# FINITE ELEMENT ANALYSIS OF A BENDING MOMENT FORMULATION FOR THE VIBRATION PROBLEM OF A NON-HOMOGENEOUS TIMOSHENKO BEAM

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Abstract. In this paper we analyze a low-order finite element method for approximating the vibration frequencies and modes of a non-homogeneous Timoshenko beam. We consider a formulation in which the bending moment is introduced as an additional unknown. Optimal order error estimates are proved for displacements, rotations, shear stress and bending moment of the vibration modes, as well as a double order of convergence for the vibration frequencies. These estimates are independent of the beam thickness, which leads to the conclusion that the method is locking free. For its implementation, displacements and rotations can be eliminated leading to a well posed generalized matrix eigenvalue problem for which the computer cost of its solution is similar to that of other classical formulations. We report numerical experiments which allow us to assess the performance of the method.

## 1. Introduction

This paper deals with the analysis of a finite element method to compute the vibration modes of an elastic non-homogeneous beam modeled by Timoshenko equations. Structural components with continuous and discontinuous variations of the geometry and of the physical parameters are common in buildings and bridges as well as in aircraft, cars, ships, etc. For that reason, it is important to know the vibration frequencies and modes of this kind of structures. This problem can be formulated as a spectral problem whose eigenvalues and eigenfunctions are related with the vibration frequencies and modes, respectively.

The Timoshenko theory to date is one of the most used models to approximate the deformation of a thin or moderately thick elastic beam [5, 9, 12, 18, 20, 26, 30]). It is well understood that standard finite elements applied to this model lead to wrong results when the thickness of the beam is small due to the so called *locking* phenomenon. To avoid locking, the most used techniques since long ago are based on reduced integration or mixed formulations (see [2]).

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In this paper, we present a rigorous analysis of a low-order finite element method to compute the vibration frequencies and modes of a non-homogeneous Timoshenko beam, by means of a mixed bending moment formulation. A similar method was recently introduced and analyzed for load problems in [23].

One advantage of such a formulation is that the bending moment and the shear stress are computed directly and not by means of a post-process, which might produce loss of accuracy. Moreover, the fact that these two quantities appear explicitly in the formulation could be useful to apply it to coupled problems in which the coupling involve these quantities. Another motivation for considering this one-dimensional problem is that it constitutes a stepping stone towards the more challenging goal of devising finite element spectral approximations for Reissner–Mindlin plates based on bending moments formulations. Let us remark that this kind of formulations have been recently proposed and analyzed in different frameworks for instance in the following references [1, 4, 6, 10, 11].

Numerical analysis of eigenvalue problems arising from the computation of the vibration modes for thin structures are not too many; among them we mention [15, 16, 17, 24, 25], where MITC-like methods for computing the vibration and buckling modes of beams and plates were analyzed. One reason for this is that the extension of mathematical results from load to vibration problems is not quite straightforward for mixed methods. In fact, Boffi et al. [7, 8] showed that eigenvalue problems for mixed formulations show peculiar features that make them substantially different from the same methods applied to the corresponding source problems. In particular, they showed that the standard inf-sup and ellipticity in the kernel conditions, which ensure convergence for the mixed formulation of source problems, are not enough to attain the same goal in the corresponding eigenvalue problem. Among the existing techniques to solve the vibration problem of Timoshenko beams, we can mention [21] where a mixed formulation in terms of displacement, rotation and shear stress has been proposed and analyzed for Timoshenko rods (which are of course applicable to Timoshenko beams).

In this paper, we consider the vibration problem for an elastic beam. We follow the approach proposed in [23] for the load problem. We introduce the bending moment together with the shear stress as new unknowns in the model (we note that the former usually represents a quantity of major interest in engineering applications), which together with the rotation and the transverse displacement lead us to a mixed variational formulation. Then, we introduce a solution operator whose eigenvalues are the reciprocals of the scaled squares of the vibration frequencies of the beam. For the numerical approximation, we use piecewise linear and continuous finite elements for the bending moments and shear stress and piecewise constants for the transverse displacement and the rotations. To study the convergence of the proposed method and obtain error estimates, we adapt the classical theory developed for non-compact operators in [13, 14]. We obtain optimal order error estimates in terms of the mesh size $h$ for the approximation of the vibration modes and a double order for the vibration frequencies. These estimates are fully independent of the beam thickness, which allows us to conclude that the method is locking-free.

Since we have included as additional variables the bending moment and the shear stress, one could think at first sight that the resulting method will be significantly more expensive than the classical ones, which are based only in displacement and rotation variables. However, we show that these last two variables can be eliminated in the resulting discrete problem without additional cost, which leads to an eigenvalue problem of the same size and sparseness as those of the classical methods.

The outline of the paper is as follows. In Section 2, we recall the vibration problem for a Timoshenko beam. In Section 3 we develop the mathematical analysis of the vibration problem. With this aim, we introduce a linear operator whose spectrum is related with the solution of the vibration problem. The resulting spectral problem is shown to be well posed. Its eigenvalues and eigenfunctions are proved to converge to the corresponding ones of the limit problem as the thickness of the plate goes to zero, which corresponds to a an Euler-Bernoulli beam model. We also prove in this section a regularity result for the eigenfunctions. In Section 4 we introduce the finite element discretization of the spectral problem and the discrete solution operator and prove some auxiliary results. In Section 5 we prove that the proposed numerical scheme provides a correct spectral approximation. We also establish error estimates for the eigenvalues and eigenfunctions. Finally, we present in Section 6 a set of numerical experiments to assess the performance of the method in order to confirm that the experimental rates of convergence are in accordance with the theory and to show that the method is completely locking-free. We also show in this section how the displacement and the rotation variables can be eliminated from the discrete eigenvalue problem, reducing its dimension to one half without affecting the sparseness, symmetry and positive definiteness of the matrices.

We use standard notations for Sobolev spaces, norms and seminorms. For $l \geq 0$ and I an open interval, $\|\cdot\|_{l,\mathrm{I}}$ stands for the norm of the Hilbertian Sobolev space $H^l(\mathrm{I})$, with the convention $H^0(\mathrm{I}) := L^2(\mathrm{I})$. Moreover, $\mathcal{D}(\mathrm{I})$ denotes the space of infinitely differentiable functions with compact support contained in I. Additionally, we will denote with $C$ a generic positive constant, possibly different at different occurrences, buy always independent of the beam thickness $t$ and the mesh parameter $h$ which will be introduced in the next sections.

Finally, given a linear bounded operator $T : X \to X$, defined on a Hilbert space $X$, we denote its spectrum by $\mathrm{sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and by $\boldsymbol{\rho}(T) := \mathbb{C} \setminus \mathrm{sp}(T)$ the resolvent set of $T$. Moreover, for any $z \in \boldsymbol{\rho}(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ denotes the resolvent operator of $T$ corresponding to $z$.

## 2. Timoshenko beam model

Let us consider an elastic beam which satisfies the Timoshenko hypotheses for the admissible displacements. We assume that the geometry and the physical parameters of the beam may change along the axial direction. The deformation of the beam is described in terms of the transverse displacement $w$ and the rotation of the transverse fibers $\beta$.

The equations for the vibration problem of a clamped Timoshenko beam reads as follows (see [27, 28, 29]):

*Find* $\omega > 0$ *and* $0 \neq (\beta, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} E\mathbb{I}\beta'\eta' + \int_{\mathrm{I}} GAk_c(\beta - w')(\eta - v') = \omega^2 \left( \int_{\mathrm{I}} \rho A w v + \int_{\mathrm{I}} \rho \mathbb{I} \beta \eta \right) \qquad \forall (\eta, v) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}), \tag{2.1}$$

where $\mathrm{I} := (0, L)$, $L$ being the length of the beam, $\omega$ is the angular vibration frequency, $E$ is the Young modulus, $\mathbb{I}$ the moment of inertia of the cross-section, $A$ the area of the cross-section, $\rho$ the mass density, $G := E/(2(1+\nu))$ the shear modulus, with $\nu$ being the Poisson ratio, and $k_c$ a correction factor. We consider that $E$, $\mathbb{I}$, $A$, $\rho$, $k_c$ and $\nu$ are piecewice smooth functions of the axial coordinate $x \in \mathrm{I}$, the most usual case being when all those coefficients are piecewise constant. Moreover, primes denote derivatives with respect to the axial coordinate $x$.

It is well known that standard finite element procedures, used in formulations such as (2.1) for very thin structures, are subject to numerical locking, a phenomenon induced by the difference of magnitude between the coefficients in front of the different terms (see [2]). The appropriate framework for analyzing this is obtained by rescaling formulation (2.1) so as to identify a family of problems with a well-posed limit as the thickness becomes infinitely small. With this aim, we introduce the following nondimensional parameter, characteristic of the thickness of the beam:

$$t^2 := \frac{1}{L}\int_{\mathrm{I}} \frac{\mathbb{I}}{AL^2}\,dx, \tag{2.2}$$

which we assume may take values in the range $(0, t_{\max}]$.

We define

$$\lambda := \frac{\omega^2}{t^2}, \quad \hat{\mathbb{I}} := \frac{\mathbb{I}}{t^3}, \quad \text{and} \quad \hat{A} := \frac{A}{t},$$

and assume that $\hat{\mathbb{I}}$ and $\hat{A}$ are bounded above and below far from zero by constants independent of the parameter $t$. Let us remark that, for instance, for a beam of rectangular section $b \times d$ with $b$ being a fixed length and $d$ the thickness of the beam, these values are constant and independent of $d$: $\hat{A} = 2\sqrt{3}bL$ and $\hat{\mathbb{I}} = 2\sqrt{3}bL^3$.

