SETTLING VELOCITIES OF PARTICULATE SYSTEMS PART 17. SETTLING VELOCITIES OF INDIVIDUAL SPHERICAL PARTICLES IN POWER-LAW NON-NEWTONIAN FLUIDS

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Abstract. An explicit theorical equation, compared with empirical published results, is presented to calculate the terminal settling velocities of spherical particles in Power-Law Non-Newtonian fluids for Reynolds Numbers less than 1000

Keywords: Sedimentation, Settling of spherical particles, Power-Law fluids, Drag coefficient, Terminal velocity

1. Introduction

In the last decades, many efforts have been made to determine the influence of non-Newtonian rheological properties on the relative motion of solids through fluids. Relevant examples are the flow of non-Newtonian fluids through packed and fluidized beds and the sedimentation of particle suspension.

With modern methods of size reduction, size particles reaches values of just a few microns, and along with the fact that many ores have high clays contain, it is common to find slurries that behave as Pseudo-plastic fluids (Abulnaga 2002). Moreover, for the design of mineral processing equipments it is often necessary to calculate the fluid dynamic drag on solid particles (Chhabra and Richardson 2008). The Bingham model is the most used for design slurry pipelines (high shear rate) while the Power law is more suitable for situations where the shear rate is low, which is the case of the thickening process (Concha 2014).

It is well-known that the most important parameter describing particles moving in fluids is its settling velocity, which is a function of the physical properties of the particles and the rheological behavior of the fluid. It is useful to express this velocity as a function of two dimensionless numbers, the drag coefficient and the Reynolds number.

Non-Newtonian fluids present a series of characteristics including plasticity, yield stress and time-dependent behavior. Models of different complexity are available to represent the relationship between shear stress and shear rate of these fluids. The most simple is the so-called power law model, which will be used in this paper.

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Several experimental studies have been performed to predict the settling velocities of spherical particles in non-Newtonian fluids. One of these studies, Kelessidis (2004), produced an explicit equation relating the size of a sphere with its settling velocity with good approximation in a wide range of Reynolds Number of practical interest in engineering by fitting parameters using experimental data.

The objective of this contribution is to provide an explicit theoretical settling velocity equation based on an extension of the theory of boundary layer over a sphere for the flow of a non-Newtonian fluid.

1.1. **Newtonian Fluids.** Newtonian fluids are represented by the general constitutive equation:

$$\tau = \mu \dot{\gamma} \tag{1}$$

where τ is the shear stress, μ is a constant, called shear viscosity, and $\dot{\gamma}$ is the shear rate.

The drag coefficient C_D and the Reynolds Number Re for a sedimenting particle in a Newtonian fluid are:

$$C_D = \frac{4}{3}\frac{\Delta\rho d g}{\rho_f u_\infty^2} \qquad Re = \frac{\rho_f u_\infty d}{\mu} \tag{2}$$

where ρ_f is the fluid density, d and u_∞ are respectively the sphere diameter and its settling velocity in an unbounded fluid, $\Delta\rho$ is the difference of the solid and fluid densities and g is the gravitational constant.

At low Reynolds numbers the relation between the drag coefficient, the Reynolds number, and the terminal settling velocity in an unbounded fluid are given by:

$$C_D = \frac{24}{Re} \qquad u_\infty = \frac{\Delta\rho g d^2}{18\mu} \tag{3}$$

Concha and Almendra (1979) considered the flow of a Newtonian fluid over a spherical particle at high Reynolds number, where the inertial and viscous forces are of the same order of magnitude and the flow may be divided in two parts, internal viscous flow near the sphere surface an external non-viscous flow. In the external flow, Euler's equation is valid, and the velocity and pressure distributions can be obtained from Bernoulli equations:

$$u_1 = \frac{3}{2}u_0 \sin\theta \qquad \text{and} \qquad p = \frac{1}{2}\rho_f u_0^2\left[1-\left(\frac{u_1}{u_0}\right)^2\right] \tag{4}$$

Where u_1 and u_0 are the velocities in the potential flow and in the unperturbed velocity fields, respectively. θ is the angle of spherical coordinate measured from the front stagnation point of the sphere and p is the pressure. Beyond the separation point the pressure is constant with a value of $p_b = -0.4$ (Tomatika and Amai, 1938). The thickness δ of the viscous boundary layer over a sphere was given by MacDonald (1954) as:

$$\frac{\delta}{R} = \frac{9.95}{Re^2},$$

where R is the sphere radius. Consider a sphere of radius a, involving the particle of radius R and boundary layer of depth δ. Since the effect of the viscosity is confined within the

