Maximizing and minimizing quasiconvex functions: related properties, existence and optimality conditions via radial epiderivatives*

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Abstract

This paper deals with maximization and minimization of quasiconvex functions in a finite dimensional setting. Firstly, some existence results on closed convex sets, possibly containing lines, are presented. Necessary or sufficient optimality conditions are derived in terms of radial epiderivatives. Finally, some attempts to define asymptotic functions under quasiconvexity are also outlined. Several examples illustrating the applicability of our results are shown.

Key words Maximization problem; Minimization problem; Quasiconvexity; Radial epiderivative; Asymptotic functions.

Mathematics subject classification 2010 90C30; 90C26

1 Introduction

Maximizing a quasiconvex function has received much less attention than minimizing it. We propose to supplement with new results, specially on related properties concerning existence of solutions on the boundary, and more precisely, based on extremality.

In [7] were introduced the notions of lower and upper radial epiderivatives for vector-valued functions. We specialize some of the results of that paper to the real-valued functions to provide more precise and new results in this setting. A further analysis based on [7] is carried out in [17].

*The research for the first author was supported in part by CONICYT-Chile through FONDECYT 112-0980 and BASAL Projects, CMM, Universidad de Chile.

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Concerning maximization, there are a few works about existence of solutions, we refer to the interesting papers [14] and [21] whose results are improved and generalized in many directions in the present paper. On the other hand, various attempts to define appropiate asymptotic functions under quasiconvexity appear in the literature, see [1, 19] and references therein. We propose some notions derived from the computation of the asymptotic cone of sub-level sets.

The paper is organized as follows. In Section 2 some basic definitions and properties on contingent and radial cones are recalled, as well as those of asymptotic functions. In particular, a new characterization of explicit quasiconvexity is established. Section 3 contains two subsections. The first one introduces two main assumptions, which allow us to obtain main results on existence of solution for maximization problems under lower semicontinuity, by exploiting a representation for closed and convex sets possibly containing lines. Several interesting properties of this problem are also established, and the particular case of quadratic functions is also discussed. The second Subsection is devoted to derive sufficient or necessary optimality conditions in terms of lower or upper radial epiderivatives. In Section 4, the minimization problem, under quasiconvexity, is revisited, and a new characterization of the nonemptiness (and boundedness) of the solution set is provided. Finally Section 5 proposes several possible notions of asymptotic functions in the quasiconvex case, and analyze their links with the standard one.

2 Basic definitions and preliminaries

We restrict ourselves to finite dimensional spaces, even if some of the results remain valid in any normed vector space.
Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} \doteq \mathbb{R} \cup \{\pm\infty\}$, we are now interested in the problem

$$\max_{x \in \mathbb{R}^n} f(x), \tag{1}$$

or

$$\min_{x \in \mathbb{R}^n} f(x). \tag{2}$$

As usual in convex analysis we use the following notations.

$$\operatorname{dom} f \doteq \{x \in \mathbb{R}^n : \ f(x) < +\infty\}, \ \operatorname{epi} f \doteq \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leq t\},$$

$$S_\lambda(f) \doteq \{x \in \mathbb{R}^n : \ f(x) \leq \lambda\}, \ \hat{S}_\lambda(f) \doteq \{x \in \mathbb{R}^n : \ f(x) < \lambda\}.$$

Given $x, y \in \mathbb{R}^n$, denote by $]x, y[$ the open segment joining x and y, i.e., without the end points x and y. In case $n = 1$, that denotes the open interval.

Definition 2.1. *We say that* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is:*

- (i) *convex if* epi f *is a convex set in* $\mathbb{R}^n \times \mathbb{R}$, *i. e., for all* x, y *satisfying* $f(x) < +\infty$, $f(y) < +\infty$, *one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall\ t \in\ ]0,1[.$$

- (ii) *semistrictly quasiconvex if for all* $x, y \in \mathbb{R}^n$, $f(x) \neq f(y)$, *one has* $f(z) < \max\{f(x), f(y)\}$ *for all* $z \in\ ]x, y[$*;*

- (iii) *quasiconvex if for every* $x, y \in \mathbb{R}^n$, *one has* $f(z) \leq \max\{f(x), f(y)\}$ *for all* $z \in\ ]x, y[$. *In other words, if each sublevel set,* $S_\lambda(f)$, *is convex for all* $\lambda \in \mathbb{R}$, *or equivalently,* $\hat{S}_\lambda(f)$ *is convex for all* $\lambda \in \mathbb{R}$*;*

- (iv) *explicitely quasiconvex if it is quasiconvex and semistrictly quasiconvex.*

When f is quasiconvex, we deduce that dom f is convex since

$$\text{dom } f = \bigcup_{k \in \mathbb{N}} \Big\{ x \in \mathbb{R}^n : \ f(x) \leq k \Big\}$$

and $S_f(\lambda_1) \subseteq S_f(\lambda_2)$ provided $\lambda_1 \leq \lambda_2$. Moreover, a semistrictly quasiconvex function may have a nonconvex domain: in fact take $f(x) = 0$ if $x \neq 0$, $f(0) = +\infty$. Simple examples show there in no relationship between quasiconvexity and semistrict quasiconvexity. However, under lower semicontinuity, semistrict quasiconvexity implies quasiconvexity.

Functions satisfying Part (b) of the next result were termed pseudoconvex in [10]. The equivalence is new and represents a unification of two concepts in a single implication.

Proposition 2.2. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be any function. The following asertions are equivalent:*

- (a) f *is quasiconvex and semistrictly quasiconvex, i. e., explicitely quasiconvex;*

- (b) $x, y \in \mathbb{R}^n$, $t \in\ ]0,1[$, $f(y + t(x-y)) \geq f(x) \Longrightarrow f(y) \geq f(y + t(x-y))$.

Proof. $(a) \Longrightarrow (b)$: Set $y_t = y + t(x-y)$. Assume that $f(y_t) \geq f(x)$, we must check that $f(y) \geq f(y_t)$. We distinguish various situations.

- If either $f(x) = f(y) = -\infty$ or $f(x) = -\infty$ and $-\infty < f(y) \leq +\infty$, then obviously $f(y_t) \leq f(y)$ by quasiconvexity.

- If $f(x) = +\infty$ then $f(y_t) = +\infty$ by assumption. Suppose that $f(y) < +\infty (= f(x))$. By the semistrict quasiconvexity we reach a contradiction. Hence $f(y) = +\infty$ and the desired inequality is proved.
- If $f(x) \in \mathbb{R}$ and $f(y_t) = +\infty$, then $f(y) = +\infty$ by quasiconvexity.
- Assume now $f(x) \in \mathbb{R}$ and $f(y_t) \in \mathbb{R}$. In case $f(y) = +\infty$ we are done. Assume on the contrary that $f(y) < f(y_t)$. If $f(y_t) > f(x)$ then $f(y_t) > \max\{f(x), f(y)\}$ contradicting the quasiconvexity of f. Therefore $f(y_t) = f(x)$ by assumption. Hence $f(y) < f(x) = f(y_t)$ which implies, by the semistrict quasiconvexity, $f(y_t) < f(x)$ contradicting a previous inequality.

$(b) \Longleftarrow (a)$: It is straightforward. □

Given any set C in $\mathbb{R}^n$, $\overline{C}$ will denote its closure; conv C its convex hull; ∂C its topological boundary; int K its (topological) interior. We first recall some basic notions.

Definition 2.3. *Given any nonempty set $C \subseteq \mathbb{R}^n$, $\bar{x} \in \overline{C}$, we define the following cones:*

(i) the contingent cone of C (or tangent cone of Bouligand) at $\bar{x}$, denoted by $T(C;\bar{x})$, is the set of all $v \in \mathbb{R}^n$ such that there exist sequences $t_k \downarrow 0$ and $v_k \to v$ with $\bar{x} + t_k v_k \in C$ for all $k \in \mathbb{N}$.

(ii) the closed radial cone of C at $\bar{x}$, denoted by $R(C;\bar{x})$, is the set of all $v \in \mathbb{R}^n$ such that there exist sequences $t_k > 0$ and $v_k \to v$ and $\bar{x} + t_k v_k \in C$ for all $k \in \mathbb{N}$.

(iii) the interiorly radial cone of C at $\bar{x}$, denoted by $R^i(C;\bar{x})$, is the set of all $v \in \mathbb{R}^n$ such that there exists $\varepsilon > 0$ satisfying $\bar{x} + tv' \in C$ for all $t > 0$, $\|v' - v\| < \varepsilon$.

By a cone we mean a set K satisfying $\lambda K \subseteq K$ for all $\lambda \geq 0$, so $0 \in K$.

Remark 2.4. (a) *Some of the equivalent definitions for $T(C;\bar{x})$ are the following:*
$v \in T(C;\bar{x})$ if and only if there exist sequences $t_n > 0$ and $v_k \to v$ such that $t_k v_k \to 0$ and $\bar{x} + t_k v_k \in C$ for all $k \in \mathbb{N}$;
$v \in T(C;\bar{x})$ if and only if there exist sequences $t_k > 0$ and $x_k \in C$ such that $x_k \to \bar{x}$ and $t_k(x_k - \bar{x}) \to v$.
(b) *$T(C;\bar{x})$ and $R(C;\bar{x})$ are non-empty closed cones.*
(c) *$T(C;\bar{x}) = R(C;\bar{x})$ for all $\bar{x} \in C$ whenever C is a convex set.*
(d) *$R^i(C;\bar{x})$ is an open set whenever $\bar{x}$ is a boundary point of C. Furthermore, $\lambda R^i(C;\bar{x}) \subseteq R^i(C;\bar{x})$ for all $\lambda > 0$. Indeed $R^i(C;\bar{x}) = \mathbb{R}^n \setminus R(\mathbb{R}^n \setminus C;\bar{x})$. Moreover*

$$v \in R^i(C;\bar{x}) \iff \exists\, \varepsilon > 0 \text{ such that } v' \in \bigcap_{t>0} t(C - \bar{x}), \quad \|v' - v\| < \varepsilon$$

$$\Longleftrightarrow v \in \text{int}\left(\bigcap_{t>0} t(C-\bar{x})\right).$$

We define the asymptotic cone of $C \subseteq \mathbb{R}^n$ as the closed cone

$$C^{\infty} = \Big\{v \in \mathbb{R}^n : \exists\, t_k \downarrow 0, \exists\, x_k \in C,\ t_k x_k \to v\Big\}.$$

We set $\emptyset^{\infty} = \emptyset$. The term "recession cone" is used when the set is convex.
A closed set C it said to be radiant at $\bar{x} \in C$ if there exists $\delta \in\ ]0,1]$ such that $\bar{x} + t(x-\bar{x}) \in C$ for all $x \in C$ and all $t \in\ ]0,\delta]$.
We recall that a subset C is starshaped with respect to $\bar{x} \in C$, if $\bar{x} + t(x-\bar{x}) \in C$ for all $t \in [0,1]$ and all $x \in C$. Thus, one immediately sees that every closed set which is starshaped with respect to a point in the set, is radiant at the same point. In particular, closed convex sets are radiant at any point belonging to the set. Let C be radiant at $\bar{x} \in C$, it is proven in [4, 5] that

$$C^{\infty} = \bigcap_{t>0} t(C-\bar{x}). \tag{3}$$

Consequently, in the case when C is closed and convex, given any $\bar{x} \in C$, we get

$$C^{\infty} = \Big\{v \in \mathbb{R}^n : \ \bar{x} + tv \in C \ \ \forall\, t > 0\Big\} = \bigcap_{t>0} t(C-\bar{x}),$$

which is independent of $\bar{x} \in C$. For general sets, we have

$$\bigcap_{t>0} t(C-\bar{x}) \subseteq C^{\infty}.$$

Hence $R^i(C;\bar{x}) \subseteq \text{int}(C^{\infty})$ for all $\bar{x} \in C$. The following proposition summarizes the previous results.

