f_v^U(x_{|U}) = 1 = x_v$. Hence, we have proved that $x_{|U}$ is a fixed point of f^U, that is,

$$S \in \mathrm{Fix}(G[U]). \tag{6}$$

Let $v \in \mathcal{N}_G(S) - U$ and let us proved that $x_v = 0$. By the definition of $\mathcal{N}_G(S)$, v belongs to a component C of G^+ such that $C \cap N_G(S) \neq \emptyset$ and $C \subseteq \mathcal{N}_G(S)$. Actually, $C \subseteq \mathcal{N}_G(S) - U$ since otherwise G has a positive edge between U and $V - U$. Thus there exists a path $u, w_1, w_2, \ldots, w_k, v$ with $u \in S$ and $w_1, w_2, \ldots, w_k, v \in C$. Since uw_1 is negative and $x_u = 1$ we have $f_{w_1}(x) = 0$ thus $x_{w_1} = 0$ and we deduce from Proposition 2 that $x_v = 0$. Thus we have $\mathbf{1}(x) \cap (\mathcal{N}_G(S) - U) = \emptyset$. In other words,

$$\mathbf{1}(x) = S \cup S'. \tag{7}$$

Let $v \in U'$. Suppose that $x_v = 0 = f_v(x)$, and suppose, for a contradiction that $f_v^{U'}(x_{|U'}) = 1$. Since there is no positive edge between U and $V - U$, and no positive edge between $\mathcal{N}_G(S)$ and $V - \mathcal{N}_G(S)$, we deduce that there is no positive edge between U' and $V - U'$. Thus there exists a negative edge uv with $u \in V - U'$ and $x_1 = 1$. But then $u \in S$, thus $v \in \mathcal{N}_G(S)$, a contradiction. Consequently, $f_v^{U'}(x_{|U'}) = 0 = x_v$. Now, if $x_v = 1$ then $f_v(x) = 1 \leq f_v^{U'}(x_{|U'})$ thus $f_v^{U'}(x_{|U'}) = 1 = x_u$. Hence, we have proved that $x_{|U'}$ is a fixed point of $f^{U'}$, that is,

$$S' \in \mathrm{Fix}(G - U - \mathcal{N}_G(S)). \tag{8}$$

According to (6), (7) and (8)), we have $\mathbf{1}(x) \in \mathrm{Fix}(G, U)$. □

In the following, if S is a subset of V and $\mathcal{S}$ is a subset of the power set of V, then

$$S \sqcup \mathcal{S} = \{S \cup S' \mid S' \in \mathcal{S}\}.$$

Lemma 8. *Let C be is a non-trivial connected component of G^+. If $G[C]$ has no negative edge then*

$$\mathrm{Fix}(G) = \mathrm{Fix}(G - C) \cup \big(C \sqcup \mathrm{Fix}(G - \mathcal{N}_G(C))\big)$$

and otherwise

$$\mathrm{Fix}(G) = \mathrm{Fix}(G - C).$$

Proof. If $G[C]$ has no negative edge then by Proposition 2 we have $\mathrm{Fix}(G[C]) = \{\emptyset, C\}$ and from Lemma 7 we deduce that

$$\begin{aligned}
\mathrm{Fix}(G) &= \mathrm{Fix}(G, C) \\
&= \{S \cup S' \mid S \in \{\emptyset, C\},\ S' \in \mathrm{Fix}(G - C - \mathcal{N}_G(S))\} \\
&= \{\emptyset \cup S' \mid S' \in \mathrm{Fix}(G - C)\} \cup \{C \cup S' \mid S' \in \mathrm{Fix}(G - \mathcal{N}_G(C))\} \\
&= \mathrm{Fix}(G - C) \cup \big(C \sqcup \mathrm{Fix}(G - \mathcal{N}_G(C))\big).
\end{aligned}$$

If $G[C]$ has a negative edge then by Proposition 2 we have $\mathrm{Fix}(G[C]) = \{\emptyset\}$ and proceeding as above we get $\mathrm{Fix}(G) = \mathrm{Fix}(G - C)$. □

The proof of Lemma 2 is a straightforward application of the above decomposition tools.

Proof of Lemma 2. Suppose that G^+ has a non-trivial component C such that $G[C]$ has only positive edges. Let ab be one of these edges, and let G' be the signed graph obtained from G by making negative each edge of $G[C]$ excepted ab. To prove that lemma, it is clearly sufficient to prove that $\mathrm{fix}(G') \geq \mathrm{fix}(G)$ (because then, the process of reduction of positive components into a single positive edges can be repeated until that each positive component G^+ reduces to a single sportive edge). According to Lemma 8 we have

$$\text{fix}(G) \leq \text{fix}(G - C) + \text{fix}(G - \mathcal{N}_G(C)) \tag{9}$$

and

$$\text{fix}(G') = \text{fix}(G' - \{a, b\}) + \text{fix}(G' - \mathcal{N}_{G'}(\{a, b\})). \tag{10}$$

Since $G - C = G' - C$ and $\{a, b\} \subseteq C$, $G - C$ is an induced subgraph of $G' - \{a, b\}$, thus according to Lemma 6 we have

$$\text{fix}(G - C) \leq \text{fix}(G' - \{a, b\}). \tag{11}$$

Since $\{a, b\} \subseteq C$ and since each positive edge of G' is a positive edge of G, we have $\mathcal{N}_{G'}(\{a, b\}) \subseteq \mathcal{N}_G(C)$. Since $G - C = G' - C$ we deduce that $G - \mathcal{N}_G(C)$ is an induced subgraph of $G' - \mathcal{N}_{G'}(\{a, b\})$. Thus according to Lemma 6 we have

$$\text{fix}(G - \mathcal{N}_G(C) \leq \text{fix}(G' - \mathcal{N}_{G'}(\{a, b\})). \tag{12}$$

The lemma follows from (9), (10), (11) and (12). □

3.3 Proof of Lemma 3 (suppression of positive edges)

In all this section, G is a simple signed graph with vertex set V in which the set of positive edges is a matching. Let G' be any simple signed graph obtained from G by making negative a positive edge e. In this section, we study the variation of the number of fixed sets under this transformation. We will prove that if G has no induced copy of C_4 that contains e and no other positive edge, then $\text{fix}(G) \leq \text{fix}(G')$, and in any case $\text{fix}(G) \leq (3/4)\,\text{fix}(G')$. The control of the variation of fixed sets is a little bit technical. This is why we decompose the proof in several steps. We first assume that (i) there is a unique positive edge in G and that (ii) this positive edge is in the neighborhood of each vertex (cf. Lemma 9). Then, thanks to an additional decomposition property (cf. Lemma 10), we suppress condition (ii). We finally suppress condition (i) to get the general statement (cf. Lemma 11).