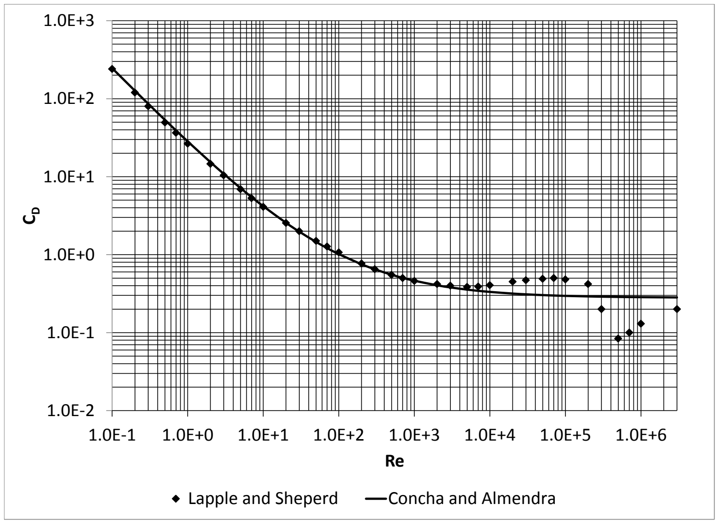

FIGURE 1. Drag coefficient for a sphere in Newtonian fluids calculated with equation (6), together with standard drag coeffient versus Reynolds number given by Lapple and Shepherd (1940)

boundary layer, the drag force F_D on the sphere of radius a can be obtained by integrating the drag force in Euler's regime:

$$F_D = 2\pi a^2 \left[\int_0^{\theta_s} p \sin\theta \mathrm{d}(\sin\theta) + \int_{\theta_s}^{\pi} p \sin\theta \mathrm{d}(\sin\theta) \right], \tag{5}$$

where S is the surface of the sphere of radius a. Substituting the values of u_1, p and p_b in (5) and integrating, the following result is obtained for the drag coefficient defined by $C_D = 2F_D/(\rho_f u_\infty^2 \pi R^2)$:

$$C_D = C_0 \left(1 + \frac{\delta_0}{Re^{1/2}} \right)^2, \quad \text{with} \quad C_0 = 0.284 \quad \text{and} \quad \delta_0 = 9.06 \tag{6}$$

Figure 1 gives the drag coefficient calculated with equation (6) together with standard drag coefficient versus Reynolds number given by Lapple and Shepherd (1940). A good agreement can be observed up to $Re = 10000$.

In the same article, Concha and Almendra (1979) presented an explicit equation for the sedimentation of spherical particles in Newtonian fluids. A dimensionless settling velocity u_∞^* was related to a dimensionless sphere size d^*, covering Stokes and Newton's regime. Indeed, due to the existing relationship between C_D and Re, the following further dimensionless relationship may be defined:

$$C_D Re^2 = \left(\frac{4}{3} \frac{\Delta\rho\rho_f g}{\mu^2} \right) d^3; \qquad \frac{Re}{C_D} = \left(\frac{3}{4} \frac{\rho_f^2}{\Delta\rho\mu g} \right) u_\infty^3 \tag{7}$$

If two parameters P and Q, dependent on the densities of the solid ρ_s and the fluid ρ_f, the density difference $\Delta\rho$, the viscosity μ and the acceleration of gravity g, are define in the form:

$$P = \left(\frac{4}{3}\frac{\mu^2}{\Delta\rho\rho_f g}\right)^{1/3}; \qquad Q = \left(\frac{4}{3}\frac{\Delta\rho\mu g}{\rho_f^2}\right)^{1/3}$$

equations (7) become

$$C_D Re^2 = \left(\frac{d}{P}\right)^3 := (d^*)^3; \qquad \frac{Re}{C_D} = \left(\frac{u}{Q}\right)^3 = (u_\infty^*)^3 \tag{8}$$

where d^* and u_∞^* are the dimensionless diameter and dimensionless settling velocity of the particle respectively. The product of $C_D Re^2$ with Re/C_D yields:

$$Re = d^* \cdot u_\infty^* \tag{9}$$

Substitution of (8) and (9) into (6) gives the quadratic equation:

$$u_\infty^* d^* + \delta_0 \left(u_\infty^* d^*\right)^{1/2} - \frac{d^{*3/2}}{C_0^{1/2}} = 0, \tag{10}$$

the solution of which is:

$$u_\infty^* = \frac{1}{4}\frac{\delta_0^2}{d^*}\left(\left(1 + \frac{4d^{*3/2}}{C_0^{1/2}\delta_0^2}\right)^{1/2} - 1\right)^2. \tag{11}$$