Proposition 2.5. *Let $C \subseteq \mathbb{R}^n$ be a closed set, $\bar{x} \in C$. If C is radiant at $\bar{x}$, then*

$$R^i(C;\bar{x}) = \text{int}\left(\bigcap_{t>0} t(C-\bar{x})\right) = \text{int}(C^{\infty}).$$

When C is convex, $R^i(C;\bar{x})$ is independent of $\bar{x}$.

By $\overline{\text{cone}}\, A$ we denote the smallest closed cone containing A, which is the closure of the smallest cone containing A. More precisely,

$$\overline{\text{cone}}\ A = \overline{\text{cone}\ (A)} \quad \text{and} \quad \text{cone}\ A = \bigcup_{t \geq 0} tA.$$

The next proposition justifies the term "closed radial" for the set $R(C;\bar{x})$.

Proposition 2.6. *Given any nonempty set C and $\bar{x} \in C$, we have*
(a) $R(C;\bar{x}) = \overline{\mathrm{cone}}(C - \bar{x})$.
(b) $R(C;\bar{x}) = T(C;\bar{x})$ *provided C is starshaped with respect to $\bar{x}$.*

Proof. Part (a) follows directly from the very definition of $R(C;\bar{x})$ and by noticing that $\overline{\mathrm{cone}(C - \bar{x})} = \overline{\mathrm{cone}}(C - \bar{x})$. Part (b) is Corollary 4.11 in [15]. □

Let us introduce another notion that will be useful in subsequent sections. Given any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of f is that function $f^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that epi $f^\infty = (\text{epi } f)^\infty$. In case f is convex and lsc, it is known that the following representation holds:

$$f^\infty(v) = \sup_{t>0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}, \quad (\bar{x} \in \mathrm{dom}\ f).$$

For some basic properties of asymptotic functions and its uses, we refer to the monograph [2].

3 Maximizing quasiconvex functions

In contrast with the minimization of quasiconvex functions, maximization problems enjoys some special properties: one of them refers to its optimal value: it is the same if one takes the supremum on the convex hull of the constraints set. If additionally the function is lower semicontinuous (lsc), that optimal value, if it is achieved, one may expect it lies on the boundary of that set. This property and others are summarized in the next proposition which is well-known in the literature. Before, some notations are required for convenience of the reader.

Set

$$\alpha \doteq \sup_{x \in K} f(x), \quad \alpha_0 \doteq \sup_{x \in \mathrm{conv}\ K} f(x), \quad \bar{\alpha} \doteq \sup_{x \in \overline{K}} f(x).$$

It is not difficult to check the following result.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be any function with* dom $f \neq \emptyset$, *and $K \subseteq \mathbb{R}^n$ be any nonempty set. The following assertions hold:*

(a) assume that f is quasiconvex, then $\alpha = \alpha_0$ and if α_0 is achieved then α is also achieved in K;

(b) assume that f is lsc, then $\alpha = \bar{\alpha}$, and f is continuous at every $\bar{x} \in \mathrm{argmax}_K f$.

3.1 Reduction to the boundary, existence and related properties

We now analyze when the optimal values on the constraint set and on its boundary coincides. Afterwards, we introduce two main assumptions under which those optimal values are equal to that when the supremum is taken on its extremal points and extremal directions.

By the previous proposition we may assume that K is closed and convex since otherwise we subsititute K by its closed convex hull if f is quasiconvex and lsc. Nevertheless, we do not impose lower semicontinuity in most of the results in this section, except when existence is claimed.

Lemma 3.2. *Let $K \subseteq \mathbb{R}^n$ be any nonempty convex closed set with* $\text{int } K \neq \emptyset$ *satisfying $K^\infty \cup (-K^\infty) \neq \mathbb{R}^n$. Let $f : K \to \overline{\mathbb{R}}$ be quasiconvex, then*

$$\sup_{x \in K} f(x) = \sup_{x \in \partial K} f(x).$$

Proof. Set

$$\hat{\alpha} \doteq \sup_{x \in \partial K} f(x).$$

Then $\hat{\alpha} \leq \alpha$. Assume that $\hat{\alpha} < \alpha < +\infty$ and take $t \in \mathbb{R}$ such that $\hat{\alpha} < t < \alpha$. Then, there exists $x \in K$ such that $\hat{\alpha} < t < f(x)$. Thus $x \notin \partial K$, and therefore $x \in \text{int } K$. Take $0 \neq v$ such that $v \notin K^\infty$, $v \notin -K^\infty$. Then, there exists $\bar{t} > 0$ satisfying $x \pm \bar{t}v \notin K$. Set

$$t_0 \doteq \inf\{t > 0 : \ x + tv \notin K\}, \ t_1 \doteq \inf\{t > 0 : \ x - tv \notin K\}.$$

Then, $t_0 > 0$, $x + t_0 v \in \partial K$ and $t_1 > 0$, $x - t_1 v \in \partial K$. Thus

$$t < f(x) \leq \max\{f(x + t_0 v), f(x - t_1 v)\} \leq \hat{\alpha},$$

a contradiction.

Assume that $\alpha = +\infty$ with $\hat{\alpha} < +\infty$. Thus, there exists $\bar{x} \in K$ such that $\hat{\alpha} < f(\bar{x})$. This implies that $\bar{x} \notin \partial K$, and so $\bar{x} \in \text{int } K$. We proceed as before to obtain $\bar{x} \in \ ]x_1, x_2[$, for some $x_1, x_2 \in \partial K$. The quasiconvexity gives $f(\bar{x}) \leq \hat{\alpha}$, a contradiction. Hence $\hat{\alpha} = +\infty$.

Assume now that $\hat{\alpha} = -\infty$. Then $f(x) = -\infty$ for all $x \in \partial K$. Take any $x' \in \text{int } K$, then one can find $x_1, x_2 \in \text{bd } K$ satisfying $x' \in \ ]x_1, x_2[$. Thus $f(x') = -\infty$, which together with the hypothesis give $f(x) = -\infty$ for all $x \in K$, i.e., $\alpha = -\infty$. □

Example 3.3.

(a) *Let us consider* $K = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_2 \geq |x_1|\}$, *and*

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } (x_1, x_2) = (0,0); \\ 1, & \text{if } (x_1, x_2) \neq (0,0). \end{cases}$$

Clearly f *is lsc in* K *and quasiconvex. It is easy to check that*

$$\max_{x \in K} f(x) = \max_{x \in \partial K} f(x) = 1.$$

(b) *The function* $f(x) = 1$, *if* $x < 0$ *and* $f(x) = 0$ *if* $0 \leq x \leq 1$, *shows that the previous lemma may be false if the assumption on* K^∞ *is not satisfied. In fact, we get* $\sup_K f = 1 > 0 = \sup_{\partial K} f$.

(c) *The previous theorem allows to deal with sets* K *containing lines, so that* K *may have no extremal points. To see that, simply take the function* $f(x_1, x_2) = \dfrac{|x_2|}{1 + |x_2|}$, $K = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_2 \in \mathbb{R}, \; |x_1| \leq 1\}$.

In order to introduce our main assumption, we recall the following decomposition due to Hirsch-Hoffman, [14, Theorem 2.6]. By such a theorem, we can write

$$K = L + \text{conv}(\text{ext } M) + M^\infty, \tag{4}$$

where L is the unique (up to translations) linear subspace associated to the extreme affine subspace of K; $M = K \cap L^\perp$ is the $L^\perp$-section of K with M^∞ being pointed. This decomposition generalizes that due to Klee, valid for closed and convex sets containing no lines, [16], where $L = \{0\}$.
In what follows, given a convex cone P, extd P denotes the set of extreme directions of P: here $q \in \text{extd } P$ if, and only if $q \in P \setminus \{0\}$ and for all $q_1, q_2 \in P$ such that $q = q_1 + q_2$ we actually have $q_1, q_2 \in \mathbb{R}_{++} q$.

Thus our first main assumption read as follows.
Assumption (A): There exists $l_0 \in L$ such that

$$f(l + e + v) \leq f(l_0 + e + v), \quad \forall \; l \in L, \quad \forall \; e \in \text{ext } M, \text{and } \forall \; v \in \text{extd } M^\infty.$$

We will see that, in case of quadratic functions (Proposition 3.10), Assumption (A) may be written in a more handle way.

The first main theorem is obtained valid for any closed convex sets K, possibly containing lines.

Theorem 3.4. *Let* $K \subseteq \mathbb{R}^n$ *be any nonempty closed convex set and* $f : K \to \mathbb{R}$ *be a quasiconvex function. If Assumption* (A) *is satisfied, then*

$$\sigma \doteq \sup_{x \in K} f(x) = \sup_{z \in \text{ext } M + (\text{extd } M^\infty) \cup \{0\}} f(l_0 + z). \tag{5}$$

Proof. By (4), for every point $x \in K$ there exist $l_x \in L$, $p_x \in \text{conv}(\text{ext } M)$, and $v_x \in M^\infty$ such that $x = l_x + p_x + v_x$. On the other hand, since M^∞ is pointed, $M^\infty = \text{conv}(\{0\} \cup \text{extd } M^\infty)$. Thus $v_x = \sum_{j=1}^{p} \beta_j m_j$, where, $0 \leq \beta_j$, $m_j \in (\text{extd } M^\infty) \cup \{0\}$, for $j = 1, \dots, p$, and $\sum_{j=1}^{p} \beta_j = 1$. In addition, $p_x = \sum_{i=1}^{q} \alpha_i e_i$, where, $0 \leq \alpha_i$, $e_i \in \text{ext } M$, for $i = 1, \dots, q$, and $\sum_{i=1}^{q} \alpha_i = 1$. By quasiconvexity

$$f(x) = f(l_x + p_x + v_x) \leq \max_{i,j} f(l_x + e_i + m_j) = f(l_x + e_{i_0} + m_{j_0}) \leq f(l_0 + e_{i_0} + m_{j_0})$$

$$\leq \sup_{z \in \text{ext } M + (\text{extd } M^\infty) \cup \{0\}} f(l_0 + z) \leq \sup_{z \in K} f(z).$$

Since x was arbitrary in K, we get the desired result. □

The previous result needs not to be true if Assumption **(A)** fails.