Lemma 9. *Suppose that G has a unique positive edge, say ab, and $V = N_G(a) \cup N_G(b)$. Let G' be the simple signed graph obtained from G by making ab negative. Then*

$$\text{fix}(G) \leq \text{fix}(G') + 1,$$

and if G has no induced copy of C_4 containing ab, then

$$\text{fix}(G) \leq \text{fix}(G').$$

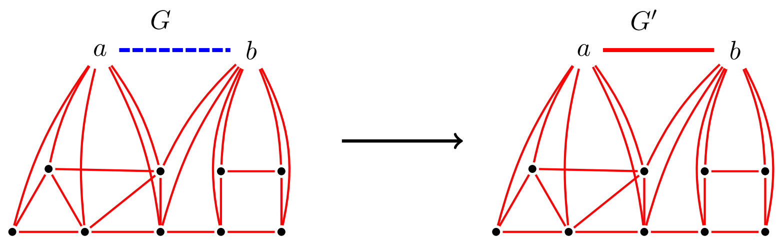


Figure 4: Example of transformation in Lemma 9.

Proof. According to Lemma 8 we have

$$\mathrm{Fix}(G) = \mathrm{Fix}(G - \{a, b\}) \cup \big(\{a, b\} \sqcup \mathrm{Fix}(G - \mathcal{N}_G(\{a, b\})\big)$$

Since $V = N_G(\{a, b\}) \subseteq \mathcal{N}_G(\{a, b\})$, $G - \mathcal{N}_G(\{a, b\})$ is empty, and since $G - \{a, b\}$ has only negative edges we deduce that

$$\mathrm{fix}(G) = \mathrm{mis}(G - \{a, b\}) + 1. \tag{13}$$

Also, since G' has only negative edges (see Figure 4) we have,

$$\mathrm{fix}(G') = \mathrm{mis}(G). \tag{14}$$

Let $A = N_G(a) - N_G(b)$, $B = N_G(b) - N_G(a)$ and $C = N_G(a) \cap N_G(b)$ (see Figure 5). Note that G has an induced copy of C_4 containing ab if and only if there is an edge between A and B. We consider four cases.

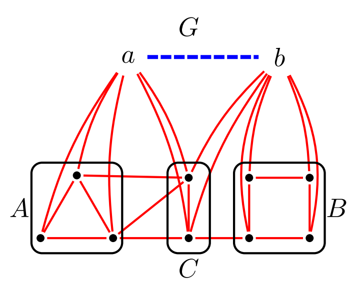


Figure 5: Example of sets A, B and C described in proof of Lemma 9.

1. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Let S be a maximal independent set of $G - \{a, b\}$. If S intersects C, or S intersects both A and B, then a and b have at least one neighbor in S, thus S is a maximal independent set of G. Otherwise it is easy to check that either S intersects A and $S \cup \{b\}$ is a maximal independent set of G, or S intersects

B and $S \cup \{a\}$ is a maximal independent set of G. Consequently, the following map is an injection from $\mathrm{Mis}(G - \{a, b\})$ to $\mathrm{Mis}(G)$:

$$S \mapsto \begin{cases} S & \text{if } S \cap C \neq \emptyset \quad \text{or} \quad (S \cap A \neq \emptyset \text{ and } S \cap B \neq \emptyset) \qquad (i) \\ S \cup \{b\} & \text{if } S \cap C = \emptyset \quad \text{and} \quad (S \cap A \neq \emptyset \text{ and } S \cap B = \emptyset) \qquad (ii) \\ S \cup \{a\} & \text{if } S \cap C = \emptyset \quad \text{and} \quad (S \cap A = \emptyset \text{ and } S \cap B \neq \emptyset) \qquad (iii) \end{cases}$$

Thus $\mathrm{mis}(G - \{a, b\}) \leq \mathrm{mis}(G)$ and we deduce from (13) and (14) that $\mathrm{fix}(G) \leq \mathrm{fix}(G') + 1$. Furthermore, if G has no induced copy of C_4 containing ab, then there is no edge between A and B, and thus cases (ii) and (iii) are not possible. Thus $\mathrm{Mis}(G - \{a, b\}) \subseteq \mathrm{Mis}(G)$ and since G has at least one maximal independent set containing a and one maximal independent set containing b, we deduce that $\mathrm{mis}(G - \{a, b\}) \leq \mathrm{mis}(G) - 2$ (see Figure 6). Using (13) and (14) we get $\mathrm{fix}(G) \leq \mathrm{fix}(G')$ (see Figure 6).

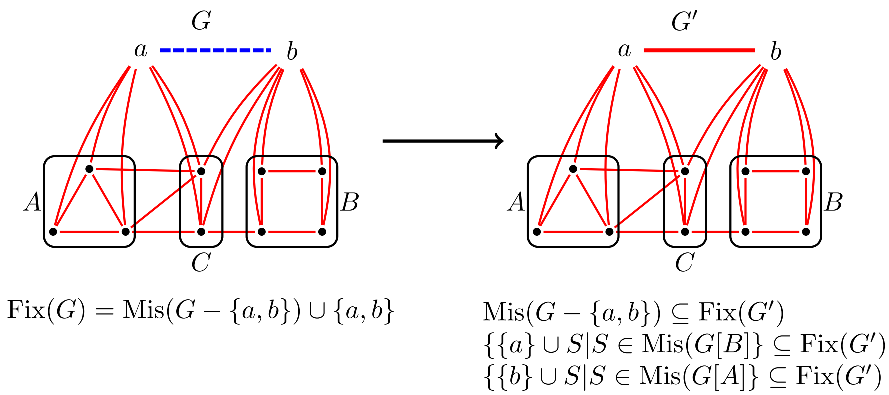


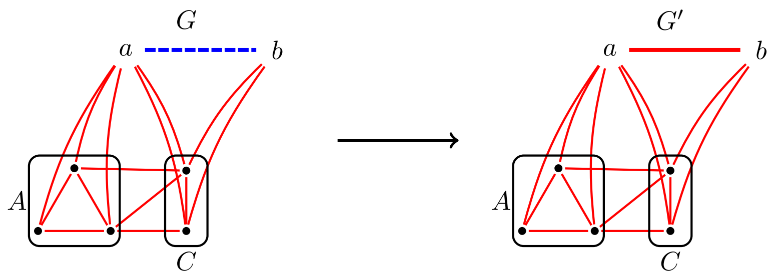
Figure 6: Relationships between the set of fixed points and maximal independent sets as described in proof of Lemma 9.