We also define

$$\mathbb{E} := E\hat{\mathbb{I}}, \qquad \kappa := G\hat{A}k_c, \qquad \mathrm{J} := \rho\hat{\mathbb{I}} \quad \text{and} \quad \mathrm{P} := \rho\hat{A},$$

so that provided the physical coefficients $E, \nu$ and $\rho$ are bounded above and below far from zero, we immediately obtain that there exist strictly positive constants $\overline{\mathbb{E}}, \underline{\mathbb{E}}, \overline{\kappa}, \underline{\kappa}, \overline{\mathrm{P}}, \underline{\mathrm{P}}, \overline{\mathrm{J}}$ and $\underline{\mathrm{J}}$ independent of $t$ such that

$$\begin{cases} \overline{\mathbb{E}} \geq \mathbb{E} \geq \underline{\mathbb{E}} > 0 & \forall x \in \mathrm{I}, \\ \overline{\kappa} \geq \kappa \geq \underline{\kappa} > 0 & \forall x \in \mathrm{I}, \\ \overline{\mathrm{P}} \geq \mathrm{P} \geq \underline{\mathrm{P}} > 0 & \forall x \in \mathrm{I}, \\ \overline{\mathrm{J}} \geq \mathrm{J} \geq \underline{\mathrm{J}} > 0 & \forall x \in \mathrm{I}. \end{cases} \tag{2.3}$$

Then, problem (2.1) can be equivalently written as follows:

*Find $\lambda > 0$ and $0 \neq (\beta, w) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ such that*

$$\int_{\mathrm{I}} \mathbb{E}\beta'\eta' + \frac{1}{t^2}\int_{\mathrm{I}} \kappa(\beta - w')(\eta - v') = \lambda\left(\int_{\mathrm{I}} \mathrm{P}wv + t^2\int_{\mathrm{I}} \mathrm{J}\beta\eta\right) \qquad \forall (\eta, v) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}). \tag{2.4}$$

It is easy to check that, as a consequence of (2.3), for each $t > 0$, the bilinear form on the left hand side of (2.4) is elliptic with an ellipticity constant independent of $t$.

Furthermore, because of the assumption on the physical and geometrical parameters, we have that $\mathbb{E}$, $\kappa$, $\mathrm{P}$ and $\mathrm{J}$ are piecewise smooth. More precisely, we assume that there exists a partition $0 = s_0 < \cdots < s_n = L$ of the interval I, with $s_1, \ldots, s_{n-1}$ being the points of discontinuity of $\mathbb{E}$, $\kappa$, P or J, such that if we denote by $S_i := (s_{i-1}, s_i)$, then, $\mathbb{E}_i := \mathbb{E}|_{S_i} \in W^{1,\infty}(S_i)$, $\kappa_i := \kappa|_{S_i} \in W^{1,\infty}(S_i)$, $\mathrm{P}_i := \mathrm{P}|_{S_i} \in W^{1,\infty}(S_i)$ and $\mathrm{J}_i := \mathrm{J}|_{S_i} \in W^{1,\infty}(S_i)$, $i = 1, \ldots, n$.

In this paper we will consider a bending moment formulation of the spectral problem (2.4). With this end, we introduce the scaled bending moment $\sigma := \mathbb{E}\beta'$ and shear stress $\gamma := t^{-2}\kappa(\beta - w')$ as new unknowns in the model and test (2.4) with $\eta, v \in \mathcal{D}(\mathrm{I})$ to obtain that $-\sigma' + \gamma = \lambda t^2 \mathrm{J}\beta$ and $\gamma' = \lambda \mathrm{P} w$.

Thus, problem (2.4) can be equivalently written as follows:

$$\begin{cases} \sigma = \mathbb{E}\beta' & \text{in} \quad \mathrm{I}, \\ -\sigma' + \gamma = \lambda t^2 \mathrm{J}\beta & \text{in} \quad \mathrm{I}, \\ \gamma = t^{-2}\kappa(\beta - w') & \text{in} \quad \mathrm{I}, \\ \gamma' = \lambda \mathrm{P} w & \text{in} \quad \mathrm{I}, \\ w(0) = \beta(0) = w(L) = \beta(L) = 0. \end{cases} \tag{2.5}$$

We introduce the following spaces that will be used in the sequel:

$$\mathbb{H} := H^1(\mathrm{I}) \times H^1(\mathrm{I}) \quad \text{and} \quad \mathbb{Q} := L^2(\mathrm{I}) \times L^2(\mathrm{I}).$$

We endow each space as well as $\mathbb{H} \times \mathbb{Q}$ with the corresponding product norm.

Testing the equations in (2.5) with adequate functions and integrating by parts, we obtain the following variational formulation of this problem:

*Find* $\lambda > 0$ *and* $0 \neq ((\sigma, \gamma), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$\int_{\mathrm{I}} \frac{\sigma\tau}{\mathbb{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa} + \int_{\mathrm{I}} \beta(\tau' - \xi) - \int_{\mathrm{I}} w\xi' = 0 \quad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.6}$$

$$\int_{\mathrm{I}} \eta(\sigma' - \gamma) - \int_{\mathrm{I}} v\gamma' = -\lambda \left( t^2 \int_{\mathrm{I}} \mathrm{J}\beta\eta \int_{\mathrm{I}} \mathrm{P} w v \right) \quad \forall (\eta, v) \in \mathbb{Q}. \tag{2.7}$$

We write this mixed problem in a more compact form as follows:

*Find* $\lambda > 0$ *and* $0 \neq ((\sigma, \gamma), (\beta, w)) \in \mathbb{H} \times \mathbb{Q}$ *such that*

$$a((\sigma, \gamma), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{2.8}$$

$$b((\sigma, \gamma), (\eta, v)) = -\lambda r((\beta, w), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}, \tag{2.9}$$

where the bilinear forms $a : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, $b : \mathbb{H} \times \mathbb{Q} \to \mathbb{R}$ and $r : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ are defined by

$$a((\sigma, \gamma), (\tau, \xi)) := \int_{\mathrm{I}} \frac{\sigma\tau}{\mathbb{E}} + t^2 \int_{\mathrm{I}} \frac{\gamma\xi}{\kappa}, \tag{2.10}$$

$$b((\tau, \xi), (\eta, v)) := \int_{\mathrm{I}} \eta(\tau' - \xi) - \int_{\mathrm{I}} v\xi', \tag{2.11}$$

and

$$r((\beta, w), (\eta, v)) := \left( t^2 \int_{\mathrm{I}} \mathrm{J}\beta\eta + \int_{\mathrm{I}} \mathrm{P} w v \right), \tag{2.12}$$

for all $(\sigma, \gamma), (\tau, \xi) \in \mathbb{H}$ and $(\beta, w), (\eta, v) \in \mathbb{Q}$.

It is easy to check that the so called *continuous kernel*

$$\mathbb{K} := \{(\tau, \xi) \in \mathbb{H} : \; b((\tau, \xi), (\eta, v)) = 0 \quad \forall (\eta, v) \in \mathbb{Q}\},$$

is given in this case by

$$\mathbb{K} = \{(\tau, \tau') : \tau \in \mathbb{P}_1(\mathrm{I})\}.$$

The following lemmas, which have been proved in [23, Lemmas 2.1 and 2.2] show that the *ellipticity in the kernel* and *inf-sup* classical conditions of mixed problems holds true for (2.8)–(2.9).

**Lemma 2.1.** *There exists $\alpha > 0$ independent of $t$ such that*

$$a((\tau,\xi),(\tau,\xi)) \geq \alpha \|(\tau,\xi)\|_{\mathbb{H}}^2 \qquad \forall (\tau,\xi) \in \mathbb{K}.$$

**Lemma 2.2.** *There exists $C > 0$ independent of $t$ such that*

$$\sup_{0\neq(\tau,\xi)\in\mathbb{H}} \frac{b((\tau,\xi),(\eta,v))}{\|(\tau,\xi)\|_{\mathbb{H}}} \geq C\|(\eta,v)\|_{\mathbb{Q}} \qquad \forall (\eta,v) \in \mathbb{Q}.$$

**Remark 2.1.** *We note that the eigenvalues of problem* (2.8)–(2.9) *are strictly positive. Indeed, it is easy to check that*

$$\lambda = \frac{a((\sigma,\gamma),(\sigma,\gamma))}{r((\beta,w),(\beta,w))} \geq 0;$$

*moreover $\lambda = 0$ implies $(\sigma,\gamma) = 0$, so that from* (2.8) *and Lemma 2.2, we have that* $(\beta, w) = 0$.

The goal of this paper is to propose and analyze a finite element method to solve the spectral problem (2.8)–(2.9) and to obtain accurate approximations of the eigenvalues $\lambda$ (from which we obtain the angular vibration frequencies $\omega$ of the beam) and the associated eigenfunctions.

## 3. Analysis of the spectral problem

Before introducing the numerical method, we define the linear operator corresponding to the source problem associated with the spectral problem (2.8)–(2.9) and prove some properties that will be useful for the subsequent convergence analysis:

*Given $(g,f) \in \mathbb{Q}$, find $((\hat\sigma,\hat\gamma),(\hat\beta,\hat w)) \in \mathbb{H}\times\mathbb{Q}$ such that*

$$a((\hat\sigma,\hat\gamma),(\tau,\xi)) + b((\tau,\xi),(\hat\beta,\hat w)) = 0 \qquad \forall (\tau,\xi) \in \mathbb{H}, \tag{3.1}$$

$$b((\hat\sigma,\hat\gamma),(\eta,v)) = -r((g,f),(\eta,v)) \qquad \forall (\eta,v) \in \mathbb{Q}. \tag{3.2}$$

As a consequence of Lemmas 2.1 and 2.2, this problem is well posed (see, for instance, [19, Section II.1.1]) and there exists a constant $C > 0$, independent of $t$, such that

$$\|\hat w\|_{0,\mathrm{I}} + \|\hat\beta\|_{0,\mathrm{I}} + \|\hat\sigma\|_{1,\mathrm{I}} + \|\hat\gamma\|_{1,\mathrm{I}} \leq C(t^2\|g\|_{0,\mathrm{I}} + \|f\|_{0,\mathrm{I}}) \leq C\|(g,f)\|_{\mathbb{Q}}.$$

Thus, we are able to introduce the following bounded linear operator, which is called the *solution operator*:

$$T_t : \mathbb{Q} \to \mathbb{Q},$$
$$(g,f) \mapsto (\hat\beta,\hat w).$$

It is easy to check that $(\mu,(\beta,w))$, with $\mu \neq 0$, is an eigenpair of $T_t$ (i.e., $T_t(\beta,w) = \mu(\beta,w)$) if and only if there exist $(\sigma,\gamma) \in \mathbb{H}$ such that, for $\lambda = 1/\mu$, $(\lambda,(\sigma,\gamma),(\beta,w))$ is a solution of problem (2.8)–(2.9). We recall that these eigenvalues are strictly positive (cf. Remark 2.1). Our aim is to approximate the smallest eigenvalues of problem (2.8)–(2.9), which correspond to the largest eigenvalues of the operator $T_t$.

