An augmented mixed–primal finite element method for a coupled flow–transport problem*

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Abstract

In this paper we analyze the coupling of a scalar nonlinear convection-diffusion problem with the Stokes equations where the viscosity depends on the distribution of the solution to the tranport problem. An augmented variational approach for the fluid flow coupled with a primal formulation for the transport model is proposed. The resulting Galerkin scheme yields an augmented mixed–primal finite element method employing Raviart-Thomas spaces of order k for the Cauchy stress, and continuous piecewise polynomials of degree $\leq k+1$ for the velocity and also for the scalar field. The classical Schauder and Brouwer fixed point theorems are utilized to establish existence of solution of the continuous and discrete formulations, respectively. Then, sufficiently small data allow us to prove uniqueness and to derive optimal a priori error estimates. Finally, we report a few numerical tests confirming the predicted rates of convergence, and illustrating the performance of a linearized method based on Newton-Raphson iterations; and we apply the proposed framework in the simulation of thermal convection and sedimentation-consolidation processes.

Key words: Stokes equations, nonlinear transport problem, augmented mixed–primal formulation, fixed point theory, thermal convection, sedimentation-consolidation process, finite element methods, a priori error analysis.

Mathematics subject classifications (2000): 65N30, 65N12, 76R05, 76D07, 65N15.

1 Introduction

We are interested in studying mixed finite element approximations to simulate the transport of a species density in an immiscible fluid. Depending on the nature of the species, this problem can be relevant to a number of practical engineering applications including natural and thermal convection, aluminum production, chemical distillation processes, formation of fog, impedance tomography, motion of biomembranes, semiconductors, granular flows, and so on. In this paper we pay particular attention to the steady state regime in the phenomenon of sedimentation-consolidation of particles (see e.g. [3, 4, 23]),

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where the sought physical quantities in the model include the velocity of the flow and the local solids concentration. On the other hand, it is well known that other variables, such as the principal components of the fluid or solids stress tensors, are of great interest in this context (see e.g. [5, Chapter 3]). In a more general sense, the need of obtaining accurate approximations of additional fields has motivated the successful derivation of a wide range of formulations in the framework of Stokes and Navier-Stokes equations, including for instance, stress-velocity formulations (see [6, 10, 14, 18] and the references therein). They feature the clear advantage (with respect to classical velocity-pressure formulations) that these auxiliary quantities of interest are computed directly, without resorting to any kind of post-process of the velocity field by numerical differentiation, which may induce an important loss in accuracy. The attached difficulties are that the finite dimensional spaces involved in the resulting discrete formulation must be properly selected in order to satisfy the corresponding inf-sup condition [2], and that approximation of stresses may become quite expensive if adequate finite elements are not used.

Now, concerning the problem we are interested in here, we realize that, in order to be able to analyze the solvability of a mixed formulation for the fluid flow coupled with a primal method for the transport model, we require $\mathrm{H}^1(\Omega)$ smoothness for the components of both the fluid velocity and its discrete approximation. However, since the usual mixed approach is only able to guarantee that they live in $\mathrm{L}^2(\Omega)$, in this paper we follow [12] (see also [11], [14]) and propose an augmented mixed scheme in which the stress stays in its original space $\mathbb{H}(\mathbf{div};\Omega)$, but the velocity components lie now in the smaller space $\mathrm{H}^1(\Omega)$. The above means that we will approximate each row of the fluid Cauchy stress tensor with Raviart-Thomas elements of order k, whereas the velocity and scalar field (which will represent a solids concentration, or temperature, depending on the specific application) will be approximated with continuous piecewise polynomials of degree $\leq k+1$. The existence of solution of the continuous and associated Galerkin schemes is established by a combination of fixed point arguments with the well-know Lax-Milgram theorem and a classical result on bijective monotone operators. In addition, the assumption of sufficiently small data allows us to conclude uniqueness of solution and to derive optimal a priori error estimates.

Outline

We have organized the contents of this paper as follows. The remainder of this section introduces some standard notations and functional spaces. In Section 2 we first describe the boundary value problem of interest and then slightly simplify it by eliminating the pressure unknown in the fluid. Next, in Section 3, we introduce and analyze the continuous formulation, which is defined by an augmented mixed approach for the fluid flow coupled with a primal method for the transport equation. The necessity of augmentation is clearly justified, and the solvability analysis is based on a fixed point strategy that makes use of the Lax-Milgram and Schauder theorems together with a well-known result on monotone operators. We prove existence of solution and for sufficiently small data we derive uniqueness. The associated Galerkin scheme is introduced in Section 4 by employing Raviart-Thomas elements for the stress, and continuous piecewise polynomial approximations for the velocity and concentration. Here the solvability is established by applying now the Brouwer fixed point theorem and analogous arguments to those employed in Section 3. In Section 5 we assume again sufficiently small data and, applying a suitable Strang-type estimate for nonlinear problems, provide optimal a priori error estimates. Finally, in Section 6 we present numerical examples illustrating the good performance of the mixed-primal method and confirming the theoretical rates of convergence.

Preliminaries

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n = 2, 3$ a given bounded domain with polyhedral boundary $\Gamma = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}, \mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega}$$

is a Hilbert space. As usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. Also, for any vector field $\boldsymbol{v} = (v_i)_{i=1,n}$ we set

$$\nabla \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}\boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}.$$

2 The model problem

The following system of partial differential equations describes the stationary state of the transport of species ϕ in an immiscible fluid occupying the domain Ω:

$$\begin{aligned} \boldsymbol{\sigma} &= \mu(\phi)\,\nabla\boldsymbol{u} - p\,\mathbb{I}, \quad -\mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f}\phi, \quad \operatorname{div}\boldsymbol{u} = 0, \\ \tilde{\boldsymbol{\sigma}} &= \vartheta(|\nabla\phi|)\,\nabla\phi - \phi\boldsymbol{u} - \gamma(\phi)\boldsymbol{k}, \quad -\operatorname{div}\tilde{\boldsymbol{\sigma}} = g, \end{aligned} \tag{2.1}$$

where the sought quantities are the Cauchy fluid stress $\boldsymbol{\sigma}$, the local volume-average velocity of the fluid $\boldsymbol{u}$, the pressure p, and the local concentration of species ϕ. For sake of clarity in the presentation, we will restrict ourselves to a specific physical scenario corresponding to the process of sedimentation-consolidation of a mixture (many forms of that problem can be found in [4]). We assume that μ, ϑ, and γ are nonlinear scalar functions of ϕ (kinematic effective viscosity, diffusion term modeling e.g. sediment compressibility, and one dimensional flux describing hindered settling, respectively), and $\boldsymbol{k}$ is a vector pointing in the direction of gravity. In addition, ϑ is of class C^1 and we assume that there exist positive constants μ_1, μ_2, γ_1, γ_2, ϑ_1, and ϑ_2, such that

$$\mu_1 \le \mu(s) \le \mu_2 \quad \text{and} \quad \gamma_1 \le \gamma(s) \le \gamma_2 \qquad \forall\, s \in \mathrm{R}, \tag{2.2}$$

$$\vartheta_1 \le \vartheta(s) \le \vartheta_2 \quad \text{and} \quad \vartheta_1 \le \vartheta(s) + s\,\vartheta'(s) \le \vartheta_2 \qquad \forall\, s \ge 0. \tag{2.3}$$

Note that (2.2) and the first assumption in (2.3) guarantee, in particular, that the corresponding Nemytsky operators, say U for μ, defined by $U(\phi)(x) := \mu(\phi(x))$ $\forall\,\phi \in \mathrm{L}^2(\Omega)$, $\forall\, x \in \Omega$ a.e., and analogously for ϑ, γ, μ^{-1}, ϑ^{-1}, and γ^{-1}, are all well defined and continuous from $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2(\Omega)$. Furthermore, it is easy to show (see, e.g. [19, Theorem 3.8]) that the assumptions in (2.3) imply Lipschitz-continuity and strong monotonicity of the nonlinear operator induced by ϑ. We will go back to this fact later on in Section 3. The driving force of the mixture also depends on the local fluctuations of the concentration, so the right hand side of the second equation in (2.1) is linear with respect to ϕ, and $\boldsymbol{f} \in \mathbf{L}^\infty(\Omega)$ and $g \in \mathrm{L}^2(\Omega)$ are given functions. Finally, given $\boldsymbol{u}_\mathrm{D} \in \mathbf{H}^{1/2}(\Gamma_\mathrm{D})$, the following mixed boundary conditions complement (2.1):

$$\boldsymbol{u} = \boldsymbol{u}_\mathrm{D} \quad \text{on} \quad \Gamma_\mathrm{D}\,, \quad \boldsymbol{\sigma\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{N}\,, \quad \phi = 0 \quad \text{on} \quad \Gamma_\mathrm{D}\,, \quad \text{and} \quad \tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_\mathrm{N}\,. \tag{2.4}$$

On the other hand, it is easy to see that the first and third equations in (2.1) are equivalent to

$$\boldsymbol{\sigma} \,=\, \mu(\phi)\,\nabla\boldsymbol{u} \,-\, p\,\mathbb{I} \quad \text{and} \quad p + \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,,$$

which permits us to eliminate the pressure p from the first equation. Consequently, we arrive at the following coupled system:

$$\begin{gathered}
\frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^\mathrm{d} \,=\, \nabla\boldsymbol{u} \quad \text{in} \quad \Omega\,, \quad -\,\mathbf{div}\,\boldsymbol{\sigma} \,=\, \boldsymbol{f}\phi \quad \text{in} \quad \Omega\,,\\
\tilde{\boldsymbol{\sigma}} \,=\, \vartheta(|\nabla\phi|)\,\nabla\phi \,-\, \phi\boldsymbol{u} \,-\, \gamma(\phi)\boldsymbol{k} \quad \text{in} \quad \Omega\,, \quad -\,\mathrm{div}\,\tilde{\boldsymbol{\sigma}} \,=\, g \quad \text{in} \quad \Omega\,,\\
\boldsymbol{u} \,=\, \boldsymbol{u}_\mathrm{D} \quad \text{on} \quad \Gamma_\mathrm{D}\,, \quad \boldsymbol{\sigma\nu} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{N}\,,\\
\phi \,=\, 0 \quad \text{on} \quad \Gamma_\mathrm{D}\,, \quad \text{and} \quad \tilde{\boldsymbol{\sigma}}\cdot\boldsymbol{\nu} \,=\, 0 \quad \text{on} \quad \Gamma_\mathrm{N}\,.
\end{gathered} \tag{2.5}$$

We remark here that the incompressibility constraint is implicitly present in the first equation of (2.5), that is in the constitutive equation relating $\boldsymbol{\sigma}$ and $\boldsymbol{u}$.

3 The continuous formulation

3.1 The augmented mixed–primal formulation

We first observe that the homogeneous Neumann boundary condition for $\boldsymbol{\sigma}$ on Γ_N (cf. second relation of (2.4)) suggests to introduce the space

$$\mathbb{H}_N(\mathbf{div};\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_\mathrm{N}\Big\}\,.$$

Hence, multiplying the first equation of (2.5) by $\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega)$, integrating by parts, and using the Dirichlet boundary condition for $\boldsymbol{u}$ (cf. third row of (2.5)), we obtain

$$\int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^\mathrm{d} : \boldsymbol{\tau}^\mathrm{d} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\boldsymbol{\tau} = \langle\boldsymbol{\tau\nu}, \boldsymbol{u}_\mathrm{D}\rangle_{\Gamma_\mathrm{D}} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega)\,,$$

where $\langle\cdot,\cdot\rangle_{\Gamma_\mathrm{D}}$ is the duality pairing between $\mathbf{H}^{-1/2}(\Gamma_\mathrm{D})$ and $\mathbf{H}^{1/2}(\Gamma_\mathrm{D})$. In addition, the equilibrium equation is then rewritten as

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\boldsymbol{\sigma} \,=\, -\int_\Omega \boldsymbol{f}\phi\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega)\,.$$

On the other hand, the Dirichlet boundary condition for ϕ (cf. fourth row of (2.5)) motivates the introduction of the space

$$\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega) := \Big\{ \psi \in \mathrm{H}^1(\Omega) : \quad \psi = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}} \Big\},$$

for which, thanks to the generalized Poincaré inequality, there exists $c_p > 0$, depending only on Ω and Γ_{D}, such that

$$\|\psi\|_{1,\Omega} \le c_p\, |\psi|_{1,\Omega} \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,. \tag{3.1}$$

Therefore , given $\phi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$, we arrive at the following mixed formulation for the flow: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} a_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}) &= \langle \boldsymbol{\tau\nu}, \boldsymbol{u}_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega), \\ b(\boldsymbol{\sigma}, \boldsymbol{v}) &= -\int_\Omega \boldsymbol{f}\phi \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{L}^2(\Omega), \end{aligned} \tag{3.2}$$

where $a_\phi : \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbb{H}_N(\mathbf{div};\Omega) \to \mathbb{R}$ and $b : \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \to \mathbb{R}$ are bounded bilinear forms defined as

$$a_\phi(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \int_\Omega \frac{1}{\mu(\phi)}\, \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}, \qquad b(\boldsymbol{\tau}, \boldsymbol{v}) := \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\boldsymbol{\tau},$$

for $\boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega)$ and $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$.