2. Suppose that $A \neq \emptyset$ and $B = \emptyset$. Let S be a maximal independent set of $G - \{a, b\}$. If S intersects C then S is clearly a maximal independent set of G. Otherwise, S intersects A and $S \cup \{b\}$ is then a maximal independent set of G. Thus, the following map is an injection from $\mathrm{Mis}(G - \{a, b\})$ to $\mathrm{Mis}(G)$:

$$S \mapsto \begin{cases} S & \text{if } S \cap C \neq \emptyset \\ S \cup \{b\} & \text{otherwise} \end{cases}$$

Since G has at least one maximal independent set containing a, we deduce that $\mathrm{mis}(G - \{a, b\}) \leq \mathrm{mis}(G) - 1$. Using (13) and (14) we get $\mathrm{fix}(G) \leq \mathrm{fix}(G')$ (see Figure 7).

3. Suppose that $A = \emptyset$ and $B \neq \emptyset$. We prove as in case 2 that $\mathrm{fix}(G) \leq \mathrm{fix}(G')$.

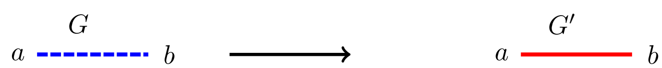


$\mathrm{Fix}(G) = \mathrm{Mis}(G - \{a, b\}) \cup \{a, b\}$

$\mathrm{Fix}(G') = \Big\{\{a\}\Big\} \cup \{S | S \in \mathrm{mis}(G - \{a, b\}), S \cap W \neq \emptyset\}$
$\cup \{S \cup \{b\} | S \in \mathrm{mis}(G - \{a, b\}), S \cap C = \emptyset\}$

Figure 7: Set of fixed points as described in Lemma 9 when $A = \emptyset$ or $B = \emptyset$.

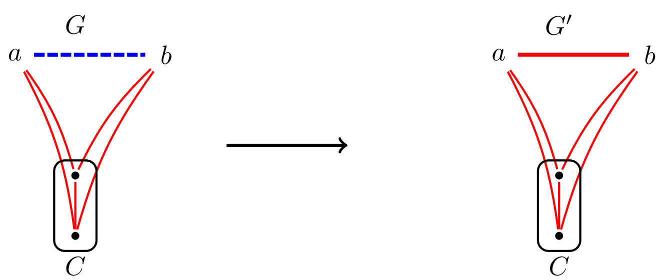
4. Suppose that $A = \emptyset$ and $B = \emptyset$. If $C = \emptyset$ then G reduces to a single positive edge and G' reduced to a single negative edge and thus $\mathrm{fix}(G) = \mathrm{fix}(G') = 2$ (see Figure 8).



$\mathrm{Fix}(G) = \Big\{\{a, b\}, \emptyset\Big\}$ $\qquad$ $\mathrm{Fix}(G') = \Big\{\{a\}, \{b\}\Big\}$

Figure 8: Set of fixed points as described in Lemma 9 when $A = \emptyset$, $B = \emptyset$ and $C = \emptyset$.

So suppose that $C \neq \emptyset$. Then $G - \{a, b\} = G[C]$ and it is clear that $\mathrm{Mis}(G - \{a, b\}) \subseteq \mathrm{Mis}(G)$. As in the first case, since G has at least one maximal independent set containing a and one maximal independent set containing b, we deduce that $\mathrm{mis}(G - \{a, b\}) \leq \mathrm{mis}(G) - 2$. Using (13) and (14) we get $\mathrm{fix}(G) \leq \mathrm{fix}(G')$ (see Figure 9).



$\mathrm{Fix}(G) = \mathrm{Mis}(G - \{a, b\}) \cup \Big\{\{a, b\}\Big\}$ $\qquad$ $\mathrm{Fix}(G') = \mathrm{Mis}(G - \{a, b\}) \cup \Big\{\{a\}\Big\} \cup \Big\{\{b\}\Big\}$

Figure 9: Set of fixed points as described in Lemma 9 when $A = \emptyset$, $B = \emptyset$ and $C \neq \emptyset$.

$\square$

The next decomposition property is a rather technical step which will allows us to suppress the condition $V = N_G(a) \cup N_G(b)$ from the previous proposition.

Lemma 10. *Suppose that G has a positive edge ab such that a and b are not adjacent to other positive edges. Let X be the set of vertices containing a, b and all the vertices of $N_G(a) \cup N_G(b)$ adjacent to only negative edges. Let G' be the simple signed graph obtained from G by making ab negative. If $U = V - X$ then*

$$\mathrm{fix}(G) - \mathrm{fix}(G, U) \leq \mathrm{fix}(G') - \mathrm{fix}(G', U).$$

Proof. Let $x \in \mathrm{Fix}(G) - \mathrm{Fix}(G, U)$. Let $S = \mathbf{1}(x) \cap U$ and $U' = V - U - \mathcal{N}_G(S)$. Since $\mathbf{1}(x)$ and $\mathcal{N}_G(S) - S$ are disjointed, we have $\mathbf{1}(x) = S \cup S'$ with $S' = \mathbf{1}(x) \cap U'$. Let f, f^U and $f^{U'}$ be the conjunctive networks of G, $G[U]$ and $G[U']$ respectively.

Claim 1. *$S \notin \mathrm{Fix}(G[U])$.*

Proof of Claim. Since $S \cup S' \notin \mathrm{Fix}(G, U)$, it is sufficient to prove that $S' \in \mathrm{Fix}(G[U'])$, which is equivalent to prove that $x_{|U'}$ is a fixed point of $f^{U'}$. Let $v \in U'$. Since $f_v(x) \leq f^{U'}_v(x_{|U'})$ if $x_v = 1$ then $f^{U'}_v(x) = 1$. So suppose that $x_v = 0$. Then G has a negative edge uv with $x_u = 1$ or a positive edge uv with $x_u = 0$. Suppose first that G has a negative edge uv with $x_u = 1$. If $u \in S$ then $v \in N_G(S)$ and this is not possible since $v \in U'$. Thus $u \in S' \subseteq U'$ and we deduce that $f^{U'}_v(x_{|U'}) = 0$. Suppose now that G has a positive edge uv with $x_u = 0$. Since there is no positive edge between U and $V - U$ and no positive edge between $\mathcal{N}_G(S)$ and $V - \mathcal{N}_G(S)$, there is no positive edges between U' and $V - U'$. Therefore $u \in U'$ and we deduce that $f^{U'}_v(x_{|U'}) = 0$. Thus in every case $f^{U'}_v(x_{|U'}) = x_v$ and this proves the claim. □