Example 3.5. *Take* $f(x_1, x_2) = x_1$ *and* $K = \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 \leq x_2 \leq 1\}$. *Thus* $L = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = 0\}$, $K \cap L^\perp = \{(0, x_2) \in \mathbb{R}^2 : \ 0 \leq x_2 \leq 1\}$, $\text{ext}(K \cap L^\perp) = \{(0,0), (0,1)\}$, *and* $\text{extd}(K \cap L^\perp) = \emptyset$. *For any fixed* $l_0 \in L$, *we obtain*

$$+\infty = \sup_{(x_1,x_2) \in K} f(x_1, x_2) > \sup_{(x_1,x_2) \in \text{ext}(K \cap L^\perp)} f(l_0 + (x_1, x_2)) > -\infty.$$

Obviously there is no $l_0 \in L$ *satisfying Assumption* (*A*) *as one can check it directly.*

The following simple corollary is obtained.

Corollary 3.6. *Assume, in addition to the hypothesis of Theorem 3.4, that* $\text{ext}(K \cap L^\perp)$ *and* $\text{extd}(K^\infty \cap L^\perp)$ *are finite sets* (*in particular if* K *is a polyhedron*). *Then, the supremum on the right hand-side of* (5) (*so* σ) *is achieved.*

The next example shows an instance where Assumption (A) holds only for $l_0 \neq 0$.

Example 3.7. *We consider*

$$f(x_1, x_2) = \begin{cases} x_1, & \text{if } \ x_1 < 1, \\ 1 & \text{if } \ x_1 \geq 1 \end{cases}$$

and $K = \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 \leq x_2 \leq 1\}$. *Observe that every* (x_1, x_2) *in* K *can be written as* $(x_1, x_2) = (x_1, 0) + (1 - x_2)(0, 0) + x_2(0, 1)$. *Here* $(0,0)$ *and* $(0,1)$ *are extreme points of* $M = K \cap L^\perp = \{(0, x_2) \in \mathbb{R}^2 : \ 0 \leq x_2 \leq 1\}$, *and so* $M^\infty = \{0\}$, *where* $L = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = 0\}$. *By chosing* $l_0 = (p, 0)$ *with* $p \geq 1$, *one can check that Assumption (A) holds, and that it fails for* $p < 1$.

Contrary to the preceding example, next proposition shows that the class of convex functions satisfies Assumption (A), with equality, for $l_0 = 0$.

Proposition 3.8. *Let $L \subseteq \mathbb{R}^n$ be a linear subspace, $M \subseteq \mathbb{R}^n$ be a convex closed set having extreme points.*

(*a*) *If $\varphi : L \to \mathbb{R}$ is a convex function satisfying $\sigma_L \doteq \sup_{x \in L} \varphi(x) < +\infty$, then φ is constant on L.*

(*b*) *If $\varphi : M \to \mathbb{R}$ is a convex function satisfying $\sigma_0 \doteq \sup_{z \in \text{ext } M + (\text{extd } M^\infty) \cup \{0\}} \varphi(z) < +\infty$, then*

$$\varphi(e + v) \le \varphi(e) \quad \forall\, e \in \text{ext } M,\ \forall\, v \in \text{extd } M^\infty.$$

Proof. (*a*): Let $l_1, l_2 \in L$ and assume that $\varphi(l_1) < \varphi(l_2)$. By writing

$$l_2 = t\Big(\frac{1}{t} l_2 - \frac{1-t}{t} l_1\Big) + (1-t) l_1,\ 0 < t < 1,$$

the convexity implies that $\varphi(l_2) \le t\sigma_L + (1-t)\varphi(l_1)$. Thus $\frac{1}{t}(\varphi(l_2) - \varphi(l_1)) + \varphi(l_1) \le \sigma_L$, which yields a contradiction once we let t goes to 0.
(*b*): Take $\lambda > 1$, and write $e + v$ as a convex combination of e and $e + \lambda v$. Then, for $\alpha = 1 - \frac{1}{\lambda}$, we obtain

$$\varphi(e + v) \le \alpha\varphi(e) + (1 - \alpha)\varphi(e + \lambda v),$$

which implies that

$$\varphi(e + v) - \varphi(e) + \frac{1}{\lambda}\varphi(e) \le \frac{1}{\lambda}\sigma_0.$$

By letting $\lambda \to +\infty$, the result follows. □

The following corollary generalizes Theorem 1 in [21], which requires that K contains no lines.

Corollary 3.9. *Let $K \subseteq \mathbb{R}^n$ be any nonempty closed convex set, and $f : K \to \mathbb{R}$ be a convex function. Then,*

$$\sigma_0 \doteq \sup_{z \in L + \text{ext } M + (\text{extd } M^\infty) \cup \{0\}} f(z) < +\infty \iff \sup_{x \in K} f(x) = \sup_{x \in \text{ext } M} f(x) < +\infty.$$

Furthermore,

(*a*) *if $\sigma_0 < +\infty$,* ext M *is compact and f is upper semicontinuous on* ext M*, then σ is achieved.*

(*b*) *if* K *contains no lines, then*

$$\sup_{z \in \text{ext } K + (\text{extd } K^\infty) \cup \{0\}} f(z) < +\infty \Longleftrightarrow \sup_{x \in K} f(x) = \sup_{x \in \text{ext } K} f(x) < +\infty.$$

Proof. It results as a direct application of Theorem 3.4 and Proposition 3.8. □

Example 3.5 shows that (*a*) of the previous result is not necessarily true if $\sigma_0 = +\infty$.

In what follows we consider the quadratic case, and find more precise formulations for Assumption **(A)** (part (a) of the next proposition) or for assumptions on σ_0 (part (*b*) and (*c*) of the same proposition). Let us consider the quadratic function

$$f(x) = \frac{1}{2} x^\top A x + a^\top x + \alpha, \tag{6}$$

with $A = A^\top$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. By using the representation $K = L + \text{conv}(\text{ext } M) + M^\infty$, we can write for any $x \in K$, $x = l + e + v$, with $l \in L$, $e \in \text{conv}(\text{ext } M)$ and $v \in M^\infty$.

Next result provides an equivalent condition to Assumption (A) when f is quadratic. Observe that necessary and sufficient conditions for the quasiconvexity of homogenous quadratic functions f may be found in [18], whereas for general C^2 functions the reader may consult [3].

Proposition 3.10. *Let* f *be a quadratic function as in* (6)*; let* $V, W \subseteq \mathbb{R}^n$ *be nonempty sets;* $z \in \mathbb{R}^n$*, and* $l_0 \in L$ *fixed.*

(*a*) *One has*

$$f(l + z) \le f(l_0 + z) \quad \forall\, l \in L \Longleftrightarrow l^\top A l \le 0 \quad \forall\, l \in L \text{ and } \nabla f(l_0 + z) \in L^\perp. \tag{7}$$

(*b*) *If* $\sigma_0 \doteq \sup_{z \in V + W} f(z) < +\infty$ *and* W *is a cone, then*

$$v^\top A v \le 0 \quad \forall\, v \in W, \text{ and}$$
$$[v^\top A v = 0,\ v \in W \Longrightarrow \nabla f(e)^\top v \le 0 \quad \forall\, e \in V].$$

(*c*) *If* $\sigma_0 \doteq \sup_{z \in V + W} f(z) < +\infty$ *and* V *is a subspace, then*

$$v^\top A v \le 0 \quad \forall\, v \in V, \text{ and}$$
$$[v^\top A v = 0,\ v \in V \Longrightarrow \nabla f(e)^\top v = 0 \quad \forall\, e \in W].$$

Proof. (*a*): It is a consequence of the identity:

$$f(l + z) = f(l_0 + z) + \nabla f(l_0 + z)^\top (l - l_0) + \frac{1}{2} (l - l_0)^\top A (l - l_0).$$

Assertions (*b*) and (*c*) follow from the equality

$$f(e + v) = f(e) + \nabla f(e)^\top v + \frac{1}{2} v^\top A v \le \sigma_0, \quad \forall\, e \in V,\ \forall\, v \in W.$$

□

We point out that for quadratic objective functions on asymptotically linear closed convex sets, various characterizations for the nonemptiness of the set of minimizers were given in [9].

We observe that Assumption **(A)** is useless if K contains no line since in this case L must be $\{0\}$, and so $K = \text{conv}(\text{ext } K) + K^\infty$. The following second main assumption arises:

Assumption (B): f satisfies

$$f(e+v) \leq f(e), \quad \forall\ e \in \text{ext } K, \text{and } \forall\ v \in \text{extd } K^\infty.$$

Thus, the next theorem is obtained when K contains no lines. It encompasses the classical linear case $f(x) = a^\top x$ and $K = \{x \in \mathbb{R}^n :\ Ax \leq b,\ x \geq 0\}$.

Theorem 3.11. *Let $K \subseteq \mathbb{R}^n$ be any nonempty closed and convex set that contains no lines; $f : K \to \mathbb{R}$ be quasiconvex satisfying Assumption* **(B)**. *Then*

$$\sup_{x \in K} f(x) = \sup_{x \in \text{ext } K} f(x).$$

Proof. By (4) (with $L = \{0\}$), for every element $x \in K$ there exist $p_x \in \text{conv}(\text{ext } M)$, and $v_x \in M^\infty$ such that $x = p_x + v_x$. Since M^∞ is pointed, $M^\infty = \text{conv}(\{0\} \cup \text{extd } M^\infty)$. Thus $v_x = \displaystyle\sum_{j=1}^{p} \beta_j m_j$, where, $0 \leq \beta_j$, $m_j \in (\text{extd } M^\infty) \cup \{0\}$, for $j = 1, \ldots, p$, and $\displaystyle\sum_{j=1}^{p} \beta_j = 1$. In addition, $p_x = \displaystyle\sum_{i=1}^{q} \alpha_i e_i$, where, $0 \leq \alpha_i$, $e_i \in \text{ext } M$, for $i = 1, \ldots, q$, and $\displaystyle\sum_{i=1}^{q} \alpha_i = 1$. By quasiconvexity and Assumption (B), we obtain

$$f(x) = f(p_x + v_x) \leq \max_{i,j} f(e_i + m_j) = f(e_{i_0} + m_{j_0}) \leq f(e_{i_0})$$

$$\leq \sup_{z \in \text{ext } M} f(z) \leq \sup_{z \in K} f(z).$$

Since x was arbitrary in K, the proof is completed. ☐

Example 3.12. *Consider $K = \mathbb{R}^2_+$ and*

$$f(x_1, x_2) = \begin{cases} 1, & \textit{if} \quad (x_1, x_2) = (0,0), \\ 0, & \textit{if} \quad (x_1, x_2) \neq (0,0). \end{cases}$$

Obviously f is quasiconvex in K (but not in $\mathbb{R}^n$); ext $K = \{(0,0)\}$ *and* extd $K = (\mathbb{R}_{++} \times \{0\}) \cup (\{0\} \times \mathbb{R}_{++})$. *Thus, Assumption* **(B)** *holds, and so Theorem 3.11 applies.*

The following proposition describes, in some sense, a recursive scheme on finding maximizers located on the boundary starting from the interior.