In turn, given $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$, and using the homogeneous Neumann boundary condition for $\tilde{\boldsymbol{\sigma}}$ (cf. fourth row of (2.5)), we deduce that the primal formulation for the concentration equation becomes: Find $\phi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$A_{\boldsymbol{u}}(\phi, \psi) = \int_\Omega \gamma(\phi)\, \boldsymbol{k} \cdot \nabla\psi + \int_\Omega g\psi \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,, \tag{3.3}$$

where

$$A_{\boldsymbol{u}}(\phi, \psi) := \int_\Omega \vartheta(|\nabla\phi|)\, \nabla\phi \cdot \nabla\psi - \int_\Omega \phi\, \boldsymbol{u} \cdot \nabla\psi \qquad \forall\, \phi, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,. \tag{3.4}$$

At this point we observe that the assumption on μ given in (2.2) and the well known Babuška-Brezzi theory suffice to show that (3.2) is well-posed (see, e.g. [17, Theorem 2.1] for details). However, in order to deal with the analysis of (3.3), and particularly to estimate the second term defining $A_{\boldsymbol{u}}$, we would require $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$. In fact, we now recall from [22, eq. (2.20)] that Hölder's inequality and standard Sobolev embeddings estimates yield the existence of a positive constant $c(\Omega)$, depending only on Ω, such that

$$\left| \int_\Omega \varphi\, \boldsymbol{v} \cdot \nabla\psi \right| \le c(\Omega)\, \|\varphi\|_{1,\Omega}\, \|\boldsymbol{v}\|_{1,\Omega}\, |\psi|_{1,\Omega} \qquad \forall\, \varphi, \psi \in \mathrm{H}^1(\Omega)\,, \quad \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,. \tag{3.5}$$

Furthermore, while the exact solution of (3.2) actually satisfies $\nabla\boldsymbol{u} = \dfrac{1}{\mu(\phi)}\, \boldsymbol{\sigma}^{\mathrm{d}}$ in $\mathcal{D}'(\Omega)$, which implies that $\boldsymbol{u}$ does belong to $\mathbf{H}^1(\Omega)$, the foregoing distributional identity does not necessarily extend to the discrete setting of (3.2), and hence the aforementioned difficulty would appear again when trying to analyze the Galerkin scheme associated to (3.3). Therefore, in order to circumvent this inconvenience, we proceed similarly as in [12, Section 3] and incorporate into (3.2) the following redundant Galerkin terms

$$\begin{aligned}
\kappa_1 \int_\Omega \left(\nabla \boldsymbol{u} - \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^{\mathrm{d}}\right) : \nabla \boldsymbol{v} &= 0 && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,, \\
\kappa_2 \int_\Omega \mathbf{div}\boldsymbol{\sigma} \cdot \mathbf{div}\boldsymbol{\tau} &= -\kappa_2 \int_\Omega \boldsymbol{f}\phi \cdot \mathbf{div}\boldsymbol{\tau} && \forall\, \boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega)\,, \\
\kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}\cdot\boldsymbol{v} &= \kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v} && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,,
\end{aligned} \tag{3.6}$$

where $(\kappa_1, \kappa_2, \kappa_3)$ is a vector of positive parameters to be specified later. Notice that the first and third equations in (3.6) implicitly require the velocity $\boldsymbol{u}$ to live in $\mathbf{H}^1(\Omega)$. In this way, instead of (3.2), we consider from now on the following augmented mixed formulation: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ such that

$$B_\phi((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \,=\, F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)\,, \tag{3.7}$$

where

$$\begin{aligned}
B_\phi((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \,:=\, & a_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, b(\boldsymbol{\tau}, \boldsymbol{u}) \,-\, b(\boldsymbol{\sigma}, \boldsymbol{v}) \,+\, \kappa_1 \int_\Omega \left(\nabla \boldsymbol{u} \,-\, \frac{1}{\mu(\phi)}\,\boldsymbol{\sigma}^{\mathrm{d}}\right) : \nabla \boldsymbol{v} \\
& +\, \kappa_2 \int_\Omega \mathbf{div}\boldsymbol{\sigma} \cdot \mathbf{div}\boldsymbol{\tau} \,+\, \kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}\cdot\boldsymbol{v}\,,
\end{aligned} \tag{3.8}$$

and

$$F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \,:=\, \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} \,+\, \int_\Omega \boldsymbol{f}\phi \cdot \boldsymbol{v} \,-\, \kappa_2 \int_\Omega \boldsymbol{f}\phi \cdot \mathbf{div}\boldsymbol{\tau} \,+\, \kappa_3 \int_{\Gamma_{\mathrm{D}}} \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{v}\,. \tag{3.9}$$

We remark in advance that the well-posedness of (3.7) is proved below in Section 3.3. Moreover, since the unique solution of (3.2) is obviously a solution of (3.7) as well, we will conclude that both continuous problems share the same unique solution.

In this way, the augmented mixed-primal formulation of our original coupled problem (2.5) reduces to (3.7) and (3.3), that is: Find $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$\begin{aligned}
B_\phi((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \,&=\, F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)\,, \\
A_{\boldsymbol{u}}(\phi, \psi) \,&=\, \int_\Omega \gamma(\phi)\,\boldsymbol{k} \cdot \nabla\psi \,+\, \int_\Omega g\psi \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,.
\end{aligned} \tag{3.10}$$

3.2 A fixed point strategy

Having proposed the alternative formulation (3.7), whose continuous and discrete solutions have second components living in $\mathbf{H}^1(\Omega)$, we are able now to take a second look at (3.3). More precisely, given $\phi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ and the corresponding solution $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ of (3.7), we can set, instead of (3.3), the modified primal formulation: Find $\widetilde{\phi} \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$A_{\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) \,=\, G_\phi(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,, \tag{3.11}$$

where

$$G_\phi(\widetilde{\psi}) \,:=\, \int_\Omega \gamma(\phi)\,\boldsymbol{k} \cdot \nabla\widetilde{\psi} \,+\, \int_\Omega g\,\widetilde{\psi} \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,. \tag{3.12}$$

The well-posedness of this nonlinear problem will also be addressed in Section 3.3. Alternatively, one could also deal, instead of (3.11), with the linear problem: Find $\widetilde{\phi} \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ such that

$$A_{\phi,\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\psi}) \,=\, G_\phi(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,,$$

where

$$A_{\phi,\boldsymbol{u}}(\widetilde{\phi},\widetilde{\psi}) \,:=\, \int_\Omega \vartheta(|\nabla\phi|)\,\nabla\widetilde{\phi}\cdot\nabla\widetilde{\psi} \,-\, \int_\Omega \widetilde{\phi}\,\boldsymbol{u}\cdot\nabla\widetilde{\psi} \qquad \forall\,\widetilde{\phi},\,\widetilde{\psi}\in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,.$$

Nevertheless, and for easiness of the analysis, throughout the rest of the paper we stay with (3.11).

Hence, the description of problems (3.7) and (3.11) suggests a fixed point strategy to analyze (3.10). Indeed, let $\mathbf{S}:\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\longrightarrow \mathbb{H}_N(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ be the operator defined by

$$\mathbf{S}(\phi) \,=\, (\mathbf{S}_1(\phi),\mathbf{S}_2(\phi)) \,:=\, (\boldsymbol{\sigma},\boldsymbol{u}) \,\in\, \mathbb{H}_N(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega) \qquad \forall\,\phi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,,$$

where $(\boldsymbol{\sigma},\boldsymbol{u})$ is the unique solution of (3.7) with the given ϕ. In turn, let $\widetilde{\mathbf{S}}:\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\times\mathbf{H}^1(\Omega)\longrightarrow \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\phi,\boldsymbol{u}) \,:=\, \widetilde{\phi} \qquad \forall\,(\phi,\boldsymbol{u}) \,\in\, \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\times\mathbf{H}^1(\Omega)\,,$$

where $\widetilde{\phi}$ is the unique solution of (3.11) with the given $(\phi,\boldsymbol{u})$. Then, we define the operator $\mathbf{T}:\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\longrightarrow\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ by

$$\mathbf{T}(\phi) \,:=\, \widetilde{\mathbf{S}}(\phi,\mathbf{S}_2(\phi)) \qquad \forall\,\phi\,\in\,\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,,$$

and realize that solving (3.10) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $\phi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ such that

$$\mathbf{T}(\phi) \,=\, \phi\,. \tag{3.13}$$

3.3 Well-posedness of the uncoupled problems

In this section we show that the uncoupled problems (3.7) and (3.11) are in fact well-posed. We begin by recalling (see, e.g. [2]) that $\mathbb{H}(\mathbf{div};\Omega) \,=\, \mathbb{H}_0(\mathbf{div};\Omega)\oplus\mathbb{R}\,\mathbb{I}$, where

$$\mathbb{H}_0(\mathbf{div};\Omega) \,:=\, \left\{\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div};\Omega): \quad \int_\Omega \mathrm{tr}(\boldsymbol{\zeta})=0\right\}\,.$$

More precisely, for each $\boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div};\Omega)$ there exist unique $\boldsymbol{\zeta}_0 := \boldsymbol{\zeta} - \left\{\dfrac{1}{n|\Omega|}\displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\right\}\mathbb{I}\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $d \,:=\, \dfrac{1}{n|\Omega|}\displaystyle\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\in\mathbb{R}$, such that $\boldsymbol{\zeta} \,=\, \boldsymbol{\zeta}_0 + d\mathbb{I}$. The following three lemmas from [2], [14], and [12], which concern the above decomposition and an equivalence of norm, will be employed to show the well-posedness of (3.7) for a given ϕ.

Lemma 3.1 *There exists $c_1 = c_1(\Omega) > 0$ such that*

$$c_1\,\|\boldsymbol{\tau}_0\|^2_{0,\Omega} \leq \|\boldsymbol{\tau}^\mathrm{d}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega} \qquad \forall\,\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I}\in\mathbb{H}(\mathbf{div};\Omega)\,,$$

with $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $c\in\mathbb{R}$.

Proof. See [2, Proposition 3.1]. □

Lemma 3.2 *There exists $c_2 = c_2(\Omega,\Gamma_\mathrm{N}) > 0$ such that*

$$c_2\,\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} \leq \|\boldsymbol{\tau}_0\|^2_{\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I}\in\mathbb{H}_N(\mathbf{div};\Omega),$$

with $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $c\in\mathbb{R}$.

Proof. See [14, Lemma 2.2]. □

Lemma 3.3 *There exists $c_3 = c_3(\Omega, \Gamma_D) > 0$ such that*

$$|\boldsymbol{v}|_{1,\Omega}^2 + \|\boldsymbol{v}\|_{0,\Gamma_D}^2 \geq c_3 \|\boldsymbol{v}\|_{1,\Omega}^2 \qquad \forall \boldsymbol{v} \in \mathbf{H}^1(\Omega).$$

Proof. It corresponds to a slight modification of the proof of [12, Lemma 3.3]. □

Furthermore, for sake of the subsequent analysis we will also require some Lipschitz continuity-type assumptions for γ and μ. More precisely, we assume that there exist positive constants L_γ and L_μ such that

$$|\gamma(s) - \gamma(t)| \leq L_\gamma |s - t| \qquad \forall s, t \in \mathrm{R}, \tag{3.14}$$

and

$$\sup_{x\in\Omega} |\mu(\phi(x)) - \mu(\psi(x))| \leq L_\mu \|\phi - \psi\|_{0,\Omega} \qquad \forall \phi, \psi \in \mathrm{H}^1_{\Gamma_D}(\Omega). \tag{3.15}$$

Note that while (3.15) could seem a restrictive assumption, it actually becomes a reasonable requirement if μ is also supposed to satisfy the analogue of (3.14), and then it is redefined by piecewise mean values. More precisely, assume now that there exists a positive constant c_μ such that

$$|\mu(s) - \mu(t)| \leq c_\mu |s - t| \qquad \forall s, t \in \mathrm{R},$$

and let $\Big\{\Omega_1, \Omega_2, \dots, \Omega_N\Big\}$ be a fixed partition of Ω. Then, redefining

$$\widetilde{\mu}(\phi(x)) := \frac{1}{|\Omega_j|} \int_{\Omega_j} \mu(\phi(z))\, dz \qquad \forall x \in \Omega_j,$$

we obtain that for some $i \in \{1, ..., N\}$ there holds

$$\begin{aligned} \sup_{x\in\Omega} |\widetilde{\mu}(\phi(x)) - \widetilde{\mu}(\psi(x))| &= \max_{j\in\{1,\dots,N\}} \left| \frac{1}{|\Omega_j|} \int_{\Omega_j} \Big\{\mu(\phi(z)) - \mu(\psi(z))\Big\}\, dz \right| \\ = \left| \frac{1}{|\Omega_i|} \int_{\Omega_i} \Big\{\mu(\phi(z)) - \mu(\psi(z))\Big\}\, dz \right| &\leq \frac{c_\mu}{|\Omega_i|^{1/2}} \|\phi - \psi\|_{0,\Omega_i} \leq \frac{c_\mu}{|\Omega_i|^{1/2}} \|\phi - \psi\|_{0,\Omega}, \end{aligned}$$

which shows that $\widetilde{\mu}$ satisfies (3.15) with $L_\mu := c_\mu \max_{j\in\{1,\dots,N\}} |\Omega_j|^{-1/2}$.

We now begin the solvability analysis of the uncoupled problems with the following result.

Lemma 3.4 *Assume that $\kappa_1 \in \left(0, \frac{2\delta\mu_1}{\mu_2}\right)$ with $\delta \in (0, 2\mu_1)$, and that $0 < \kappa_2, \kappa_3$. Then, for each $\phi \in \mathrm{H}^1_{\Gamma_D}(\Omega)$ the problem (3.7) has a unique solution $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \boldsymbol{u}) \in H := \mathbb{H}_N(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$. Moreover, there exists $C_S > 0$, independent of ϕ, such that*

$$\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, \boldsymbol{u})\|_H \leq C_S \Big\{ \|\boldsymbol{u}_D\|_{1/2,\Gamma_D} + \|\boldsymbol{f}\|_{\infty,\Omega} \|\phi\|_{1,\Omega} \Big\} \qquad \forall \phi \in \mathrm{H}^1_{\Gamma_D}(\Omega). \tag{3.16}$$

Proof. We first observe from (3.8) that, given $\phi \in \mathrm{H}^1_{\Gamma_D}(\Omega)$, B_ϕ is clearly a bilinear form. Next, applying the Cauchy-Schwarz inequality, the lower bound for μ (cf. (2.2)), and the trace theorem (with constant c_0), we also obtain from (3.8) that

$$\begin{aligned} |B_\phi((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v}))| &\leq \frac{1}{\mu_1} \|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \|\boldsymbol{u}\|_{0,\Omega} \|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega} \|\mathbf{div}\boldsymbol{\sigma}\|_{0,\Omega} \\ + \quad \kappa_1 |\boldsymbol{u}|_{1,\Omega} |\boldsymbol{v}|_{1,\Omega} &+ \frac{\kappa_1}{\mu_1} \|\boldsymbol{\sigma}^{\mathrm{d}}\|_{0,\Omega} |\boldsymbol{v}|_{1,\Omega} + \kappa_2 \|\mathbf{div}\boldsymbol{\sigma}\|_{0,\Omega} \|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega} + c_0^2 \kappa_3 \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega}. \end{aligned}$$

It follows that there exists a positive constant, denoted $\|B\|$ and depending on μ_1, κ_1, κ_2, κ_3, and c_0, such that

$$|B_\phi((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v}))| \,\le\, \|B\|\,\|(\boldsymbol{\sigma},\boldsymbol{u})\|_H\,\|(\boldsymbol{\tau},\boldsymbol{v})\|_H \qquad \forall\,(\boldsymbol{\sigma},\boldsymbol{u}),\,(\boldsymbol{\tau},\boldsymbol{v})\,\in\,H\,, \tag{3.17}$$

and hence B_ϕ is bounded independently of $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$.

