From a stochastic Becker-Döring model to the Lifschitz-Slyozov equation with boundary value*

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Abstract: We investigate the connection between two classical models of phase transition phenomena, the (discrete size) stochastic Becker-Döring equations and the (continuous size) deterministic Lifshitz-Slyozov equation. For general coefficients and initial data, we introduce a scaling parameter and show that the empirical measure associated to the stochastic Becker-Döring system converges in law to the weak solution of the Lifshitz-Slyozov equation when the parameter goes to 0. Contrary to previous studies, we use a weak topology that includes the boundary of the state space allowing us to rigorously derive a boundary value for the Lifshitz-Slyozov model in the case of incoming characteristics. It is the main novelty of this work and it answers to a question that has been conjectured or suggested by both mathematicians and physicists. We emphasize that the boundary value depends on a particular scaling (as opposed to a modeling choice) and is the result of a separation of time scale and an averaging of fast (fluctuating) variables.

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Contents

Summary

We deal with the convergence in law of the stochastic Becker-Döring process to the Lifschitz-Slyozov partial differential equation, up to a small scaling parameter. The former is a probabilistic model for the lengthening/shrinking dynamics of a finite number and discrete size clusters, while the latter is seen as its *infinite* number and continuous size extension. In the Becker-Döring model, the clusters are assumed to increase or decrease their size (number of particles in a cluster) by addition or subtraction of only one single particle at a time (stepwise coagulation and fragmentation) without regarding the space structure. More precisely, in this model, the transitions are assumed to be Markovian and actually related to some random Poisson point measures. The lengthening rates depend on the size, the number of clusters of this size and the number of free particles throught a Law of Mass Action. The fragmentation rates depend on the size and the number of clusters of this size, through a spontaneous shricking (exponential law). The evolution of the configuration of the system is then described thanks to its empirical measure. It starts with a finite number of clusters and particles. So that, the state space of the model is finite (but possibly large) and bounded by the number of particles and clusters of all possible sizes up to the maximal one (given by the total number of particles in the system).

Under an appropriate scaling of the rates parameters, the number of monomers and the sizes of clusters, we construct a rescaled measure-valued stochastic process from the empirical measure of the Becker-Döring model. We prove the convergence in law of this process towards a measure solution of the Lifschitz-Slyozov equation. This equation is of transport type with a nonlinear flux coupling the particle variable. The necessity of prescribing a boundary value at the minimal size naturally appears in the case of incoming characteristics. The value of the latter is still an open-debated question for this continuous model. The probabilistic approach of this work allows us to rigorously derive a boundary value as a result of a particular scaling (as opposed to a modeling choice) of the original discrete model. The proof of this result is mainly based on an adiabatic procedure, the boundary condition being the result of a separation of time scale and an averaging of a fast (fluctuating) variable.

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1. Introduction

The self-assembly of macromolecules and particles is a fundamental process in many physical and chemical systems. Although particle nucleation and assembly have been studied for many decades, the interest in this field has been recently intensified due to engineering, biotechnological and imaging advances at the nanoscale level. Hence, this area of research is still very active [51]. Applications range from industrial material design, physics, chemistry to biology. In particular, the understanding of a large class of biological phenomena, such as the rare protein assembly in neuro-degenerative diseases, requires to develop stochastic self-assembly model. The interested reader is referred to [39, 48], the introduction in [61] and references therein.

The mathematical study of theoretical models for self-assembly have a long story. Often, these models consider the mean-field concentrations of clusters for each possible discrete size (number of particles in a cluster) and describe their evolution using the so-called Law of Mass-Action. Probably one of the most common model used is the celebrated Becker-Döring model in its deterministic version. The well-posedness theory and long-time behaviour have been extensively studied, see *e.g.* [3, 4, 11, 53, 60]. For a review of these results, we refer to [46], while in [15, 59] the reader will find connection to other mass-action deterministic coagulation-fragmentation models. Nevertheless many open-questions still remain, particularly on the long-time behaviour, as the computation of the rate of convergence towards equilibrium [13, 30] or the precise and rigorous description of the transient metastability phenomena [8, 14, 22, 23, 32, 44, 55].

Probabilistic approaches have been also investigated as general finite-particle stochastic coagulation models introduced in [37, 38] or the stochastic counterpart of the Becker-Döring model, by which we start. But, for instance, the latter has received much less attention than their deterministic analogues. Initial analyses, numerical simulations and interesting open-questions have been raised in [6, 52] on this model. More recently, stationary states and first passage times have been partly characterized in [20, 61], emphasizing striking finite-size effects that arise in the stochastic the Becker-Döring model. We also mention an interesting work in [47] which relates stochastic modeling to metastability.

On the other side, instead of a discrete size, the clusters can be described by a continuous size (radius, length, *etc*). In this case an equivalent of the Becker-Döring model would be the celebrated too Lifschitz-Slyozov model. Theoretical analyses of this equation have also been extensively developed. Particularly, the well-posedness has been laid down in [28, 34, 35, 43], while the long-time behaviour has been analyzed theoretically in [16, 17] and numerically in [12, 56]. We finally mention a review [42] on the Lifschitz-Slyozov theory together with open-questions.

An interesting problem is to link mathematically the stochastic and deterministic models and/or the discrete with the continuous-size models. This leads to many questions, particularly on the domain of validity of each model, the scaling law between them, *etc*. Some of them found rigorous answers, here, we intent to go further in this direction.

The link between stochastic and deterministic coagulation models is studied since the review [1]. The seminal work [31] consists of deriving a law of large number (mean-field limit) for discrete-size general stochastic coagulation-fragmentation models, including the stochastic Becker-Döring model as a particular case. This approach is useful to derive results on existence of solutions of the deterministic model, and on explosion (gelation) times. Since then, to the best of our knowledge, much of the works related to stochastic models focus on the pure coagulation model, *e.g.* [25, 26].

Discrete and continuous-size models have been linked and studied within the context of deterministic models. Two main approaches are used. The first considers the large time behaviour of the Becker-Döring model, and relates the dynamics of large clusters to solutions of various version of Lifschitz-Slyozov equations. It is the so-called theory of Ostwald ripening, see [40, 41, 45, 54, 57, 58]. A second approach considers an initial condition with a large excess of particles. Then, an appropriate re-scaling of the initial condition and the rate functions leads to solutions "closed" to the Lifschitz-Slyozov dynamics, see [18, 21, 36].

Here, we will follow the latter approach, but with a stochastic model. Indeed, our approach is intended to define a general scaling between the stochastic Becker-Döring and the deterministic Lifschitz-Slyozov. It seems it is the first time a rigorous link from discrete-stochastic to continuous-deterministic is proposed. Our method recovers similar results known yet in the pure deterministic case [18, 36] and a general existence result for a large class of rates. The novelty of our result is that we rigorously identify, for general scaling, a boundary-value in the Lifschitz-Slyozov equation. It was conjectured *e.g.* in [18, 48] but never proved. Historically, there was no need of boundary-value in Lifschitz-Slyozov since the problem was wellposed under physical assumptions (when small clusters tend to fragment). But, recent applications in Biology have raised this problem to include nucleation in this equation, for instance in [29, 48]. The originality of the work resides in the proof too in order to identify the boundary. Indeed, we carefully introduce particular measure spaces and their topology. We adapt to our context the tools developed in [33] about averaging to obtain the limit of some fast (fluctuating) variables. Our results are illustrated by simulations at the ends.

Finally, note that such links between the discrete-size and continuous-size models may also have interest for numerical schemes to solve the latter model (see [5]). We also believe that our study may be helpful to understand large deviation phenomena on the stochastic Becker-Döring model.

Organization of the paper We start by introducing the stochastic Becker-Döring model in the next Section 2. Then Section 3 is devoted to a measure-valued formulation of the model known as the empirical measure. We introduce a definition of the scaling law and the statement of our two main results: convergence to vague and weak solution (with boundary value). The martingale problem of both the original and the rescaled problem are highlighted in Section

4. Technical results on moment estimates and tightness properties are grouped in Section 5 in order to prepare the proof of the main results. We emphasize in this section the introduction of a particular (occupation) measure containing the information on the boundary value. In Section 6 are the two most important theorems which yield the main results. First an identification of the limit equation in its general form with abstract boundary value, second we identify the boundary value to the stationary measure of a modified (deterministic) Becker-Döring system. We then conclude by numerical illustration of the theoritical results in Section 7 and a discussion which relates other scaling in Section 8.

2. The stochastic Becker-Döring process

We consider a finite stochastic version of the Becker-Döring model. As previously introduced in [6, 20, 52, 61], we can define such process as a Markov chain on a finite subset of a lattice. Choose an integer $i_0 \geq 2$ and a (possibly random, but almost surely finite) parameter $M \in \mathcal{N}_{i_0}$, where $\mathcal{N}_j := \{i \in \mathbb{N} : i \geq j\}$ for any $j \geq 1$, that gives the total mass of the system. The state space of the process is given by

$$\mathcal{E} := \left\{ (P_i)_{i\in\mathcal{N}_{i_0}} \subset \mathbb{N} \, : \, \sum_{i\geq i_0} iP_i \leq M \right\} .$$

For each configuration $(P_i)_{i\in\mathcal{N}_{i_0}} \in \mathcal{E}$, the number P_i represents the quantity of clusters consisting of i particles, while $C = M - \sum_{i\geq i_0} iP_i$, is the number of *free* particles. This quantity is non-negative by virtue of the definition of the state space $\mathcal{E}$. In the Becker-Döring model, clusters can increase or decrease their size one-by-one, by capturing (aggregation process) or shedding (fragmentation process) one particle. The set of kinetics reactions that we consider can be resumed by

$$\begin{aligned} i_0 C \quad &\underset{l_0(P_{i_0})}{\overset{k_0(C)}{\rightleftharpoons}} \quad P_{i_0} \,, \\ C + P_i \quad &\underset{b_0(i+1)P_{i+1}}{\overset{a_0(i)CP_i}{\rightleftharpoons}} \quad P_{i+1} \,, \quad i \geq i_0 \,. \end{aligned} \tag{1}$$

The first reaction is the formation/destruction of an cluster of the minimal size i_0. The second reaction occurs for any $i \geq i_0$ and is the aggregation-fragmentation process between clusters of two successive sizes. This set of kinetics reactions (1) completely defines a Markov chain on $\mathcal{E}$. Let us briefly explain how we build the transition matrix from the reaction (1). For the forward aggregation reaction, the transition is

$$\Big(C, P_{i_0}, \cdots, P_i, P_{i+1}, \cdots\Big) \mapsto \Big(C-1, P_{i_0}, \cdots, P_i - 1, P_{i+1}+1, \cdots\Big) ,$$

and occurs with a rate given by $a_0(i)CP_i$ while for the backward fragmentation reaction, the transition is

$$\Big(C, P_{i_0}, \cdots, P_i, P_{i+1}, \cdots\Big) \mapsto \Big(C+1, P_{i_0}, \cdots, P_i + 1, P_{i+1}-1, \cdots\Big) ,$$

at a rate given by $b_0(i+1)P_{i+1}$. Equivalently, for the formation of an cluster with minimal size, the transitions is

$$\Big(C, P_{i_0}, \cdots, P_i, P_{i+1}, \cdots\Big) \mapsto \Big(C - i_0, P_{i_0} + 1, \cdots, P_i, P_{i+1}, \cdots\Big),$$

occuring at a rate $k(C)$ and for the destruction of such an cluster, the transition is

$$\Big(C, P_{i_0}, \cdots, P_i, P_{i+1}, \cdots\Big) \mapsto \Big(C + i_0, P_{i_0} - 1, \cdots, P_i, P_{i+1}, \cdots\Big),$$

and occurs at a rate $l_0(P_{i_0})$.

The Markov chain is well-defined as long as the sum of all rates is finite, and up to the minimal explosion time (the limit of the transition times). The well-posedness of the model is then guaranteed by

Assumption 1. *We suppose that the formation and destruction rates vanish when there are not enough reactants, i.e.*

$$k_0(c) = 0\,, \quad \forall c < i_0\,.$$

$$l_0(0) = 0\,.$$

Moreover, all aggregation and fragmentation rates are non-negative, i.e. for any $i \geq i_0$*,*

$$a_0(i) \geq 0\,, \quad b_0(i+1) \geq 0\,.$$

Indeed, with such conditions, and when $(P_i(0))_{i\in\mathcal{N}_{i_0}} \in \mathcal{E}$, it is then trivial to see that for any time $t \geq 0$ up to the minimal explosion time, $(P_i(t))_{i\in\mathcal{N}_{i_0}}$ belongs to $\mathcal{E}$. But, $(P_i(t))_{i\in\mathcal{N}_{i_0}}$ can be re-written as a Markov chain in a finite state space ($\mathrm{Card}(\mathcal{E}) < \infty$), for which existence for all times is guaranteed (no explosion in finite time). A crucial property of this model is also to preserve the mass balance property (because each transition preserves it together with Assumption 1)

$$\sum_{i\geq i_0} iP_i(t) + C(t) \equiv M. \tag{2}$$

On the set of kinetics reactions (1), we emphasize that we have chosen a Law of Mass-Action for the aggregation and fragmentation of clusters of size larger than i_0. The non-negative functions a_0 and b_0, defined on $\mathcal{N}_{i_0}$ and $\mathcal{N}_{i_0+1}$, stand respectively for the aggregation and fragmentation constant reaction rates (that may depend on the size of the cluster). For the formation and destruction rate of an cluster of the minimal size i_0, we choose a generalized law, given by arbitrary functions $C \mapsto k_0(C)$ and $P_{i_0} \mapsto l_0(P_{i_0})$ that satisfy Assumption 1. This choice is motivated by the fact that these two latter reactions will be re-scaled further differently from the others.

Remark 1. If $i_0 = 2$ with $k_0(x) = a_0(1)x(x-1)$ and $l_0(x) = b_0(2)x$ we recover the stochastic Becker-Döring model with the Law of Mass-Action up to the first size. See the discussion in Section 8 for corresponding results in this case.

3. Scaling law and main results

Notations For the remainder we introduce few classical notations we will use for sake of clarity. First, $\mathcal{C}$ denotes the space of continuous functions. Similarly, $\mathcal{C}_b$, $\mathcal{C}_c$ and $\mathcal{C}_0$ are the spaces of continuous functions which are, respectively, bounded, supportly compact and vanishing at boundary (seen as a closure of $\mathcal{C}_c$). We denote by $\mathcal{C}^k$ the functions having k continuous derivatives (up to $k = \infty$). Similarly for the other spaces the k derivatives have the same regularity.

For a Polish space E, we denote by $\mathcal{M}(E)$ the set of non-negative Radon measures on E, $\mathcal{M}_b(E)$ the set of non-negative and finite Radon measures on E and $\mathcal{P}(E)$ the probability measures. For any $\nu \in \mathcal{M}_b(E)$ and φ a real-valued measurable function on E, we write

$$\langle \nu, \varphi \rangle_E = \int_E \varphi(x)\nu(dx)\,.$$

When no doubt remains on the measurable space E, we will simply write $\langle \nu, \varphi \rangle$ instead of $\langle \nu, \varphi \rangle_E$.

3.1. The measure-valued stochastic Becker-Döring process

The model described in Section 2 can be studied using classical tools from Markov chains, such as stochastic equations, Chapman-Kolmogorov equations, first-passage time analysis, *etc*. As our objective is in particular to investigate the limit as $M \to \infty$ in (2) (large numbers) and to recover a weak form of a deterministic partial differential equation, it is preferable to use a measure-valued stochastic process approach. The advantage is to get a fixed state space while performing the limit $M \to \infty$. To that, we consider the set

$$\mathcal{M}_\delta(\mathcal{N}_{i_0}) := \left\{ \sum_{i=1}^n \delta_{x_i} \,:\, n \geq 0,\, (x_1, \ldots, x_n) \in \mathcal{N}_{i_0}^n \right\} \subset \mathcal{M}_b([i_0, +\infty))\,.$$

We represent the population of clusters, with the following measure at time $t \geq 0$

$$\mu_t = \sum_{i \geq i_0} P_i(t)\delta_i \in \mathcal{M}_\delta(\mathcal{N}_{i_0})\,. \tag{3}$$

where $(P_i(t))_{i\in\mathcal{N}_{i_0}}$ is the Markov chain described in Section 2 by the set of kinetics reactions (1), with finite mass initial condition given by $(P_i(0))_{i\in\mathcal{N}_{i_0}} \in \mathcal{E}$. The solution $P_i(t)$ represents the number of clusters of size i at time $t \geq 0$, and may be given now by $P_i(t) = \langle \mu_t, 1_i \rangle$ where $y \mapsto 1_i(y)$ is the function equals to 1 for $y = i$ and 0 elsewhere. This point of view defines $(\mu_t)_{t\geq 0}$ as a measure-valued stochastic process that entirely contains the information of the system. We define below, first the probabilistic objects we use and then the stochastic differential equation satisfied by the empirical measure (3)

Definition 1 (Probabilistic objects)**.** Let $i_0 \in \mathbb{N}^*$ and $(\Omega, \mathcal{F}, \mathbb{P})$ a sufficiently large probability space. $\mathbb{E}[\cdot]$ denotes the expectation. We define on this space four independent random Poisson point measures

i) The nucleation Poisson point measure $Q_1(dt, du)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity
$$\mathbb{E}[Q_1(dt, du)] = dtdu\,.$$
ii) The de-nucleation Poisson point measure $Q_2(dt, du)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity
$$\mathbb{E}[Q_2(dt, du)] = dtdu\,.$$
iii) The aggregation Poisson point measure $Q_3(dt, du, di)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{N}_{i_0}$ with intensity
$$\mathbb{E}[Q_3(dt, du, di)] = dtdu\ \#_{i_0}(di)\,.$$
iv) The fragmentation Poisson point measure $Q_4(dt, du, di)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{N}_{i_0+1}$ with intensity
$$\mathbb{E}[Q_4(dt, du, di)] = dtdu\ \#_{i_0+1}(di)\,.$$

where dt and du are Lebesgue measures on $\mathbb{R}^+$, and $\#_j(di)$ is the counting measure on $\mathcal{N}_j$. Moreover, we define two more independent (from the above) random elements

v) The initial distribution μ_{in} is a $\mathcal{M}_b([i_0, +\infty))$-valued random variable such that *a.s.* μ_{in} belongs to $\mathcal{M}_\delta(\mathcal{N}_{i_0})$ and $\langle \mu_{in}, \text{Id}\rangle$ is finite, where Id is the identity function.
vi) The initial quantity of particles C_{in} is a $\mathbb{R}_+$-valued random variable (*a.s.* finite).

Finally, we define the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ associated to the Poisson point measure such that μ_{in} and C_{in} are F_t-measurable.

Now we give a definition of measure formulation of the Becker-Döring model.

Definition 2 (Measure-valued stochastic Becker-Döring process)**.** Assume the probabilistic objects of Definition 1 are given, and that the rate functions satisfy Assumption 1. A *measure-valued stochastic Becker-Döring process* (abbreviated by SBD process) is a $\mathcal{M}_b([i_0, +\infty))$-valued stochastic process $\mu = (\mu_t)_{t\geq 0}$ that satisfies *a.s.* and for all $t \geq 0$

$$\begin{aligned}
\mu_t = \mu_{\text{in}} \quad &+ \int_0^t \int_{\mathbb{R}_+} \delta_{i_0} \mathbf{1}_{\left\{u \leq k_0(C_{s-})\right\}} Q_1(ds, du) \\
&- \int_0^t \int_{\mathbb{R}_+} \delta_{i_0} \mathbf{1}_{\left\{u \leq l_0(\langle \mu_{s-}, 1_{i_0}\rangle)\right\}} Q_2(ds, du) \\
&+ \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0}} (\delta_{i+1} - \delta_i)\, \mathbf{1}_{\left\{u \leq a_0(i) C_{s-} \langle \mu_{s-}, 1_i\rangle\right\}} Q_3(ds, du, di) \\
&- \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0+1}} (\delta_i - \delta_{i-1})\, \mathbf{1}_{\left\{u \leq b_0(i) \langle \mu_{s-}, 1_i\rangle\right\}} Q_4(ds, du, di)\,,
\end{aligned} \tag{4}$$

with the balance law given, *a.s.* for all $t \geq 0$, we have $C_t \geq 0$ and

$$C_t + \langle \mu_t, \mathrm{Id} \rangle = M \,, \tag{5}$$

where $C_t \geq 0$.

Remark 2. The total mass M is a random element defined by $M := C_{\mathrm{in}} + \langle \mu_{\mathrm{in}}, \mathrm{Id} \rangle$ and is *a.s.* finite. A solution μ satisfies $\mu_t \in \mathcal{M}_\delta(\mathcal{N}_{i_0})$ *a.s.* and for all $t \geq 0$.

The existence result and the martingale problem associated to this process is described in Proposition 1 in Section 4.1. We emphasize that this stochastic process is still evolving in a finite state space that is a subset of $\mathcal{M}_\delta(\mathcal{N}_{i_0})$, for which all properties on non-explosion, generator and martingale properties are trivial.

3.2. Definition of the scaling and the associate process

We introduce a (small) parameter $\varepsilon > 0$ in the system and we make explicit the dependence of the other parameters on ε. We consider the sequences (indexed by $\varepsilon > 0$) of parameters $\{a_0^\varepsilon\}$, $\{b_0^\varepsilon\}$, $\{k_0^\varepsilon\}$ and $\{l_0^\varepsilon\}$, all satisfying Assumption 1 for each $\varepsilon > 0$. Also, we introduce a sequence $\{i_0^\varepsilon\}$.

For each $\varepsilon > 0$, in Definition 1, we replace i_0 by i_0^ε and we consider a sequence of *a.s.* finite initial quantity of particles $\{\tilde{C}_{\mathrm{in}}^\varepsilon\}$ and a sequence of initial distribution $\{\tilde{\mu}_{\mathrm{in}}^\varepsilon\}$ that are $\mathcal{M}_b([i_0^\varepsilon, +\infty))$-valued random variables and such that for all $\varepsilon > 0$, $\langle \tilde{\mu}_{\mathrm{in}}^\varepsilon, \mathrm{Id} \rangle$ is *a.s.* finite. We associate the canonical filtration $(\tilde{\mathcal{F}}_t^\varepsilon)_{t \geq 0}$ as in Definition 1. Then, we may apply an obvious ε-version of Definition 2. Thus, for all $\varepsilon > 0$, we denote by $\tilde{\mu}^\varepsilon$ a measure-valued SBD process, in the sense of Definition 2, associated to this set of parameters that belongs to the state space

$$\hat{\mathcal{M}}^\varepsilon := \left\{ \nu \in \mathcal{M}_\delta(\mathcal{N}_{i_0^\varepsilon}) \,:\, \langle \nu, \mathrm{Id} \rangle \leq \tilde{M}^\varepsilon \right\} ,$$

where $\tilde{M}^\varepsilon := \tilde{C}_{\mathrm{in}}^\varepsilon + \langle \tilde{\mu}_{\mathrm{in}}^\varepsilon, \mathrm{Id} \rangle$.

Given that solution $\tilde{\mu}^\varepsilon$, we perform a general scaling with respect to the number, the size, and the time by

$$\mu_t^\varepsilon := \sum_{i \geq i_0^\varepsilon} \varepsilon^\alpha \langle \tilde{\mu}_{\varepsilon^\gamma t}^\varepsilon, 1_i \rangle \delta_{\varepsilon^\beta i}, \tag{6}$$

for some $\alpha, \beta, \gamma \geq 0$ (to be specified latter). This scaling yields the two relations:

$$\langle \mu_t^\varepsilon, 1_{\varepsilon^\beta i} \rangle = \varepsilon^\alpha \langle \tilde{\mu}_{\varepsilon^\gamma t}^\varepsilon, 1_i \rangle \,, \text{ and, } \langle \mu_t^\varepsilon, \mathrm{Id} \rangle = \varepsilon^{\alpha+\beta} \langle \tilde{\mu}_{\varepsilon^\gamma t}^\varepsilon, \mathrm{Id} \rangle \,.$$

Moreover, we allow a specific scaling of the particle variable, given by

$$C_t^\varepsilon := \varepsilon^\theta \tilde{C}_{\varepsilon^\gamma t}^\varepsilon$$

for some $\theta > 0$. It is then natural to define

$$M^\varepsilon := \varepsilon^{\alpha+\beta} \tilde{M}^\varepsilon = \varepsilon^{\alpha+\beta-\theta} C_{\mathrm{in}}^\varepsilon + \langle \mu_{\mathrm{in}}^\varepsilon, \mathrm{Id} \rangle \,.$$

Then, we define the following rescaled functions

$$\begin{aligned} a^\varepsilon(\varepsilon^\beta i) &:= \varepsilon^A a_0^\varepsilon(i), \quad \forall i \in \mathcal{N}_{i_0^\varepsilon}, \\ b^\varepsilon(\varepsilon^\beta i) &:= \varepsilon^B b_0^\varepsilon(i), \quad \forall i \in \mathcal{N}_{i_0^\varepsilon+1}, \\ k^\varepsilon(\varepsilon^\theta c) &:= \varepsilon^K k_0^\varepsilon(c), \quad \forall c \in \mathbb{N}, \\ l^\varepsilon(\varepsilon^\alpha p) &:= \varepsilon^L l_0^\varepsilon(p), \quad \forall p \in \mathbb{N}, \end{aligned} \tag{7}$$

for some A, B, K, L not necessarily non-negative (to be specified latter). Finally, we define the rescaled minimal size

$$x_0^\varepsilon := \varepsilon^\beta i_0^\varepsilon \geq x_0 \geq 0\,, \quad \forall \varepsilon > 0\,,$$

for some $x_0 \geq 0$.

The aim of this work being the study of the process μ^ε in the limit ε goes to 0, we need to embed the rescaled measure (6) in a measure space independent of ε. The natural choice is

$$\mathcal{X}([x_0, +\infty)) := \Big\{ \nu \in \mathcal{M}_b([x_0, +\infty)) \,:\, \langle \nu, \mathrm{Id} \rangle < +\infty \Big\}\,.$$

When no doubt remains we drop the explicit dependence on $[x_0, +\infty)$. Clearly, for each $\varepsilon > 0$ and $t \geq 0$, we have $\mu_t^\varepsilon \in \mathcal{X}$. The evolution equation on μ^ε is postponed in Section 4.2. Finally, for each $\varepsilon > 0$, we always denote $(\mu_t^\varepsilon)_{t \geq 0}$ by μ^ε.

3.3. Convergence towards the Lifschitz-Slyozov equation

Before stating the main results of this work, we introduce the assumptions required to obtain the convergence of $\{\mu^\varepsilon\}$, the sequence of measure-valued SBD processes constructed by (6), towards a measure solution of the Lifschitz-Slyozov equation (including a boundary value).