A plot of u_∞^* versus d^* is given in figure 2 togheter with data of drag coefficients from Lapple and Shepherd (1940).

1.2. **Non-Newtonian Fluids.** Non Newtonian fluids may be represented by a variety of models, the two most commons are Bingham Plastic and Power Law models. The Bingham Plastic model is represented by the equation

$$\tau = \tau_y + K\dot{\gamma}$$

where τ_y is the yield stress and K is the plastic viscosity. In the case of the Power Law model, the constitutive equation is:

$$\tau = m\dot{\gamma}^n$$

where n is the power law index and m is called the consistence index. The Reynolds number Re_M for the flow of a Power-Law fluid over a sphere was defined by Metzner and Reed (1955):

$$Re_M = \frac{\rho u_\infty^{2-n} d^n}{m}.$$

Much attention has been placed in studying the drag behavior of solid spheres in Non-Newtonian fluids, consequently extensive amount of information is available. Chhabra (2007) presented in his book an excellent review of the state of the art on this topic. Especially of interest is a summary of experimental data for spheres falling in Power-Law fluids for Reynolds Number less than 1000 (figure 4).

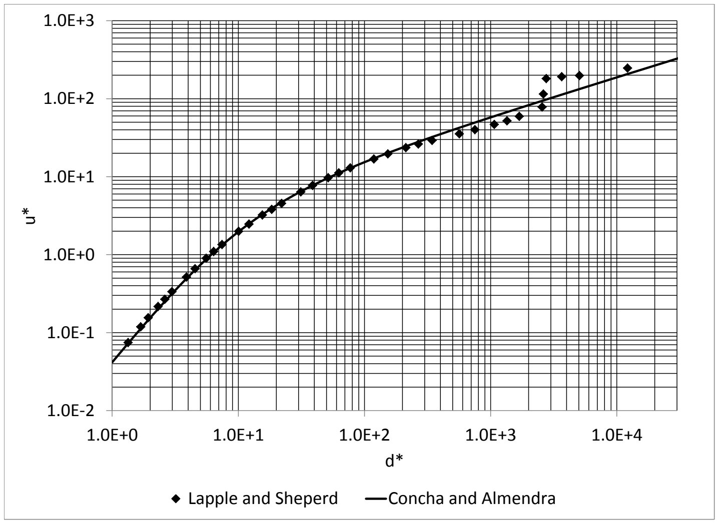

FIGURE 2. Dimensionless velocity versus dimensionless diameter for the sedimentation of spheres according to the equation (11). Circles uses standard data of drag coefficient from Lapple and Shepherd (1940).

At low Reynolds numbers, using dimensional arguments, it can be proved that the relation between the drag coefficient and the Reynolds numbers Re_M has the same form that in the case of a Newtonian fluid but includes a correction factor X depending only on the index n:

$$C_D = X(n)\frac{24}{Re_M}. \tag{12}$$

Here, as a correction factor, we will use

$$X(n) = -1.1492n^2 + 0.8734n + 1.2778 \tag{13}$$

which was obtained by fitting the numerical solution of Tripathi et al. (1984) and of Gu and Tanner (1985) for the creeping flow of a sphere in a Power-Law fluid. Data are given in Chhabra (2007). See figure 3.