In turn, we now aim to show that B_ϕ is H-elliptic. In fact, given $(\boldsymbol{\tau},\boldsymbol{v}) \in H$, we have again from (3.8) that

$$B_\phi((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\tau},\boldsymbol{v})) = \int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\tau}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \kappa_1\,|\boldsymbol{v}|^2_{1,\Omega} - \kappa_1 \int_\Omega \frac{1}{\mu(\phi)}\,\boldsymbol{\tau}^{\mathrm{d}} : \nabla\boldsymbol{v} + \kappa_2\|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\Omega} + \kappa_3\,\|\boldsymbol{v}\|^2_{0,\Gamma_\mathrm{D}}\,,$$

which, using the bounds for μ (cf. (2.2)), the Young inequality, and Lemmas 3.1, 3.2, and 3.3, and taking δ, κ_1, κ_2, and κ_3 as stated in the hypotheses, yields

$$\begin{aligned}
B_\phi((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\tau},\boldsymbol{v})) \,&\ge\, \left(\frac{1}{\mu_2} - \frac{\kappa_1}{2\delta\mu_1}\right)\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} \,+\, \kappa_2\|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\Omega} \,+\, \kappa_1\left(1-\frac{\delta}{2\mu_1}\right)|\boldsymbol{v}|^2_{1,\Omega} \,+\, \kappa_3\,\|\boldsymbol{v}\|^2_{0,\Gamma_\mathrm{D}}\\
&\ge\, c_1\alpha_1\,\|\boldsymbol{\tau}_0\|^2_{0,\Omega} \,+\, \frac{\kappa_2}{2}\,\|\mathbf{div}\boldsymbol{\tau}\|^2_{0,\Omega} \,+\, \kappa_1\left(1-\frac{\delta}{2\mu_1}\right)|\boldsymbol{v}|^2_{1,\Omega} \,+\, \kappa_3\,\|\boldsymbol{v}\|^2_{0,\Gamma_\mathrm{D}}\\
&\ge\, \alpha_2\,\|\boldsymbol{\tau}_0\|^2_{\mathbf{div};\Omega} \,+\, \alpha_3\left\{|\boldsymbol{v}|^2_{1,\Omega} \,+\, \|\boldsymbol{v}\|^2_{0,\Gamma_\mathrm{D}}\right\}\\
&\ge\, c_2\alpha_2\,\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} \,+\, c_3\alpha_3\,\|\boldsymbol{v}\|^2_{1,\Omega}\,,
\end{aligned}$$

where $\alpha_1 := \min\left\{\left(\frac{1}{\mu_2} - \frac{\kappa_1}{2\delta\mu_1}\right), \frac{\kappa_2}{2}\right\}$, $\alpha_2 := \min\left\{c_1\alpha_1, \frac{\kappa_2}{2}\right\}$, and $\alpha_3 := \min\left\{\kappa_1\left(1-\frac{\delta}{2\mu_1}\right), \kappa_3\right\}$. In this way, defining $\alpha := \min\{c_2\alpha_2, c_3\alpha_3\}$, which depends on μ_1, μ_2, δ, κ_1, κ_2, κ_3, c_1, c_2, and c_3, we conclude that

$$B_\phi((\boldsymbol{\tau},\boldsymbol{v}),(\boldsymbol{\tau},\boldsymbol{v})) \,\ge\, \alpha\,\|(\boldsymbol{\tau},\boldsymbol{v})\|^2_H \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v})\,\in\,H\,, \tag{3.18}$$

thus confirming the H-ellipticity of B_ϕ independently of $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ as well. In particular, choosing the feasible values $\delta = \mu_1$ and $\kappa_1 = \frac{\mu_1^2}{\mu_2}$, and then taking $\kappa_2 = 2\left(\frac{1}{\mu_2} - \frac{\kappa_1}{2\delta\mu_1}\right)$ and $\kappa_3 = \kappa_1\left(1-\frac{\delta}{2\mu_1}\right)$, we find that $\kappa_2 = \frac{1}{\mu_2}$, $\kappa_3 = \frac{\mu_1^2}{2\mu_2}$, and $\alpha = \frac{1}{2\mu_2}\min\{c_1c_2, c_2, c_3\mu_1^2\}$.

Next, given $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, we look at the functional F_ϕ (cf. (3.9)), which is certainly linear. Then, using the Cauchy-Schwarz inequality and the trace estimates in $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{H}^1(\Omega)$, with constants 1 and c_0, respectively, we deduce that for each $(\boldsymbol{\tau},\boldsymbol{v}) \in H$ there holds

$$\begin{aligned}
|F_\phi(\boldsymbol{\tau},\boldsymbol{v})| \,\le\, \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi\|_{0,\Omega}\left\{\|\boldsymbol{v}\|_{0,\Omega} + \kappa_2\,\|\mathbf{div}\boldsymbol{\tau}\|_{0,\Omega}\right\}\\
+\quad c_0\,\kappa_3\,\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}\,\|\boldsymbol{v}\|_{1,\Omega}\,,
\end{aligned}$$

which provides the existence of a positive constant, denoted $\|F\|$ and depending on κ_2, κ_3, and c_0, such that

$$|F_\phi(\boldsymbol{\tau},\boldsymbol{v})| \,\le\, \|F\|\left\{\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi\|_{0,\Omega}\right\}\|(\boldsymbol{\tau},\boldsymbol{v})\| \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in H\,. \tag{3.19}$$

The foregoing inequality shows the boundedness of F_ϕ with

$$\|F_\phi\| \,\le\, \|F\|\left\{\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi\|_{0,\Omega}\right\}. \tag{3.20}$$

Finally, a straightforward application of the Lax-Milgram Lemma (see, e.g. [15, Theorem 1.1]), proves that, for each $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, problem (3.7) has a unique solution $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \boldsymbol{u}) \in H$. Moreover, the corresponding continuous dependence result together with the estimates (3.18) and (3.19) give

$$\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, \boldsymbol{u})\|_H \,\leq\, \frac{1}{\alpha}\,\|F_\phi\|_{H'} \,\leq\, C_S \left\{ \|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi\|_{0,\Omega} \right\},$$

with $C_S := \frac{\|F\|}{\alpha}$, thus completing the proof. □

We now establish the unique solvability of the nonlinear problem (3.11).

Lemma 3.5 *Let $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ such that $\|\boldsymbol{u}\|_{1,\Omega} < \frac{\vartheta_1}{c_p\, c(\Omega)}$ (cf. (2.3), (3.1), (3.5)). Then, there exists a unique $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi, \boldsymbol{u}) \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ solution of (3.11), and there holds*

$$\|\widetilde{\mathbf{S}}(\phi, \boldsymbol{u})\|_{1,\Omega} \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\leq\, \frac{c_p^2}{(\vartheta_1 - c_p c(\Omega)\|\boldsymbol{u}\|_{1,\Omega})} \left\{ \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \right\}. \tag{3.21}$$

Proof. We begin by recalling from [19, Theorem 3.8] that the nonlinear operator induced by the first term defining $A_{\boldsymbol{u}}$ (cf. (3.4)) is strongly monotone and Lipschitz-continuous with constants ϑ_1 and $\widetilde{\vartheta}_2 := \max\left\{\vartheta_2, 2\vartheta_2 - \vartheta_1\right\}$ (cf. (2.3)), respectively. It follows, using also the Cauchy-Schwarz inequality, (3.5), and (3.1), that for all $\widetilde{\varphi}$, $\widetilde{\psi} \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ there holds

$$\begin{aligned}
& A_{\boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\varphi} - \widetilde{\psi}) \,-\, A_{\boldsymbol{u}}(\widetilde{\psi}, \widetilde{\varphi} - \widetilde{\psi}) \\
&\qquad = \quad \int_\Omega \left\{ \vartheta(|\nabla\widetilde{\varphi}|)\nabla\widetilde{\varphi} - \vartheta(|\nabla\widetilde{\psi}|)\nabla\widetilde{\psi} \right\} \cdot \nabla(\widetilde{\varphi} - \widetilde{\psi}) - \int_\Omega (\widetilde{\varphi} - \widetilde{\psi})\boldsymbol{u} \cdot \nabla(\widetilde{\varphi} - \widetilde{\psi}) \\
&\qquad \geq \quad \vartheta_1\,|\widetilde{\varphi} - \widetilde{\psi}|^2_{1,\Omega} \,-\, c(\Omega)\,\|\widetilde{\varphi} - \widetilde{\psi}\|_{1,\Omega}\,\|\boldsymbol{u}\|_{1,\Omega}\,|\widetilde{\varphi} - \widetilde{\psi}|_{1,\Omega} \\
&\qquad \geq \quad \left\{ \vartheta_1 \,-\, c_p\,c(\Omega)\,\|\boldsymbol{u}\|_{1,\Omega} \right\} |\widetilde{\varphi} - \widetilde{\psi}|^2_{1,\Omega} \\
&\qquad \geq \quad c_p^{-2} \left\{ \vartheta_1 \,-\, c_p\,c(\Omega)\,\|\boldsymbol{u}\|_{1,\Omega} \right\} \|\widetilde{\varphi} - \widetilde{\psi}\|^2_{1,\Omega},
\end{aligned}$$

which shows that $A_{\boldsymbol{u}}$ is strongly monotone with constant $\widetilde{\alpha}_{\boldsymbol{u}} := c_p^{-2}\left\{\vartheta_1 \,-\, c_p\,c(\Omega)\,\|\boldsymbol{u}\|_{1,\Omega}\right\}$. In turn, proceeding similarly, we find that for all $\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{\rho} \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ there holds

$$\begin{aligned}
\left|A_{\boldsymbol{u}}(\widetilde{\varphi}, \widetilde{\rho}) \,-\, A_{\boldsymbol{u}}(\widetilde{\psi}, \widetilde{\rho})\right| \,&=\, \left| \int_\Omega \left\{ \vartheta(|\nabla\widetilde{\varphi}|)\nabla\widetilde{\varphi} - \vartheta(|\nabla\widetilde{\psi}|)\nabla\widetilde{\psi} \right\} \cdot \nabla\widetilde{\rho} \,-\, \int_\Omega (\widetilde{\varphi} - \widetilde{\psi})\boldsymbol{u} \cdot \nabla\widetilde{\rho} \right| \\
&\leq \quad \widetilde{\vartheta}_2\,|\widetilde{\varphi} - \widetilde{\psi}|_{1,\Omega}\,|\widetilde{\rho}|_{1,\Omega} \,+\, c(\Omega)\,\|\widetilde{\varphi} - \widetilde{\psi}\|_{1,\Omega}\,\|\boldsymbol{u}\|_{1,\Omega}\,|\widetilde{\rho}|_{1,\Omega} \\
&\leq \quad \left\{ \widetilde{\vartheta}_2 \,+\, c(\Omega)\,\|\boldsymbol{u}\|_{1,\Omega} \right\} \|\widetilde{\varphi} - \widetilde{\psi}\|_{1,\Omega}\,\|\widetilde{\rho}\|_{1,\Omega},
\end{aligned}$$

which proves that $A_{\boldsymbol{u}}$ is Lipschitz-continuous with constant $\widetilde{L}_{\boldsymbol{u}} := \widetilde{\vartheta}_2 \,+\, c(\Omega)\,\|\boldsymbol{u}\|_{1,\Omega}$. Therefore, a direct application of a classical result on the bijectivity of monotone operators (see, e.g. [21, Theorem 3.3.23]) implies the existence of a unique solution $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi, \boldsymbol{u}) \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ of (3.11). Moreover, applying the strong monotonicity of $A_{\boldsymbol{u}}$ to $\widetilde{\varphi} = \widetilde{\phi}$ and $\widetilde{\psi} = 0$, and noting from (3.4) that $A_{\boldsymbol{u}}(0, \cdot) = 0$, we deduce that

$$\widetilde{\alpha}_{\boldsymbol{u}}\,\|\widetilde{\phi}\|^2_{1,\Omega} \,\leq\, A_{\boldsymbol{u}}(\widetilde{\phi}, \widetilde{\phi}) \,=\, G_\phi(\widetilde{\phi}),$$

which gives $\widetilde{\alpha}_{\boldsymbol{u}}\,\|\widetilde{\phi}\|_{1,\Omega} \,\leq\, \|G_\phi\|$. Finally, using the Cauchy-Schwarz inequality and the upper bound of γ (cf. (2.2)), it follows from (3.12) that $\|G_\phi\| \,\leq\, \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}$, which yields (3.21) and finishes the proof. □

A simple corollary of the above lemma, which removes the dependence on $\boldsymbol{u}$ of the strong monotonicity constant of $A_{\boldsymbol{u}}$ and of the estimate (3.21), is given as follows.

Lemma 3.6 *Let $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ and $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ such that $\|\boldsymbol{u}\|_{1,\Omega} < \frac{\vartheta_1}{2\,c_p\,c(\Omega)}$ (cf. (2.3), (3.1), (3.5)). Then, there exists a unique $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi,\boldsymbol{u}) \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ solution of (3.11), and there holds*

$$\|\widetilde{\mathbf{S}}(\phi,\boldsymbol{u})\|_{1,\Omega} \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\le\, \frac{2\,c_p^2}{\vartheta_1}\left\{\gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}\right\}. \tag{3.22}$$

Proof. It follows directly from the proof of Lemma 3.5. Note in particular that the strong monotonicity of $A_{\boldsymbol{u}}$ holds with the constant $\widetilde{\alpha} := \frac{\vartheta_1}{2\,c_p^2}$. Further details are omitted. □

We end this section by remarking that the restriction on $\|\boldsymbol{u}\|_{1,\Omega}$ in Lemma 3.6 could also have been taken as $\|\boldsymbol{u}\|_{1,\Omega} < \delta\frac{\vartheta_1}{c_p\,c(\Omega)}$ with any $\delta \in (0,1)$. However, we have chosen $\delta = \frac{1}{2}$ for simplicity and because it yields a joint maximization of the constant $\widetilde{\alpha}$ and the upper bound for $\|\boldsymbol{u}\|_{1,\Omega}$.

3.4 Solvability analysis of the fixed point equation

Having established in the previous section the well-posedness of the uncoupled problems (3.7) and (3.11), which confirms that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$, and $\mathbf{T}$ (cf. Section 3.2) are well defined, we now address the solvability analysis of the fixed point equation (3.13). For this purpose, in what follows we verify the hypotheses of the Schauder fixed point theorem, which is stated as follows (see, e.g. [8, Theorem 9.12-1(b)]).

Theorem 3.7 *Let W be a closed and convex subset of a Banach space X and let $T : W \to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

We begin the analysis with the following result.

Lemma 3.8 *Given $r > 0$, we let W be the closed and convex subset of $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ defined by*

$$W \,:=\, \left\{\phi \,\in\, \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega): \quad \|\phi\|_{1,\Omega} \,\le\, r\right\},$$

and assume that the data satisfy

$$\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} < \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \text{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\le\, \frac{\vartheta_1\,r}{2\,c_p^2}. \tag{3.23}$$

Then $\mathbf{T}(W) \subseteq W$.

Proof. Given $\phi \in W$, we get from (3.16) (cf. Lemma 3.4) that

$$\|\mathbf{S}(\phi)\|_H \,=\, \|(\boldsymbol{\sigma},\boldsymbol{u})\|_H \,\le\, C_S\left\{\,\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega}\right\},$$

and hence, thanks to the first restriction in (3.23), we observe that $\boldsymbol{u} = \mathbf{S}_2(\phi)$ satisfies the hypotheses of Lemma 3.6. Moreover, the corresponding estimate (3.22) gives

$$\|\mathbf{T}(\phi)\|_{1,\Omega} \,=\, \|\widetilde{\mathbf{S}}(\phi,\boldsymbol{u})\|_{1,\Omega} \,\le\, \frac{2\,c_p^2}{\vartheta_1}\left\{\gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}\right\},$$

which, due to the second inequality in (3.23), proves that $\mathbf{T}(\phi) \in W$, thus finishing the proof. □

Next, we aim to prove the continuity and compactness properties of $\mathbf{T}$, which basically will be direct consequences of the following two lemmas providing the continuity of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, respectively.