Assumption 2 (Convergence of the parameters)**.** *We assume that*

$$\{x_0^\varepsilon\} \text{ converges towards } x_0 \geq 0\,. \tag{H1}$$

$$\{M^\varepsilon\} \text{ converges a.s. to a deterministic value } m > 0\,. \tag{H2}$$

Assumption 3 (Convergence of the rate functions)**.** *Assume that there exist two functions k and l from $\mathbb{R}_+$ to $\mathbb{R}_+$, and two continuous functions a and b from $[x_0, +\infty)$ to $\mathbb{R}_+$. In addition, we assume that k is locally bounded. Then, we suppose that*

$$\{k^\varepsilon\} \textit{ converges uniformly on any compact set of } [0,+\infty) \textit{ towards } k\,. \tag{H3}$$

$$\begin{aligned}&\{l^\varepsilon\} \textit{ converges uniformly on any compact set of } [0,+\infty) \textit{ towards } l \textit{ and}\\ &\qquad \exists K_l > 0 \textit{ s.t. } l^\varepsilon(x) \le K_l x,\ \forall x \in [x_0,+\infty) \textit{ and } \forall \varepsilon > 0\,.\end{aligned} \tag{H4}$$

$$\begin{aligned}&\{a^\varepsilon\} \textit{ converges uniformly on any compact set of } [x_0,+\infty) \textit{ towards } a \textit{ and}\\ &\qquad \exists K_a > 0 \textit{ s.t. } a^\varepsilon(x) \le K_a(1+x),\ \forall x \in [x_0,+\infty) \textit{ and } \forall \varepsilon > 0\,.\end{aligned} \tag{H5}$$

$$\begin{aligned}&\{b^\varepsilon\} \textit{ converges uniformly on any compact set of } [x_0,+\infty) \textit{ towards } b \textit{ and}\\ &\qquad \exists K_b > 0 \textit{ s.t. } b^\varepsilon(x) \le K_b(1+x),\ \forall x \in [x_0,+\infty) \textit{ and } \forall \varepsilon > 0\,.\end{aligned} \tag{H6}$$

Remark 3. First, we will widely use a direct consequence of (H3). That is, from the convergence of k^ε toward a locally bounded function k, it entails

$$\exists K_k > 0,\ \text{s.t.} \sup_{\varepsilon>0} \sup_{x\in[0,2m]} |k^\varepsilon(x)| \le K_k\,. \tag{8}$$

Remark that (H4) to (H6) entail for all $x \ge 0$, $l(x) \le K_l x$, and for all $x \ge x_0$, $a(x) \le K_a(1+x)$, $b(x) \le K_b(1+x)$.

From now on and for the remainder, we will use some notations on the scaling exponents, to be more readable, that are:

$$\begin{aligned}&\lambda_k = \alpha + \gamma - K, &&\lambda_l = \alpha + \gamma - L,\\ &\lambda_a = \gamma + \beta - A - \theta, &&\lambda_b = \gamma + \beta - B,\\ &\lambda_c = \theta - \alpha - \beta\end{aligned} \tag{9}$$

Assumption 4 (Scaling hypothesis)**.** *We assume that the scaling exponents* (9) *satisfy*

$$\lambda_a = \lambda_b = \lambda_k = \lambda_l = \lambda_c = 0. \tag{H7}$$

The method we use here to prove the convergence needs a uniform control on superlinear moments of μ^ε. To that, let us introduce the set $\mathcal{U}_1$ of nonnegative functions Φ, convex and belonging to $\mathcal{C}^1([0,+\infty)) \cap W^{2,\infty}_{loc}((0,+\infty))$ such that $\Phi(0) = 0$, Φ' is concave and $\Phi'(0) \ge 0$ and the set $\mathcal{U}_\infty$ of nonnegative increasing convex functions Φ such that

$$\lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty\,.$$

We denote by $\mathcal{U}_{1,\infty} := \mathcal{U}_1 \cap \mathcal{U}_\infty$. These functions have remarkable properties when conjugate to the structure of the SBD equation and provide important estimates as in the deterministic case, see for instance [35].

Remark 4. Any function $x \mapsto x^{1+\eta}$ with $\eta \in (0,1)$ belongs to the set $\mathcal{U}_{1,\infty}$. The functions Φ are morally a moment slightly greater than 1.

Assumption 5 (Initial measure)**.** *We assume that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\langle \mu_{\mathrm{in}}^{\varepsilon}, \mathbf{1}\rangle\right] < +\infty \,. \tag{H8}$$

Moreover, there exist Φ_1 *and* Φ_2 *belonging to* $\mathcal{U}_{1,\infty}$ *such that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\langle \mu_{\mathrm{in}}^{\varepsilon}, \Phi_1\rangle\right] < +\infty \,, \tag{H9}$$

and

$$\sup_{\varepsilon>0} \mathbb{E}\left[\Phi_2(\langle \mu_{\mathrm{in}}^{\varepsilon}, \mathbf{1}\rangle)\right] < +\infty \,. \tag{H10}$$

We are now ready to present the first result giving the tightness of the process $\{\mu^\varepsilon\}$ and that any accumulation point is a measure solution to the Lifshitz-Slyozov equation in a sense called *vague*.

Theorem 1. *Let* μ^ε *constructed thanks to* (6) *for each* $\varepsilon > 0$. *Assume that Assumptions 2 to 5 hold and that* $\mu_{\mathrm{in}}^{\varepsilon}$ *converges in* $\mathcal{P}(w-\mathcal{X})$ *towards a deterministic* μ_{in}. *Then,* $\{\mu^\varepsilon\}$ *converges along an appropriate subsequence to* μ *in* $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$ *as* $\varepsilon \to 0$. *The limit* μ *belongs to* $\mathcal{C}(\mathbb{R}_+, w-\mathcal{X})$ *and is a vague solution of the Lifshitz-Slyozov equation, that is a.s. for all* $\varphi \in \mathcal{C}_c^1((x_0,+\infty))$ *and* $t \geq 0$

$$\langle \mu_t, \varphi\rangle = \langle \mu_{\mathrm{in}}, \varphi\rangle + \int_0^t \int_{x_0}^{\infty} \varphi'(x)(a(x)c_s - b(x))\mu_s(dx)ds \,, \tag{10}$$

where $c_t = m - \langle \mu_t, \mathrm{Id}\rangle \geq 0$.

Remark 5. Here, $w-\mathcal{X}$ (or alternatively $(\mathcal{X}, w)$ in the remainder) denotes the space $\mathcal{X}$ equipped with the weak topology (in fact a *weak* $-*$) of the convergence $\langle \nu^\varepsilon, (\mathbf{1}+\mathrm{Id})\varphi\rangle \to \langle \nu, (\mathbf{1}+\mathrm{Id})\varphi\rangle$ for all $\varphi \in \mathcal{C}_b([x_0,+\infty))$, as described in Appendix A.1. Since $(\mathcal{X}, w)$ is a Polish space, see Lemma A.1, we consider the space $\mathcal{D}(\mathbb{R}_+, w-\mathcal{X})$ of right-continuous functions from $\mathbb{R}_+$ to $\mathcal{X}$ having a left limit at each time (càdlàg) equipped with the Skorohod topology which is a Polish space too (see [24] for more details). Thus the space $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$ is the space of probability measures on the space $\mathcal{D}(\mathbb{R}_+, w-\mathcal{X})$. The convergence of $\{\mu^\varepsilon\}$ has to be understood as the classical convergence in law or distribution of random variables, see [7].

This result will be a direct consequence of Theorem 3 stated further in Section 6. This equation is known to be well-posed (uniqueness) in the case of "outgoing characteristics". Indeed, this theorem is limited by the fact the test functions do not account for the boundary value in x_0. Thanks to the result given by Collet and Goudon in [17, Theorem 3] it readily follows:

Corollary 1. *In addition to the hypotheses of Theorem 1, assume that a and b belong to $C^1([x_0,+\infty))$. For any $T>0$ such that the limit c satisfy*

$$a(0)\sup_{t\in[0,T]} c_t - b(0) \le 0\,,$$

the equation (10) *has a unique solution μ in $\mathcal{C}([0,T], w-\mathcal{X})$, hence the whole sequence $\{\mu^\varepsilon\}$ converges in $\mathcal{P}(\mathcal{D}([0,T], w-\mathcal{X}))$ to μ.*

This corollary does not include cases where a and b behave as a power law ($x \to x^\eta$) with γ in $(0,1)$, as it is usual. Note a better result of uniqueness is disponible in [34] for density solution.

Of course, we are interested in the case of "incoming characteristics" when a boundary condition is necessary for the well-posedness. To treat the boundary term we will need more information on the behavior of the rate functions a and b near x_0. More precisely, we suppose that: the limit functions a and b behave as a power-law function near x_0, that the functions a^ε and b^ε evolve in a similar way close to x_0, and that the convergence of x_0^ε towards x_0 is sufficiently fast.

Assumption 6 (Behavior of the rate functions near x_0)**.** *We suppose there exist $r_a,\ r_b \ge 0$, and $\overline{a},\ \overline{b} > 0$ such that*

$$a(x_0+x) \underset{x\to 0}{\sim} \overline{a}x^{r_a} \ \text{ and } \ b(x_0+x) \underset{x\to 0}{\sim} \overline{b}x^{r_b}\,, \tag{H11}$$

and that

$$\begin{aligned} \frac{a^\varepsilon(x_0^\varepsilon+\varepsilon^\beta n) - a(x_0^\varepsilon+\varepsilon^\beta n)}{\varepsilon^{\beta r_a}} \underset{\varepsilon\to 0}{\to} 0\,,\\ \frac{b^\varepsilon(x_0^\varepsilon+\varepsilon^\beta n) - b(x_0^\varepsilon+\varepsilon^\beta n)}{\varepsilon^{\beta r_b}} \underset{\varepsilon\to 0}{\to} 0\,. \end{aligned} \tag{H12}$$

Moreover, we suppose there exists $x_0^1 > 0$ such that

$$x_0^\varepsilon = x_0 + \varepsilon^\beta x_0^1 + o(\varepsilon^\beta),. \tag{H13}$$

Remark 6. The two last hypotheses are trivial in the case $a^\varepsilon \equiv a$, $b^\varepsilon \equiv b$ and $i_0^\varepsilon \equiv i_0$.

Before stating the second theorem we introduce a critical threshold which will be debated below, namely

$$\rho := \lim_{x\to x_0} \frac{b(x)}{a(x)} \in [0,+\infty]\,. \tag{11}$$

The result reads:

Theorem 2. *In addition to the hypotheses of Theorem 1, assume that Assumption 6 holds with $\min(r_a,r_b) < 1$. Then, on any time interval $[t_0,t_1]$ such that $c_t > \rho$ for all $t\in[t_0,t_1]$, the limit μ is a weak solution of the Lifschitz-Slyozov equation, that is a.s. for all $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t\in[t_0,t_1]$*

$$\langle \mu_t, \varphi \rangle = \langle \mu_{t_0}, \varphi \rangle + \int_{t_0}^{t} \int_{x_0}^{\infty} \varphi'(x)(a(x)c_s - b(x))\mu_s(dx)ds + \int_{t_0}^{t} \varphi(x_0)k(c_s)ds \,, \quad (12)$$

where $c_t = m - \langle \mu_t, \mathrm{Id} \rangle > \rho$.

Let us do a sort of zoology of our condition. If $0 \le r_a < r_b < 1$, the aggregation term is stronger than the fragmentation and clusters of critical size can growth for all time since $\rho = 0$. If $0 \le r_a = r_b < 1$, it is a limit case and $\rho = \overline{b}/\overline{a}$. The nucleation occurs when enough particles is supplied. Note in the case $\rho > m$, we are always in the case $c_t \le m < \rho$. While if $0 \le r_b < r_a < 1$, the fragmentation is stronger than the aggregation in 0 and Theorem 2 is nothing compared to Theorem 1 since $\rho = +\infty$. The case r_a and r_b greater than 1 is related, either an outgoing case or a case where no boundary condition is needed. The latter corresponds to the case clusters of critical size cannot growth in finite time, see Proposition 10.

The uniqueness of this latter theorem is left. The measure formulation together with the regularity of the coefficients near x_0 make the problem difficult to treat. But we believe, at least for power law our result contain all the "incoming characteristic" cases. If yes, the Lipshitz-Slyozov equation is well-defined on $\mathbb{R}_+$ just by combining the vague solution on the interval where $c_t < \rho$ and weak when $c_t > \rho$. It remains to treat the case where $c_t = \rho$, by continuity it should works if it occurs a countable number of times.

Finally, we mention the boundary condition can be interpreted as a flux condition in the case of a density solution, that is $\mu_t = f(t,x)dx$. The problem formally reads,

$$\partial_t f + \partial_x[(a(x)c_t - b(x))f(t,x)] = 0 \,, \qquad \text{on } [0,T] \times [x_0, +\infty) \,,$$
$$\lim_{x \to x_0}(a(x)c_t - b(x))f(t,x) = k(c_t) \,, \qquad \text{on } [0,T] \,.$$

The reader interested in this problem and its uniqueness should probably refer to the works by Boyer [10]. Clearly, our condition differs from [16] where it was conjectured a boundary given by

$$(a(0)c_t + b(0))f(t,0) = k(c_t) \,,$$

when $c_t \ge b(0)/a(0)$.

4. Equations and martingale properties

In this section, we detail the generator and the martingale problem associated to the original measure-valued stochastic process and its rescaled version.

4.1. The original process

Proposition 1 (Existence of the measure-valued SBD process)**.** *Assume the probabilistic objects of Definition 1 are given, and that the rate functions are consistent i.e. Assumption 1 holds. Then, there exists a unique measured-valued stochastic Becker-Döring process μ in the sense of Definition 2 on $\mathbb{R}_+$ (for any time). In particular, a.s. for all $t \geq 0$, μ_t belongs to the state space*

$$\hat{\mathcal{M}} := \{\nu \in \mathcal{M}_\delta(\mathcal{N}_{i_0}) \, : \, \langle \nu, \mathrm{Id} \rangle \leq M\} \, ,$$

and we have

$$\sup_{t \in \mathbb{R}_+} \langle \mu_t, \mathbf{1} \rangle \leq \frac{M}{i_0}, \quad a.s.$$

Moreover, μ is a Markov process whose infinitesimal generator $\mathcal{L}$ is given, for all $\nu \in \mathcal{M}_\delta$ and for all locally bounded measurable function ψ from $\mathcal{M}_b([i_0, +\infty))$ to $\mathbb{R}$, by

$$\begin{aligned} \mathcal{L}\psi(\nu) \quad = \quad & [\psi(\nu + \delta_{i_0}) - \psi(\nu)] \, k_0(C) \\ & + [\psi(\nu - \delta_{i_0}) - \psi(\nu)] \, l_0(\langle \nu, 1_{i_0} \rangle) \\ & + \sum_{i \geq i_0} [\psi(\nu + \delta_{i+1} - \delta_i) - \psi(\nu)] \, a_0(i) C \langle \nu, 1_i \rangle \\ & + \sum_{i \geq i_0 + 1} [\psi(\nu - \delta_i + \delta_{i-1}) - \psi(\nu)] \, b_0(i) \langle \nu, 1_i \rangle \, , \end{aligned}$$

where $C = M - \langle \nu, \mathrm{Id} \rangle$. Finally μ is also an $\mathcal{X}$-valued stochastic process which has a.s. sample paths in $\mathcal{D}(\mathbb{R}_+, w - \mathcal{X})$.

Proof. Note that if μ is a measure-valued SBD process in the sense of Definition 2, then for all measurable locally bounded function $\psi : \mathcal{M}_b([i_0, +\infty)) \to \mathbb{R}$ and all $t \geq 0$,

$$\psi(\mu_t) = \psi(\mu_{\mathrm{in}}) + \sum_{s \leq t} \psi(\mu_s) - \psi(\mu_{s^-}) \, ,$$

where the sum is finite over the stopping time. We deduce from the above relation, that, *a.s.* and for all $t \geq 0$,

$$\begin{aligned}\psi(\mu_t) &= \psi(\mu_{\text{in}}) \\ &+ \int_0^t \int_{\mathbb{R}_+} [\psi(\mu_{s^-} + \delta_{i_0}) - \psi(\mu_{s^-})] \mathbf{1}_{\left\{u \leq k_0(C_{s^-})\right\}} Q_1(ds, du) \\ &+ \int_0^t \int_{\mathbb{R}_+} [\psi(\mu_{s^-} - \delta_{i_0}) - \psi(\mu_{s^-})] \\ &\qquad\qquad \times \mathbf{1}_{\left\{u \leq l_0(\langle \mu_{s^-}, 1_{i_0} \rangle)\right\}} Q_2(ds, du) \\ &+ \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0}} [\psi(\mu_{s^-} + \delta_{i+1} - \delta_i) - \psi(\mu_{s^-})] \\ &\qquad\qquad \times \mathbf{1}_{\left\{u \leq a_0(i) C_{s^-} \langle \mu_{s^-}, 1_i \rangle\right\}} Q_3(ds, du, di) \\ &+ \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0+1}} [\psi(\mu_{s^-} - \delta_i + \delta_{i-1}) - \psi(\mu_{s^-})] \\ &\qquad\qquad \times \mathbf{1}_{\left\{u \leq b_0(i) \langle \mu_{s^-}, 1_i \rangle\right\}} Q_4(ds, du, di) \, ,\end{aligned} \tag{13}$$

which allow us to identify the infinitesimal generator. Moreover, note that for M *a.s.* bounded, the sums in the infinitesimal generator are finite sums (up to $i = M$ for the first one and $i = M - 1$ for the second), and μ_t stays in a finite state space. Then the integrability of the martingale is trivial for any measurable function. □

Corollary 2. *Under the assumptions of Proposition 1, for all φ measurable and locally bounded on $[i_0, +\infty)$, the measure-valued stochastic Becker-Döring process μ satisfies*

$$\langle \mu_t, \varphi \rangle = \langle \mu_{\text{in}}, \varphi \rangle + \mathcal{V}_t^{\varphi} + \mathcal{O}_t^{\varphi} \, ,$$

where $\mathcal{V}_t^{\varphi}$ is the finite variation defined by

$$\begin{aligned}\mathcal{V}_t^{\varphi} := &\int_0^t \varphi(i_0)(k_0(C_s) - l_0(\langle \mu_s, 1_{i_0} \rangle)) ds \\ &+ \int_0^t \sum_{i \geq i_0} (\varphi(i+1) - \varphi(i)) \left(a_0(i) C_s \langle \mu_s, 1_i \rangle - b_0(i+1) \langle \mu_s, 1_{i+1} \rangle\right) ds \, ,\end{aligned}$$

and $\mathcal{O}_t^{\varphi}$ is a $L^2 - (\mathcal{F}_t)_{t \geq 0}$ martingale starting from 0 with (predictable) quadratic variation

$$\begin{aligned}\langle \mathcal{O}^{\varphi} \rangle_t = &\int_0^t \varphi(i_0)^2 (k_0(C_s) + l_0(\langle \mu_s, 1_{i_0} \rangle)) ds \\ &+ \int_0^t \sum_{i \geq i_0} (\varphi(i+1) - \varphi(i))^2 \left(a_0(i) C_s \langle \mu_s, 1_i \rangle + b_0(i+1) \langle \mu_s, 1_{i+1} \rangle\right) ds \, .\end{aligned}$$

Proof. It is a direct consequence of Proposition 1 using the function $\psi(\nu) := \langle \nu, \varphi \rangle$ and by identification of the martingale term thanks to Itô formula. □

4.2. The rescaled process

From Equations (4)-(5) satisfied by $\tilde{\mu}^\varepsilon$ and the scaling properties (6) to (7), we immediatly derive the equation on μ^ε: *a.s.* for all $t \geq 0$,

$$\begin{aligned} \mu_t^\varepsilon \quad = \quad & \mu_{\rm in}^\varepsilon + \int_0^t \int_{\mathbb{R}_+} \varepsilon^\alpha \delta_{\varepsilon^\beta i_0^\varepsilon} \mathbf{1}_{\left\{u \leq k^\varepsilon(C_{s-}^\varepsilon)/\varepsilon^K\right\}} Q_1(\varepsilon^\gamma ds, du) \\ & - \int_0^t \int_{\mathbb{R}_+} \varepsilon^\alpha \delta_{\varepsilon^\beta i_0^\varepsilon} \mathbf{1}_{\left\{u \leq l^\varepsilon(\langle \mu_{s-}^\varepsilon, 1_{\varepsilon^\beta i_0^\varepsilon}\rangle)/\varepsilon^L\right\}} Q_2(\varepsilon^\gamma ds, du) \\ & + \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0^\varepsilon}} \varepsilon^\alpha \left(\delta_{\varepsilon^\beta i + \varepsilon^\beta} - \delta_{\varepsilon^\beta i}\right) \\ & \qquad \times \mathbf{1}_{\left\{u \leq a^\varepsilon(\varepsilon^\beta i) C_{s-}^\varepsilon \langle \mu_{s-}^\varepsilon, 1_{\varepsilon^\beta i}\rangle / \varepsilon^{A+\alpha+\theta}\right\}} Q_3(\varepsilon^\gamma ds, du, di) \\ & - \int_0^t \int_{\mathbb{R}_+ \times \mathcal{N}_{i_0^\varepsilon+1}} \varepsilon^\alpha \left(\delta_{\varepsilon^\beta i} - \delta_{\varepsilon^\beta i - \varepsilon^\beta}\right) \\ & \qquad \times \mathbf{1}_{\left\{u \leq b^\varepsilon(\varepsilon^\beta i) \langle \mu_{s-}^\varepsilon, 1_{\varepsilon^\beta i}\rangle / \varepsilon^{B+\alpha}\right\}} Q_4(\varepsilon^\gamma ds, du, di) \,, \end{aligned} \tag{14}$$

such that *a.s.*, for all $t \geq 0$, we have $C_t^\varepsilon \geq 0$ and

$$\varepsilon^{-\lambda_c} C_t^\varepsilon + \langle \mu_t^\varepsilon, \mathrm{Id}\rangle = M^\varepsilon \,. \tag{15}$$

The next proposition and its corollary readily follows as in the previous section.

Proposition 2. *Let μ^ε constructed thanks to* (6) *for each $\varepsilon > 0$. Then, μ^ε is an $\mathcal{X}$-valued stochastic process which has a.s. sample paths in $\mathcal{D}(\mathbb{R}_+, w - \mathcal{X})$. Moreover, for each $\varepsilon > 0$, a.s. and for all $t \geq 0$, μ_t^ε belongs to the state space*

$$\hat{\mathcal{M}}^\varepsilon := \left\{ \nu \in \mathcal{X} \,:\, \nu = \sum_{i=1}^n \varepsilon^\alpha \delta_{\varepsilon^\beta y_i} \,,\ (y_1, \ldots, y_n) \in \mathcal{N}_{i_0^\varepsilon} \,,\ \langle \mu^\varepsilon, \mathrm{Id}\rangle \leq M^\varepsilon \right\},$$

and we have

$$\sup_{t \in \mathbb{R}_+} \langle \mu_t^\varepsilon, \mathbf{1}\rangle \leq \frac{M^\varepsilon}{x_0^\varepsilon}, \quad a.s.$$

μ^ε is also a Markov proces whose infinitesimal generator $\mathcal{L}^\varepsilon$ is given, for all

$\nu \in \mathcal{X}$ and ψ locally bounded on $\mathcal{X}$ to $\mathbb{R}$, by

$$\begin{aligned}\mathcal{L}^\varepsilon\psi(\nu) = \varepsilon^{\lambda_k}\frac{\psi(\nu+\varepsilon^\alpha\delta_{\varepsilon^\beta i_0^\varepsilon})-\psi(\nu)}{\varepsilon^\alpha}k^\varepsilon(C) \\ + \varepsilon^{\lambda_l}\frac{\psi(\nu-\varepsilon^\alpha\delta_{\varepsilon^\beta i_0^\varepsilon})-\psi(\nu)}{\varepsilon^\alpha}l^\varepsilon(\langle\nu, 1_{\varepsilon^\beta i_0^\varepsilon}\rangle) \\ + \varepsilon^{\lambda_a}\int_{x_0^\varepsilon}^{+\infty}\frac{\psi(\nu+\varepsilon^\alpha\delta_{x+\varepsilon^\beta}-\varepsilon^\alpha\delta_x)-\psi(\nu)}{\varepsilon^{\alpha+\beta}}a^\varepsilon(x)C\nu(dx) \\ + \varepsilon^{\lambda_b}\int_{x_0^\varepsilon+\varepsilon^\beta}^{+\infty}\frac{\psi(\nu-\varepsilon^\alpha\delta_x+\varepsilon^\alpha\delta_{x-\varepsilon^\beta})-\psi(\nu)}{\varepsilon^{\alpha+\beta}}b^\varepsilon(x)\nu(dx)\,,\end{aligned}$$

where $C=\varepsilon^{\lambda_c}(M^\varepsilon - \langle\nu, \mathrm{Id}\rangle)$. For such ψ, the process

$$\psi(\mu_t^\varepsilon) - \psi(\mu_{\mathrm{in}^\varepsilon}) - \int_0^t \mathcal{L}^\varepsilon\psi(\mu_s^\varepsilon)ds$$

is a $L^1 - (\mathcal{F}_t^\varepsilon)_{t\geq 0}$ martingale starting from 0.

Remark 7. The filtration $(\mathcal{F}_t^\varepsilon)_{t\geq 0}$ for each $\varepsilon > 0$ can be easily deduce from the construction of the rescaled measure (6).