Experiments of settling of spheres in Non-Newtonian fluids at higher Reynolds Number have been performed producing data also expressed as drag coefficient C_D versus Reynolds Number Re_M . Figure 4 gives a correlation for many experimental data in the range of $1 \leq Re_M \leq 1000$ (Chhabra 2007). The solid line corresponds to equation (14) (Chhabra 2007):

$$C_D = \left(2.25Re_M^{-0.31} + 0.36Re_M^{0.06}\right)^{3.45} \tag{14}$$

Several researchers measured terminal settling velocity of spheres at low Reynolds Number, however, the agreement between theory and experiments is not completely satisfactory

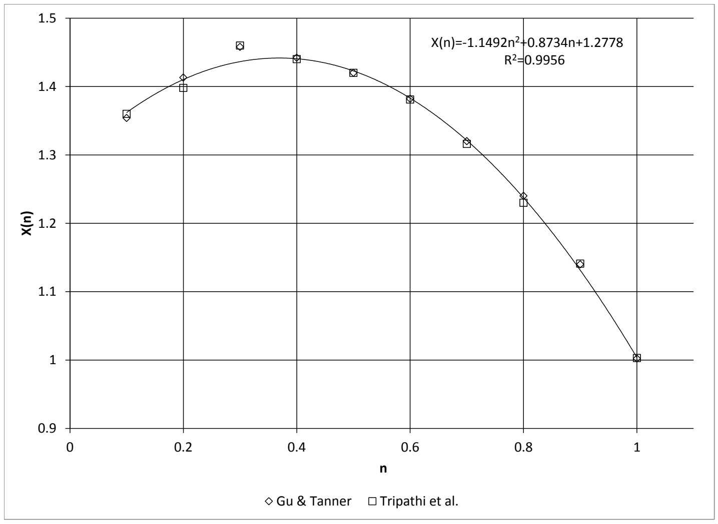

FIGURE 3. Parameter X as a function of the power index n

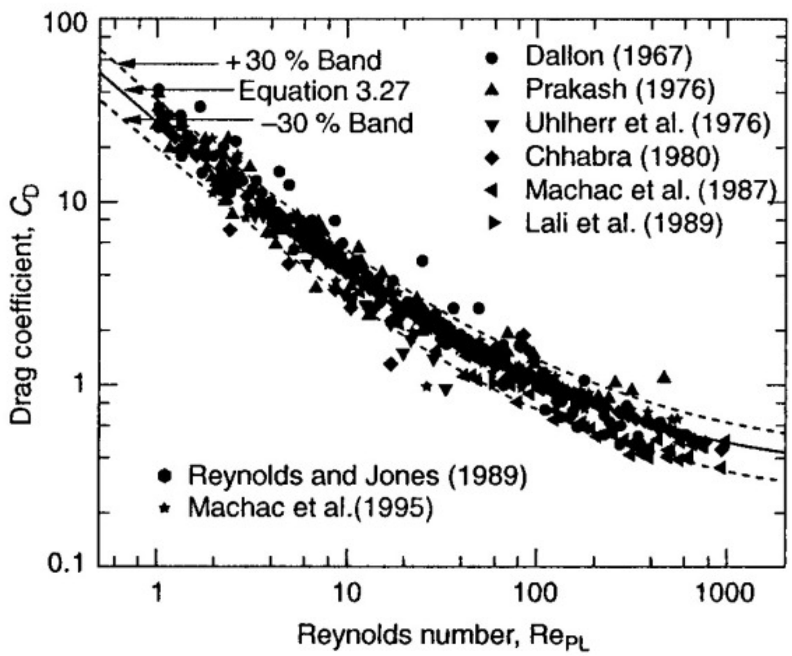

FIGURE 4. Drag coefficient versus plastic Reynolds numbers for experimental data of 8 different authors, from Chhabra (2007)

(Chhabra 2007). Empirical relations have been developed to predict the terminal settling velocity based on experimental results (such as Koziol and Glowacki 1988, Chhabra and Peri 1991, Darby 1996, the list being far from complete). An interesting work was done by Kelessidis (2004), where he extended the empirical approach of Turton and Clark (1987) for Newtonian fluids to Non-Newtonian fluids obtaining an explicit equation to predict settling velocity and he compared its predictions with several experimental results from literature in the range $0.1 \leq Re_M \leq 1000$.