Lemma 3.9 *There exists a positive constant C, depending on μ_1, κ_1, κ_2, L_μ, and α (cf.* (2.2), (3.6), (3.15), (3.18)*), such that*

$$\|\mathbf{S}(\phi)-\mathbf{S}(\psi)\|_H \,\le\, C\Big\{\|\boldsymbol{f}\|_{\infty,\Omega} + \|\mathbf{S}_1(\psi)\|_{0,\Omega}\Big\}\|\phi-\psi\|_{0,\Omega} \quad \forall\,\phi,\,\psi\,\in\,\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,. \tag{3.24}$$

Proof. Given ϕ, $\psi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, we let $(\boldsymbol{\sigma},\boldsymbol{u})=\mathbf{S}(\phi)$ and $(\boldsymbol{\zeta},\boldsymbol{w})=\mathbf{S}(\psi)$, that is

$$B_\phi((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) \,=\, F_\phi(\boldsymbol{\tau},\boldsymbol{v}) \quad\text{and}\quad B_\psi((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\tau},\boldsymbol{v})) \,=\, F_\psi(\boldsymbol{\tau},\boldsymbol{v}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v})\in H\,.$$

It follows, using the ellipticity of B_ϕ (cf. (3.18)) and then subtracting and adding the expression $F_\psi((\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w}))=B_\psi((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w}))$, that

$$\begin{aligned}\alpha\,\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\|_H^2 \,&\le\, B_\phi((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})) \,-\, B_\phi((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w}))\\ &=\,(F_\phi-F_\psi)\big((\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\big) \,+\, (B_\psi-B_\phi)\big((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\big)\,.\end{aligned} \tag{3.25}$$

Then, according to the definition of F_ϕ (cf. (3.9)), and applying the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\Big|(F_\phi-F_\psi)\big((\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\big)\Big| \,&=\, \Big|\int_\Omega \boldsymbol{f}(\phi-\psi)\cdot(\boldsymbol{u}-\boldsymbol{w}) \,-\, \kappa_2\int_\Omega \boldsymbol{f}(\phi-\psi)\cdot\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\zeta})\Big|\\ &\le\; \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi-\psi\|_{0,\Omega}\Big\{\|\boldsymbol{u}-\boldsymbol{w}\|_{0,\Omega} \,+\, \kappa_2\,\|\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\zeta})\|_{0,\Omega}\Big\}\\ &\le\; \big(1+\kappa_2^2\big)^{1/2}\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi-\psi\|_{0,\Omega}\,\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\|_H\,.\end{aligned} \tag{3.26}$$

In turn, it follows easily from (3.8) that

$$(B_\psi-B_\phi)\big((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\big) \,=\, \int_\Omega\left\{\frac{\mu(\phi)-\mu(\psi)}{\mu(\phi)\,\mu(\psi)}\right\}\boldsymbol{\zeta}^\mathtt{d} : \Big\{(\boldsymbol{\sigma}-\boldsymbol{\zeta})^\mathtt{d} - \kappa_1\,\nabla(\boldsymbol{u}-\boldsymbol{w})\Big\}\,,$$

which, thanks to the lower bound of μ (cf. (2.2)) and its Lipschitz-continuity type assumption (3.15), yields

$$\Big|(B_\psi-B_\phi)\big((\boldsymbol{\zeta},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\big)\Big| \,\le\, \frac{L_\mu\,\big(1+\kappa_1^2\big)^{1/2}}{\mu_1^2}\,\|\phi-\psi\|_{0,\Omega}\,\|\boldsymbol{\zeta}\|_{0,\Omega}\,\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\zeta},\boldsymbol{w})\|_H\,. \tag{3.27}$$

In this way, inequalities (3.25), (3.26), and (3.27) imply (3.24) and complete the proof. □

Lemma 3.10 *Let $\widetilde{\alpha} := \frac{\vartheta_1}{2\,c_p^2}$ be the strong monotonicity constant provided in the proof of Lemma* 3.6*. Then, there exists a positive constant $\widetilde{C}$, depending on $\widetilde{\alpha}$, $c(\Omega)$, and L_γ (cf.* (3.5), (3.14)*), such that for all $(\phi,\boldsymbol{u})$, $(\varphi,\boldsymbol{w})\,\in\,\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\times\mathbf{H}^1(\Omega)$, with $\|\boldsymbol{u}\|_{1,\Omega}$, $\|\boldsymbol{w}\|_{1,\Omega} < \frac{\vartheta_1}{2\,c_p\,c(\Omega)}$, there holds*

$$\|\widetilde{\mathbf{S}}(\phi,\boldsymbol{u})-\widetilde{\mathbf{S}}(\varphi,\boldsymbol{w})\|_{1,\Omega} \,\le\, \widetilde{C}\Big\{\|\boldsymbol{k}\|\,\|\phi-\varphi\|_{0,\Omega} \,+\, \|\widetilde{\mathbf{S}}(\varphi,\boldsymbol{w})\|_{1,\Omega}\,\|\boldsymbol{u}-\boldsymbol{w}\|_{1,\Omega}\Big\}\,. \tag{3.28}$$

Proof. Given $(\phi,\boldsymbol{u})$, $(\varphi,\boldsymbol{w})$ as stated, we let $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi,\boldsymbol{u})$ and $\widetilde{\varphi} := \widetilde{\mathbf{S}}(\varphi,\boldsymbol{w})$, that is (cf. (3.11))

$$A_{\boldsymbol{u}}(\widetilde{\phi},\widetilde{\psi}) \,=\, G_\phi(\widetilde{\psi}) \quad\text{and}\quad A_{\boldsymbol{w}}(\widetilde{\varphi},\widetilde{\psi}) \,=\, G_\varphi(\widetilde{\psi}) \qquad \forall\,\widetilde{\psi}\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)\,.$$

It follows, according to the strong monotonicity of $A_{\boldsymbol{u}}$ with constant $\widetilde{\alpha}$, and then subtracting and adding $G_\varphi(\widetilde{\phi}-\widetilde{\psi}) \,=\, A_{\boldsymbol{w}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\psi})$, that

$$\begin{aligned}\widetilde{\alpha}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}^2 \,&\leq\, A_{\boldsymbol{u}}(\widetilde{\phi},\widetilde{\phi}-\widetilde{\varphi}) \,-\, A_{\boldsymbol{u}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi})\\ &=\quad G_\phi(\widetilde{\phi}-\widetilde{\varphi}) - G_\varphi(\widetilde{\phi}-\widetilde{\varphi}) \,+\, A_{\boldsymbol{w}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\psi}) - A_{\boldsymbol{u}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi})\\ &=\quad \int_\Omega \big(\gamma(\phi)-\gamma(\psi)\big)\,\boldsymbol{k}\cdot\nabla(\widetilde{\phi}-\widetilde{\psi}) \,+\, \int_\Omega \widetilde{\varphi}\,(\boldsymbol{u}-\boldsymbol{w})\cdot\nabla(\widetilde{\phi}-\widetilde{\psi})\,,\end{aligned}$$

where the last equality has employed the definitions given by (3.4) and (3.12). Then, applying the Lipschitz-continuity of γ (cf. (3.14)), the Cauchy-Schwarz inequality, and the estimate (3.5), we deduce from the foregoing equation that

$$\widetilde{\alpha}\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}^2 \,\leq\, \Big\{L_\gamma\,\|\boldsymbol{k}\|\,\|\phi-\psi\|_{0,\Omega} \,+\, c(\Omega)\,\|\widetilde{\varphi}\|_{1,\Omega}\,\|\boldsymbol{u}-\boldsymbol{w}\|_{1,\Omega}\Big\}\,|\widetilde{\phi}-\widetilde{\psi}|_{1,\Omega}\,,$$

which gives (3.28) and finishes the proof. □

The following result is a straightforward corollary of Lemmas 3.9 and 3.10.

Lemma 3.11 *Given $r>0$, we let $W \,:=\, \Big\{\phi \,\in\, \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega): \quad \|\phi\|_{1,\Omega} \,\leq\, r\Big\}$, and assume that*

$$\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \textit{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then, with the constants C and $\widetilde{C}$ from Lemmas 3.9 and 3.10, for all ϕ, $\varphi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ there holds

$$\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega} \,\leq\, \Big\{\widetilde{C}\,\|\boldsymbol{k}\| \,+\, C\,\widetilde{C}\,\|\mathbf{T}(\varphi)\|_{1,\Omega}\,\Big(\|\boldsymbol{f}\|_{\infty,\Omega}+\|\mathbf{S}_1(\varphi)\|_{0,\Omega}\Big)\Big\}\,\|\phi-\varphi\|_{0,\Omega}\,. \tag{3.29}$$

Proof. It suffices to recall from Section 3.2 that $\mathbf{T}(\phi) \,=\, \widetilde{\mathbf{S}}(\phi,\mathbf{S}_2(\phi)) \quad \forall\,\phi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$, and then apply Lemmas 3.8, 3.9, and 3.10. □

The announced properties of $\mathbf{T}$ are proved now.

Lemma 3.12 *Given $r>0$, we let $W \,:=\, \Big\{\phi \,\in\, \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega): \quad \|\phi\|_{1,\Omega} \,\leq\, r\Big\}$, and assume that*

$$\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \textit{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then, $\mathbf{T}: W \longrightarrow W$ is continuous and $\overline{\mathbf{T}(W)}$ is compact.

Proof. The continuity of $\mathbf{T}$ follows directly from (3.29). In turn, let $\{\phi_k\}_{k\in\mathrm{N}}$ be a sequence of W, which is clearly bounded. It follows that there exist a subsequence $\{\phi_k^{(1)}\}_{k\in\mathrm{N}} \subseteq \{\phi_k\}_{k\in\mathrm{N}}$ and $\phi\in\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ such that $\phi_k^{(1)} \stackrel{w}{\longrightarrow} \phi$. Then, since the injection $i: \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega) \longrightarrow \mathrm{L}^2(\Omega)$ is compact, we deduce that $\phi_k^{(1)} \longrightarrow \phi$ in $\mathrm{L}^2(\Omega)$, which, thanks again to (3.29), implies that $\mathbf{T}(\phi_k^{(1)}) \longrightarrow \mathbf{T}(\phi)$ in $\mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$. This proves the compactness of $\overline{\mathbf{T}(W)}$ and finishes the proof. □

Finally, the main result of this section is given as follows.

Theorem 3.13 *Given* $r > 0$*, we let* $W := \Big\{\phi \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega): \quad \|\phi\|_{1,\Omega} \le r\Big\}$*, and assume that*

$$\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} + r\,\|\boldsymbol{f}\|_{\infty,\Omega} < \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \text{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| + \|g\|_{0,\Omega} \le \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then the augmented mixed-primal problem (3.10) *has at least one solution* $(\boldsymbol{\sigma},\boldsymbol{u},\phi) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega)$ *with* $\phi \in W$*, and there holds*

$$\|\phi\|_{1,\Omega} \le \frac{2\,c_p^2}{\vartheta_1\,r}\Big\{\gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| + \|g\|_{0,\Omega}\Big\} \tag{3.30}$$

and

$$\|(\boldsymbol{\sigma},\boldsymbol{u})\|_H \le C_S\Big\{\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} + \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi\|_{1,\Omega}\Big\}\,. \tag{3.31}$$

Moreover, if the data $\boldsymbol{k}$*,* $\boldsymbol{f}$*, and* $\boldsymbol{u}_\mathrm{D}$ *are sufficiently small so that, with the constants* C *and* $\widetilde{C}$ *from Lemmas* 3.9 *and* 3.10*, there holds*

$$\widetilde{C}\,\|\boldsymbol{k}\| + C\,\widetilde{C}\,r\Big\{\big(1 + r\,C_S\big)\,\|\boldsymbol{f}\|_{\infty,\Omega} + C_S\,\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}}\Big\} < 1\,, \tag{3.32}$$

then the solution ϕ *is unique in* W*.*

Proof. According to the equivalence between (3.10) and the fixed point equation (3.13), and thanks to the previous Lemmas 3.8 and 3.12, the existence of solution is just a straightforward application of the Schauder fixed point theorem (cf. Theorem 3.7). In turn, the estimates (3.30) and (3.31) follow from (3.16) (cf. Lemma 3.4) and (3.22) (cf. Lemma 3.6). Furthermore, given another solution $\varphi \in W$ of (3.13), the estimates $\|\mathbf{T}(\varphi)\|_{1,\Omega} = \|\varphi\|_{1,\Omega} \le r$ and $\|\mathbf{S}_1(\varphi)\|_{0,\Omega} \le C_S\Big\{\|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} + \|\boldsymbol{f}\|_{\infty,\Omega}\,\|\varphi\|_{1,\Omega}\Big\}$ confirm (3.32) as a sufficient condition for concluding, together with (3.29), that $\phi = \varphi$. $\square$

4 The Galerkin scheme

In this section we introduce and analyze the Galerkin scheme of the augmented mixed-primal problem (3.10). To this end, we now let $\mathcal{T}_h$ be a regular triangulation of Ω by triangles K (resp. tetrahedra K in R^3) of diameter h_K, and define the meshsize $h := \max\Big\{h_K: \quad K \in \mathcal{T}_h\Big\}$. In addition, given an integer $k \ge 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$ be the space of polynomial functions on K of degree $\le k$, and define the corresponding local Raviart-Thomas space of order k as

$$\mathrm{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\,\boldsymbol{x}\,,$$

where, according to the notations described in Section 1, $\mathbf{P}_k(K) = [\mathrm{P}_k(K)]^n$, and $\boldsymbol{x}$ is the generic vector in R^n. Then, we introduce the finite element subspaces approximating the unknowns $\boldsymbol{\sigma}$, $\boldsymbol{u}$, and ϕ, respectively, as the global Raviart-Thomas space of order k, and the corresponding Lagrange spaces given by the continuous piecewise polynomials of degree $\le k+1$, that is

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_N(\mathbf{div};\Omega): \quad \mathbf{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_K \in \mathrm{RT}_k(K) \quad \forall\,\mathbf{c} \in \mathrm{R}^n\,, \quad \forall\,K \in \mathcal{T}_h\Big\}\,, \tag{4.1}$$

$$\mathbf{H}_h^{\boldsymbol{u}} := \Big\{\boldsymbol{v}_h \in \mathbf{C}(\Omega): \quad \boldsymbol{v}_h|_K \in \mathbf{P}_{k+1}(K) \quad \forall\,K \in \mathcal{T}_h\Big\}\,, \tag{4.2}$$

$$\mathrm{H}_h^{\phi} := \Big\{\psi_h \in C(\Omega) \cap \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega): \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\,K \in \mathcal{T}_h\Big\}\,. \tag{4.3}$$