Proof. Let us only remark that, similarly to (13), we have for all ψ measurable locally bounded from $\mathcal{X}$ to $\mathbb{R}$,

$$\begin{aligned}\psi(\mu_t^\varepsilon) &= \psi(\mu_{\mathrm{in}}^\varepsilon) \\ &+ \int_0^t\int_{\mathbb{R}_+}[\psi(\mu_{s^-}^\varepsilon+\varepsilon^\alpha\delta_{\varepsilon^\beta i_0^\varepsilon})-\psi(\mu_{s^-}^\varepsilon)]\mathbf{1}_{\left\{u\leq k^\varepsilon(C_{s^-}^\varepsilon)/\varepsilon^K\right\}}Q_1(\varepsilon^\gamma ds, du) \\ &+ \int_0^t\int_{\mathbb{R}_+}[\psi(\mu_{s^-}^\varepsilon-\varepsilon^\alpha\delta_{\varepsilon^\beta i_0^\varepsilon})-\psi(\mu_{s^-}^\varepsilon)] \\ &\qquad\qquad \times \mathbf{1}_{\{u\leq l^\varepsilon(\langle\mu_{s^-}^\varepsilon, 1_{\varepsilon^\beta i_0^\varepsilon}\rangle)/\varepsilon^L\}}Q_2(\varepsilon^\gamma ds, du) \\ &+ \int_0^t\int_{\mathbb{R}_+\times\mathcal{N}_{i_0^\varepsilon}}[\psi(\mu_{s^-}^\varepsilon+\varepsilon^\alpha\delta_{\varepsilon^\beta(i+1)}-\varepsilon^\alpha\delta_{\varepsilon^\beta i})-\psi(\mu_{s^-}^\varepsilon)] \\ &\qquad\qquad \times \mathbf{1}_{\left\{u\leq a^\varepsilon(\varepsilon^\beta i)C_{s^-}^\varepsilon\langle\mu_{s^-}^\varepsilon, 1_{\varepsilon^\beta i}\rangle/\varepsilon^{A+\alpha+\theta}\right\}}Q_3(\varepsilon^\gamma ds, du, di) \\ &+ \int_0^t\int_{\mathbb{R}_+\times\mathcal{N}_{i_0^\varepsilon+1}}[\psi(\mu_{s^-}^\varepsilon-\varepsilon^\alpha\delta_{\varepsilon^\beta i}+\varepsilon^\alpha\delta_{\varepsilon^\beta(i-1)})-\psi(\mu_{s^-}^\varepsilon)] \\ &\qquad\qquad \times \mathbf{1}_{\left\{u\leq b^\varepsilon(\varepsilon^\beta i)\langle\mu_{s^-}^\varepsilon, 1_{\varepsilon^\beta i}\rangle/\varepsilon^{B+\alpha}\right\}}Q_4(\varepsilon^\gamma ds, du, di)\,.\end{aligned} \tag{16}$$

□

We apply this result to the functions $\langle\cdot, \varphi\rangle$, from $\mathcal{X}$ to $\mathbb{R}$ with φ measurable and bounded from $[x_0, +\infty)$ to $\mathbb{R}$ which will be usefull to identify the limit equation (the convergence towards the Lifschitz-Slyozov equation). With such test fonctions, we obtain:

Corollary 3. *Let μ^ε construct thanks to* (6) *for each $\varepsilon > 0$ and φ a bounded measurable real-valued function on $[x_0, +\infty)$. Then, a.s. and for all $t \geq 0$*

$$\langle \mu_t^\varepsilon, \varphi \rangle = \langle \mu_{\mathrm{in}}^\varepsilon, \varphi \rangle + \mathcal{V}_t^{\varepsilon,\varphi} + \mathcal{O}_t^{\varepsilon,\varphi} \,, \tag{17}$$

where $\mathcal{V}_t^{\varepsilon,\varphi}$ is the finite-variation part of $\langle \mu_t^\varepsilon, \varphi \rangle$ given by

$$\begin{aligned} \mathcal{V}_t^{\varepsilon,\varphi} = & \int_0^t \varphi(x_0^\varepsilon) \left[\varepsilon^{\lambda_k} k^\varepsilon(C_s^\varepsilon) - \varepsilon^{\lambda_l} l^\varepsilon(\langle \mu_s^\varepsilon, 1_{\varepsilon^\beta i_0^\varepsilon} \rangle) \right] ds \\ & + \int_0^t \int_{x_0^\varepsilon}^{+\infty} \varepsilon^{\lambda_a} \Delta_\varepsilon(\varphi) a^\varepsilon(x) C_s^\varepsilon \mu_s^\varepsilon(dx)\, ds \\ & - \int_0^t \int_{x_0^\varepsilon + \varepsilon^\beta}^{+\infty} \varepsilon^{\lambda_b} \tau_\varepsilon \Delta_\varepsilon(\varphi)(x) b^\varepsilon(x) \mu_s^\varepsilon(dx)\, ds \,. \end{aligned} \tag{18}$$

where τ_ε is the ε^β-translation, $\tau_\varepsilon f = f(\cdot - \varepsilon^\beta)$, and $\Delta_\varepsilon(\varphi)$ is the ε^β-discrete derivative of φ given by

$$\Delta_\varepsilon(\varphi) = \frac{\varphi(x + \varepsilon^\beta) - \varphi(x)}{\varepsilon^\beta} \,.$$

Moreover, $\mathcal{O}_t^{\varepsilon,\varphi}$ is a $L^2 - (\mathcal{F}_t^\varepsilon)_{t \geq 0}$ martingale starting from 0 with (predictable) quadratic variation :

$$\begin{aligned} \langle \mathcal{O}^{\varepsilon,\varphi} \rangle_t = & \, \varepsilon^\alpha \int_0^t \varphi(x_0^\varepsilon)^2 \left[\varepsilon^{\lambda_k} k^\varepsilon(C_s^\varepsilon) + \varepsilon^{\lambda_l} l^\varepsilon(\langle \mu_s^\varepsilon, 1_{\varepsilon^\beta i_0^\varepsilon} \rangle) \right] ds \\ & + \varepsilon^{\alpha+\beta} \int_0^t \int_{x_0^\varepsilon}^{+\infty} \varepsilon^{\lambda_a} \left(\Delta_\varepsilon(\varphi)(x) \right)^2 a^\varepsilon(x) C_s^\varepsilon \mu_s^\varepsilon(dx)\, ds \\ & + \varepsilon^{\alpha+\beta} \int_0^t \int_{x_0^\varepsilon + \varepsilon^\beta}^{+\infty} \varepsilon^{\lambda_b} \left(\tau_\varepsilon \Delta_\varepsilon(\varphi)(x) \right)^2 b^\varepsilon(x) \mu_s^\varepsilon(dx)\, ds \end{aligned} \tag{19}$$

We attempt to pass to the limit in (17) when enough compactness is available. We want the finite variation (18) to converge to the weak form of the Lifshitz-Slyozov operator (including boundary value) and the martingale (19) to vanish (its quadratic variation) to recover a weak formulation of the deterministic problem at the limit in (17). For that, we need moment estimates and tightness properties to obtain the compactness in the appropriate space. These are the results presented in the next section.

5. Estimations and technical results

5.1. Moment estimates

The proof of convergence of $\{\mu^\varepsilon\}$, the sequence of measure-valued SBD processes constructed in Section 3.2, will rely on compactness arguments (or tightness). These are achieved, in particular, thanks to moment estimates that are uniform with respect to ε. In this section, we provide the appropriate estimates that shall be necessary in the next sections. We suppose all along this section (except if it is mentioned) that

$$\lambda_a, \lambda_b, \lambda_k, \lambda_l, \lambda_c \geq 0\,, \tag{20}$$

which is weakest than (H7). It allows us some flexibility if other scaling are investigated.

Our first proposition provides a control of the L^p-norm of the total mass of the measure, namely $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$, then a L^∞ control of the particle C_t^ε and of the first x-moment $\langle \mu_t^\varepsilon, \mathrm{Id}\rangle$.

Proposition 3. *Let μ^ε constructed by* (6) *for each $\varepsilon > 0$. Assume that* (H1) *to* (H3) *and* (20) *hold. Then, for all $T > 0$ and as $\varepsilon \to 0$, we have a.s. that*

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} C_t^\varepsilon < +\infty\,, \tag{21}$$

and

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathrm{Id}\rangle < +\infty\,. \tag{22}$$

Moreover, if (H8) *holds, then*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\right] < +\infty\,. \tag{23}$$

And if additionally there exists $p \in \mathbb{N}^*$ such that $\sup_{\varepsilon>0} \mathbb{E}\left[(\langle \mu_{\mathrm{in}}^\varepsilon, \mathbf{1}\rangle)^p\right] < +\infty$, then for all $q \in \mathbb{N}^*$ and $q \leq p$

$$\sup_{\varepsilon>0} \mathbb{E}\left[\sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle^q\right] < +\infty\,.$$

Proof. First, remark that the conservation of mass (15) yields for all $t \in [0,T]$ and $\varepsilon > 0$

$$C_t^\varepsilon \leq \varepsilon^{\lambda_c} M^\varepsilon\,.$$

Then, the convergence of M^ε in (H2) ensures that for ε small enough, we have *a.s.* $M^\varepsilon \leq 2m$ and thus *a.s.* $C_t^\varepsilon \leq 2m$ for all $t \in [0,T]$ which is uniform in ε, giving (21). We can similarly show (22).

Let us now prove the estimation (23). From the stochastic differential equation (14) on μ^ε, dropping the non-positive terms, we have

$$\langle \mu_t^\varepsilon, \mathbf{1}\rangle \leq \langle \mu_{\rm in}^\varepsilon, \mathbf{1}\rangle + \int_0^t \int_{\mathbb{R}_+} \varepsilon^\alpha \mathbf{1}_{\left\{u \leq k^\varepsilon(C_{s^-}^\varepsilon)/\varepsilon^K\right\}} Q_1(\varepsilon^\gamma ds, du)\,. \tag{24}$$

We remark that, for ε small enough, due to (8) and the previous bound on C_t^ε we have

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} k^\varepsilon(C_t^\varepsilon) \leq K_k\,. \tag{25}$$

Thus, in (24), taking the sup on $[0,T]$

$$\sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle \leq \langle \mu_{\rm in}^\varepsilon, \mathbf{1}\rangle + \int_0^T \int_{\mathbb{R}_+} \varepsilon^\alpha \mathbf{1}_{\{u\leq K_k/\varepsilon^K\}} Q_1(\varepsilon^\gamma ds, du).$$

Thanks to the uniform L^1 bound on the initial moment (H8) represented here by K^1, we conclude by taking the mean that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\right] \leq K^1 + \varepsilon^{\lambda_k} K_k T\,.$$

Now we consider the case when $p > 1$. From the stochastic differential equation (16), taking $\psi(\mu) = \langle \mu, \mathbf{1}\rangle^p$ it follows (dropping again the non-positive terms)

$$\begin{aligned}(\langle \mu_t^\varepsilon, \mathbf{1}\rangle)^p &\leq (\langle \mu_{\rm in}^\varepsilon, \mathbf{1}\rangle)^p \\ &+ \int_0^t \int_{\mathbb{R}_+} \Big[(\langle \mu_{s^-}^\varepsilon, \mathbf{1}\rangle + \varepsilon^\alpha)^p - (\langle \mu_{s^-}^\varepsilon, \mathbf{1}\rangle)^p\Big] \mathbf{1}_{\left\{u\leq k^\varepsilon(C_{s^-}^\varepsilon)/\varepsilon^K\right\}} Q_1(\varepsilon^\gamma ds, du).\end{aligned}$$

Using that there exist $c_p > 0$ such that for all $x \geq 0$ we have $(x+\varepsilon^\alpha)^p - x^p \leq c_p(\varepsilon^{p\alpha} + \varepsilon^\alpha x^{p-1})$, taking the sup on $[0,T]$ and then the mean, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} (\langle \mu_t^\varepsilon, \mathbf{1}\rangle)^p\right] \leq K^p + \varepsilon^{\lambda_k} c_p K_k \mathbb{E}\left[\varepsilon^{(p-1)\alpha} + \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle^{p-1}\right]\,,$$

where K^p is the initial L^p bound. A recursive argument on p and the inclusion of the L^p spaces allow us to conclude the proof. □

Remark 8. We here mention that the hypothesis $\lambda_c < 0$ is known as the Lifschitz-Slyozov-Wagner case. In this case the estimations obtained through the mass conservation do not work. One should probably keep some negative terms, and find a suitable lower bound on them in order to get similar estimates, as in [36].

The next proposition provides the propagation of an extra x-moment of the rescaled measure-valued SBD process so that it controls the tail at infinity. It will be necessary to prove the tightness in $(\mathcal{X}, w)$.

Proposition 4. *Let μ^ε constructed by* (6) *for each $\varepsilon > 0$. Assume that* (H1) *to* (H3)*,* (H5)*,* (H8)*,* (H9) *and* (20) *hold. In particular, let $\Phi_1 \in \mathcal{U}_{1,\infty}$ given by* (H9)*. Then, for all $T > 0$ and as $\varepsilon \to 0$, we have*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \Phi_1\rangle\right] < +\infty\,.$$

Proof. Let $T > 0$. We will use the same strategy as in the proof of Proposition 3, but using the Φ_1 given by (H9). As Φ_1 is nonnegative and increasing (because $\Phi_1(x) \le x\Phi_1'(x)$ by convexity of Φ_1 and $\Phi_1(0) = 0$, so that Φ_1' is non-negative), we may also drop the non-positive terms, to obtain

$$\begin{aligned}\langle \mu_t^\varepsilon, \Phi_1\rangle \le \langle \mu_{\rm in}^\varepsilon, \Phi_1\rangle + \int_0^t \int_{\mathbb{R}_+} \varepsilon^\alpha \Phi_1(x_0^\varepsilon) \mathbf{1}_{\left\{u\le k^\varepsilon(C_{s-}^\varepsilon)/\varepsilon^K\right\}}\, Q_1(\varepsilon^\gamma ds, du)\\ + \int_0^t \int_{\mathbb{R}_+\times \mathcal{N}_{i_0^\varepsilon}} \varepsilon^\alpha \left(\Phi_1(\varepsilon^\beta i + \varepsilon^\beta) - \Phi_1(\varepsilon^\beta i)\right) \mathbf{1}_{\left\{u\le a^\varepsilon(\varepsilon^\beta i) C_{s-}^\varepsilon \langle \mu_{s-}^\varepsilon, 1_{\varepsilon^\beta i}\rangle/\varepsilon^{A+\alpha+\theta}\right\}}\\ \times Q_3(\varepsilon^\gamma ds, du, di)\end{aligned}$$

Then, using the convexity of Φ_1, the concavity of Φ_1' and then its non-increasing right derivative (denoted $\Phi_{1,r}''$), we have, for all $i \ge i_0^\varepsilon$

$$\Phi_1(\varepsilon^\beta i + \varepsilon^\beta) - \Phi_1(\varepsilon^\beta i) \le \varepsilon^\beta \Phi_1'(\varepsilon^\beta i + \varepsilon^\beta) \le \varepsilon^\beta \left(\Phi_1'(\varepsilon^\beta i) + \varepsilon^\beta \Phi_{1,r}''(0)\right)\,.$$

Taking the supremum in time, and then the expectation, this entails

$$\begin{aligned}\mathbb{E}\left[\sup_{\sigma\in(0,t)} \langle \mu_\sigma^\varepsilon, \Phi_1\rangle\right] \le \mathbb{E}\left[\langle \mu_{\rm in}^\varepsilon, \Phi_1\rangle\right] + \varepsilon^{\lambda_k} \Phi_1(x_0^\varepsilon) K_k T\\ + 2m\varepsilon^{\lambda_a} \int_0^t \mathbb{E}\left[\sup_{\sigma\in(0,s)} \int_{\mathbb{R}_+} (\Phi_1'(x) + \varepsilon^\beta \Phi_{1,r}''(0)) a^\varepsilon(x) \mu_\sigma^\varepsilon(dx)\right] ds\,, \quad (26)\end{aligned}$$

where we used again that for ε small enough we have (25). Since $\Phi_1 \in \mathcal{U}_{1,\infty}$ we have $x\Phi_1'(x) \le 2\Phi_1(x)$ for all $x \ge 0$ by [34, Lemma A.1]. Thus for any $R > 0$ and thanks to (H5) on a^ε we obtain

$$\begin{aligned}\int_0^{+\infty} \Phi_1'(x) a^\varepsilon(x) \mu_t^\varepsilon(dx) = \int_0^R \Phi_1'(x) a^\varepsilon(x) \mu_t^\varepsilon(dx) + \int_R^{+\infty} \Phi_1'(x) a^\varepsilon(x) \mu_t^\varepsilon(dx)\\ \le K_a \int_0^R (1+x)\Phi_1'(x) \mu_t^\varepsilon(dx) + K_a\left(\frac{1}{R} + 1\right) \int_R^{+\infty} x\Phi_1'(x) \mu_t^\varepsilon(dx)\\ \le K_a \left(\sup_{x\in(0,R)} \Phi_1'(x)\right) \langle \mu_t^\varepsilon, \mathbf{1}\rangle + 2K_a \langle \mu_t^\varepsilon, \Phi_1\rangle + \frac{2K_a}{R} \langle \mu_t^\varepsilon, \Phi_1\rangle\,.\end{aligned}$$

Thus, there exists a constant K which depends on Φ_1, R, the uniform bound (23) on $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$ and on K_a such that

$$\mathbb{E}\left[\sup_{\sigma\in(0,s)}\int_0^{+\infty}\Phi_1'(x)a^\varepsilon(x)\mu_\sigma^\varepsilon(dx)\right]\le K\left(1+\mathbb{E}\left[\sup_{\sigma\in(0,s)}\langle\mu_\sigma^\varepsilon,\Phi_1\rangle\right]\right). \quad (27)$$

Also, there exists a constant, still denoted by K, which depends on the uniform bound (23) on $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$, the uniform bound (22) on $\langle \mu_t^\varepsilon, \mathrm{Id}\rangle$ and on K_a such that

$$\mathbb{E}\left[\sup_{\sigma\in(0,s)}\int_0^{+\infty}a^\varepsilon(x)\mu_\sigma^\varepsilon(dx)\right]\le K. \quad (28)$$

Finally, combining (26), (27) and (28), there exists a constant, still denoted by K (independent of ε when small enough), such that for all $t\in[0,T]$

$$\mathbb{E}\left[\sup_{\sigma\in(0,t)}\langle\mu_\sigma^\varepsilon,\Phi_1\rangle\right]\le K\left(1+\int_0^t\mathbb{E}\left[\sup_{\sigma\in(0,s)}\langle\mu_\sigma^\varepsilon,\Phi_1\rangle)\right]ds\right).$$

We get the desired estimation using the Gronwall lemma.

□

We will also need a superlinear control on the total mass $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$ avoiding explosion. It will notably be useful to treat the boundary condition.

Proposition 5. *Let μ^ε be constructed by* (6) *for each $\varepsilon>0$. Assume that* (H1) *to* (H3)*,* (H5)*,* (H8)*,* (H10) *and* (20) *hold. In particular, let $\Phi_2\in\mathcal{U}_{1,\infty}$ given by* (H10)*. Then, for all $T>0$ and as $\varepsilon\to0$, we have*

$$\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{t\in[0,T]}\Phi_2(\langle\mu_t^\varepsilon,\mathbf{1}\rangle)\right]<+\infty.$$

Proof. Let $\psi(\mu)=\Phi_2(\langle\mu,\mathbf{1}\rangle)$ in (16), and using that Φ_2 is increasing we drop the non-negative terms

$$\Phi_2(\langle\mu_t^\varepsilon,\mathbf{1}\rangle)\le\Phi_2(\langle\mu_{\mathrm{in}}^\varepsilon,\mathbf{1}\rangle)$$
$$+\int_0^t\int_{\mathbb{R}_+}\Phi_2(\langle\mu_{s-}^\varepsilon,\mathbf{1}\rangle+\varepsilon^\alpha)-\Phi_2(\langle\mu_{s-}^\varepsilon,\mathbf{1}\rangle)]\mathbf{1}_{\{u\le k^\varepsilon(C_{s-}^\varepsilon)/\varepsilon^K\}}\,Q_1(\varepsilon^\gamma ds,du).$$

Then the proof follows by the same arguments as Proposition 4. □

5.2. Tightness of the rescaled process

The aim of this section is to prove the following tightness property of the family $\{\mu^\varepsilon\}$ of $\mathcal{X}$-valued processes.

Proposition 6. *Let* $(\mu^\varepsilon)_{t\geq 0}$ *be constructed by* (6) *for each* $\varepsilon > 0$. *Assume that* (H1) *to* (H6), (H8), (H9) *and* (20) *hold. Then* $\{\mu^\varepsilon\}$ *is tight in* $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$ *and* $\{C^\varepsilon\}$ *is tight in* $\mathcal{P}(\mathcal{D}(\mathbb{R}_+,\mathbb{R}_+))$. *Moreover, any accumulation point* μ *of* $\{\mu^\varepsilon\}$ *belongs a.s. to* $\mathcal{C}(\mathbb{R}_+, w-\mathcal{X})$.

The proof of this result rests on the Aldous criterion for tightness [7, p 176]. It is in two parts, first the compact containment condition and then the equicontinuity (in the sense of càdlàg). For the former we need to make explicit a weakly compact of $\mathcal{X}$.

Lemma 1. *Under the same assumptions as Proposition 6, for all* $T > 0$ *and* η *sufficiently small, there exists a compact* $\mathcal{K}_{\eta,T}$ *of* $(\mathcal{X}, w)$ *such that*

$$\mathbb{P}\left(\{\mu_t^\varepsilon \in \mathcal{K}_{\eta,T} \,:\, 0 \leq t \leq T\}\right) \geq 1-\eta\,.$$

Proof. Let $\eta \in (0,1)$ and $T > 0$. Thanks to Propositions 3 and 4, we define three constants

$$C^1_{\eta,T} = 3\sup_\varepsilon \mathbb{E}\left[\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \Phi_1\rangle\right]/\eta,\ C^2_{\eta,T} = 3\sup_\varepsilon \mathbb{E}\left[\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \mathrm{Id}\rangle\right]/\eta,$$

and,

$$C^3_{\eta,T} = 3\sup_\varepsilon \mathbb{E}\left[\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\right]/\eta.$$

We then introduce the weakly relatively compact set of $\mathcal{X}$, by Lemma A.2,

$$\mathcal{K}_{\eta,T} = \left\{\mu \in \mathcal{X} : \langle \mu, \Phi_1\rangle \leq C^1_{\eta,T},\ \langle \mu, \mathrm{Id}\rangle \leq C^2_{\eta,T},\ \langle \mu, \mathbf{1}\rangle \leq C^3_{\eta,T}\right\}.$$

We have, by Markov's inequality, that

$$\mathbb{P}\left(\left\{\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \Phi_1\rangle \geq C^1_{\eta,T}\right\} \cup \left\{\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \mathrm{Id}\rangle \geq C^2_{\eta,T}\right\} \cup \left\{\sup_{t\in(0,T)} \langle \mu_t^\varepsilon, \mathbf{1}\rangle \geq C^3_{\eta,T}\right\}\right) \leq \eta\,,$$

providing the desired compact containment condition. □

The second step of the proof gives the equicontinuity property of the process on the càdlàg space. To that we introduce a metric $d_\mathcal{X}$ equivalent to the weak convergence on $\mathcal{X}$. We follow [27] to define a sequence $\{\varphi_k\}$ of functions in $\mathcal{C}^1_b([x_0,+\infty))$ such that $\|\varphi_k\|_\infty + \|\varphi_k'\|_\infty \leq 1$ and for all $(\nu_1,\nu_2) \in \mathcal{X}\times\mathcal{X}$,

$$d_\mathcal{X}(\nu_1,\nu_2) = \sum_{k\geq 1} 2^{-k} |\langle(\mathbf{1}+\mathrm{Id})\cdot\nu_1, \varphi_k\rangle - \langle(\mathbf{1}+\mathrm{Id})\cdot\nu_2,\varphi_k\rangle|\,. \tag{29}$$

Lemma 2. *Under the same assumptions as Proposition 6, for all $\eta > 0$, there exists $h > 0$ such that*

$$\sup_{\varepsilon} \sup_{t} \sup_{s\in(0,h)} \mathbb{E}\left[d_{\mathcal{X}}(\mu^{\varepsilon}_{t+s}, \mu^{\varepsilon}_{t})\right] \leq \eta\,.$$

Proof. Let $\{\varphi_k\}$ be as described above. For any $k \in \mathbb{N}$, we define $\psi_k(x) = (1+x)\varphi_k(x)$ and then the approximation $\psi_{k,R}(x) = (1+x)\varphi_k(x)$ if $x < R$ and $\psi_{k,R}(x) = (1+R)\varphi_k(x)$ otherwise. Then, we get for all $h < T$ and $t \in [0, T-h]$ with $s \in (0,h)$,

$$\begin{aligned}|\langle(\mathbf{1}+\mathrm{Id})\cdot\mu^{\varepsilon}_{t+s}, \varphi_k\rangle - \langle(\mathbf{1}+\mathrm{Id})\cdot\mu^{\varepsilon}_{t}, \varphi_k\rangle| \\ \leq |\langle\mu^{\varepsilon}_{t+s}, \psi_{k,R}\rangle - \langle\mu^{\varepsilon}_{t}, \psi_{k,R}\rangle| + 2\sup_{t\in[0,T]}|\langle\mu^{\varepsilon}_{t}, (\mathrm{Id}-R)\varphi_k\mathbf{1}_{[R,+\infty)}\rangle|\,.\end{aligned} \tag{30}$$

Since $\|\varphi_k\|_\infty + \|\varphi_k'\|_\infty \leq 1$, we obtain

$$\begin{aligned}&\sup_{\varepsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|\langle\mu^{\varepsilon}_{t}, (\mathrm{Id}-R)\varphi_k\mathbf{1}_{[R,+\infty)}\rangle|\right] \\ &\qquad\leq \left(\sup_{x>R}\frac{x}{\Phi_1(x)}\right)\sup_{\varepsilon}\mathbb{E}\left[\sup_{t\in[0,T]}\langle\mu^{\varepsilon}_{t}, \Phi_1\rangle\right].\end{aligned}$$

By Propostion 4 and since Φ_1 belongs to $\mathcal{U}_\infty$, for $\eta > 0$ it exists R large enough such that

$$\sup_{\varepsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|\langle\mu^{\varepsilon}_{t}, (\mathrm{Id}-R)\varphi_k\mathbf{1}_{[R,+\infty)}\rangle|\right] \leq \frac{\eta}{4}\,. \tag{31}$$

Moreover, we have with the notations of Corollary 3

$$\begin{aligned}&|\langle\mu^{\varepsilon}_{t+s}, \psi_{k,R}\rangle - \langle\mu^{\varepsilon}_{t}, \psi_{k,R}\rangle| \\ &\qquad\leq |\mathcal{V}^{\varepsilon,\psi_{k,R}}_{t+s} - \mathcal{V}^{\varepsilon,\psi_{k,R}}_{t}| + \sup_{s\in(0,h)}|\mathcal{O}^{\varepsilon,\psi_{k,R}}_{t+s} - \mathcal{O}^{\varepsilon,\psi_{k,R}}_{t}|\,.\end{aligned}$$

Then taking expectation, applying the Cauchy-Schwarz inequality in the martingal term and then the Burkholder-Davis-Gundy inequality [49, Theorem 48, p. 193], both on the martingale term, we get

$$\begin{aligned}&\sup_{s\in(0,h)}\mathbb{E}\left[|\langle\mu^{\varepsilon}_{t+s}, \psi_{k,R}\rangle - \langle\mu^{\varepsilon}_{t}, \psi_{k,R}\rangle|\right] \\ &\qquad\leq \sup_{s\in(0,h)}\mathbb{E}\left[\left|\mathcal{V}^{\varepsilon,\psi_{k,R}}_{t+s} - \mathcal{V}^{\varepsilon,\psi_{k,R}}_{t}\right|\right] + \mathbb{E}\left[\left|\langle\mathcal{O}^{\varepsilon,\psi_{k,R}}\rangle_{t+h} - \langle\mathcal{O}^{\varepsilon,\psi_{k,R}}\rangle_{t}\right|\right]\end{aligned}$$

Again, since $\|\varphi_k\|_\infty + \|\varphi_k'\|_\infty \leq 1$ and by (H3) to (H6) with (8), for all $s \in (0,h)$, we obtain

$$\begin{aligned}\left|\mathcal{V}^{\varepsilon,\psi_{k,R}}_{t+s} - \mathcal{V}^{\varepsilon,\psi_{k,R}}_{t}\right| &\leq \left(K_k\varepsilon^{\lambda_k} + K_l\varepsilon^{\lambda_l}\sup_{[0,T]}\langle\mu^{\varepsilon}_{t}, \mathbf{1}\rangle\right)(1+R)h \\ &\quad + \left(2mK_a\varepsilon^{\lambda_a} + K_b\varepsilon^{\lambda_b}\right)\left[\sup_{[0,T]}\langle\mu^{\varepsilon}_{t}, \mathbf{1}\rangle + \sup_{[0,T]}\langle\mu^{\varepsilon}_{t}, \mathrm{Id}\rangle\right](2+R)h\,,\end{aligned}$$

and,

$$\left|\langle \mathcal{O}^{\varepsilon,\psi_{k,R}}\rangle_{t+h} - \langle \mathcal{O}^{\varepsilon,\psi_{k,R}}\rangle_t\right| \leq \varepsilon^{\alpha}\left(K_k\varepsilon^{\lambda_k} + K_l\varepsilon^{\lambda_l}\sup_{[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle\right)(1+R)^2 h$$

$$+ \varepsilon^{\alpha+\beta}\left(2mK_a\varepsilon^{\lambda_a} + K_b\varepsilon^{\lambda_b}\right)\left[\sup_{[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle + \sup_{[0,T]}\langle \mu_t^\varepsilon, \mathrm{Id}\rangle\right](2+R)^2 h\,.$$

Thus, using estimates in Proposition 3 and under (20), we deduce that it exists h (small enough) such that:

$$\sup_{s\in(0,h)} \mathbb{E}\left[|\langle \mu_{t+s}^\varepsilon, \psi_{k,R}\rangle - \langle \mu_t^\varepsilon, \psi_{k,R}\rangle|\right] \leq \frac{\eta}{2}\,, \tag{32}$$

where the estimation is uniform in ε and t. We conclude by combining (31) and (32) into (30).