2. Explicit equation for the Drag Coefficient versus Reynolds Number in the range of $0 \leq Re_M \leq 1000$

The type of fluid, Newtonian or non Newtonian, should affect the drag coefficient and the thickness of the boundary layer, then we propose $C_0(n)$ and $\delta_0(n)$ with $0.5 \leq n \leq 1$. Concha and Almendra's equation (1979) for the drag coefficient can now be written in the form:

$$C_D = C_0(n)\left(1 + \frac{\delta_0(n)}{Re_M^{1/2}}\right)^2 \tag{15}$$

It is assumed that $C_0(n) = C_0 Y(n)$ and $\delta_0(n) = \delta_0 Z(n)$, where $Y(n)$ and $Z(n)$ are correction factors. Replacing in (15) we obtain

$$C_D = C_0 Y(n)\left(1 + \frac{\delta_0 Z(n)}{Re_M^{1/2}}\right)^2$$

It must be noted that these factors are not independent each other, since in the creeping flow limit ($Re_M \to 0$) we have

$$C_D = \frac{C_0 \delta_0^2 Y(n) Z(n)^2}{Re_M} = X(n)\frac{24}{Re_M}$$

which corresponds to equation (12). An equivalent formulation, more useful for fitting data publicated in literature is

$$C_D = C_0\left(\tilde{Y}(n) + \frac{\delta_0 X(n)^{1/2}}{Re_M^{1/2}}\right)^2 \tag{16}$$

where $Z(n)\sqrt{Y(n)} = \sqrt{X(n)}$ and $\sqrt{Y(n)} = \tilde{Y}(n)$. The function $\tilde{Y}(n)$ can be obtained from values of the settling velocities publicated in the literature. Here we consider the data of Kelessidis (2003), Kelessidis and Mpandelis (2004), Miura et al. (2001), Pinelli and Magelli (2001), obtaining

$$\tilde{Y}(n) = 0.2058\exp(1.5843n) \tag{17}$$

Making $n = 1$ in eq.(16) the expression for the newtonian fluid is recovered. Figure (5) shows the agreement between the equation (14) proposed by Chhabra and equation (16) with $n = 1$.

Figure 6 shows experimental data of the drag coefficient versus Reynolds number from several authors and prediction with equation (16) for several values of the power index n in the range $0.5 \leq n \leq 1$.

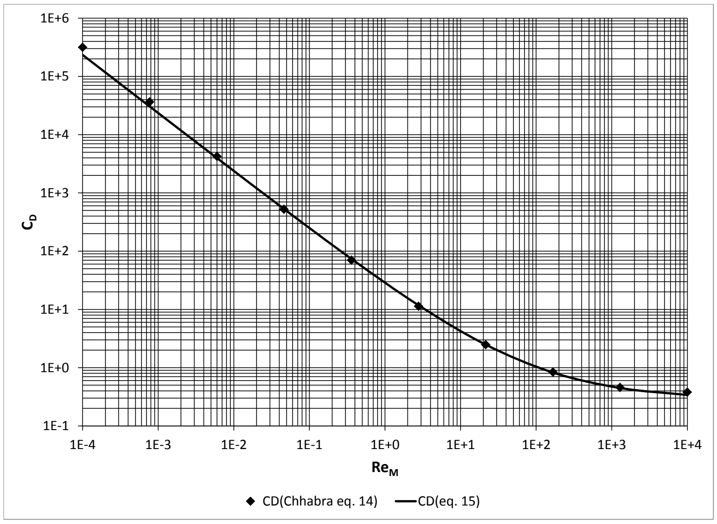

FIGURE 5. Drag coefficient for a sphere according to equations (14) and (15).

Using the same arguments presented for a Newtonian fluid, we can define the dimensionless terms $C_D Re_M^{\frac{2}{2-n}}$ and Re_M/C_D^n, which have the characteristics of being dependent, in addition to n, the first only to the particle size and the second particle velocity. A direct calculation yields:

$$C_D Re_M^{\frac{2}{2-n}} = \left(\frac{4}{3}\frac{\Delta\rho d g}{\rho_f u_\infty^2}\right)\left(\frac{\rho_f u_\infty^{2-n} d^n}{m}\right)^{\frac{2}{2-n}} = \left(\frac{4}{3}\frac{\Delta\rho g \rho_f^{\frac{n}{2-n}}}{m^{\frac{2}{2-n}}}\right) d^{\frac{2+n}{2-n}}$$

$$\frac{Re_M}{C_D^n} = \left(\frac{\rho_f u_\infty^{2-n} d^n}{m}\right)\left(\frac{3}{4}\frac{\rho_f u_\infty^2}{\Delta\rho d g}\right)^n = \left(\left(\frac{3}{4}\right)^n \frac{\rho_f^{n+1}}{m\Delta\rho^n g^n}\right) u_\infty^{2+n}$$