In this way, the underlying Galerkin scheme, given by the discrete counterpart of (3.10), reads: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\phi}$ such that

$$\begin{aligned} B_{\phi_h}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,&=\, F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\,, \\ A_{\boldsymbol{u}_h}(\phi_h, \psi_h) \,&=\, \int_\Omega \gamma(\phi_h)\, \boldsymbol{k} \cdot \nabla \psi_h \,+\, \int_\Omega g \psi_h \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,. \end{aligned} \tag{4.4}$$

Throughout the rest of this section we adopt the discrete analogue of the fixed point strategy introduced in Section 3.3. Hence, we now let $\mathbf{S}_h : \mathrm{H}_h^{\phi} \longrightarrow \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ be the operator defined by

$$\mathbf{S}_h(\phi_h) \,=\, (\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h)) \,:=\, (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \qquad \forall\, \phi_h \in \mathrm{H}_h^{\phi}\,,$$

where $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ is the unique solution of

$$B_{\phi_h}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,=\, F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \quad \forall\, (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\,, \tag{4.5}$$

with B_{ϕ_h} and F_{ϕ_h} being defined by (3.8) and (3.9), respectively, with $\phi = \phi_h$. In addition, we let $\widetilde{\mathbf{S}}_h : \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\boldsymbol{u}} \longrightarrow \mathrm{H}_h^{\phi}$ be the operator defined by

$$\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h) \,:=\, \widetilde{\phi}_h \qquad \forall\, (\phi_h, \boldsymbol{u}_h) \,\in\, \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\boldsymbol{u}}\,,$$

where $\widetilde{\phi}_h \in \mathrm{H}_h^{\phi}$ is the unique solution of

$$A_{\boldsymbol{u}_h}(\widetilde{\phi}_h, \widetilde{\psi}_h) \,=\, G_{\phi_h}(\widetilde{\psi}_h) \qquad \forall\, \widetilde{\psi}_h \in \mathrm{H}_h^{\phi}\,, \tag{4.6}$$

with $A_{\boldsymbol{u}_h}$ and G_{ϕ_h} being defined by (3.4) and (3.12), respectively, with $\boldsymbol{u} = \boldsymbol{u}_h$ and $\phi = \phi_h$. Finally, we define the operator $\mathbf{T}_h : \mathrm{H}_h^{\phi} \longrightarrow \mathrm{H}_h^{\phi}$ by

$$\mathbf{T}_h(\phi_h) \,:=\, \widetilde{\mathbf{S}}_h(\phi_h, \mathbf{S}_{2,h}(\phi_h)) \qquad \forall\, \phi_h \,\in\, \mathrm{H}_h^{\phi}\,,$$

and realize that (4.4) can be rewritten, equivalently, as: Find $\phi_h \in \mathrm{H}_h^{\phi}$ such that

$$\mathbf{T}_h(\phi_h) \,=\, \phi_h\,. \tag{4.7}$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.5) and (4.6) are well-posed. Indeed, it is easy to see that the respective proofs are almost verbatim of the continuous analogues provided in Section 3.3, and hence we simply state the corresponding results as follows.

Lemma 4.1 *Assume that* $\kappa_1 \in \left(0, \frac{2\delta\mu_1}{\mu_2}\right)$ *with* $\delta \in (0, 2\mu_1)$*, and that* $0 < \kappa_2, \kappa_3$*. Then, for each* $\phi_h \in \mathrm{H}_h^{\phi}$ *the problem* (4.5) *has a unique solution* $\mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$*. Moreover, with the same constant* $C_S > 0$ *from Lemma* 3.4*, there holds*

$$\|\mathbf{S}_h(\phi_h)\|_H \,=\, \|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \,\le\, C_S \Big\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\, \|\phi_h\|_{1,\Omega} \Big\} \qquad \forall\, \phi_h \,\in\, \mathrm{H}_h^{\phi}\,.$$

Proof. It suffices to see that for each $\phi_h \in \mathrm{H}_h^{\phi}$, B_{ϕ_h} is elliptic on $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$ with the same constant α from Lemma 3.4 (cf. (3.18)), and that $\|F_{\phi_h}\|_{\left(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\right)'}$ is bounded as in (3.20) with ϕ_h in place of ϕ. The rest of the proof is a direct application of the Lax-Milgram lemma. □

Lemma 4.2 *Let* $\phi_h \in \mathrm{H}_h^\phi$ *and* $\boldsymbol{u}_h \in \mathbf{H}_h^{\boldsymbol{u}}$ *such that* $\|\boldsymbol{u}_h\|_{1,\Omega} < \frac{\vartheta_1}{2\,c_p\,c(\Omega)}$ *(cf.* (2.3), (3.1), (3.5)*). Then, there exists a unique* $\widetilde{\phi}_h := \widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h) \in \mathrm{H}_h^\phi$ *solution of* (4.6)*, and there holds*

$$\|\widetilde{\mathbf{S}}_h(\phi_h, \boldsymbol{u}_h)\|_{1,\Omega} \,=\, \|\widetilde{\phi}_h\|_{1,\Omega} \,\leq\, \frac{2\,c_p^2}{\vartheta_1} \left\{ \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \right\}.$$

Proof. It basically follows by observing that, under the assumption on $\|\boldsymbol{u}_h\|_{1,\Omega}$, $A_{\boldsymbol{u}_h}$ becomes Lipschitz-continuous and strongly monotone on $\mathrm{H}_h^\phi \times \mathrm{H}_h^\phi$ with the constants $\widetilde{L}_{\boldsymbol{u}_h} := \widetilde{\vartheta}_2 \,+\, c(\Omega)\,\|\boldsymbol{u}_h\|_{1,\Omega}$ and $\widetilde{\alpha} := \frac{\vartheta_1}{2\,c_p^2}$ given in the proofs of Lemmas 3.5) and 3.6, respectively, and then applying again [21, Theorem 3.3.23]. In addition, the fact that $\|G_\phi\|$ is bounded independently of ϕ (cf. proof of Lemma 3.5), confirms the same upper bound for $\|G_{\phi_h}\|_{\left(\mathrm{H}_h^\phi\right)'}$. □

We now aim to show the solvability of (4.4) by analyzing the equivalent fixed point equation (4.7). To this end, in what follows we verify the hypotheses of the Brouwer fixed point theorem, which is given as follows (see, e.g. [8, Theorem 9.9-2]).

Theorem 4.3 *Let* W *be a compact and convex subset of a finite dimensional Banach space* X *and let* $T : W \to W$ *be a continuous mapping. Then* T *has at least one fixed point.*

We begin with the discrete version of Lemma 3.8.

Lemma 4.4 *Given* $r > 0$*, we let* $W_h := \left\{ \phi_h \,\in\, \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \,\leq\, r \right\}$*, and assume that*

$$\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \text{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then $\mathbf{T}_h(W_h) \,\subseteq\, W_h$.

Proof. It is a straightforward consequence of Lemmas 4.1 and 4.2. □

Next, utilizing discrete analogues of Lemmas 3.9 and 3.10 (which for sake of space saving are not specified here), we can prove the discrete version of Lemma 3.11.

Lemma 4.5 *Given* $r > 0$*, we let* $W_h := \left\{ \phi_h \,\in\, \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \,\leq\, r \right\}$*, and assume that*

$$\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \text{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then, with the constants C *and* $\widetilde{C}$ *from Lemmas* 3.9 *and* 3.10*, for all* $\phi_h,\, \varphi_h \in \mathrm{H}_h^\phi$ *there holds*

$$\|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{1,\Omega} \,\leq\, \left\{ \widetilde{C}\,\|\boldsymbol{k}\| + C\,\widetilde{C}\,\|\mathbf{T}_h(\varphi_h)\|_{1,\Omega} \left(\|\boldsymbol{f}\|_{\infty,\Omega} + \|\mathbf{S}_{1,h}(\varphi_h)\|_{0,\Omega} \right) \right\} \|\phi_h - \varphi_h\|_{0,\Omega}\,. \quad (4.8)$$

Consequently, since the foregoing lemma confirms the continuity of $\mathbf{T}_h$, we conclude, thanks to the Brouwer fixed point theorem (cf. Theorem 4.3) and Lemmas 4.4 and 4.5, the main result of this section.

Theorem 4.6 *Given* $r > 0$*, we let* $W_h := \left\{ \phi_h \,\in\, \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \,\leq\, r \right\}$*, and assume that*

$$\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, r\,\|\boldsymbol{f}\|_{\infty,\Omega} \,<\, \frac{\vartheta_1}{2\,C_S\,c_p\,c(\Omega)} \qquad \text{and} \qquad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{\vartheta_1\,r}{2\,c_p^2}\,.$$

Then the Galerkin scheme (4.4) *has at least one solution* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\phi}$ *with* $\phi_h \in W_h$, *and there holds*

$$\|\phi_h\|_{1,\Omega} \,\le\, \frac{2\, c_p^2}{\vartheta_1\, r} \left\{ \gamma_2\, |\Omega|^{1/2}\, \|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \right\}$$

and

$$\|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \,\le\, C_S \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \,+\, \|\boldsymbol{f}\|_{\infty,\Omega}\, \|\phi_h\|_{1,\Omega} \right\}.$$

Moreover, if the data $\boldsymbol{k}$, $\boldsymbol{f}$, *and* $\boldsymbol{u}_{\mathrm{D}}$ *are sufficiently small so that, with the constants* C *and* $\widetilde{C}$ *from Lemmas* 3.9 *and* 3.10, *there holds*

$$\widetilde{C}\, \|\boldsymbol{k}\| \,+\, C\, \widetilde{C}\, r \left\{ \big(1 + r\, C_S\big)\, \|\boldsymbol{f}\|_{\infty,\Omega} \,+\, C_S\, \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \right\} \,<\, 1\,,$$

then the solution ϕ_h *is unique in* W_h.

5 A priori error analysis

Given $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)$ with $\phi \in W$, and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathrm{H}_h^{\phi}$ with $\phi_h \in W_h$, solutions of (3.10) and (4.4), respectively, we now aim to derive a corresponding a priori error estimate. For this purpose, we now recall from (3.10) and (4.4), that the above means that

$$\begin{aligned} B_\phi((\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\tau}, \boldsymbol{v})) \,&=\, F_\phi(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)\,, \\ B_{\phi_h}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,&=\, F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\,, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} A_{\boldsymbol{u}}(\phi, \psi) \,&=\, G_\phi(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega)\,, \\ A_{\boldsymbol{u}_h}(\phi_h, \psi_h) \,&=\, G_{\phi_h}(\psi_h) \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,. \end{aligned} \tag{5.2}$$

Next, we recall from [16] a Strang-type lemma, which will be utilized in our subsequent analysis.

Lemma 5.1 *Let* H *be a Hilbert space,* $F \in H'$, *and* $\mathbf{A} : H \to H'$ *a nonlinear operator. In addition, let* $\{H_n\}_{n\in N}$ *be a sequence of finite dimensional subspaces of* H, *and for each* $n \in N$ *consider a nonlinear operator* $\mathbf{A}_n : H_n \to H_n'$ *and a functional* $F_n \in H_n'$. *Assume that the family* $\{\mathbf{A}\} \cup \{\mathbf{A}_n\}_{n\in N}$ *is uniformly Lipschitz continuous and strongly monotone with constants* $\Lambda_{\mathtt{LC}}$ *and* $\Lambda_{\mathtt{SM}}$, *respectively. In turn, let* $u \in H$ *and* $u_n \in H_n$ *such that*

$$[\mathbf{A}(u), v] \,=\, [F, v] \qquad \forall\, v \in H \quad \textit{and} \quad [\mathbf{A}_n(u_n), v_n] \,=\, [F_n, v_n] \qquad \forall\, v_n \in H_n\,,$$

where $[\cdot,\cdot]$ *denotes the duality pairings of both* $H' \times H$ *and* $H_n' \times H_n$. *Then for each* $n \in N$ *there holds*

$$\begin{aligned} \|u - u_n\|_H \,\le\, \Lambda_{\mathtt{ST}} \Bigg\{ &\sup_{\substack{w_n \in H_n \\ w_n \neq \mathbf{0}}} \frac{\big|\, [F, w_n] - [F_n, w_n] \,\big|}{\|w_n\|_H} \\ &+ \inf_{\substack{v_n \in H_n \\ v_n \neq \mathbf{0}}} \left(\|u - v_n\|_H \,+\, \sup_{\substack{w_n \in H_n \\ w_n \neq \mathbf{0}}} \frac{\big|\, [\mathbf{A}(v_n), w_n] - [\mathbf{A}_n(v_n), w_n] \,\big|}{\|w_n\|_H} \right) \Bigg\}\,, \end{aligned}$$

with $\Lambda_{\mathtt{ST}} := \Lambda_{\mathtt{SM}}^{-1} \max\Big\{1, \Lambda_{\mathtt{SM}} + \Lambda_{\mathtt{LC}}\Big\}$.

Proof. It is a particular case of [16, Theorem 6.4]. □

We begin our analysis by denoting as usual

$$\mathrm{dist}\big(\phi, \mathrm{H}_h^\phi\big) \,:=\, \inf_{\varphi_h \in \mathrm{H}_h^\phi} \|\phi - \varphi_h\|_{1,\Omega},$$

and

$$\mathrm{dist}\big((\boldsymbol{\sigma}, \boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\big) \,:=\, \inf_{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}} \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_H \,.$$

Then, we have the following result concerning $\|\phi - \phi_h\|_{1,\Omega}$.