□

Finally, we finish by the continuity of the limit process.

Lemma 3. *Under the same assumptions as Proposition 6. Let μ be an accumulation point of $\{\mu^\varepsilon\}$ in $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$. Then, μ a.s. belongs to $\mathcal{C}(\mathbb{R}_+, w-\mathcal{X})$.*

Proof. Using the stochastic differential equation (16), we obtain for all φ with $\|\varphi\|_\infty + \|\varphi'\|_\infty \leq 1$ that

$$\begin{aligned}
&|\langle(\mathbf{1}+\mathrm{Id})\cdot\mu_s^\varepsilon, \varphi\rangle - \langle(\mathbf{1}+\mathrm{Id})\cdot\mu_{s^-}^\varepsilon, \varphi\rangle| \\
&\leq \varepsilon^{\alpha}\Big(|(1+x_0^\varepsilon)\varphi(x_0^\varepsilon)| + \sup_{x\in(x_0^\varepsilon, M^\varepsilon/\varepsilon^\alpha)}|(1+x+\varepsilon^\beta)\varphi(x+\varepsilon^\beta) - (1+x)\varphi(x)|\Big) \\
&\qquad\qquad \leq \varepsilon^{\alpha}(1+2x_0) + \varepsilon^{\alpha+\beta}\Big(\frac{2m}{\varepsilon^\alpha} + \varepsilon^\beta\Big) + \varepsilon^{\alpha+\beta}\,.
\end{aligned}$$

We deduce that for all $T\geq 0$, we have *a.s.* $\lim_{\varepsilon\to 0}\sup_{s\in[0,T]} d_{\mathcal{X}}(\mu_s^\varepsilon, \mu_{s^-}^\varepsilon) = 0$. This concludes the proof.

□

Proof of Proposition 6. The tighness of $\{\mu^\varepsilon\}$ readily follows from Lemma 1 and Lemma 2 which are the Aldous criterion of tightness given in [7, p 176]. The continuity is a direct consequence of Lemma 3. The tightness of $\{C^\varepsilon\}$ is by its definition an immediate consequence. □

5.3. Tightness of the boundary term

While trying to pass to the limit in the generator in $(\mathcal{X}, w)$ it makes appear a term $\langle \mu_t^\varepsilon, 1_{x_0^\varepsilon}\rangle$ in (18), for which we need to prove also a tightness property. However, when looking at the time evolution equation of $\langle \mu_t^\varepsilon, 1_{x_0^\varepsilon}\rangle$, it appears that such a term may evolve at a faster time scale than μ_ε, viewed as a measure in $\mathcal{X}$. We use ideas from [33], and separate the action of μ^ε as a measure on

$[x_0,+\infty)$ and the evaluation at a given fixed size. But, the equation on $\langle \mu_t^\varepsilon, 1_{x_0^\varepsilon}\rangle$ involves $\langle \mu_t^\varepsilon, 1_{x_0^\varepsilon+\varepsilon^\beta}\rangle$, the latter involves $\langle \mu_t^\varepsilon, 1_{x_0^\varepsilon+2\varepsilon^\beta}\rangle$, *etc.* Thus, we need to consider together all the evaluations of the measure at points $i\varepsilon^\beta$. That is, for all ε and all t, we define the sequence $p_t^\varepsilon = (p_{n,t}^\varepsilon)_{n\in\mathbb{N}}$ by

$$p_{n,t}^\varepsilon = \langle \mu_t^\varepsilon, 1_{\varepsilon^\beta(i_0^\varepsilon+n)}\rangle \,. \tag{33}$$

These sequences belong for all ε to the space ℓ_1^+, the non-negative cone of the space of summable sequences, that is

$$\ell_1^+ := \left\{ (q_n)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}} \,:\, \|q\|_{\ell_1} := \sum_{n\geq 0} q_n < +\infty \right\}.$$

We also remark that, for each ε, this sequence is by definition compactly supported between 0 and $M^\varepsilon/\varepsilon^{\alpha+\beta} - i_0^\varepsilon$. For the remainder we let in ℓ_1^+ the canonical sequences $(\mathbf{1}_k)_{k\in\mathbb{N}}$ be defined for all k by $\mathbf{1}_{k,n}=0$ for all $n\neq k$ and $\mathbf{1}_{k,k}=1$.

The following proposition and corollary are then immediate.

Proposition 7. *Let p^ε given by* (33) *for each $\varepsilon>0$. Then, p^ε is a ℓ_1^+-valued stochastic process which has sample paths a.s. in $\mathcal{D}(\mathbb{R}_+, v-\ell_1^+)$. Its infinitesimal generator is defined for all measurable and locally bounded g and compactly supported $q\in\ell_1^+$ by*

$$\begin{aligned}\mathcal{H}^\varepsilon g(q) &= \varepsilon^{\lambda_k}\frac{g(q+\varepsilon^\alpha\mathbf{1}_0)-g(q)}{\varepsilon^\alpha}k^\varepsilon(C^\varepsilon) + \varepsilon^{\lambda_l}\frac{g(q-\varepsilon^\alpha\mathbf{1}_0)-g(q)}{\varepsilon^\alpha}l^\varepsilon(q_0)\\ &\quad + \varepsilon^{\lambda_a-(1-r_a)\beta}\sum_{n\geq 0}\frac{g(q+\varepsilon^\alpha(\mathbf{1}_{n+1}-\mathbf{1}_n))-g(q)}{\varepsilon^\alpha}\frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_a\beta}}C^\varepsilon q_n\\ &\quad + \varepsilon^{\lambda_b-(1-r_b)\beta}\sum_{n\geq 1}\frac{g(q-\varepsilon^\alpha(\mathbf{1}_{n}-\mathbf{1}_{n-1}))-g(q)}{\varepsilon^\alpha}\frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_b\beta}}q_n\,,\end{aligned} \tag{34}$$

where $C^\varepsilon = \varepsilon^{\lambda_c}(M^\varepsilon - \varepsilon^\beta\sum_{n\geq 0}(n+i_0^\varepsilon)q_n)$. Moreover, for such a function g,

$$g(p_t^\varepsilon) - g(p_{\text{in}}^\varepsilon) - \int_0^t \mathcal{H}^\varepsilon g(p_s^\varepsilon)\,ds$$

is a $L^1-(\mathcal{F}_t^\varepsilon)_{t\geq 0}$ martingale.

Remark 9. $v-\ell_1^+$, or alternatively (ℓ_1^+, v), stands for ℓ_1^+ equipped with the vague topology,*i.e.* the topology of the convergence, $q^\varepsilon\to q$ in $v-\ell_1^+$ if and only if $\langle q^\varepsilon, u\rangle_{\ell^1}\to\langle q,u\rangle_{\ell^1}$ for all $u\in\ell_0$ (the sequences vanishing at infinity). We recall, for any $q\in\ell_1^+$ and any bounded sequence u, we write $\langle q,u\rangle_{\ell^1} := \sum_{n\in\mathbb{N}} q_n u_n$. Remark, the space is not the Banach space, as usual, but is a Polish space (consider for instance its canonical homeomorphism with $(\mathcal{M}_b(\mathbb{N}), v)$).

Remark 10. The scaling exponents r_a and r_b in Equation (34) are those given by Assumption 6 and ensure that both $a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))/\varepsilon^{r_a\beta}$ and $b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))/\varepsilon^{r_b\beta}$ stay bounded and converge to positive values as $\varepsilon\to 0$. Hence the exponents

$\lambda_a - (1-r_a)\beta$, $\lambda_b - (1-r_b)\beta$ are really proper time scales of the infinitesimal generator $\mathcal{H}^\varepsilon$, which is then made of terms with potentially different scalings (distinct from λ_k, λ_l). This implies that the main part of the generator depends on the values of the exponents and particularly on the values of r_a and r_b. In particular, for $\min(r_a, r_b) < 1$, the time scale of the infinitesimal generator $\mathcal{H}^\varepsilon$ is faster than the time scale of the generator $\mathcal{L}^\varepsilon$ of μ_ε.

Similarly to Corollary 3, we consider the functions $\langle \cdot, u\rangle_{\ell_1} : q \mapsto \langle q, u\rangle_{\ell_1}$ for u in ℓ_0 and we deduce

Corollary 4. *Let p^ε be defined by (33) for each $\varepsilon > 0$. For all $u \in \ell_0$*

$$\langle p^\varepsilon_t, u\rangle_{\ell^1} = \langle p^\varepsilon_{\text{in}}, u\rangle_{\ell^1} + \mathcal{V}^{\varepsilon,u}_t + \mathcal{O}^{\varepsilon,u}_t\,, \tag{35}$$

where $\mathcal{V}^{\varepsilon,u}_t$ is the finite variation part given by

$$\begin{aligned}\mathcal{V}^{\varepsilon,u}_t = \int_0^t &\left(\varepsilon^{\lambda_k} k^\varepsilon(C^\varepsilon_t) - \varepsilon^{\lambda_l} l^\varepsilon(p^\varepsilon_{0,s})\right) u_0 ds \\ &+ \int_0^t \varepsilon^{\lambda_a-(1-r_a)\beta} \sum_{n\geq 0} (u_{n+1}-u_n) \frac{a^\varepsilon(\varepsilon^\beta(i^\varepsilon_0+n))}{\varepsilon^{r_a\beta}} C p^\varepsilon_{n,s}\, ds \\ &\qquad - \int_0^t \varepsilon^{\lambda_b-(1-r_b)\beta} \sum_{n\geq 1} (u_n-u_{n-1}) \frac{b^\varepsilon(\varepsilon^\beta(i^\varepsilon_0+n))}{\varepsilon^{r_b\beta}} p^\varepsilon_{n,s}\, ds\,,\end{aligned}$$

with $C^\varepsilon_t = \varepsilon^{\lambda_c}(M^\varepsilon - \sum_{n\geq 0} \varepsilon^\beta(n+i^\varepsilon_0)p^\varepsilon_{n,t})$. Moreover, $\mathcal{O}^{\varepsilon,u}_t$ is a $L^2 - (\mathcal{F}^\varepsilon_t)_{t\geq 0}$ martingale of (predictable) quadratic variation

$$\begin{aligned}\langle \mathcal{O}^{\varepsilon,u}\rangle_t = \int_0^t &\varepsilon^\alpha \left(\varepsilon^{\lambda_k} k^\varepsilon(C^\varepsilon_s) + \varepsilon^{\lambda_l} l^\varepsilon(p^\varepsilon_{0,s})\right) u_0^2\, ds \\ &+ \int_0^t \varepsilon^{\lambda_a+\alpha-(1-r_a)\beta} \sum_{n\geq 0} (u_{n+1}-u_n)^2 \frac{a^\varepsilon(\varepsilon^\beta(i^\varepsilon_0+n))}{\varepsilon^{r_a\beta}} C^\varepsilon_s p^\varepsilon_{n,s}\, ds \\ &\qquad + \int_0^t \varepsilon^{\lambda_b+\alpha-(1-r_b)\beta} \sum_{n\geq 1} (u_n-u_{n-1})^2 \frac{b^\varepsilon(\varepsilon^\beta(i^\varepsilon_0+n))}{\varepsilon^{r_b\beta}} p^\varepsilon_{n,s}\, ds\,.\end{aligned}$$

Again, when $\lambda_a = \lambda_b = 0$ (which will be required to obtain a non-trivial limit for μ^ε), it clearly appears that the process p^ε evolves at a faster time scale than μ^ε if $\min(r_a, r_b) < 1$. Using ideas from [33], we want to prove a tightness result for the sequence $\{p^\varepsilon\}$ in a space that *do not see* the fast variations of p^ε. For this, we consider the subspace $\mathcal{Y}$ of non-negative measures on $\mathbb{R}_+ \times \ell_1^+$ given by

$$\begin{aligned}\mathcal{Y} := \Bigg\{\Theta \in \mathcal{M}(\mathbb{R}_+ \times \ell_1^+) \,:\, \forall t \geq 0,\ \Theta([0,t]\times \ell_1^+) = t\,, \\ \text{and} \int_{[0,t]\times\ell_1^+} (1 + \|q\|_{\ell_1})\, \Theta(dq\times ds) < +\infty\Bigg\}.\end{aligned}$$

When ℓ_1^+ is equipped with the vague topology, we can endow $\mathcal{Y}$ with a weak$^\#$ topology, see Appendix A.2, such that $(\mathcal{Y}, w^\#)$ is a Polish space. We define the occupation measure: for any $\varepsilon > 0$, any Borel set A of $\mathbb{R}_+$ and B of (ℓ_1^+, v),

$$\Gamma^\varepsilon(A \times B) = \int_A \mathbf{1}_{\{p_s^\varepsilon \in B\}} \, ds \,, \tag{36}$$

with p^ε defined in (33). Thus, Γ^ε belongs to $\mathcal{Y}$ for each $\varepsilon > 0$. The following proposition states the relative compactness of $\{\Gamma^\varepsilon\}$.

Proposition 8. *Let Γ^ε be defined by* (36) *for each $\varepsilon > 0$ and assume* (H1) *to* (H3)*,* (H8)*,* (H10) *and* (20) *hold. Then $\{\Gamma^\varepsilon\}$ is tight in $\mathcal{P}(w^\# - \mathcal{Y})$.*

Proof. We start by proving that for all $t \geq 0$, the restriction of Γ^ε to $[0,t] \times \ell_1^+$ belongs (uniformly in ε) to a compact of

$$\mathcal{Y}_t := \left\{ \Theta \in \mathcal{M}_b([0,t] \times \ell_1^+) \, : \, \forall t \geq 0, \, \int_{[0,t]\times \ell_1^+} \|q\|_{\ell_1} \, \Theta(dq \times ds) < +\infty \right\},$$

for the weak topology defined in Appendix A.1. Indeed, by definition we have $\|p_t^\varepsilon\|_{\ell_1} = \langle \mu_t^\varepsilon, \mathbf{1} \rangle$, and we get

$$\int_{[0,t]\times\ell_1^+} (1 + \|q\|_{\ell_1} + \Phi_2(\|q\|_{\ell_1}))\Gamma^\varepsilon(ds \times dq) \leq t \left(1 + \sup_{s\in[0,t]} \langle \mu_s^\varepsilon, \mathbf{1} \rangle + \sup_{s\in[0,t]} \Phi_2(\langle \mu_s^\varepsilon, \mathbf{1} \rangle) \right).$$

Thanks to Propositions 3 and 5, we easily check that

$$\sup_{\varepsilon>0} \mathbb{E} \left[\int_{[0,t]\times\ell_1^+} (1 + \|q\|_{\ell_1} + \Phi_2(\|q\|_{\ell_1}))\Gamma^\varepsilon(ds \times dq) \right] < +\infty \,.$$

The latter yields, by Lemma A.2 (in Appendix) and remarking that the bounded subset of (ℓ_1^+, v) are relatively compact, that for all $t \geq 0$ the sequence $\{\Gamma^{\varepsilon,t}\}$ belongs to a weak compact set $\mathcal{K}_t$ of $(\mathcal{Y}_t, w^\#)$.

Now, we let a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = +\infty$ and we let $\eta > 0$. We construct a sequence $\{C_{k,\eta}\}$ of positive constants such that

$$\sup_{\varepsilon>0} \mathbb{E} \left[\int_{[0,t_k]\times\ell_1^+} (1 + \|q\|_{\ell_1} + \Phi_2(\|q\|_{\ell_1}))\Gamma^\varepsilon(ds \times dq) \right] \leq \eta \, C_{k,\eta} 2^{-k} \,.$$

We then define the weak compact set $\mathcal{K}_{t_k,\eta}$ of $(\mathcal{Y}_t, w^\#)$ consisting of measures $\Theta \in \mathcal{M}_b([0,t_k] \times \ell_1^+)$ such that

$$\int_{[0,t_k]\times\ell_1^+} (1 + \|q\|_{\ell_1} + \Phi_2(\|q\|_{\ell_1}))\Theta(ds \times dq) \leq C_{k,\eta} \,.$$

It follows by Markov inequality that

$$\mathbb{P}\left\{\Gamma^{\varepsilon,t_k} \in \mathcal{K}^c_{t_k,\eta}\right\} \leq \frac{\sup_{\varepsilon>0} \mathbb{E}\left[\int_{[0,t_k]\times\ell_1^+}(1+\|q\|_{\ell_1}+\Phi_2(\|q\|_{\ell_1}))\Gamma^\varepsilon(ds\times dq)\right]}{C_{k,\eta}} \leq \eta 2^{-k}\,.$$

Thus, letting $\mathcal{K}_\eta = \{\Theta \in \mathcal{Y}(\mathbb{R}_+ \times \ell_1^+) \,:\, \forall k \geq 0,\ \Theta^{t_k} \in \mathcal{K}_{t_k,\eta}\}$, we obtain that

$$\mathbb{P}\left\{\Gamma^\varepsilon \in \mathcal{K}^c_\eta\right\} \leq \sum_{k\geq 0} \mathbb{P}\left\{\Gamma^{\varepsilon,t_k} \in \mathcal{K}^c_{t_k,\eta}\right\} \leq \eta\,.$$

As $\mathcal{K}_\eta$ defines a compact of $(\mathcal{Y}, w^\#)$ for any $\eta > 0$, by Lemma A.5, this proves the tightness of $\{\Gamma^\varepsilon\}$. □

When $\min(r_a, r_b) \geq 1$, the process p^ε evolves at the same time scale than μ^ε and we are able to obtain a stronger result which is the tightness of $\{p^\varepsilon\}$ in $\mathcal{D}(\mathbb{R}^+, v-\ell_1^+)$.

Proposition 9. *Let p^ε be defined by* (33) *for each $\varepsilon > 0$ and assume* (H1) *to* (H4)*,* (H8) *and* (20) *hold. Moreover, suppose that Assumption 6 holds with $\min(r_a, r_b) \geq 1$. Then $\{p^\varepsilon\}$ is tight in $\mathcal{P}(\mathcal{D}(\mathbb{R}^+, v-\ell_1^+))$ and any accumulation point p belongs to $\mathcal{C}(\mathbb{R}_+, v-\ell_1^+)$.*

Proof. By definition, $\|p_t^\varepsilon\|_{\ell_1} = \langle \mu_t^\varepsilon, \mathbf{1}\rangle$. We recall that the bounded subset of (ℓ_1^+, v) are relatively compact. Thus, thanks to the moment estimates on the moment $\sup_{t\in[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle$ in Proposition 3, we already have the compact containment condition, as a reformulation of Proposition 8. Namely, for all $T > 0$ and $\eta > 0$, there exists a compact $\mathcal{K}_{\eta,T}$ in (ℓ_1^+, v) such that

$$\mathbb{P}\left\{p_t^\varepsilon \in \mathcal{K}_{\eta,T},\ 0 \leq t \leq T\right\} \geq 1-\eta\,.$$

Now, we aim to apply the Roelly-Copoletta criterion on tightness in the space $(\mathcal{M}_b(\mathbb{N}), v)$, see [50, Theorem 2.1], by the natural homeomorphism between this space and (ℓ_1^+, v). It remains to check that for any compactly supported $u \in \ell_0$, and for $u \equiv \mathbf{1}$ (the sequence equal to 1 everywhere), we have that $\langle p^\varepsilon, u\rangle_{\ell_1}$ is tight on $\mathcal{P}(\mathcal{D}(\mathbb{R}^+, \mathbb{R}^+))$. To that, let u be a sequence with compact support denoted by I_u. From Equation (35), we have, for all $h, T > 0$ and $0 \leq t \leq T-h$,

$$\begin{aligned}\left|\mathcal{V}^{\varepsilon,u}_{t+h} - \mathcal{V}^{\varepsilon,u}_t\right| &\leq h\|u\|_\infty\Big(\varepsilon^{\lambda_k}K_k + \varepsilon^{\lambda_l}K_l \sup_{t\in[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle\Big)\\ &\quad + 4m\|u\|_\infty \varepsilon^{\lambda_a+(r_a-1)\beta} \sup_{n\in I_u}\left|\frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_a\beta}}\right| \sup_{t\in[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle\\ &\quad + 2\|u\|_\infty \varepsilon^{\lambda_b+(r_b-1)\beta} \sup_{n\in I_u}\left|\frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_b\beta}}\right| \sup_{t\in[0,T]}\langle \mu_t^\varepsilon, \mathbf{1}\rangle\,.\end{aligned}$$

and

$$|\langle \mathcal{O}^{\varepsilon,u}\rangle_{t+h} - \langle \mathcal{O}^{\varepsilon,u}\rangle_t| \leq \varepsilon^\alpha h \|u\|_\infty^2 \left(\varepsilon^{\lambda_k} K_k + \varepsilon^{\lambda_l} K_l \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\right)$$
$$+ 4m\varepsilon^\alpha \|u\|_\infty^2 \varepsilon^{\lambda_a + (r_a-1)\beta} \sup_{n\in I_u} \left| \frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_a\beta}} \right| \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle$$
$$+ 2\varepsilon^\alpha \|u\|_\infty^2 \varepsilon^{\lambda_b + (r_b-1)\beta} \sup_{n\in I_u} \left| \frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_b\beta}} \right| \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\,.$$

Thus, by the moment estimates of Proposition 3, (H11), (H12) and the Aldous criterion [7, p 176], we have that $\langle p^\varepsilon, u\rangle_{\ell_1}$ is tight in $\mathcal{P}(\mathcal{D}(\mathbb{R}^+, \mathbb{R}^+))$. A similar property holds as well for $u \equiv \mathbf{1}$, as

$$\left|\mathcal{V}_{t+h}^{\varepsilon,u} - \mathcal{V}_t^{\varepsilon,u}\right| \leq h\Big(\varepsilon^{\lambda_k} K_k + \varepsilon^{\lambda_l} K_l \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\Big)\,,$$

and,

$$|\langle \mathcal{O}^{\varepsilon,u}\rangle_{t+h} - \langle \mathcal{O}^{\varepsilon,u}\rangle_t| \leq \varepsilon^\alpha h \left(\varepsilon^{\lambda_k} K_k + \varepsilon^{\lambda_l} K_l \sup_{t\in[0,T]} \langle \mu_t^\varepsilon, \mathbf{1}\rangle\right)\,.$$

It concludes the tightness. The continuity follows by the same arguments as Lemma 3. □

6. Convergence and limit problem

At this point, we already know, thanks to Propositions 6 and 8, there exists a sequence $\{\varepsilon_n\}$ converging to 0 as n goes to infinity, such that $\{\mu^{\varepsilon_n}\}$ and $\{\Gamma^{\varepsilon_n}\}$ converge resp. to μ in $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$ and Γ in $\mathcal{P}(w^\# - \mathcal{Y})$. In the remainder the sequence $\{\varepsilon_n\}$ is still denoted by ε for simplicity. The aim of this section is to identify the limit problem, *i.e.* to recover the main results stated at the beginning. As far as possible, we will discuss on the role and importance played by the limit Γ and its identification.

6.1. The weak limit

This section is devoted to an intermediate theorem, but nonetheless central, which identifies the equation satisfied by the limit μ. As we will see, it is a Lifschitz-Slyozov equation (in the weak sense) with boundary terms depending in particular on integrals against Γ ("averages").

Theorem 3. *Let μ^ε be defined by* (6) *and the occupation measure Γ^ε by* (36), *for each $\varepsilon > 0$. Assume that Assumptions* (H1) *to* (H10) *hold and that $\{\mu_{\text{in}}^\varepsilon\}$ converges towards a deterministic measures μ_{in} in $\mathcal{P}(w-\mathcal{X})$. Then, $\{\mu^\varepsilon\}$ and $\{\Gamma^\varepsilon\}$ converge along an appropriate subsequence as $\varepsilon \to 0$, respectively, to μ in*

$\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w-\mathcal{X}))$ and Γ in $\mathcal{P}(w^\# - \mathcal{Y})$. The limit μ belongs to $\mathcal{C}(\mathbb{R}_+, w-\mathcal{X})$ and we have a.s., for all $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t \geq 0$

$$\langle \mu_t, \varphi\rangle = \langle \mu_{\text{in}}, \varphi\rangle + \int_0^t \int_{x_0}^\infty \varphi'(x)(a(x)c_s - b(x))\mu_s(dx)\, ds + \int_0^t \varphi(x_0)k(c_s)\, ds$$
$$- \varphi(x_0)\int_{[0,t]\times\ell_1^+} l(q_0)\Gamma(dq\times ds) + \varphi'(x_0)b(x_0)\int_{[0,t]\times\ell_1^+} q_0\Gamma(dq\times ds), \quad (37)$$

where $c_t = m - \langle \mu_t, \text{Id}\rangle \geq 0$ and q_0 is the first component of the variable $q \in \ell_1^+$.

The second term on the right hand side is the classical drift (in weak form) in the Lifschitz-Slyozov equation. Moreover, it clearly appears that the terms with Γ contributes to the boundary value. A simple computation, taking $\varphi = \mathbf{1}$, shows that the terms in $\varphi(x_0)$ account for the number of clusters with critical size x_0 which are created. While the one in $\varphi'(x_0)$ gives a mass to these clusters taking $\varphi = \text{Id}$. Remark here, the proof of Theorem 1 is a direct consequence of the result given above. Indeed, taking $\varphi \in C_c^1((x_0,+\infty))$ vanishes the boundary terms, μ is a solution in the *vague* sense.