Claim 2. *If $S' = \{a, b\}$ then $S \in \mathrm{Fix}(G[U])$.*

Proof of Claim. Suppose that $S' = \{a, b\}$. Let $v \in U$. Since $f_v(x) \leq f^U_v(x_{|U})$, if $x_v = 1$ then $f^U_v(x) = 1$. So suppose that $x_v = 0$. We consider two cases. Suppose first that v is adjacent to a positive edge, say uv. Since $x_v = 0$ we have $f_u(x) = 0 = x_u$, and since G has no positive edge between U and $V - U$, we have $u \in U$ and we deduce that $f^U_v(x_{|U}) = 0$. Now, suppose that v is not adjacent to a positive edge. Then G has a negative edge uv with $x_u = 1$. If $u \in S' = \{a, b\}$ then $v \in X$ since v is adjacent to no positive edge, a contradiction. Therefore $u \in S \subseteq U$ and we deduce that $f^U_v(x_{|U}) = 0$. Thus $f^U_v(x_{|U}) = x_v$ in every case, and this proves the claim. □

Claim 3. *$S \cup S' \in \mathrm{Fix}(G - \{a, b\})$.*

Proof of Claim. Since every vertex in $U' - \{a, b\}$ is adjacent to a or b by a negative edge, if $\{a, b\} \subseteq S'$ then $S' = \{a, b\}$, which is not possible by Claims 1 and 2. We deduce that S and $\{a, b\}$ are disjoint. By Lemma 8 we have

$$S \cup S' \in \mathrm{Fix}(G) = \mathrm{Fix}(G - \{a, b\}) \cup \big(\{a, b\} \sqcup \mathrm{Fix}(G - \mathcal{N}_G(\{a, b\}))\big).$$

and the claim follows. $\square$

We are now in position to prove the lemma. We have already proved that

$$\mathrm{Fix}(G) - \mathrm{Fix}(G, U) \subseteq \mathrm{Fix}(G - \{a, b\}). \tag{15}$$

Since $G' - \{a, b\} = G - \{a, b\}$ we have

$$\mathrm{Fix}(G', V - \{a, b\}) = \{S \cup S' \mid S \in \mathrm{Fix}(G - \{a, b\}),\ S' \in \mathrm{Fix}(G'[\{a, b\}] - \mathcal{N}_{G'}(S))\}. \tag{16}$$

By Lemma 5 we have $\mathrm{Fix}(G', V - \{a, b\}) \subseteq \mathrm{Fix}(G')$, and we deduce from (15) and (16) that there exists a maps $x \mapsto x'$ from $\mathrm{Fix}(G) - \mathrm{Fix}(G, U)$ to $\mathrm{Fix}(G')$ such that

$$\mathbf{1}(x) \subseteq \mathbf{1}(x') \subseteq \mathbf{1}(x) \cup \{a, b\}.$$

Thus $x \mapsto x'$ is an injection and by Claim 1, we have

$$\mathbf{1}(x') \cap U = \mathbf{1}(x) \cap U \notin \mathrm{Fix}(G[U]) = \mathrm{Fix}(G'[U]).$$

So $x' \notin \mathrm{Fix}(G', U)$ and we deduce that $x \mapsto x'$ is an injection from $\mathrm{Fix}(G) - \mathrm{Fix}(G, U)$ to $\mathrm{Fix}(G') - \mathrm{Fix}(G', U)$. Thus $|\mathrm{Fix}(G) - \mathrm{Fix}(G, U)| \leq |\mathrm{Fix}(G') - \mathrm{Fix}(G', U)|$. By Lemma 5 we have $\mathrm{Fix}(G, U) \subseteq \mathrm{Fix}(G)$ and $\mathrm{Fix}(G', U) \subseteq \mathrm{Fix}(G')$ and the lemma follows. $\square$

Figure 10 is an example illustrating the previous lemma.

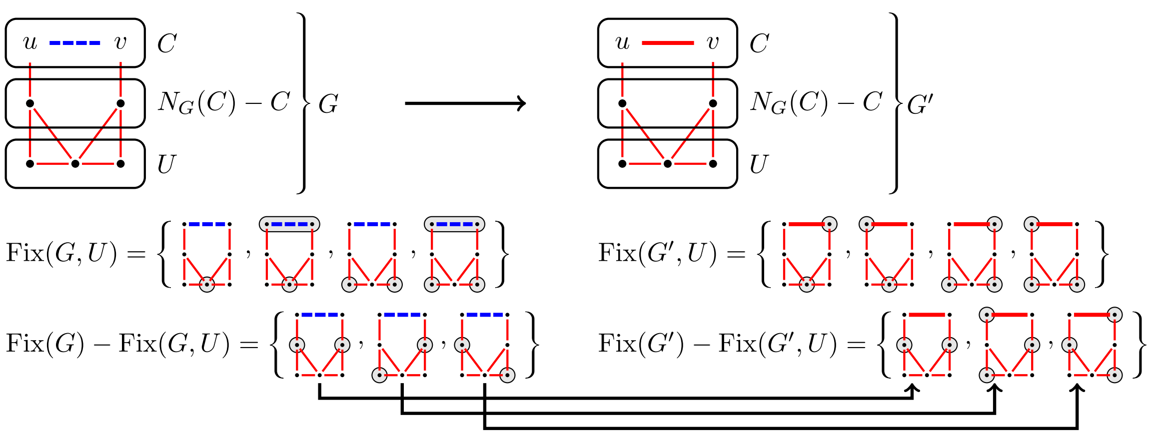


Figure 10: Illustrative example of Lemma 10.

We are now in position to conclude the proof with the following quantitative version of Lemma 3.