Theorem 3.13. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex and proper function, let $K \subseteq \text{dom } f$ be closed convex with nonempty interior, and $\bar{x} \in \text{int } K$ such that $f(\bar{x}) = \alpha \in \mathbb{R}$. If f is not constant on K then there exists $v \neq 0$ such that either there exists $\bar{t} > 0$ satisfying $\bar{x} + \bar{t}v \in \partial K$ and $f(\bar{x} + \bar{t}v) = \alpha$, or*

$$\bar{x} + tv \in K \quad \textit{and} \ f(\bar{x} + tv) = \alpha \quad \forall\ t > 0.$$

Proof. By hypothesis there is an open set $U \subseteq K$ with $\bar{x} \in U$ and there is $x_1 \in K$ such that $f(x_1) < f(\bar{x})$. Take $x_2 \in U$ of the form $x_2 = \bar{x} - t(x_1 - \bar{x})$ for some $t > 0$. Clearly $f(x_2) = f(\bar{x})$ since otherwise, by quasiconvexity we reach a contradiction. By quasiconvexity again, $f(x) = f(\bar{x})$ for all $x \in [\bar{x}, x_2]$. We have two posibilities:
i. there exists $s > 1$ such that $\bar{x} + s(x_2 - \bar{x}) \notin K$. This implies that $\bar{x} + \bar{s}(x_2 - \bar{x}) \in \text{bd } K$ where $\bar{s} \doteq \inf\{s > 0 : \ \bar{x} + s(x_2 - \bar{x}) \notin K\} \geq 1$. Obviously $f(\bar{x} + \bar{s}(x_2 - \bar{x})) = f(\bar{x})$.
ii. $\bar{x} + s(x_2 - \bar{x}) \in K$ for all $s > 1$. In which case $f(\bar{x} + s(x_2 - \bar{x})) = f(\bar{x})$ for all $s > 0$. □

Corollary 3.14. *Let f, K be as in the previous theorem with* $\text{int } K \neq \emptyset$. *Assume that $\alpha = f(\bar{x})$ for some $\bar{x} \in K$. If f is not constant on K and explicitely quasiconvex (in particular convex), then $\bar{x} \in \partial K$.*

Proof. This is a consequence of the previous result. □

Remark 3.15. *The quasiconvexity of f is not enough for the validity of the previous result as the next function shows: take $K = [-2, 2]$, $f(x) = 0$ if $|x| \leq 1$, $f(x) = 1$ if $2 \geq |x| > 1$.*

The following characterization is easy to check.

Proposition 3.16. *Let $K \subseteq \mathbb{R}^n$ be nonempty, convex and bounded. Let $f : K \to \mathbb{R}$ quasiconvex. Then the following assertions are equivalent:*

(*a*) $\sup_{x \in K} f(x) = +\infty$;

(*b*) *f cannot be extended to all $\mathbb{R}^n$ as a quasiconvex finite-valued function.*

Proof. Take a polytope (compact and polyhedral) D containing K and denote by x_i, $i = 1, \ldots, m$, the extreme points of D. Then, if (a) is assumed we have

$$+\infty = \sup_{x \in K} f(x) \leq \sup_{x \in D} f(x) \leq \max_{1 \leq i \leq m} \{f(x_i)\} < +\infty,$$

reaching a contradiction, so that (b) holds. Conversely, if on the contrary $\sup_K f = \lambda < +\infty$, we consider $\tilde{f}(x) = f(x)$ if $x \in K$, and $f(x) = \lambda$ elsewhere. Then $\tilde{f}$ is quasiconvex and finite-valued defined on $\mathbb{R}^n$, which cannot happen if (b) is assumed. □

As a counterpart to the preceding result, we have the following lemma whose proof is straightforward.

Lemma 3.17. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be quasiconvex and $K \subseteq \mathbb{R}^n$ be a bounded closed and convex set. Then,*

(a) $\sup_{x \in K} f(x) < +\infty$;

(b) *if, in addition f is usc on K, there exists an extreme point of K, $x_0 \in K$ satisfyng $\alpha = f(x_0)$.*

Example 3.18. *Let us consider*

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 0, \ x_2 \geq 0, \ x_2 \leq 2 - x_1\},$$

and

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = 0 \\ \dfrac{x_2}{x_1}, & \text{if } x_1 \neq 0. \end{cases}$$

Clearly f is lsc in K, but not in $\mathbb{R}^2$: indeed f is not lsc at $\bar{x} = (0,1)$, simply take $x_k = (-\frac{1}{k}, 1)$; then $f(x_k) = -k$ and $f(\bar{x}) > \lim f(x_k) = -\infty$.
Since $f(1/k, 1/\sqrt{k}) = \sqrt{k}$ for all $k \in \mathbb{N}$, $\sup_K f = +\infty$. Even more, $f(x, 2-x) = \dfrac{2}{x} - 1$, for all $x > 0$. Thus,

$$\sup_{x \in K} f(x) = \sup_{x \in \partial K} f(x) = +\infty.$$

If f was quasiconvex then, as a consequence of the previous lemma, $\sup_K f < +\infty$, which is false. Hence f is not quasiconvex in K. This is also verified by taking $x = (0,1)$, $y = (1,1)$ and $x_0 = \frac{1}{2}(x+y) = (\frac{1}{2}, 1)$, since $2 = f(x_0) > \max\{f(x), f(y)\} = 1$.

Example 3.19. *Let us consider*

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } x_2 = 0 \\ \dfrac{x_1^2}{x_2}, & \text{if } x_2 \neq 0. \end{cases}$$

Clearly f is lsc and semistrictly quasiconvex in K. Take

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 2x_2, \ x_2 \geq x_1^4\}.$$

Since $f(1/k, 1/k^4) = k^2$ for all $k \in \mathbb{N}$ suficiently large, $\sup_K f = +\infty$. Even more,

$$\sup_{x \in K} f(x) = \sup_{x \in \partial K} f(x) = +\infty.$$

Hence, by Lemma 3.17 f is not quasiconvex in $\mathbb{R}^2$, and by Proposition 3.16, f cannot be extended to all $\mathbb{R}^2$ as a finite-valued quasiconvex function.

3.2 Optimality conditions via radial epiderivatives

We now recall the optimality conditions established in [7] for vector-valued functions, and state in the scalar case. The results to be presented here supplement those appearing in [7] and [17]. Such optimality conditions will be derived without quasiconvexity assumptions by providing more precise formulations for constrained or unconstrained optimization problems. To that end some notations are in order.

Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\bar{x}$ with $\bar{x} \in f^{-1}(\mathbb{R})$, let us consider the following functions:

$$f'_-(\bar{x}; u) \doteq \inf_{t>0} \frac{f(\bar{x}+tu) - f(\bar{x})}{t}, \quad f'_+(\bar{x}; u) \doteq \sup_{t>0} \frac{f(\bar{x}+tu) - f(\bar{x})}{t}.$$

The "*lower radial epiderivative*" of f at $\bar{x}$, $\underline{D}_e^R f(\bar{x}; \cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$\underline{D}_e^R f(\bar{x}; u) = \liminf_{u' \to u} \inf_{t>0} \frac{f(\bar{x}+tu') - f(\bar{x})}{t} \doteq \sup_{\varepsilon>0} \inf_{\|u'-u\|<\varepsilon} \inf_{t>0} \frac{f(\bar{x}+tu') - f(\bar{x})}{t},$$

that is, $\underline{D}_e^R f(\bar{x}; \cdot)$ is the lower semicontinuous hull function of $f'_-(\bar{x}; \cdot)$, which is the largest lower semicontinuous function below $f'_-(\bar{x}; \cdot)$.

The "*upper radial epiderivative*" of f at $\bar{x}$, $\overline{D}_e^R f(\bar{x}; \cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$\overline{D}_e^R f(\bar{x}; u) = \limsup_{u' \to u} \sup_{t>0} \frac{f(\bar{x}+tu') - f(\bar{x})}{t} \doteq \inf_{\varepsilon>0} \sup_{\|u'-u\|<\varepsilon} \sup_{t>0} \frac{f(\bar{x}+tu') - f(\bar{x})}{t}.$$

It is easy to check that

$$\underline{D}_e^R f(\bar{x}; u) = -\overline{D}_e^R (-f)(\bar{x}; u). \tag{8}$$

We also obtain

$$\text{epi } f'_+(\bar{x}; \cdot) = \bigcap_{t>0} t\Big(\text{epi } f - (\bar{x}, f(\bar{x})\Big), \tag{9}$$

which implies that

$$R^i(\text{epi } f; (\bar{x}, f(\bar{x}))) = \text{int}\Big(\text{epi } f'_+(\bar{x}; \cdot)\Big). \tag{10}$$

Moreover,

$$\text{epi } f'_+(\bar{x}; \cdot) = (\text{epi } f)^\infty = \text{epi } f^\infty, \tag{11}$$

provided f is lsc and epi f is radiant at $(\bar{x}, f(\bar{x}))$, by Proposition 2.5, and therefore $f'_+(\bar{x}; \cdot)$ is lsc. Actually $f'_+(\bar{x}; \cdot)$ is lsc under the solely assumption of lower semicontinuity of f by virtue of Proposition 3.28.

We see that when f is a quadratic function:

$$f(x) = \frac{1}{2} x^\top A x + a^\top x + \alpha, \quad A = A^\top, \ a \in \mathbb{R}^n, \ \alpha \in \mathbb{R},$$

easy computations show that

$$f'_+(\bar{x}; u) = \begin{cases} \nabla f(\bar{x})^\top u & \text{if } \ u^\top A u \leq 0 \\ +\infty & \text{if } \ u^\top A u > 0, \end{cases}$$

and $\overline{D}_e^R f(\bar{x}; u) = f'_+(\bar{x}, u)$ for all $u \in \mathbb{R}^n$ satisfying $|u^\top Au| > 0$.

From the very definition we have

$$\underline{D}_e^R f(\bar{x}; u) \le f'_-(\bar{x}; u) \le \frac{f(\bar{x} + tu) - f(\bar{x})}{t} \le f'_+(\bar{x}; u) \le \overline{D}_e^R f(\bar{x}; u), \quad u \in \mathbb{R}^n, \quad t > 0.$$

This implies that

$$f(\bar{x} + tu) \ge f(\bar{x}) + t\underline{D}_e^R f(\bar{x}; u) \;\; \forall\, t > 0, \; \forall\, u \in \mathbb{R}^n;$$

$$f(\bar{x} + tu) \le f(\bar{x}) + t\overline{D}_e^R f(\bar{x}; u) \;\; \forall\, t > 0, \; \forall\, u \in \mathbb{R}^n.$$

Furthermore, we also infer

$$\text{epi } \overline{D}_e^R f(\bar{x}; \cdot) \subseteq \text{epi } f'_+(\bar{x}; \cdot). \tag{12}$$

The proof of the next proposition is straightforward, so omitted.