The proof of this theorem relies on the identification of the limit through a functional, that stands for the limit model, studied along the process. The deterministic nature of the problem follows from the fact the martingale vanishes at the limit. Thus, for any given $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t \geq 0$, we define for all $(\nu,\Theta) \in \mathcal{D}(\mathbb{R}_+, w-\mathcal{X}) \times \mathcal{Y}$, the functional

$$F_t^\varphi(\nu,\Theta) = \langle \nu_t, \varphi\rangle - \langle \nu_{\text{in}}, \varphi\rangle - D_t^\varphi(\nu) - B_t^\varphi(\nu) - \tilde{B}_t^\varphi(\Theta)\,, \quad (38)$$

where D_t^φ denotes the drift, B_t^φ and $\tilde{B}_t^\varphi$ the boundary terms, respectively given by

$$\begin{aligned} D_t^\varphi(\nu) &= \int_0^t \Big((m - \langle \nu_s, \text{Id}\rangle)\langle \nu_s, a\varphi'\rangle - \langle \nu_s, b\varphi'\rangle \Big) ds\,, \\ B_t^\varphi(\nu) &= \varphi(x_0)\int_0^t k(m - \langle \nu_s, \text{Id}\rangle)\, ds\,, \\ \tilde{B}_t^\varphi(\Theta) &= -\varphi(x_0)\int_{[0,t)\times\ell^1} l(q_0)\Theta(dq\times ds) \\ &\qquad + \varphi'(x_0)b(x_0)\int_{[0,t)\times\ell^1} q_0\Theta(dq\times ds)\,. \end{aligned}$$

We aim to prove that the limit (μ,Γ) satisfies $\mathbb{E}\left[|F_t^\varphi(\mu,\Gamma)|\right] = 0$. We start by few lemmas.

Lemma 4. *For all $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t \geq 0$, the function F_t^φ is continuous at any point (ν,Θ) such that $(\nu,\Theta) \in \mathcal{C}(\mathbb{R}_+, w-\mathcal{X}) \times \mathcal{Y}$.*

Proof. From their own definition, it appears clearly that D_t^φ and B_t^φ are continuous on $\mathcal{D}(\mathbb{R}_+, w-\mathcal{X})$, and that $\tilde{B}_t^\varphi$ is continuous on $(\mathcal{Y}, w^\#)$. Moreover, the continuity of $t \mapsto \langle \nu_t, \varphi\rangle$ entails the continuity of the application $\tilde{\nu} \mapsto \langle \tilde{\nu}_t, \varphi\rangle$ at ν. □

Lemma 5. *Under the same assumptions as Theorem 3, for all* $\varphi \in \mathcal{C}_b^1([x_0, +\infty))$ *and all* $t \geq 0$*, the family* $\{|F_t^{\varphi}(\mu^{\varepsilon}, \Gamma^{\varepsilon})|\}$ *is uniformly integrable.*

Proof. Let us fix $t > 0$ and φ in $C_b^1([x_0, +\infty))$. The uniform integrability of the family $\{|F_t^{\varphi}(\mu^{\varepsilon}, \Gamma^{\varepsilon})|\}_{\varepsilon}$ is equivalent to the existence of a function Ψ in $\mathcal{U}_{\infty}$ such that

$$\sup_{\varepsilon} \mathbb{E}\left[\Psi(|F_t^{\varphi}(\mu^{\varepsilon}, \Gamma^{\varepsilon})|)\right] < +\infty \,,$$

by [24, Proposition 2.2 in Appendixes]. Thus we aim to show such a property. Remark, by definition (38), we obtain

$$|F_t^{\varphi}(\mu_t^{\varepsilon}, \Gamma^{\varepsilon})| \leq |\langle \mu_t^{\varepsilon}, \varphi \rangle| + |\langle \mu_{\text{in}}^{\varepsilon}, \varphi \rangle| + |D_t^{\varphi}(\mu^{\varepsilon})| + |B_t^{\varphi}(\mu^{\varepsilon})| + |\tilde{B}_t^{\varphi}(\Gamma^{\varepsilon})| \,.$$

Therefore, using similar bound as in Section 5 and in particular the fact that *a.s.* $\langle \mu^{\varepsilon}, \mathrm{Id} \rangle \leq 2m$ (for ε small enough), there exists a constant K depending on t, $\|\varphi\|_{\infty}$, $\|\varphi'\|_{\infty}$, m, K_k, K_l, K_a, K_b and $b(x_0)$ such that

$$|F_t^{\varphi}(\mu_t^{\varepsilon}, \Gamma^{\varepsilon})| \leq tK \left(1 + \langle \mu_{\text{in}}^{\varepsilon}, \mathbf{1} \rangle + \langle \mu_t^{\varepsilon}, \mathbf{1} \rangle + \frac{1}{t}\int_0^t \langle \mu_s^{\varepsilon}, \mathbf{1} \rangle \, ds \right) . \tag{39}$$

Consider $\Phi_2 \in \mathcal{U}_{1,\infty}$ given by Proposition 5. We divide both sides of Inequality (39) by $4K$ and using the convexity Φ_2 we get

$$\Phi_2\left(\frac{1}{4tK}|F_t^{\varphi}(\mu_t^{\varepsilon}, \Gamma^{\varepsilon})|\right) \leq \frac{1}{4}\left(\Phi_2(1) + \Phi_2(\langle \mu_{\text{in}}^{\varepsilon}, \mathbf{1} \rangle) + \Phi_2(\langle \mu_t^{\varepsilon}, \mathbf{1} \rangle) + \sup_{s \in [0,t]} \Phi_2(\langle \mu_s^{\varepsilon}, \mathbf{1} \rangle)\right) ,$$

where the last terms follows after the use Jensen's inequality. By (H10) and Proposition 5 it is immediate that

$$\sup_{\varepsilon > 0} \mathbb{E}\left[\Phi_2\left(\frac{1}{4tK}|F_t^{\varphi}(\mu_t^{\varepsilon}, \Gamma^{\varepsilon})|\right)\right] < +\infty \,.$$

Since the map $x \mapsto \Phi_2(x/4tK)$ remains in $\mathcal{U}_{\infty}$, it ends the proof for $t > 0$. The case $t = 0$ is trivial. □

Before stating the last lemma which will achieve the proof of Theorem 3, we introduce a technical result that will be usefull to treat the convergence of some terms.

Lemma 6. *Let* $\{\nu^{\varepsilon}\}$ *be a sequence of* $\mathcal{D}(\mathbb{R}_+, w - \mathcal{X})$ *such that there exists a function* $\Phi_1 \in \mathcal{U}_{1,\infty}$ *satisfying for any* $T > 0$

$$\sup_{\varepsilon > 0} \mathbb{E}\left[\sup_{t \in [0,T]} \langle \nu_t^{\varepsilon}, \mathbf{1} + \Phi_1 \rangle\right] < +\infty \,.$$

Consider a sequence $\{\varphi^\varepsilon\}$ in $C([x_0,+\infty))$ such that there exists a constant $K > 0$ with $\varphi^\varepsilon(x) \leq K(1+x)$ for all x and $\varepsilon > 0$. If $\{\varphi^\varepsilon\}$ converges towards a function φ uniformly on the compact sets, then for all $t \in [0,T]$

$$\mathbb{E}\left[\int_0^t |\langle \nu_s^\varepsilon, \varphi^\varepsilon - \varphi\rangle| ds\right] \longrightarrow 0\,, \quad \varepsilon \to 0\,.$$

Proof. We use the same ideas as in the proof of Lemma 2. Let $T > 0$, $R > 0$, $\varepsilon > 0$ and $t \in [0,T]$, we write

$$|\langle \nu_t^\varepsilon, \varphi^\varepsilon - \varphi\rangle| \leq |\langle \nu_t^\varepsilon, (\varphi^\varepsilon - \varphi)\mathbf{1}_{x<R}\rangle| + 2K\langle \nu_t^\varepsilon, (1+x)\mathbf{1}_{x\geq R}\rangle\,.$$

Thus,

$$\mathbb{E}\left[\int_0^t |\langle \nu_s^\varepsilon, \varphi^\varepsilon - \varphi\rangle|\, ds\right] \leq T \sup_{x\leq R} |\varphi^\varepsilon(x) - \varphi(x)|\, \mathbb{E}\left[\sup_{t\in[0,T]} \langle \nu_t^\varepsilon, \mathbf{1}\rangle\right] + 2KT\left(\sup_{x\geq R} \frac{1+x}{\Phi_1(x)}\right) \mathbb{E}\left[\sup_{t\in[0,T]} \langle \nu_t^\varepsilon, \Phi_1\rangle\right]\,.$$

We conclude using the moment estimates and taking the lim sup in $\varepsilon \to 0$ the first term on the right-hand side goes to 0. Finally letting $R \to +\infty$ the second term goes to 0 with the property fulfilled by Φ_1. □

Lemma 7. *Under the same assumptions as Theorem 3, for all $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t \geq 0$*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[|F_t^\varphi(\mu^\varepsilon, \Gamma^\varepsilon)|\right] = 0\,. \tag{40}$$

Proof. First, we remark that by Equation (17),

$$F_t^\varphi(\mu^\varepsilon, \Gamma^\varepsilon) = \mathcal{O}_t^{\varepsilon,\varphi} + R_t^{\varepsilon,\varphi}\,,$$

where $R_t^{\varepsilon,\varphi} = \mathcal{V}_t^{\varepsilon,\varphi} - D_t^\varphi(\mu^\varepsilon) - B_t^\varphi(\mu^\varepsilon) - \tilde{B}_t^\varphi(\Gamma^\varepsilon) = \sum_{i=1}^8 R_t^{\varepsilon,\varphi,i}$ with the terms corresponding to the drift:

$$R_t^{\varepsilon,\varphi,1} = (M^\varepsilon - m)\int_0^t \langle \mu_s^\varepsilon, a\varphi'\rangle\, ds\,,$$

$$R_t^{\varepsilon,\varphi,2} = \int_0^t (M^\varepsilon - \langle \mu_s^\varepsilon, \mathrm{Id}\rangle)\langle \mu_s^\varepsilon, (a^\varepsilon - a)\varphi'\rangle\, ds\,,$$

$$R_t^{\varepsilon,\varphi,3} = \int_0^t \langle \mu_s^\varepsilon, (b - b^\varepsilon)\varphi'\rangle\, ds\,,$$

$$R_t^{\varepsilon,\varphi,4} = \int_0^t (M^\varepsilon - \langle \mu_s^\varepsilon, \mathrm{Id}\rangle)\langle \mu_s^\varepsilon, (\Delta_\varepsilon(\varphi) - \varphi')a^\varepsilon\rangle\, ds\,,$$

$$R_t^{\varepsilon,\varphi,5} = \int_0^t \langle \mu_s^\varepsilon, (\varphi' - \tau_\varepsilon\Delta_\varepsilon(\varphi))b^\varepsilon\mathbf{1}_{(x_0^\varepsilon+\varepsilon^\beta,+\infty)}\rangle\, ds\,,$$

and to the boundary:

$$R_t^{\varepsilon,\varphi,6} = \int_0^t \varphi(x_0^\varepsilon) k^\varepsilon (M^\varepsilon - \langle \mu_s^\varepsilon, \mathrm{Id}\rangle) - \varphi(x_0) k(m - \langle \mu_s^\varepsilon, \mathrm{Id}\rangle)\, ds\,,$$

$$R_t^{\varepsilon,\varphi,7} = \int_{[0,t)\times\ell^1} \varphi(x_0) l(q_0) \Gamma^\varepsilon(dq \times ds) - \int_{[0,t)\times\ell^1} \varphi(x_0^\varepsilon) l^\varepsilon(q_0)\Gamma^\varepsilon(dq\times ds)\,,$$

$$R_t^{\varepsilon,\varphi,8} = \int_{[0,t)\times\ell^1} \varphi'(x_0^\varepsilon) b^\varepsilon(x_0^\varepsilon) q_0 \Gamma^\varepsilon(dq\times ds) - \int_{[0,t)\times\ell^1} \varphi'(x_0) b(x_0) q_0 \Gamma^\varepsilon(dq\times ds)\,.$$

From the Burkholer inequality, we have

$$\mathbb{E}\left[|\mathcal{O}_t^{\varepsilon,\varphi}|\right] \leq \left(\mathbb{E}\left[|\mathcal{O}_t^{\varepsilon,\varphi}|^2\right]\right)^{1/2} \leq \left(\mathbb{E}\left[\langle \mathcal{O}^{\varepsilon,\varphi}\rangle_t\right]\right)^{1/2}.$$

Starting back from Equation (19) with $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$, we have

$$\mathbb{E}\left[\langle \mathcal{O}^{\varepsilon,\varphi}\rangle_t\right] \leq t\varepsilon^\alpha ||\varphi||_\infty \left(K_k + K_l \sup_{\varepsilon>0} \mathbb{E}\left[\sup_{s\in[0,t]} \langle \mu_s^\varepsilon, \mathbf{1}\rangle\right]\right)$$
$$+ t||\varphi'||_\infty \varepsilon^{\alpha+\beta}(2mK_a + K_b) \sup_{\varepsilon>0}\left(\mathbb{E}\left[\sup_{s\in[0,t]}\langle \mu_s^\varepsilon,\mathbf{1}\rangle\right] + \mathbb{E}\left[\sup_{s\in[0,t]}\langle\mu_s^\varepsilon, \mathrm{Id}\rangle)\right]\right).$$

Thus, using the moment estimates in Proposition 3,

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[|\mathcal{O}_t^{\varepsilon,\varphi}|\right] = 0.$$

Then, the expectation of the remainder $|R_t^{\varepsilon,\varphi,1}|$ to $|R_t^{\varepsilon,\varphi,5}|$ go to 0 by the convergence of a^ε and b^ε in (H5) and (H6), the convergence of M^ε in (H2), the moment estimates in Proposition 3 and the above result in Lemma 6. The remainder $|R_t^{\varepsilon,\varphi,6}|$ converges to 0 thanks to (H3) together with estimates in Proposition 3.

For the two last remainders $|R_t^{\varepsilon,\varphi,7}|$ and $|R_t^{\varepsilon,\varphi,8}|$, we use a similar strategy as in Lemma 6. For instance, for any $R>0$,

$$\left|\int_{[0,t]\times\ell_1^+} \varphi(x_0) l(q_0)\Gamma^\varepsilon(dq\times ds) - \int_{[0,t]\times\ell_1^+} \varphi(x_0^\varepsilon) l^\varepsilon(q_0)\Gamma^\varepsilon(dq\times ds)\right| \leq$$
$$2t||\varphi||_\infty \left(\sup_{x\leq R}|l(x)-l^\varepsilon(x)| + 2K_l\left(\sup_{x>R}\frac{x}{\Phi_2(x)}\right)\sup_{s\in[0,t]}\Phi_2(p_{0,t}^\varepsilon))\right),$$

and we conclude by Proposition 5 remarking that

$$\sup_{\varepsilon>0}\mathbb{E}\left[\sup_{t\in[0,T]}\Phi_2(p_{0,t}^\varepsilon)\right] \leq \sup_{\varepsilon>0}\mathbb{E}\left[\sup_{t\in[0,T]}\Phi_2(\langle\mu_t^\varepsilon,\mathbf{1}\rangle)\right] < +\infty\,.$$

This proves that (40) holds. □

Proof of Theorem 3. By Propositions 6 and 8 it follows that along an appropriate subsequence, $\{\mu^\varepsilon\}$ and $\{\Gamma^\varepsilon\}$, converge as $\varepsilon \to 0$, respectively, to μ in $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w - \mathcal{X}))$ and $\mathcal{P}(w^\# - \mathcal{Y})$ with $\mu \in \mathcal{C}(\mathbb{R}_+, w - \mathcal{X})$. Thus $\{(\mu^\varepsilon, \Gamma^\varepsilon)\}$ converges in law along the same subsequence and by Lemma 4, it readily follows that $\{|F_t^\varphi(\mu^\varepsilon, \Gamma^\varepsilon)|\}$ converges in law towards $\{|F^\varphi(\mu, \Gamma)|\}$. By the uniform integrability in Lemma 5 together with [7, Chap. 1, Theorem 3.5] it yields

$$\mathbb{E}\left[|F_t^\varphi(\mu^\varepsilon, \Gamma^\varepsilon)|\right] \to_{\varepsilon \to 0} \mathbb{E}\left[|F_t^\varphi(\mu, \Gamma)|\right] .$$

Thus, by Lemma 7, for all $\varphi \in C_b^1([x_0, +\infty))$ and $t \geq 0$ we have

$$F_t^\varphi(\mu, \Gamma) = 0 , \quad a.s.$$

Moreover, by Proposition 6 we have $\{C^\varepsilon\}$ is tight in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ and converges (along the same subsequence, up to a modification) to a non-negative c for which it is easy to show that it belongs to $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$. By the same arguments as above, continuity, uniform intergability and identification we obtain, for all $t \geq 0$, that

$$c_t = m - \langle \mu_t, \mathrm{Id} \rangle \geq 0 .$$

Finally, it remains *a.s.*, for all φ_k as given for the metric $d_\mathcal{X}$ in (29) and $\tau \in \mathbb{Q}^+$, the limit μ satisfies (37) by construction of a set of probability 0 as the countable union of probability 0 sets. By time continuity of μ and density of the $\{\varphi_k\}$ it ends the proof of Theorem 3.

□

6.2. Identification of the occupation measure

Theorem 3 lacks of information because it does not provide any information on Γ. In this section we aim to identify this measure thanks to a particular limit of the generator $\mathcal{H}^\varepsilon$ defined in (34) and more precisely to its unique stationary measure when it is possible.

To that, we focus on p^ε, defined by (33), through its infinitesimal generator $\mathcal{H}^\varepsilon$. As we saw, for each $\varepsilon > 0$ the processes μ^ε and p^ε are compactly supported. However the same property is not expected at the limit. Contrary to proposition 7, it requires to make the infinitesimal generator act on functions allowing us to consider sequences in the whole space ℓ_1^+, not only compactly supported. Therefore, we introduce the domain $\mathcal{G}$ defined as

$$\mathcal{G} := \{g : \ell_1 \to \mathbb{R} \,:\, \exists N \geq 1,\ \exists G \in \mathcal{G}_N,\ g(v) = G(v_0, \ldots, v_{N-1}),\ \forall v \in \ell_1\} ,$$

where

$$\mathcal{G}_N := \left\{G \in \mathcal{C}^2(\mathbb{R}^N) \,:\, G(0) = 0 \text{ and } \partial_n G \in \mathcal{C}_c^1(\mathbb{R}^N)\right\} .$$

Remark, ∂_n denotes the partial derivatives with respect to the n^{th} variables.

Now, contrary to Proposition 7, using the idea of [33], we see the infinitesimal generator $\mathcal{H}^\varepsilon$ as an operator coupling the action of p^ε and μ^ε. In order to do that, we define, assuming (H7), for all g in the domain $\mathcal{G}$, and for all $(\nu, q) \in \mathcal{X} \times \ell_1^+$, the operator

$$[\widetilde{\mathcal{H}}^\varepsilon g](\nu,q) := \frac{g(q+\varepsilon^\alpha \mathbf{1}_0) - g(q)}{\varepsilon^\alpha} k^\varepsilon(C^\varepsilon) + \frac{g(q-\varepsilon^\alpha \mathbf{1}_0) - g(q)}{\varepsilon^\alpha} l^\varepsilon(q_0)$$
$$+ \varepsilon^{-(1-r_a)\beta} \sum_{n\geq 0} \frac{g(q+\varepsilon^\alpha(\mathbf{1}_{n+1} - \mathbf{1}_n)) - g(q)}{\varepsilon^\alpha} \frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_a \beta}} C^\varepsilon q_n$$
$$+ \varepsilon^{-(1-r_b)\beta} \sum_{n\geq 1} \frac{g(q-\varepsilon^\alpha(\mathbf{1}_{n} - \mathbf{1}_{n-1})) - g(q)}{\varepsilon^\alpha} \frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_b \beta}} q_n \,,$$

where C^ε is now replaced by $M^\varepsilon - \langle \nu, \mathrm{Id} \rangle$ contrary to the previous definition of $\mathcal{H}^\varepsilon$ in (34). The operator $\widetilde{\mathcal{H}}^\varepsilon$ is well-defined on the whole domain $\mathcal{G}$ since: for all g in $\mathcal{G}$, there exists a $M \geq 0$ such that

$$g(q+\varepsilon^\alpha(\mathbf{1}_{n+1} - \mathbf{1}_n)) = g(q-\varepsilon^\alpha(\mathbf{1}_{n} - \mathbf{1}_{n-1})) = g(q)\,,$$

for all $q \in \ell_1^+$ and $n \geq M$. It readily follows from its definition that, for all $g \in \mathcal{G}$,

$$g(p_t^\varepsilon) = g(p_0^\varepsilon) + \int_0^t [\widetilde{\mathcal{H}}^\varepsilon g](\mu_s^\varepsilon, p_s^\varepsilon)\, ds + \mathcal{O}_t^{\varepsilon, g} \tag{41}$$

where $\mathcal{O}_t^{\varepsilon\, g}$ is a martingale. Remark that, taking $r := \min(r_a, r_b) < 1$ and multiplying this generator by $\varepsilon^{(1-r)\beta}$, at the limit some terms will vanish depending on the value of r_a and r_b. The latter depend on the behaviour of a and b around x_0. Indeed, a direct consequence of Assumption 6 is that: for all $n \in \mathbb{N}$,

$$\frac{a^\varepsilon(x_0^\varepsilon + \varepsilon^\beta n)}{\varepsilon^{\beta r_a}} \underset{\varepsilon\to 0}{\to} a_n := \overline{a}\,(x_0^1 + n)^{r_a}, \quad \frac{b^\varepsilon(x_0^\varepsilon + \varepsilon^\beta n)}{\varepsilon^{\beta r_b}} \underset{\varepsilon\to 0}{\to} b_n := \overline{b}\,(x_0^1 + n)^{r_b}\,. \tag{42}$$

with $\{a_n\}$ and $\{b_n\}$ positives since $x_0^1 > 0$ by (H13). We are in position to define the limit operator: for all g in $\mathcal{G}$ and $(\nu, q) \in \mathcal{X} \times \ell_1^+$,

$$[\widetilde{\mathcal{H}}\, g](\nu,q) := \sum_{n\geq 0} Dg[q](1_n)(J_{n-1}(\nu,q) - J_n(\nu,q))$$
$$= \sum_{n\geq 0} Dg[q](1_{n+1} - 1_n) J_n(\nu,q))\,, \tag{43}$$

where Dg is the Fréchet derivative of g, by convention $J_{-1} = 0$ and for all $n \geq 0$

$$J_n(\nu,q) := \begin{cases} a_n c q_n - b_{n+1} q_{n+1}\,, & \text{if } r_a = r_b \leq 1\,, \\ a_n c q_n\,, & \text{if } r_a < r_b\,, \text{ and } r_a \leq 1\,, \\ -b_{n+1} q_{n+1}\,, & \text{if } r_b < r_a\,, \text{ and } r_b \leq 1\,. \end{cases} \tag{44}$$

with $c = m - \langle \nu, \mathrm{Id} \rangle$, the constant m arising from (H2). Note the similarity with the classical Becker-Döring fluxes for the deterministic equations. The first result

Theorem 4. *In addition to the hypotheses of Theorem 3, assume that Assumption 6 holds with* $\min(r_a, r_b) < 1$*. Then, with probability one, for all* t *and all* $g \in \mathcal{G}$

$$\int_{[0,t]\times \ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_s, q)\Gamma(dq \times ds) = 0\,.$$

Moreover, let ρ *be defined by* (11)*. On a time interval* $[t_0, t_1]$ *such that the limit* $c_t > \rho$ *for all* $t_0 \le t \le t_1$*, the measure* Γ *is a.s. in* $\mathcal{P}(w^{\#} - \mathcal{Y})$ *equal to* $ds \times \delta_0$ *where* δ_0 *denotes the Dirac measure at the sequence of* ℓ_1^+ *equals to* 0 *everywhere.*

We here note that Theorem 2 readily follows from this result combining to Theorem 3.

Remark 11. This theorem informs us that the limit Γ vanishes the limit generator $\widetilde{\mathcal{H}}$. So we are able to completely identify Γ only when we can ensure the operator $\widetilde{\mathcal{H}}$ has a unique stationary measure. It makes clear that this operator is connected to a constant-particle Becker-Döring system. If we investigate the stationary solutions of the generator through its dynamic, there is two cases: either the time-dependent solution trends to an equilibrium or the solution escapes to infinity (larger and larger clusters are formed), see for instance [3, 4, 11]. Surprisingly, we cannot identify the stationary measure in the first case since the equilibrium is parametrized by its mass which is unknown here (it is *not* $\langle \mu_t, \mathbf{1} \rangle$). It provides an infinity of stationary solutions and one can show (see Appendix B) that the support of the stationary measure belongs to the set of all possible stationary solutions and not only one. On the other hand, when there is no equilibrium, the solution vaguely converges to the unique zero-solution which provides an identification between the stationary measure and the long time solution of the Becker-Döring system. In this case, p_n^ε, for a fix n, which is a very small cluster, goes to 0 when ε goes to 0. In contrary, larger and larger clusters in n are formed, which induce at the limit clusters of size $x > x_0$. This is the case when we have an identifiable boundary condition.

Let us introduce some lemmas before the proof of Theorem 4. First, note the Fréchet derivative of a function in $\mathcal{G}$ can be expressed in a simpliest way. For any $g \in \mathcal{G}$, there exists an integer N and a function G in $\mathcal{C}^2(\mathbb{R}^N)$ by definition such that $g(v) = G(v_0, \ldots, v_{N-1}),\ \forall v \in \ell_1$. Hence, the Fréchet derivative Dg is, for all q in ℓ_1^+ and v in ℓ_1,

$$Dg[q](v) = \sum_{n=0}^{N} \partial_n\, G(q_1, \ldots, q_N)\ v_n\,, \tag{45}$$

and so $Dg[q](1_n) = \mathbf{1}_{0\le n\le N}\ \partial_n\, G(q_1, \ldots, q_N)$. This shows that the generator $\widetilde{\mathcal{H}}$ is well-defined on $\mathcal{G}$ since the sum is actually finite.

We now state a lemma on the convergence of the generators $\varepsilon^{(1-r)\beta}\widetilde{\mathcal{H}}^\varepsilon$ to $\widetilde{\mathcal{H}}$ along the processes p^ε and μ^ε.