Lemma 11 (Quantitative version of Lemma 3)**.** *Suppose that G has a positive edge ab such that a and b are adjacent to no other positive edge. Let X be the set of vertices containing a, b and all the vertices of $N_G(a) \cup N_G(b)$ adjacent to only negative edges. Let Ω be the set of $S \in \mathrm{Fix}(G - X)$ such that $G[X] - \mathcal{N}_G(S)$ has an induced copy of C_4 containing ab. Let G' be the simple signed graph obtained from G by making ab negative. Then*

$$\mathrm{fix}(G) \leq \mathrm{fix}(G') + |\Omega| \leq \frac{3}{2}\mathrm{fix}(G').$$

Remark 2. *If $\mathrm{fix}(G) > \mathrm{fix}(G')$ then Ω is not empty and we deduce that $G[X]$ has an induced copy C of C_4 containing ab. By the definition of X, a and b are the only vertices of C adjacent to a positive edge. This is why this lemma implies Lemma 3.*

Proof of Lemma 11. Let $U = V - X$ and for every $S \in \mathrm{Fix}(G[U])$, let $G_S = G[X] - \mathcal{N}_G(S)$ and $G'_S = G'[X] - \mathcal{N}_{G'}(S)$. Since $G[U] = G'[U]$ we have

$$\begin{aligned} \mathrm{Fix}(G, U) &= \{S \cup S' \mid S \in \mathrm{Fix}(G[U]),\ S' \in \mathrm{Fix}(G_S)\} \\ \mathrm{Fix}(G', U) &= \{S \cup S' \mid S \in \mathrm{Fix}(G[U]),\ S' \in \mathrm{Fix}(G'_S)\}. \end{aligned}$$

Thus

$$\mathrm{fix}(G, U) - \mathrm{fix}(G', U) = \sum_{S \in \mathrm{Fix}(G[U])} \mathrm{fix}(G_S) - \mathrm{fix}(G'_S).$$

For every $S \in \mathrm{Fix}(G[U])$ we have $X \cap \mathcal{N}_G(S) = X \cap \mathcal{N}_{G'}(S)$ thus G'_S is obtained from G_S by making ab negative. Thus according to Lemma 9, we have

$$\begin{aligned} \forall S \in \Omega, &\qquad \mathrm{fix}(G_S) - \mathrm{fix}(G'_S) \leq 1 \\ \forall S \notin \Omega, &\qquad \mathrm{fix}(G_S) - \mathrm{fix}(G'_S) \leq 0. \end{aligned}$$

Thus

$$\mathrm{fix}(G, U) - \mathrm{fix}(G', U) \leq |\Omega|,$$

and using Lemma 10 we obtain

$$\mathrm{fix}(G) - \mathrm{fix}(G') \leq \mathrm{fix}(G, U) - \mathrm{fix}(G', U) \leq |\Omega|.$$

Furthermore, for every $S \in \Omega$, G'_S has only negative edges, and contains ab. Thus it has a maximal independent set containing a, say S_a, and a maximal independent set containing b, say S_b. Then $S \cup S_a$ and $S \cup S_b$ are distinct elements of $\mathrm{Fix}(G', U)$. We deduce that $2|\Omega| \leq \mathrm{fix}(G', U) \leq \mathrm{fix}(G')$, and the proposition follows. $\square$

4 Proof of Theorem 2

Let G be a simple signed graph with vertex V and edge set E. Let C be a set of vertices such that $G[C]$ is connected and $|C| \geq 2$. We denote by G/C the simple signed graph obtained from G by contracting C into a single vertex c, and by adding a negative edge cc'. Formally: the vertex set of G/C is $V = (V - C) \cup \{c, c'\}$ where c and c' are new vertices; the edge set is $(\{\nu(v)\nu(u) \mid uv \in E\} - \{cc\}) \cup \{cc'\}$, where ν is the function that maps every vertex in $V - C$ to itself, and every vertex in C to the new vertex c; an edge uv of G/C is negative if $u = c$ or $v = c$ and it has the same sign as in G otherwise.

Lemma 12. *If C is a non-trivial connected component of G^+ then*

$$\text{fix}(G) \leq \text{fix}(G/C),$$

and the upper bound is reached if $G[C]$ has no negative edge.

Proof. Let

$$S_c = \{S \in \text{Fix}(G/C) \mid c \in S\}$$
$$S_{\bar{c}} = \{S \in \text{Fix}(G/C) \mid c \notin S\}.$$

Since c is adjacent to only negative edges, we have

$$S_c = \{c\} \sqcup \text{Fix}\big((G/C) - c - \mathcal{N}_{G/C}(c)\big).$$

and since there are only negative edges between C and $V - C$ we have

$$(G/C) - c - \mathcal{N}_{G/C}(c) = G - \mathcal{N}_G(C).$$

Thus

$$|S_c| = \text{fix}(G - \mathcal{N}_G(C)).$$

Now, since c' has c as unique neighbor, and since cc' is negative, we have

$$S_{\bar{c}} = \{S \in \text{Fix}(G/C) \mid c' \in S\} = \{c'\} \sqcup \text{Fix}(G/C - \{c, c'\}).$$

Since $G/C - \{c, c'\} = G - C$ we deduce that

$$|S_{\bar{c}}| = \text{fix}(G - C).$$

Thus

$$\text{fix}(G - C) + \text{fix}(G - \mathcal{N}_G(G)) = |S_{\bar{c}}| + |S_c| = \text{fix}(G/C)$$

and the lemma is then an obvious application of Lemma 8. □

Proof of Theorem 2. Let G be an unsigned graph, and let σ be a repartition of signs in G such that $\mathrm{fix}(G_\sigma) = \mathrm{fix}(G, \mathbb{C})$ and such that the number of positive edges in G_σ is minimal for this property. According to Lemmas 1 and 2, G_σ is a simple signed graph in which the positive edges form a matching, say $u_1v_1, \dots, u_kv_k$. Let $H^0 = G_\sigma$ and for $1 \le \ell \le k$, let $H^\ell = H^{\ell-1}/\{u_k, v_k\}$. By Lemma 12 we have

$$\mathrm{fix}(G, \mathbb{C}) = \mathrm{fix}(G_\sigma) = \mathrm{fix}(H^0) = \mathrm{fix}(H^1) = \cdots = \mathrm{fix}(H^k)$$

and since H_k has no positive edges, $\mathrm{fix}(H^k) = \mathrm{mis}(H^k)$. Since the underlying unsigned graph of H_k is a member of $\mathcal{H}'(G)$ this proves that

$$\mathrm{fix}(G, \mathbb{C}) = \mathrm{mis}(H^k) \le \mathrm{mis}(\mathcal{H}'(G)).$$