Proposition 3.20. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be any function and $\bar{x} \in f^{-1}(\mathbb{R})$. Then*

(*a*) $\underline{D}_e^R f(\bar{x}; u) = +\infty \Longrightarrow f'_-(\bar{x}; u) = +\infty \Longleftrightarrow f(\bar{x} + tu) = +\infty \; \forall\, t > 0$;

(*b*) $f'_-(\bar{x}; 0) = 0 = f'_+(\bar{x}; 0)$;

(*c*) $f'_-(\bar{x}; \lambda u) = \lambda f'_-(\bar{x}; u), \quad f'_+(\bar{x}; \lambda u) = \lambda f'_+(\bar{x}; u) \;\; \forall\, \lambda > 0, \; \forall\, u \in \mathbb{R}^n$;

(*d*) $\overline{D}_e^R f(\bar{x}; u) = -\infty, \; u \ne 0 \Longrightarrow f'_+(\bar{x}; u) = -\infty \Longleftrightarrow f(\bar{x} + tu) = -\infty \; \forall\, t > 0$;

(*e*) $\overline{D}_e^R f(\bar{x}; u) \le 0, \; \forall\, u \ne 0, \;\; \Longrightarrow f(x) < +\infty \;\; \forall\, x \in \mathbb{R}^n$.

The next result is a consequence of (*d*) of the previous proposition.

Proposition 3.21. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be any function and $\bar{x} \in f^{-1}(\mathbb{R})$. Then, if either f is lsc at $\bar{x}$ or $f(y) > -\infty$ for all $y \in \mathbb{R}^n$, then $f'_+(\bar{x}; u) > -\infty$ for all $u \in \mathbb{R}^n$ and so also $\overline{D}_e^R f(\bar{x}; u) > -\infty$ for all $u \in \mathbb{R}^n$.*

Part of the following theorem is Corollary 3.4 in [7], which still remains valid for extended real-valued functions.

Theorem 3.22. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\bar{x} \in f^{-1}(\mathbb{R})$. Then*

(*a*) *For all $u \in \mathbb{R}^n$,*

$$\begin{aligned} \underline{D}_e^R f(\bar{x}; u) &= \inf\Big\{v \in \mathbb{R} : \; (u, v) \in R(\text{epi } f; (\bar{x}, f(\bar{x}))\Big\} \\ &= \sup\Big\{v \in \mathbb{R} : \; (u, v) \in R^i(\text{hyp } f; (\bar{x}, f(\bar{x}))\Big\}. \end{aligned}$$

(*b*) *For all* $u \in \mathbb{R}^n$,

$$\begin{aligned}\overline{D}_e^R f(\bar{x};u) &= \sup\Big\{v \in \mathbb{R}: \ (u,v) \in R(\text{hyp } f;(\bar{x},f(\bar{x}))\Big\} \\ &= \inf\Big\{v \in \mathbb{R}: \ (u,v) \in R^i(\text{epi } f;(\bar{x},f(\bar{x}))\Big\}.\end{aligned}$$

(*c*) epi $(\underline{D}_e^R f(\bar{x};\cdot)) = R(\text{epi } f;(\bar{x},f(\bar{x})))$;

(*d*) hyp $(\overline{D}_e^R f(\bar{x};\cdot)) = R(\text{hyp } f;(\bar{x},f(\bar{x})))$.

Proof. For (a) and (b) we refer to Corollary 3.4 in [7]. We now prove only (c), being the other entirely similar.
(c): From (a) we obtain $R(\text{epi } f;(\bar{x},f(\bar{x}))) \subseteq \text{epi } (\underline{D}_e^R f(\bar{x};\cdot))$. Take any $(u,v) \in \text{epi } (\underline{D}_e^R f(\bar{x};\cdot))$. If on the contrary $(u,v) \notin R(\text{epi } f;(\bar{x},f(\bar{x})))$, we get

$$(u,v) \in R^i(\mathbb{R}^n \times \mathbb{R} \setminus \text{epi } f;(\bar{x},f(\bar{x}))) \subseteq R^i(\text{hyp } f;(\bar{x},f(\bar{x})))$$

by Remark 2.4(d). Since the last set is open, a contradiction is reached by virtue of (a). □

What follows collects the homogeneity properties of the radial epiderivatives, their proofs are straightforward.

Proposition 3.23. *Let* $f:\mathbb{R}^n \to \overline{\mathbb{R}}$, $\bar{x} \in f^{-1}(\mathbb{R})$, *we have*

(*a*) $\underline{D}_e^R f(\bar{x};\lambda u) = \lambda \underline{D}_e^R f(\bar{x};u) \quad \forall\ \lambda > 0,\ \forall\ u \in \mathbb{R}^n$;

(*b*) $\underline{D}_e^R f(\bar{x};0) = 0$ *if and only if* $-\infty < \underline{D}_e^R f(\bar{x};u) \quad \forall\ u \in \mathbb{R}^n$;

(*c*) $\overline{D}_e^R f(\bar{x};\lambda u) = \lambda \overline{D}_e^R f(\bar{x};u) \quad \forall\ \lambda > 0,\ \forall\ u \in \mathbb{R}^n$;

(*d*) $\overline{D}_e^R f(\bar{x};0) = 0$ *if and only if* $\overline{D}_e^R f(\bar{x};u) < +\infty \quad \forall\ u \in \mathbb{R}^n$.

Next theorem may be considered a Fermat-type result, and it follows easily from the definition of upper radial epiderivative.

Theorem 3.24. *Let* $f:\mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ *be any function with* $\bar{x} \in f^{-1}(\mathbb{R})$. *Then*

$$\bar{x} \in \operatorname*{argmax}_{\mathbb{R}^n} f \Longleftrightarrow \overline{D}_e^R f(\bar{x};u) \leq 0 \ \ \forall\ u \in \mathbb{R}^n.$$

When dealing with constrained maximization problems, is usual to consider the function $\tilde{f}:\mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $\tilde{f}(x) = f(x)$ if $x \in K$, and $\tilde{f}(x) = -\infty$ elsewhere. Obviously

$$\sup_{x \in \mathbb{R}^n} \tilde{f}(x) = \sup_{x \in K} f(x).$$

A relationship between the upper radial epiderivatives of f and $\tilde{f}$ is shown next.

Lemma 3.25. *Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be closed; $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ and $\bar{x} \in K \cap f^{-1}(\mathbb{R})$. Then, if $-\infty < \overline{D}_e^R \tilde{f}(\bar{x}; u) \leq +\infty$ then $u \in R(K; \bar{x})$ and $\overline{D}_e^R f(\bar{x}; u) \geq \overline{D}_e^R \tilde{f}(\bar{x}; u)$.*

Proof. We consider first the case when $\overline{D}_e^R \tilde{f}(\bar{x}; u) \in \mathbb{R}$. By a characterization $(u, \overline{D}_e^R \tilde{f}(\bar{x}; u)) \in R(\text{hyp } \tilde{f}; (\bar{x}; f(\bar{x})))$.
Then, $\exists\, t_k > 0,\ \exists\, u_k \to u,\ \exists\, v_k \to \overline{D}_e^R \tilde{f}(\bar{x}; u)$ such that $(\bar{x}, f(\bar{x})) + t_k(u_k, v_k) \in \text{hyp } \tilde{f}$. By definition of $\tilde{f}$,

$$\bar{x} + t_k u_k \in K,\ f(\bar{x} + t_k u_k) \geq f(\bar{x}) + t_k v_k \quad \forall\, k \in \mathbb{N}.$$

It follows that $u \in R(K; \bar{x})$. Take any $\varepsilon > 0$ and k_ε such that $\|u_k - u\| < \varepsilon\ \forall\, k \geq k_\varepsilon$. Then

$$\sup_{\|u'-u\|<\varepsilon} \sup_{t>0} \frac{f(\bar{x} + tu') - f(\bar{x})}{t} \geq \frac{f(\bar{x} + t_k u_k) - f(\bar{x})}{t_k} \geq v_k \quad \forall\, k \geq k_\varepsilon.$$

Hence, letting $k \to +\infty$, we obtain $\overline{D}_e^R f(\bar{x}; u) \geq \overline{D}_e^R \tilde{f}(\bar{x}; u)$, which is the desired result.
A similar reasoning proves also that if $\overline{D}_e^R \tilde{f}(\bar{x}; u) = +\infty$ then $\overline{D}_e^R f(\bar{x}; u) = +\infty$, completing the proof. □

By using the previous result and Theorem 3.24, we get the following sufficient or necessary optimality condition when a constraint set is present. The sufficient part is new; whereas the necessary already appears (as a particular case) in [7].

Theorem 3.26. *Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be any set, and $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be any function with $\bar{x} \in K \cap f^{-1}(\mathbb{R})$. Let us consider the following assertions:*

(*a*) $\overline{D}_e^R f(\bar{x}; u) \leq 0\ \forall\, u \in R(K; \bar{x})$;

(*b*) $\bar{x} \in \underset{K}{\text{argmax}}\ f$;

(*c*) $\overline{D}_e^R f(\bar{x}; u) \leq 0\ \forall\, u \in R^i(K; \bar{x})$ *and* $\underline{D}_e^R f(\bar{x}; u) \leq 0\ \forall\, u \in R(K; \bar{x})$.

Then, $(a) \Longrightarrow (b) \Longrightarrow (c)$.