Lemma 8. *Under the same assumptions as Theorem 4, we have, for all $T>0$ and g in $\mathcal{G}$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \varepsilon^{(1-r)\beta}[\widetilde{\mathcal{H}}^\varepsilon g](\mu_s^\varepsilon,p_s^\varepsilon)-[\widetilde{\mathcal{H}}g](\mu_s^\varepsilon,p_s^\varepsilon)\,ds\right|\right]=0\,.$$

Proof. We start with the case $r_a=r_b$ so that the fluxes J_n in (44) are in the more general form. Let us fix $T>0$ and g in $\mathcal{G}$. Remark first that, thanks to (45) and by Taylor's theorem, there exists a positive constant K_g such that for all q in ℓ_1^+ and v in ℓ_1, we have the following bounds

$$|Dg[q](v)|\le K_g\|v\|_{\ell_1},\quad \left|\frac{g(q+\varepsilon^\alpha v)-g(q)}{\varepsilon^\alpha}-Dg[q](v)\right|\le K_g\,\varepsilon^\alpha\|v\|_{\ell_1}^2\,,$$

and, therefore

$$\left|\frac{g(q+\varepsilon^\alpha v)-g(q)}{\varepsilon^\alpha}\right|\le K_g\|v\|_{\ell_1}(1+\varepsilon^\alpha\|v\|_{\ell_1})\,.$$

From the definition of $\widetilde{\mathcal{H}}^\varepsilon$ and $\widetilde{\mathcal{H}}$ it readily follows that for all $s\in[0,T]$

$$\left|\varepsilon^{(1-r)\beta}[\widetilde{\mathcal{H}}^\varepsilon\,g](\mu_s^\varepsilon,p_s^\varepsilon)-[\widetilde{\mathcal{H}}\,g](\mu_s^\varepsilon,p_s^\varepsilon)\right|\le I_{1,s}^\varepsilon+\sum_{n=0}^N I_2^\varepsilon(n)p_{n,s}^\varepsilon+\sum_{n=1}^N I_3^\varepsilon(n)p_{n,s}^\varepsilon\,,$$

with

$$\begin{aligned}
I_{1,s}^\varepsilon&=\varepsilon^{(1-r)\beta}K_g(1+\varepsilon^\alpha)(|k^\varepsilon(C_s^\varepsilon)|+|l^\varepsilon(p_{0,s}^\varepsilon)|)\,,\\
I_2^\varepsilon(n)&=\Bigg|\frac{g(p_{n,s}^\varepsilon+\varepsilon^\alpha(\mathbf{1}_{n+1}-\mathbf{1}_n))-g(p_{n,s}^\varepsilon)}{\varepsilon^\alpha}\frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_a\beta}}C_s^\varepsilon\\
&\qquad\qquad -Dg[p_{n,s}^\varepsilon](1_{n+1}-1_n)a_nc_s\Bigg|\,,\\
I_3^\varepsilon(n)&=\Bigg|\frac{g(p_{n,s}^\varepsilon-\varepsilon^\alpha(\mathbf{1}_{n}-\mathbf{1}_{n-1}))-g(p_{n,s}^\varepsilon)}{\varepsilon^\alpha}\frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon+n))}{\varepsilon^{r_b\beta}}\\
&\qquad\qquad -Dg[p_{n,s}^\varepsilon](1_{n-1}-1_n)b_n\Bigg|\,.
\end{aligned}$$

Following the method of estimation as in the proof of Proposition 3 together with the definition of p_ε we get

$$I_{1,s}^\varepsilon\le\varepsilon^{(1-r)\beta}K_g(1+\varepsilon^\alpha)(K_k+K_l\langle\mu_s^\varepsilon,\mathbf{1}\rangle)\,.$$

Then, for all n in $\mathbb{N}$, we have $I_2^\varepsilon(n) \le I_2^{\varepsilon\prime}(n) + I_2^{\varepsilon\prime\prime}(n) + I_2^{\varepsilon\prime\prime\prime}(n)$ with

$$
\begin{aligned}
I_2^{\varepsilon\prime}(n) = &\left| \frac{g(p^\varepsilon_{n,s} + \varepsilon^\alpha(\mathbf{1}_{n+1} - \mathbf{1}_n)) - g(p^\varepsilon_{n,s})}{\varepsilon^\alpha} - Dg[p^\varepsilon_{n,s}](1_{n+1} - 1_n) \right| \\
&\qquad \times \frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_a\beta}} C_s^\varepsilon \,, \\
I_2^{\varepsilon\prime\prime}(n) = &\ \left| Dg[p^\varepsilon_{n,s}](1_{n+1} - 1_n) \right| \left| \frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_a\beta}} - a_n \right| C_s^\varepsilon \,, \\
I_2^{\varepsilon\prime\prime\prime}(n) = &\ \left| Dg[p^\varepsilon_{n,s}](1_{n+1} - 1_n) \right| a_n \left| M^\varepsilon - m \right| .
\end{aligned}
$$

For ε small enough, we then end up with the bound

$$
I_2^\varepsilon(n) \le K_g \left(16\, \varepsilon^\alpha\, a_n m + 2m \left| \frac{a^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_a\beta}} - a_n \right| + a_n \left| M^\varepsilon - m \right| \right) .
$$

In a similar way, we obtain for $n \ge 1$

$$
I_3^\varepsilon(n) \le K_g \left(8\, \varepsilon^\alpha\, b_n + \left| \frac{b^\varepsilon(\varepsilon^\beta(i_0^\varepsilon + n))}{\varepsilon^{r_b\beta}} - b_n \right| \right) .
$$

Finally, from the above estimates and using that $p^\varepsilon_{n,s} \le \langle \mu^\varepsilon_s, \mathbf{1} \rangle$ by definition, recalling the convergence of M^ε in (H2), the convergence of the a^ε's and b^ε's in (42), and by the moment estimates in Proposition 3, it concludes the proof for $r_a = r_b$. In the other cases, the proof is similar. □

Before proving Theorem 4 and more precisely in order to use the convergence of $(\mu^\varepsilon, \Gamma^\varepsilon)$ towards (μ, Γ), a last lemma of continuity is necessary.

Lemma 9. *For all g in $\mathcal{G}$ and all $t \ge 0$, the function*

$$
(\nu, \Theta) \mapsto \int_{[0,t]\times\ell_1^+} [\tilde{\mathcal{H}}\, g](\nu_s, q) \Theta(ds \times dq)
$$

is continuous at any point of $\mathcal{C}(\mathbb{R}_+, w - \mathcal{X}) \times \mathcal{Y}$.

Proof. Let us fix g in $\mathcal{G}$, $t \ge 0$ and a point (ν, Θ) of $\mathcal{C}(\mathbb{R}_+, w - \mathcal{X}) \times \mathcal{Y}$. For any sequence $(\nu^\varepsilon, \Theta^\varepsilon)$ converging to (ν, Θ) when ε goes to 0, we have

$$
\begin{aligned}
&\left| \int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\nu^\varepsilon_s, q) \Theta^\varepsilon(ds \times dq) - \int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\nu_s, q) \Theta(ds \times dq) \right| \\
&\quad \le \left| \int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\nu_s, q) \Theta^\varepsilon(ds \times dq) - \int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\nu_s, q) \Theta(ds \times dq) \right| \\
&\qquad + \int_{[0,t]\times\ell_1^+} \left| [\widetilde{\mathcal{H}}\, g](\nu^\varepsilon_s, q) - [\widetilde{\mathcal{H}}\, g](\nu_s, q) \right| \Theta^\varepsilon(ds \times dq) . \qquad (46)
\end{aligned}
$$

First, the convergence of the first term on the right-hand side of(46) to 0 is due to the convergence of Θ^ε to Θ in $\mathcal{Y}$ for the $weak^\#$ topology (see Appendix) since we have the bound

$$|[\widetilde{\mathcal{H}}\, g](\nu_s, q)| \leq 2K_g\Big(\sup_{0\leq n\leq N} a_n(m + \sup_{s\in[0,t]} \langle \nu_s, \mathrm{Id}\rangle) + \sup_{1\leq n\leq N+1} b_n\Big) \sum_{n=0}^{N+1} q_n \,.$$

where the constant K_g is the same as in the previous proof.

Consider now the second term. For all $n \in \mathbb{N}$, $s \in [0,t]$ and q in ℓ_1^+, the following bound on the flux J_n is obtained

$$|J_n(\nu_s^\varepsilon, q) - J_n(\nu_s, q)| \leq a_n |\langle \nu_s^\varepsilon, \mathrm{Id}\rangle - \langle \nu_s, \mathrm{Id}\rangle|\, q_n \,.$$

Therefore, we get

$$\Big|[\widetilde{\mathcal{H}}\, g](\nu_s^\varepsilon, q) - [\widetilde{\mathcal{H}}\, g](\nu_s, q)\Big| \leq 2K_g \Big(\sup_{0\leq n\leq N} a_n\Big)\Big(\sup_{s\in[0,t]} |\langle \nu_s^\varepsilon, \mathrm{Id}\rangle - \langle \nu_s, \mathrm{Id}\rangle|\Big) \sum_{n=0}^{N} q_n \,.$$

This inequality in particular shows the continuity of the map $\tilde\nu \mapsto [\widetilde{\mathcal{H}}\, g](\tilde\nu, q)$ for all g in $\mathcal{G}$ and q in ℓ_1^+ and gives the convergence to 0 of the second term on the right-hand side in (46) when ε goes to 0. The result is proved. □

We are now in position to identify the limit Γ and prove Theorem 4.

Proof of Theorem 4. We may rewrite (41) as

$$\begin{aligned}\varepsilon^{(1-r)\beta}\mathcal{O}_t^{\varepsilon,g} &= \varepsilon^{(1-r)\beta}(g(p_t^\varepsilon) - g(p_0^\varepsilon)) - \int_0^t [\widetilde{\mathcal{H}}\, g](\mu_s^\varepsilon, p_s^\varepsilon) - e_t^\varepsilon \\ &= \varepsilon^{(1-r)\beta}(g(p_t^\varepsilon) - g(p_0^\varepsilon)) - \int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_s^\varepsilon, q)\Gamma^\varepsilon(ds\times dq) - e_t^\varepsilon\end{aligned}$$

with

$$e_t^\varepsilon = \int_0^t \varepsilon^{(1-r)\beta}[\widetilde{\mathcal{H}}^\varepsilon u](\mu_s^\varepsilon, p_s^\varepsilon) - [\widetilde{\mathcal{H}}\, g](\mu_s^\varepsilon, p_s^\varepsilon)\, ds$$

Thus, for all $T > 0$, we have by Lemma 8 that $\mathbb{E}\left[\sup_{t\in[0,T]}|e_t^\varepsilon|\right] \to 0$ when $\varepsilon \to 0$. We may check that the limit

$$\int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_s, q)\Gamma(ds\times dq)\,,$$

obtained by Lemma 9, is a martingale, which is continuous and of bounded variations and hence must be constant, in fact equal to 0. Thus, for each $g \in \mathcal{G}$ and $t \geq 0$

$$\int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_s, q)\Gamma(ds\times dq) = 0\,, \quad a.s.$$

Using [33, Lemma 1.4] with a slight adaptation along the proof, it exists $(\gamma_t)_{t\geq 0}$ a $\mathcal{P}(\ell_1^+)$-valued optional process, s.t. for all $t \geq 0$ and $B \in \mathcal{B}(\ell_1^+)$

$$\Gamma([0,t] \times B) = \int_{[0,t]} \gamma_s(B)ds\,, \quad a.s.$$

and since the functions $q \in \ell_1^+ \mapsto q_i$ are Γ-integrable it readily follows that for all $t \geq 0$ and $g \in \mathcal{G}$

$$\int_{[0,t]\times\ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_s, q)\gamma_s(dq)ds = 0, \quad a.s.$$

Hence, by separability of $\mathcal{G}$ (see Lemma A.6), with probability one, we have

$$\int_{\ell_1^+} [\widetilde{\mathcal{H}}\, g](\mu_t, q)\gamma_t(dq) = 0, \quad a.e\ t \geq 0 \text{ and } \forall g \in \mathcal{G}\,.$$

Then, thanks to Proposition B.2 in Appendix, for a fixed $\nu \in \mathcal{X}$ such that $c = m - \langle \nu, \mathrm{Id}\rangle > \rho$, the operator $[\tilde{\mathcal{H}}\cdot](\nu,\cdot)$ has a unique stationary distribution $\pi_\nu = \delta_0$ in $\mathcal{P}(v-\ell_1^+)$, the Dirac measure at the sequence equals to 0 everywhere, *i.e.* satisfying

$$\int_{\ell_1^+} [\widetilde{\mathcal{H}}g](\nu, q)\pi_\nu(dq) = 0\,, \quad \forall g \in \mathcal{G}\,.$$

Therefore, on a time interval $[t_0, t_1]$ such that $c_s = m - \langle \mu_s, \mathrm{Id}\rangle > \rho$, we can conclude that the process $(\gamma_s)_{s\in[t_0,t_1]}$ is deterministic and equals to δ_0, and finally, $\Gamma = ds \times \delta_0$. This proves the result. □

We finish this section by mentioning an original behaviour of the solution. When $\min(r_a, r_b) \geq 1$ the inverse of the rate functions a and b are not integrable in x_0. Hence, depending on c_t is greater or smaller than ρ, a cluster of size x_0 would take an infinite time to reach a size $x > x_0$ or at the reverse a cluster of size $x > x_0$ would take an infinite time to go back to x_0. The boundary and the rest of the clusters are completely disconnected. We are in a case where no boundary condition is needed and Theorem 1 is sufficient to define the solution. Nevertheless we can quantify the value of $\mu_t(x_0)$. Indeed Proposition 9 provides a better convergence of $\{p^\varepsilon\}$ than the one given by the occupation measure. In fact, formally, the occupation measure charges the limit p, that is $\Gamma = \delta_{p_t} \times dt$ because of the better tightness obtained. We do not give a proof of the following result which mainly relies on the same arguments as in the proof of Theorem 4. The result reads:

Proposition 10. *Let μ^ε constructed by* (6) *and p^ε defined in* (33)*, for each $\varepsilon > 0$. Assume that Assumptions* (H1) *to* (H10) *hold in addition to* (H11) *with $\min(r_a, r_b) \geq 1$. Assume also $\{\mu_{\mathrm{in}}^\varepsilon\}$ converges towards a deterministic measures μ_{in} in $\mathcal{P}(w\text{–}\mathcal{X})$. Then, $\{\mu^\varepsilon\}$ and $\{p^\varepsilon\}$ converge along an appropriate subsequence as $\varepsilon \to 0$, respectively, to μ in $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, w - \mathcal{X}))$ and p in $\mathcal{P}(\mathcal{D}(\mathbb{R}_+, v - \ell_1^+))$.*

The limit μ belongs to $\mathcal{C}(\mathbb{R}_+, w-\mathcal{X})$ and p belongs to $\mathcal{C}(\mathbb{R}_+, v-\ell_1^+)$. Moreover, they satisfy for all $\varphi \in \mathcal{C}_b^1([x_0,+\infty))$ and $t \geq 0$

$$\langle \mu_t, \varphi\rangle = \langle \mu_{\rm in}, \varphi\rangle + \int_0^t \int_{x_0}^\infty \varphi'(x)(a(x)c_s - b(x))\mu_s(dx)ds \\ + \int_0^t \varphi(x_0)\left[k(c_s) - l(p_{0,s})\right] + \varphi'(x_0)b(x_0)p_{0,s}ds,$$

where $c_t = m - \langle \mu_t, \mathrm{Id}\rangle \geq 0$ and $p_0 \in \mathcal{C}(\mathbb{R}_+)$ is the first component of the solution $p \in \mathcal{C}(\mathbb{R}_+, v-\ell_1^+)$ to the Becker-Döring model, i.e. for all $u \in \ell_c$ and $t \geq 0$

$$\langle p_t, u\rangle_{\ell_1} = \langle p_{\rm in}\,, u\rangle_{\ell_1} + \int_0^t \Big[u_0(k(c_s) - l(p_{0,s}) - J_0(\mu_s, p_s)) \\ + \sum_{n\geq 1} u_n(J_{n-1}(\mu_s,p_s) - J_n(\mu_s,p_s))\Big]ds\,.$$

7. Numerical examples

We here illustrate the theoretical results we obtained by numerical simulations, with the comparison between simulations of the rescaled stochastic Becker-Döring model and simulations of the deterministic limiting model. Sample paths (trajectories) of the rescaled stochastic Becker-Döring model are simulated by a discrete-event simulation, using the next reaction method (see [2], code available upon request). We use this code to illustrate the behavior of the rescaled stochastic process as $\varepsilon \to 0$, by plotting single trajectories for given small ε. For the deterministic model, we use two approaches. When the choice of coefficients allows to reduce the Lifshitz-Slyozov model to a set of ordinary differential equations, we simulate the reduced ordinary differential equations by a standard explicit Euler Scheme. For the general case, we simulate the transport-like equation using an explicit up-wind scheme (code available upon request). The Lifshitz-Slyozov model can be reduced to two ordinary differential equations on c_t (or $\langle \mu_t^\varepsilon, \mathrm{Id}\rangle$) and $\langle \mu_t, \mathbf{1}\rangle$ when $a(x) = \overline{a}x^{r_a}$, $b(x) = \overline{b}x^{r_b}$ with $r_a, r_b \in \{0,1\}$.

Below, we want to illustrate several aspects of our theoretical results.

- Firstly, the behavior of the boundary term p_0^ε as $\varepsilon \to 0$. We verify that this quantity converges to 0 as $\varepsilon \to 0$ when the characteristics are incoming in $x = x_0$, and converges to a positive value when the characteristics exit the domain. We also illustrate the fast variations of p_0^ε compared to the measure μ_t^ε in $\mathcal{X}$ (for instance quantity such as $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$ or C_t^ε).
- Secondly, we compare the behavior of C_t^ε and $\langle \mu_t^\varepsilon, \mathbf{1}\rangle$ with the deterministic limit. We show that the behavior of the particle variable is faithfully reproduced in all cases by the limiting model, and the total number of clusters is only well approached by the limiting model in the case of incoming characteristics, in agreement with the different convergence theorems obtained either in the vague or the weak topology.

- Thirdly, we show the good agreement between the profile μ_t^ε and its limiting value μ_t, and highlight the discrepancies around $x = x_0$ due to a vague convergence rather than a weak convergence in the case of outgoing characteristics.
- Finally, we point interesting (numerical) deviations of the stochastic model from the deterministic model. Starting with a pure-particleic initial condition, in the case of outgoing characteristics, the limiting deterministic model predicts that no cluster can be formed. But, for $\varepsilon > 0$, in the stochastic model, if one cluster eventually reaches a critical threshold, it will start growing very rapidly. This is an analog scenario to an initial condition which is a stable fixed point of a deterministic model, in the presence of a second stable fixed point (bistability). Occasionally, the stochastic process may escape the first stable fixed point, and rapidly reach the second one, as in phase-transition phenomena.

The choice of rate coefficients $k(c)$, $l(p)$, $a(x)$ and $b(x)$, the total mass m and initial conditions μ_0 are summarized in Table 1. For the stochastic model, we use an ε-interpolation μ_0^ε of μ_0, such that $\mu_0^\varepsilon \to \mu_0$ as $\varepsilon \to 0$. When we detail our numerical results, we constantly refer to the cases depicted in Table 1, where essentially rate coefficients $a(x), b(x)$ and initial condition are varied. Case I used constant rate coefficients, with either incoming or outgoing characteristics. In Cases II and IV, $a(0) = 0$ and $b(0) > 0$ so that characteristics are always outgoing. Finally in case III, $a(0) > 0$ and $b(0) = 0$ so that characteristics are always incoming.

TABLE 1

Summary of numerical simulations. In all cases, $i_0^\varepsilon = 2$, $k(c) = c^2$ and $l(p) = p$. Reaction rates are taken independent of ε. ν is a normalized ($\langle \nu, \mathrm{Id}\rangle$=1) truncated Gaussian centered on $x = 0.5$. The scaling exponents are chosen as follows: $\alpha = \beta = 1$, $A = -1$, $B = 1$, $\gamma = 0$ and $K = L = 1$.

Case		Rates			Initial value			Flux		Figure	Video*
		$a(x)$	$b(x)$	ρ	m	c_0	μ_0	In	Out		
I	a	1	2	2	1	1	0		✓	1	
	b				3	1	2ν	✓	✓	2	Yes
	c				1	0.5	0.5ν		✓		Yes
	d				3	3	0	✓	✓	3 4	Yes
II	a	x	1	$+\infty$	3	3	0		✓	7	Yes
	b				3	1	2ν		✓	6	Yes
III		1	x	0	3	3	0	✓		5	Yes
IV	a	$x^{1/3}$	1	$+\infty$	3	3	0		✓	8	Yes
	b				3	2	ν		✓		Yes

*Videos are disponible at http://yvinec.perso.math.cnrs.fr/video.html

7.1. Boundary Term

When characteristics are outgoing, we are not able to recover the value of the boundary term. We proved in Theorem 3 that such a term was tight in a particular topology, and followed a fast subsystem given by the generator in Equation (43). But we could not identify uniquely the stationary states of such a fast subsystem. We found instead that the number of stationary states are potentially infinite, parametrized by a normalization factor given by $\sum_i p_i$. For constant rate coefficients, the stationary states of Equation (43) are simply given by

$$p_0(t) = \big(1 - \frac{ac_t}{b}\big) \sum_{i \geq 0} p_i(t). \tag{47}$$

In some cases, as in case Ia, we observe fast "variations" around the value predicted by the stationary state (47) when we choose $\sum_i p_i(t) = \langle \mu_t, 1 \rangle$ (Figure 1). Note that in such a case, the particle variable stays constant (data not shown).

In other cases, as in case Ib, we observe that this heuristic breaks down, and the goodness of the approximation value of the fast variable is depending on time. Heuristically, for small times, the dimer variable equilibrates with the small clusters sizes, why for larger times, it equilibrates with all sizes (total cluster number). See Figure 2.

When the characteristics are incoming at $x = x_0$, we are able to recover the value of the boundary term, and Theorem 2 predicts that the boundary term is zero. In the Case Id, we check that p_2^ε converges to 0 as $\varepsilon \to 0$ (Figure 3). A second order approximation of the fast subsystem indicates that

$$p_0^\varepsilon \approx \varepsilon^\beta \frac{k(c)}{ac - b}, \tag{48}$$

which is verified numerically. Such a value is obtained as the equilibrium value of the (fast) system:

$$\begin{cases} \dfrac{d}{dt} p_0^\varepsilon = \varepsilon^\beta \Big(k(c) - l(p_0^\varepsilon(t)) \Big) - \Big(acp_0^\varepsilon(t) - bp_1^\varepsilon(t) \Big) \\ \dfrac{d}{dt} p_i^\varepsilon = \Big(acp_{i-1}^\varepsilon(t) - bp_i^\varepsilon(t) \Big) - \Big(acp_i^\varepsilon(t) - bp_{i+1}^\varepsilon(t) \Big), \ i \geq 1\,. \end{cases} \tag{49}$$

We also observe the fast variations of p_0^ε compared to μ^ε. Note that after a time of approximately 1, c^ε is going below through the threshold ρ, and the "characteristics" are now exiting the domain (see Figure 4), which explains why p_0^ε seems to behave differently.

7.2. Moments

For constant or linear coefficients, we can directly compared the moment $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$, and $\langle \mu_t^\varepsilon, \mathrm{Id} \rangle = m^\varepsilon - C_t^\varepsilon$ with the solution of ordinary differential equations. For instance, in the constant coefficient case (case I), with incoming characteristics, the Lifshitz-Slyozov equations together with the boundary conditions (see Equation (12)) can be reduced to

$$\begin{aligned} \frac{d}{dt} \langle \mu_t, \mathbf{1} \rangle &= k(c_t) \\ \frac{d}{dt} \langle \mu_t, \mathrm{Id} \rangle &= (ac_t - b) \langle \mu_t, \mathbf{1} \rangle\,. \end{aligned} \tag{50}$$

We show in Figure 4 the good agreement between the stochastic simulation and the numerical solution of the ordinary differential equations (50). Note that around time 1, C_t^ε is going below through the threshold ρ and the “characteristics” are now outgoing, and the solution deviates from the ODEs.

A similar picture is obtained in the case III, with $a(x) = 1$, $b(x) = x$. In such case, the characteristics are always incoming, and the moment equations obtained from the Lifshitz-Slyozov model (Equation (12)) are

$$\begin{aligned} \frac{d}{dt}\langle \mu_t, \mathbf{1}\rangle &= k(c_t) \\ \frac{d}{dt}\langle \mu_t, \mathrm{Id}\rangle &= ac_t\langle \mu_t, \mathbf{1}\rangle - b\langle \mu_t, \mathrm{Id}\rangle \,. \end{aligned} \tag{51}$$

We illustrate in Figure 5 the very good match between the numerical solution of Equations (51) and the stochastic simulations.

For outgoing characteristics, we were not able to obtain a formulation of the Lifshitz-Slyozov equations in the weak topology, so that the moment equations cannot be straighforwardly deduced. For instance, on the case II, one could heuristically think that the Lifshitz-Slyozov equations reduce to

$$\begin{aligned} \frac{d}{dt}\langle \mu_t, \mathbf{1}\rangle &= k(c_t) - l(p_0) \\ \frac{d}{dt}\langle \mu_t, \mathrm{Id}\rangle &= ac_t\langle \mu_t, \mathrm{Id}\rangle - b\langle \mu_t, \mathbf{1}\rangle + bp_0 \,. \end{aligned} \tag{52}$$

One could not uniquely identify the boundary term p_0, but we know however that this term is tight in a particular topology, and the numerical simulations indicate that it does converge indeed as $\varepsilon \to 0$. Using the boundary value obtained from the stochastic simulations (with $\varepsilon = 0.0005$), we show below in Figure 6 that the stochastic simulation and the numerical solution of the ordinary differential equations (52) are very close. We also display the moment evolution calculated with the numerical solution of the full Lifshitz-Slyozov model, which also shows a perfect match for the particle evolution.

7.3. Profile

We provide on supplementary data videos of the time evolution of the stochastic point measures μ^ε and the numerical solution of the Lifshitz-Slyozov equations for the cases Ib,Ic,Id, IIb, III and IVb. The overall behavior is a good match, away from the boundary $x = 0$ for the outgoing characteristic case, which is explained by a vague convergence only. In particular, in case Id, the size distribution is well approximated by the Lifshitz-Slyozov model as soon as $c_t^\varepsilon > \rho$, while it is well approximated for all times in cases III. In all other cases, Ib,Ic, IIb and IVb, the characteristics are outgoing and the approximation are valid only away from the boundary.

7.4. Deviations

We illustrate two similar situations where the finite stochastic model largely deviates from its limiting deterministic model. In both Cases IIa and IVa, we are in a situation where the characteristics are outgoing but the drift is positive for a size greater than a critical size X_c that depends on the particle variable. Hence, in Case IIa, the critical size equals to $X_c = \frac{1}{c}$, while in Case IVa, $X_c = \frac{1}{c^3}$. When starting with no cluster, i.e. $c(0) = m$, we are in a stable situation: indeed, in such a case, no cluster can be formed by the deterministic Lifhsitz-Slyozov model. But in a stochastic simulation, if eventually a cluster reaches the critical size (with may happen with a very low probability as $\varepsilon \to 0$), then this cluster will now have a strong tendency to grow as the deterministic drift is positive. This is typical of a bistable scenario (or metastability). We illustrate this on Figures 7 and 8 by showing the time evolution of the particle variable, and on supplementary material with the time evolution of the profile in both cases.