Now, let $H \in \mathcal{H}'(G)$ such that $\mathrm{mis}(H) = \mathrm{mis}(\mathcal{H}'(G))$. Then there exists disjoint subsets of vertices $C_1, \dots, C_k$ and a sequence of graphs $H_0, \dots, H_k$ with $H^0 = G$ and $H^k = H$ such that $H^{\ell-1}[C_\ell]$ is connected and $H^\ell = H^{\ell-1}/C_\ell$ for all $1 \le \ell \le k$. For $1 \le \ell \le k$, let σ_ℓ be the repartition of signs in H^ℓ such that $\sigma_\ell(uv)$ is positive if and only if $u, v \in C_p$ for some $\ell < p \le k$. In this way we have $H^\ell_{\sigma_\ell} = H^{\ell-1}_{\sigma_{\ell-1}}/C_\ell$ for $1 \le \ell \le k$, and by Lemma 12 we have

$$\mathrm{fix}(G_{\sigma_0}) = \mathrm{fix}(H^0_{\sigma_0}) = \mathrm{fix}(H^1_{\sigma_1}) = \cdots = \mathrm{fix}(H^k_{\sigma_k}).$$

Since $H^k_{\sigma_k}$ has only negative edges, we deduce that

$$\mathrm{fix}(G, \mathbb{C}) \ge \mathrm{fix}(G_{\sigma_0}) = \mathrm{mis}(H^k) = \mathrm{mis}(H) = \mathrm{mis}(\mathcal{H}'(G)).$$

□

5 Proof of Theorem 3

Let us begin with an easy complexity result, proved with a straightforward reduction to SAT similar to the one introduced in [12].

Proposition 3. *Given a graph G and a subset U of its vertices, it is NP-hard to decide if G has a maximal independent set disjoint from U.*

Proof. Let ϕ be a CNF-formula with variables $x_1, \dots, x_n$ and clauses $C_1, \dots, C_k$. Let G be the graph defined as follows. The vertices of G are the positive literals $x_1, \dots, x_n$, the negative literals $\overline{x_1}, \dots, \overline{x_n}$, and the clause $C_1, \dots, C_k$. The edges are defined as follows: there is an edge connecting any two contradict literal x_i and $\overline{x_i}$, and each clause C_i is adjacent to all literals it contains. It is then clear that ϕ is satisfiable if and only if G has a maximal independent set disjoint from the set of clauses. □

According to this proposition, the following lemma is a good step for proving Theorem 3.

Lemma 13. *Let G be a graph and let U a non-empty subset of vertices. Let $\tilde{G}$ be the graph obtained from G by adding four additional vertices a, b, c, d, the edges ab, bc, cd, da, and an edge av for every vertex $v \in U$. Suppose that $\tilde{G}$ has a unique induced copy of C_4, the one induced by $\{a, b, c, d\}$. Then the following are equivalent:*

1. $\mathrm{fix}(\tilde{G}, \mathbb{C}) = \mathrm{mis}(\tilde{G})$,
2. $\mathrm{mis}(\mathcal{H}(\tilde{G})) = \mathrm{mis}(\tilde{G})$,
3. *G has no maximal independent set disjoint from U.*

Proof. We first need the following claim.

Claim 1. *We have*

$$\mathrm{mis}(\tilde{G}) \leq 2\mathrm{mis}(G) + \mathrm{mis}(G - U)$$

and the bound is reached if and only if G has a maximal independent set disjoint from U.

Proof of Claim. Let

$$\begin{aligned} M_{\bar{c}} &= \{S \in \mathrm{Mis}(\tilde{G}) \mid c \notin S\} \\ M_{ca} &= \{S \in \mathrm{Mis}(\tilde{G}) \mid c \in S,\ a \in S\} \\ M_{c\bar{a}} &= \{S \in \mathrm{Mis}(\tilde{G}) \mid c \in S,\ a \notin S\} \end{aligned}$$

Clearly, these three sets form a partition of $\mathrm{Mis}(G)$, and it is easy to see that

$$\begin{aligned} M_{\bar{c}} &= \{b, d\} \sqcup \mathrm{Mis}(G) \\ M_{ca} &= \{c, a\} \sqcup \mathrm{Mis}(G - U) \\ M_{c\bar{a}} &= \{c\} \sqcup \{S \in \mathrm{Mis}(G) \mid S \cap U \neq \emptyset\} \end{aligned}$$

So $|M_{\bar{c}}| = \mathrm{mis}(G)$ and $|M_{ca}| = \mathrm{mis}(G - U)$ and $|M_{c\bar{a}}| \leq \mathrm{mis}(G)$. Since $|M_{c\bar{a}}| = \mathrm{mis}(G)$ if and only if G has no maximal independent set disjoint from U, the claim is proved. □

Now, let $\tilde{G}_{ab}$ be the simple signed graph obtained from $\tilde{G}$ by labeling ab with a positive sign and all the other edges by a negative sign. Let $\tilde{G}_{bc}$, $\tilde{G}_{cd}$ and $\tilde{G}_{da}$ be defined similarly.

Claim 2. *We have*

$$\mathrm{fix}(\tilde{G}_{ab}) = \mathrm{fix}(\tilde{G}_{bc}) = \mathrm{fix}(\tilde{G}_{cd}) = \mathrm{fix}(\tilde{G}_{da}) = 2\mathrm{mis}(G) + \mathrm{mis}(G - U).$$

Proof of Claim. Since $\tilde{G}_{ab}$ as ab has unique positive edge, we deduce from Lemma 8 that

$$\mathrm{fix}(\tilde{G}_{ab}) = \mathrm{mis}(G_1) + \mathrm{mis}(G_2) \quad \text{with} \quad \begin{cases} G_1 = \tilde{G} - \{a, b\} \\ G_2 = \tilde{G} - N_{\tilde{G}}(\{a, b\}). \end{cases}$$

Since G_1 is the disjoint union of the edge cd and G, we have $\mathrm{mis}(G_1) = 2\mathrm{mis}(G)$. Furthermore, since $N_{\tilde{G}}(\{a, b\}) = \{a, b, c, d\} \cup U$, we have $G_2 = G - U$. So $\mathrm{mis}(\tilde{G}_{ab}) = 2\mathrm{mis}(G) + \mathrm{mis}(G - U)$, and by symmetry $\mathrm{mis}(\tilde{G}_{da}) = \mathrm{mis}(\tilde{G}_{ab})$.