Lemma 5.2 *Let* $\widetilde{C}_{\mathtt{ST}} := \widetilde{\alpha}^{-1} \max\big\{1, \widetilde{\alpha} + \widetilde{L}\big\}$, *with* $\widetilde{\alpha} := \frac{\vartheta_1}{2\,c_p^2}$ *and* $\widetilde{L} := \widetilde{\vartheta}_2 + \frac{\vartheta_1}{2\,c_p}$. *Then there holds*

$$\begin{aligned} \|\phi - \phi_h\|_{1,\Omega} \,\le\, \widetilde{C}_{\mathtt{ST}} \Bigg\{ & L_\gamma \,\|\boldsymbol{k}\|\, \|\phi - \phi_h\|_{0,\Omega} \,+\, c(\Omega)\, \|\phi\|_{1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \\ + \quad & \bigg(1 \,+\, c(\Omega)\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\bigg)\, \mathrm{dist}(\phi, \mathrm{H}_h^\phi) \Bigg\} \,. \end{aligned} \tag{5.3}$$

Proof. We first observe from Lemmas 3.5, 3.6, and 4.2, that the nonlinear operators $A_{\boldsymbol{u}}$ and $A_{\boldsymbol{u}_h}$ are both strongly monotone and Lipschitz-continuous on their corresponding spaces with constants $\widetilde{\alpha}$ and $\widetilde{L}$, respectively. Then, by applying the abstract Lemma 5.1 to the context (5.2), we find that

$$\begin{aligned} \|\phi - \phi_h\|_{1,\Omega} \,\le\, \widetilde{C}_{\mathtt{ST}} \Bigg\{ & \sup_{\substack{\psi_h \in \mathrm{H}_h^\phi \\ \psi_h \neq \mathbf{0}}} \frac{\big|\, G_\phi(\psi_h) - G_{\phi_h}(\psi_h)\,\big|}{\|\psi_h\|_{1,\Omega}} \\ + \quad & \inf_{\substack{\varphi_h \in \mathrm{H}_h^\phi \\ \varphi_h \neq \mathbf{0}}} \Bigg(\|\phi - \varphi_h\|_{1,\Omega} \,+\, \sup_{\substack{\psi_h \in \mathrm{H}_h^\phi \\ \psi_h \neq \mathbf{0}}} \frac{\big|\, A_{\boldsymbol{u}}(\varphi_h, \psi_h) - A_{\boldsymbol{u}_h}(\varphi_h, \psi_h)\,\big|}{\|\psi_h\|_{1,\Omega}} \Bigg) \Bigg\} \,. \end{aligned} \tag{5.4}$$

Next, we proceed similarly as in the proof of Lemma 3.10 to estimate each term in the foregoing equation involving a supremum. In fact, according to the definition of G_ϕ (cf. (3.12)), and applying the same arguments from that proof, we readily see that

$$\sup_{\substack{\psi_h \in \mathrm{H}_h^\phi \\ \psi_h \neq \mathbf{0}}} \frac{\big|\, G_\phi(\psi_h) - G_{\phi_h}(\psi_h)\,\big|}{\|\psi_h\|_{1,\Omega}} \,\le\, L_\gamma \,\|\boldsymbol{k}\|\, \|\phi - \phi_h\|_{0,\Omega} \,. \tag{5.5}$$

In turn, it is clear from the definition of $A_{\boldsymbol{u}}$ (cf. (3.4)) and the estimate (3.5) that for each $\varphi_h \in \mathrm{H}_h^\phi$ there holds

$$\begin{aligned} \sup_{\substack{\psi_h \in \mathrm{H}_h^\phi \\ \psi_h \neq \mathbf{0}}} \frac{\big|\, A_{\boldsymbol{u}}(\varphi_h, \psi_h) - A_{\boldsymbol{u}_h}(\varphi_h, \psi_h)\,\big|}{\|\psi_h\|_{1,\Omega}} & \,\le\, c(\Omega)\, \|\varphi_h\|_{1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \\ & \,\le\, c(\Omega)\, \|\phi - \varphi_h\|_{1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \,+\, c(\Omega)\, \|\phi\|_{1,\Omega}\, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} \,. \end{aligned} \tag{5.6}$$

In this way, replacing (5.5) and (5.6) back into (5.4), we arrive at (5.3) and end the proof. □

The following lemma provides a preliminary estimate for the error $\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H$.

Lemma 5.3 *Let $C_{\mathtt{ST}} := \alpha^{-1} \max\big\{1, \alpha + \|B\|\big\}$, where $\|B\|$ and α are the boundedness and ellipticity constants, respectively, of the bilinear forms B_ϕ (cf. (3.17), (3.18)). Then there holds*

$$
\begin{aligned}
&\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \\
&\quad\le\ C_{\mathtt{ST}} \Bigg\{ \left(\big(1 + \kappa_2^2\big)^{1/2} \|\boldsymbol{f}\|_{\infty,\Omega} \ +\ \frac{L_\mu\, (1 + \kappa_1^2)^{1/2}}{\mu_1^2}\, \|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega} \right) \|\phi - \phi_h\|_{0,\Omega} \\
&\qquad +\ \left(1 + \frac{L_\mu\, (1 + \kappa_1^2)^{1/2}}{\mu_1^2}\, \|\phi - \phi_h\|_{0,\Omega} \right) \operatorname{dist}\big((\boldsymbol{\sigma}, \boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\big) \Bigg\}.
\end{aligned}
\tag{5.7}
$$

Proof. By applying the abstract Lemma 5.1 to the context (5.1), we obtain

$$
\begin{aligned}
&\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H \\
&\quad\le\ C_{\mathtt{ST}} \Bigg\{ \sup_{\substack{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \\ (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \neq \mathbf{0}}} \frac{\big|\, F_\phi(\boldsymbol{\tau}_h, \boldsymbol{v}_h) - F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \,\big|}{\|(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_H} \\
&\qquad +\ \inf_{\substack{(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \\ (\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \neq \mathbf{0}}} \Bigg(\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_H \\
&\qquad +\ \sup_{\substack{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \\ (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \neq \mathbf{0}}} \frac{\big|\, B_\phi((\boldsymbol{\zeta}_h, \boldsymbol{w}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) - B_{\phi_h}((\boldsymbol{\zeta}_h, \boldsymbol{w}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,\big|}{\|(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_H} \Bigg) \Bigg\}.
\end{aligned}
\tag{5.8}
$$

Then, proceeding analogously as in the proof of Lemma 3.9, we easily deduce that

$$
\sup_{\substack{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \\ (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \neq \mathbf{0}}} \frac{\big|\, F_\phi(\boldsymbol{\tau}_h, \boldsymbol{v}_h) - F_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \,\big|}{\|(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_H} \ \le\ \big(1 + \kappa_2^2\big)^{1/2} \|\boldsymbol{f}\|_{\infty,\Omega} \|\phi - \phi_h\|_{0,\Omega}\,,
$$

and

$$
\begin{aligned}
&\sup_{\substack{(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \\ (\boldsymbol{\tau}_h, \boldsymbol{v}_h) \neq \mathbf{0}}} \frac{\big|\, B_\phi((\boldsymbol{\zeta}_h, \boldsymbol{w}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) - B_{\phi_h}((\boldsymbol{\zeta}_h, \boldsymbol{w}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h)) \,\big|}{\|(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_H} \\
&\qquad\le\ \frac{L_\mu\, (1 + \kappa_1^2)^{1/2}}{\mu_1^2}\, \|\phi - \phi_h\|_{0,\Omega} \|\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega} \\
&\qquad\le\ \frac{L_\mu\, (1 + \kappa_1^2)^{1/2}}{\mu_1^2}\, \|\phi - \phi_h\|_{0,\Omega} \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega} \ +\ \|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega} \Big\}.
\end{aligned}
$$

Finally, by replacing the foregoing inequalities into (5.8), we get (5.7), which ends the proof. □

We now combine the inequalities provided by Lemmas 5.2 and 5.3 to derive the Céa estimate for the total error $\|\phi - \phi_h\|_{1,\Omega} + \|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_H$. To this end, and in order to simplify the subsequent writing, we introduce the following constants

$$
C_1 \ :=\ \widetilde{C}_{\mathtt{ST}}\, L_\gamma\,, \quad C_2 \ :=\ \widetilde{C}_{\mathtt{ST}}\, c(\Omega)\, r\, C_{\mathtt{ST}}\, (1 + \kappa_1^2)^{1/2}\,, \quad C_3 \ :=\ \widetilde{C}_{\mathtt{ST}}\, c(\Omega)\, r\, C_{\mathtt{ST}}\, C_S\, \frac{L_\mu\, (1 + \kappa_1^2)^{1/2}}{\mu_1^2}\,.
$$

Hence, by replacing the bound for $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$ given by (5.7) into the second term on the right hand side of (5.3), recalling that $\|\phi\|_{1,\Omega} \le r$, employing the bound for $\|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega}$ provided by (3.16), and performing some algebraic manipulations, we can assert that

$$\begin{aligned}
\|\phi - \phi_h\|_{1,\Omega} \,\le\, & \Big\{ C_1\,\|\boldsymbol{k}\| \,+\, \big(C_2 \,+\, r\,C_3\big)\,\|\boldsymbol{f}\|_{\infty,\Omega} \,+\, C_3\,\|\boldsymbol{u}_{\rm D}\|_{1/2,\Gamma_{\rm D}} \Big\}\,\|\phi - \phi_h\|_{0,\Omega} \\
& + \widetilde{C}_{\tt ST}\,c(\Omega)\,r\,C_{\tt ST}\left(1 + \frac{L_\mu\,(1+\kappa_1^2)^{1/2}}{\mu_1^2}\,\|\phi - \phi_h\|_{0,\Omega}\right) {\rm dist}\big((\boldsymbol{\sigma},\boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\big) \\
& + \widetilde{C}_{\tt ST}\Big(1 \,+\, c(\Omega)\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}\Big)\,{\rm dist}(\phi, {\rm H}_h^{\phi})\,.
\end{aligned} \tag{5.9}$$

Note here that the expressions in (5.9) multiplying the distances are already controlled by constants, parameters, and data only. In fact, $\|\phi\|_{0,\Omega}$ and $\|\phi_h\|_{0,\Omega}$ are certainly bounded by r, whereas $\|\boldsymbol{u}\|_{1,\Omega}$ and $\|\boldsymbol{u}_h\|_{1,\Omega}$ are also estimated according to (3.16). As a consequence of the foregoing discussion, we can establish the following result providing the requested Céa estimate.

Theorem 5.4 *Assume that the data $\boldsymbol{k}$, $\boldsymbol{f}$, and $\boldsymbol{u}_{\rm D}$ are sufficiently small so that*

$$C_1\,\|\boldsymbol{k}\| \,+\, \big(C_2 \,+\, r\,C_3\big)\,\|\boldsymbol{f}\|_{\infty,\Omega} \,+\, C_3\,\|\boldsymbol{u}_{\rm D}\|_{1/2,\Gamma_{\rm D}} \,<\, \frac{1}{2}\,. \tag{5.10}$$

Then, there exist positive constants C_4 and C_5, depending only on parameters, data, and other constants, all them independent of h, such that

$$\|\phi - \phi_h\|_{1,\Omega} \,+\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \,\le\, C_4\,{\rm dist}\big((\boldsymbol{\sigma},\boldsymbol{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}\big) \,+\, C_5\,{\rm dist}(\phi, {\rm H}_h^{\phi})\,. \tag{5.11}$$

Proof. The estimate for $\|\phi - \phi_h\|_{1,\Omega}$ follows straightforwardly from (5.9) and (5.10), and then, the replacement of it back into (5.7) completes the proof. □

We remark at this point that, if one assumes for a moment that there holds

$$\|\phi - \phi_h\|_{0,\Omega} \,\le\, c\,h^{\epsilon}\,\|\phi - \phi_h\|_{1,\Omega}\,, \tag{5.12}$$

for some positive constants c and ϵ, independent of h, then the first term on the right hand side of (5.9) implies that, instead of the restriction on the data given by (5.10), we would obtain the Céa estimate (5.11) for h sufficiently small. While in Section 6 below we provide some numerical evidences of (5.12) with $\epsilon = 1$, the question whether it can be proved or not is an open problem.

We end this section with the corresponding rates of convergence of our Galerkin scheme (4.4).

Theorem 5.5 *In addition to the hypotheses of Theorems* 3.13, 4.6, *and* 5.4, *assume that there exists $s > 0$ such that $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, $\boldsymbol{u} \in \mathbf{H}^{1+s}(\Omega)$, and $\phi \in {\rm H}^{1+s}(\Omega)$. Then, there exists $\widehat{C} > 0$, independent of h, such that, with the finite element subspaces defined by* (4.1), (4.2), *and* (4.3), *there holds*

$$\begin{aligned}
& \|\phi - \phi_h\|_{1,\Omega} \,+\, \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_H \\
& \qquad \le \quad \widehat{C}\,h^{\min\{s,k+1\}}\Big\{\|\boldsymbol{\sigma}\|_{s,\Omega} \,+\, \|\mathbf{div}\boldsymbol{\sigma}\|_{s,\Omega} \,+\, \|\boldsymbol{u}\|_{1+s,\Omega} \,+\, \|\phi\|_{1+s,\Omega}\Big\}\,.
\end{aligned} \tag{5.13}$$

Proof. It follows directly from the Céa estimate (5.11) and the approximation properties of $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\boldsymbol{u}}$, and ${\rm H}_h^{\phi}$ (cf. [2, 7, 15]). □

6 Numerical results

We illustrate the performance of our mixed–primal finite element method with some numerical tests. We first study the accuracy of the approximations by manufacturing an exact solution of the nonlinear problem (2.1) defined on $\Omega = (0,1)^2$. We introduce the coefficients $\mu(\phi) = (1-c\phi)^{-2}$, $\gamma(\phi) = c\phi(1-c\phi)^2$, $\vartheta(|\nabla\phi|) = m_1 + m_2(1+|\nabla\phi|^2)^{m_3/2-1}$, and the source terms on the right hand sides are adjusted in such a way that the exact solutions are given by the smooth functions

$$\phi(x_1,x_2) = b - b\exp(-x_1(x_1-1)x_2(x_2-1)), \quad \boldsymbol{u}(x_1,x_2) = \begin{pmatrix} \sin(2\pi x_1)\cos(2\pi x_2) \\ -\cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix},$$

$$\boldsymbol{\sigma}(x_1,x_2) = 2\pi \begin{pmatrix} \dfrac{\cos(2\pi x_1)\cos(2\pi x_2)}{(1-bc+bce^{-x_1(x_1-1)x_2(x_2-1)})^2} & \dfrac{-\sin(2\pi x_1)\sin(2\pi x_2)}{(1-bc+bce^{-x_1(x_1-1)x_2(x_2-1)})^2} \\ \dfrac{\sin(2\pi x_1)\sin(2\pi x_2)}{(1-bc+bce^{-x_1(x_1-1)x_2(x_2-1)})^2} & \dfrac{-\cos(2\pi x_1)\cos(2\pi x_2)}{(1-bc+bce^{-x_1(x_1-1)x_2(x_2-1)})^2} \end{pmatrix} - (x_1^2 - x_2^2)\mathbb{I},$$

for $(x_1,x_2) \in \overline{\Omega}$. We take $b = 15, c = m_1 = m_2 = 1/2, m_3 = 3/2$ and set $\Gamma_{\mathrm{D}} = \partial\Omega$, where ϕ vanishes and $\boldsymbol{u}_D$ is imposed accordingly to the exact solution. The mean value of $\operatorname{tr}\boldsymbol{\sigma}_h$ over Ω is fixed via a penalization strategy. As defined above, the scalar field ϕ is bounded in Ω and so the coefficients are also bounded. In particular we have $\mu_1 = 0.99$ and $\mu_2 = 3.35$. Therefore, and as suggested by Lemma 3.4, the stabilization constants are chosen as $\kappa_1 = \mu_1^2/\mu_2 = 0.2976$, $\kappa_2 = 1/\mu_2 = 0.2985$, and $\kappa_3 = \kappa_1/2 = 0.1488$.