This operator is self-adjoint with respect to the inner product $r(\cdot,\cdot)$ in $\mathbb{Q}$. In fact, given $(g,f),(\tilde g,\tilde f) \in \mathbb{Q}$, let $((\hat\sigma,\hat\gamma),(\hat\beta,\hat w)),((\tilde\sigma,\tilde\gamma),(\tilde\beta,\tilde w)) \in \mathbb{H}\times\mathbb{Q}$ be the solutions to problem (3.1)–(3.2), with right hand side $(g, f)$ and $(\tilde{g}, \tilde{f})$, respectively, so that $T_t(g, f) = (\hat{\beta}, \hat{w})$ and $T_t(\tilde{g}, \tilde{f}) = (\tilde{\beta}, \tilde{w})$. Then, using the symmetry of the bilinear forms $a(\cdot, \cdot)$ and $r(\cdot, \cdot)$, we have

$$\begin{aligned} r((g, f), T_t(\tilde{g}, \tilde{f})) =& r((g, f), (\tilde{\beta}, \tilde{w})) \\ =& -\Big( a((\hat{\sigma}, \hat{\gamma}), (\tilde{\sigma}, \tilde{\gamma})) + b((\tilde{\sigma}, \tilde{\gamma}), (\hat{\beta}, \hat{w})) + b((\hat{\sigma}, \hat{\gamma}), (\tilde{\beta}, \tilde{w})) \Big) \\ =& r((\tilde{g}, \tilde{f}), (\hat{\beta}, \hat{w})) \\ =& r(T_t(g, f), (\tilde{g}, \tilde{f})). \end{aligned}$$

The operator $T_t$ is also compact. To prove this we resort to the following additional regularity result, which has been proved in [23, Proposition 2.1].

**Proposition 3.1.** *Given* $(g, f) \in \mathbb{Q}$. *Let* $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ *be the unique solution to problem* (3.1)–(3.2). *Then, there exists a constant* $C > 0$ *independent of* $t$, $g$ *and* $f$ *such that*

$$\|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \|\hat{\gamma}\|_{1,\mathrm{I}} \leq C \|(g, f)\|_{\mathbb{Q}}.$$

Hence, as a consequence of the compact inclusion $H^1(\mathrm{I}) \hookrightarrow L^2(\mathrm{I})$, $T_t$ is a compact operator. Then, we know that the spectrum of $T_t$ satisfies $\mathrm{sp}(T_t) = \{0\} \cup \{\mu_n : n \in \mathbb{N}\}$, where $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of positive eigenvalues which converges to zero, the multiplicity of each non-zero eigenvalue being finite. Moreover, additional regularity of the eigenfunctions holds as a consequence of the following improved form of Proposition 3.1, which has been proved in [23, Remark 2.1].

**Proposition 3.2.** *Let* $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ *be the solution of problem* (3.1)–(3.2). *If* $g|_{S_i}, f|_{S_i} \in H^1(S_i), i = 1, \ldots, n$, *then, there exists* $C > 0$ *independent of* $t$ *such that*

$$\begin{aligned} \|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \left( \sum_{i=1}^{n} \|\hat{\sigma}''\|^2_{0,S_i} \right)^{1/2} + \|\hat{\gamma}\|_{1,\mathrm{I}} + \left( \sum_{i=1}^{n} \|\hat{\gamma}''\|^2_{0,S_i} \right)^{1/2} \\ \leq C \left( \|g\|^2_{0,\mathrm{I}} + \|f\|^2_{0,\mathrm{I}} + \sum_{i=1}^{n} \left( \|g'\|^2_{0,S_i} + \|f'\|^2_{0,S_i} \right) \right)^{1/2}. \end{aligned}$$

As a consequence of this result and Proposition 3.1, we easily obtain the following additional regularity for the eigenfunctions of problem (2.8)–(2.9).

**Corollary 3.1.** *Let* $(\lambda, (\sigma, \gamma, \beta, w))$ *be a solution of problem* (2.8)–(2.9). *Then, there exists* $C > 0$ *independent of* $t$ *such that*

$$\begin{aligned} \|w\|_{1,\mathrm{I}} + \|\beta\|_{1,\mathrm{I}} + \|\sigma\|_{1,\mathrm{I}} + \left( \sum_{i=1}^{n} \|\sigma''\|^2_{0,S_i} \right)^{1/2} + \|\gamma\|_{1,\mathrm{I}} + \left( \sum_{i=1}^{n} \|\gamma''\|^2_{0,S_i} \right)^{1/2} \\ \leq C \lambda \|(\beta, w)\|_{\mathbb{Q}}. \end{aligned}$$

The rest of this section is devoted to prove the convergence of the operator $T_t$ as $t$ goes to zero. For this purpose, we introduce the *limit problem* that we write as follows:

*Given $f \in L^2(\mathrm{I})$, find $((\sigma_0,\gamma_0),(\beta_0,w_0)) \in \mathbb{H}\times\mathbb{Q}$ such that*

$$\int_{\mathrm{I}} \frac{\sigma_0\tau}{\mathbb{E}} + \int_{\mathrm{I}} \beta_0(\tau'-\xi) - \int_{\mathrm{I}} w_0\xi' = 0 \qquad \forall(\tau,\xi)\in\mathbb{H}, \tag{3.3}$$

$$\int_{\mathrm{I}} \eta(\sigma_0'-\gamma_0) - \int_{\mathrm{I}} v\gamma_0' = -\int_{\mathrm{I}} \mathrm{P} f v \qquad \forall(\eta,v)\in\mathbb{Q}. \tag{3.4}$$

This is a mixed formulation of the load problem of an Euler-Bernoulli beam. Repeating the arguments used in the proof of [23, Theorem 2.3], we have that problem (3.3)–(3.4) is well posed. Moreover, the proof of Proposition 3.1 holds for $t=0$, too. Thus, the solution of problem (3.3)–(3.4) satisfies the following regularity result: There exists a constant $C>0$ independent of $f$ such that

$$\|w_0\|_{1,\mathrm{I}} + \|\beta_0\|_{1,\mathrm{I}} + \|\gamma_0\|_{1,\mathrm{I}} + \|\sigma_0\|_{1,\mathrm{I}} \le C\|f\|_{0,\mathrm{I}}. \tag{3.5}$$

Now, let $T_0$ be the following bounded linear operator:

$$\begin{aligned} T_0 &: \mathbb{Q}\to\mathbb{Q},\\ &(g,f)\mapsto(\beta_0,w_0). \end{aligned}$$

We note that, because of (3.5), $T_0$ is a compact operator. Moreover, it is self-adjoint. In fact, essentially the same arguments used to prove that $T_t$ is self-adjoint holds for $T_0$, too. Also the arguments of Remark 2.1 hold in this case and allow us to show that the eigenvalues of $T_0$ has to be strictly positive. So, the spectrum also satisfies $\mathrm{sp}(T_0) = \{0\}\cup\{\mu_n^0 : n\in\mathbb{N}\}$, where $\{\mu_n^0\}_{n\in\mathbb{N}}$ is a sequence of positive eigenvalues which converges to zero, the multiplicity of each non-zero eigenvalue being finite.

The following lemma states the convergence in norm of $T_t$ to $T_0$.

**Lemma 3.1.** *There exists a positive constant $C$ independent of $t$ such that*

$$\|(T_t - T_0)(g,f)\|_{\mathbb{Q}} \le Ct\|(g,f)\|_{\mathbb{Q}}.$$

*Proof.* Substracting (3.3)–(3.4) from (3.1)–(3.2), we obtain

$$\int_{\mathrm{I}} \frac{(\hat\sigma-\sigma_0)\tau}{\mathbb{E}} + \int_{\mathrm{I}} (\hat\beta-\beta_0)(\tau'-\xi) - \int_{\mathrm{I}} (\hat w - w_0)\xi' = -t^2\int_{\mathrm{I}} \frac{\hat\gamma\xi}{\kappa} \quad \forall(\tau,\xi)\in\mathbb{H},$$

$$\int_{\mathrm{I}} \eta((\hat\sigma'-\sigma_0') - (\hat\gamma-\gamma_0)) - \int_{\mathrm{I}} v(\hat\gamma'-\gamma_0') = -t^2\int_{\mathrm{I}} \mathrm{J} g\eta \quad \forall(\eta,v)\in\mathbb{Q}.$$

Testing the system above with $\tau=\hat\sigma-\sigma_0$, $\xi=\hat\gamma-\gamma_0$, $\eta=\hat\beta-\beta_0$ and $v=\hat w-w_0$ and subtracting the resulting equations, we obtain

$$\int_{\mathrm{I}} \frac{(\hat\sigma-\sigma_0)^2}{\mathbb{E}} = t^2\int_{\mathrm{I}} \mathrm{J} g(\hat\beta-\beta_0) - t^2\int_{\mathrm{I}} \frac{\hat\gamma(\hat\gamma-\gamma_0)}{\kappa}.$$

Thus, by using (2.3), Proposition 3.1 and (3.5), we have

$$\begin{aligned} \|\hat\sigma-\sigma_0\|_{0,\mathrm{I}}^2 &\le Ct^2(\|g\|_{0,\mathrm{I}}\|\hat\beta-\beta_0\|_{0,\mathrm{I}} + \|\hat\gamma\|_{0,\mathrm{I}}\|\hat\gamma-\gamma_0\|_{0,\mathrm{I}})\\ &\le Ct^2\Big(\|g\|_{0,\mathrm{I}}(\|\hat\beta\|_{0,\mathrm{I}} + \|\beta_0\|_{0,\mathrm{I}}) + \|\hat\gamma\|_{0,\mathrm{I}}(\|\hat\gamma\|_{0,\mathrm{I}} + \|\gamma_0\|_{\mathrm{I}})\Big)\\ &\le Ct^2\|(g,f)\|_{\mathbb{Q}}^2. \end{aligned}$$

Now, we use Lemma 2.2, (3.1), (3.3), (2.3) and the above inequality, to obtain

$$\begin{aligned}\|(\hat{\beta},\hat{w})-(\beta_0,w_0)\|_{\mathbb{Q}} &\le C \sup_{0\neq(\tau,\xi)\in\mathbb{H}} \frac{b((\tau,\xi),(\hat{\beta}-\beta_0,\hat{w}-w_0))}{\|(\tau,\xi)\|_{\mathbb{H}}}\\ &= C \sup_{0\neq(\tau,\xi)\in\mathbb{H}} \frac{-\displaystyle\int_{\mathrm{I}}\frac{(\hat{\sigma}-\sigma_0)\tau}{\mathbb{E}} - t^2\int_{\mathrm{I}}\frac{\hat{\gamma}\xi}{\kappa}}{\|(\tau,\xi)\|_{\mathbb{H}}}\\ &\le Ct\|(g,f)\|_{\mathbb{Q}},\end{aligned}$$

which allows us to complete the proof. □

As a consequence of this lemma, standard properties of separation of isolated parts of the spectrum (see, for instance [22]) yield the following result.