Similarly, since $\tilde{G}_{bc}$ has bc as unique positive edge, we deduce from Lemma 8 that

$$\mathrm{fix}(\tilde{G}_{bc}) = \mathrm{mis}(G_1) + \mathrm{mis}(G_2) \quad \text{with} \quad \begin{cases} G_1 = \tilde{G} - \{b, c\} \\ G_2 = \tilde{G} - N_{\tilde{G}}(\{b, c\}). \end{cases}$$

In G_1, d has a as unique neighbor. Thus, the number of maximal independent sets of G_1 containing d is $\mathrm{mis}(G_1 - \{a, d\}) = \mathrm{mis}(G)$; and the number of maximal independent sets of G_1 not containing d is equal to the number of maximal independent sets containing a, which is $\mathrm{mis}(G_1 - a - U - d) = \mathrm{mis}(G - U)$. Since $N_{\tilde{G}}(\{b, c\}) = \{a, b, c, d\}$, we have $G_2 = G - U$ thus $\mathrm{mis}(\tilde{G}_{bc}) = 2\mathrm{mis}(G) + \mathrm{mis}(G - U)$. By symmetry $\mathrm{mis}(\tilde{G}_{cd}) = \mathrm{mis}(\tilde{G}_{bc})$. □

Let σ be a repartition of sign in $\tilde{G}$ such that $\mathrm{fix}(\tilde{G}_\sigma) = \mathrm{fix}(\tilde{G}, \mathbb{C})$, and such that the number of positive edges in $\tilde{G}_\sigma$ is minimal for this property. If $\mathrm{fix}(\tilde{G}_\sigma) = \mathrm{mis}(\tilde{G})$ then $\mathrm{fix}(\tilde{G}_{ab}) \leq \mathrm{mis}(\tilde{G})$ and we deduce from Claims 1 and 2 that G has no maximal independent set disjoint from U. Otherwise, $\mathrm{fix}(\tilde{G}_\sigma) > \mathrm{mis}(\tilde{G})$ and since $\tilde{G}$ has a unique induced copy of C_4 and we deduce from Lemmas 3 that $\tilde{G}_\sigma$ has a unique positive edge $e \in \{ab, bc, cd, da\}$. Thus by Claim 2 we have

$$\mathrm{mis}(\tilde{G}) < \mathrm{fix}(\tilde{G}_\sigma) = 2\mathrm{mis}(G) + \mathrm{mis}(G - U)$$

and we deduce from Claim 1 that G has a maximal independent set disjoint from U. This shows the equivalence between the point 1 and 3 in the statement.

To conclude the proof, it is sufficient to show that $\mathrm{mis}(\mathcal{H}(\tilde{G})) = \mathrm{fix}(\tilde{G}_\sigma)$. By Theorem 2, we have

$$\mathrm{mis}(\mathcal{H}(\tilde{G})) \leq \mathrm{mis}(\mathcal{H}'(\tilde{G})) = \mathrm{fix}(\tilde{G}_\sigma)$$

so we only need to prove that

$$\mathrm{mis}(\mathcal{H}(\tilde{G})) \geq \mathrm{fix}(\tilde{G}_\sigma).$$

If $\mathrm{fix}(\tilde{G}_\sigma) = \mathrm{mis}(\tilde{G})$ then $\mathrm{fix}(\tilde{G}_\sigma) \leq \mathrm{mis}(\mathcal{H}(\tilde{G}))$ since $\tilde{G} \in \mathcal{H}(\tilde{G})$. Otherwise, $\mathrm{fix}(\tilde{G}_\sigma) > \mathrm{mis}(\tilde{G})$ and as above we deduce that $\tilde{G}_\sigma$ has a unique positive edge $e \in \{ab, bc, cd, da\}$. Let H_e be the graph obtained from $\tilde{G}$ by contracting e, and let $H'_e = G/e$ be obtain from G by contracting e into a single vertex c, and by adding an edge cc'. It is easy to check that in every case we have $\mathrm{mis}(H_e) = \mathrm{mis}(H'_e)$, and by Lemma 12 we have $\mathrm{mis}(H'_e) = \mathrm{fix}(\tilde{G}_\sigma)$. Since $H_e \in \mathcal{H}(\tilde{G})$ we deduce that $\mathrm{mis}(\mathcal{H}(\tilde{G})) \geq \mathrm{fix}(\tilde{G}_\sigma)$. □

According to Lemma 13, to prove Theorem 3, it is sufficient to prove the following strengthening of Proposition 3.

Lemma 14. *Let G be a graph and let U be a non-empty subset of its vertices. Let $\tilde{G}$ be obtained from G and U as in Lemma 13. It is NP-hard to decide if G has a maximal independent set disjoint from U, even if $\tilde{G}$ has an unique induced copy of C_4.*

Proof. We follow the proof of Proposition 3, using however a slightly more involved arguments. Let ϕ be a CNF-formula with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_k$. Let $G(\phi)$ be the graph defined as follows: the vertex set is the union of the following sets

$$\begin{aligned} X &= \{x_1, \ldots, x_n\} \\ \overline{X} &= \{\overline{x_1}, \ldots, \overline{x_n}\} \\ Y &= \{y_1, \ldots, y_n\} \\ \overline{Y} &= \{\overline{y_1}, \ldots, \overline{y_n}\} \\ \mathcal{L} &= \{L_1, \ldots, L_n\} \\ \overline{\mathcal{L}} &= \{\overline{L_1}, \ldots, \overline{L_n}\} \\ \mathcal{C} &= \{C_1, \ldots, C_k\}. \end{aligned}$$

The edge set is defined by: for all $1 \leq i \leq n$, $x_i y_i$, $\overline{x_i}\,\overline{y_i}$, $x_i L_i$, $\overline{x_i} L_i$, $y_i \overline{L_i}$, and $\overline{y_i}\,\overline{L_i}$ are edges; for all $1 \leq i < j \leq k$, $C_i C_j$ is an edge; for all $1 \leq i \leq n$ and $1 \leq j \leq k$, $L_i C_j$ is an edge; for all $1 \leq i \leq n$ and $1 \leq j \leq k$, $x_i C_j$ is an edge if x_i is a positive literal of the clause C_j; and finally, for all $1 \leq i \leq n$ and $1 \leq j \leq k$, $\overline{x_i} C_j$ is an edge if $\overline{x_i}$ is a negative literal of the clause C_j (see an illustration of $G(\phi)$ in Figure 11). We set