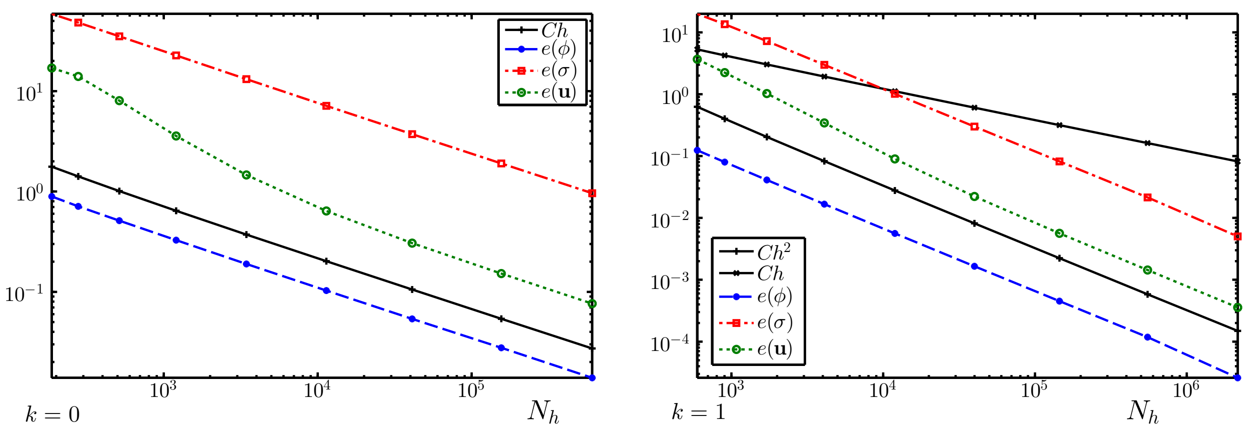

Figure 1: Example 1: Computed errors $e(\phi), e(\boldsymbol{\sigma}), e(\boldsymbol{u})$ associated to the mixed–primal approximation versus the number of degrees of freedom N_h for $\mathrm{RT}_0 - P_1 - P_1$ and $\mathrm{RT}_1 - P_2 - P_2$ finite elements (left and right, respectively). See values in Table 1.

The domain is partitioned into quasi-uniform meshes with $2^n + 3$, $n = 0, 1, \ldots, 8$ vertices on each side of the domain. The convergence of the approximate solutions is assessed by computing errors in the respective norms and experimental rates, that we define as usual

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega}, \quad e(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}, \quad e(\phi) := \|\phi - \phi_h\|_{1,\Omega},$$
$$r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/\hat{e}(\boldsymbol{\sigma}))}{\log(h/\hat{h})}, \quad r(\boldsymbol{u}) := \frac{\log(e(\boldsymbol{u})/\hat{e}(\boldsymbol{u}))}{\log(h/\hat{h})}, \quad r(\phi) := \frac{\log(e(\phi)/\hat{e}(\phi))}{\log(h/\hat{h})},$$

where e and $\hat{e}$ denote errors computed on two consecutive meshes of sizes h and $\hat{h}$, respectively. Notice that these errors are computed between the finite element approximation and the corresponding

N_h	h	$e(\phi)$	$r(\phi)$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\frac{\\|\phi-\phi_h\\|_{0,\Omega}}{h\\|\phi-\phi_h\\|_{1,\Omega}}$	iter
Augmented $\mathrm{RT}_0 - P_1 - P_1$ scheme									
187	0.353553	0.891473	–	58.80212	–	16.97841	–	0.297779	8
278	0.282843	0.711188	0.970463	48.21425	0.852938	13.99512	1.014962	0.270623	7
514	0.202031	0.512189	0.975540	35.19082	0.935794	8.041585	1.424675	0.277445	7
1202	0.128565	0.327347	0.990462	22.67913	0.972039	3.573343	1.579459	0.281382	7
3442	0.074432	0.189813	0.997142	13.16677	0.994888	1.461483	1.563582	0.283007	6
11378	0.040406	0.103089	0.999241	7.138732	1.002043	0.639297	1.235346	0.283558	6
41074	0.021107	0.053859	0.999801	3.722753	1.002661	0.305779	1.113577	0.283726	6
155762	0.010795	0.027705	0.999948	1.904552	1.002240	0.152283	1.034021	0.283795	6
606322	0.005460	0.013933	0.999987	0.961174	1.001041	0.076408	1.010863	0.283789	6
Augmented $\mathrm{RT}_1 - P_2 - P_2$ scheme									
595	0.353553	0.123752	–	19.88141	–	3.675443	–	0.0862117	7
903	0.282843	0.079988	1.955574	13.55213	1.717465	2.237812	2.223581	0.0847032	6
1711	0.202031	0.041028	1.984189	7.213065	1.874291	1.026756	2.215637	0.0805605	6
4095	0.128565	0.016689	1.990120	2.989083	1.949025	0.343355	2.223416	0.0772966	6
11935	0.074432	0.005607	1.995567	1.012340	1.981522	0.089977	2.150313	0.0754936	6
39903	0.040406	0.001654	1.998442	0.299392	1.994287	0.022247	2.187332	0.0747908	6
144991	0.021107	0.000451	1.999545	0.081778	1.998531	0.005629	2.116371	0.0745706	6
551775	0.010795	0.000118	1.999836	0.021401	1.999468	0.001439	2.034801	0.0749512	6
2164783	0.005460	0.000026	1.999935	0.005014	2.006076	0.000357	2.013878	0.0742895	6

Table 1: Example 1: Convergence history and Newton iteration count for the mixed–primal $\mathrm{RT}_k - P_{k+1} - P_{k+1}$ approximations of the coupled problem, $k = 0, 1$. Here N_h stands for the number of degrees of freedom associated to each triangulation $\mathcal{T}_h$.

interpolate of the exact solution. Values and plots of errors and corresponding rates associated to $\mathrm{RT}_k - P_{k+1} - P_{k+1}$ approximations with $k = 0$ and $k = 1$ are summarized in Table 1 and Figure 1, respectively, where we observe convergence rates of $O(h^{k+1})$ for stresses, velocities and the scalar field in the relevant norms. These findings are in agreement with the theoretical error bounds of Section 5 (cf. (5.13)). In addition, we also depict the quotient between the error L^2−norm and h times the error H^1−norm of ϕ, which remains bounded and therefore reflects the conjectured estimate (5.12). A Newton-Raphson algorithm with a tolerance of 1E-08 has been applied to the resolution of the nonlinear problem (4.4), and at each iteration the linear systems resulting from the linearization were solved by means of the multifrontal massively parallel solver (MUMPS [1]). We mention that an average number of 7 Newton steps were required to reach the desired tolerance. All remaining examples were carried out using $k = 0$, i.e., lowest-order Raviart-Thomas finite element approximations for the rows of the Cauchy stress tensor, and piecewise linear approximations of velocity components and the scalar field ϕ. The augmented mixed–primal approximations computed on a mesh of 37249 vertices and 74496 elements are depicted in Figure 2.

In our second example we assess the capability of a 3D implementation by carrying out the benchmark test of thermal convection on the cube $\Omega = (0, 1)^3$ (see e.g. [13, 20]). The relevant equations, here written in terms of stresses $\boldsymbol{\sigma}$, velocities $\boldsymbol{u}$, and *temperature* ϕ correspond to the Boussinesq approximation and can be readily recovered from (2.5) by setting $g = 0$, $\boldsymbol{f}\phi = \frac{1}{\rho}(0, \phi - 1, 0)^{\mathtt{t}}$,

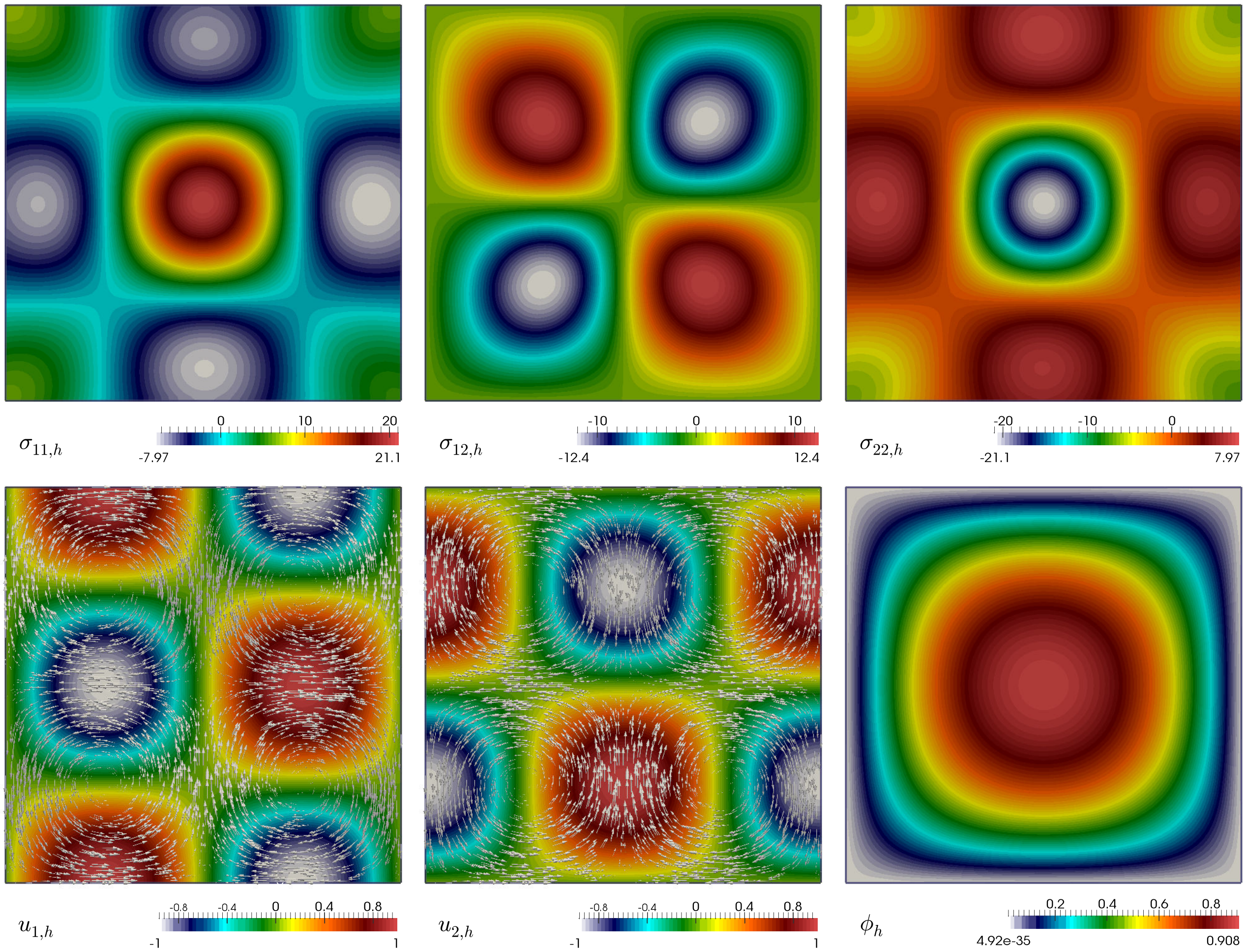

Figure 2: Example 1: $\mathrm{RT}_0 - P_1 - P_1$ approximation of stress components $\boldsymbol{\sigma}_h$ (top panels), velocity components $\boldsymbol{u}_h$ (with vector directions, bottom left and center, respectively), and scalar field ϕ_h (bottom right) solving (3.10). The mesh has 37249 vertices and 74496 triangular elements.

$\mu(\phi) = \mathrm{Re}^{-1} = (\mathrm{Ra}/\mathrm{Pr})^{-1/2}$, $\vartheta(\phi) = (\mathrm{Re}\,\mathrm{Pr})^{-1}$, $\gamma(\phi) = 0$, where $\mathrm{Pr} = 0.71$, Ra =1E05, and $\rho = 0.1$ are the Prandtl (ratio between the viscous and thermal diffusions), Rayleigh (only parameter remaining after nondimensionalization of the Boussinesq approximation), and overheat ratio coefficients, respectively. Notice that this problem is linear, except for the convection term. Even if the problem setting does not coincide exactly with the case analyzed previously, our goal is to illustrate the applicability of the present coupling strategy in diverse scenarios. In fact, if we redefine $\boldsymbol{f} := \frac{1}{\rho}(0,1,0)^{\mathtt{t}}$, then the functional (3.9) will eventually contain two additional terms independent of $\boldsymbol{f}$, and all the subsequent continuous and discrete analysis would remain unchanged after replacing $\boldsymbol{f}\phi$ by $\boldsymbol{f}\phi - \boldsymbol{f}$.

The stabilization constants are chosen as $\kappa_1 = \mu$, $\kappa_2 = 1/\mu$, and $\kappa_3 = \mu/2$. As boundary data we impose $\boldsymbol{u}_D = \mathbf{0}$ on the whole $\partial\Omega$, whereas we put $\phi = (2-\rho)/2$ at $x_1 = 0$ and $\phi = (2+\rho)/2$ at $x_1 = 1$. On the remainder of $\partial\Omega$ we impose zero-flux conditions for ϕ, that is $\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu} = 0$. The domain is discretized on a mesh $\mathcal{T}_h$ of 46656 vertices and 271950 tetrahedra, and we represent the field quantities of interest in Figure 3. From these plots we can observe a satisfactory qualitative agreement with respect to published data (see e.g. [9, 13, 20]).