Defining two parameters P_n and Q_n in the form:

$$P_n = \left(\frac{3}{4}\frac{m^{\frac{2}{2-n}}}{\Delta\rho g \rho_f^{\frac{n}{2-n}}}\right)^{\frac{2-n}{2+n}}, \quad Q_n = \left(\left(\frac{4}{3}\right)^n \frac{m\Delta\rho^n g^n}{\rho_f^{n+1}}\right)^{\frac{1}{2+n}}$$

we obtain:

$$C_D Re_M^{\frac{2}{2-n}} = \left(\frac{d}{P_n}\right)^{\frac{2+n}{2-n}}, \quad \frac{Re_M}{C_D^n} = \left(\frac{u_\infty}{Q_n}\right)^{2+n}$$

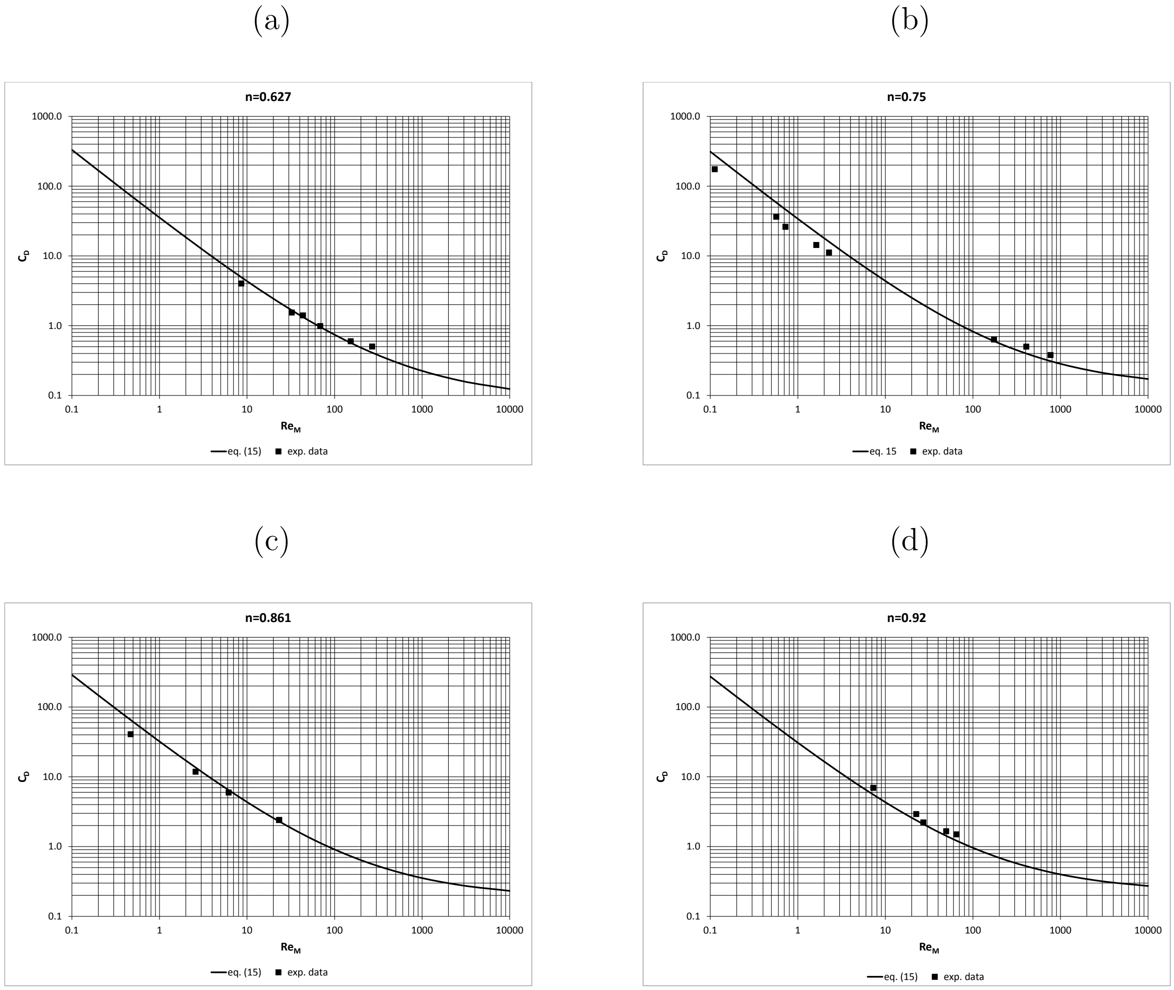

FIGURE 6. Comparison of experimental values of Plastic Reynolds Numbers versus Drag Coefficients with predictions of equation (15) for the settling of spheres in Power Law Non-Nwtonian fluids. Data by Kelessidis (2003), Kelessidis and MPandelis (2004) Miura et al. (2001) and, Pinelli and Magelli (2001).