**Lemma 3.2.** *Let $\mu_0 > 0$ be an eigenvalue of $T_0$ of multiplicity $m$. Let $D$ be any disc in the complex plane centered at $\mu_0$ and containing no other element of the spectrum of $T_0$. Then, for $t$ small enough, $D$ contains exactly $m$ eigenvalues of $T_t$ (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_0 > 0$ of $T_0$ is a limit of eigenvalues $\mu$ of $T_t$, as $t$ goes to zero.*

## 4. Spectral approximation

We will study in this section, the numerical approximation of the eigenvalue problem (2.8)–(2.9). With this aim, first we consider a family of partitions of I,

$$\mathcal{T}_h : 0 = x_0 < \cdots < x_N = L,$$

which are all refinements of the initial partition $0 = s_0 < \cdots < s_n = L$. Recall that $s_1, \ldots, s_{n-1}$ are the points of discontinuity of any of the coefficients, $\mathbb{E}$, $\kappa$, P or J. We denote $\mathrm{I}_j := (x_{j-1}, x_j)$, $j = 1, \ldots, N$, and the largest subinterval length is denoted $h := \max_{1\le j\le N}(x_j - x_{j-1})$. Notice that for any mesh $\mathcal{T}_h$, each $\mathrm{I}_j$ is contained in one of the subinterval $S_i$, $i = 1, \ldots, n$, where the physical coefficients are smooth.

We consider the space of piecewise linear continuous finite elements:

$$W_h := \{\xi_h \in H^1(\mathrm{I}) \ :\ \xi_h|_{\mathrm{I}_j} \in \mathbb{P}_1(\mathrm{I}_j),\ j = 1, \ldots, N\}.$$

For $\xi \in H^1(\mathrm{I})$ let $\mathcal{L}_h\xi \in W_h$ be its Lagrange interpolant. We recall that

$$\|\xi - \mathcal{L}_h\xi\|_{1,\mathrm{I}} \le Ch\left(\sum_{j=1}^{N}\|\xi''\|_{0,\mathrm{I}_j}^2\right)^{1/2} \qquad \forall \xi|_{\mathrm{I}_j} \in H^2(\mathrm{I}_j),\ j = 1, \ldots, N. \tag{4.1}$$

We will also consider the space of piecewise constant functions:

$$Z_h := \{v_h \in L^2(\mathrm{I}) \ :\ v_h|_{\mathrm{I}_j} \in \mathbb{P}_0(\mathrm{I}_j),\ j = 1, \ldots, N\},$$

and the $L^2$-projector onto $Z_h$:

$$\begin{aligned}&\mathcal{P}_h : L^2(\mathrm{I}) \to Z_h,\\ &\qquad v \mapsto \mathcal{P}_h v \in Z_h \ :\ \int_{\mathrm{I}}(v - \mathcal{P}_h v)q_h = 0 \quad \forall q_h \in Z_h.\end{aligned}$$

It is well known that

$$\|v - \mathcal{P}_h v\|_{0,\mathrm{I}} \le Ch|v|_{1,\mathrm{I}} \qquad \forall v \in H^1(\mathrm{I}). \tag{4.2}$$

Defining $\mathbb{H}_h := W_h \times W_h$ and $\mathbb{Q}_h := Z_h \times Z_h$, the discretization of problem (2.8)–(2.9) reads as follows:

*Find* $\lambda_h > 0$ *and* $0 \neq ((\sigma_h, \gamma_h), (\beta_h, w_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ *such that*

$$a((\sigma_h, \gamma_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\beta_h, w_h)) = 0 \qquad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{4.3}$$

$$b((\sigma_h, \gamma_h), (\eta_h, v_h)) = -\lambda_h r((\beta_h, w_h), (\eta_h, v_h)) \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{4.4}$$

As in the continuous case, we introduce for the analysis the *discrete solution operator*

$$\begin{aligned} T_{th} :\ & \mathbb{Q} \to \mathbb{Q} \\ & (g, f) \mapsto (\hat{\beta}_h, \hat{w}_h), \end{aligned}$$

where $((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h)) \in \mathbb{H}_h \times \mathbb{Q}_h$ is the solution of the corresponding discrete source problem:

$$a((\hat{\sigma}_h, \hat{\gamma}_h), (\tau_h, \xi_h)) + b((\tau_h, \xi_h), (\hat{\beta}_h, \hat{w}_h)) = 0 \qquad \forall (\tau_h, \xi_h) \in \mathbb{H}_h, \tag{4.5}$$

$$b((\hat{\sigma}_h, \hat{\gamma}_h), (\eta_h, v_h)) = -r((g, f), (\eta_h, v_h)) \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h. \tag{4.6}$$

It is easy to check that the *discrete kernel*

$$\mathbb{K}_h := \{(\tau_h, \xi_h) \in \mathbb{H}_h :\ b((\tau_h, \xi_h), (\eta_h, v_h)) = 0 \quad \forall (\eta_h, v_h) \in \mathbb{Q}_h\},$$

coincides with the continuous one $\mathbb{K} = \{(\tau, \tau') : \tau \in \mathbb{P}_1(\mathrm{I})\}$. Therefore, the ellipticity estimate from Lemma 2.1 holds true for $(\tau_h, \xi_h) \in \mathbb{K}_h$ with the same constant $\alpha$ independent of $t$ and $h$. Moreover, the discrete *inf-sup* condition

$$\sup_{0 \neq (\tau_h, \xi_h) \in \mathbb{H}_h} \frac{b((\tau_h, \xi_h), (\eta_h, v_h))}{\|(\tau_h, \xi_h)\|_{\mathbb{H}}} \geq C \|(\eta_h, v_h)\|_{\mathbb{Q}} \qquad \forall (\eta_h, v_h) \in \mathbb{Q}_h \tag{4.7}$$

holds true with a positive constant $C$ independent of $t$ and $h$ (see [23, Lemma 3.2]). Consequently, the discrete mixed problem (4.5)–(4.6) has a unique solution and there holds

$$\|(\hat{\sigma}_h, \hat{\gamma}_h)\|_{\mathbb{H}} + \|(\hat{\beta}_h, \hat{w}_h)\|_{\mathbb{Q}} \leq C \|(g, f)\|_{\mathbb{Q}}, \tag{4.8}$$

once more with a positive constant $C$ independent of $t$ and $h$. Hence, $T_{th}$ is a well defined bounded linear operator.

**Remark 4.1.** *The above estimate can be improved as follows:*

$$\|(\hat{\sigma}_h, \hat{\gamma}_h)\|_{\mathbb{H}}^2 + \|(\hat{\beta}_h, \hat{w}_h)\|_{\mathbb{Q}}^2 \leq C \left( t^2 \int_{\mathrm{I}} \mathrm{J} |g|^2 + \int_{\mathrm{I}} \mathrm{P} |f|^2 \right), \tag{4.9}$$

*always with a positive constant $C$ independent of $t$ and $h$. In fact, this follows easily from taking into account the particular form of the right hand side of problem* (4.5)–(4.6) *and using, for instance,* [19, Remark II.1.3].

As in the continuous case, $(\mu_h, (\beta_h, w_h))$, with $\mu_h \neq 0$, is an eigenpar of $T_{th}$ if and only if there exists $(\sigma_h, \gamma_h) \in \mathbb{H}_h$ such that, for $\lambda_h = 1/\mu_h$, $(\lambda_h, (\sigma_h, \gamma_h, \beta_h, w_h))$ is a solution of problem (4.3)–(4.4). Moreover, the same arguments used for $T_t$ allow us to show that the operator $T_{th}$ is self-adjoint with respect to the inner product $r(\cdot, \cdot)$.

Our next goal is to obtain a spectral characterizaton for problem (4.3)–(4.4):

**Lemma 4.1.** *The variational problem* (4.3)–(4.4) *has exactly* $\dim \mathbb{Q}_h$ *eigenvalues, repeated according to their respective multiplicities. All of them are real and positive.*

*Proof.* Taking particular bases of the discrete spaces, problem (4.3)–(4.4) can be written in matrix form as follows:

$$
\begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & -\mathbf{D} & -\mathbf{E} \\ \mathbf{E}^{\mathrm{t}} & -\mathbf{D}^{\mathrm{t}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{E}^{\mathrm{t}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\sigma}}_h \\ \vec{\boldsymbol{\gamma}}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix} = -\lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\sigma}}_h \\ \vec{\boldsymbol{\gamma}}_h \\ \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix}, \tag{4.10}
$$

where $\vec{\boldsymbol{\sigma}}_h$, $\vec{\boldsymbol{\gamma}}_h$, $\vec{\boldsymbol{\beta}}_h$ and $\vec{\boldsymbol{w}}_h$ denote the vectors whose entries are the components in those basis of $\sigma_h$, $\gamma_h$, $\beta_h$ and $w_h$, respectively.