Proof. $(a) \Longrightarrow (b)$: We have to prove that $\overline{D}_e^R \tilde{f}(\bar{x}; u) \leq 0$ for all $u \in \mathbb{R}^n$. In case $\overline{D}_e^R \tilde{f}(\bar{x}; u) = -\infty$ there nothing to do, otherwise we apply the previous lemma to conclude that (b) holds.
$(b) \Longrightarrow (c)$: To prove the first inequality, given any $u \in R^i(K; \bar{x})$, we get the existence of $\varepsilon > 0$ such that $u' \in \bigcap_{t>0} t(K - \bar{x})$ for every $u' \in \mathbb{R}^n$, $\|u' - u\| \leq \varepsilon$. This yields $\overline{D}_e^R f(\bar{x}; u) \leq 0$. For the second inequality, given any $t_k > 0$, $u_k \to u$ such that $\bar{x} + t_k u_k \in K$ for all $k \in \mathbb{N}$, we obtain $f(\bar{x} + t_k u_k) \leq f(\bar{x})$. Thus $\underline{D}_e^R f(\bar{x}; u) \leq 0$. This completes the proof of the theorem. □

Example 3.27. *(i) Let* $f(x) = -\sqrt{x}$ *if* $x \geq 0$ *and* $f(x) = -\infty$ *elsewhere. Then,* $f'_+(0;u) = 0$ *if* $u \geq 0$, $f'_+(0;u) = -\infty$ *if* $u < 0$*; and therefore* $\overline{D}_e^R f(0;u) = 0$ *if* $u \geq 0$, $\overline{D}_e^R f(0;u) = -\infty$ *elsewhere. Morevore,* $f'_-(0;u) = -\infty$ *if* $u \neq 0$, $f'_-(0;0) = 0$*, and thus* $\underline{D}_e^R f(0;u) = -\infty$ *for all* $u \in \mathbb{R}$.
(ii) Consider now $f(x) = -\sqrt{|x|}$, $x \in \mathbb{R}$*. Here we obtain* $f'_+(0;u) = 0$ *for* $u \in \mathbb{R}$ *and so* $\overline{D}_e^R f(0;u) \equiv 0$*. On the other hand,* $f'_-(0;u) = -\infty$ *if* $u \neq 0$, $f'_-(0;0) = 0$*, and thus* $\underline{D}_e^R f(\bar{x};u) = -\infty$ *for* $u \in \mathbb{R}$.
(iii) Take $f(x) = \sqrt{1+x^2}$ *if* $x > 0$, $f(0) = 0$ *and* $f(x) = -\infty$ *elsewhere. Then,* $f'_-(0;u) = u$ *if* $u \geq 0$, $f'_-(0;u) = -\infty$ *elsewhere, and so* $\underline{D}_e^R(0;u) = u$ *if* $u > 0$ *and* $\underline{D}_e^R(0;u) = -\infty$ *if* $u \leq 0$*; whereas* $f'_+(0;u) = -\infty$ *if* $u < 0$, $f'_+(0;u) = +\infty$ *if* $u > 0$ *and* $f'_+(0;0) = 0$*. Thus* $\overline{D}_e^R f(0;u) = +\infty$ *if* $u \geq 0$, $\overline{D}_e^R f(0;u) = -\infty$ *if* $u < 0$.
(iv) Take $f(x) = -\sqrt{1+x^2}$ *if* $x \neq 0$, $f(0) = 0$*. Then,* $f'_-(0;u) = -\infty$ *if* $u \neq 0$, $f'_-(0;0) = 0$*, and so* $\underline{D}_e^R(0;u) \equiv -\infty$*; whereas* $f'_+(0;u) = -|u| = \overline{D}_e^R f(0;u)$.

Proposition 3.28. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be any function and* $\bar{x} \in f^{-1}(\mathbb{R})$*. Then,*

(a) *if* f *is lsc (resp. usc) then* $f'_+(\bar{x};\cdot)$ *is also lsc (resp. usc);*

(b) *if* f *is quasiconvex then* $f'_+(\bar{x};\cdot)$ *is also quasiconvex.*

Proof. We only prove (b). Let u_1, u_2 such that $f'_+(\bar{x}, u_i) < +\infty$, $i = 1,2$. Then $f(\bar{x} + tu_i) < +\infty$ for all $t > 0$, $i = 1,2$. On the other hand,

$$\begin{aligned} \frac{f(\bar{x} + t(\lambda u_1 + (1-\lambda)u_2)) - f(\bar{x})}{t} &= \frac{f((\bar{x} + tu_1)\lambda + (\bar{x} + tu_2)(1-\lambda)) - f(\bar{x})}{t} \\ &\leq \max_{i=1,2} \frac{f(\bar{x} + tu_i) - f(\bar{x})}{t}. \end{aligned}$$

Thus,

$$f'_+(\bar{x}; \lambda u_1 + (1-\lambda)u_2) \leq \sup_{t>0} \max_{i=1,2} \frac{f(\bar{x} + tu_i) - f(\bar{x})}{t} = \max_{i=1,2} \sup_{t>0} \frac{f(\bar{x} + tu_i) - f(\bar{x})}{t},$$

which means that $f'_+(\bar{x};\cdot)$ is quasiconvex. □

Proposition 3.29. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\bar{x} \in f^{-1}(\mathbb{R})$, $\bar{y} = f(\bar{x})$*, we have*

(a) $\operatorname{int}\Big(\bigcap_{t>0} t\big(\text{epi } f - (\bar{x},\bar{y})\big)\Big) \subseteq \text{epi } (\overline{D}_e^R f(\bar{x};\cdot)) \subseteq \bigcap_{t>0} t\big(\text{epi } f - (\bar{x},\bar{y})\big)$;

(b) $\operatorname{int}\Big(\bigcap_{t>0} t\big(\text{hyp } f - (\bar{x},\bar{y})\big)\Big) \subseteq \text{hyp } (\underline{D}_e^R f(\bar{x};\cdot)) \subseteq \bigcap_{t>0} t\big(\text{hyp } f - (\bar{x},\bar{y})\big)$.

Proof. Let us prove only (a), since (b) is entirely similar.
(a): From Theorem 3.22 it follows that $R^i(\text{epi } f; (\bar{x},\bar{y})) \subseteq \text{epi } \overline{D}_e^R f(\bar{x};\cdot)$. The conclusion follows once we apply (9), (10) and (12). □

From (a) in the previous proposition it follows that for $\bar{x} \in f^{-1}(\mathbb{R})$,

$$f^{\infty}(u) \leq \overline{D}_e^R f(\bar{x}; u) \quad \forall\, u \in \mathbb{R}^n.$$

We now express epi f in terms of epi $\overline{D}_e^R f(x; \cdot)$ for x varying in dom f.

Proposition 3.30. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be any function,* dom $f \neq \emptyset$*. If* $\overline{D}_e^R f(x'; 0) = 0$*, for all* $x' \in \operatorname{dom} f$*, then*

$$\operatorname{epi} f \subseteq \bigcup_{x \in f^{-1}(\mathbb{R})} [\operatorname{epi} \overline{D}_e^R f(x; \cdot) + (x, f(x))].$$

Consequently,

$$\operatorname{epi} f = \bigcup_{x \in f^{-1}(\mathbb{R})} [\operatorname{epi} \overline{D}_e^R f(x; \cdot) + (x, f(x))].$$

Proof. Let $(x', \lambda') \in \operatorname{epi} f$. Then $f(x') \in \mathbb{R}$. We write $(x', \lambda') = (0, \lambda' - f(x')) + (x', f(x'))$, and by noticing that $(0, \lambda' - f(x')) \in \operatorname{epi} \overline{D}_e^R f(x; \cdot)$ since $\overline{D}_e^R f(x'; 0) = 0 \leq \lambda' - f(x')$, the desired result is reached.
The other inclusion follows from (a) of Proposition 3.29. □

We end this section by establishing another representation for the function $f'_+(\bar{x}; \cdot)$ for a given $\bar{x} \in f^{-1}(\mathbb{R})$.

Proposition 3.31. *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\bar{x} \in f^{-1}(\mathbb{R})$*. Set*

$$C \doteq \bigcap_{t>0} t(\operatorname{epi} f - (x, f(x)).$$

Then

- (a) $(u, v) \in C \Longrightarrow (u, v + p) \in C$ *for all* $p \geq 0$;
- (b) *we have*

$$f'_+(\bar{x}; v) = \inf \Big\{ v \in \mathbb{R} : \ (u, v) \in \bigcap_{t>0} t(\operatorname{epi} f - (x, f(x)) \Big\}.$$

Proof. (a): It is straightforward.
(b): Denote by $g(u)$ the left hand-side of the equality in (b). Obviously $C \subseteq \operatorname{epi} g$.
Let $(u, \lambda) \in \operatorname{epi} g$. If $g(u) = -\infty$ then $(u, r) \in C$ for all $r \in \mathbb{R}$, from which the desired result is obtained. We now assume that $g(u) \in \mathbb{R}$. For every $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \mathbb{R}$ such that $(u, \lambda_\varepsilon) \in C$ and $\lambda_\varepsilon < g(u) + \varepsilon$. Thus $(u, g(u) + \varepsilon) = (u, \lambda_\varepsilon + g(u) + \varepsilon - \lambda_\varepsilon) \in C$ by (a). Hence,

$$f(x + tu) \leq f(x) + tg(u) + t\varepsilon \ \ \forall\, t > 0, \ \forall\, \varepsilon > 0.$$

Letting ε goes to zero, we obtain $f(x+tu) \leq f(x) + tg(u)$ for all $t > 0$, i.e., $(u, g(u)) \in C$. Now $(u, \lambda) = (u, g(u) + \lambda - g(u)) \in C$ by (a), which completes the proof of epi $g \subseteq C$. □

4 Minimizing quasiconvex functions

Having introduced the notion of asymptotic cone and function, we analyze the minimization problem without any coercivity assumption, which means the solution set may be unbounded. Given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\text{dom } f \neq \emptyset$, let us now consider the problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{13}$$

Set

$$R \doteq \{v \in \mathbb{R}^n : \; f^\infty(v) \leq 0\}.$$

As expected, one must be interested in studying the behaviour of f along directions belonging to R, which, in essence, are obtained as limit of sequences $\dfrac{x_k}{\|x_k\|}$ with $\{x_k\}$ being any unbounded minimizing sequence. Thus, the condition we impose on f along those directions is expressed in condition (C_1):

(C_1) : if the sequence $x_k \in \text{dom } f$, $\|x_k\| \to +\infty$ is such that $\frac{x_k}{\|x_k\|} \to v$ with $v \in R$, and for all $y \in \text{dom } f$ it exists k_y such that

$$f(y) \geq f(x_k) \;\text{ when }\; k \geq k_y,$$

then there exist u and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \leq f(x_{\bar{k}})$.

(C_2) : if the sequence $x_k \in \text{dom } f$, $\|x_k\| \to +\infty$ then there exist u and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \leq f(x_{\bar{k}})$.

Remark 4.1. *(i) First of all notice that a sequence satisfying the premises of condition (C_1) has to be minimizing, that is,*

$$\lim_{k \to +\infty} f(x_k) = \inf_{x \in \mathbb{R}^n} f(x).$$

(ii) Condition (C_1) is satisfied vacuously if f is coercive in the sense that all of its level sets are bounded, or, if $R = \{0\}$, since there are no sequences satisfying the premises of (C_1).

Next theorem establish various characterizations of the nonemptiness of the solution set to problem (13). It is a revisited version of that appearing in [2, 19] without any convexity assumption.

Theorem 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc with* $\text{dom } f \neq \emptyset$*. Then, the following assertions are equivalent:*

(a) $\underset{\mathbb{R}^n}{\text{argmin}}\, f \neq \emptyset$*;*

(*b*) (C_1) *is satisfied;*

(*c*) (C_2) *is satisfied.*

Hence, under any of the previous assumptions, $R = \{v \in \mathbb{R}^n : \ f^{\infty}(v) = 0\}$.