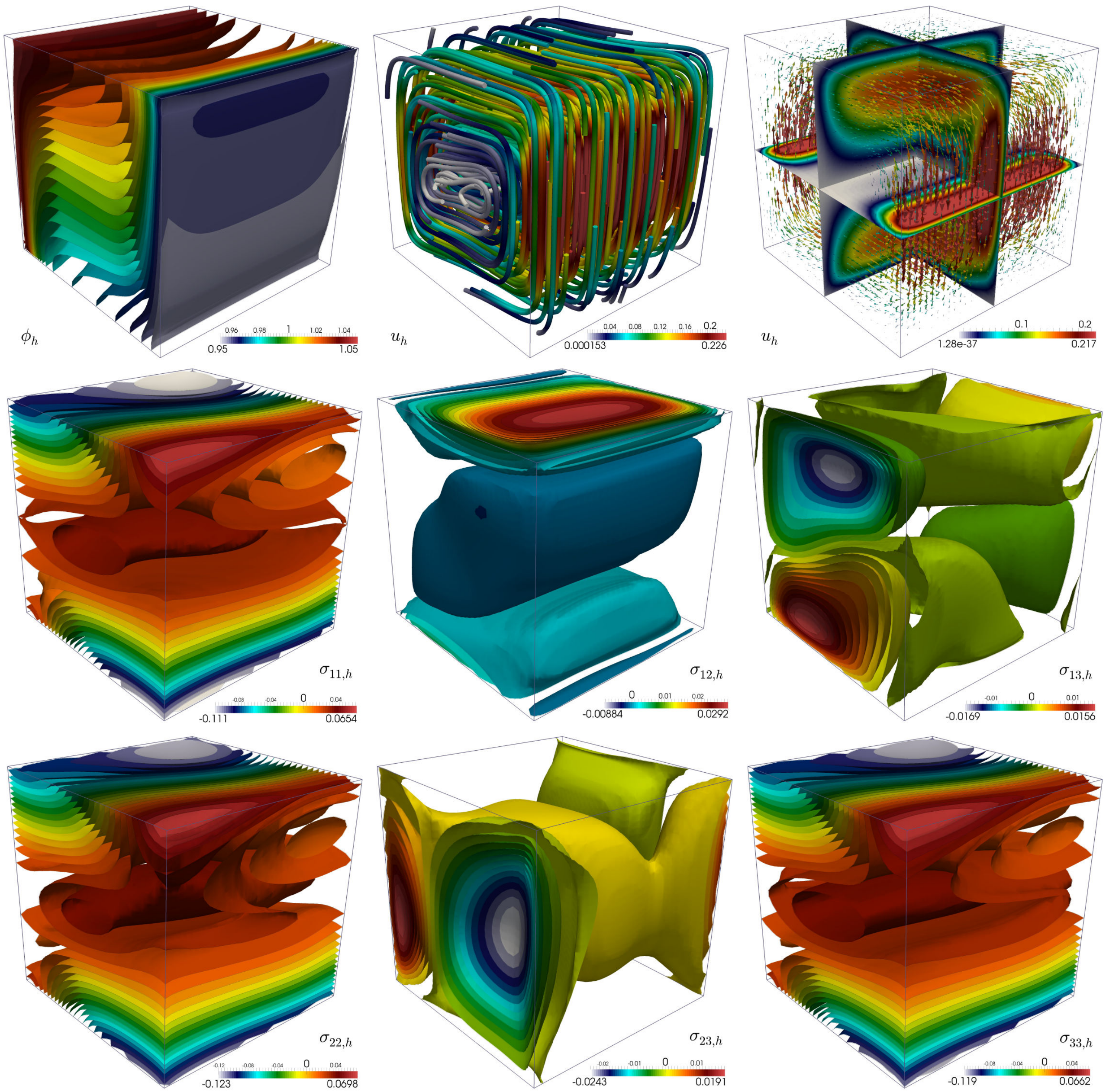

Figure 3: Example 2: Computed temperature iso-surfaces (top left) and velocity streamlines and vectors colored by magnitude (top center and right, respectively) and principal components of the Cauchy stress (center and bottom rows) for the thermal cavity test.

Moreover, Figure 4 reports on the mid-plane ($x_3 = 0.5$) profiles and a comparison with respect to values described in [13], including the average Nusselt number associated to a plane S (at fixed x_1) and computed as $\mathrm{Nu} = \int_S \mathrm{Pr}\,\mathrm{Re}\, u_1\phi - \partial_1\phi$. Our findings, after an average of 9 Newton iterations to reach a tolerance of 1E-08, satisfactorily match the benchmark data in terms of maximum and minimum velocities and temperature profiles at the symmetry lines $x_1 = 0.5$ and $x_2 = 0.5$. More quantitative comparisons are also presented in Table 2, where we have collected some outputs of interest for different values of the Rayleigh number. For larger Rayleigh numbers, an homotopy (or continuation) method was carried out on the Rayleigh number in order to ensure convergence of the algorithm.

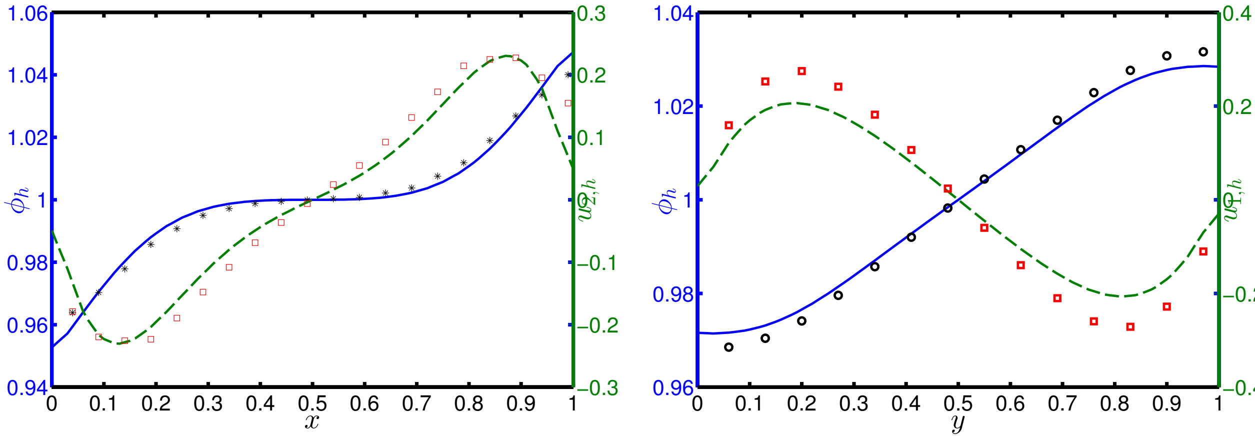

Figure 4: Example 2: Temperature profiles (solid blue, left axis) and velocity components (dashed green, right axis) at $x_3 = 0.5$, and comparison with respect to benchmark solutions.

	Ra	Nu	$\max(\lvert\hat{u}_{1,h}\rvert)$	$\max(\lvert\hat{u}_{2,h}\rvert)$	x_1^∞	x_2^∞
Computed	10^3	1.134	0.129	0.131	0.176	0.845
[9]	10^3	1.117	0.136	0.138	0.178	0.813
[13]	10^3	–	0.132	0.131	0.200	0.833
Computed	10^4	2.030	0.195	0.229	0.121	0.819
[9]	10^4	2.054	0.192	0.234	0.119	0.823
[13]	10^4	2.100	0.201	0.225	0.117	0.817
Computed	10^5	4.321	0.145	0.244	0.064	0.843
[9]	10^5	4.337	0.153	0.261	0.066	0.855
[13]	10^5	4.361	0.147	0.247	0.065	0.855

Table 2: Example 2: Outputs of interest (Nusselt number, maximum value of the normalized horizontal velocity on the mid-plane attained at $(0.5, x_2^\infty, 0.5)$, and maximum value of the normalized vertical velocity and its position $(x_1^\infty, 0.5, 0.5)$ on the central horizontal plane, respectively) for different values of the Rayleigh number, and comparison with respect to values from [9, 13].

Our last example focuses on the simulation of the steady state of a clarifying-thickening process. The basin, the different boundaries of the geometry, and the generated volumetric mesh consisting of 64135 vertices and 370597 tetrahedra are sketched in Figure 5. The size of the mesh and the finite element choice (row-wise Raviart-Thomas approximations for stresses and piecewise linear elements for velocity components and concentration) implies that at each Newton step we solve for a total of 2515211 degrees of freedom. The nonlinear functions of the concentration are taken as in [4]: $\mu(\phi) = (1-\phi/\phi_{\max})^{-2.5}$, $\gamma(\phi) = u_\infty(1+\phi(1-\phi/\phi_{\max}))^2$, $\vartheta(\phi) = \frac{\gamma(\phi)\sigma_0\alpha(\phi/\phi_c)^{\alpha-1}}{\phi\phi_c G\Delta\rho} + u_\infty$ and the source terms are $\boldsymbol{f} = (0,0,-G)^{\mathtt{t}}$, $g = 0$. The physical values assumed by the concentration (it remains bounded between 0 and $\phi_{\max}$) imply that the viscosity, hindered flux, and compressibility coefficients

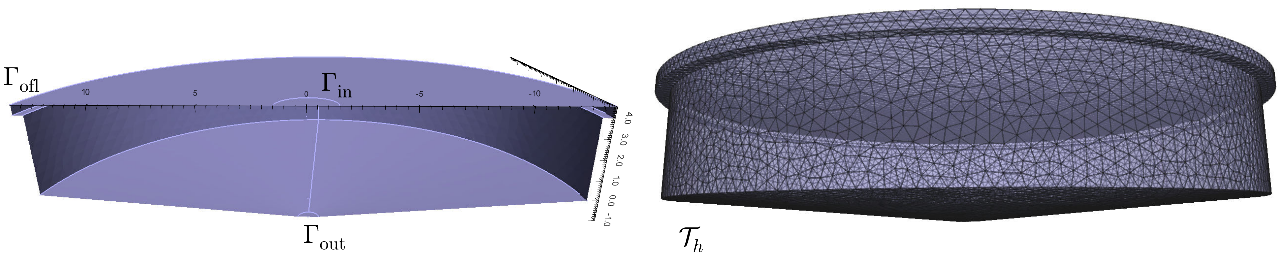

Figure 5: Example 3: Geometry of the clarifier-thickener unit (left panel) and tetrahedral mesh $\mathcal{T}_h$ with 64135 vertices and 370597 elements (right panel).

satisfy (2.2)-(2.3) with $\mu_1 = 1, \mu_2 = 2.7, \gamma_1 = u_\infty, \gamma_2 = 1.15u_\infty, \vartheta_1 = 4.28, \vartheta_2 = 29.74$. However, notice that ϑ depends explicitly on ϕ and not on the concentration gradient, which was not addressed in the solvability analysis of the model problem.

Boundary conditions are set as follows: Concentration and velocities are fixed on the inlet disc $\Gamma_{\rm in}$ according to $\phi = \phi_{\rm in}$ and $\boldsymbol{u} = \boldsymbol{u}_{\rm in} = (0, 0, -u_{3,\rm in})^{\tt t}$. At the outlet disk $\Gamma_{\rm out}$ we prescribe $\boldsymbol{u} = \boldsymbol{u}_{\rm out} = (0, 0, -u_{3,\rm out})^{\tt t}$, at the overflow annulus we do not constraint the velocity field, and on the remainder of $\partial\Omega$ we put no slip boundary data for the velocity and zero-flux conditions for the concentration. Model parameters are set as $u_{3,\rm in}$ =1.29E-02, $u_{3,\rm out}$ =2.54E-03, $\Delta\rho = 1562$, $\phi_{\max} = 0.9$, $\phi_c = 0.1$, u_∞ =2.2E-03, $G = 9.81$, $\phi_{\rm in} = 0.08$, $\alpha = 5$, and σ_0 =5E-02. We mimic the behavior of a transient simulation by adding a mass term $\eta\phi$ to the concentration equation, with η =1E-03. Such a modification does not entail a major change in the analysis: it suffices to replace the part of the flux $\phi\boldsymbol{u}$ by $\phi(\boldsymbol{u} + \eta)$.

According to the bounds of the viscosity, the stabilization parameters were set as $\kappa_1 = \kappa_2 = 0.4784$, and $\kappa_3 = 0.2392$. We mention that 8 Newton iterations were needed to achieve a tolerance of 1E-07 for the energy norm of the incremental approximations. The numerical results are depicted in Figure 6 (we show half of the tank for visualization purposes), including concentration profile, velocity vectors, pressure approximation (computed in terms of the trace of the Cauchy stress), and velocity components. We can observe that the material is removed from the unit at the boundary $\Gamma_{\rm out}$ with concentration $\phi \approx 0.24$, which agrees with the results in e.g. [3, Example 3].

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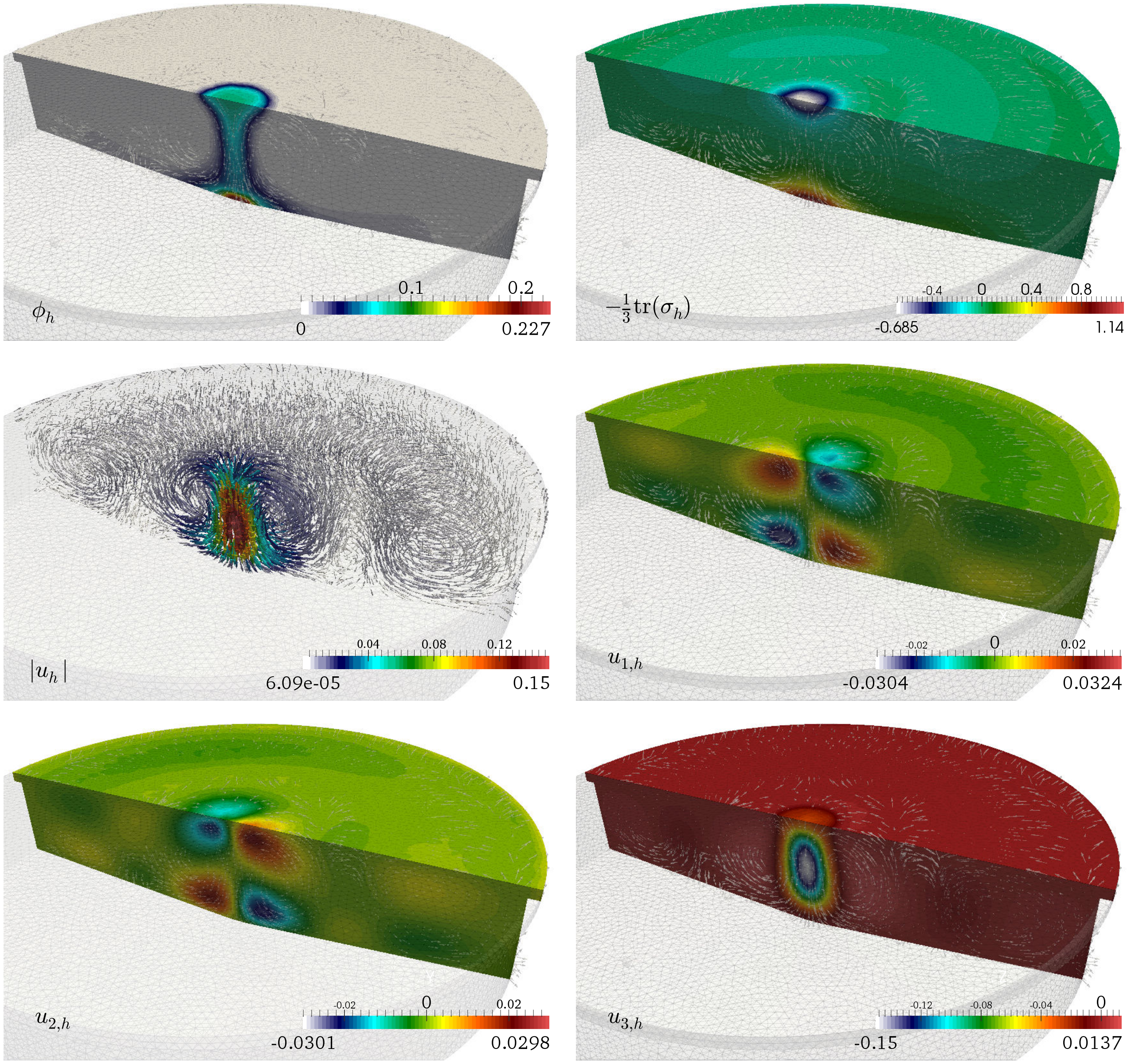

Figure 6: Example 3: Simulation of a clarifier-thickener unit. From left-top: Approximated concentration profile, opposite of the trace of the Cauchy stress tensor (which corresponds to the suggested approximation of the pressure field), and velocity components.

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