Now we define the following matrices

$$
\mathbf{R} := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}, \quad \mathbf{S} := \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ -\mathbf{D} & -\mathbf{E} \end{bmatrix}, \quad \mathbf{T} := \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix},
$$

and vectors

$$
\vec{\boldsymbol{u}}_h := \begin{bmatrix} \vec{\boldsymbol{\sigma}}_h \\ \vec{\boldsymbol{\gamma}}_h \end{bmatrix}, \quad \vec{\boldsymbol{v}}_h := \begin{bmatrix} \vec{\boldsymbol{\beta}}_h \\ \vec{\boldsymbol{w}}_h \end{bmatrix},
$$

to rewrite (4.10) as follows:

$$
\begin{bmatrix} \mathbf{R} & \mathbf{S} \\ \mathbf{S}^{\mathrm{t}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{v}}_h \end{bmatrix} = -\lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{u}}_h \\ \vec{\boldsymbol{v}}_h \end{bmatrix}. \tag{4.11}
$$

The system above is equivalent to solve

$$
\begin{aligned} \mathbf{R}\vec{\boldsymbol{u}}_h + \mathbf{S}\vec{\boldsymbol{v}}_h &= \mathbf{0} \\ \mathbf{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h &= -\lambda_h \mathbf{T}\vec{\boldsymbol{v}}_h. \end{aligned}
$$

Since $\mathbf{A}, \mathbf{C}, \mathbf{P}$ and $\mathbf{Q}$ are scaled mass matrices, it is easy to check that all of them, as well as $\mathbf{R}$ and $\mathbf{T}$, are symmetric and positive definite (although not uniformly in $t$) and hence invertible. Thus, from the first equation above we have that $\vec{\boldsymbol{u}}_h = -\mathbf{R}^{-1}\mathbf{S}\vec{\boldsymbol{v}}_h$ and substituting this into the second equation we obtain:

$$
(\mathbf{S}^{\mathrm{t}}\mathbf{R}^{-1}\mathbf{S})\vec{\boldsymbol{v}}_h = \lambda_h \mathbf{T}\vec{\boldsymbol{v}}_h. \tag{4.12}
$$

Conversely, if $(\lambda_h, \vec{\boldsymbol{v}}_h)$ is an eigenpar of the above problem, by defining $\vec{\boldsymbol{u}}_h := -\mathbf{R}^{-1}\mathbf{S}\vec{\boldsymbol{v}}_h$, we have that $(\lambda_h, (\vec{\boldsymbol{u}}_h, \vec{\boldsymbol{v}}_h))$ is an eigenpar of (4.11).

The eigenvalue problem (4.12) is well posed because $\mathbf{T}$ is symmetric and positive definite. The same holds true for $\mathbf{S}^{\mathrm{t}}\mathbf{R}^{-1}\mathbf{S}$. In fact, this matrix is clearly symmetric and semipositive definite. Moreover, it is positive definite because $\mathbf{S}\vec{\boldsymbol{v}}_h = \mathbf{0}$ implies $\vec{\boldsymbol{v}}_h = \mathbf{0}$, as a consequence of (4.7). Then, the generalized eigenvalue problem is well posed and all its eigenvalues are real and positive. Therefore, the number of eigenvalues of problem (4.10) equals the number of eigenvalues of this problem, namely $\dim \mathbb{Q}_h$, and we complete the proof. □

In order to prove that the solutions of the discrete problem (4.5)–(4.6) converge to those of the continuous problem (3.1)–(3.2), the standard procedure would be to show that $T_{th}$ converges in norm to $T_t$ as $h$ goes to zero. However, such a proof does not seem straightforward in our case. In fact, $\|(T_t - T_{th})(g, f)\|_{\mathbb{Q}}$ is bounded for $g$ and $f$ piecewise smooth as follows:

$$
\|(T_t - T_{th})(g, f)\|_{\mathbb{Q}} \leq Ch \left( \|g\|_{0,\mathrm{I}}^2 + \|f\|_{0,\mathrm{I}}^2 + \sum_{i=1}^{n} \left( \|g'\|_{0,S_i}^2 + \|f'\|_{0,S_i}^2 \right) \right)^{1/2},
$$

but the last terms above are not bounded in general by $\|(g, f)\|_{\mathbb{Q}}$. To circumvent this drawback, we will resort instead to the spectral theory from [13] and [14].

In spite of the fact that the main use of this theory is when $T_t$ is a noncompact operator, it can also be applied to compact $T_t$ and we will show that in our case it works.

The remainder of this section is devoted to prove the following properties which will be used in the next section:

P1. $\|T_t - T_{th}\|_h := \sup_{0 \neq (g_h, f_h) \in \mathbb{Q}_h} \dfrac{\|(T_t - T_{th})(g_h, f_h))\|_{\mathbb{Q}}}{\|(g_h, f_h)\|_{\mathbb{Q}}} \to 0 \quad \text{as} \quad h \to 0.$

P2. $\forall (\eta, v) \in \mathbb{Q}, \inf_{(\eta_h, v_h) \in \mathbb{Q}_h} \|(\eta, v) - (\eta_h, v_h)\|_{\mathbb{Q}} \to 0 \quad \text{as} \quad h \to 0.$

Property P2 is a consequence of the fact that piecewise constant functions are dense in $L^2(\mathrm{I})$. Regarding property P1, we have the following result.

**Lemma 4.2.** *Property P1 holds true; moreover, there exists a constant $C > 0$ independent of $t$ and $h$ such that*

$$\|T_t - T_{th}\|_h \leq Ch.$$

*Proof.* Given $(g_h, f_h) \in \mathbb{Q}_h$, let $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H} \times \mathbb{Q}$ and $((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ be the solutions of problems (3.1)–(3.2) and (4.5)–(4.6), respectively, in both cases with $(g, f) = (g_h, f_h)$. Therefore, $(\hat{\beta}, \hat{w}) = T_t(g_h, f_h)$ and $(\hat{\beta}_h, \hat{w}_h) = T_{th}(g_h, f_h)$.

The same arguments used in the proof of Proposition 3.2 (see [23, Remark 2.1]) allow us to show that there exists a constant $C > 0$, independent of $t$, $g_h$ and $f_h$, such that

(4.13)
$$\|\hat{w}\|_{1,\mathrm{I}} + \|\hat{\beta}\|_{1,\mathrm{I}} + \|\hat{\sigma}\|_{1,\mathrm{I}} + \left(\sum_{j=1}^{N} \|\hat{\sigma}''\|^2_{0,\mathrm{I}_j}\right)^{1/2} + \|\hat{\gamma}\|_{1,\mathrm{I}} + \left(\sum_{j=1}^{N} \|\hat{\gamma}''\|^2_{0,\mathrm{I}_j}\right)^{1/2} \leq C\|(g_h, f_h)\|_{\mathbb{Q}},$$

where we have also used that $g'_h|_{\mathrm{I}_j} = f'_h|_{\mathrm{I}_j} = 0$, because $g_h$ and $f_h$ are piecewise constant. On the other hand, since problem (4.5)–(4.6) is just the finite element discretization of problem (3.1)–(3.2), using again the results from [23] (in particular, Theorem 3.3), we have that

$$\begin{aligned}
\|(T_t - T_{th})(g_h, f_h)\|_{\mathbb{Q}} \leq & \|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\hat{\sigma}_h, \hat{\gamma}_h), (\hat{\beta}_h, \hat{w}_h))\|_{\mathbb{H} \times \mathbb{Q}} \\
\leq & C \inf_{((\tau_h, \xi_h), (\eta_h, v_h)) \in \mathbb{H}_h \times \mathbb{Q}_h} \|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\tau_h, \xi_h), (\eta_h, v_h))\|_{\mathbb{H} \times \mathbb{Q}} \\
\leq & C\|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\mathcal{L}_h\hat{\sigma}, \mathcal{L}_h\hat{\gamma}), (\mathcal{P}_h\hat{\beta}, \mathcal{P}_h\hat{w}))\|_{\mathbb{H} \times \mathbb{Q}} \\
\leq & Ch\|(g_h, f_h)\|_{\mathbb{Q}},
\end{aligned}$$

where, for the last inequality we have used the error estimates (4.1) and (4.2) together with the additional regularity result (4.13). Thus, the proof follows from the definition of $\|T_t - T_{th}\|_h$ and the above estimate. □

## 5. Convergence and error estimates.

In this section we will adapt the arguments from [13, 14] to prove convergence of our spectral approximation as well as to obtain error estimates for the approximate eigenvalues and eigenfunctions. With this end, we will use the following results.

**Lemma 5.1.** *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \operatorname{sp}(T_0) = \emptyset$. Then, there exist strictly positive constants $t_0$ and $C$ such that, $\forall t < t_0$, $F \cap \operatorname{sp}(T_t) = \emptyset$ and*

$$\|R_z(T_t)\| := \sup_{0 \neq (\eta,v) \in \mathbb{Q}} \frac{\|R_z(T_t)(\eta,v)\|_{\mathbb{Q}}}{\|(\eta,v)\|_{\mathbb{Q}}} \leq C \quad \forall z \in F.$$

*Proof.* We consider the mapping $z \to \|(zI - T_0)^{-1}\|$, which is continuous for all $z \in \boldsymbol{\rho}(T_0)$. It is clear that this mapping goes to zero as $|z| \to \infty$. Hence, if $F \subset \boldsymbol{\rho}(T_0)$ is a closed subset, then the mapping above attains its maximum. Let $\hat{C} := \max_{z \in F} \|(zI - T_0)^{-1}\|$; there holds

$$\|(zI - T_0)(\eta,v)\|_{\mathbb{Q}} \geq \frac{1}{\hat{C}} \|(\eta,v)\|_{\mathbb{Q}} \quad \forall (\eta,v) \in \mathbb{Q} \quad \forall z \in F.$$

Next, we observe that

$$\|(zI - T_0)(\eta,v)\|_{\mathbb{Q}} \leq \|(zI - T_t)(\eta,v)\|_{\mathbb{Q}} + \|(T_t - T_0)(\eta,v)\|_{\mathbb{Q}}.$$