8. Discussion

The link between the discrete size Becker-Döring model and the continuous size Lifschitz-Slyozov model has already been studied within the context of deterministic model by [18, 36]. We used a similar approach to those previous studies, in the sense that we introduced a scaling parameter, linked to the initial number of particles, and investigated the limit when this scaling parameter tends to zero. The main difference is that in both studies [18, 36], the authors obtained convergence results in a vague topology, that is the topology of convergence against compactly supported test functions. The authors in [36] were able to extend the convergence in the weak topology of $L^1(xdx)$, which do not see the boundary as well (as the weight x vanishes at the boundary $x = 0$). Thus, these results were restricted in practice to cases where the characteristics exit the domain (for which uniqueness do not require specification of the boundary term).

Concerning the regularity imposed on the rate coefficients, we essentially have the same hypotheses as [36]. However, our choice of scalings slightly differs. Let us explain this in detail now. In our model, considering $i_0 = 2$ and the law of mass-action for the nucleation reaction rates k, l (see Remark 1) gives

$$\begin{aligned} k_0^\varepsilon(c) &= a_0^\varepsilon(1)c(c-1) \\ l_0^\varepsilon(\langle \mu, 1_2 \rangle) &= b_0^\varepsilon(2)\langle \mu, 1_2 \rangle \,. \end{aligned}$$

Allowing specific scalings for $a_0^\varepsilon(1)$ and $b_0^\varepsilon(2)$ (that potentially differ from the scalings on $(a_0^\varepsilon(i))_{i\geq 2}$ and $(b_0^\varepsilon(i))_{i\geq 3}$), we define (see the link with Equation (7))

$$\begin{aligned} a^\varepsilon(\varepsilon) &:= \varepsilon^{A_1} a_0^\varepsilon(1), \\ b^\varepsilon(2\varepsilon) &:= \varepsilon^{B_1} b_0^\varepsilon(2). \end{aligned}$$

Thus, still with reference to the scaling given in Equation (7),

$$\begin{aligned} k^\varepsilon(c) &= \varepsilon^K k_0^\varepsilon(\varepsilon^{-\theta} c) = \varepsilon^{K-2\theta-A_1} a^\varepsilon(\varepsilon) c(c - \varepsilon^\theta) \,, \\ l^\varepsilon(p) &= \varepsilon^L l_0^\varepsilon(\varepsilon^{-\alpha} p) = \varepsilon^{L-\alpha-B_1} b^\varepsilon(2\varepsilon) p \,. \end{aligned}$$

This implies that (under the convergence of k^ε, l^ε, a^ε, b^ε in Assumptions 3 and 6)

$$\begin{aligned} K &\geq A_1 + 2\theta - r_a\beta, \\ L &\geq B_1 + \alpha - r_b\beta. \end{aligned}$$

With $\gamma = 0$, to satisfy the scaling relations (H7), we then need

$$\begin{aligned} A_1 &\leq -\alpha - 2\beta + r_a\beta = A - 2\beta + r_a\beta, \\ B_1 &\leq r_b\beta = B - \beta + r_b\beta. \end{aligned}$$

Hence, with a mass-action law hypothesis, the scaling relation (H7) are satisfied if both nucleation and de-nucleation rates are slowing down compared to the aggregation and fragmentation rates, respectively. In the previous works [18, 36], only the first aggregation rate (nucleation rate) $a(1)$ is slowing down; this condition being crucial to prevent explosion due to the nucleation term, as in the moment estimates property (Proposition 3), and allowing to get the compact containment condition. The slow down of the first fragmentation rates (de-nucleation rates) allows to derive the "equicontinuity" part of tightness property (Proposition 6). Such a part has been only proved for compactly supported functions in the previous works [18, 36], without the extra scaling of the de-nucleation rate.

Our framework allows us to investigate different scalings as well (with λ_K, λ_L or λ_C non zero for instance). The case $\lambda_C < 0$ is linked to the so-called Lifshitz-Slyozov-Wagner model, where the particle variable is a fast variable and instantaneously averaged to conserve the mass relationship. This case has been treated in [36], where the convergence of the particle variable towards

$$\int b(x)\mu_t(dx) / \int a(x)\mu_t(dx)$$

occurred in the weak-$*$ topology on $L^\infty(0,T)$. We conjecture that the same is true in our model, however we were not able to find suitable moment estimates on C_t^ε to prove it. Also, we conjecture that some similar limiting models may be derived for $\lambda_K, \lambda_L < 0$ with specific boundary conditions.

Appendix A: Topology and Compactness

In the sequel E is a Polish space (a separable topological space which is completely metrizable) and we consider its underlying Borel σ-algebra $\mathcal{B}(E)$.

A.1. The space $\mathcal{X}$

Let $(h_i)_{i\geq 1}$ be a countable sequence (possibly finite) of nonnegative real-valued measurable function on E. We define

$$\mathcal{X}(E) := \left\{ \nu \in \mathcal{M}_b(E) \,:\, \langle \nu, h_i \rangle < +\infty\,,\ \forall i \geq 1 \right\},$$

which is equipped with the weak topology, denoted by $(\mathcal{X}, w)$ or alternatively $w - \mathcal{X}$, *i.e.* the coarsest topology that makes continuous $\nu \mapsto \langle \nu, \varphi \rangle$ and $\nu \mapsto \langle \nu, h_i \varphi \rangle$ for all $\varphi \in \mathcal{C}_b(E)$ and $i \geq 1$. Remark, for all $i \geq 1$ and $\nu \in \mathcal{X}(E)$, we can define the density measure $h_i \cdot \nu \in \mathcal{M}_b(E)$ by such that $\langle h_i \cdot \nu, \varphi \rangle = \langle \nu, h_i \varphi \rangle$ for any $\varphi \in \mathcal{C}_b(E)$, see [9, Chap. IX §2.2].

Lemma A.1. *The space $(\mathcal{X}(E), w)$ is a Polish space. Let $\rho_{\mathcal{X}}$ defined, for any $(\nu, \mu) \in \mathcal{X}(E) \times \mathcal{X}(E)$, by*

$$\rho_{\mathcal{X}}(\nu, \mu) := \sum_{i \geq 0} 2^{-(i+1)} (1 \wedge \rho(h_i \cdot \nu, h_i \cdot \mu)) ,$$

where ρ is the Prohorov metric on $\mathcal{M}_b(E)$ and the convention $h_0 = \mathbf{1}$. Then, $\rho_{\mathcal{X}}$ is a complete metric equivalent to the weak topology on $\mathcal{X}(E)$.

See for instance [7, Section 6] for a definition of the Prohorov metric.

Proof. The properties of $(\mathcal{X}(E), w)$ derive from its identification to the space $\{\nu \in \mathcal{M}_b \, : \, h_i \cdot \nu \in \mathcal{M}_b(E) \, , \; \forall i \geq 0\}$. □

Lemma A.2 (A criterion of weakly relatively compactness in $\mathcal{X}$)**.** *Let $\mathcal{K}$ be a subset of $\mathcal{X}$. Suppose it exists H a nonngative measurable function on E such that, for all $n > 0$ the sets $K_n = H^{-1}([0, n])$ are compact in E. Suppose moreover it exists $n_0 \geq 0$ such that for all $i \geq 0$, and $x \in K_{n_0}^c$, we have $h_i(x) \leq H(x)$. Assume further that it exists $\Phi \in \mathcal{U}_\infty$ (defined in Section 3.3) such that*

$$\sup_{\nu \in \mathcal{K}} \langle \nu, \mathbf{1} + H + \Phi(H) \rangle < +\infty \, .$$

Then, $\mathcal{K}$ is relatively compact in $(\mathcal{X}(E), w)$. Moreover, let $\{\nu^\varepsilon\}$ a sequence in $\mathcal{K}$ and assume that h_i is continuous for some $i \geq 1$, then up to a subsequence it exists $\nu \in \mathcal{X}(E)$ such that $\nu^\varepsilon \to \nu$ in the weak topology and as $\varepsilon \to 0$ and

$$\langle \nu^\varepsilon, h_i \rangle \to_{\varepsilon \to 0} \langle \nu, h_i \rangle \, .$$

Proof. The aim is to link these bounds to a criterion of weakly relatively compactness in $\mathcal{M}_b(E)$. Let ν in $\mathcal{K}$, then for $n \geq n_0$ and for all $i \geq 1$

$$\begin{aligned} \langle (\mathbf{1} + h_i) \cdot \nu, \mathbf{1}_{K_n^c} \rangle = \int_{K_n^c} \frac{1}{H} H + \frac{h_i}{\Phi(H)} \Phi(H) \nu(dx) \\ \leq \left(\frac{1}{n} + \sup_{y \geq n} \frac{y}{\Phi(y)} \right) \sup_{\nu \in \mathcal{K}} \langle \nu, H + \Phi(H) \rangle \, . \end{aligned}$$

When $n \to +\infty$ the right hand side goes to 0. It yields $(\mathbf{1} + h_i) \cdot \nu$ belongs to a weakly relatively compact set of $\mathcal{M}_b(E)$, see [9, Chap. IX §5.5, Theorem 2]. Let $\{\nu^\varepsilon\}$ be a sequence of $\mathcal{K}$, there exist $\mu_i \in \mathcal{M}_b(E)$ and a subsequence (still indexed by ε) such that $\{(\mathbf{1} + h_i) \cdot \nu^\varepsilon\}$ weakly converges to μ_i in $\mathcal{M}_b(E)$. Hence, by a diagonal process, for all $i \geq 1$ and for any $\varphi \in \mathcal{C}_b(E)$,

$$\langle (\mathbf{1} + h_i) \cdot \nu^\varepsilon, \varphi \rangle \to \langle \mu_i, \varphi \rangle \, .$$

Since $\varphi = (\mathbf{1} + h_i)^{-1}$ is a continuous bounded function, it yields in particular, for all $i \geq 0$, $\nu^\varepsilon \to \nu_i := (\mathbf{1}+h_i)^{-1} \cdot \mu_i$ in $\mathcal{M}_b(E)$. By the uniqueness of the limit, we define $\nu = \nu_i$. It readily follows that $\nu \in \mathcal{X}(E)$ and $\nu^\varepsilon \to \nu$ in $w - \mathcal{X}(E)$. The last remark comes from the fact we can take $\varphi = h_i/(1 + h_i)$ which is a bounded and continuous function. □

Let us details two classical examples about the control of x-moments and a more complex applications useful for our purpose.

Example 1. Let us take $E = [0, +\infty)$, the functions $h_1 = H = \mathrm{Id}$. It readily follows a compact criterion for the measure space $\mathcal{X}([0, +\infty))$ defined by $\{\nu \in \mathcal{M}_b([0, +\infty)) \, : \, \langle \nu, \mathrm{Id} \rangle < +\infty\}$.

Example 2. Let us take $E = [0, +\infty)$, the functions $h_i : x \mapsto x^i$ for $i = 1, \dots, p$ and $H = h_p$. We have for all $x > 1$ that $h_i(x) \leq H(x)$. It readily follows a compact criterion for $\mathcal{X}(E) := \{\nu \in \mathcal{M}_b(E) \, : \, \langle \nu, h_i \rangle < +\infty \, , \; i = 1, \dots, p\}$. Remark that Φ can be chosen as $x \mapsto x^{p+1}$.

Example 3. Let us take $E = \ell_1^+$ with the vague topology (see Remark 9), the functions $h_i(q) = q_i$ and $H(q) = \|q\|_{\ell_1}$. The h_i are continuous and the pre-image by H of any bounded set is compact in E for the vague topology. It readily follows from the previous lemma a compact criterion for $\mathcal{X}(E) := \{\nu \in \mathcal{M}_b(E) \, : \, \langle \nu, h_i \rangle < +\infty \, , \; \forall i \geq 0\}$. Remark that, if $\nu^\varepsilon \to \nu$ in $w - \mathcal{X}$ then

$$\int_{\ell_1^+} q_i \nu^\varepsilon(dq) \to \int_{\ell_1^+} q_i \nu(dq) \, , \; \forall i \geq 0 \, .$$

But since the norm is not continuous for the vague topology it appears clearly that we cannot hope the convergence of $\int_{\ell_1^+} \|q\| \nu^\varepsilon(dq)$ to $\int_{\ell_1^+} \|q\| \nu(dq)$.

A.2. The space $\mathcal{Y}$

We proceed here to a slight adaptation of [33]. Let $(h_i)_{i \geq 1}$ be a sequence of measurable function on a Polish space E, and for any $t \geq 0$ we consider $\mathcal{X}([0, t] \times E)$ defined similarly to the previous section as a subset of $\mathcal{M}_b([0, t] \times E)$. Now, we consider the space

$$\mathcal{Z}(\mathbb{R}_+ \times E) := \Big\{ \Theta \in \mathcal{M}(\mathbb{R}_+ \times E) \; : \; \forall t \geq 0, \, \Theta^t \in \mathcal{X}([0, t] \times E) \Big\} \, ,$$

where Θ^t denotes the restriction of Θ to $[0, t] \times E$. We endow this space with the metric $\rho_{\mathcal{Z}}$ given, for any Θ and Γ belonging to $\mathcal{Z}(\mathbb{R}_+ \times E)$, by

$$\rho_{\mathcal{Z}}(\Theta, \Gamma) = \int_0^\infty e^{-t} \, 1 \wedge \rho_{\mathcal{X}}^t(\Theta^t, \Gamma^t) \, dt \, ,$$

where $\rho_{\mathcal{X}}^t$ is the modified Prohorov metric on $\mathcal{X}([0, t] \times E)$. This metric defines the weak$^\#$ topology on $\mathcal{Z}(\mathbb{R}_+ \times E)$ and the space is denoted by $(\mathcal{Z}(\mathbb{R}_+ \times E), w^\#)$. Note that a sequence $\{\Theta^\varepsilon\} \subset \mathcal{Z}$ converges in $\rho_{\mathcal{Z}}$ if and only if $\{\Theta^{\varepsilon, t}\}$ converges in $\rho_{\mathcal{X}}^t$ for almost every t. The next three lemmas follow [19, Appendix 2.6] and [33].

Lemma A.3. *The space $\mathcal{Z}(\mathbb{R}_+ \times E)$ equipped with the weak$^{\#}$ topology is a Polish space.*

Lemma A.4. *The subspace of $\mathcal{Z}(\mathbb{R}_+ \times E)$ given by*

$$\mathcal{Y}(\mathbb{R}_+ \times E) := \{\Theta \in \mathcal{Z}(\mathbb{R}_+ \times E) \,:\, \Theta([0,t] \times E) = t\}$$

and equipped with the topology induced by $\rho_{\mathcal{Z}}$ is a Polish space.

Proof. We just remark that $\mathcal{Y}(\mathbb{R}_+ \times E)$ is a closed suset of $\mathcal{Z}(\mathbb{R}_+ \times E)$. Indeed, if $\{\Theta^\varepsilon\}$ converges to Θ in $\rho_{\mathcal{Z}}$, then $\{\Theta^{\varepsilon,t}\} \to \Theta^t$ in ρ_t if and only if $\Theta^\varepsilon([0,t]\times E) \to \Theta([0,t] \times E)$ as $\varepsilon \to 0$. □

Lemma A.5 (A criterion of weakly relatively compactness in $\mathcal{Y}$)**.** *Let $\{t_k\}$ be a non-decreasing sequence in $\mathbb{R}_+$ such that $\lim_{k\to+\infty} t_k = +\infty$. Then, the set*

$$\Big\{\Theta \in \mathcal{Y}(\mathbb{R}_+ \times E) \,:\, \forall k,\ \exists \text{ weak compact } \mathcal{K}_k \subset \mathcal{X}([0,t] \times E),\ \Theta^{t_k} \in \mathcal{K}_k,\ \forall \nu\Big\}$$

is a compact of $(\mathcal{Y}, \rho_{\mathcal{Z}})$.

A.3. The domain $\mathcal{G}$

The space $\mathcal{G}$ is defined as

$$\mathcal{G} := \{g : \ell_1 \to \mathbb{R} \,:\, \exists N \geq 1,\ \exists G \in \mathcal{G}_N,\ g(v) = G(v_0, \ldots, v_{N-1}),\ \forall v \in \ell_1\}$$

where

$$\mathcal{G}_N := \left\{G \in \mathcal{C}^2(\mathbb{R}^N) \,:\, G(0) = 0 \text{ and } \partial_n G \in \mathcal{C}^1_c(\mathbb{R}^N)\right\} ,$$

with ∂_n the partial derivative with respect to the n^{th} variable.

Lemma A.6. *The space $\mathcal{G}$ as a subset of $\mathcal{C}(\ell_1, \mathbb{R})$ (for the vague topology of ℓ_1, see Remark 9) with the topology of the uniform convergence is separable.*

Proof. First, by definition of the space $\mathcal{G}$, it can be noticed that

$$\mathcal{G} = \bigcup_{N\in\mathbb{N}^*} \widetilde{\mathcal{G}}^N ,$$

where $\widetilde{\mathcal{G}}^N$ is made of the functions in $\mathcal{G}$ related to the functions in $\mathcal{G}_N$. Therefore, the separability of $\mathcal{G}$ follows from the separability of each $\widetilde{\mathcal{G}}^N$.

Let us fix $N \geq 1$ and show that $\widetilde{\mathcal{G}}^N$ is separable. We start by recalling that the space $\mathcal{C}^1_c(\mathbb{R}^N)$ is separable as the following countable union of separable spaces

$$\mathcal{C}^1_c(\mathbb{R}^N) = \bigcup_{M\in\mathbb{N}^*} \mathcal{C}^1_c\left(\overline{\mathbb{B}}_{\mathbb{R}^N}(0, M)\right) ,$$

where $\overline{\mathbb{B}}_{\mathbb{R}^N}(0, M)$ is the closed ball of radius M and center 0. Therefore, a dense countable collection $(F_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{C}^1_c(\mathbb{R}^N)$ can be constructed with respect to this union. We then define the countable family given by the functions $(\widetilde{F}_{i_0,\dots,i_{N-1}})_{(i_0,\dots,i_{N-1})\in\mathbb{N}^N}$ of $\widetilde{\mathcal{G}}^N$ with

$$\widetilde{F}_{i_0,\dots,i_{N-1}}(v) = \int_0^1 \sum_{n=0}^{N-1} F_{i_n}(tv_0,\dots,tv_{N-1})\ v_n\,dt\,,$$

for all v in ℓ_1.

We now consider a function g in $\widetilde{\mathcal{G}}^N$ with its associate function $G \in \mathcal{G}_N$. Since the partial derivatives of G are compactly supported, there exists $M \in \mathbb{N}^*$ such that for all $0 \le n \le N-1$, $\operatorname{supp}\,\partial_n G \subset \overline{\mathbb{B}}_{\mathbb{R}^N}(0,M)$. For all $\varepsilon > 0$, by separability of $C_c^1(\mathbb{R}^N)$, there exist $i_0,\dots,i_{N-1}$ such that

$$\|\partial_n G - F_{i_n}\|_\infty \le \frac{\epsilon}{NM} \quad \text{and} \quad \operatorname{supp} F_{i_n} \subset \overline{\mathbb{B}}_{\mathbb{R}^N}(0,M)\,,$$

for all $0 \le n \le N-1$. Then, since we have for all v such that $(v_0,\dots,v_{N-1}) \in \overline{\mathbb{B}}_{\mathbb{R}^N}(0,M)$

$$g(v) = G(v_0,\dots,v_{N-1}) = \int_0^1 \sum_{n=0}^{N-1} \partial_n G(tv_0,\dots,tv_{N-1})\ v_n\,dt\,,$$

we directly compute that

$$\begin{aligned}
&|g(v) - \widetilde{F}_{i_0,\dots,i_{N-1}}(v)| \\
&\qquad = \Big| \int_0^1 \sum_{n=0}^{N-1} (\partial_n G(tv_0,\dots,tv_{N-1}) - F_{i_n}(tv_0,\dots,tv_{N-1}))\ v_n\,dt \Big| \\
&\qquad \le \int_0^1 \sum_{n=0}^{N-1} |\partial_n G(tv_0,\dots,tv_{N-1}) - F_{i_n}(tv_0,\dots,tv_{N-1}))\ v_n|\,dt \\
&\qquad\qquad \le \epsilon\,.
\end{aligned}$$

All the partial derivatives $\partial_n G$ and the functions F_{i_n} being supported inside $\overline{\mathbb{B}}_{\mathbb{R}^N}(0,M)$, for all v such that $(v_0,\dots,v_{N-1}) \in \mathbb{R}^N \setminus \overline{\mathbb{B}}_{\mathbb{R}^N}(0,M)$, we thus have

$$g(v) = g\Big(\frac{M}{\|v\|_N}\ v\Big) \quad \text{and} \quad \widetilde{F}_{i_0,\dots,i_{N-1}}(v) = \widetilde{F}_{i_0,\dots,i_{N-1}}\Big(\frac{M}{\|v\|_N}\ v\Big)\,,$$

where $\|v\|_N^2 = \sum_{n=0}^{N-1} |v_n|^2$. Finally, we can conclude that

$$\|g - \widetilde{F}_{i_0,\dots,i_{N-1}}\|_\infty \le \varepsilon$$

and that the family $(\widetilde{F}_{i_0,\dots,i_{N-1}})_{(i_0,\dots,i_{N-1})\in\mathbb{N}^N}$ is dense in $\widetilde{\mathcal{G}}^N$. This ends the proof. □

Appendix B: Stationary states and measures for Becker-Döring

The aim of this appendix is to investigate the stationary measures of a modified Becker-Döring model represented for a given $\nu \in \mathcal{X}$ by an operator $[\widetilde{\mathcal{H}}\cdot](\nu,\cdot)$ defined by (43).

A stationary measure of such a model is a probability measure π on ℓ_1^+ solution of

$$\int_{\ell_1^+} [\widetilde{\mathcal{H}}g](\nu,q)\pi(dq) = 0\,, \quad \forall g \in \mathcal{G}\,. \tag{53}$$

We start by studying the stationary states of $[\widetilde{\mathcal{H}}\cdot](\nu,\cdot)$, that is, the sequences $\overline{q} \in \ell_1^+$ satisfying

$$\begin{aligned}[\widetilde{\mathcal{H}}g](\nu,\overline{q}) = 0\,, &\quad \forall g \in \mathcal{G}\\ \Leftrightarrow \sum_{n\geq 0} Dg[\overline{q}](1_n)(J_{n-1}(\nu,\overline{q}) - J_n(\nu,\overline{q})) = 0\,, &\quad \forall g \in \mathcal{G}\,,\end{aligned} \tag{54}$$

where $J_{-1} = 0$ and the Becker-Döring fluxes J_n for $n \geq 0$ are given by (44) and are recalled in the next proposition.

Proposition B.1. *Let $\nu \in \mathcal{X}$ such that $c = m - \langle \nu, \mathrm{Id}\rangle \geq 0$, the exponents r_a, r_b, the coefficients $\overline{a}$, $\overline{b}$ given by Assumption 6, ρ defined by* (11) *and the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\geq 1}$ by* (42) *.*

1. *In the case $r_a < r_b$, $r_a < 1$, the Becker-Döring fluxes $J_n(\nu,q) = a_n c q_n$ for all $n \in \mathbb{N}$. If $c > \rho = 0$, then the unique stationary state is*
$$\overline{q}_n = 0\,, \quad \forall n \geq 0\,.$$
2. *In the case $r_b < r_a$, $r_b < 1$, the Becker-Döring fluxes $J_n(\nu,q) = -b_{n+1}q_{n+1}$ for all $n \in \mathbb{N}$. Then the stationary states are all given by*
$$\overline{q}_n = 0\,,\ \forall n \geq 1\,,\ \textit{and}\ \ \overline{q}_0 \geq 0\,.$$
In particular, $\overline{q}_0 = \|\overline{q}\|_{\ell_1}$.
3. *In the case $r_a = r_b < 1$, the Becker-Döring fluxes $J_n(\nu,q) = a_n c q_n - b_{n+1}q_{n+1}$ for all $n \in \mathbb{N}$. Denoting $Q_n = (\prod_{i=0}^{n-1} a_i/b_{i+1})$ for all $n \in \mathbb{N}^*$ and $Q_0 = 1$, we have*
$$1/\rho = \limsup_{n\to+\infty} Q_n^{1/n} = \frac{\overline{a}}{\overline{b}}\,.$$
Moreover, if $0 \leq c < \rho$, then the stationary states are all given by
$$\overline{q}_n = (Q_n c^n)\overline{q}_0\,,\ \forall n \geq 1\,,\ \textit{and}\ \ \overline{q}_0 \geq 0\,.$$
In particular $\|\overline{q}\|_{\ell_1} = (\sum_{n\geq 0} Q_n c^n)\overline{q}_0$.

4. In the case $r_a = r_b < 1$, if $c > \rho$, then the unique stationary state is

$$\overline{q}_n = 0\,, \quad \forall n \geq 0\,.$$

Proof of Proposition B.1. We start by proving point 1. By (54), the state $\overline{q}$ is stationary if for all $g \in \mathcal{G}$,

$$\sum_{n\geq 0} Dg[\overline{q}](1_n)(J_{n-1}(\nu,\overline{q}) - J_n(\nu,\overline{q})) = 0\,.$$

In particular, applying for some functions g depending on only one term of sequences of ℓ_1^+, that is, $g(q) = G(q_n)$ with $G \in \mathcal{C}_c^2$ for instance for a fixed n, we obtain

$$J_n(\nu,\overline{q}) = 0\,, \quad \forall n\,.$$

Since for any n we have $J_n = a_n c q_n$ with $a_n > 0$ and $c > \rho = 0$ it readily follows that $\overline{q}_n = 0$. Conversely, the zero sequence is a stationary state.

Point 2. proceeds in the same way. Indeed, as previously, we have $J_n(\nu,\overline{q}) = 0$ for all $n \geq 0$. But as $J_n = -b_{n+1}\overline{q}_{n+1}$ with $b_{n+1} > 0$ it follows that for all $n \geq 1$, $\overline{q}_n = 0$. Therefore it remains a degree of freedom on $\overline{q}_0$ which has to be non-negative since $\overline{q}$ belongs to ℓ_1^+.

We now end up with the points 3. and 4. in which $r = r_a = r_b$. As above, we have $J_n(\nu,\overline{q}) = a_n c\overline{q}_n - b_{n+1}\overline{q}_{n+1} = 0$ for all $n \geq 0$ with $a_n > 0$ and $b_{n+1} > 0$. We clearly get that

$$\overline{q}_{n+1} = \frac{a_n}{b_{n+1}} c\overline{q}_n\,,$$

and thus, by induction, for all $n \geq 0$

$$\overline{q}_n = (Q_n c^n)\overline{q}_0\,.$$

Let us then prove that the radius of convergence of the series $\sum Q_n c^n$ is $\rho = \overline{b}/\overline{a}$. By the Cauchy-Hadamard Rule, this radius is $1/\limsup_{n\to+\infty} Q_n^{1/n}$. Note that, since the a_n's and the b_n's are given by (42), the term Q_n can be written for $n \geq 1$ as

$$Q_n = \prod_{i=0}^{n-1} \frac{a_i}{b_{i+1}} = \left(\frac{\overline{a}}{\overline{b}}\right)^n \frac{(x_0^1)^r \cdots (x_0^1 + n - 1)^r}{(x_0^1 + 1)^r \cdots (x_0^1 + n)^r} = \left(\frac{\overline{a}}{\overline{b}}\right)^n \frac{(x_0^1)^r}{(x_0^1 + n)^r}\,.$$

We thus immediately obtain that $\limsup_{n\to+\infty} Q_n^{1/n} = \overline{a}/\overline{b} = 1/\rho$ and the result is proved.