Proof. $(b) \Longrightarrow (a)$: for every $k \in \mathbb{N}$, set $K_k = \{x \in \mathbb{R}^n : \ \|x\| \leq k\}$. We may suppose, without loss of generality, that $K_k \cap \operatorname{dom} f \neq \emptyset$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let x_k be a solution to the problem

$$\min_{x \in K_k} \ f(x).$$

Take $x_k \in \operatorname{argmin}\{\|x\| : \ x \in S_k\}$, where S_k is the set of minimizers of h on K_k. Let us prove that $\{x_k\}$ is bounded. If not, we may assume without loss of generality, that $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to v$. For fixed $y \in \operatorname{dom} f$, take $k_0 \in \mathbb{N}$ sufficiently large ($k_0 > \|y\|$) in such a way that $f(x_k) \leq f(y)$ for every $k \geq k_0$. This implies $f^{\infty}(v) \leq 0$ and thus $v \in R$. By assumption (C_1), there exist $u \in \mathbb{R}^n$ and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \leq f(x_{\bar{k}})$, which cannot happen since $x_{\bar{k}}$ is of minimal norm. Consequently, $\{x_k\}$ is bounded. We may suppose that $x_k \to \bar{x}$, and since f is lsc,

$$f(\bar{x}) \leq \liminf_{k \to +\infty} f(x_k) = \inf_{\mathbb{R}^n} f,$$

where the equality follows from the choice of x_k. Hence $\bar{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$.
$(c) \Longrightarrow (b)$: it is straightfoward.
$(a) \Longrightarrow (c)$: take simply $u = \bar{x} \in \operatorname{argmin}_{\mathbb{R}^n} f$. □

A trivial condition implying (f_1) is that when $R = \{0\}$. In such a case the boundedness of the solution set is obtained as next corollary shows.

Corollary 4.3. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be lsc with* $\operatorname{dom} f \neq \emptyset$. *If* $R = \{0\}$ *then* $\operatorname{argmin}_{\mathbb{R}^n} f$ *is non-empty and compact.*

Proof. The nonemptiness of $\operatorname{argmin}_{\mathbb{R}^n} f$ follows from the previous theorem; and the unboundedness of such a set implies the existence of $0 \neq v \in R$, which is impossible. □

Unfortunately one cannot expect the reverse implication in the preceding corollary holds: an exampe is given next. This is due to the fact the cone R is too large. To propose a substitute for that cone is the scope of what follows.

Example 4.4. *The function* $f(x) = \sqrt{x}$, $x \in [0, +\infty[$ *and* $f(x) = +\infty$ *elsewhere, shows that the reverse implication in the preceding theorem fails to be true, since in this case* $R = [0, +\infty[$.

In order to get some characterizations of the nonemptiness and boundedness of the set of minimizers, we will impose quasiconvexity and introduce an alternative set for R. An insight is provided by the next proposition.

Proposition 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be quasiconvex and lsc with* dom $f \neq \emptyset$. *Then, assuming $S_\lambda(f) \neq \emptyset$, $\lambda \in \mathbb{R}$, one obtains*

$$(S_\lambda(f))^\infty = \bigcap_{y\in\mathbb{R}^n} \Big\{v \in \mathbb{R}^n : \ f(y+tv) \le \max\{f(y), \lambda\}, \quad \forall\, t > 0\Big\} \tag{14}$$

$$= \bigcap_{y\in S_\lambda(f)} \Big\{v \in \mathbb{R}^n : \ f(y+tv) \le \lambda \ \ \forall\, t > 0\Big\} \tag{15}$$

$$= \Big\{v \in \mathbb{R}^n : \ f(x+tv) \le \lambda \ \ \forall\, t > 0\Big\} \ \ (x \in S_\lambda(f)). \tag{16}$$

In case $\lambda = f(\bar{x})$ with $\bar{x} \in$ dom f,

$$\big(S_{f(\bar{x})}(f)\big)^\infty = \Big\{v \in \mathbb{R}^n : \ f(\bar{x}+tv) \le f(\bar{x}) \ \ \forall\, t > 0\Big\}. \tag{17}$$

Proof. Since $S_\lambda(f)$ is convex and closed, we have $v \in (S_\lambda(f))^\infty$ if, and only if $y + tv \in S_\lambda(f)$ for all $t > 0$ and all $y \in S_\lambda(f)$. This means that $f(y+tv) \le \lambda$ for all $t > 0$. This proves equalities (15) and (16).
Take any $v \in (S_\lambda(f))^\infty$, we have the existence of $t_k \downarrow 0$ and $x_k \in S_\lambda(f)$ such that $t_k x_k \to v$. For fixed $y \in \mathbb{R}^n$ and $t > 0$, we obtain, for all k sufficiently large,

$$f((1 - tt_k)y + tt_k x_k) \le \max\{f(y), f(x_k)\} \le \max\{f(y), \lambda\}.$$

By lower semicontinuity, $f(y+tv) \le \max\{f(y), \lambda\}$, which together to any of the above equalities lead to (14) and (17). □

As a consequence of the previous proposition (under quasiconvexity and lower semicontinuity of f), if $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$, we get

$$\begin{aligned}(\operatorname*{argmin}_{\mathbb{R}^n} f)^\infty &= \Big(\bigcap_{y\in\mathbb{R}^n} \Big\{x \in K : \ f(x) \le f(y)\Big\}\Big)^\infty \\ &= \bigcap_{y\in\mathbb{R}^n} \ \bigcap_{x\in S_{f(y)}(f)} \Big\{v \in \mathbb{R}^n : \ f(x+tv) \le f(y) \ \ \forall\, t > 0\Big\} \\ &= \bigcap_{y\in\mathbb{R}^n} \Big\{v \in \mathbb{R}^n : \ f(y+tv) \le f(y) \ \ \forall\, t > 0\Big\}.\end{aligned}$$

This motivates the definition of the function $f_q^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$f_q^\infty(v) \doteq \sup_{\substack{x\in \mathrm{dom}\ f\\ t>0}} \frac{f(x+tv) - f(x)}{t}, \tag{18}$$

and set

$$R_q \doteq \Big\{ v \in \mathbb{R}^n : \ f_q^\infty(v) \le 0 \ \Big\}.$$

Obviously if f is lsc and quasiconvex then f_q^∞ is lsc and quasiconvex by Proposition 3.28, and therefore, R_q is a closed convex cone. Such cones have been introduced in [7, 8] for general equilibrium problems. Observe also that $f_q^\infty = f^\infty$ whenever f is convex and lsc.

We now establish the expected characterization, which enhances Corollary 4.3.

Theorem 4.6. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be lsc and quasiconvex with* dom $f \neq \emptyset$. *Then, the following assertions are equivalent each other*

(a) $\operatorname*{argmin}_{\mathbb{R}^n} f$ *is nonempty and bounded*;

(b) $R_q = \{0\}$;

(c) $f_q^\infty(v) > 0 \ \forall \ v \neq 0$.

Proof. The equivalences follow from the previous proposition. □

This theorem applies to the function $f(x) = \sqrt{x}$ if $x \ge 0$ and $f(x) = +\infty$ elsewhere. By simple computation, one gets $f^\infty(v) = 0$ if $v \ge 0$ and $f^\infty(v) = +\infty$ elsewhere, but $f_q^\infty(0) = 0$ and $f_q^\infty(v) = +\infty$ if $v \neq 0$. This implies that $R_q = \{0\}$. Now, let us take the function $f(x) = \dfrac{x^2}{1+x^2}$, $x \in \mathbb{R}$. Here $f^\infty(v) \equiv 0$, $R = \mathbb{R}$, whereas $R_q = \{0\}$.

In order to deal with minimization problems having possibly unbounded solution set, we shall use the following condition:

(C_q) : if the sequence $x_k \in$ dom f, $\|x_k\| \to +\infty$ is such that $\frac{x_k}{\|x_k\|} \to v$ with $v \in R_q$ and for all $y \in$ dom f it exists k_y such that

$$f(y) \ge f(x_k) \ \text{ when } \ k \ge k_y,$$

then there exist u and $\bar{k}$ such that $\|u\| < \|x_{\bar{k}}\|$ satisfying $f(u) \le f(x_{\bar{k}})$.

As for assumption (C_1), condition (C_q) is satisfied vacuously if f is coercive, or, if $R_q = \{0\}$, since there are no sequences satisfying the premises of (C_q).

Theorem 4.7. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be quasiconvex and lsc with* dom $f \neq \emptyset$. *Then, the following assertions are equivalent:*

(a) $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$;

(b) (C_q) *is satisfied.*

In such a case, $R_q = \{v \in \mathbb{R}^n : \ f_q^\infty(v) = 0\}$.

Proof. We proceed as in the proof of Theorem 4.2. It only remains to check $(b) \Longrightarrow (a)$. Let us take the same sequence $\{x_k\}$ as constructed in that proof. We assume that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to v$, We now show that $v \in \mathbb{R}_q$. Indeed, for fixed $x \in \operatorname{dom} f$, we get $f(x_k) \leq f(x)$ for all k sufficiently large. Given $t > 0$, by quasiconvexity, one obtains

$$f\left((1 - \frac{t}{\|x_k\|})x + \frac{t}{\|x_k\|}x_k\right) \leq f(x), \quad \forall\ k \geq k_0.$$

By lsc,

$$f(x + tv) \leq \liminf_{k \to +\infty} f\left((1 - \frac{t}{\|x_k\|})x + \frac{t}{\|x_k\|}x_k\right) \leq f(x),$$

which implies $f_q^\infty(v) \leq 0$, which means $v \in R_q$. By applying assumption (f_q), we arrive to a contradiction. Hence $\{x_k\}$ is bounded, and so standard arguments show that any limit point of $\{x_k\}$ is a minimizer for f. □

We end this section by stating a necessary and sufficient optimality condition derived from Theorem, valid for minimization: let $\bar{x} \in f^{-1}(\mathbb{R})$

$$\bar{x} \in \operatorname*{argmin}_{\mathbb{R}^n} f \Longleftrightarrow \underline{D}_e^R f(\bar{x}; u) \geq 0 \ \ \forall\ u \in \mathbb{R}^n.$$

5 Attempts to define asymptotic functions in the quasiconvex case

Proposition 4.5 allows us to propose some kind of asymptotic function in the quasiconvex case, which keeps much more information than usual notion of asymptotic function. Other attempts have been discussed in [1, 19]. Thus we introduce the following functions, whenever $S_\lambda(f) \neq \emptyset$, $\lambda \in \mathbb{R}$:

$$f^\infty(v; \lambda) \doteq \sup_{x \in S_\lambda(f)} \sup_{t > 0} \frac{f(x + tv) - \lambda}{t};$$

$$f^\infty(v; x, \lambda) \doteq \sup_{t > 0} \frac{f(x + tv) - \lambda}{t}.$$

It is easy to check that for all $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$ and $\bar{x} \in S_\lambda(f)$:

- $f^\infty(0; \lambda) = 0 = f^\infty(0; \bar{x}, \lambda) = 0$;
- $f^\infty(tv; \lambda) = tf^\infty(v; \lambda)$ for all $t > 0$ and all $v \in \mathbb{R}^n$;

- $f^{\infty}(tv;\bar{x},\lambda) = tf^{\infty}(v;\bar{x},\lambda)$ for all $t>0$, $v\in\mathbb{R}^n$.

The following is an easy consequence of the definition, so the proof is omitted.