Moreover, according to Lemma 3.1, there exists $t_0 > 0$ such that, for all $t < t_0$,

$$\|(T_t - T_0)(\eta,v)\|_{\mathbb{Q}} \leq \frac{1}{2\hat{C}} \|(\eta,v)\|_{\mathbb{Q}} \qquad \forall (\eta,v) \in \mathbb{Q}.$$

Therefore, for all $(\eta,v) \in \mathbb{Q}$, for all $z \in F$ and for all $t < t_0$,

$$\|(zI - T_t)(\eta,v)\|_{\mathbb{Q}} \geq \|(zI - T_0)(\eta,v)\|_{\mathbb{Q}} - \|(T_t - T_0)(\eta,v)\|_{\mathbb{Q}} \geq \frac{1}{2\hat{C}} \|(\eta,v)\|_{\mathbb{Q}}.$$

Consequently, $z$ is not an eigenvalue of $T_t$. Moreover, $z \neq 0$, because $0 \notin \boldsymbol{\rho}(T_0)$. Hence, since the spectrum of $T_t$ consists of eigenvalues and $\mu = 0$, we have that $z \notin \operatorname{sp}(T_t)$, so that $(zI - T_t)$ is invertible for all $t < t_0$ and for all $z \in F$. Moreover, from the above inequality, we have that

$$\|R_z(T_t)\| = \|(zI - T_t)^{-1}\| \leq 2\hat{C}$$

and we conclude the proof. □

The following result shows that $R_z(T_{th}|_{\mathbb{Q}_h})$ is bounded on any closed subset of the complex plane not intersecting $\operatorname{sp}(T_0)$, provided $t$ and $h$ are small enough. Here and thereafter, $h_0$ and $t_0$ denote small positive constants, not necessarily the same at each occurrence.

**Lemma 5.2.** *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \operatorname{sp}(T_0) = \emptyset$. Then, there exist strictly positive constants $h_0$, $t_0$ and $C$ such that, $\forall h < h_0$ and $\forall t < t_0$, $F \cap \operatorname{sp}(T_{th}) = \emptyset$ and*

$$\|R_z(T_{th})\|_h \leq C \quad \forall z \in F.$$

*Proof.* Let $F$ be a closed set such that $F \cap \operatorname{sp}(T_0) = \emptyset$. As an immediate consequence of Lemma 5.1, we have that for all $(\eta,v) \in \mathbb{Q}$, for all $z \in F$ and for all $t < t_0$,

$$\|(\eta,v)\|_{\mathbb{Q}} \leq C \|(zI - T_t)(\eta,v)\|_{\mathbb{Q}}.$$

Now, from Lemma 4.2, we have that there exists $h_0 > 0$ such that for all $h < h_0$

$$\|(T_t - T_{th})(\eta_h,v_h)\|_{\mathbb{Q}} \leq \frac{1}{2C} \|(\eta_h,v_h)\|_{\mathbb{Q}} \quad \forall (\eta_h,v_h) \in \mathbb{Q}_h.$$

Then, for $(\eta_h,v_h) \in \mathbb{Q}_h$ and $z \in F$, we have

$$\|(zI - T_{th})(\eta_h,v_h)\|_{\mathbb{Q}} \geq \|(zI - T_t)(\eta_h,v_h)\|_{\mathbb{Q}} - \|(T_t - T_{th})(\eta_h,v_h)\|_{\mathbb{Q}} \geq \frac{1}{2C} \|(\eta_h,v_h)\|_{\mathbb{Q}}.$$

Since $\mathbb{Q}_h$ is finite dimensional, we deduce that $(zI - T_{th})$ is invertible and, hence, $z \notin \mathrm{sp}(T_{th})$. Moreover

$$\|R_z(T_{th})\|_h = \|(zI - T_{th})^{-1}\|_h \leq 2C \quad \forall z \in F$$

and we complete the proof. □

An equivalent form of the first assertion of this theorem is that any open set of the complex plane containing $\mathrm{sp}(T_0)$, also contains $\mathrm{sp}(T_{th})$ for $h$ and $t$ small enough.

The eigenvalues $\mu$ of $T_t$ are tipically simple and converges to simple eigenvalues $T_0$ as $t$ tends to zero. Because of this, we state our results only for eigenvalues of $T_t$ converging to a simple eigenvalue of $T_0$ as $t$ goes to zero.

Let $\mu_0 \neq 0$ be an eigenvalue of $T_0$ with multiplicity $m = 1$. Let $D$ be a closed disk centered at $\mu_0$ with boundary $\Gamma$ such that $0 \notin D$ and $D \cap \mathrm{sp}(T_0) = \{\mu_0\}$. Let $t_0 > 0$ be small enough so that, for all $t < t_0$, $D$ contains only one eigenvalue $\mu$ of $T_t$, which we already know is simple (cf. Lemma 3.2). Let $\mathcal{E}$ be the eigenspace of $T_t$ corresponding to $\mu$.

According to Lemma 5.2 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0$, $\Gamma \subset \boldsymbol{\rho}(T_{th})$. Moreover, proceeding as in [13, Section 2], from properties P1 and P2 it follows that for $h$ small enough $T_{th}$ has exactly one eigenvalue $\mu_h \in D$. Let $\mathcal{E}_h$ be the eigenspace of $T_{th}$ associated to $\mu_h$. The theory in [14] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of $T_{th}$ to those of $T_0$ as $h$ and $t$ go to zero. However, our goal is not this one, but to prove that $\mu_h$ converges to $\mu$ as $h$ goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [14].

Let $\Pi_h : \mathbb{Q} \to \mathbb{Q}$ be defined for all $(\eta, v) \in \mathbb{Q}$ by $\Pi_h(\eta, v) = (\mathcal{P}_h\eta, \mathcal{P}_h v) \in \mathbb{Q}_h$, with $\mathcal{P}_h$ being the $L^2$-projector defined in the previous section. The properties of $\mathcal{P}_h$ lead to analogous properties for $\Pi_h$; for instance, $\Pi_h$ is bounded uniformly on $h$, namely, $\|\Pi_h(\eta, v)\|_{\mathbb{Q}} \leq \|(\eta, v)\|_{\mathbb{Q}}$. Moreover, the error estimate (4.2) holds for $\Pi_h$ too:

$$\|\Pi_h(\eta, v) - (\eta, v)\|_{\mathbb{Q}} \leq Ch(|\eta|_{1,\mathrm{I}} + |v|_{1,\mathrm{I}}) \qquad \forall (\eta, v) \in \mathbb{H}. \tag{5.1}$$

Next, we define

$$B_{th} := T_{th}\Pi_h : \mathbb{Q} \to \mathbb{Q}_h \hookrightarrow \mathbb{Q}.$$

We observe that $B_{th}$ and $T_{th}$ have the same non-zero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result analogous to [14, Lemma 1].

**Lemma 5.3.** *There exist strictly positive constants $h_0$, $t_0$ and $C$ such that*

$$\|R_z(B_{th})\| \leq C \qquad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

*Proof.* It is essentially identical to that of Lemma 5.2 from [24]. □

Next, we introduce

- $E_t : \mathbb{Q} \to \mathbb{Q}$, the spectral projector of $T_t$ corresponding to the isolated eigenvalue $\mu$, namely,

$$E_t := \frac{1}{2\pi i} \int_\Gamma R_z(T_t)\, dz;$$

- $F_{th} : \mathbb{Q} \to \mathbb{Q}$, the spectral projector of $B_{th}$ corresponding to the eigenvalue $\mu_h$, namely,

$$F_{th} := \frac{1}{2\pi i} \int_{\Gamma} R_z(B_{th})\, dz.$$

As a consequence of Lemma 5.3, the spectral projectors $F_{th}$ are bounded uniformly in $h$ and $t$ for $h$ and $t$ small enough. Notice that $E_t(\mathbb{Q})$ is the eigenspace of $T_t$ associated to $\mu$ and $F_{th}(\mathbb{Q})$ is the eigenspace of $B_{th}$ (and hence of $T_{th}$) associated to $\mu_h$.

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces $Y$ and $Z$ of $\mathbb{Q}$:

$$\hat{\delta}(Y, Z) := \max\left\{\delta(Y, Z), \delta(Z, Y)\right\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y \in Y \\ \|y\|_{\mathbb{Q}}=1}} \left( \inf_{z \in Z} \|y - z\|_{\mathbb{Q}} \right).$$

The following results will be used to prove convergence of the eigenspaces.

**Lemma 5.4.** *There exist positive constants $h_0$, $t_0$ and $C$ such that, for all $h < h_0$ and for all $t < t_0$,*

$$\|(E_t - F_{th})|_{E_t(\mathbb{Q})}\| \leq \|(T_t - B_{th})|_{E_t(\mathbb{Q})}\| \leq Ch.$$

*Proof.* The proof of the first inequality follows from Lemmas 5.1 and 5.3 and the same arguments as Lemma 3 from [14]. For the other inequality, let $(\beta, w) \in E_t(\mathbb{Q})$. We have

$$\begin{aligned}
\|(T_t - B_{th})(\beta, w)\|_{\mathbb{Q}} \leq& \|(T_t - T_t\Pi_h)(\beta, w)\|_{\mathbb{Q}} + \|(T_t\Pi_h - B_{th})(\beta, w)\|_{\mathbb{Q}} \\
\leq& \|T_t\| \|(I - \Pi_h)(\beta, w)\|_{\mathbb{Q}} + \|(T_t - T_{th})\Pi_h(\beta, w)\|_{\mathbb{Q}} \\
\leq& Ch(|\beta|_{1,\mathrm{I}} + |w|_{1,\mathrm{I}}) + Ch\|\Pi_h(\beta, w)\|_{\mathbb{Q}} \\
\leq& Ch\|(\beta, w)\|_{\mathbb{Q}},
\end{aligned}$$

where we have used (5.1), Lemma 4.2 and Corollary 3.1. □

Now, we are in position to prove an optimal order error estimate for the eigenspaces.

**Theorem 5.1.** *There exist positive constants $h_0$, $t_0$ and $C$ such that, for all $h < h_0$ and for all $t < t_0$,*

$$\hat{\delta}\left(F_{th}(\mathbb{Q}), E_t(\mathbb{Q})\right) \leq Ch.$$

*Proof.* The proof follows by using Lemma 5.4 and arguing exactly as in the proof of [14, Theorem 1]. □

In what follows, we state a preliminary suboptimal error estimate for $|\mu - \mu_h|$ that will be used in the sequel but which will be improved below (cf. Theorem 5.2).