And, as $\overline{q}$ has to belongs to ℓ_1^+, if $0 \leq c < \rho$ the series is convergent and we obtain point 3. If now $c > \rho$ the series is not convergent so the unique solution is null-sequence, giving point 4.

□

We can now proceed to the identification of the stationary measures of a modified Becker-Döring model but, unfortunately, only in the cases 1. and 4. of the previous proposition.

Proposition B.2. *Let $\nu \in \mathcal{X}$ such that $c = m - \langle \nu, \mathrm{Id} \rangle > 0$. In the cases 1. and 4. of Proposition B.1, the unique stationary measure of the modified Becker-Döring model represented by the operator $[\widetilde{\mathcal{H}}\cdot](\nu, \cdot)$ is the Dirac measure δ_0 at the null-sequence.*

Remark B.1. In the cases 2. and 3. of Proposition B.1, there is no uniqueness of the stationary states vanishing all the fluxes but an infinite collection parametrized by the first component $\overline{q}_0$. Because of this, there is also no uniqueness of the stationary measures of the associate modified Becker-Döring model. Indeed, following the proof of Proposition B.2 here below, we can only conclude that, in these cases, a stationary measure is supported on all the stationary states. For instance, any probability measure, convex combination of Dirac measures at stationary states, is a stationary measure. This particularly implies that we are not able to identify the limit of the occupation measures Γ in Theorem 4 in the cases 2. or 3.

Before proving this result, we state a useful lemma requiring the introduction of a new space of functions from ℓ_1^+ to $\mathbb{R}$. We denote by $\overline{\mathcal{G}}$ the set of functions f from ℓ_1^+ to $\mathbb{R}$ such that there exist $N' \geq 0$ and a function F in $\mathcal{C}_c^1(\mathbb{R}^{N'})$ satisfying $f(q) = F(q_0, \ldots, q_{N'-1})$ for all q in ℓ_1^+. This space can be understood as the set of functions obtained by taking the Fréchet Derivative of a function g in $\mathcal{G}$ applied to a canonical sequence $\mathbf{1}_n$ for a given n, that is, $Dg[q](1_n)$.

Lemma B.1. *Let V be a continuous function from $v - \ell_1^+$ to $\mathbb{R}$ such that there exist $N \geq 0$ and a continuous function $\overline{V}$ from $\mathbb{R}^N$ to $\mathbb{R}$ with $V(q) = \overline{V}(q_0, \ldots, q_{N-1})$ for all q in ℓ_1^+. A probability measure π satisfying*

$$\int_{\ell_1^+} f(q) V(q)\, \pi(dq) = 0\,, \quad \forall f \in \overline{\mathcal{G}}\,, \tag{55}$$

is supported on $Z(V) := \{q \mid V(q) = 0\}$.

Proof. First note that all the measures supported on $Z(V)$ satisfy (55). Conversely let us prove that a measure π such that (55) holds in supported on $\mathcal{Z}(V)$. We introduce $\Omega = \operatorname{supp} \pi \cap Z(V)^c$ with $Z(V)^c = \ell_1^+ \setminus Z(V)$. We recall that the space ℓ_1^+ is endowed with the vague topology and is metrizable as $(\mathcal{M}_b(\mathbb{N}), v)$.

We start by assuming that the interior of Ω is nonempty, ie $\mathring{\Omega} \neq \emptyset$, and let us fix an element q^1 in $\mathring{\Omega}$. By definition $V(q^1)$ is either positive or negative. We here suppose that $V(q^1) > 0$ (the other case is similar). Since the function V is continuous, there exists $r_1 > 0$ such that V is positive on $\overline{\mathbb{B}}(q^1, r_1) \subset \mathring{\Omega}$, the closed ball of radius r_1 and center q_1. We then consider a function f in $\overline{\mathcal{G}}$ such that

$$\begin{cases} f(q) > 0 & \text{for all } q \text{ in the open ball } \mathbb{B}(q^1, r_1/2)\,, \\ f(q) \geq 0 & \text{for all } q \text{ in } \overline{\mathbb{B}}(q^1, r_1)\,, \\ f(q) = 0 & \text{otherwise}\,. \end{cases}$$

Applying (55) with f, we have

$$0 = \int_{\ell_1^+} f(q) V(q) \pi(dq) = \int_{\overline{\mathbb{B}}(q^1, r_1)} f(q) V(q) \pi(dq)\,.$$

Since $f(q)V(q) \geq 0$ for all $q \in \overline{\mathbb{B}}(q^1, r_1)$ and $f(q)V(q) > 0$ on $\mathbb{B}(q^1, r_1/2)$, there is a contradiction. Thus, the set Ω has an empty interior and is therefore discrete. The measure π restricted to Ω can be written as

$$\pi_{|\Omega} = \sum_{i \in I} \lambda_i \delta_{q^i} ,$$

with the q^i's in Ω and $\lambda_i \geq 0$. Now using a test function f in $\overline{\mathcal{G}}$ such that $f(q^i) = V(q^i)$ and $f(q^j) = 0$ for all $j \neq i$, we can deduce that $\lambda_i = 0$. This proves the result. □

We now in position to prove Proposition B.2.

Proof of Proposition B.2. Assume that π is a stationary measure, that is, satisfying (53). For all $i \in \mathbb{N}$ and $f \in \overline{\mathcal{G}}$, we consider the function g^i in $\mathcal{G}$ such that for all q in ℓ_1^+

$$Dg^i[q](1_n) = f(q)\mathbf{1}_{n=i} ,$$

that is,

$$g^i(q) = \int_0^1 f(tq) q_i \, dt .$$

Applying (53) with $g = g^i$, we get

$$\int_{\ell_1^+} f(q)(J_{i-1}(\nu, q) - J_i(\nu, q))\pi(dq) = 0 .$$

and thus, the measure π satisfies

$$\int_{\ell_1^+} f(q)(J_{i-1}(\nu, q) - J_i(\nu, q))\pi(dq) = 0 , \quad \forall f \in \overline{\mathcal{G}} \text{ and } i \in \mathbb{N} .$$

Since for $i = 0$, we have

$$\int_{\ell_1^+} f(q) J_0(\nu, q)\pi(dq) = 0 , \quad \forall f \in \overline{\mathcal{G}} ,$$

we can deduce by induction that, for all n in $\mathbb{N}$ and f in $\overline{\mathcal{G}}$,

$$\int_{\ell_1^+} f(q) J_n(\nu, q)\pi(dq) = 0 .$$

Finally, applying Lemma B.1 with $V = J_n(\nu, \cdot)$ for all n in $\mathbb{N}$, the measure π is supported on the sequences of ℓ_1^+ vanishing all the fluxes J_n and by Proposition B.1, the result follows. □

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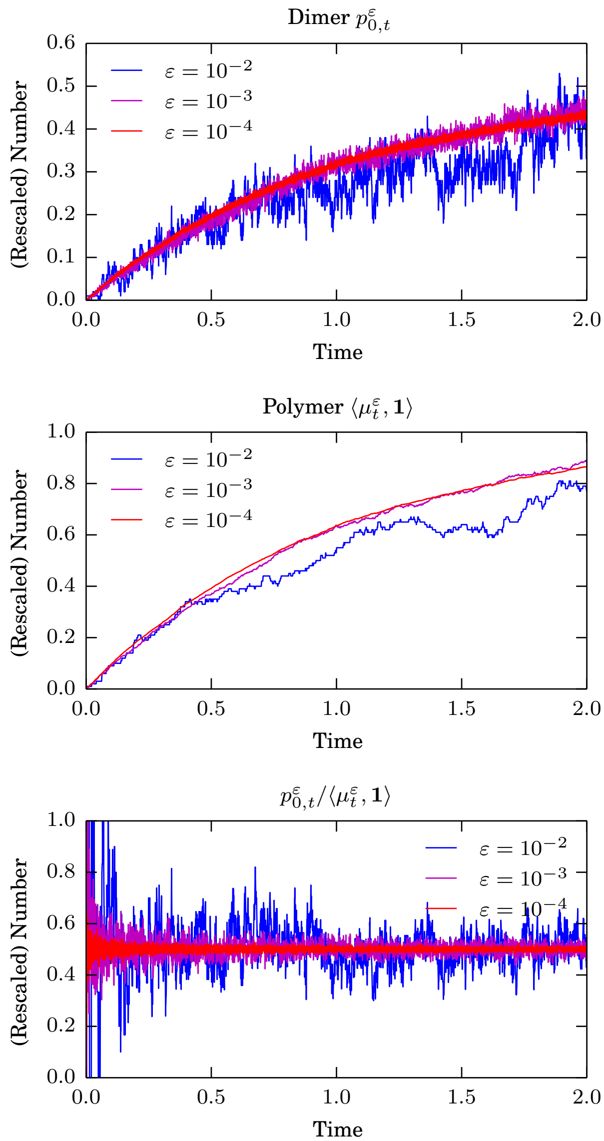

FIG 1. *Case Ia: constant rate coefficients a and b with outgoing characteristics (see Table 1). We plot the time evolution of the number of dimers* p_0^{ε} *(top), the total cluster number* $\langle \mu_t^{\varepsilon}, \mathbf{1} \rangle$ *(middle) and the ratio between both (down) for* $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$ *(see legend). The Dimer variable has faster variations than the total cluster number, and the ratio between both seems to fluctuate very quickly around a fixed value. Such a value (here* 0.5*) is compatible with the heuristic value* $(1 - \frac{ac_t}{b}$ *derived in Equation* (47)*. Note that in this simulation the particle variable stays constant.*

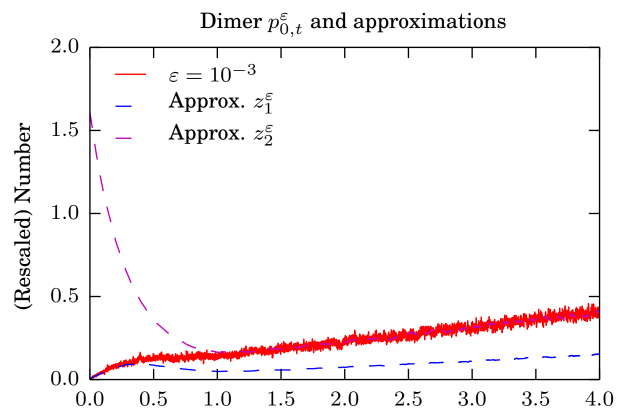

FIG 2. *Case Ib: constant rate coefficients a and b with outgoing characteristics (see Table 1). We plot the time evolution of the number of dimers p^{ε}_2 together with two different approximations (see legend). The approximations of the dimer variable are calculated according to the equilibrium value of the fast system (see Theorem 4 and Equation (47)). As explained in the main text, such an equilibrium value is not unique and depends on a normalization factor. We use for the first approximation $z^{\varepsilon}_1(t) = (1 - aC^{\varepsilon}_t/b)\sum_{i=0}^{20} p^{\varepsilon}_i(t)$, and for the second one $z^{\varepsilon}_2(t) = (1 - aC^{\varepsilon}_t/b)\sum_{i\geq 0} p^{\varepsilon}_i(t)$. We illustrate here how delicate can be the choice of the normalization factor. In small time, a correct choice seems the number of "small" clusters (the 20 first sizes here), while in large time it seems to be the total cluster number $\langle \mu^{\varepsilon}_t, \mathbf{1}\rangle$.*

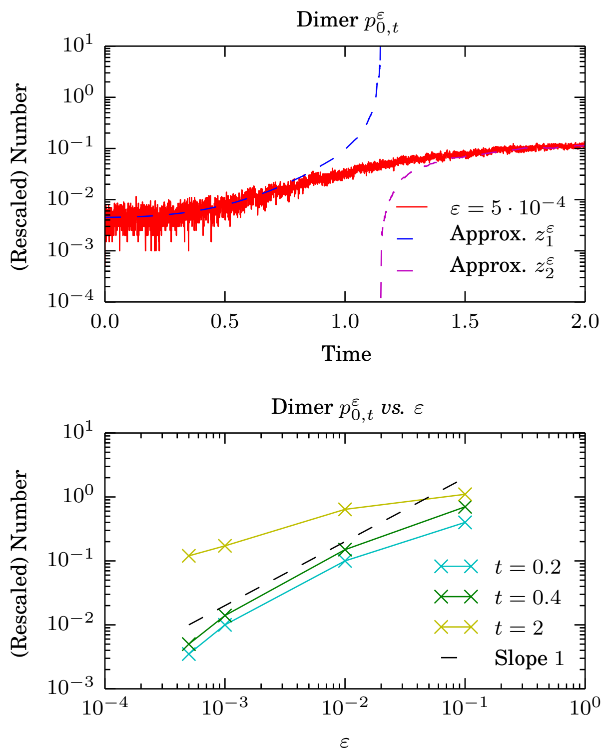

FIG 3. *Case Id: constant rate coefficients a and b with initially incoming characteristics (see Table 1). On top, we plot the time evolution of the number of dimers p_0^ε (colored plain transparent lines) together with numerical approximations (see legend). Down, we plot the number of dimer p_0^ε as a function of ε at different times (see legend). On top, the approximation of the dimer variable is calculated according to the equilibrium value of the fast subsystem (see Theorem 4). When $C_t^\varepsilon > \rho$, such an equilibrium value is predicted to be zero in the limit $\varepsilon \to 0$. We use here a (heuristically) second order approximation, by keeping track of the small nucleation rates in the fast subsystem, given by $z_1^\varepsilon = \varepsilon^\beta k(C_t^\varepsilon)/(aC_t^\varepsilon - b)$ (see Equations (48) and (49)). When $C_t^\varepsilon < \rho$, we use the equilibrium value of the fast subsystem (47) with the normalization factor given by the total cluster number, $z_2^\varepsilon(t) = (1 - aC_t^\varepsilon/b)\sum_{i\geq 0} p_i^\varepsilon(t)$. In log scale, both approximations diverge when C_t^ε crosses the threshold ρ (see also Figure 4). Down, we observe a clear convergence towards 0 of p_0^ε as $\varepsilon \to 0$ (at speed ε^β), as soon as $C_t^\varepsilon > \rho$ (i.e. at times $t = 0.2, 0.4$).*

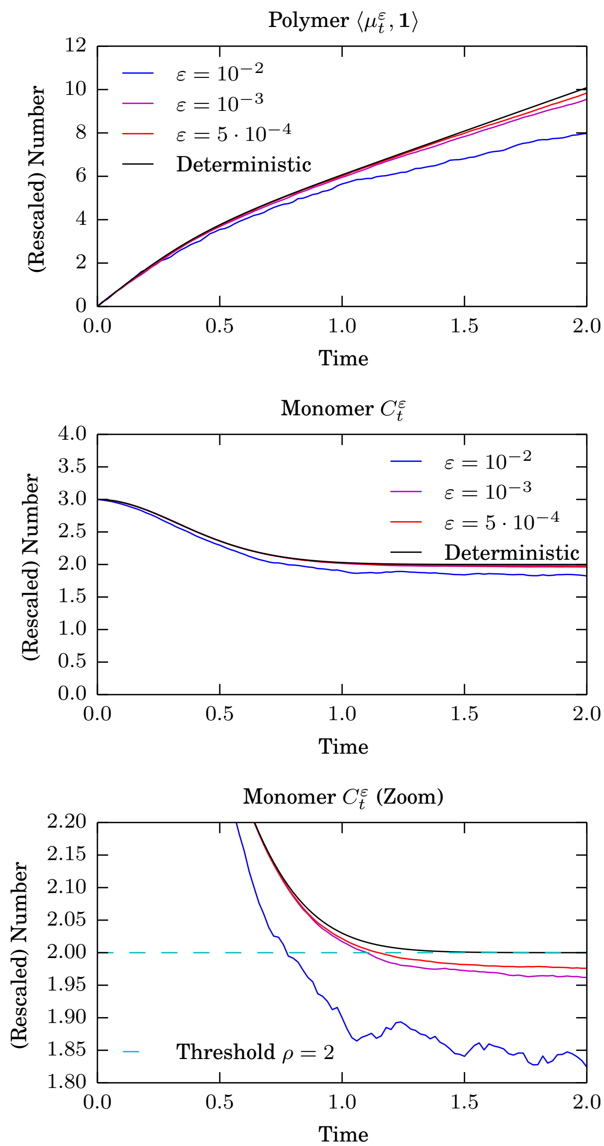

FIG 4. *Case Id) constant rate coefficients a and b with initially incoming characteristics (see Table 1). We plot the time evolution of the total number of clusters* $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$ *(top) and the number of particles* C_t^ε *(middle and down) for differents* ε *(see legend), together with the deterministic solution of the moment equations* (50) *(in black). The moment equations give a very good approximation of the stochastic solution. Down, we zoom around the value* $\rho = 2$ *and illustrate the fact that* C_t^ε *crosses the threshold* ρ *in finite time, while the deterministic limit* c_t *does not. We also observe that the deterministic approximation is slightly worst after the time* C_t^ε *crosses the threshold* ρ*.*

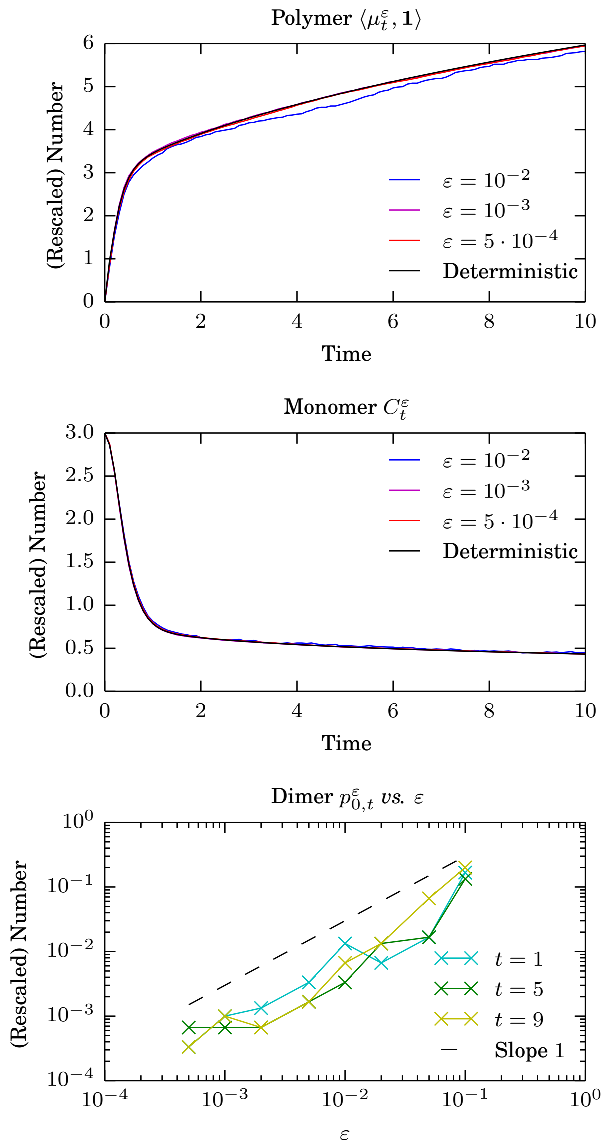

Fig 5. *Case III: rate coefficients with* $a(0) > 0$ *and* $b(0) = 0$*, so that the characteristics are always incoming (see Table 1). We plot the time evolution of the total number of clusters* $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$ *(top) and the number of particles* C_t^ε *(middle), for differents* ε *(see legend), together with the deterministic solution of the limit moment equations* (51) *(in black). Down, we plot the number of dimers* p_0^ε *as a function of* ε *at different times (see legend). We confirm the very good agreement of the deterministic limit solution given by the moment equations, and the fact that* p_0^ε *converges to* 0 *as* $\varepsilon \to 0$ *at speed* ε^β.

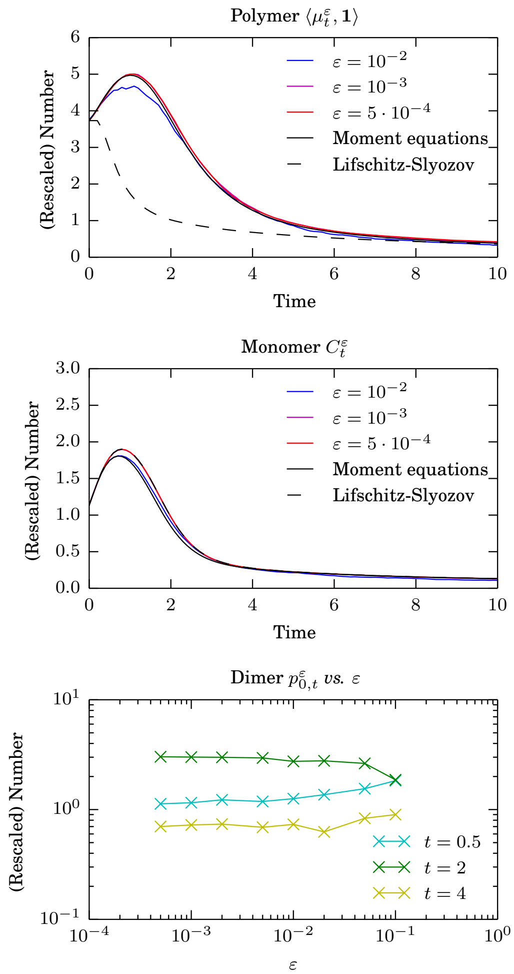

Fig 6. *Case IIb: rate coefficients with $a(0) = 0$ and $b(0) > 0$, so that the characteristics are always outgoing (see Table 1). We plot the time evolution of the total number of clusters $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$ (top) and the number of particles C_t^ε (middle), for differents ε (see legend), together with the deterministic solution of the limit moment equations (52) (in black, plain lines) and of the full Lifshitz-Slyozov equation obtained in Theorem 1 (in black, dashed lines). Down, we plot the number of dimers p_0^ε as a function of ε at different times (see legend). To solve numerically the moment equations (52), we use the boundary value p_0 obtained from p_0^ε in the stochastic simulations (with $\varepsilon = 0.0005$). We observe first that the boundary value p_0^ε seems to converge to a positive value as $\varepsilon \to 0$. Using that limit value, the moment equations faithfully reproduce the time evolution of the moments $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$, C_t^ε. The Lifshitz-Slyozov equation provides an even better fit for the time evolution of particles C_t^ε, but fails to reproduce the total number of clusters, as predicted by the vague convergence obtained in Theorem 1.*

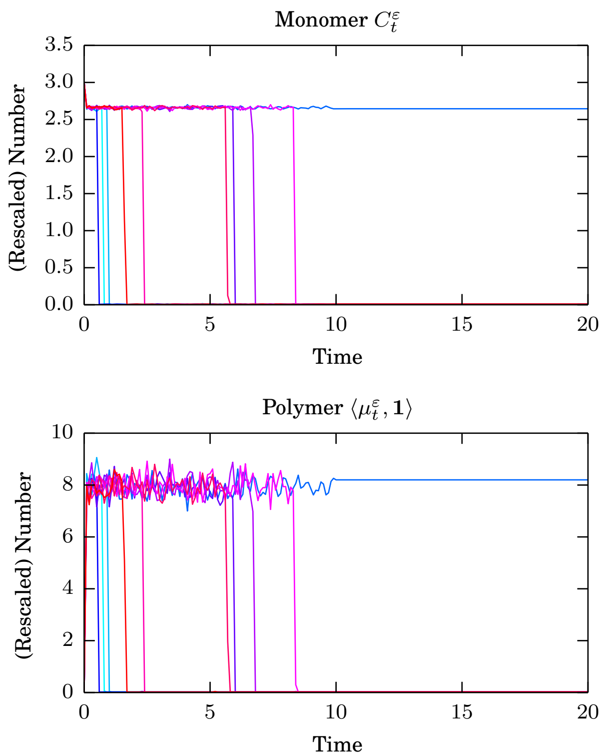

FIG 7. *Case IIa: rate coefficients with $a(0) = 0$ and $b(0) > 0$, so that the characteristics are always outgoing. Pure particleic initial condition with $c(0) = m$ (see Table 1). We plot the time evolution of the number of particles C_t^ε (top) and the total number of clusters $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$ (down) for ten independent trajectories with $\varepsilon = 2.10^{-2}$. We observe that the numerical solutions largely differ from one to each other, mostly by the time at which the number of particles (and clusters) goes down. This time corresponds to the time a cluster of size greater than the critical size X_c has been formed (see main text and video in supplementary materials).*

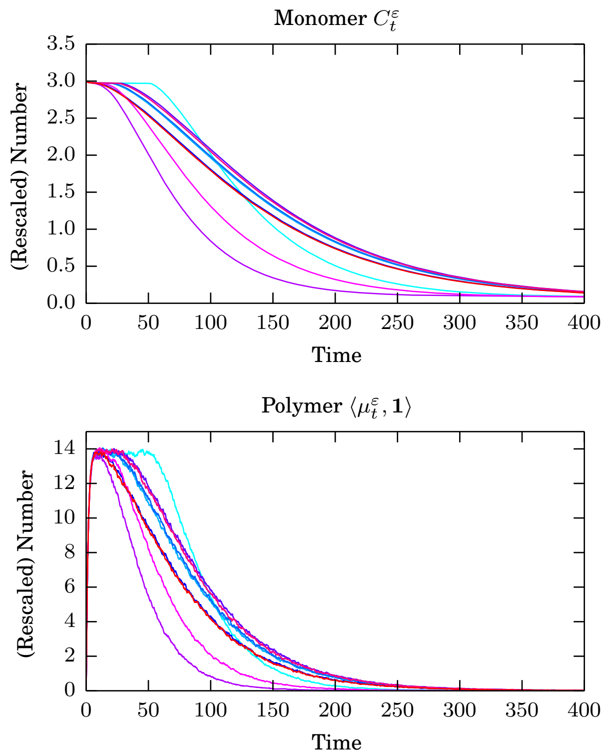

FIG 8. *Case IVa: rate coefficients with $a(0) = 0$ and $b(0) > 0$, so that the characteristics are always outgoing. Pure particleic initial condition with $c(0) = m$ (see Table 1). We plot the time evolution of the number of particles C_t^ε (top) and the total number of clusters $\langle \mu_t^\varepsilon, \mathbf{1} \rangle$ (down) for ten independent trajectories with $\varepsilon = 7.10^{-4}$. We observe that the numerical solutions largely differ from one to each other, mostly by the time at which the number of particles (and clusters) goes down. This time corresponds to the time a cluster of size greater than the critical size X_c has been formed. For three trajectories among the ten, the speed of decay of the particles (and the number of clusters) variable is significantly higher than the rest of the trajectories. This is due to the fact that in such case, two critical clusters have been formed (see main text and video in supplementary materials).*