Finally, defining the dimensionless diameter d^* and the dimensionless velocity u^*_∞ in the form:

$$d^* = \frac{d}{P_n}, \quad u^*_\infty = \frac{u_\infty}{Q_n}$$

yields

$$Re_{PL} = d^{*n} u_\infty^{*(2-n)}, \qquad C_D = \frac{d^*}{u_\infty^{*2}} \tag{18}$$

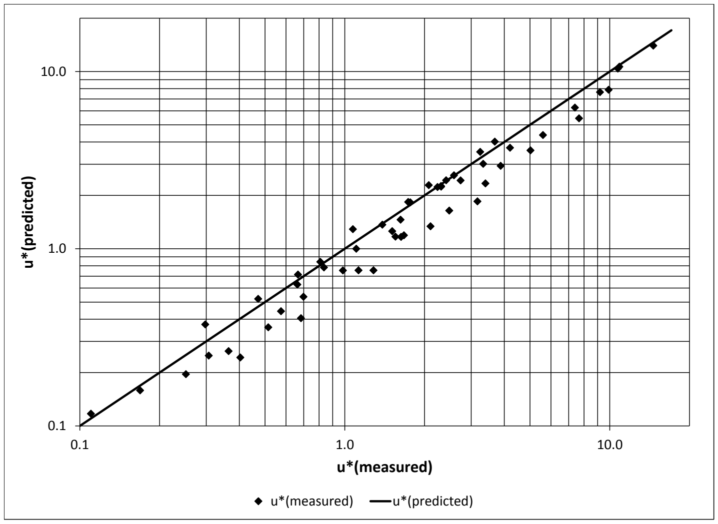

Figure 7. Comparison of predicted with measured dimensionless settling velocity for Power Law fluids for several authors.

Replacing (18) into (15) leads to the non-linear algebraic equation of u^*_∞ as a function of d^*:

$$\tilde{Y}(n)u^*_\infty d^{*n} + \delta_0 X(n)^{\frac{1}{2}} d^{*\frac{n}{2}} u^{*\frac{n}{2}}_\infty - d^{*n+\frac{1}{2}} C_0^{-\frac{1}{2}} = 0 \tag{19}$$

Notice that for n=1, equation (19) reproduces equation (10) for Newtonian fluids. Unfortunately, we can not obtain a closed-form solution of (19), but it can be solved numerically. To avoid this problem, in eq. (19), we approximate $u^{*n/2}_\infty$ by $u^{*1/2}_\infty$ getting a quadratic equation for (19) which solution is:

$$u^*_\infty = \frac{1}{4}\frac{\delta_0^2 X(n)}{\tilde{Y}(n)^2 d^{*n}}\left[\left(1 + \frac{4\tilde{Y}(n)d^{*n+1/2}}{X(n)C_0^{1/2}\delta_0^2}\right)^{1/2} - 1\right]^2. \tag{20}$$

To evaluate the prediction quality of equation (20), we compare their results with the experimental values reported by Kelessidis (2004), Kelessidis and Mpandelis (2004), Miura et. al. (2001) and Pinelli and Magelli (2001). This comparison is made in figure 7. It is clear that equation (20) for the settling velocity of spheres in Newtonian fluids may be used safely for non-Newtonian power law fluids if the correction factors $X(n)$ and $\tilde{Y}(n)$ are incorporated.

Conclusions

A theoretical explicit equation for the Drag Coefficient and the Settling velocities of spherical particles in power-law fluids was developed based on previous work of one of the authors. The effect of the non Newtonian character of the fluid was expressed as empirical relationship obtained from publish data by defining parameters $X(n)$ and $Y(n)$ related to the drag coefficient and the thickness of the boundary layer over the sphere. There is a good agreement between the proposed expression and experimental data.

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