**Lemma 5.5.** *There exist strictly positive constants $h_0$, $t_0$ and $C$ such that, for $h < h_0$ and $t < t_0$,*

$$|\mu - \mu_h| \leq Ch.$$

*Proof.* The proof follows by repeating the arguments used in [24] to derive Lemma 5.6 from this reference. □

Since the eigenvalue $\mu$ of $T_t$ corresponds to an eigenvalue $\lambda = 1/\mu$ of problem (2.8)-(2.9), Lemma 5.5 leads to an error estimate for the approximation of $\lambda$ as well. However, the order of convergence $O(h)$ in this lemma is not optimal. The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues, but it is interesting by itself, too. In fact, it shows optimal order convergence for the bending moment and shear stress of the vibration modes.

**Lemma 5.6.** *Let $(\lambda, (\sigma, \gamma, \beta, w))$ and $(\lambda_h, (\sigma_h, \gamma_h, \beta_h, w_h))$ be the solutions of problems (2.8)–(2.9) and (4.3)–(4.4), respectively, with $\|(\beta, w)\|_{\mathbb{Q}} = \|(\beta_h, w_h)\|_{\mathbb{Q}} = 1$ and such that*

$$\|\beta - \beta_h\|_{0,\mathrm{I}} + \|w - w_h\|_{0,\mathrm{I}} \leq Ch. \tag{5.2}$$

*Then, for $h$ and $t$ small enough,*

$$\|\sigma - \sigma_h\|_{1,\mathrm{I}} + \|\gamma - \gamma_h\|_{1,\mathrm{I}} \leq Ch. \tag{5.3}$$

*Proof.* Let $((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) \in \mathbb{H}\times\mathbb{Q}$ be the solution of the following auxiliary problem:

$$a((\hat{\sigma}, \hat{\gamma}), (\tau, \xi)) + b((\tau, \xi), (\hat{\beta}, \hat{w})) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H}, \tag{5.4}$$

$$b((\hat{\sigma}, \hat{\gamma}), (\eta, v)) = -\lambda_h r((\beta_h, w_h), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}. \tag{5.5}$$

Notice that problem (4.3)–(4.4) can be seen as a discretization of the load problem above. The arguments in the proof of Lemma 4.2 can be repeated, considering $g_h = \lambda_h \beta_h$ and $f_h = \lambda_h w_h$, to show that

$$\|((\hat{\sigma}, \hat{\gamma}), (\hat{\beta}, \hat{w})) - ((\sigma_h, \gamma_h), (\beta_h, w_h))\|_{\mathbb{H}\times\mathbb{Q}} \leq Ch\lambda_h \|(\beta_h, w_h)\|_{\mathbb{Q}} \leq Ch\lambda. \tag{5.6}$$

the last inequality because $\lambda_h \to \lambda$ as a consequence of Lemma 5.5.

On the other hand, susbtracting (2.8)–(2.9) from (5.4)–(5.5), we obtain

$$a((\sigma - \hat{\sigma}, \gamma - \hat{\gamma}), (\tau, \xi)) + b((\tau, \xi), (\beta - \hat{\beta}, w - \hat{w})) = 0 \qquad \forall (\tau, \xi) \in \mathbb{H},$$

$$b((\sigma - \hat{\sigma}, \gamma - \hat{\gamma}), (\eta, v)) = -r((\lambda\beta - \lambda_h\beta_h, \lambda w - \lambda_h w_h), (\eta, v)) \qquad \forall (\eta, v) \in \mathbb{Q}.$$

As a consequence of Lemmas 2.1 and 2.2, the problem above has a unique solution (see, for instance, [19, Section II.1.1]) and there exists $C > 0$ such that

$$\begin{aligned}
\|\sigma - \hat{\sigma}\|_{1,\mathrm{I}} + \|\gamma - \hat{\gamma}\|_{1,\mathrm{I}} \leq\ & C(\|\lambda\beta - \lambda_h\beta_h\|_{0,\mathrm{I}} + \|\lambda w - \lambda_h w_h\|_{0,\mathrm{I}}) \\
\leq\ & C(\lambda\|\beta - \beta_h\|_{0,\mathrm{I}} + |\lambda - \lambda_h|\|\beta_h\|_{0,\mathrm{I}} \\
& \quad + \lambda\|w - w_h\|_{0,\mathrm{I}} + |\lambda - \lambda_h|\|w_h\|_{0,\mathrm{I}}) \\
\leq\ & Ch,
\end{aligned}$$

the last inequality because of (5.2) and Lemma 5.5.

Finally, from the above inequality and (5.6) we obtain (5.3) and the proof is complete. □

Now we are in a position to prove a double order of convergence for the eigenvalues.

**Theorem 5.2.** *There exist strictly positive constants $h_0$, $t_0$ and $C$ such that, for $h < h_0$ and $t < t_0$,*

$$|\lambda - \lambda_h| \leq Ch^2.$$

*Proof.* Let $(\lambda, (\sigma, \gamma, \beta, w))$ and $(\lambda_h, (\sigma_h, \gamma_h, \beta_h, w_h))$ be as in Lemma 5.6. Then, we write problem (2.8)–(2.9) and problem (4.3)–(4.4) as follows:

$$A((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) = -\lambda B((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)),$$

$$A((\sigma_h, \gamma_h, \beta_h, w_h), (\tau_h, \xi_h, \eta_h, v_h)) = -\lambda_h B((\sigma_h, \gamma_h, \beta_h, w_h), (\tau_h, \xi_h, \eta_h, v_h)),$$

where the bilinear forms $A$ and $B$ are defined by

$$A((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) := a((\sigma, \gamma), (\tau, \xi)) + b((\tau, \xi), (\beta, w)) + b((\sigma, \gamma), (\eta, v)),$$
$$B((\sigma, \gamma, \beta, w), (\tau, \xi, \eta, v)) := r((\beta, w), (\eta, v)).$$

Let $U := (\sigma, \gamma, \beta, w)$ and $U_h := (\sigma_h, \gamma_h, \beta_h, w_h)$. Then, it is easy to check the following identity (see, for instance, [3, Lemma 9.1]):

$$(\lambda - \lambda_h) B(U_h, U_h) = A(U - U_h, U - U_h) + \lambda B(U - U_h, U - U_h).$$

Now, since $B(U_h, U_h) = t^2 \int_{\mathrm{I}} \mathrm{J} \beta_h^2 + \int_{\mathrm{I}} \mathrm{P} w_h^2$ and $(\sigma_h, \gamma_h, \beta_h, w_h)$ can be seen as the solution of problem (4.5)–(4.6) with data $(g, f) = \lambda_h (\beta_h, w_h)$, as a consequence of Remark 4.1, we have that

$$B(U_h, U_h) \geq \frac{1}{C \lambda_h^2} \|(\beta_h, w_h)\|_{\mathbb{Q}}^2 = \frac{1}{C \lambda_h^2}.$$

Since $\lambda_h \to \lambda$ and $\lambda > 0$, for $h$ small enough

$$B(U_h, U_h) \geq \frac{1}{2 C \lambda^2},$$

the right hand side being a positive constant independent of $h$ and $t$. Hence from Theorem 5.1 and Lemma 5.6, we obtain

$$|\lambda - \lambda_h| \leq C h^2$$

and the proof is complete. □

## 6. Numerical results

We report in this section the results of some numerical tests computed with a MATLAB code implementing the finite element method described above. For all the tests we have considered a clamped beam of length $L$ and uniform meshes of $N$ elements, with different values of $N$.

In all the tests, we have used the following physical parameters:

- Young modulus: $E = 2.1 \times 10^6$ Kgf/cm$^2$, (1 Kgf= 980 kg/cm$^2$),
- Poisson ratio: $\nu = 0.3$,
- Density: $\rho = 7.85 \times 10^{-3}$ kg/cm$^3$,
- Correction factor: $k_c = 1$.

6.1. **Implementation.** The generalized eigenvalue problem that has to be solved has been written in matrix form into the proof of Lemma 4.1 (cf. (4.10)). This is a degenerate matrix generalized eigenvalue problem since none of the matrices is positive definite. Therefore, its solution would need of some specialized software. Alternativaly, problem (4.10) has been equivalently written as (4.12), where both matrices are symmetric and positive definite. However, on the left hand side there is a full matrix, because $\mathbf{R}^{-1}$ is full too. Therefore, (4.12) is not appropiate for the computer solution of the problem, either.

Instead, we proceed from (4.11) as follows: From the second equation $\mathbf{T}\vec{\boldsymbol{v}}_h = -\frac{1}{\lambda_h}\mathbf{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h$ and, since $\mathbf{T}$ is invertible, $\vec{\boldsymbol{v}}_h = -\frac{1}{\lambda_h}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{t}}\vec{\boldsymbol{u}}_h$. Substituting this into the first equation of (4.11) we arrive at

$$(\mathbf{S}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{t}})\vec{\boldsymbol{u}}_h = \lambda_h \mathbf{R}\vec{\boldsymbol{u}}_h. \tag{6.7}$$

Matrix $\mathbf{R}$ is symmetric and positive definite, whereas $(\mathbf{S}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{t}})$ is symmetric and semipositive definite. Thus, this generalized eigenvalue problem can be solved with standard software. Moreover, since $\mathbf{T}$ is formed by two mass matrices with piecewise constant elements, it is diagonal. Hence to compute $\mathbf{T}^{-1}$ is completely inexpensive and the matrix $(\mathbf{S}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{t}})$ result as sparse as $\mathbf{R}$. The only minor drawback is that the eigenvalue problem (6.7) has the spurious eigenvalue $\lambda_h = 0$ with multiplicity 2. Since $\mathbf{T}^{-1}$ is positive definite, the eigenspace associated to $\lambda_h = 0$ is the kernel of $\mathbf{S}$. Using the standard basis of the finite element spaces $W_h$ (piecewise linear and continuous elements) and $Z_h$ (piecewise constant elements) it is posible to prove that if $\vec{\boldsymbol{u}}_h = (\vec{\boldsymbol{\sigma}}_h, \vec{\boldsymbol{\gamma}}_h)^{\mathrm{t}}$ with $\vec{\boldsymbol{\sigma}}_h$ and $\vec{\boldsymbol{\gamma}}_h$ being the vector of nodal components of $\sigma_h \in W_h$ and $\gamma_h \in W_h$, respectively, thus $(\mathbf{S}\vec{\boldsymbol{u}}_h)_i = \int_{\mathrm{I}_j}(\sigma_h' - \gamma_h)$, $i = 1, \dots, N$. Therefore, $\vec{\boldsymbol{u}}_h \in \ker \mathbf{S}$ implies that either $\gamma_h = 0$ and $\sigma_h$ is constant or $\gamma_h$ is constant and $\sigma_h' = \gamma_h$. Thus, the eigenspace of $\lambda_h = 0$ in problem (6.7) is spanned by $(1, 0) \in \mathbb{H}_h$ and $(x, 1) \in \mathbb{H}_h$.