$$U = \mathcal{C} \cup \mathcal{L} \cup \overline{\mathcal{L}}.$$

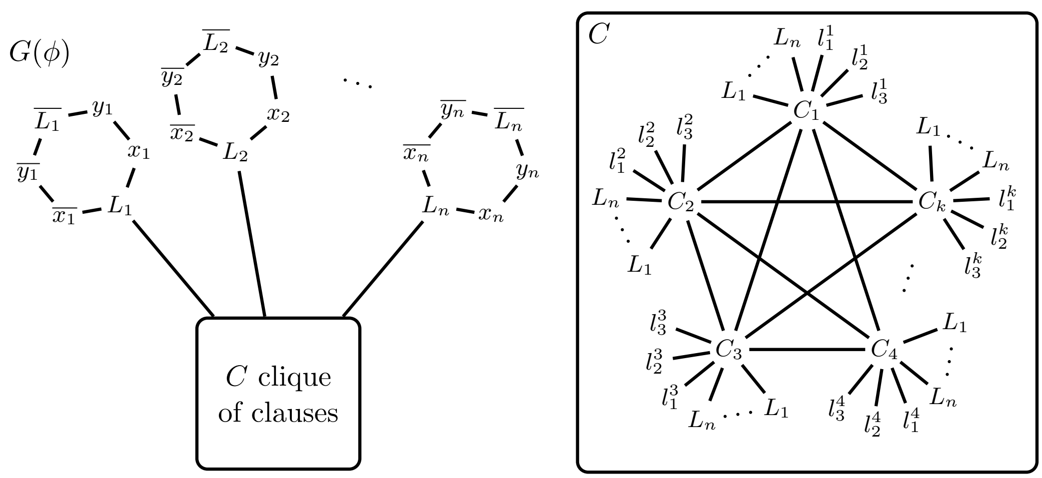


Figure 11: Illustration of $G(\phi)$.

Claim 1. *ϕ is satisfiable if and only if there exists $S \in \mathrm{Mis}(G)$ such that $S \cap U = \emptyset$.*

Proof of Claim. Suppose that $S \in \mathrm{Mis}(G)$ and $S \cap U = \emptyset$. Consider the assignment defined by $x_i = 1$ is $x_i \in S$ and $x_i = 0$ otherwise. Let C_j be any clause of ϕ. Since $S \cap U = \emptyset$, there exists $1 \le i \le n$ such that x_iC_j is an edge with $x_i \in S$, or such that $\overline{x_i}C_j$ is an edge with $\overline{x_i} \in S$. If x_iC_j is an edge with $x_i \in S$ then x_i is a positive literal of the clause C_j and $x_i = 1$ thus the clause C_j is made true by the assignment. Suppose now that $\overline{x_i}C_j$ is an edge with $\overline{x_i} \in S$. Then $\overline{y_i} \notin S$ and since $\overline{L_i}$ has a neighbor in S, we deduce that $y_i \in S$, and consequently, $x_i \notin S$. Thus $x_i = 0$, and since $\overline{x_i}C_j$ is an edge, $\overline{x_i}$ is a negative literal of C_j, and thus C_j is made true by the assignment. Hence, every clauses is made true, thus ϕ is satisfiable.

Suppose now that there exists a assignment that makes ϕ true, and let

$$S = \{x_i, \overline{y_i} \,|\, x_i = 1,\ 1 \le i \le n\} \cup \{\overline{x_i}, y_i \,|\, x_i = 0,\ 1 \le i \le n\}$$

Then S is an independent set disjoint from U, and it is maximal if every vertex in U has a neighbor in S. For $1 \le i \le n$, since either x_i or $\overline{x_i}$ is in S, the vertex L_i has a neighbor in S, and since either y_i or $\overline{y_i}$ is in S, the vertex $\overline{L_i}$ has also a neighbor in S. Now let C_j be any clause of ϕ. Since C_j is made true by the assignment, it contains a positive literal x_i with $x_i = 1$ or a negative literal $\overline{x_i}$ with $x_i = 0$. In the first case, $x_i \in S$ and x_iC_j is an edge, and in the second case, $\overline{x_i} \in S$ and $\overline{x_i}C_j$ is an edge. Thus $S \in \mathrm{Mis}(G)$. □

Claim 2. *G has no induced copy of C_4.*

Proof of Claim. Suppose, for a contradiction, that G has an induced copy of C_4 with vertices $W = \{v_1, v_2, v_3, v_4\}$ given in the order. If $v_1 = \overline{L_i}$ for some $1 \le i \le n$, then $\{v_2, v_4\} = \{y_i, \overline{y_i}\}$ and since $\overline{L_i}$ is the unique common neighbor of y_i and $\overline{y_i}$, there is a contradiction. We deduce that $W \cap \overline{\mathcal{L}} = \emptyset$. Since $\{x_i, \overline{L_i}\}$ is the neighborhood of y_i we deduce that $W \cap Y = \emptyset$, and similarly, $W \cap \overline{Y} = \emptyset$. Hence,

$$W \cap (Y \cup \overline{Y} \cup \overline{\mathcal{L}}) = \emptyset.$$

Suppose now that $v_1 = L_i$ for some $1 \le i \le n$. If $v_2 = C_j$ for some $1 \le j \le k$ then $v_3 \in X \cup \overline{X} \cup \mathcal{C} \cup \mathcal{L}$ and we deduce that $v_3 \in X \cup \overline{X}$ since otherwise v_1v_3 is an edge. So $v_4 \in \mathcal{C} \cup \mathcal{L}$ and we deduce that v_2v_4 is an edge, a contradiction. Thus $v_2 \in \{x_i, \overline{x_i}\}$. Suppose that $v_2 = x_i$, the other case being similar. Then $v_3 \in \mathcal{C} \cup \{y_i\}$, and thus $v_3 \in \mathcal{C}$, so v_1v_3 is an edge, a contradiction. We deduce that $v_1 \notin \mathcal{L}$, and thus $U \cap \mathcal{L} = \emptyset$. Thus $W \subseteq X \cup \overline{X} \cup \mathcal{C}$. Since $X \cup Y$ is an independent set, and since $G[\mathcal{C}]$ is complete, we easily obtain a contradiction. □

Let $\tilde{G}$ be obtained from G and U as in Lemma 13.

Claim 3. *$\tilde{G}$ has a unique induced copy of C_4, the one induced by $\{a, b, c, d\}$.*

Proof of Claim. Suppose for a contradiction that $\tilde{G}$ has an induced copy of C_4 with vertices $W = \{v_1, v_2, v_3, v_4\}$ given in the order, and suppose that $W \ne \{a, b, c, d\}$. Since G has no induced copy of C_4, we deduce that $W \cap \{a, b, c, d\} = \{a\}$. Suppose, without loss of generality, that $a = v_1$. Then $v_2, v_4 \in U$, and since two distinct vertices in $\mathcal{L} \cup \overline{\mathcal{L}}$ have no common neighbor in G, we deduce that $v_2, v_4 \in \mathcal{C}$. But then $v_2 v_4$ is an edge of $\tilde{G}$, a contradiction. □

□

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