Proposition 5.1. *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be any function,* dom $f\neq\emptyset$*. Then, for all $\lambda\in\mathbb{R}$ such that $S_\lambda(f)\neq\emptyset$*

(*a*) $$\text{epi } f^{\infty}(\cdot;\lambda) = \bigcap_{x\in S_\lambda(f)}\bigcap_{t>0} t\left(\text{epi } f-(x,\lambda)\right);$$
$$\text{epi } f^{\infty}(\cdot;x,\lambda) = \bigcap_{t>0} t(\text{epi } f-(x,\lambda)) \quad (x\in S_\lambda(f));$$
$$\text{epi } f_q^{\infty} = \bigcap_{x\in\text{dom } f}\bigcap_{t>0} t(\text{epi } f-(x,f(x)).$$

(*b*) $$f^{\infty}(v;\lambda) = \sup_{x\in S_\lambda(f)} f^{\infty}(v;x,\lambda);$$

(*c*) $$f^{\infty}(v;\lambda) \leq \sup_{x\in S_\lambda(f)}\sup_{t>0}\frac{f(x+tv)-f(x)}{t} \leq f_q^{\infty}(v);$$

(*d*) $f'_+(\bar{x};v)\leq f^{\infty}(v;\bar{x})$ *for all* $\bar{x}\in$ dom f.

The reverse inequality in (*d*) of the previous proposition is not true in general as the next example shows.

Example 5.2. *Take the semistrictly quasiconvex and continuous function $f:\mathbb{R}\to\mathbb{R}$ defined by*

$$f(x) = \begin{cases} 0, & \textit{if} \quad x\leq 1,\\ \sqrt{x-1} & \textit{if} \quad x\geq 1.\end{cases}$$

Easy computations shows that

$$f'_+(0;1) = \sup_{t>0}\frac{f(t)-f(0)}{t} \leq \sup_{t>0}\sqrt{\frac{t-1}{t}} = 1;$$

whereas

$$f^{\infty}(1;0) = \sup_{t>0}\sup_{x\in S_{f(0)}(f)}\frac{f(x+t)-f(0)}{t} = \sup_{t>0}\sup_{\substack{x+t>1\\ x\leq 1}}\frac{\sqrt{x+t-1}}{t} \geq \sup_{t>0}\frac{\sqrt{t}}{t} = +\infty.$$

Proposition 4.5 yields the following result in terms of the different functions f^{∞}.

Proposition 5.3. *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be quasiconvex and lsc with* dom $f\neq\emptyset$*. Let $\lambda\in\mathbb{R}$ and $x\in S_\lambda(f)$. Then*

(*a*) $(S_\lambda(f))^{\infty} = \Big\{v\in\mathbb{R}^n:\ f^{\infty}(v;\lambda)\leq 0\Big\} = \Big\{v\in\mathbb{R}^n:\ f^{\infty}(v;x,\lambda)\leq 0\Big\};$

(*b*) $S_\lambda(f)$ is bounded $\iff$ $f^{\infty}(v;\lambda)>0\ \ \forall\, v\neq 0 \iff f^{\infty}(v;x,\lambda)>0\ \ \forall\, v\neq 0.$

Proof. (a) follows from the preceding proposition, and (b) is a consequence of (a). □

In case $\lambda = f(\bar{x})$ for some $\bar{x} \in \operatorname{dom} f$, we simply write

$$f^{\infty}(v;\bar{x}) \doteq f^{\infty}(v;f(\bar{x})) = \sup_{x \in S_{f(\bar{x})}(f)} \sup_{t>0} \frac{f(x+tv)-f(\bar{x})}{t}.$$

Under the presence of convexity on f, we recapture the usual notion of asymptotic function defined previously.

Proposition 5.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc and convex with* $\operatorname{dom} f \neq \emptyset$*. Then, for all $v \in \mathbb{R}^n$ and all $\bar{x} \in \operatorname{dom} f$,*

$$f^{\infty}(v;\bar{x}) = f^{\infty}(v) = f_q^{\infty}(v),$$

where f^{∞} is the asymptotic function defined before.

Proof. By definition we have, for all $\bar{x} \in \operatorname{dom} f$ and all $v \in \mathbb{R}^n$

$$f^{\infty}(v;\bar{x}) = \sup_{t>0} \sup_{x \in S_{f(\bar{x})}(f)} \frac{f(x+tv)-f(\bar{x})}{t} \geq \sup_{t>0} \frac{f(\bar{x}+tv)-f(\bar{x})}{t} = f^{\infty}(v).$$

Moreover, $x \in S_{f(\bar{x})}(f)$ implies $-f(\bar{x}) \leq -f(x)$. Therefore,

$$f^{\infty}(v;\bar{x}) = \sup_{t>0} \sup_{x \in S_{f(\bar{x})}(f)} \frac{f(x+tv)-f(\bar{x})}{t} \leq \sup_{t>0} \sup_{x \in S_{f(\bar{x})}(f)} \frac{f(x+tv)-f(x)}{t} =$$

$$= \sup_{x \in S_{f(\bar{x})}(f)} \sup_{t>0} \frac{f(x+tv)-f(x)}{t} = f^{\infty}(v),$$

which proves the first equality. The second one is straightforward. □

In what follows, we say that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is coercive if each of its level subset, $S_{\lambda}(f)$ is bounded, or equivalently $f(x) \to +\infty$ as $\|x\| \to +\infty$. Next result requires no assumption on f.

Proposition 5.5. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any function with* $\operatorname{dom} f \neq \emptyset$*. The following assertions hold:*

(a) $f_q^{\infty}(v) \geq f^{\infty}(v)\ \forall\ v \in \mathbb{R}^n$.

(b) *If $S_{\lambda}(f)$ is nonempty and bounded for some $\lambda \in \mathbb{R}$, then $f^{\infty}(v;\lambda) > 0\ \ \forall\ v \neq 0$.*

(c) $f^{\infty}(v) > 0,\ \ \forall\ v \neq 0 \Longrightarrow f$ *is coercive* $\Longrightarrow f^{\infty}(v;\lambda) > 0,\ \forall\ v \neq 0,\ \forall\ \lambda \in \mathbb{R}$, $S_{\lambda}(f) \neq \emptyset \Longrightarrow f_q^{\infty}(v) > 0,\ \ \forall\ v \neq 0$.

Proof. (a): If $(v,t) \in \text{epi } f_q^\infty$ then

$$(\bar{x}, f(\bar{x})) + \lambda(v,t) \in \text{epi } f, \quad \forall\, \lambda > 0,$$

which means that $(v,t) \in (\text{epi } f)^\infty = \text{epi } f^\infty$.

(b): Let $\bar{x} \in S_\lambda(f)$ and $v \neq 0$. Obviously there exists a sequence $t_k \to +\infty$ as $k \to +\infty$ such that $\bar{x} + t_k v \notin S_\lambda(f)$ for all $k \in \mathbb{N}$. Thus,

$$f^\infty(v;\lambda) = \sup_{t>0} \sup_{x \in S_\lambda(f)} \frac{f(x+tv)-\lambda}{t} \geq \sup_{k\in\mathbb{N}} \frac{f(\bar{x}+t_k v)-\lambda}{t_k} > 0.$$

(c): The first implication is a well known fact, and the second follows from (b) and Proposition 5.1(c). ☐

Example 5.6. *This example shows the reverse implication in* (a) *in the previous proposition is not true in general:*

$$f(x) = \begin{cases} \dfrac{x}{1+x}, & \text{if} \quad x \geq 0, \\ +\infty, & \text{if} \quad x < 0. \end{cases} \qquad f^\infty(v) = \begin{cases} 0, & \text{if} \quad v \geq 0, \\ +\infty, & \text{if} \quad v < 0. \end{cases}$$

$$f^\infty(v;\bar{x}) = f'_+(\bar{x};v) = \begin{cases} \dfrac{1}{(1+\bar{x})^2}v, & \text{if} \quad v \geq 0, \\ +\infty, & \text{if} \quad v < 0. \end{cases} \qquad (\bar{x} \geq 0).$$

$$f_q^\infty(v) = \begin{cases} 1, & \text{if} \quad v > 0, \\ 0, & \text{if} \quad v = 0, \\ +\infty, & \text{if} \quad v < 0. \end{cases}$$

The continuity of $f'_+(\bar{x};\cdot)$ for extended real-valued functions f defined on $\mathbb{R}$ is expected out of 0, as shows next.

Proposition 5.7. *Let* $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ *be any function with* $\bar{x} \in f^{-1}(\mathbb{R})$. *Then,* $f'_+(\bar{x};\cdot)$ *is continuous in* $\mathbb{R} \setminus \{0\}$. *Consequently* $f'_+(\bar{x};u) = \overline{D}_e^R f(\bar{x};u)$ *for all* $u \neq 0$.

Proof. Let $u < 0$. Since $f'_+(\bar{x};u) = -uf'_+(\bar{x};-1)$, we have

$$f'_+(\bar{x};u) = \pm\infty \Longleftrightarrow f'_+(\bar{x};-1) = \pm\infty,$$

proving the continuity at $u < 0$ in case $f'_+(\bar{x};u) = \pm\infty$. Assume that $u_0 < 0$ and $f'_+(\bar{x};u_0) \in \mathbb{R}$, then for $-\infty < u < 0$,

$$f'_+(\bar{x};u) = f'_+(\bar{x};u_0\frac{u}{u_0}) = \frac{u}{u_0} f'_+(\bar{x};u_0) = \frac{f'_+(\bar{x};u_0)}{u_0}u.$$

This concludes with the proof that $f'_+(\bar{x};\cdot)$ is continuous at every $u < 0$. For $u > 0$ we proceed in a similar way. Thus the continuity in $\mathbb{R} \setminus \{0\}$ follows. ☐

The next example shows that one cannot expect the continuity of $f'_+(\bar{x};\cdot)$ at 0.

Example 5.8. *Take the function $f(x) = x^2$, then for any fixed $\bar{x} \in \mathbb{R}$, $f'_+(\bar{x};u) = +\infty$ for all $u \neq 0$ and $f'_+(\bar{x};0) = 0$; whereas $f^\infty(u) = f'_+(\bar{x};u)$ and $\overline{D}_e^R f(\bar{x};u) = +\infty$ for all $u \in \mathbb{R}$. Hence, even if f is convex, we may have $f^\infty \neq \overline{D}_e^R f(\bar{x};\cdot)$.*

The following result provides a full description of the boundedness of a sublevel set of quasiconvex and lsc functions defined on the real-line.

Theorem 5.9. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be quasiconvex and lsc at $\bar{x} \in f^{-1}(\mathbb{R})$. Then,*

$$\begin{aligned} S_{f(\bar{x})}(f) \text{ is bounded } &\iff f'_+(\bar{x};u) > 0 \ \forall\ u \neq 0 \iff \overline{D}_e^R f(\bar{x};u) > 0 \ \forall\ u \neq 0 \\ &\iff f^\infty(u;\bar{x}) > 0 \ \ \forall\ u \neq 0. \end{aligned}$$

Proof. The first equivalence is straightforward. The remaining implications are consequences of Propositions 5.3 and 5.7. ☐

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