6.2. **Test 1: Uniform beam with analytical solution.** The aim of this first test is to validate the computer code by solving a problem with known analytical solution. With this purpose, we will compare the exact vibration frequencies of a uniform clamped beam as that shown in Figure 1 (undeformed beam) with those computed with the method analyzed in this paper.

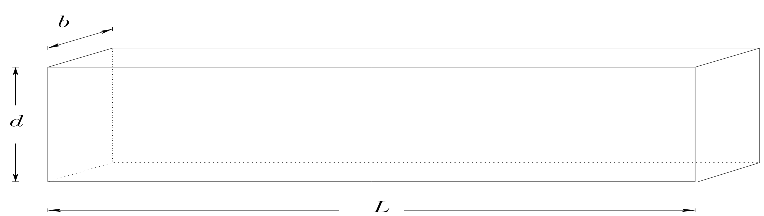



FIGURE 1. Undeformed uniform beam.

We note also that for this kind of beam, we have that $\mathbb{I} = \frac{bd^3}{12}$ and $A = bd$ are constant.

In Table 1 we report the three lowest vibration frequencies computed by our method with four different meshes ($N = 16, 32, 64, 128$). We have taken $L = 120$ cm and a square cross section of side-length $b = d = 20$ cm. The table includes computed orders of convergence and the exact vibration frequencies.

TABLE 1. Angular vibration frequencies of a uniform beam.

| Mode | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | Order | Exact |
|---|---|---|---|---|---|---|
| $\omega_1^h$ | 4017.49 | 4000.74 | 3996.84 | 3995.90 | 2.1 | 3995.61 |
| $\omega_2^h$ | 9778.27 | 9644.64 | 9613.68 | 9606.23 | 2.1 | 9603.80 |
| $\omega_3^h$ | 170614.73 | 16621.41 | 16520.22 | 16495.89 | 2.1 | 16487.94 |

It can be seen from Table 1 that the computed frequencies converge to the exact ones with an optimal quadratic order.

6.3. **Test 2: Beam with a smoothly varying cross-section.** In this test we apply the method analyzed in this paper to a beam of rectangular section with smoothly varying thickness. With this purpose, we consider a beam as that shown in Figure 2.

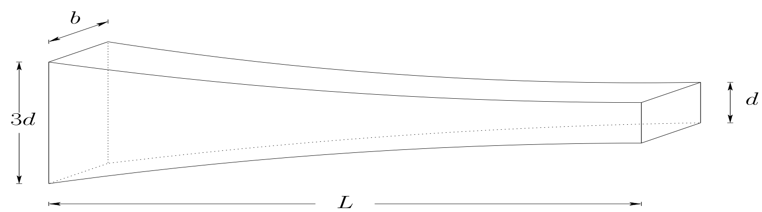



FIGURE 2. Smoothly varying cross-section beam.

Let $b$ and $d$ be as shown in Figure 2. We have taken $L = 100$, $b = 3$ and $d = 3$ cm. The equation of the top and botton surfaces of the beam are

$$z = \pm\frac{150d}{2x + 100}, \quad 0 \le x \le 100.$$

Hence, the area of the cross section and the moment of inertia are given by

$$A(x) = \frac{900d}{2x + 100}, \qquad \mathbb{I}(x) = \frac{1}{4}\left(\frac{300d}{2x + 100}\right)^3, \qquad 0 \le x \le 100.$$

In Table 2 we report the four lowest vibration frequencies computed by our method with four different meshes ($N = 16, 32, 64, 128$). The table includes computed orders of convergence as well as more accurate values obtained by means of a least-squares fitting.

TABLE 2. Angular vibration frequencies of a beam with a smoothly varying cross-section.

| Mode | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | Order | Extrap. |
|---|---|---|---|---|---|---|
| $\omega_1^h$ | 1674.8167 | 1667.2007 | 1665.2819 | 1664.8012 | 2.03 | 1664.6419 |
| $\omega_2^h$ | 4382.5912 | 4308.8768 | 4290.4391 | 4285.8294 | 2.03 | 4284.3014 |
| $\omega_3^h$ | 8432.5758 | 8139.6797 | 8067.2309 | 8049.1697 | 2.03 | 8043.1848 |
| $\omega_4^h$ | 13875.8820 | 13078.9166 | 12884.6634 | 12836.4208 | 2.03 | 12820.4405 |

It can be seen from Table 2 that the computed vibration frequencies also converge with an optimal quadratic order as predicted by the theoretical results.

We show in Figure 3 the deformed beam for the four lowest vibration modes.

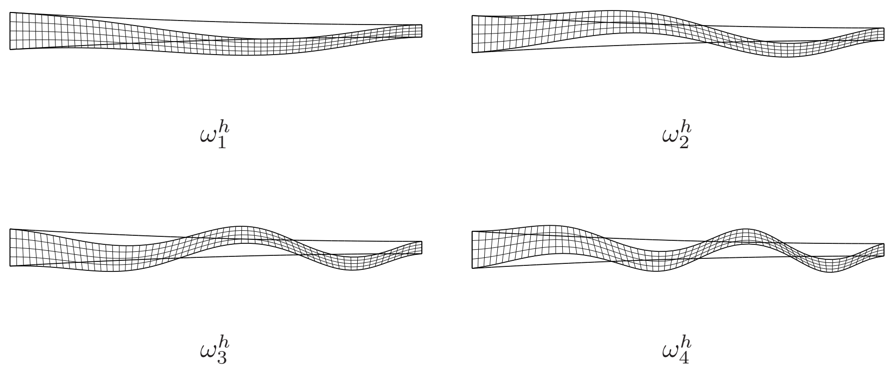



FIGURE 3. Smoothly varying cross-section beam; four vibration modes with lowest frequency.

6.4. **Test 3: Rigidly joined beams.** The aim of this test is to apply the method analyzed in this paper to a beam with area varying discontinuously along its axis. With this purpose, we consider a composed beam formed by two rigidly joined beams as shown in Figure 4. Moreover, we will assess the performance of the method as the thickness $d$ approaches to zero to check that the proposed method is thoroughly locking-free.

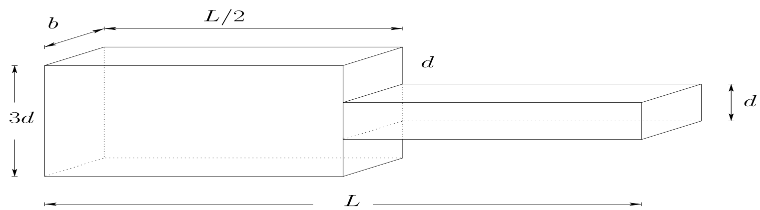



FIGURE 4. Rigidly joined beams.

Let $b$ and $d$ be as shown in Figure 4. We have taken $L = 100$ and $b = 3$, so that the area of the cross section and the moment of inertia are:

$$A(x) = \begin{cases} 9d, & 0 \leq x \leq 50, \\ 3d, & 50 < x \leq 100, \end{cases} \qquad \mathbb{I}(x) = \begin{cases} \frac{27d^3}{4}, & 0 \leq x \leq 50, \\ \frac{d^3}{4}, & 50 < x \leq 100. \end{cases}$$

We have used uniform meshes with an even number $N$ of elements, so that the point $x = L/2$ is always a node of the mesh as required by the theory.

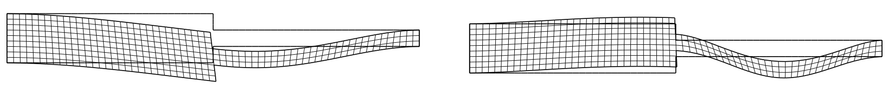
In Table 3 we present the results for the lowest computed rescaled eigenvalue $\lambda_1^h = \left(\omega_1^h/t\right)^2$, with varying thickness $d$ and different meshes. According to (2.2), the nondimensional parameter $t$ is given in this case by $t^2 = \frac{5d^2}{8L^2}$. Again, we have computed the orders of convergence and more accurate extrapolated values by means of a least-squares fitting.

The results from Table 3 show clearly that the method does not deteriorate when the thickness parameter becomes small, thus we may conclude that the method is locking-free.

TABLE 3. Lowest rescaled eigenvalue $\lambda_h^1$ (multiplied by $10^{-10}$) of a composed beam with varying thickness $d$.

| Thickness | $N = 16$ | $N = 32$ | $N = 64$ | $N = 128$ | Order | Extrap. |
|---|---|---|---|---|---|---|
| $d = 4$ | 4.72371 | 4.68871 | 4.67989 | 4.67768 | 2.03 | 4.67695 |
| $d = 0.4$ | 5.00424 | 4.96518 | 4.95534 | 4.95288 | 2.03 | 4.95207 |
| $d = 0.04$ | 5.00724 | 4.96813 | 4.95829 | 4.95582 | 2.03 | 4.95500 |
| $d = 0.004$ | 5.00727 | 4.96816 | 4.95831 | 4.95585 | 2.03 | 4.95503 |

Finally, we show in Figure 5 the deformed beam for the two lowest frequency vibration modes.



$\omega_1^h$ $\omega_2^h$

FIGURE 5. Rigidly joined beams; two lowest frequency vibration modes.

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